PROJECTION AND RAMIFICATION

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1. Introduction

If $X \subset \mathbf{P}^N$ is an n-dimensional smooth projective variety, not contained in any hyperplane, and if $L \subset \mathbf{P}^N$ is a general $(N - n - 1)$-dimensional plane, then the linear projection away from L

$$p_L : X \longrightarrow \mathbf{P}^n$$

is a finite surjective map with ramification divisor

$$R_L \subset X.$$

Our overall objective is to understand the assignment $L \mapsto R_L$. To pose our study more precisely, observe that the Riemann-Hurwitz formula gives an equivalence of divisor classes

$$R_L \sim R := K_X + (n+1)H$$

where H and K_X denote the hyperplane and canonical classes of X. We wish to understand basic properties of the *projection-ramification* rational map:

$$\begin{aligned} \rho_X : \mathbf{G}(N - n - 1, N) &\dashrightarrow |R| \\ L &\mapsto R_L \end{aligned}$$

A simple argument shows that ρ_X is a linear projection of $\mathbf{G}(N - n - 1, N)$ in its Plucker embedding. When X is a rational normal curve of degree N (over a field of characteristic zero), the map ρ_X is defined everywhere, and can be naturally identified with the *Wronskian* determinant, which assigns to a pencil $\langle p(x, y), q(x, y)\rangle$ of degree N binary forms the degree $2N - 2$ divisor on $\mathbf{P}^1$ defined by the determinant $p_x q_y - p_y q_x$. In this sense, our main object of study is a generalization of the classical Wronskian.

When X has dimension ≥ 2 however, ρ_X may not be defined on the entire Grassmannian, posing a serious challenge. This failure of regularity of ρ_X is equivalent, essentially by definition, to the existence of a linear space $L_0 \subset \mathbf{P}^N$ of dimension $N - n - 1$ which happens to be incident to *every* tangent plane of X.

Our primary focus centers around two questions:

(1) Is ρ_X generically finite onto its image?
(2) Is ρ_X ever dominant, and if so what is its degree?

Due to usual complications in positive characteristic, we work throughout over a characteristic zero field.

1.1. Results.

We first address the question of *maximal variation* which, to our knowledge, first appeared in work of Flenner and Manaresi [FM98]:

If X is a smooth variety, is ρ_X generically finite onto its image?

We answer the question affirmatively, at least for "most" varieties:

Theorem A. *Assume $X \subset \mathbf{P}^N$ is a non-degenerate variety over a field of characterstic zero, such that the dual variety $X^* \subset \mathbf{P}^{N*}$ is a hypersurface. Then ρ_X is generically finite onto its image.*

Our second result shows that maximal variation does not always hold:

Theorem B. *There exist smooth, non-degenerate rational normal scrolls $X^n \subset \mathbf{P}^N$ of every dimension $n \geq 4$ such that the projection-ramification map ρ_X is not generically finite onto its image.*

Our third structural result highlights the special role played by varieties of minimal degree:

Theorem C. *Suppose* $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$ *is a smooth, non-degenerate projective variety, with* $\rho_{\mathrm{X}} : \mathbf{G} \dashrightarrow |\mathrm{R}|$ *the projection-ramification map. Then*

$$\dim \mathbf{G} \leq \dim |\mathrm{R}|,$$

with equality holding if and only if X *is a variety of minimal degree.*

Theorem C naturally shifts our attention to rational normal scrolls. In particular, it is natural to wonder how common it is for a rational normal scroll to satisfy $\deg \rho_{\mathrm{X}} = 0$. Our fourth result states, roughly, that among balanced scrolls, this phenomenon only occurs in "low degree."

Theorem D. *Let* $\mathrm{X} = \mathbf{P}(\mathrm{V}) \subset \mathbf{P}^{\mathrm{N}}$ *be a rational normal scroll, where* V *is a balanced rank* n *ample vector bundle on* $\mathbf{P}^1$.

If $\deg \mathrm{V} \geq (n-1)(2n-1)+1$, *then* $\deg \rho_{\mathrm{X}} > 0$.

Theorem D is Theorem 5.15 in the text. It is proven via degeneration, using the theory of limit linear series.

1.2. **The projection-ramification enumerative problem.** Theorem C uncovers a large array of enumerative problems:

Problem 1.1. For each variety of minimal degree $\mathrm{X}^n \subset \mathbf{P}^{\mathrm{N}}$, determine the degree of ρ_{X}.

Problem 1.1 rapidly increases in difficulty with n. When X is a rational normal curve in $\mathbf{P}^{\mathrm{N}}$, it is well-known and easy to see that

$$\deg \rho_{\mathrm{X}} = \mathrm{C}_{\mathrm{N}}$$

where $\mathrm{C}_{\mathrm{N}} = \frac{(2\mathrm{N}-2)!}{\mathrm{N}!(\mathrm{N}-1)!}$ is the Catalan number [Gol91], as follows from the simple fact that for any non-degenerate smooth curve $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$, rational or otherwise, ρ_{X} is a *regular* linear projection of the Grassmannian known as the *Wronski map*. This immediately yields the equality $\deg \rho_{\mathrm{X}} = \deg \mathbf{G}(\mathrm{N}-2, \mathrm{N}) = \frac{(2\mathrm{N}-2)!}{\mathrm{N}!(\mathrm{N}-1)!}$ in this case.

Problem 1.1 is completely open in dimensions $n \geq 2$ – the complexity of the base scheme of ρ_{X} effectively blocks any straightforward application of the excess intersection formula. A search of the classical literature combined with some new arguments allows us to extract some answers to Problem 1.1 in higher dimensions. For the ease of the reader, we collect all known examples here:

Theorem E. *The following hold:*

1. *If* $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$ *is a rational normal curve, then* $\deg \rho_{\mathrm{X}} = \frac{(2\mathrm{N}-2)!}{\mathrm{N}!(\mathrm{N}-1)!}$.
2. *If* $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$ *is a quadric hypersurface, then* ρ_{X} *is an isomorphism.*
3. *If* $\mathrm{X} = \mathbf{P}^k \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^{2k+1}$ *is the Segre embedding, then* ρ_{X} *is birational.*
4. *If* $\mathrm{X} \subset \mathbf{P}^5$ *is the Veronese surface, then* $\deg \rho_{\mathrm{X}} = 3$.
5. *If* $\mathrm{X} \subset \mathbf{P}^5$ *is a general quartic surface scroll, then* $\deg \rho_{\mathrm{X}} = 2$.
6. *If* $\mathrm{X} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(k+1)) \subset \mathbf{P}^{k+3}$ *is the surface scroll with most imbalanced splitting type, then* ρ_{X} *is birational.*
7. *If* $\mathrm{X} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(k+1)) \subset \mathbf{P}^{k+5}$ *is the threefold scroll with most imbalanced splitting type, then* ρ_{X} *is birational.*

We now provide some remarks to clarify these theorems and to compare them with known results from the literature.

Remark 1.2. (1) *Theorem A is proven inductively by arguing that a general linear space* L *which is incident to* X *must be an isolated point in its fiber under* ρ_{X}. *Note that we do not assume* X *to be smooth in the statement of Theorem A. This requires us to be more precise about what is meant by the projection-ramification map* ρ_{X}, *but to state the* *conclusion informally: If we move a generically chosen complementary dimensional plane* $\mathrm{L} \subset \mathbf{P}^{\mathrm{N}}$*, then the corresponding ramification locus* $\mathrm{R}_{\mathrm{L}} \subset \mathrm{X}$ *also moves.*

(2) *The hypothesis in Theorem A is sufficient, but not necessary: If* $\mathrm{X} = \mathbf{P}^k \times \mathbf{P}^1 \subset \mathbf{P}^{2k+1}$, $k > 1$ *is the Segre embedding, then* X^* *is not a hypersurface, yet* ρ_{X} *is generically finite onto its image. In fact, Theorem E states in this case* ρ_{X} *is birational.*

(3) *The rational normal scrolls in Theorem B include some which are balanced. In lieu of Theorem C, we can rephrase the statement as: There exist rational normal scrolls for which* $\deg \rho_{\mathrm{X}} = 0$*. A natural problem emerges – to identify all rational normal scrolls with this property.*

Each example we produce is "explained by automorphisms" in the following sense: The action of $\mathrm{Aut}(\mathrm{X})$ *on the Grassmannian* $\mathbf{G}$ *has trivial generic stabilizer, yet its action on* $|\mathrm{R}|$ *has a positive dimensional generic stabilizer. It would be interesting to know if* $\deg \rho_{\mathrm{X}} = 0$ *is always caused by automorphisms.*

(4) *As Flenner and Manaresi point out in the paper* [FM04, Remark 2.6]*, their example from a previous paper* [FM98, Example 4.8] *does not provide an example of the failure of maximal variation of ramification divisors. Therefore Theorem B provides the first known examples of varieties with non-maximal variation of ramification divisors.*

(5) *To our knowledge, the following observation accounts for our collective understanding regarding the variation of ramification divisors (especially for* X *smooth) up until the writing of this paper:*

Observation 1. If X is a variety such that ρ_{X} is a regular map, then ρ_{X} is finite.

We call a variety X compressible *if* ρ_{X} *is not a regular map. Compressibility is equivalent to the existence of a linear space* L_0 *of dimension* $\mathrm{N} - n - 1$ *such that the projection* $p_{\mathrm{L}_0} : \mathrm{X} \dashrightarrow \mathbf{P}^n$ *is not dominant, thereby "compressing"* X*. Otherwise, we say* X *is* incompressible*.*

It is easy to see that (in char. 0*) every smooth hypersurface and every smooth, non-degenerate curve is incompressible. The cubic surface scroll* $\mathrm{X} \subset \mathbf{P}^4$ *serves as a first example of a compressible surface: projection from its directrix line is not dominant.*

Even so, Theorem A applies to all non-degenerate smooth surfaces, as the dual variety of such a surface is automatically divisorial. In light of Theorem B, the case of threefolds becomes particularly interesting – by the classification of varieties with small dual varieties, it suffices to investigate the question of maximal variation of ramification divisors for $\mathbf{P}^2$*-bundles over curves.*

(6) *A version of Theorem A is proven in Flenner and Manaresi's paper* [FM98, Theorem 3.4]*, under the assumption that* X *obeys a strong geometric condition: for every* $(\mathrm{N}-n-1)$*-plane* $\mathrm{L} \subset \mathbf{P}^{\mathrm{N}}$*, the join variety* $\mathrm{J}(\mathrm{L}, \mathrm{X})$ *equals* $\mathbf{P}^{\mathrm{N}}$*. If we additionally assume* X *to be smooth, this property directly implies the regularity of* ρ_{X} *and hence falls under Observation 1.*

If $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$ *is any smooth non-degenerate variety, then a sufficiently high degree Veronese re-embedding of* X *will have divisorial dual variety* X^**, and hence will satisfy the hypothesis of Theorem A. On the other hand, it is easy to see that a sufficiently high degree Veronese re-embedding of* X *is compressible, meaning Observation 1 does not apply.*

(7) *Flenner and Manaresi observe in* [FM98, Remark 3.5] *that for a smooth, non-degenerate variety* $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$*, the ampleness of the twisted normal bundle* $\mathrm{N}_{\mathrm{X}/\mathbf{P}^{\mathrm{N}}}(-1)$ *forces* X *to be incompressible, and hence such an* X *falls under the umbrella of Observation 1. In particular, Observation 1 applies to all non-degenerate smooth complete intersections.*

1.3. **Comparison with related work.**

1.3.1. *Wronski maps and Shapiro conjecture.* If $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$ is a non-degenerate smooth curve, then it immediately follows that ρ_{X} is regular. As we mentioned before, ρ_{X} is called the *Wronski map*. If we further assume $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$ is a rational normal curve, then we get the very well-studied finite projection:

$$\rho : \mathbf{G}(1, \mathrm{N}) \longrightarrow \mathbf{P}^{2\mathrm{N}-2}.$$

The geometry surrounding such Wronski maps has received a lot of attention thanks to the Shapiro conjecture which states that the preimage of any point in $\mathbf{P}^{2\mathrm{N}-2}$ corresponding to $2\mathrm{N}-2$ real points consists entirely of real points in $\mathbf{G}(1, \mathrm{N})$ (see [Sot00]; the conjecture was resolved by Eremenko and Gabrielov in [EG02]). When placed in this context, our Theorem C potentially sets the stage for a vast higher-dimensional generalization of the body of work around the Shapiro conjecture.

1.3.2. *Catalan numbers vs. Hurwitz numbers, and Chisini's conjecture.* Our Theorem C points to an interesting difference between enumerative problems concerning ramification divisors and branch divisors. Arguably, the most central enumerative problem concerning branch divisors, originating in the work of Hurwitz, is to compute the number of branched covers of the projective line with specified branch set in $\mathbf{P}^1$. As is well-known, these Hurwitz numbers exhibit remarkable structure.

In contrast, the calculation of the number of (equivalence classes of) degree N rational functions on $\mathbf{P}^1$ ramified at a prescribed set of $2\mathrm{N}-2$ general points is elementary, yielding the Catalan numbers.

In higher dimensions, however, the analogue of the Hurwitz enumerative problem becomes less interesting, thanks to Chisini's conjecture, now Kulikov's theorem [Kul99], which asserts that a branched cover $\mathrm{S} \longrightarrow \mathbf{P}^2$ with *generic branching* is uniquely determined by its branch divisor $\mathrm{B} \subset \mathbf{P}^2$, with a few finite exceptions. The corresponding enumerative problem regarding ramification divisors persists in all dimensions, thanks to Theorem C, and poses a significant challenge. We end our paper with empirical evidence suggesting that the occurrence of the combinatorial Catalan numbers is not an isolated event – there appears to be a mysterious combinatorial structure intertwined with the projection-ramification enumerative problems emerging from Theorem C.

2. Proof of Theorem A

Our proof of Theorem A proceeds inductively by showing that a general $(\mathrm{N}-n-1)$-plane which is incident to X is an isolated point in its fiber under ρ_{X}. In order to execute this strategy, we find it easier to work with abstract pairs (X, L) of a variety and a line bundle. Before proceeding, we fix some notation.

2.1. **Notation and conventions.** We work over an algebraically closed field k of characteristic 0 (We use Bertini's theorem and generic smoothness. We also appeal to the Kodaira Vanishing theorem.) By a *proper variety*, we mean a proper, integral, finite-type k-scheme. For any scheme X, we let X^s denote its smooth locus. If F is a coherent sheaf, we let $\mathrm{P}(\mathrm{F})$ denote its sheaf of principal parts. We will let $e : \mathrm{H}^0(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{P}(\mathrm{F})$ denote the natural evaluation morphism – we suppress the dependence on F. If s is a global section of a locally free sheaf, we let $v(s)$ denote the vanishing scheme of s. If L is a line bundle, we let $|\mathrm{L}|$ denote the projective space $\mathbf{P}(\mathrm{H}^0(\mathrm{L}))$. If $\mathrm{L} \longrightarrow \mathrm{Y}$ is a line bundle on a smooth variety, and $s \in \mathrm{H}^0(\mathrm{Y}, \mathrm{L})$ is a section, then the *singular scheme* $\mathrm{Sing}(v(s))$ of s is the vanishing scheme of $e(s) \in \mathrm{H}^0(\mathrm{Y}, \mathrm{P}(\mathrm{L}))$; if K, the kernel sheaf of

$e : \mathrm{H}^0(\mathrm{Y}, \mathrm{L}) \otimes \mathcal{O}_\mathrm{Y} \longrightarrow \mathrm{P}(\mathrm{L})$, is locally free, then $\mathrm{Sing}(v(s))$ is the largest closed subscheme $\mathrm{T} \subset \mathrm{Y}$ such that $s : \mathcal{O}_\mathrm{T} \longrightarrow \mathrm{H}^0(\mathrm{Y}, \mathrm{L}) \otimes \mathcal{O}_\mathrm{T}$ factors through $\mathrm{K}|_\mathrm{T}$.

2.2. **Non-defective line bundles.** Throughout, X will be a proper variety of dimension n. Rather than working with centers of projections in $\mathbf{P}^\mathrm{N}$, we will work with $n+1$ dimensional vector spaces of sections of the line bundle $\mathrm{L} = \mathcal{O}_\mathrm{X}(1)$. Furthermore, since we will ultimately want to blow up points on X, we will work in a setting where the line bundle L is not necessarily assumed to be very ample. The exact positivity property which we need is *non-defectivity*. We study this property in this section.

Definition 2.1. A line bundle $\mathrm{L} \longrightarrow \mathrm{X}$ is *non-defective* if for a general point $x \in \mathrm{X}$, the vanishing scheme of a general element of $\mathrm{H}^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_x^2)$ has an isolated singularity at x.

The *conormal variety* (of a line bundle $\mathrm{L} \longrightarrow \mathrm{X}$) will play an important role. To define it, we first note that there is a non-empty open set $\mathrm{U} \subset \mathrm{X}$ over which the kernel K of

$$e : \mathrm{H}^0(\mathrm{X}, \mathrm{L}) \otimes \mathcal{O}_\mathrm{U} \longrightarrow \mathrm{P}(\mathrm{L})|_\mathrm{U}$$

is locally free. The conormal variety, which we denote simply by P, is defined to be the closure of the projectivization $\mathbf{P}(\mathrm{K}|_\mathrm{U}) \subset \mathrm{X} \times |\mathrm{L}|$.

Proposition 2.2. *Let* $\mathrm{L} \longrightarrow \mathrm{X}$ *be a non-defective line bundle. If* $h^0(\mathrm{L}) \geq n+2$*, then the conormal variety* P *has dimension* $\dim \mathrm{P} = \dim |\mathrm{L}| - 1$*. If* $h^0(\mathrm{L}) \leq n+1$*, then* P *is empty.*

Proof. Since P is a subscheme of $\mathrm{X} \times |\mathrm{L}|$, it has a projection $\Sigma : \mathrm{P} \longrightarrow |\mathrm{L}|$ defined by $(x, s) \mapsto s$.

Over a nonempty open subset $\mathrm{U} \subset \mathrm{X}$, the kernel K of $e : \mathrm{H}^0(\mathrm{X}, \mathrm{L}) \otimes \mathcal{O}_\mathrm{U} \longrightarrow \mathrm{P}(\mathrm{L})|_\mathrm{U}$ is locally free of some rank r, and P is its projectivization. Observe that $r \geq \dim |\mathrm{L}| - n$; the statement of the proposition amounts to showing that if $r > 0$ then $r = \dim |\mathrm{L}| - n$.

If $s \in |\mathrm{L}|$ is any element, then we can view $\Sigma^{-1}(s)$ as a subscheme of X. We then have:

$$\Sigma^{-1}(s) \cap \mathrm{U} = \mathrm{Sing}(v(s|_\mathrm{U})).$$

So suppose $r > 0$. In particular, P is non-empty. Since L is non-defective, a general point $(x, s) \in \mathrm{P}$ is such that x is an isolated point in $\mathrm{Sing}(v(s))$. Therefore, we conclude that $\Sigma : \mathrm{P} \longrightarrow |\mathrm{L}|$ is generically finite onto its image, hence $\dim \mathrm{P} \leq \dim |\mathrm{L}|$.

To summarize, $\dim \mathrm{P} = \dim |\mathrm{L}|$ holds if and only if $\mathrm{P} \longrightarrow |\mathrm{L}|$ is surjective, and otherwise $\dim \mathrm{P} = \dim |\mathrm{L}| - 1$. Now we argue the former possibility is impossible by using Bertini's theorem. Let $\mathrm{B} \subset \mathrm{X}$ denote the base scheme of the linear series $|\mathrm{L}|$, and set $\widetilde{\mathrm{X}} = \mathrm{Bl}_\mathrm{B}\, \mathrm{X}$. Then, the linear series $|\mathrm{L}|$, viewed as a subseries of $|\pi^*\mathrm{L} - \mathrm{E}|$ on $\widetilde{\mathrm{X}}$ is basepoint free. The conormal variety

$$\widetilde{\mathrm{P}} := \overline{\{(x, \mathrm{D}) \mid x \in \widetilde{\mathrm{X}}^{sm} \text{ general, } x \in \mathrm{Sing}\, \mathrm{D}, \mathrm{D} \in |\mathrm{L}|\}} \subset \widetilde{\mathrm{X}} \times |\mathrm{L}|$$

would also dominate $|\mathrm{L}| \subset |\pi^*\mathrm{L} - \mathrm{E}|$ under second projection. By Bertini's theorem, the only way this can happen is if the locus $\widetilde{\mathrm{P}}^{sing} \subset \widetilde{\mathrm{P}}$ lying over $\mathrm{Sing}\, \widetilde{\mathrm{X}}$ dominated $|\mathrm{L}|$. Since we already know $\dim \widetilde{\mathrm{P}} \leq \dim |\mathrm{L}|$, $\widetilde{\mathrm{P}}^{sing}$ would then contain an irreducible component of $\widetilde{\mathrm{P}}$ contradicting the irreducibility of $\widetilde{\mathrm{P}}$. Hence, $\mathrm{P} \longrightarrow |\mathrm{L}|$ could not be dominant, and we are done. □

Proposition 2.3. *Suppose* $\mathrm{L} \longrightarrow \mathrm{X}$ *is a line bundle with* $h^0(\mathrm{X}, \mathrm{L}) \geq n+2$*, and let* P *denote its conormal variety. The projection* $\Sigma : \mathrm{P} \longrightarrow |\mathrm{L}|$ *is generically finite onto its image if and only if* L *is non-defective.*

Proof. The proof closely follows the reasoning in the proof of Proposition 2.2, and hence we leave the details to the reader. □

Proposition 2.4. *Let* $\mathrm{L}\longrightarrow\mathrm{X}$ *be a non-defective line bundle, with* $h^0(\mathrm{X},\mathrm{L}) \geq n+2$*. If* $x \in \mathrm{X}$ *is a general point, then a general element* $s \in \mathrm{H}^0(\mathrm{X}, \mathrm{L}\otimes\mathfrak{m}_x^2)$ *has an ordinary double point singularity at* x*.*

Proof. As in the proof of Proposition 2.2, we know that the projection $\Sigma : \mathrm{P}\longrightarrow|\mathrm{L}|$ is generically finite onto its image. Since our ground field has characteristic zero, at a general point $(x, s) \in \mathrm{P} \subset \mathrm{X} \times |\mathrm{L}|$, the projection Σ is a local immersion. In other words, (1) $x \in \mathrm{Sing}(v(s))$ is isolated, and (2) with scheme structure inherited from $\mathrm{Sing}(v(s))$, x is a reduced point. (Note: we are allowed to assume $x \in \mathrm{X}^s$.)

These two properties imply that $v(s)$ possesses an ordinary double point at x: choose local coordinates $(x_1, ..., x_n)$ so that the complete local ring $\widehat{\mathcal{O}}_{\mathrm{X},x}$ is isomorphic to $k[[x_1, ..., x_n]]$. Our claim then follows from the next statement, which we leave to the reader to check: Let $s = s(x_1, ..., x_n) \in \mathfrak{m}_x^2$ be a power series. Then the partial derivatives $\partial_1 s, ..., \partial_n s$ induce linearly independent elements in $\mathfrak{m}_x/\mathfrak{m}_x^2$ if and only if the tangent cone of s is a non-degenerate quadratic cone. □

Proposition 2.5. *If* $\mathrm{L}\longrightarrow\mathrm{X}$ *is a non-defective line bundle with* $h^0(\mathrm{L}) \geq n+1$*, then the global sections of* L *separate tangent vectors at a general point* $x \in \mathrm{X}$*, i.e.*

$$e_x : \mathrm{H}^0(\mathrm{X}, \mathrm{L}) \otimes k(x)\longrightarrow\mathrm{P}(\mathrm{L}) \otimes k(x)$$

is surjective for $x \in \mathrm{X}$ *general.*

Proof. This follows from the proof of Proposition 2.2: Let K be the kernel sheaf of $e : \mathrm{H}^0(\mathrm{L}) \otimes \mathcal{O}_{\mathrm{X}}\longrightarrow\mathrm{P}(\mathrm{L})$. Then it was shown in the proof of Proposition 2.2 that the generic rank of K is $r = h^0(\mathrm{X}, \mathrm{L}) - (n + 1)$, which in turn implies that e is generically surjective. □

Proposition 2.6. *Let* $\mathrm{L}\longrightarrow\mathrm{X}$ *be a non-defective line bundle, let* $x \in \mathrm{X}$ *be a general point, let* $\pi : \widetilde{\mathrm{X}}\longrightarrow\mathrm{X}$ *the blow up of* X *at* x*, and let* E *denote the exceptional divisor of* π*. Further assume that* $h^0(\mathrm{L}) \geq n+2$*.*

Then $\pi^*\mathrm{L}(-\mathrm{E})\longrightarrow\widetilde{\mathrm{X}}$ *is also non-defective.*

Proof. As a consequence of Proposition 2.2, at a general point $y \in \mathrm{X}$, we get

$$h^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_y^2) = h^0(\mathrm{X}, \mathrm{L}) - (n + 1).$$

By non-defectivity of L, at the general point $y \in \mathrm{X}^s$ there is an element $t_1 \in \mathrm{H}^0(\mathrm{X}, \mathfrak{m}_y^2 \otimes \mathrm{L})$ such that $v(t_1)$ has an isolated singularity at y.

Now we can choose a second general point $x \in \mathrm{X}^s \setminus v(t_1)$. Notice that by Proposition 2.5, L separates tangent vectors at x, and therefore we get an isomorphism

$$\mathrm{H}^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_x) \simeq \mathrm{H}^0(\widetilde{\mathrm{X}}, \pi^*\mathrm{L}(-\mathrm{E})).$$

Furthermore, we can choose x so that the following hold:

(1) $h^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_y^2 \cdot \mathfrak{m}_x) = h^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_y^2) - 1$, and
(2) a general nonzero element $t \in \mathrm{H}^0(\mathrm{X}, \mathrm{L}\otimes\mathfrak{m}_y^2 \cdot \mathfrak{m}_x)$ has an isolated singularity at y, provided it exists.

With x chosen as above, these two conditions imply that the conormal variety $\widetilde{\mathrm{P}}$ for the line bundle $\pi^*\mathrm{L}(-\mathrm{E})\longrightarrow\widetilde{\mathrm{X}}$ over $\widetilde{\mathrm{X}} = \mathrm{Bl}_x \mathrm{X}$ has the expected dimension according to Proposition 2.2, and, provided t exists, that the point (y, t) is a point in $\widetilde{\mathrm{P}}$ around which the projection $\Sigma : \widetilde{\mathrm{P}}\longrightarrow|\pi^*\mathrm{L}(-\mathrm{E})|$ is quasi-finite onto its image. We conclude using Proposition 2.3. □

2.3. Projections.

Definition 2.7. A *projection* is a pair
$$(\mathrm{L}, \mathrm{V})$$
where $\mathrm{L}\longrightarrow\mathrm{X}$ is a line bundle and $\mathrm{V} \subset \mathrm{H}^0(\mathrm{X}, \mathrm{L})$ is a subspace of dimension $n+1$.

Definition 2.8. A projection (L, V) is *properly ramified* if the evaluation homomorphism $e : \mathrm{V} \otimes \mathcal{O}_{\mathrm{X}}\longrightarrow\mathrm{P}(\mathrm{L})$ is an isomorphism over a general point in X. If (L, V) is properly ramified, its *ramification divisor*
$$\mathrm{R}(\mathrm{V}) \subset \mathrm{X}^s$$
is the scheme defined by the determinant of $e : \mathrm{V} \otimes \mathcal{O}_{\mathrm{X}^s}\longrightarrow\mathrm{P}(\mathrm{L})|_{\mathrm{X}^s}$.

If $\mathrm{L}\longrightarrow\mathrm{X}$ is fixed, then the set of all projections (L, V) is parametrized by the Grassmannian
$$\mathrm{G}(n+1, \mathrm{H}^0(\mathrm{X}, \mathrm{L})).$$

Corollary 2.9. *Suppose* $\mathrm{L}\longrightarrow\mathrm{X}$ *is a non-defective line bundle satisfying* $h^0(\mathrm{X}, \mathrm{L}) \geq n+1$*. Then there exists a properly ramified projection* (L, V)*.*

Proof. This follows immediately from Proposition 2.5. □

Next we define the ramification divisor of a projection. If $\mathrm{L}\longrightarrow\mathrm{X}$ is a line bundle over an n-dimensional variety X, then the sheaf $\mathrm{P}(\mathrm{L})$ is locally free of rank $n+1$ over X^s, and $\wedge^{n+1}\mathrm{P}(\mathrm{L})|_{\mathrm{X}^s} \simeq \mathrm{K}_{\mathrm{X}^s} \otimes \mathrm{L}^{\otimes n+1}$.

Therefore, by applying $\wedge^{n+1}$ to $e : \mathrm{H}^0(\mathrm{X}, \mathrm{L}) \otimes \mathcal{O}_{\mathrm{X}^s}\longrightarrow\mathrm{P}(\mathrm{L})|_{\mathrm{X}^s}$, we get a map of vector spaces of global sections:
$$\wedge^{n+1}\mathrm{H}^0(\mathrm{X}, \mathrm{L})\longrightarrow\mathrm{H}^0(\mathrm{X}^s, \mathrm{K}_{\mathrm{X}^s} \otimes \mathrm{L}^{\otimes n+1}). \tag{2.1}$$

Definition 2.10. Let $\mathrm{L}\longrightarrow\mathrm{X}$ be a line bundle which separates tangent vectors at a general point of X. The *projection-ramification* map for L is the composite:
$$\rho_{(\mathrm{X},\mathrm{L})} : \mathrm{G}(n+1, \mathrm{H}^0(\mathrm{L})) \hookrightarrow \mathbf{P}(\wedge^{n+1}\mathrm{H}^0(\mathrm{L})) \dashrightarrow \mathbf{P}(\mathrm{H}^0(\mathrm{X}^s, \mathrm{K}_{\mathrm{X}^s} \otimes \mathrm{L}^{\otimes n+1}))$$
rationally defined by $(\mathrm{L}, \mathrm{V}) \mapsto \mathrm{R}(\mathrm{V})$. Here the dashed arrow comes from 2.1.

Corollary 2.9 implies that the projection-ramification map $\rho_{(\mathrm{X},\mathrm{L})}$ exists whenever $\mathrm{L}\longrightarrow\mathrm{X}$ is non-defective and $h^0(\mathrm{X}, \mathrm{L}) \geq n+1$.

2.4. Proof of main result.

We proceed to the proof of Theorem A. More specifically, we prove Theorem 2.12 below, which is more precise and implies Theorem A.

Definition 2.11. A projection (L, V) is *isolated* if it is properly ramified and if $[\mathrm{V}] \in \mathrm{G}(n+1, \mathrm{H}^0(\mathrm{X}, \mathrm{L}))$ is an isolated point in the corresponding fiber of $\rho_{(\mathrm{X},\mathrm{L})}$.

Our main next objective is to show:

Theorem 2.12. *If* $\mathrm{L}\longrightarrow\mathrm{X}$ *is a non-defective line bundle satisfying* $h^0(\mathrm{X}, \mathrm{L}) \geq n+2$*, then there exists an isolated projection* (L, V)*.*

Lemma 2.13. *Let* $\mathrm{L}\longrightarrow\mathrm{X}$ *be a non-defective line bundle with* $h^0(\mathrm{L}) \geq n+2$*, and let* $x \in \mathrm{X}$ *be a general point.*

Suppose $\mathrm{V} \subset \mathrm{H}^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_x)$ *is a general* $(n+1)$*-dimensional subspace. Then the ramification divisor* $\mathrm{R}(\mathrm{V})$ *has an ordinary double point singularity at* x*.*

Proof. Using Proposition 2.5 and Proposition 2.4, we can assume V has a basis $(s_1, ..., s_n, t)$ such that:

(1) $(s_1, ..., s_n)$ generate $\mathrm{L} \otimes (\mathfrak{m}_x/\mathfrak{m}_x^2)$ and
(2) $v(t)$ has an ordinary double point singularity at x.

Let $\widehat{\mathcal{O}}_{\mathrm{X},x}$ denote the completion of the local ring at $x \in \mathrm{X}$ along its maximal ideal. Upon trivializing L, we may regard s_i and t as elements of $\widehat{\mathcal{O}}_{\mathrm{X},x}$, and also assume that $\widehat{\mathcal{O}}_{\mathrm{X},x} = k[[s_1, ..., s_n]]$. A local matrix representative for the evaluation map

$$e : \mathrm{V} \otimes \widehat{\mathcal{O}}_{\mathrm{X},x} \longrightarrow \mathrm{P(L)} \otimes \widehat{\mathcal{O}}_{\mathrm{X},x}$$

is given by

$$\begin{pmatrix} s_1 & s_2 & \dots & t \\ 1 & 0 & \dots & \partial_1 t \\ 0 & 1 & \dots & \partial_2 t \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \partial_n t \end{pmatrix} \tag{2.2}$$

Here, ∂_i denotes $\frac{d}{ds_i}$. The determinant of 2.2 is

$$t - \sum_i s_i \partial_i t,$$

and is an analytic local equation for the ramification divisor R(V) near x. It is then immediately clear that R(V) shares the same tangent cone as $v(t)$ at x, and hence the proposition follows. □

Lemma 2.14. *Maintain the same setting as in Lemma 2.13, except now assume* $\mathrm{V} \subset \mathrm{H}^0(\mathrm{X}, \mathrm{L})$ *has a basis of the form* $(u, a_1, ..., a_{n-1}, b)$ *where*

(1) u *does not vanish at* x*,*
(2) $a_1, ..., a_{n-1}$ *all vanish at* x *and induce linearly independent elements in* $\mathrm{L} \otimes (\mathfrak{m}_x/\mathfrak{m}_x^2)$*, and*
(3) $v(b)$ *has an ordinary double point at* x*.*

Then $x \in \mathrm{R(V)}$ *and* R(V) *is smooth at* x*.*

Proof. That $x \in \mathrm{R(V)}$ is clear since $\mathrm{V} \cap \mathrm{H}^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_x^2) \neq 0$.

Again we work in the completion $\widehat{\mathcal{O}}_{x,\mathrm{X}}$, and after trivializing L, we assume $u, a_1, ..., b$ are elements of $\widehat{\mathcal{O}}_{x,\mathrm{X}}$.

We choose an element $z \in \widehat{\mathcal{O}}_{x,\mathrm{X}}$ in such a way that $(a_1, ..., a_{n-1}, z)$ form a system of coordinates, i.e. $\widehat{\mathcal{O}}_{x,\mathrm{X}} \simeq k[[a_1, ..., a_{n-1}, z]]$.

With respect to these coordinates, a local matrix representative for

$$e : \mathrm{V} \otimes \widehat{\mathcal{O}}_{x,\mathrm{X}} \longrightarrow \mathrm{P(L)} \otimes \widehat{\mathcal{O}}_{x,\mathrm{X}}$$

has the form

$$\begin{pmatrix} u & a_1 & a_2 & \dots & b \\ \partial_1 u & 1 & 0 & \dots & \partial_1 b \\ \partial_2 u & 0 & 1 & \dots & \partial_2 b \\ \vdots & \vdots & \vdots & \vdots & \\ \partial_z u & 0 & 0 & \dots & \partial_z b \end{pmatrix} \tag{2.3}$$

The determinant of 2.3 is the analytic local equation for R(V):

$$\bar{u} \cdot \partial_z b \pm \partial_z u \cdot \bar{b}$$

Here, for any $r \in \widehat{\mathcal{O}}_{x,\mathrm{X}}$ we set

$$\bar{r} := r - a_1 \partial_1 r - a_2 \partial_2 r - ... - z \partial_z r.$$

Since $b \in \mathfrak{m}_x^2$, we get $\bar{b} \in \mathfrak{m}_x^2$ and so $\partial_z b \in \mathfrak{m}_x$. Furthermore, $\bar{u}$ is a unit. Therefore, $\mathrm{R(V)}$ is singular at x if and only if:

$$\partial_z b \in \mathfrak{m}_x^2,$$

but this would mean that the tangent cone of $v(b)$ is degenerate, contrary to our third assumption. So $\mathrm{R(V)}$ is smooth at x. □

Now we are ready for the proof of Theorem 2.12.

Proof of Theorem 2.12. We induct on $h^0(\mathrm{L})$. First, assume $h^0(\mathrm{X}, \mathrm{L}) = n+2$ and choose $x \in \mathrm{X}$ a general point. Then $\mathrm{V} = \mathrm{H}^0(\mathrm{X}, \mathfrak{m}_x \otimes \mathrm{L})$ satisfies the conditions in Lemma 2.13 and in particular V is properly ramified and $\mathrm{R(V)}$ has an ordinary double point at x.

Suppose we have a 1-parameter family of $(n+1)$-dimensional subspaces $\mathrm{V}_c \subset \mathrm{H}^0(\mathrm{X}, \mathrm{L})$ parametrized by points c of a smooth pointed curve $(\mathrm{C}, 0)$ such that

(1) $\mathrm{V}_0 = \mathrm{V}$
(2) $\mathrm{V}_c \neq \mathrm{V}_0$ for $c \in \mathrm{C}$, $c \neq 0$.

We must prove $\mathrm{R}(\mathrm{V}_c) \neq \mathrm{R(V)}$ for a general point $c \in \mathrm{C}$.

By upper semicontinuity,

$$\dim \left(\mathrm{V}_c \cap \mathrm{H}^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_x^2)\right) \leq 1.$$

If $\dim \mathrm{V}_c \cap \mathrm{H}^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_x^2) = 0$, then $x \notin \mathrm{R}(\mathrm{V}_c)$ and therefore $\mathrm{R}(\mathrm{V}_c) \neq \mathrm{R(V)}$. Otherwise, if $\dim \mathrm{V}_c \cap \mathrm{H}^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_x^2) = 1$ then a nonzero section $b \in \mathrm{V}_c \cap \mathrm{H}^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_x^2)$ must also possess an ordinary double point singularity at x, as this is an open condition on families of singularities.

Since $\mathrm{V}_c \neq \mathrm{V}_0 = \mathrm{H}^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m})$, there exists a section $u \in \mathrm{V}_c$ not vanishing at x. This implies that V_c obeys the hypotheses in Lemma 2.14. But then the conclusion of Lemma 2.14 implies $\mathrm{R}(\mathrm{V}_c) \neq \mathrm{R}(\mathrm{V}_0)$ as the former is smooth while the latter is singular at x, establishing our initial case.

Next suppose $h^0(\mathrm{X}, \mathrm{L}) > n+2$, and again choose a general point $x \in \mathrm{X}$ so that Proposition 2.6 holds. Choose a general $(n+1)$-dimensional subspace

$$\mathrm{W} \subset \mathrm{H}^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_x)$$

so as to obey the conditions in Lemma 2.13. By induction hypothesis and Proposition 2.6, the projection $(\pi^*\mathrm{L}(-\mathrm{E}), \mathrm{W})$ is isolated, where $\pi : \widetilde{\mathrm{X}} \longrightarrow \mathrm{X}$ is the blow up at x.

So assume we have a 1-parameter family W_c parameterized by a pointed curve $(\mathrm{C}, 0)$ satisfying

(1) $\mathrm{W}_0 = \mathrm{W}$,
(2) $\mathrm{W}_c \neq \mathrm{W}_0$ for $c \in \mathrm{C}$ general,
(3) W_c is not contained in $\mathrm{H}^0(\mathrm{X}, \mathrm{L} \otimes \mathfrak{m}_x)$ for $c \in \mathrm{C}$ general.

We must verify that $\mathrm{R}(\mathrm{W}_c) \neq \mathrm{R}(\mathrm{W}_0)$. Again by upper semicontinuity and openness (among singularities) of possession of ordinary double point, we find that W_c meets the conditions of Lemma 2.14. We conclude $\mathrm{R}(\mathrm{W}_c) \neq \mathrm{R}(\mathrm{W}_0)$ since the former is smooth while the latter is singular at x, completing the argument. □

Recall that if $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$ is a projective variety then its dual variety $\mathrm{X}^* \subset \mathbf{P}^{\mathrm{N}*}$ is the image of the conormal variety P (associated to the line bundle $\mathcal{O}_{\mathrm{X}}(1)$) under the projection $\Sigma : \mathrm{P} \longrightarrow |\mathcal{O}_{\mathrm{X}}(1)| \simeq \mathbf{P}^{\mathrm{N}*}$. We immediately get Theorem A:

Corollary 2.15. *Let $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$ be a non-degenerate projective variety such that the dual variety $\mathrm{X}^* \subset \mathbf{P}^{\mathrm{N}*}$ is a hypersurface. Then $\rho_{\mathrm{X}, \mathcal{O}_{\mathrm{X}}(1)}$ is generically finite onto its image.*

Proof. Indeed, since $\mathcal{O}_X(1)$ separates tangent vectors at a general point of X, the condition that X^* is a hypersurface implies that the projection $\Sigma : P \longrightarrow |\mathcal{O}_X(1)|$ is generically finite onto X^*, and hence $\mathcal{O}_X(1)$ is non-defective by Proposition 2.3. Thus, Theorem 2.12 applies. $\square$

3. Proof of Theorem B

Our next objective is to prove Theorem B by exhibiting some examples. Before doing so, we pose the general problem of maximal variation for rational normal scrolls in explicit affine coordinates.

3.1. **The generalized Wronski map in affine coordinates.** Fix variables $x_1, \dots, x_n, t$.

Definition 3.1. Let $\underline{d} = (d_1, \dots, d_n)$ denote an n-tuple of degrees. We define $V(\underline{d})$ to be the vector space of forms $\sum_{i=1}^n p_i(t)x_i$, where $\deg p_i \le d_i$.

Remark 3.2. *$V(\underline{d})$ is simply the space of global sections of the line bundle $\mathcal{O}_{\mathbf{P}E}(1)$ on the scroll $\mathbf{P}E$ over $\mathbf{P}^1$, where $E = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_n)$.*

Next, if $v_1 \wedge \cdots \wedge v_{n+1} \in \bigwedge^{n+1} V(\underline{d})$ is any pure tensor, we set

$$Wr(v_1 \wedge \cdots \wedge v_{n+1}) := \det \begin{pmatrix} — & v_1 & — & v_1' \\ — & v_2 & — & v_2' \\ \vdots & \vdots & \vdots & \vdots \\ — & v_{n+1} & — & v_{n+1}' \end{pmatrix} \in V(\underline{e}) \tag{3.1}$$

where $\underline{e} = (e_1, \dots, e_n)$ is given by $e_i = d_i - 2 + \sum_{j=1}^n d_j$.

Definition 3.3. The induced map

$$Wr_{\underline{d}} : Gr(n+1, V(\underline{d})) \dashrightarrow \mathbf{P}V(\underline{e}) \tag{3.2}$$

is called the *Wronskian* map.

Remark 3.4. *$Wr_{\underline{d}}$ is equal to the projection-ramification map for the scroll $X = \mathbf{P}E$.*

The dimensions of source and target of the Wronskian map are equal, hence we may pose the general question:

Problem 3.5. For which degree vectors $\underline{d} = (d_1, ..., d_n)$ is the Wronskian map $Wr_{\underline{d}}$ dominant?

In the next section, we show that Problem 3.5 is genuinely interesting by demonstrating that $Wr_{\underline{d}}$ fails to be dominant for degree vectors of the form $(1, 1, 1, \dots, k+1)$, $n \ge 4$.

3.2. **Proof of Theorem B.** Let $E = \mathcal{O}(1)^{n-1} \oplus \mathcal{O}(k+1)$ be the vector bundle over $\mathbf{P}^1$, and set $X = \mathbf{P}E$. We will prove:

Theorem 3.6. *The projection-ramification map for the embedding of X given by $\mathcal{O}_E(1)$ is not dominant once $k(n-3) > 1$.*

Remark 3.7. *The basic phenomenon underlying this example is: a general point in the source Grassmannian has trivial $Aut(X)$-stabilizer, yet every point of $|R|$ has positive dimensional stabilizer.*

Remark 3.8. *If $k = 1$ and $n \ge 5$, then X is a balanced scroll. Therefore, the non-dominance of projection-ramification is not directly connected to the eccentricity of the splitting type of a scroll. Rather, among balanced scrolls, non-dominance of ρ_X happens only in "low" degree – see $\boxed{\textit{ToDo: CITE ANAND THEOREM}}$.*

As an immediate corollary, we get a result concerning Grassmannians in their Plucker embeddings. Recall that an n dimensional variety $X \subset \mathbf{P}^N$ is *compressible* if there exists a $(N-n-1)$-dimensional linear space $\Lambda \subset \mathbf{P}^N$ with the property that the projection $p_\Lambda : X \dashrightarrow \mathbf{P}^n$ is not dominant.

Proof of Theorem 3.6. We will show that the general element in the ramification divisor class $|R|$ has a positive dimensional stabilizer under the action of $Aut(X)$. We leave it to the reader to check that the Grassmannian $\mathbf{G}$ does not have generic stabilizer under Aut(X). The theorem then follows by the $Aut(X)$-equivariance of $\rho : \mathbf{G} \dashrightarrow |R|$.

In terms of the affine coordinates $(x_1, \ldots, x_n, t)$ introduced in the previous section, we find ourselves in the situation corresponding to the degree vector $\underline{d} = (1, ..., 1, k+1)$. The degree vector corresponding to ramification divisors is then $\underline{e} = (n+k+1, n+k+1, \ldots, n+2k+1)$.

In these affine coordinates, the substitutions

$$
\begin{aligned}
x_1 &\mapsto x_1 + p_1(t)x_n \\
x_2 &\mapsto x_2 + p_2(t)x_n \\
&\vdots \\
x_n &\mapsto x_n
\end{aligned}
$$

produce distinct automorphisms in $Aut(X)$ per choice of the p_i, where each p_i has degree $\leq k$.

If $\sum_{j=1}^n a_j x_j$ represents a general element of $V(\underline{e})$, then the above substitutions have the effect of replacing the coefficient $a_n(t)$ with $a_n + \sum_{j=1}^{n-1} a_j p_j$.

Now, if $(n-3)k > 1$ then the dimension of the vector space of choices for the polynomials p_i exceeds the dimension of degree $n+2k+1$ polynomials a_n. Hence there exist particular p_i's (not all zero) such that the above automorphism fixes the equation $\sum_{j=1}^n a_j x_j$. Scaling these particular p_i's by constants produces the positive dimensional stabilizer we seek. □

We obtain the following immediate corollary:

Corollary 3.9. *The Grassmannian* $G(r, k)$ *is compressible if* $5 \leq r \leq k-r$. ToDo: Is this obvious? Can we find r

4. Proof of Theorem C

ToDo: Can we make it valid in characteristic p? Come up with an argument which avoids Kodaira vanishing

In this section, we provide the simple proof of Theorem C.

Lemma 4.1. *Let* Y *be a smooth projective* m*-dimensional variety with* $m > 0$*, and let* H *be an ample divisor class on* Y *which induces a morphism*

$$Y \longrightarrow \mathbf{P}^n.$$

Then

$$h^0(K_Y + mH) \geq n - m.$$

Proof. We proceed by induction on $m = \dim Y$. First assume $m > 1$. Let $D \subset Y$ be a general divisor (smooth by Bertini's theorem) in the linear system $|H|$, and consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_Y(K_Y + (m-1)H) \longrightarrow \mathcal{O}_Y(K_Y + mH) \longrightarrow \mathcal{O}_D(K_D + mH) \longrightarrow 0.$$

The Kodaira Vanishing Theorem states that $h^1(\mathcal{O}_Y(K_Y + (m-1)H) = 0$, and hence

$$h^0(Y, \mathcal{O}_Y(K_Y + mH)) \geq h^0(D, \mathcal{O}_D(K_D + (m-1)H)).$$

Notice that $h^0(\mathcal{O}_D(H)) \geq n$, and therefore by induction we know:

$$h^0(D, \mathcal{O}_D(K_D + (m-1)H)) \geq (n-1) - (m-1) = n - m,$$

and hence the proposition follows once we settle the case $m = 1$.

But in this case, the inequality

$$h^0(\mathcal{O}_Y(K_Y + H)) \geq h^0(\mathcal{O}_Y(H)) - 2$$

follows at once from Riemann-Roch, and the fact that a degree d line bundle on Y cannot have more than $d + 1$ linearly independent global sections. □

Proof of Theorem C. We first show: If $\dim \mathbf{Gr}(n-m, n+1) \geq \dim |K_X + (m+1)H|$, then X is a variety of minimal degree. Then we argue that in the case of minimal varieties, this inequality is actually an equality.

We proceed by intersecting with a hyperplane: let $X' = X \cap H$ be a general hyperplane section of X. By combining the Kodaira vanishing theorem, adjunction, and Lemma 4.1 we get:

$$h^0(\mathcal{O}_X(K_X + (m+1)H)) - h^0(\mathcal{O}_{X'}(K_{X'} + mH)) \geq n - m.$$

Therefore, the inequality

$$\dim \mathbf{Gr}(n-m, n+1) \geq \dim |K_X + (m+1)H|$$

implies

$$\dim \mathbf{Gr}(n-m, n) \geq \dim |K_{X'} + mH|$$

and hence by the inductive hypothesis, X′ is a variety of minimal degree; thus, X is also a variety of minimal degree.

Thus we are reduced to the case $m = 1$: we leave it to the reader to translate this case into the well-known fact that a non-degenerate degree d smooth curve in $\mathbf{P}^d$ is a rational normal curve.

To complete the proof we observe that if X is a variety of minimal degree, and if $\dim \mathbf{Gr}(n-m, n+1) > \dim |K_X + (m+1)H|$, then by arguing in exactly the same way as above, we would conclude the analogous strict inequality for its iterated hyperplane slices. Again we reduce to the case of X a rational normal curve ($m = 1$), where such an inequality is clearly false. This completes the proof of Theorem C. □

5. Proof of Theorem D

In this section, we extend the projection ramification map to vector bundles on nodal curves using limit linear series. We then use degeneration to a nodal curve to prove generic maximal variation for vector bundles on smooth curves.

5.1. **Limit linear series.** A linear series on a curve of rank r, degree d, and dimension k consists of a vector bundle E on the curve of rank r and degree d, and a k-dimensional subspace of the vector space $H^0(E)$. A limit linear series is an extension of this idea to singular curves, done in a manner suitable for degeneration techniques.

Let B be a DVR with special point 0 and general point η. Let $\pi\colon X \longrightarrow B$ be a family of connected projective curves of genus g, smooth over η, and at worst nodal over 0, with non-singular total space X. Assume that the special fiber $X_0 = C$ is the nodal union of two curves C_1 and C_2 meeting at a unique point p. Then $\pi\colon X \longrightarrow B$ is a particularly simple example of an almost local smoothing family [**?**, § 2.1–2.2].

We recall the notion of a limit linear series from [**?**], where it is called linked linear series. In [**?**], Osserman defines two types of linked linear series. In our setting, where C has only two components, both notions coincide [**?**, Remark 3.4.15]. We model our definition on the definition of the type II series.

Fix the following data:

(1) positive integers r, d, and k;
(2) integers d_1, d_2, and b satisfying $d_1 + d_2 - rb = d$;
(3) maps $\theta_v \colon \mathcal{O}_X \longrightarrow \mathcal{O}_X(C_v)$ for $v = 1, 2$ vanishing precisely on C_v;
(4) integers w_1, w_2 satisfying $w_v \equiv d_v \pmod r$ and $w_1 + w_2 = d$.

The integers r, d, and k will denote the rank, the degree, and the dimension of the linear series. The tuple $w = (w_1, w_2)$ will encode the multi-degree of the vector bundle in the limit linear series, and the integers d_1 and d_2 will encode its extremal twists. The maps θ_v are unique up to an element of $\mathcal{O}_B^*$. The choice of w_v, and θ_v is entirely auxilliary; different choices give isomorphisms between the corresponding moduli stacks of limit linear series. The choice of d_v and b is also largely auxilliary; increasing them leads to open inclusions between the corresponding moduli stacks of limit linear series.

Let S be a B-scheme, and let $\mathcal{E}$ be a vector bundle on X_S of rank r and degree d. For every $n \in \mathbf{Z}$, define the vector bundle $\mathcal{E}_n$ by

$$\mathcal{E}_n = \begin{cases} \mathcal{E} \otimes \mathcal{O}_X(C_1)^n & \text{if } n \geq 0, \\ \mathcal{E} \otimes \mathcal{O}_X(C_2)^{-n} & \text{if } n < 0. \end{cases}$$

Define maps

$$\theta_n \colon \mathcal{E}_m \longrightarrow \mathcal{E}_{m+n}$$

by

$$\theta_n = \begin{cases} \theta_1^n & \text{if } n \geq 0, \\ \theta_2^{-n} & \text{if } n < 0. \end{cases}.$$

We say that $\mathcal{E}$ has multi-degree w if for every $s \in S$ mapping to $0 \in B$, the degree of $\mathcal{E}|_s$ on C_v is w_v for $v = 1, 2$. Note that, if $\mathcal{E}$ has multi-degree (w_1, w_2), then $\mathcal{E}_n$ has multi-degree $(w_1 - rn, w_2 + rn)$.

Let $n_1 \in \mathbf{Z}$ be such that

$$(w_1 - n_1 r, w_2 + n_1 r) = (d_1, d_2 - rb),$$

and $n_2 \in \mathbf{Z}$ such that

$$(w_1 - n_2 r, w_2 + n_2 r) = (d_1 - rb, d_2).$$

Observe that $n_2 - n_1 = b$.

Definition 5.1 (Special case of [?, Definition 3.3.2])**.** Let S be a B-scheme. A *limit linear series* on X_S consists of $(\mathcal{E}, V_n \mid n \in \mathbf{Z})$, where $\mathcal{E}$ is a vector bundle of rank r, degree d, and multi-degree w on X_S, and V_n is a sub-bundle of $\pi_*\mathcal{E}_n$ of rank k satisfying the following conditions.

(1) (Vanishing) For every $z \in S$ over $0 \in B$ and $v = 1, 2$, we have

$$H^0(C_v, \mathcal{E}_{n_v}|_{C_v}(-(b+1)p)) = 0.$$

(2) (Compatibility) For every $m, n \in \mathbf{Z}$, the map

$$\pi_*\theta_n \colon \pi_*\mathcal{E}_m \longrightarrow \pi_*\mathcal{E}_{m+n}$$

maps V_m to V_{m+n}.

The notion of a sub-bundle of a push-forward $\pi_*\mathcal{E}_n$ is as in [?, Definition B.2.1], namely V_n is a vector bundle with a map $V_n \longrightarrow \pi_*\mathcal{E}_n$ which remains injective after arbitrary base-change.

Remark 5.2. *In our case, the various twists of $\mathcal{E}$ are indexed by integers n. In general, the twists are indexed by a graph G_{II} that depends on the dual graph of X_0.*

Denote by $\mathcal{G}^k_{r,d,d_*,w_*}(\mathrm{X/B})$ the category fibered over the category of B-schemes whose objects over S are the limit linear series on S of rank r, degree d, and multi-degree w, and whose morphisms are isomorphisms over S, defined in the obvious way.

Definition 5.3. Let $\mathrm{S} = \operatorname{Spec}\mathrm{K}$, where K is a field, and let $\lambda = (\mathcal{E}, \mathrm{V}_n \mid n \in \mathbf{Z})$ be a limit linear series on S. We say that λ is *simple* if there exist integers $w_1, \ldots, w_k$, not necessarily distinct, and elements $v_i \in \mathrm{V}_{w_i}$ such that for every integer w, the images of $v_1, \ldots, v_k$ in V_w form a basis of V_w. Here the maps $\mathrm{V}_{w_i} \longrightarrow \mathrm{V}_w$ are as in Definition 5.1 (2) .

Remark 5.4. *By* [**?**, Lemma 3.4.14]*, it suffices to check the basis condition for* $w = n_1$ *and* $w = n_2$*.*

Let $\mathrm{M}_{r,d,w}(\mathrm{X/B})$ be the category fibered over the category of B-schemes whose objects over a B-scheme S are vector bundles $\mathcal{E}$ of rank r, degree d, and multi-degree w on $\mathrm{X_S}$, and whose morphisms are isomorphisms over S. Let $\mathrm{M}_{r,d,d_*,w_*}(\mathrm{X/B}) \subset \mathrm{M}_{r,d,w}(\mathrm{X/B})$ be the full-subcategory that parametrizes bundles satisfying the vanishing condition in Definition 5.1 (1). Then $\mathrm{M}_{r,d,w}(\mathrm{X/B})$ is an Artin stack over B, locally of finite type. By the semi-continuity of cohomology, $\mathrm{M}_{r,d,d_*,w_*}(\mathrm{X/B}) \subset \mathrm{M}_{r,d,w}(\mathrm{X/B})$ is an open substack.

Theorem 5.5 ([**?**, Theorem 3.4.7])**.** *Retain the notation above. Then* $\mathcal{G}^k_{r,d,d_*,w_*}(\mathrm{X/B})$ *is an Artin stack over* B*. The natural forgetful map*

$$\beta \colon \mathcal{G}^k_{r,d,d_*,w_*}(\mathrm{X/B}) \longrightarrow \mathrm{M}_{r,w,d_*}(\mathrm{X/B})$$

is representable by schemes, which are projective locally on the target. The locus of simple limit linear series is an open substack of $\mathcal{G}^k_{r,d,d_*,w_*}(\mathrm{X/B})$*; it has universal relative dimension at least* $k(d-k-r(g-1))$ *over* $\mathrm{M}_{r,w,d_*}(\mathrm{X/B})$*.*

The last statement implies that if λ is a simple limit linear series such that the fiber of β through λ has dimension at most $k(d-k-r(g-1))$ at λ, then β is open at λ of relative dimension exactly $k(d-k-r(g-1))$.

Although Definition 5.1 requires specifying infinitely many vector bundles V_n, for $n \in \mathbf{Z}$, specifying finitely many determines the rest. Set $\mathrm{I} = [n_1, n_2] \cap \mathbf{Z}$. Define an I-*linear series* to be the data of $(\mathcal{E}, \mathrm{V}_n \mid n \in \mathrm{I})$ satisfying the conditions (1) and (2) in Definition 5.1 whenever the subscripts lie in I.

Proposition 5.6. *The natural forgetful map from the groupoid of limit linear series to the groupoid of* I*-linear series is an equivalence.*

It is often enough to specify only the two extremal bundles, for $n = n_1$ and $n = n_2$, provided they satisfy certain compatibility conditions. This approach gives the original incarnation of the notion of limit linear series due to Eisenbud and Harris [**?**] for the rank 1 case and Teixidor i Bigas in the higher rank case [**?**].

Let $\mathcal{E}$ be a vector bundle on C of multi-degree w satisfying the vanishing condition in Definition 5.1.

Definition 5.7 (Adapted from [**?**, Definition 4.1.2])**.** Let $\mathrm{W}_v \subset \mathrm{H}^0(\mathrm{C}, \mathcal{E}_{n_v}|_{\mathrm{C}_v})$ be a k-dimensional subspace for $v = 1, 2$. We say that $(\mathcal{E}, \mathrm{W}_1, \mathrm{W}_2)$ is an *EHT limit linear series* if the following conditions are satisfied.

(1) If $a^v_1 \leq \cdots \leq a^v_k$ is the vanishing sequence for $(\mathcal{E}_{n_v}|_{\mathrm{C}_v}, \mathrm{W}_v)$ at p for $v = 1, 2$, then for every $i = 1, \ldots, k$ we have
$$a^1_i + a^2_{k+1-i} \geq b.$$

(2) There exist bases $s_1^v, \dots, s_k^v$ for W_v for $v = 1, 2$, such that s_i^v has order of vanishing a_i^v at p, and if we have $a_i^1 + a_{k+1-i}^2 = b$ for some i, then
$$\widetilde{\phi}(s_i^1) = s_{k+1-i}^2,$$
where $\widetilde{\phi}\colon \mathcal{E}_{n_1}(-a_i^1 p)|_p \longrightarrow \mathcal{E}_{n_2}(-a_{k+1-i}^2 p)|_p$ is the isomorphism obtained by taking the appropriate twist of the identity map.

We say that $(\mathcal{E}, \mathrm{W}_1, \mathrm{W}_2)$ is *refined* if all the inequalities in (1) are equalities.

Due to the vanishing condition, the restriction map
$$\mathrm{H}^0(\mathrm{C}, \mathcal{E}_{n_v}) \longrightarrow \mathrm{H}^0(\mathrm{C}_v, \mathcal{E}_{n_v}|_{\mathrm{C}_v})$$
is an isomorphism. Via this isomorphism, we sometimes treat W_v as a subspace of $\mathrm{H}^0(\mathrm{C}_v, \mathcal{E}_{n_v}|_{\mathrm{C}_v})$.

It is possible to define an a stack of EHT limit linear series so that the locus of refined EHT limit linear series forms an open substack [**?**, § 4].

Let $\lambda = (\mathcal{E}, \mathrm{V}_n \mid n \in \mathbf{Z})$ be a limit linear series on C in the sense of Definition 5.1. Set $\mathrm{W}_1 = \mathrm{V}_{n_1}$ and $\mathrm{W}_2 = \mathrm{V}_{n_2}$.

Proposition 5.8. *In the notation above, $(\mathcal{E}, \mathrm{W}_1, \mathrm{W}_2)$ is an EHT limit linear series on* C. *Conversely, given an EHT limit linear series $\mu = (\mathcal{E}, \mathrm{W}_1, \mathrm{W}_2)$, there exists a limit linear series $\lambda = (\mathcal{E}, \mathrm{V}_n)$ on* C *such that $\mathrm{W}_1 = \mathrm{V}_{n_1}$ and $\mathrm{W}_2 = \mathrm{V}_{n_2}$. Furthermore, if μ is refined, then λ is unique and simple.*

Proof. This is a point-wise version of the stack theoretic statement [**?**, Theorem 4.3.4], plus the equivalence of type I and type II series in the two component case ([**?**, Remark 3.4.15]).

The assertion about refined series follows from the proof of [**?**, Theorem 4.3.4], but it is not explicitly stated there. So we offer a proof.

Let μ be a refined EHT limit linear series. We now construct $\mathrm{V}_n \subset \mathrm{H}^0(\mathcal{E}_n)$. By Proposition 5.6, it suffices to take $n \in [n_1, n_2]$.

By composing the restriction $\mathcal{E}_n \longrightarrow \mathcal{E}_n|_{\mathrm{C}_v}$ and the inclusion $\mathcal{E}_n|_{\mathrm{C}_v} \longrightarrow \mathcal{E}_{n_v}|_{\mathrm{C}_v}$, we get a map
$$\iota\colon \mathrm{H}^0(\mathrm{C}, \mathcal{E}_n) \longrightarrow \mathrm{H}^0(\mathrm{C}_1, \mathcal{E}_{n_1}|_{\mathrm{C}_1}) \oplus \mathrm{H}^0(\mathrm{C}_2, \mathcal{E}_{n_2}|_{\mathrm{C}_2}).$$
The vanishing condition in Definition 5.1 implies that ι is an injection. The compatibility condition in Definition 5.1 implies that $\iota(\mathrm{V}_n) \subset \mathrm{W}_1 \oplus \mathrm{W}_2$. Therefore, the subspace $\mathrm{V}_n \subset \mathrm{H}^0(\mathrm{C}, \mathcal{E}_n)$ lies in the kernel of the map
$$\bar{\iota}\colon \mathrm{H}^0(\mathrm{C}, \mathcal{E}_n) \longrightarrow \mathrm{H}^0(\mathrm{C}_1, \mathcal{E}_{n_1}|_{\mathrm{C}_1})/\mathrm{W}_1 \oplus \mathrm{H}^0(\mathrm{C}_2, \mathcal{E}_{n_2}|_{\mathrm{C}_2})/\mathrm{W}_2. \tag{5.1}$$
We show that
$$\dim \ker \bar{\iota} = k. \tag{5.2}$$
Suppose $s \in \ker \bar{\iota}$. Then $\iota(s)$ is a linear combination of $(s_1^1, 0), \dots, (s_k^1, 0), (0, s_1^2), \dots, (0, s_k^2)$. Writing $\iota(s) = (s_1, s_2)$, we have
$$\operatorname{ord}_p(s_1) \geq n - n_1 \text{ and } \operatorname{ord}_p(s_2) \geq n_2 - n. \tag{5.3}$$
Recall that $a_1^v \leq \dots \leq a_k^v$ is the vanishing sequence of W_v for $v = 1, 2$. Let i the smallest such that
$$a_i^1 \geq n - n_1,$$
and $i + c$ the smallest such that
$$a_{i+c}^1 > n - n_1.$$
Since μ is a refined series and $n_2 - n_1 = b$, we get that $j = k + 1 - i$ is the largest such that
$$a_j^2 \leq n_2 - n,$$

and $j - c$ the largest such that

$$a^2_{j+c} < n_2 - n.$$

The vanishing conditions (5.3) imply that $\iota(s)$ must in fact be a linear combination of $(s^1_i, 0), \dots, (s^1_k, 0), (0, s^2_{k-i-c}), \dots$ But since s_v for $v = 1, 2$ are the restriction to C_v of a section on C, they satisfy a gluing condition at p. Write

$$\iota(s) = \sum_{\ell=i}^{k} \alpha_\ell (s^1_\ell, 0) + \sum_{\ell=j}^{k} \beta_\ell (0, s^2_\ell),$$

where α_ℓ, β_ℓ are in the base-field. By the condition (2) in Definition 5.7, the gluing condition for s_1 and s_2 is equivalent to

$$\alpha_\ell = \beta_{k+1-\ell}$$

for $\ell = i, \dots, i + c - 1$. Hence $\iota(s)$ must be a linear combination of the k elements

$$(s^1_i, s^2_{k+1-i}), \dots, (s^1_{i+c-1}, s^2_{k+2-i-c}), (s^1_{i+c}, 0), \dots, (s^1_k, 0), (0, s^2_{k+2-i}), \dots, (s^2_k, 0).$$

It follows that $\ker \bar\iota$ is k-dimensional.

Since $\ker \bar\iota$ is k-dimensional, there is a unique possible choice for V_n, namely $\mathrm{V}_n = \ker \bar\iota$. It is easy to check that with this choice, the compatibility condition in Definition 5.1 is satisfied. Therefore, we get a limit linear series λ whose associated EHT limit linear series is μ.

It remains to show that λ is simple. For $i = 1, \dots, k$, set $n_i = n - n_1 - a^1_i$, and let $s_i \in \mathrm{V}_{n_i} \subset \mathrm{H}^0(\mathrm{C}, \mathcal{E}_{n_i})$ to be the section whose image under ι is (s^1_i, s^2_{k+1-i}). Then the images of $s_1, \dots, s_k$ form a basis of $\mathrm{V}_{n_1} = \mathrm{W}_1$ and $\mathrm{V}_{n_2} = \mathrm{W}_2$. By Remark 5.4, we conclude that λ is simple. □

5.2. **Projection-ramification for nodal curves.** Let C be a smooth curve and $p \in \mathrm{C}$ a point. Let E be a vector bundle on C of rank r. The projective spaces $\mathbf{P}\mathrm{E}(np)|_p$, for $n \in \mathbf{Z}$, are canonically isomorphic to each other, so we identify them.

Suppose $\lambda \subset \mathrm{H}^0(\mathrm{C}, \mathrm{E})$ is an $(r+1)$-dimensional subspace with the vanishing sequence

$$(\underbrace{a, \dots, a}_{i}, \underbrace{a+1, \dots, a+1}_{r+1-i}), \tag{5.4}$$

for some i with $1 \le i \le r$, and $a \ge 0$. Let $\Lambda_0 \subset \mathrm{E}|_p \cong \mathrm{E}(-ap)|_p$ be the image of $\lambda(-ap)$, and $\Lambda_1 \subset \mathrm{E}|_p \cong \mathrm{E}(-(a+1)p)|_p$ the image of $\lambda(-(a+1)p)$. Then $\dim \Lambda_0 = i$ and $\dim \Lambda_1 = r + 1 - i$. Assume that Λ_0 and Λ_1 satisfy the following genericity condition

$$\dim(\Lambda_0 \cap \Lambda_1) = 1. \tag{5.5}$$

Recall that the ramification $\mathrm{R}(\lambda)$ is a section of $\mathrm{E} \otimes \det \mathrm{E} \otimes \mathrm{T}_\mathrm{C}$.

Proposition 5.9. *In the setup above,* $\mathrm{R}(\lambda)$ *vanishes to order* $(r+1)a + (r-i)$ *at* p. *Write* $\widetilde{\mathrm{R}} = \mathrm{R}(\lambda)/t^{(r+1)a+r-i}$, *where* t *is a uniformizer at* p. *Then, the one-dimensional subspace of* $\mathrm{E}|_p$ *spanned by* $\widetilde{\mathrm{R}}|_p$ *is* $\Lambda_0 \cap \Lambda_1$.

Proof. Let $\langle s_1, \dots, s_r \rangle$ be a local trivialization for E in an open set around p such that in these local coordinates, we have

$$\lambda = \{t^a s_1, \dots, t^a s_i, t^{a+1} s_i, t^{a+1} s_{i+1}, \dots, t^{a+1} s_r\}.$$

In these coordinates, $\mathrm{R}(\lambda)$ is given by

$$\begin{aligned}\mathrm{R}(\lambda) = \det &\begin{pmatrix} t^a & & & & & & at^{a-1}s_1 \\ & t^a & & & & & at^{a-1}s_2 \\ & & \ddots & & & & \vdots \\ & & & t^a & & & at^{a-1}s_i \\ & & & t^{a+1} & & & (a+1)t^a s_i \\ & & & & \ddots & & \vdots \\ & & & & & t^{a+1} & (a+1)t^a s_r \end{pmatrix} \\ &= (-1)^{r-i} t^{(r+1)a+r-i} \cdot s_i.\end{aligned}$$

Since s_i spans $\Lambda_0 \cap \Lambda_1$, the proof is complete. ∎

Let $\pi\colon \mathrm{X} \longrightarrow \mathrm{B}$ be a family as in § 5.1 with X_η smooth and $\mathrm{X}_0 = \mathrm{C}$ a nodal union $\mathrm{C} = \mathrm{C}_1 \cup_p \mathrm{C}_2$, with $g(\mathrm{C}_v) = g_v$ for $v = 1, 2$. Fix r, d, d_1, d_2, b, w_1, w_2, θ_1, and θ_2 as in § 5.1, and take $k = r+1$. Set

$$\begin{aligned} r' &= r, \\ d' &= d + r(d - 2g + 2), \\ d_1' &= d_1 + r(d_1 - 2g_1 + 1), \\ d_2' &= d_2 + r(d_2 - 2g_2 + 1), \\ b' &= b(r+1), \\ w_1' &= w_1 + r(w_1 - 2g_1 + 1), \\ w_2' &= w_2 + r(w_1 - 2g_1 + 1), \\ k' &= 1. \end{aligned}$$

Defining n_1' and n_2' analogously to n_1 and n_2, we get

$$\begin{aligned} n_1' &= n_1(1 + r), \\ n_2' &= n_2(1 + r). \end{aligned}$$

We define a rational map

$$\mathcal{R}\colon \mathcal{G}^{r+1}_{r,d,d_*,w_*}(\mathrm{X}/\mathrm{B})^{\mathrm{red}} \dashrightarrow \mathcal{G}^{1}_{r',d',d_*',w_*'}(\mathrm{X}/\mathrm{B}) \tag{5.6}$$

that extends the projection-ramification map on the generic fiber. Let $\mathcal{U} \subset \mathcal{G}^{r+1}_{r,d,d_*,w_*}(\mathrm{X}/\mathrm{B})$ be the open substack obtained by excluding the following closed loci:

(1) the closure of the locus of linear series $(\mathcal{E}, \lambda)$ on the generic fiber X_η such that $\lambda \otimes \mathcal{O}_{\mathrm{X}_\eta} \longrightarrow \mathcal{E}$ has generic rank less than r,
(2) the locus of limit linear series $\lambda = (\mathcal{E}, \mathrm{V}_n)$ on the central fiber such that the associated EHT limit linear series $\mu = (\mathcal{E}, \mathrm{W}_1, \mathrm{W}_2)$ is not refined, or does not have vanishing sequences as in (5.4), or does not satisfy the genericity condition $\dim(\Lambda_0 \cap \Lambda_1) = 1$ as in (5.5).

Let S be a B scheme with a map to $\mathcal{U}$ given by the limit linear series $(\mathcal{E}, \mathrm{V}_n)$. On X_S, we have the diagram

$$\begin{tikzcd} & \det \mathcal{E}_n^* \otimes \det \mathrm{V}_n \arrow[r, "j"] \arrow[d, "d"] & \mathrm{V}_n \otimes \mathcal{O}_{\mathrm{X}_\mathrm{S}} \arrow[r, "i"] \arrow[d, "\widetilde{i}"] & \mathcal{E}_n \arrow[d, equal] & \\ 0 \arrow[r] & \Omega_{\mathrm{X}_\mathrm{S}/\mathrm{S}} \otimes \mathcal{E}_n \arrow[r] & j_1 \mathcal{E}_n \arrow[r] & \mathcal{E}_n \arrow[r] & 0. \end{tikzcd} \tag{5.7}$$

In the top row, the map i is induced by the inclusion $\mathrm{V}_n \longrightarrow \pi_* \mathcal{E}_n$ on S, and the map j is given by $j = \wedge^r i^* \otimes \det \mathrm{V}_n$. In the bottom row, the sheaf $j_1 \mathcal{E}_n$ is the first jet bundle of $\mathcal{E}_n$ along $\mathrm{X}_\mathrm{S} \longrightarrow \mathrm{S}$, and the row is the natural jet bundle exact sequence. The map $\widetilde{i}$ is the canonical lift of i, and the map d is the unique induced map owing to $i \circ j = 0$. By composing d through the inclusion $\Omega_{\mathrm{X}_\mathrm{S}/\mathrm{S}} \longrightarrow \omega_{\mathrm{X}_\mathrm{S}/\mathrm{S}}$, and rearranging the line bundles, we obtain a map

$$\mathrm{R}_n \colon \det \mathrm{V}_n \longrightarrow \pi_*(\mathcal{E}_n \otimes \det \mathcal{E}_n \otimes \omega^*_{\mathrm{X}_\mathrm{S}/\mathrm{S}}). \tag{5.8}$$

Set $\mathcal{E}' = \mathcal{E} \otimes \det \mathcal{E} \otimes \omega^*_{\mathrm{X}_\mathrm{S}/\mathrm{S}}$. We want to say that the sections given by R_n of the various twists of $\mathcal{E}'$ define a limit linear series of dimension 1. The catch is that the twists $\mathcal{E}_n \otimes \det \mathcal{E}_n \otimes \omega^*_{\mathrm{X}_\mathrm{S}/\mathrm{S}}$ are only *some* of the twists of $\mathcal{E}'$. However, on the open set $\mathcal{U}$, this is more than enough information—just looking at the extremal twists suffices. Observe that we have $\mathcal{E}_{n_v} \otimes \det \mathcal{E}_{n_v} \otimes \omega_{\mathrm{X}_\mathrm{S}/\mathrm{S}} = \mathcal{E}'_{n'_v}$ for $v = 1, 2$.

Proposition 5.10. *In the setup above, assume that* S *is reduced. Then there is a unique simple limit linear series* $(\mathcal{E}', \mathrm{L}_n)$ *of dimension 1 such that* $\mathrm{L}_{n'_v} = \det \mathrm{V}_{n_v}$ *and the map* $\mathrm{L}_{n'_v} \longrightarrow \pi_* \mathcal{E}'_{n'_v}$ *is given by* R_n *for* $v = 1, 2$.

Proof. First, suppose S is a point mapping to $0 \in \mathrm{B}$. The key is that the two extremal sections $\mathrm{L}_{n_v} \longrightarrow \mathrm{H}^0(\mathcal{E}_{n_v} \otimes \det \mathcal{E}_{n_v} \otimes \omega^*_{\mathrm{X}_\mathrm{S}/\mathrm{S}})$ for $v = 1, 2$ define a refined EHT limit linear series. To see this, suppose $(\mathcal{E}_{n_1}|_{\mathrm{C}_1}, \mathrm{V}_{n_1})$ has vanishing sequence

$$(\underbrace{a, \dots, a}_{i}, \underbrace{a+1, \dots, a+1}_{r+1-i}),$$

at p for some i with $1 \le i \le r$ and $a \ge 0$. Then $(\mathcal{E}_{n_2}|_{\mathrm{C}_2}, \mathrm{V}_{n_2})$ has the vanishing sequence

$$(\underbrace{b-a-1, \dots, b-a-1}_{r+1-i}, \underbrace{b-a, \dots, b-a}_{i}).$$

By construction, the section R_{n_v} restricted to C_v is the ramification of $(\mathcal{E}_{n_v}|_{\mathrm{C}_v}, \mathrm{V}_{n_v})$ composed with the inclusion $\omega^*_\mathrm{C}|_{\mathrm{C}_v} \longrightarrow \Omega^*_{\mathrm{C}_v}$. Therefore, by Proposition 5.9, R_{n_1} vanishes at p to order $a'_1 = a(r+1) + (r-i) + 1$ and R_{n_2} to order $a'_2 = (b-a-1)(r+1) + i$. Since $a'_1 + a'_2 = b'$, we have the equality required in condition (1) of Definition 5.7. Note that the spaces Λ_0 and Λ_1 are exchanged when we switch from C_1 to C_2, and so their intersection $\Lambda_0 \cap \Lambda_1$ remains the same. By Proposition 5.9, after dividing by the appropriate power of the uniformizer, the sections R_{n_1} and R_{n_2} at p are proportional; they both span $\Lambda_0 \cap \Lambda_1$. Hence, we also have the gluing condition in required in (2) in Definition 5.7.

For a general S, note that for every point $s \in \mathrm{S}$, the map

$$\mathrm{H}^0(\mathrm{X}_s, \mathcal{E}'_n|_s) \longrightarrow \mathrm{H}^0(\mathrm{X}_s, \mathcal{E}'_{n_1}|_s)/\mathrm{L}_{n_1}|_s \oplus \mathrm{H}^0(\mathrm{X}_s, \mathcal{E}'_{n_2}|_s)/\mathrm{L}_{n_2}|_s$$

has kernel of dimension 1. If s lies over the generic point of Δ, then this is automatic. If s lies over the special point of Δ, then this follows from the fact that $(\mathcal{E}', \mathrm{L}_{n_1}, \mathrm{L}_{n_2})$ is a refined EHT linear series; see (5.2). Since S is reduced, we conclude that the kernel of the map

$$\pi_*(\mathcal{E}'_n) \longrightarrow \pi_*(\mathcal{E}'_{n_1})/\mathrm{L}_{n_1} \oplus \pi_*(\mathcal{E}'_{n_2})/\mathrm{L}_{n_2}$$

is a line bundle, say L_n, and the inclusion $\mathrm{L}_n \longrightarrow \pi_*(\mathcal{E}'_n)$ is a sub-bundle map. The data $(\mathcal{E}', \mathrm{L}_n)$ is the unique simple limit linear series claimed in the statement. □

Remark 5.11. *A simple limit linear series of dimension 1 on a vector bundle $\mathcal{E}'$ on C is simply a section of one of its twists that is non-zero on both components of C. From Proposition 5.9, we see that this twist is $\mathcal{E}'_{n_1+m}$ where*

$$m = (r+1)a + ar - i.$$

In particular, it is not one of the twists $\mathcal{E}'_{n_1+r(n-n_1)} = \mathcal{E}_n \otimes \det \mathcal{E}_n \otimes \omega^*$ in (5.8) that receive the image of the ramification section R_n.

Thanks to Proposition 5.10, we have a morphism

$$\mathcal{R} \colon \mathcal{U}^{\mathrm{red}} \longrightarrow \mathcal{G}^1_{r',d',d'_*,w'_*}(\mathrm{X}/\mathrm{B}) \tag{5.9}$$

defined by

$$(\mathcal{E}, \mathrm{V}_n) \mapsto (\mathcal{E}', \mathrm{L}_n).$$

5.3. **Maximal variation.** Let E be an ample vector bundle on $\mathbf{P}^1$ of rank r. Fix a point $p \in \mathbf{P}^1$. Consider the locally closed subset $\mathrm{U} \subset \mathbf{Gr}(r+1, \mathrm{H}^0(\mathrm{E}))$ consisting of linear series with vanishing sequence

$$(0, \underbrace{1, \dots, 1}_{r})$$

over p. Given $\lambda \in \mathrm{U}$, let $\widetilde{\mathrm{R}}(\lambda) \in \mathbf{P}\mathrm{H}^0(\mathrm{E} \otimes \det \mathrm{E} \otimes \mathrm{T}_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p)$ be the reduced ramification divisor; see Proposition 5.9. The assignment $\lambda \mapsto \widetilde{\mathrm{R}}(\lambda)$ gives a *reduced* projection-ramification map

$$\widetilde{\mathrm{R}} \colon \mathrm{U} \longrightarrow \mathbf{P}\mathrm{H}^0(\mathrm{E} \otimes \det \mathrm{E} \otimes \mathrm{T}_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p)) \tag{5.10}$$

between varieties of the same dimension.

Given a one-dimensional subspace $\ell \subset \mathrm{E}|_p$, define E'_ℓ by the exact sequence

$$0 \longrightarrow \mathrm{E}'_\ell \longrightarrow \mathrm{E} \longrightarrow \mathrm{E}|_p/\ell \longrightarrow 0.$$

There exists a Zariski open subset of $\mathbf{P}_{\mathrm{sub}}(\mathrm{E}|_p)$, such that for all ℓ in this set, the isomorphism class of E'_ℓ remains constant. Denote this isomorphism class by $\mathrm{E}'_{\mathrm{gen}}$.

Proposition 5.12. *If the usual projection-ramification map*

$$\mathrm{R} \colon \mathbf{Gr}(r+1, \mathrm{H}^0(\mathrm{E}'_{\mathrm{gen}})) \dashrightarrow \mathbf{P}\mathrm{H}^0(\mathrm{E}'_{\mathrm{gen}} \otimes \det \mathrm{E}'_{\mathrm{gen}} \otimes \mathrm{T}_{\mathbf{P}^1})$$

is dominant, then so is the reduced projection-ramification map

$$\widetilde{\mathrm{R}} \colon \mathrm{U} \longrightarrow \mathbf{P}\mathrm{H}^0(\mathrm{E} \otimes \det \mathrm{E} \otimes \mathrm{T}_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p)).$$

Proof. Let $\mathrm{D} \in \mathbf{P}\mathrm{H}^0(\mathrm{E} \otimes \det \mathrm{E} \otimes \mathrm{T}_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p))$ be a generic section. Let $\ell \subset \mathrm{E}|_p$ be the one-dimensional subspace defined by $\mathrm{D}|_p$, and set $\mathrm{E}' = \mathrm{E}'_\ell$. Since D is generic, we may assume $\mathrm{E}' \cong \mathrm{E}'_{\mathrm{gen}}$. We have the inclusion

$$\mathrm{E}' \otimes \det \mathrm{E}' \otimes \mathrm{T}_{\mathbf{P}^1} \longrightarrow \mathrm{E} \otimes \det \mathrm{E} \otimes \mathcal{O}(-(r-1)p) \otimes \mathrm{T}_{\mathbf{P}^1},$$

and by construction D is the image of a section $\mathrm{D}' \in \mathbf{P}\mathrm{H}^0(\mathrm{E}' \otimes \det \mathrm{E}' \otimes \mathrm{T}_{\mathbf{P}^1})$. Since R is dominant for E', there exists a sequence of subspaces $\lambda'_n \in \mathbf{Gr}(r+1, \mathrm{H}^0(\mathrm{E}'))$ such that $\mathrm{R}(\lambda'_n)$ limit to D'. Let $\lambda_n \subset \mathbf{Gr}(r+1, \mathrm{H}^0(\mathrm{E}))$ be the image of λ'_n. Then $\widetilde{\mathrm{R}}(\lambda_n)$ limit to D. Since D was generic, we get that $\widetilde{\mathrm{R}}$ is dominant. □

Corollary 5.13. *The reduced projection-ramification map is dominant for the bundles $\mathrm{E} = \mathcal{O}(1) \oplus \mathcal{O}(2)^{r-1}$ and $\mathrm{E} = \mathcal{O}(2) \oplus \mathcal{O}(3)^{r-1}$.*

Proof. Follows from Proposition 5.12 and that the projection-ramification map is dominant for $\mathrm{E}' = \mathcal{O}(1)^r$ and $\mathrm{E}' = \mathcal{O}(2)^r$. □

For $v = 1, 2$, let C_v be a smooth curve and E_v a vector bundles of rank r on C_v. Let $p_v \in \mathrm{C}_v$ be a point. Suppose $\lambda_1 \in \mathbf{Gr}(r+1, \mathrm{H}^0(\mathrm{E}_1))$ is a linear series with vanishing sequence $(0, \dots, 0, 1)$ at p_1, and $\lambda_2 \in \mathbf{Gr}(r+1, \mathrm{H}^0(\mathrm{E}_2))$ is a linear series with vanishing sequence $(0, 1, \dots, 1)$ at p_2.

Let C be the nodal union of C_1 and C_2 formed by identifying p_1 and p_2. We construct a simple limit linear series λ on C of rank r and degree $\deg \mathrm{E}_1 + \deg \mathrm{E}_2 - r$. Choose an isomorphism $\phi\colon \mathrm{E}_1(-p)|_{p_1} \longrightarrow \mathrm{E}_2|_{p_2}$ that sends the image of $\lambda_1(-p)$ in $\mathrm{E}_1(-p)|_{p_1}$ to the image of λ_2 in $\mathrm{E}_2|_{p_2}$. Let $\mathcal{E}$ be the vector bundle on C constructed by gluing $\mathrm{E}_1(-p)$ and E_2 by ϕ. Let $b = 2m$ be large enough so that $\mathrm{H}^0(\mathrm{E}_1(-mp)) = 0$ and $\mathrm{H}^0(\mathrm{E}_2(-mp)) = 0$. Set $d_1 = \deg \mathrm{E}_1 + (m-1)r$ and $d_2 = \deg \mathrm{E}_2 + mr$. Then $w_1 = d_1 - r$, so $n_1 = m$; and $w_2 = d_2$, so $n_2 = -m$. Let $\mathrm{V}_{n_1} \subset \mathrm{H}^0(\mathrm{C}, \mathcal{E}_{n_1})$ be the subspace that restricts to $\lambda_1((m-1)p) \subset \mathrm{H}^0(\mathrm{C}_1, \mathrm{E}_1((m-1)p))$ and $\mathrm{V}_{n_2} \subset \mathrm{H}^0(\mathrm{C}, \mathcal{E}_{n_2})$ the subspace that restricts to $\lambda_2(mp) \subset \mathrm{H}^0(\mathrm{C}_2, \mathrm{E}_2(mp))$. Then the vanishing sequence of V_{n_1} is $(m-1, \dots, m-1, m)$ and that of V_{n_2} is $(m, m+1, \dots, m+1)$. By the choice of ϕ, we see that the two series are compatible at the node, and hence define a refined EHT limit linear series on C. Let λ be the associated unique simple limit linear series.

Let $\mathrm{X} \longrightarrow \mathrm{B}$ be a smoothing of C, and $\mathscr{E}$ a vector bundle on X whose restriction to $\mathrm{X}_0 = \mathrm{C}$ is $\mathcal{E}$. Set $d = d_1 + d_2 - r$ and $g = g(\mathrm{C}_1) + g(\mathrm{C}_2)$.

Proposition 5.14. *Suppose λ_v is isolated in their respective projection-ramification maps, for $v = 1, 2$. Then λ is isolated in the projection-ramification map $\mathcal{R}$. Suppose, furthermore, that the dimension of the fiber of the forgetful map β at λ is $(r+1)(d-1-rg)$. Then the projection-ramification map is generically finite for the vector bundle $\mathscr{E}_\eta$ on X_η.*

Proof. The projection-ramification map for λ reduces to the projection-ramification map for λ_v (up to twists) on components C_v. So if both λ_v are isolated in the fibers of their projection-ramification maps, so is λ.

If the dimension condition holds, then β is open at λ by Theorem 5.5. In particular, λ is in the closure of $\mathbf{Gr}(r+1, \mathrm{H}^0(\mathrm{X}_\eta, \mathscr{E}_\eta))$. By the semi-continuity of fiber dimension, it follows that the projection-ramification map

$$\mathrm{R}\colon \mathbf{Gr}(r+1, \mathrm{H}^0(\mathrm{X}_\eta, \mathscr{E}_\eta)) \longrightarrow \mathbf{P}\mathrm{H}^0(\mathrm{X}_\eta, \mathscr{E}_\eta \otimes \det \mathscr{E}_\eta \otimes \omega^*_{\mathrm{X}_\eta})$$

is generically finite. □

Theorem 5.15. *Let* E *be a generic vector bundle on* $\mathbf{P}^1$ *of rank r and degree $d = a(r-1) + b(2r-1) + 1$, where a, b are positive integers. Then the projection-ramification map is generically finite, and hence dominant, for* E*. In particular, the projection-ramification map is dominant for generic* E *of degree $\geq (r-1)(2r-1)+1$.*

Proof. Let E be a generic vector bundle of rank $d \geq 0$ such that the projection-ramification map is dominant for E. Set $\mathrm{C}_1 = \mathbf{P}^1$ and $\mathrm{E}_1 = \mathrm{E}$. Take $\mathrm{C}_2 = \mathbf{P}^1$ and $\mathrm{E}_2 = \mathcal{O}(1) \oplus \mathcal{O}(2)^{r-1}$ or $\mathrm{E}_2 = \mathcal{O}(2) \oplus \mathcal{O}(3)^{r-1}$. Pick points $p_v \in \mathrm{C}_v$ for $v = 1, 2$. Let $\lambda_1 \subset \mathrm{H}^0(\mathrm{E}_1)$ be an $(r+1)$-dimensional subspace with vanishing sequence $(0, \dots, 0, 1)$ at p_1, and $\lambda_2 \subset \mathrm{H}^0(\mathrm{E}_2)$ an $(r+1)$-dimensional subspace with vanishing sequence $(0, 1, \dots, 1)$. Assume that λ_v are isolated in the respective fibers of their projection-ramification maps. Furthermore, assume that the image ℓ of λ_2 in $\mathrm{E}_2|_{p_2}$ is generic in the sense that the kernel of

$$\mathrm{E}_2 \longrightarrow \mathrm{E}_2|_{p_2}/\ell$$

is the generic vector bundle $\mathrm{E}_2^{\mathrm{gen}}$ (which will be either $\mathcal{O}(1)^r$ or $\mathcal{O}(2)^r$). Let λ be the limit linear series on $\mathrm{C} = \mathrm{C}_1 \cup \mathrm{C}_2$ constructed from λ_1 and λ_2 as above. Then, we get

$$\begin{aligned}\dim_\lambda \beta^{-1}(\beta(\lambda)) &= \dim \mathbf{Gr}(r+1, \mathrm{H}^0(\mathrm{E}_1)) + \dim \mathbf{Gr}(r+1, \mathrm{H}^0(\mathrm{E}_2^{\mathrm{gen}})) \\ &= (r+1)(\deg \mathrm{E} + \deg \mathrm{E}_2^{\mathrm{gen}} - 2) \\ &= (r+1)(\deg \mathrm{E} + \deg \mathrm{E}_2 - r - 1) \\ &= (r+1)(\deg \mathcal{E} - 1).\end{aligned}$$

Here is how the dimension count goes: it suffices to count dimensions for the refined EHT limit linear series associated to λ, since λ can be recovered uniquely from it. For the EHT limit linear series, we begin by choosing an $(r+1)$-dimensional subspace of $\mathrm{H}^0(\mathrm{E}_1)$, giving us the first term in the dimension count. This choice gives a one-dimensional subspace $\Lambda_1 \in \mathrm{E}_1|_{p_1} = \mathcal{E}|_p = \mathrm{E}_2|_{p_2}$. We must choose an $(r+1)$-dimensional subspace of $\mathrm{H}^0(\mathrm{E}_2)$ with the complementary vanishing sequence and satisfying the compatibility condition over p. These two conditions force it to be a subspace of $\mathrm{H}^0(\mathrm{E}_2')$, where E_2' is the kernel of

$$\mathrm{E}_2 \longrightarrow \mathrm{E}_2|_{p_2}/\Lambda_1.$$

Since the kernel is isomorphic to $\mathrm{E}_2^{\mathrm{gen}}$, we get the second term in the dimension count.

Let $\pi\colon \mathrm{X} \longrightarrow \mathrm{B}$ be a smoothing of B. Every vector bundle on C is the restriction of a vector bundle on X. Indeed, this is clearly true for line bundles on C, and hence for vector bundles of arbitrary rank, as these are direct sums of line bundles. Let $\mathscr{E}$ be a vector bundle on X whose restriction to C is $\mathcal{E}$. By Proposition 5.14, the projection-ramification map is generically finite, and hence dominant, for $\mathcal{E}_\eta$. By the semi-continuity of fiber dimension, the same is true for a generic vector bundle of rank r and degree $\deg \mathrm{E} + \deg \mathrm{E}_2 - r$. The two choices of E_2 give $\deg \mathrm{E}_2 - r = r - 1$ and $\deg \mathrm{E}_2 - r = 2r - 1$.

In summary, dominance for a generic bundle of rank r and degree d implies the same for a generic bundle of rank r and degree $d + r - 1$ and degree $d + 2r - 1$. Starting with the base case $d = 1$, namely $\mathrm{E} = \mathcal{O}^{r-1} \oplus \mathcal{O}(1)$, we obtain the statement by induction. ☐

5.4. **Maximal variation for $\mathcal{O}(2)^r$.** The goal of this section is to establish dominance of the projection ramification morphism for $\mathrm{E} = \mathcal{O}(2)^r$. We do this by a tangent space calculation. For simplicity, we work with inhomogeneous polynomials in $x = \mathrm{X}/\mathrm{Y}$ instead of homogeneous polynomials in X and Y.

Consider the point λ of $\mathbf{Gr}(r+1, \mathrm{H}^0(\mathrm{E}))$ represented by the $(r+1) \times r$ matrix

$$\Lambda = \begin{pmatrix} (x-a_1)^2 & 0 & \cdots & 0 \\ 0 & (x-a_2)^2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & (x-a_r)^2 \\ p_1 & p_2 & \cdots & p_r \end{pmatrix},$$

where $a_i \in \mathbf{C}$ and $p_j \in \mathrm{H}^0(\mathcal{O}(2))$. We claim that if the a_i and the p_j are generic, then the map on the tangent spaces induced by the projection-ramification construction is surjective at Λ.

Recall that if M is an $(r+1) \times r$ matrix of polynomials in x, then the ramification divisor of the projection map represented by M is given by the formula

$$\mathrm{R}(\mathrm{M}) = \det\left(\mathrm{M} \mid \xi(\mathrm{M})\right),$$

where $\xi(\mathrm{M})$ is the vector given by

$$\xi(\mathrm{M})_i = \sum_{j=1}^{r} \partial_x \mathrm{M}_{i,j} \cdot \mathrm{X}_j.$$

To do the tangent space computation, we compute the ramification divisor R for the matrix $\mathrm{M} = \Lambda + \epsilon\Delta$, where Δ is an $(r+1)\times r$ matrix of elements in $\mathrm{H}^0(\mathcal{O}(2))$, assuming $\epsilon^2 = 0$. The result will be of the form

$$\mathrm{R}(\Lambda + \epsilon\Delta) = \mathrm{R}(\Lambda) + \epsilon \mathrm{S}(\Lambda, \Delta),$$

where $\mathrm{S}(\mathrm{R}, \Delta)$ is an element of $\mathrm{H}^0(2r) \otimes \langle \mathrm{X}_1, \ldots, \mathrm{X}_r\rangle$, linear in the entries of Δ. We must show that the linear map

$$\mathrm{H}^0(\mathcal{O}(2)) \otimes \mathrm{M}_{r+1,r} \longrightarrow \mathrm{H}^0(\mathcal{O}(2r)) \otimes \langle \mathrm{X}_1, \ldots, \mathrm{X}_r\rangle$$

given by

$$\Delta \mapsto \mathrm{S}(\Lambda, \Delta)$$

is surjective.

We compute $\mathrm{R}(\Lambda + \epsilon\Delta)$ for elementary matrices Δ. Denote by $\mathrm{E}_{i,j}$ the elementary matrix with 1 at the (i,j)th place, and 0 everywhere else.

First, suppose

$$\Delta = q\mathrm{E}_{j,i},$$

where $i \neq j$, and $1 \leq j \leq r$. By direct calculation, we obtain

$$\mathrm{S}(\Lambda, \Delta) = \frac{(x-a_1)^2 \cdots (x-a_r)^2 p_j}{(x-a_i)^2 (x-a_j)^2} \cdot \left[q, (x-a_i)^2\right] \cdot \mathrm{X}_i, \tag{5.11}$$

where the notation $[a, b]$ means $ab' - a'b$.

Second, suppose

$$\Delta = q\mathrm{E}_{r,i},$$

where $1 \leq i \leq r$. Again, by direct calculation, we obtain

$$\mathrm{S}(\Lambda, \Delta) = -\frac{(x-a_1)^2 \cdots (x-a_r)^2}{(x-a_i)^2} \cdot \left[q, (x-a_i)^2\right] \cdot \mathrm{X}_i. \tag{5.12}$$

Third, suppose

$$\Delta = q\mathrm{E}_{i,i},$$

where $1 \leq i \leq r$. As before, by direct calculation, we obtain

$$\mathrm{S}(\Lambda, \Delta) = \mathrm{R}(\Lambda_i(q)), \tag{5.13}$$

where $\Lambda_i(q)$ is obtained from Λ by changing the (i,i)th entry from $(x-a_i)^2$ to q.

We want to show that the map

$$\mathrm{H}^0(\mathcal{O}(2)) \otimes \mathrm{M}_{(r+1)\times r} \longrightarrow \mathrm{H}^0(\mathcal{O}(2r)) \otimes \langle \mathrm{X}_1, \ldots, \mathrm{X}_r\rangle \tag{5.14}$$

given by

$$\Delta \mapsto \mathrm{S}(\Lambda, \Delta)$$

is surjective. Fix a i with $1 \leq i \leq r$ and consider the subspace of the domain given by

$$\mathrm{H}^0(\mathcal{O}(2)) \otimes \langle \mathrm{E}_{j,i} \mid j \neq i\rangle.$$

By our calculations above, the image of this space lies in $\mathrm{H}^0(\mathcal{O}(2r)) \otimes \mathrm{X}_i$. We begin by identifying the image. For $1 \leq j \leq r$ and $j \neq i$, set

$$\mathrm{Q}_{i,j} = \frac{(x-a_1)^2 \cdots (x-a_r)^2 p_j}{(x-a_i)^2 (x-a_j)^2}$$

and set

$$\mathrm{Q}_{i,r+1} = \frac{(x-a_1)^2 \cdots (x-a_r)^2}{(x-a_i)^2}.$$

Lemma 5.16. *For generic $p_1, \dots, p_r$, there is no non-trivial linear relation among the polynomials $\mathrm{Q}_{i,j}$ for $j \in \{1, \dots, r+1\} \setminus \{i\}$.*

Proof. Suppose we had a linear relation

$$\sum l_j \mathrm{Q}_{i,j} = 0.$$

Divide throughout by $\frac{(x-a_1)^2 \dots (x-a_r)^2}{(x-a_i)^2}$. Then we get the relation

$$\sum_{j=1}^{r} l_j \frac{p_j}{(x-a_j)^2} + l_{r+1} = 0.$$

If $l_j \neq 0$ for some j with $1 \leq j \leq r$, then we have a pole on the left hand side at $x = a_j$, but not on the right hand side; a contradiction. Therefore, we must have $l_j = 0$ for all j with $1 \leq j \leq r$, and hence also $l_{r+1} = 0$. Thus, the relation was trivial. ☐

Lemma 5.17. *The image of the map*

$$\mathrm{H}^0(\mathcal{O}(2)) \longrightarrow \mathrm{H}^0(\mathcal{O}(2))$$

given by

$$q \mapsto [q, (x-a)^2]$$

is

$$(x-a) \cdot \mathrm{H}^0(\mathcal{O}(1)).$$

Proof. Straightforward. ☐

Lemma 5.18. *The image of the map*

$$\mathrm{H}^0(\mathcal{O}(2)) \otimes \langle \mathrm{E}_{j,i} \mid j \in \{1, \dots, r+1\} \setminus \{i\} \rangle \longrightarrow \mathrm{H}^0(\mathcal{O}(2r)) \otimes \mathrm{X}_i$$

is

$$(x-a) \cdot \mathrm{H}^0(\mathcal{O}(2r-1)) \otimes \mathrm{X}_i.$$

Proof. By the computation in (5.11) and (5.12) and Lemma 5.17 the image of the map above is the same as the image of the multiplication map

$$\langle \mathrm{Q}_{i,j} \mid j \in \{1, \dots, r+1\} \setminus \{i\} \rangle \otimes (x-a) \cdot \mathrm{H}^0(\mathcal{O}(1)) \longrightarrow \mathrm{H}^0(\mathcal{O}(2r)) \otimes \mathrm{X}_i.$$

By Lemma 5.16, the map

$$\langle \mathrm{Q}_{i,j} \mid j \in \{1, \dots, r+1\} \setminus \{i\} \rangle \otimes \mathrm{H}^0(\mathcal{O}(1)) \longrightarrow \mathrm{H}^0(\mathcal{O}(2r-1))$$

is injective. Since both sides have dimension $2r$, the map is an isomorphism. The proof is now complete. ☐

By Lemma 5.18, the cokernel of the map

$$\mathrm{H}^0(\mathcal{O}(2)) \otimes \langle \mathrm{E}_{j,i} \mid j \in \{1, \dots, r+1\} \setminus \{i\} \rangle \longrightarrow \mathrm{H}^0(\mathcal{O}(2r)) \otimes \mathrm{X}_i$$

is $\mathbf{C} \otimes \mathrm{X}_i$, where the map $\mathrm{H}^0(\mathcal{O}(2r)) \longrightarrow \mathbf{C}$ is the evaluation at a_i. Putting all these maps together for various i, we get that the cokernel of the map

$$\mathrm{H}^0(\mathcal{O}(2)) \otimes \langle \mathrm{E}_{j,i} \mid j \neq i \rangle \longrightarrow \mathrm{H}^0(\mathcal{O}(2r)) \otimes \langle \mathrm{X}_1, \dots, \mathrm{X}_r \rangle$$

is $\mathbf{C} \otimes \langle \mathrm{X}_1, \dots, \mathrm{X}_r \rangle$, where the map

$$\mathrm{H}^0(\mathcal{O}(2)) \otimes \langle \mathrm{X}_1, \dots, \mathrm{X}_r \rangle \longrightarrow \mathbf{C} \otimes \langle \mathrm{X}_1, \dots, \mathrm{X}_r \rangle \tag{5.15}$$

is given on $\mathrm{H}^0(\mathcal{O}(2r)) \otimes \mathrm{X}_i$ by the evaluation at a_i.

It remains to show that the map

$$\mathrm{H}^0(\mathcal{O}(2)) \otimes \langle \mathrm{E}_{i,i} \rangle \longrightarrow \mathbf{C} \otimes \langle \mathrm{X}_1, \dots, \mathrm{X}_r \rangle, \tag{5.16}$$

obtained by composing (5.14) and (5.15), is surjective. Recall from (5.13) that the image of $q\mathrm{E}_{i,i}$ is given by $\mathrm{R}(\Lambda_i(q))$. Suppose $q = (x - a_i)l$, where $l(a_i) \neq 0$. Then, we have

$$\mathrm{R}(\Lambda_i(q))_{x=a_j} = 0$$

for $j \neq i$, and

$$\mathrm{R}(\Lambda_i(q))_{x=a_i} = \pm l(a_i) \prod_{j \neq i} (a_i - a_j)^2 p_i(a_i)\mathrm{X}_i.$$

Thus, up to scaling, $q\mathrm{E}_{i,i}$ maps to X_i under (5.16). Therefore, the map (5.16) is surjective.

6. Enumerative Problems

Our objective in this section is to prove Theorem E, and to connect some of the projection-ramification enumerative problems with another set of enumerative problems concerning moduli spaces of rational curves in projective space.

6.1. **Quadric hypersurfaces.** A smooth quadric hypersurface $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$ defined by an equation $\mathrm{F}(x_0, x_1, x_2, ...) = 0$ induces the classical *polarity isomorphism* $\mathbf{P}^{\mathrm{N}} \longleftrightarrow (\mathbf{P}^{\mathrm{N}})^{\vee}$ given by

$$p = [p_0 : p_1 : p_2 : ...] \mapsto [\partial_0\mathrm{F}(p) : \partial_1\mathrm{F}(p) : \partial_2\mathrm{F}(p) : ...]$$

where ∂_i denotes derivative with respect to the i-th variable x_i. The duality morphism is equal to ρ_{X}, and hence $\deg \rho_{\mathrm{X}} = 1$.

6.2. **The Veronese surface.** Let $\mathbf{P}^2 \simeq \mathrm{X} \subset \mathbf{P}^5$ be the Veronese surface. Then the projection-ramification morphism

$$\rho_{\mathrm{X}} : \mathbf{G}(2, 5) \dashrightarrow \mathbf{P}^9$$

assigns to a general net of conics N the cubic curve $\mathrm{C} \subset \mathbf{P}^2$ consisting of the nodes of the singular members of N. We will show that the degree of ρ_{X} is 3.

Suppose $\mathrm{N} = \langle \mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3 \rangle$ is a general net of conics in $\mathbf{P}^2$, with $\mathbf{Q}_i$ general ternary quadratic forms. For each line $\mathrm{L} \subset \mathbf{P}^2$, the net N restricts either to the complete linear series of two points on L or it restricts to a pencil. We call the latter type of line a *special* line.

Lemma 6.1. *The set of special lines $\mathrm{C}' \subset \mathbf{P}^{2*}$ is a smooth cubic equipped with a fixed-point free involution τ with quotient isomorphic to* C.

Proof. Each special line L arises from a unique singular conic of the net N, and hence possesses a conjugate line L', which defines the fixed point free involution τ.

Let $\mathrm{S} \longrightarrow \mathbf{P}^{2*}$ denote the rank 2 tautological subbundle. The forms $\mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3$ define a map of vector bundles

$$\mathcal{O}^3 \longrightarrow \mathrm{S}ym^2\mathrm{S}^*$$

The determinant of this map defines the locus of special lines. A simple Chern class calculation shows that this locus is a cubic $\mathrm{C}' \subset \mathbf{P}^{2*}$ □

If L is a special line, then L is a component of a unique singular member of the net N, and therefore has a conjugate line L'. On L, there are now three points of significance: $x = \mathrm{L} \cap \mathrm{L}'$, which is clearly a point on C, and the residual pair of points $a_{\mathrm{L}}, b_{\mathrm{L}} \in \mathrm{L} \cap \mathrm{C}$. Similarly for L'.

The quotient of C' by the fixed point free involution $\mathrm{L} \mapsto \mathrm{L}'$ is clearly isomorphic to C. Lying above the points $a_{\mathrm{L}}, b_{\mathrm{L}} \in \mathrm{C}$ are points $a'_{\mathrm{L}}, a''_{\mathrm{L}}, b'_{\mathrm{L}}, b''_{\mathrm{L}} \in \mathrm{C}'$.

Lemma 6.2. *Maintain the setting above. Then* $2a_{\mathrm{L}} \sim 2b_{\mathrm{L}} \sim 2a_{\mathrm{L}'} \sim 2b_{\mathrm{L}'}$.

Proof. Attached to a_{L} and b_{L} are the dual lines $a_{\mathrm{L}}^*, b_{\mathrm{L}}^* \subset \mathbf{P}^*$. The intersections $a_{\mathrm{L}}^* \cap \mathrm{C}'$ and $b_{\mathrm{L}}^* \cap \mathrm{C}'$ both contain the point $[\mathrm{L}] \in \mathbf{P}^{2*}$. The residual intersections with a_{L}^* and b_{L}^* precisely correspond to the elements $a_{\mathrm{L}}', a_{\mathrm{L}}''$ and $b_{\mathrm{L}}', b_{\mathrm{L}}''$ in C'. Hence, on C' we get a linear equivalence $a_{\mathrm{L}}' + a_{\mathrm{L}}'' \sim b_{\mathrm{L}}' + b_{\mathrm{L}}''$. Pushing this linear equivalence forward under the quotient map $\mathrm{C}' \longrightarrow \mathrm{C}$ gives $2a_{\mathrm{L}} \sim 2b_{\mathrm{L}}$.

Similarly, the points $[\mathrm{L}]$ and $[\mathrm{L}']$ constitute the two points in C' lying above x. Further, on C' we get the equivalence $[\mathrm{L}] + a_{\mathrm{L}}' + a_{\mathrm{L}}'' \sim [\mathrm{L}'] + a_{\mathrm{L}'}' + a_{\mathrm{L}'}''$ since both triads are collinear in $\mathbf{P}^{2*}$. Pushing this equivalence forward to C yields the equivalence $2a_{\mathrm{L}} \sim 2a_{\mathrm{L}'}$. □

Lemma 6.3. *The class $\eta = a_{\mathrm{L}} - b_{\mathrm{L}} \in \mathrm{J(C)}[2] \setminus \{0\}$ is independent of the point x and the choice of special line* L*.*

Proof. For each $x \in \mathrm{C}$ there are two special lines L, L' containing x, and two pairs of points $a_{\mathrm{L}}, b_{\mathrm{L}}$ and $a_{\mathrm{L}'}, b_{\mathrm{L}'}$ respectively.

Now, if four points $p, q, r, s \in \mathrm{C}$ satisfy $2p \sim 2q \sim 2r \sim 2s$, then it is always true that $p - q \sim r - s$, as a straightforward divisor calculation shows.

The lemma now follows by continuity, and the observation that there are only finitely many 2-torsion divisor classes on C. □

The 2-torsion class η defines a translation on C which takes a point $x \in \mathrm{C}$ to the unique point denoted $\eta(x) \in \mathrm{C}$ which is linearly equivalent to $x + \eta$. Therefore, Lemma 6.3 allows us to describe the set of special lines as the lines joining p with $\eta(p)$ for all points $p \in \mathrm{C}$.

Thanks to Lemma 6.3, we see that the projection-ramification map ρ_{X} factors as:

$$\rho_{\mathrm{X}} : \mathbf{G}(2,5) \dashrightarrow \mathrm{J}[2] \dashrightarrow \mathbf{P}^9 \tag{6.1}$$

where J[2] is the variety parametrizing pairs (C, η) with C a smooth plane cubic and $\eta \in \mathrm{J(C)}[2]$ a non-trivial 2-torsion element.

To conclude, we argue that the first map in (6.1) is birational by constructing its inverse. To that end, suppose C is a smooth plane cubic, and $\eta \in \mathrm{J(C)}[2]$ a chosen non-trivial 2-torsion element. We will create from this data a net of conics N whose set of nodes is C. Again, we think of η as a translation $\mathrm{C} \longrightarrow \mathrm{C}$ in the usual way.

For every point $p \in \mathrm{C}$, we get a line $\mathrm{L}_p \subset \mathbf{P}^2$ joining p and $\eta(p)$. In this way, we obtain a map $f : \mathrm{C} \longrightarrow \mathbf{P}^{2*}$ which is $2:1$ onto its image, since $\mathrm{L}_p = \mathrm{L}_q$ if and only if $p = q$ or $\eta(p) = q$. Using the fact that $\eta : \mathrm{C} \longrightarrow \mathrm{C}$ is fixed-point free , it is easy to see that f is also unramified. Hence, the image of f must be a smooth cubic

If $\beta \neq \eta \in \mathrm{J(C)}[2]$ is any other non-trivial 2-torsion element, the pair of points $\beta(p), \beta(\eta(p))$ span a well-defined second line L_p' containing the point p.

The collection of singular conics $\mathrm{L}_p \cup \mathrm{L}_p'$ parametrized by $p \in \mathrm{C}$ induces a map $\mathrm{C} \longrightarrow \mathbf{P}^5$, whose degree is 3, since one can check that through a general point in $\mathbf{P}^2$ there pass 3 of the lines L_p.

Hence, $\mathrm{C} \subset \mathbf{P}^2$ spans a plane, which by construction is a net of conics N whose locus of nodes is C.

6.3. **Rational curves, the differential construction, and case of Segre varieties.** Problem 1.1 connects with an intriguing collection of enumerative problems involving rational curves in projective space. We now explain these problems.

Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ be a degree k morphism. Its derivative

$$d\gamma : \mathrm{T}_{\mathbf{P}^1} \longrightarrow \gamma^*(\mathrm{T}_{\mathbf{P}^n})$$

may be viewed as a global section of the rank n vector bundle $\gamma^*(\mathrm{T}_{\mathbf{P}^n}) \otimes \mathrm{T}_{\mathbf{P}^1}^\vee$. The splitting of $\gamma^*(\mathrm{T}_{\mathbf{P}^n})$ is known to be balanced for a general morphism γ. In particular, if the divisibility

$$n \mid k$$

holds, and if we set $\ell := k + k/n - 2$, then a general γ satisfies:

$$(\gamma^*\mathrm{T}_{\mathbf{P}^n}) \otimes \mathrm{T}_{\mathbf{P}^1}^\vee \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbf{P}^1}(\ell).$$

The direct sum decomposition is not canonical, it is only defined up to the action of $\mathrm{GL}_n(k)$.

Assuming γ is an immersion, the element $d\gamma \in \mathrm{H}^0(\mathbf{P}^1, \bigoplus_{i=1}^n \mathcal{O}_{\mathbf{P}^1}(\ell))$ does not vanish anywhere, and hence defines a degree ℓ map

$$\mathrm{D}(\gamma) : \mathbf{P}^1 \longrightarrow \mathbf{P}^{n-1},$$

only well-defined up to the action of post-composition by $\mathrm{PGL}_n(k)$.

Definition 6.4. Let M_d^n denote the moduli stack parametrizing $\mathrm{PGL}_{n+1}(k)$ equivalence classes of degree d maps $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$, and let $\mathrm{U}_d^n \subset \mathrm{M}_d^n$ denote the open substack parametrizing local immersions with $\gamma^*(\mathrm{T}_{\mathbf{P}^n})$ balanced.

Remark 6.5. *Notice that* $\dim \mathrm{M}_k^n = (k+1)(n+1) - (n+1)^2 = (n+1)(k-n) = \dim \mathbf{G}(n,k)$. *Furthermore, notice that* $\mathrm{PGL}_2(k)$ *acts on* U_k^n *and* M_k^n *by pre-composition.*

Remark 6.6. *Though* M_k^n *is an Artin stack, the open substack* U_k^n *is a scheme, provided* $n \leq k$, *represented by an open subset of* $\mathbf{G}(n,k)$.

When $n \mid k$, and $\ell := k + k/n - 2$, we get the morphism of stacks:

$$\begin{aligned} \mathrm{D}_k^n : \mathrm{U}_k^n &\longrightarrow \mathrm{M}_\ell^{n-1} \\ \gamma &\longmapsto \mathrm{D}(\gamma) \end{aligned}$$

which we call the *differential construction*. Interestingly, the dimensions of the domain and codomain of the differential construction are equal, and this leads to another collection of enumerative problems:

Problem 6.7. Compute the degrees of the differential constructions $\mathrm{D}_k^n : \mathrm{U}_k^n \longrightarrow \mathrm{M}_\ell^{n-1}$.

Remark 6.8. *The maps* D_k^n *are clearly* $\mathrm{PGL}_2(k)$ *equivariant. The image of the differential construction* D_k^n *need not be the open set* U_ℓ^{n-1}. $\boxed{\textit{ToDo: Sure?}}$

The $n = k$ instances of Problem 6.7 are easy:

Proposition 6.9. *The degree of the differential construction* D_k^k *is* 1.

Proof. The space U_k^k is a single $\mathrm{PGL}_2(k)$ orbit. □

Definition 6.10. Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ be any map. We define the *point-hyperplane scroll* of γ to be

$$\mathrm{X}_\gamma := \left\{ (t, \Lambda) \mid \gamma(t) \in \Lambda \right\} \subset \mathbf{P}^1 \times (\mathbf{P}^n)^\vee$$

We denote by π_1, π_2 the projections of X_γ to $\mathbf{P}^1$ and $(\mathbf{P}^n)^\vee$ respectively. Finally, we set $\mathrm{X}_\gamma^\vee := \mathbf{P}(\gamma^*\mathrm{T}_{\mathbf{P}^n})$.

Remark 6.11. *The* $\mathbf{P}^{n-1}$*-bundle* X_γ *is isomorphic to* $\mathbf{P}(\gamma^*\mathrm{T}_{\mathbf{P}^n}^\vee)$. *Hence, for a general map* $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$, X_γ *and* $\mathrm{X}_\gamma^\vee$ *are balanced scrolls.*

Proposition 6.12. *Let* $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ *be a non constant map.*

(1) *The image of* $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ *is non-degenerate if and only if* $\pi_2 : \mathrm{X}_\gamma \longrightarrow (\mathbf{P}^n)^\vee$ *is finite; in any case,* $\deg \pi_2 = \deg \gamma$.

(2) *The ramification divisor* $\mathrm{R}(\pi_2) \subset \mathrm{X}_\gamma$ *is a smooth, codimension* 1 *subscroll of* X_γ *if and only if* γ *is an immersion.*

(3) *Assuming* γ *is an immersion, the dual section* $\mathrm{R}^\vee(\pi_2) \subset \mathrm{X}_\gamma^\vee$ *is induced by the inclusion* $d\gamma : \mathrm{T}_{\mathbf{P}^1} \hookrightarrow \gamma^* \mathrm{T}_{\mathbf{P}^n}$.

Proof. $\boxed{\text{ToDo: PROVE}}$ □

Let $\mathrm{X} = \mathbf{P}^1 \times \mathbf{P}^{n-1}$, and denote by h and f the divisor classes of the pullback of a hyperplane in $\mathbf{P}^{n-1}$ and a point in $\mathbf{P}^1$, respectively. When $n \mid k$, we see that Proposition 6.12 sets up a commuting diagram:

$$\begin{tikzcd} \mathrm{U}^n_k \arrow[r, leftrightarrow, "\text{duality}"] \arrow[d, "\mathrm{D}^n_k"] & \mathrm{PGL}_{n+1} \Big\backslash \left\{ \begin{array}{c} \text{Deg. } k \text{ maps} \\ \mathrm{X} \longrightarrow (\mathbf{P}^n)^\vee \\ \text{induced by } |h + \frac{k}{n} f| \end{array} \right\} \Big/ \mathrm{PGL}_n \arrow[d, "\rho_{\mathrm{X}}"] \\ \mathrm{M}^{n-1}_\ell \arrow[r, leftrightarrow, "\text{duality}"] & \left\{ \begin{array}{c} \text{Smooth divisors } \mathrm{R} \subset \mathrm{X} \\ \text{with div. class } |h + \ell f| \end{array} \right\} \Big/ \mathrm{PGL}_n \end{tikzcd}$$

From this, we conclude:

Proposition 6.13. *Let* $k = nm$, *and let* $\mathrm{X} \subset \mathbf{P}^{n(m+1)-1}$ *be the variety* $\mathbf{P}^1 \times \mathbf{P}^{n-1}$ *embedded by the linear series* $|h + mf|$. *Then*

$$\deg \rho_{\mathrm{X}} = \deg \mathrm{D}^n_k.$$

Corollary 6.14. *If* $\mathrm{X} \subset \mathbf{P}^{2n-1}$ *is a Segre embedding of* $\mathbf{P}^1 \times \mathbf{P}^{n-1}$, *then* $\deg \rho_{\mathrm{X}} = 1$.

Proof. The corollary follows at once from Proposition 6.13 and Proposition 6.9. □

6.4. **Quartic surface scrolls, trinodal quartics, and their perspective conics.** There are two types of quartic surface scrolls X in $\mathbf{P}^5$ – those that are balanced, isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and those which are isomorphic to F^2. As we show in explicit coordinates in the next section, in the latter case $\deg \rho_{\mathrm{X}} = 1$. The situation with the balanced scroll is much more interesting.

6.4.1. *Trinodal quartics and their perspective conics.* Aside from D^k_k, the only non-trivial differential construction we understand is

$$\mathrm{D}^2_4 : \mathrm{U}^2_4 \longrightarrow \mathrm{M}^1_4.$$

A general element $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^2$ of D^2_4 has as its image a trinodal quartic curve, which we will denote by $\mathrm{R} \subset \mathbf{P}^2$. Hence, the differential construction D^2_4 can be construed as a way of assigning a degree 4 branched cover $\mathrm{R} \longrightarrow \mathbf{P}^1$ to a trinodal plane quartic $\mathrm{R} \subset \mathbf{P}^2$. We will explore this beautiful construction in this section, and our exploration will lead to the following theorem.

Theorem 6.15. *The degree of the differential construction* D^2_4 *is* 2.

We have no doubt that the classical geometers must have known Theorem 6.15. The old paper [**?**] discusses rational quartics quite thoroughly, and one can see computations and conclusions which seem relevant. At the very least, we are confident that the *presentation* of the following material is new.

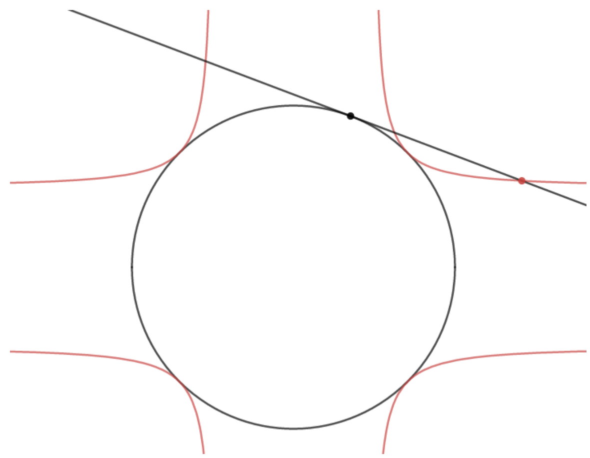

FIGURE 1. The red curve is the trinodal quartic R given by equation $\frac{1}{x^2}+\frac{1}{y^2}=1$. The black curve C, given by $x^2+y^2=4$, is one of its perspective conics. As one slides the black point once along C, the corresponding red point sweeps out R exactly one time.

Definition 6.16. Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^2$ be degree 4 map. A *perspective conic* of γ is a smooth conic $\mathrm{C} \subset \mathbf{P}^2$ which admits a parametrization $\varphi : \mathbf{P}^1 \longrightarrow \mathrm{C}$ such that for all $t \in \mathbf{P}^1$, the incidence

$$\gamma(t) \in \mathrm{T}_{\varphi(t)}\mathrm{C}$$

holds.

Remark 6.17. *Notice that the notion of perspective conic does not depend on choice of parametrization of a trinodal quartic* R.

Proposition 6.18. *Let γ be a map inducing a point in U_4^2, so that $\mathrm{X}_\gamma \simeq \mathbf{P}^1 \times \mathbf{P}^1$. Then the set of perspective conics of γ are in natural bijection with the set of square-zero sections of the point-hyperplane scroll $\pi_1 : \mathrm{X}_\gamma \longrightarrow \mathbf{P}^1$ projecting to smooth conics under $\pi_2 : \mathrm{X}_\gamma \longrightarrow (\mathbf{P}^2)^\vee$.*

Proof. A perspective conic $\varphi : \mathbf{P}^1 \longrightarrow \mathrm{C}$ induces a section $\sigma_\mathrm{C} : \mathbf{P}^1 \longrightarrow \mathrm{X}_\gamma$ defined by $t \mapsto \mathrm{T}_{\varphi(t)}\mathrm{C}$. This section $\sigma_\mathrm{C} \subset \mathrm{X}_\gamma$ projects isomorphically to the dual conic $\mathrm{C}^\vee \subset (\mathbf{P}^2)^\vee$, by the definition of being perspective. A simple divisor class computation shows that σ_C must therefore by a square-zero section.

Conversely, given a square-zero section $\sigma : \mathbf{P}^1 \longrightarrow \mathrm{X}_\gamma$ projecting to a conic $\mathrm{D} \subset (\mathbf{P}^2)^\vee$, we consider $\mathrm{D}^\vee \subset \mathbf{P}^2$. The parametrization of D induces a parametrization of its dual $\mathrm{D}^\vee$, which by design realizes the condition of being a perspective conic for γ. □

The perspective conics have the following geometric significance:

Proposition 6.19. *Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^2$ represent an element of U_4^2 , and let* C *be a perspective conic for γ. Then the divisor $\gamma^*(\mathrm{C})$ equals* 2E *for $\mathrm{E} \subset \mathbf{P}^1$ a degree* 4 *effective divisor.*

Proof. Let $\mathrm{Y} \subset \mathrm{C} \times \mathbf{P}^1$ be the correspondence defined by

$$\mathrm{Y} = \{(p,t) \mid \gamma(t) \in \mathrm{T}_p\mathrm{C}\}.$$

The projection $p_2 : \mathrm{D} \longrightarrow \mathbf{P}^1$ is clearly finite and degree 2, and the conic C being perspective means that the double cover p_2 has a section. By considering monodromy, This can only happen if the branch scheme $\mathrm{B} \subset \mathbf{P}^1$ of p_2 has the form 2E. Now, the proposition follows from the simple observation that $\mathrm{B} = \gamma^*(\mathrm{C})$. □

Therefore, perspective conics are 4-*tangent* to the trinodal quartic R – they meet R everywhere with even multiplicity. (This should be interpreted "per branch" near the singularities.)

Remark 6.20. *If* $\mathrm{R} \subset \mathbf{P}^2$ *is a trinodal quartic, and* $\mathrm{C} \subset \mathbf{P}^2$ *is a perspective conic tangent to* R *at four distinct points, then the cross-ratio of the set* $\mathrm{C} \cap \gamma(\mathbf{P}^1)$ *viewed as four points on* C *is equal to the cross-ratio of the same set viewed as four points on (the normalization of)* R.

The family of perspective conics is but one out of four families of smooth conics which are 4-tangent to R. We investigate this in the next section. (Of course, these families of conics will have singular limits, but we are emphasizing that the general member is smooth.)

6.5. **Reinterpreting the perspective conics.** Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^2$ be a quartic map, with image R. In what follows, suppose R is tri-nodal, and that none of the branches of the nodes of R are flexes. We let $\mathrm{N} = \{a, b, c\}$ denote the set of nodes of R, and we write a', a'', etc. for the two preimages of the corresponding node of R. Finally, we let $\mathrm{S} \subset \mathbf{P}^1$ be the preimage of N.

In this section, we will relate the geometry from the previous section with the 2-torsion in Pic(R)[2].

Lemma 6.21. *Let* $\mathrm{C} \subset \mathbf{P}^2$ *be a smooth conic such that* $\gamma^*(\mathrm{C})$ *is a divisor of the form* 2E *with* E *consisting of* 4 *distinct points on* $\mathbf{P}^1$. *Then* $\gamma(\mathrm{E})$ *must be supported away from* N.

Proof. In order for $\gamma^*(\mathrm{C})$ to have the form 2E with $\mathrm{E} \cap \mathrm{S}$ nonempty, the conic C must be tangent to both branches of the corresponding node. This forces C to be singular, (with the singularity of C located at the node of R). □

We call a smooth conic as in Lemma 6.21 a 4-*tangent* conic of R.

Lemma 6.22. *Let* C *be a* 4*-tangent conic to* R*, and write* $\mathrm{C} \cap \mathrm{R} = 2\mathrm{E}$. *Then the line bundle* $\mathcal{O}_{\mathrm{R}}(\mathrm{E})$ *differs from* $\mathcal{O}_{\mathrm{R}}(1)$ *by a well-defined element of* Pic(R)[2].

Proof. The conic C yields a section of $\mathcal{O}_{\mathrm{R}}(2)$, up to scaling. The equality $\mathrm{C} \cap \mathrm{R} = 2\mathrm{E}$ translates into an isomorphism $\mathcal{O}_{\mathrm{R}}(\mathrm{E})^{\otimes 2} \simeq \mathcal{O}_{\mathrm{R}}(2)$. The result follows. □

The Jacobian $\mathrm{Pic}^0(\mathrm{R})$ is isomorphic to $(\mathbf{C}^\times)^3$. This isomorphism is realized as follows. Any line bundle L on R of degree 0 pulls back to the trivial bundle $\mathcal{O}_{\mathbf{P}^1}$. To recover L, one must specify identifications of the fibers of $\mathcal{O}_{\mathbf{P}^1}$ at a' and a'', b' and b'', and c' and c''. This yields three nonzero complex numbers. In this way, the 2-torsion subgroup Pic(R)[2] is identified with $\{-1, +1\}^3 \subset (\mathbf{C}^\times)^3$.

Lemma 6.23. *As* R *varies among all generic tri-nodal quartics, the monodromy group of the set* N *is the full symmetric group* S_3.

Proof. This is well-known, so we omit it. □

As R varies, the element $(-1, -1, -1) \in$ Pic(R)[2] is unchanged by monodromy, and hence distinguished. We will see that this distinguished element corresponds (in the sense of Lemma 6.22) to the 1-parameter family of perspective conics of R. Furthermore, there are two monodromy orbits in Pic(R)[2] of size three, namely the orbits of $(-1, -1, 1)$ and $(-1, 1, 1)$. We will see the geometric significance of these orbits in terms of plane geometry.

The line bundle $\mathcal{O}_{\mathrm{R}}(1)$ is the dualizing bundle ω_{R}. By Riemann-Roch, any other degree 4 line bundle L on R has $h^0(\mathrm{R}, \mathrm{L}) = 2$.

Lemma 6.24. *Let* $\ell \subset \mathbf{P}^2$ *be a line passing through the node* $a \in \mathrm{R}$*, and let* $\eta_a \in \mathrm{Pic}^0(\mathrm{R})[2]$ *be the element with* +1*-identifications above the nodes* b *and* c*, and* –1 *identification above* a. *Then the Cartier divisor* $\ell \cap \mathrm{R}$ *is the divisor of a unique (up to scaling) section* s *of* $\mathcal{O}_{\mathrm{R}}(1)$ *and of a unique (up to scaling) section* s' *of* $\mathcal{O}_{\mathrm{R}}(1) \otimes \eta_a$.

Proof. The pullback $\gamma^*(\mathcal{O}_R(1))$ is isomorphic to $\mathcal{O}_{\mathbf{P}^1}(4)$, and the pullback map provides a fixed set of identifications of the fibers of $\mathcal{O}_{\mathbf{P}^1}(4)$ over a', a'', etc. The identifications creating $\mathcal{O}_R(1)\otimes\eta$ are the above fixed identifications, multiplied by the ±1 identifications prescribing the 2-torsion bundle η_a.

The line ℓ obviously corresponds to a unique (up to scaling) section s of $\mathcal{O}_R(1)$. The section $\gamma^*(s)$ is a quartic polynomial on $\mathbf{P}^1$ vanishing, among other places, at the points a', a''. Since the section $\gamma^*(s)$ vanishes at a' and a'', it is automatically compatible with any identifications of the line bundle $\mathcal{O}_{\mathbf{P}^1}(4)$ above a' and a'', and thus automatically descends to a section s' of the twist $\mathcal{O}_R(1)\otimes\eta_a$. □

Corollary 6.25. *Let $\eta_a \in \mathrm{Pic}(R)[2]$ be the element with* –1 *identification above the node a, and* +1 *identifications above the other nodes. Then the Cartier divisors cut out by global sections of $\mathcal{O}_R(1)\otimes\eta_a$ are precisely those cut out by lines in $\mathbf{P}^2$ passing through the point a.*

Proof. This follows from Lemma 6.24 and from the fact that $h^0(R, O_R(1)\otimes\eta_a) = 2$. □

Corollary 6.26. *Let* C *be a* smooth *conic* 4*-tangent to* R*, with* $C\cap R = 2E$*. Then* $\mathcal{O}_R(E) = O_R(1)\otimes\eta$ *where* $\eta\in \mathrm{Pic}(R)[2]$ *is an element with at least two* –1 *identifications above the nodes of* R.

Proof. This follows from Corollary 6.25 and Lemma 6.21. □

Theorem 6.27. *The system of perspective conics for* R *cuts out the linear system of divisors for* $\mathcal{O}_R(1)\otimes\eta$*, where* $\eta\in\mathrm{Pic}(R)[2]$ *is the distinguished* 2*-torsion element* (–1, –1, –1).

Proof. There is a well-defined system of perspective conics for a tri-nodal quartic R, which varies rationally with R. Indeed, by Proposition 6.18, it is given by the dual conics of the system of square-zero sections of X_γ upon choosing any parametrization γ of R. By Lemma 6.23, we conclude that the degree 4 linear system on R cut out by the perspective conics must correspond to the distinguished 2-torsion element $\eta = (–1, –1, –1)$ in the statement of the proposition. □

Now we investigate the four remaining 1-parameter families of 4-tangent conics to R. Before continuing, let R′ denote the Cremona image of R under the Cremona transformation through the three nodes N. Note that R′ is a conic. Furthermore, let $\{x, y, z\}$ denote the Cremona images of the lines joining the nodes pairwise.

Lemma 6.28. *The* 1*-parameter families of* 4*-tangent conics for* R *are in natural bijection with the families of irreducible quartics* Z *which are* 4*-tangent to the conic* R′*, and have singularities at* $\{x, y, z\}$.

Proof. The Cremona transformation through N transforms a 4-tangent conic to a rational quartic Z having the required properties. □

Let $\alpha : Q\longrightarrow\mathbf{P}^2$ denote the double cover of $\mathbf{P}^2$ branched over the curve R′, and let ι denote the involution on Q interchanging sheets. Note that Q is isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$, so it makes sense to talk about (m, n) curves.

Lemma 6.29. *Let* Z *be an irreducible tri-nodal quartic* 4*-tangent to* R′*. Then any parametrization* $\beta : \mathbf{P}^1\longrightarrow Z$ *factors through a lift* $\mathbf{P}^1 \xrightarrow{\beta'} Q \xrightarrow{\alpha} \mathbf{P}^2$*. There are two such lifts, conjugate to each other under* ι.

Proof. This follows from the fact that Z is tangent to the branch curve R′, with even order intersection multiplicity everywhere. □

For any given irreducible tri-nodal quartic Z which is 4-tangent to R′, we let Z′ denote any one of the two lifts of Z in Q. There are now only two possibilities, up to involution: Either Z′ is a smooth $(1,3)$ curve, or Z′ is a once-nodal $(2,2)$ curve.

The 4-tangent tri-nodal quartics Z we are interested in have the additional property of having singularities at the three points $\{x, y, z\}$. This constraint gives a description of the four 1-parameter families of such curves in terms of their lifts in S.

Indeed, let $x', x'',\ y', y'',\ z', z''$ denote preimages in Q of the corresponding points in $\mathbf{P}^2$. Then consider, for example, the linear system of $(2,2)$ curves, nodal at x', and passing through y', y'', z', z''. This system is 1 dimensional, and projects to a family of 4-tangent (generically tri-nodal) quartics X in $\mathbf{P}^2$. Interchanging x' and x'' in the above description yields the same family of 4-tangent quartics, due to the involution ι.

Therefore, we obtain three families in this way, labelled by the points x, y and z. In the original geometric picture of $\mathrm{R} \subset \mathbf{P}^2$, these three families must therefore correspond to elements $\eta \in \mathrm{Pic}(\mathrm{R})[2]$ with precisely two -1 identifications.

The final, distinguished family of tri-nodal 4-tangent quartics is described as follows: Consider the pencil of $(1,3)$ curves on Q passing through all six points $x', x'', y', y'', z', z''$. This pencil projects to the fourth, and final family, and after re-applying the Cremona transformation yields the family of perspective conics.

From the discussion immediately preceding Theorem 6.27, we also conclude:

Corollary 6.30. *Let* $\mathrm{R} \subset \mathbf{P}^2$ *be a tri-nodal quartic. Then through general point* $p \in \mathbf{P}^2$ *there pass two perspective conics, i.e. the* 1*-parameter family of perspective conics traces out a conic in the* $\mathbf{P}^5$ *of conics.*

Proof. Indeed, after applying the Cremona transformation through the nodes of R, we see that the perspective conics transform into the image of a pencil of $(1,3)$ curves on the double cover $\mathrm{S} \longrightarrow \mathbf{P}^2$. Therefore, for a general point in $\mathbf{P}^2$ there will be two curves in the pencil projecting through it. □

6.5.1. *The explicit differential construction for trinodal quartics.* A trinodal quartic R can be obtained as an abstract curve by identifying three pairs of points $\{a', a''\}, \{b', b''\}, \{c', c''\}$ on $\mathbf{P}^1$. These pairs can be encoded by the three binary quadratic forms (up to scale) defining them. In terms of these three quadratic forms, we will now describe the differential construction D_4^2.

In what follows, we let $\{q_1, q_2, q_3\}$ denote a point in $\mathrm{Sym}^3 \mathbf{P}\mathrm{H}^0(\mathcal{O}_{\mathbf{P}^1}(2))$.

Definition 6.31. Let

$$\nu : \mathrm{Sym}^3 \mathbf{P}\mathrm{H}^0(\mathcal{O}_{\mathbf{P}^1}(2)) \dashrightarrow \mathbf{G}(2,4)$$

denote the map given by the formula:

$$\nu\left(\{q_1, q_2, q_3\}\right) = \left\{ \begin{array}{c} \text{v. space of meromorphic 1-forms } \omega \text{ on } \mathbf{P}^1 \text{ with at worst} \\ \text{simple poles at the zeros of } q_i \text{ and with } \textit{opposite} \text{ residues} \\ \text{at the pairs of zeros of } q_i, \text{ for all } i = 1,2,3 \end{array} \right\}$$

Proposition 6.32. *The map* ν *is birational.*

Proof. Suppose a general three dimensional space $\mathrm{W} \subset \mathrm{H}^0(\mathcal{O}_{\mathbf{P}^1}(4))$ is given. Then the induced degree four map $\mathbf{P}^1 \longrightarrow \mathbf{P}\mathrm{W}^\vee$ is the normalization of a trinodal quartic R. The vector space W is naturally identified with the sections of the dualizing sheaf of R, which consist of meromorphic 1-forms on $\mathbf{P}^1$ with the properties stated in the proposition. □

Definition 6.33. Let

$$\pi : \mathrm{Sym}^3 \mathbf{P}^2 \dashrightarrow \mathbf{G}(1,4)$$

be given by the formula

$$\pi(\{q_1,q_2,q_3\}) = \left\{\begin{array}{l}\text{v. space of meromorphic 1-forms } \omega \text{ with at worst simple}\\ \text{poles at the zeros of } q_i \text{ and with } \textit{equal} \text{ residues}\\ \text{at the pairs of zeros of } q_i\text{, for all } i=1,2,3\end{array}\right\}$$

Proposition 6.34. *The rational map $\pi\circ\nu^{-1} : \mathbf{G}(2,4) \dashrightarrow \mathbf{G}(1,4)$ is the differential construction D^2_4.*

Proof. Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^2$ be a general map induced by a three dimensional vector space $\mathrm{W} \subset \mathrm{H}^0(\mathcal{O}_{\mathbf{P}^1}(4))$ having image R, and let (q_1,q_2,q_3) be $\nu^{-1}(\varphi)$. The pencil $\mathrm{D}(\gamma)$ is cut out by the perspective conics. $\boxed{\text{ToDo: Why?}}$ According to Theorem 6.27, the linear series on R cut out by perspective conics is $\mathcal{O}_{\mathrm{R}}(1)\otimes\eta$, where η is the distinguished element $(-1,-1,-1)\in \mathrm{Pic}(\mathrm{R})[2]$. If the space of sections of the line bundle $\mathcal{O}_{\mathrm{R}}(1)$ is identified with $\nu(q_1,q_2,q_3)$, then it follows that the space of sections of the twist $\mathcal{O}_{\mathrm{R}}(1)\otimes\eta$ equals $\pi(q_1,q_2,q_3)$. $\square$

Definition 6.35. Let $\{a(x,y),b(x,y)\}$ be two homogeneous quadratic polynomials with no common zeros. Their *Jacobian* is

$$\mathrm{J}(a,b) := a_x b_y - a_y b_x.$$

Note that the Jacobian vanishes precisely at the two branch points of the map $[x:y]\mapsto [a(x,y):b(x,y)]$.

Theorem 6.36. *Let $\{q_1,q_2,q_3\}\in \mathrm{Sym}^3\,\mathbf{P}^2$ have six distinct roots. Then the vector space*

$$\langle q_1\mathrm{J}(q_2,q_3), q_2\mathrm{J}(q_1,q_3), q_3\mathrm{J}(q_1,q_2)\rangle$$

is equal to $\pi(q_1,q_2,q_3)\in\mathbf{G}(1,4)$.

Proof. By $\mathrm{SL}_2(k)$-equivariance, it suffices to prove the theorem for three quadratic functions $\{xy,q_2,q_3\}$ where q_2 and q_3 are general.

Let α_1,α_2, and β_1,β_2 denote the roots of q_2,q_3. Note that these roots are assumed to be in $\mathbf{A}^1\subset\mathbf{P}^1$. We let $t=x/y$ denote the affine coordinate.

The vector space $\Pi := \pi(t,q_2(t),q_3(t))$ is equal to the vector space of forms

$$\omega = \frac{f(t)\,dt}{tq_2(t)q_3(t)},$$

with $\deg(f)\le 4$, and with the additional constraints

$$\begin{aligned}\mathrm{Res}_{\alpha_1}\omega &= \mathrm{Res}_{\alpha_2}\omega\\ \mathrm{Res}_{\beta_1}\omega &= \mathrm{Res}_{\beta_2}\omega\\ \mathrm{Res}_{0}\omega &= \mathrm{Res}_{\infty}\omega\end{aligned}$$

Since we know a priori that the space of such forms is two dimensional, we conclude in particular that there exists a nonzero $\omega\in\Pi$ which is nonzero and vanishing at α_1. However, the first residue condition then forces ω to vanish at α_2 as well. (This is clear from the geometry: an element of the pencil of perspective conics is cut out by a (possibly singular) conic in $\mathbf{P}^2$. If it contains a node, then its pullback to $\mathbf{P}^1$ must vanish at both points above the node.)

Therefore, there exists an $\omega\in\Pi$ of the form

$$\omega = \frac{(t-\alpha_1)(t-\alpha_2)g(t)\,dt}{tq_2q_3} = \frac{g(t)\,dt}{tq_3}.$$

The residue conditions at β_i, and $0, \infty$ together imply that, up to nonzero scaling,

$$g(t) = t^2 - \beta_1\beta_2.$$

The roots $\pm\sqrt{\beta_1\beta_2}$ are precisely the branch points of the map $[x : y] \longrightarrow [xy : q_3]$. Therefore, we see that ω vanishes at the roots of the quartic polynomial $q_1 j(xy, q_3)$. The theorem follows by arguing in the same manner for the two other pairs of roots.

□

Theorem 6.37. *Let* $\mathrm{X} \subset \mathbf{P}^5$ *be a balanced quartic surface scroll. Then* $\deg \rho_{\mathrm{X}} = 2$.

6.6. **Eccentric surface scrolls.** Let $\mathrm{V} = \mathcal{O}(1) \oplus \mathcal{O}(k+1)$, $\mathrm{X} = \mathbf{P}\mathrm{V}$. Choose an affine coordinate t on $\mathbf{P}^1$, and consider the projection-ramification enumerative problem for $\mathrm{X} \subset \mathbf{P}^{k+3}$. We claim:

Proposition 6.38. ρ_{X} *is birational.*

Let $\mathrm{A} = \mathrm{H}^0(\mathcal{O}_{\mathrm{X}}(1))$. This vector space will be identified with the space of expressions of the form $\ell(t)x_0 + q_{k+1}(t)x_1$, where ℓ, q_{k+1} have degrees 1 and $k+1$ respectively.

If $\mathrm{W} \subset \mathrm{A}$ is a general three dimensional vector space, then there will be three elements in W of the form

$$\begin{aligned} w_0 &= t(x_0 + q_k x_1) \\ w_\infty &= (x_0 + r_k x_1) \\ w_* &= s_{k+1} x_1 \end{aligned}$$

The Jacobian of this triple is:

$$s x_0 + \left[s(qt)' - s'(qt) - t(r's - s'r)\right] x_1 \tag{6.2}$$

Proof of Proposition 6.38. Let $r := \sigma x_0 + \tau x_1 \in \mathrm{H}^0(\mathrm{X}, \mathcal{O}(\mathrm{R}))$ be a general element, we can extract the unique vector space W such that $\rho_{\mathrm{X}}(\mathrm{W}) = r$ as follows: First, we set $s := \sigma$. Secondly, given s, the equation $[s(qt)' - s'(qt) - t(r's - s'r)] = \tau$ is a system of $2k+2$ linear equations on the $2k+2$ coefficients of the pair (q, r). We know (from our our main theorem) that this system has only finitely many solutions, hence it must have a unique solution, proving the proposition. □

6.7. **Eccentric threefold scrolls.** Now let $\mathrm{V} = \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(k+1)$, $k \geq 0$, and set $\mathrm{X} := \mathbf{P}\mathrm{V}$. View $\mathrm{X} \subset \mathbf{P}^{k+5}$ via the natural $\mathcal{O}(1)$ on X. Again, we choose affine coordinate $t \in \mathbf{P}^1$ and relative coordinates x_0, x_1, x_2 on X corresponding to the three factors of the splitting of V.

Proposition 6.39. ρ_{X} *is birational.*

Suppose $\mathrm{W} \subset \mathrm{H}^0(\mathrm{V})$ is a general 4 dimensional vector space. Then the projection $\mathrm{W} \longrightarrow \mathrm{H}^0(\mathcal{O}(1) \oplus \mathcal{O}(1))$ will be an isomorphism. Hence, there will be 4 uniquely defined elements of W of the form:

$$\begin{aligned} x_0 &+ a x_2 \\ x_1 &+ b x_2 \\ t x_0 &+ c x_2 \\ t x_1 &+ d x_2 \end{aligned}$$

where a, b, c, d are degree $\leq k+1$ polynomials in t. The Jacobian determinant for this tuple of equations is:

$$\alpha x_0 + \beta x_1 + \gamma x_2 = (d - bt)x_0 + (at - c)x_1 + \left[a't(bt - d) + b't(c - at) + c'(d - bt) + d'(at - c)\right] x_2. \tag{6.3}$$

Proof of Proposition 6.39. We replace the Grassmannian $\mathrm{G}(4, \mathrm{H}^0(\mathrm{V}))$ with the affine open subset $\mathbf{A}^{4k+8}$ parametrizing quadruples (a, b, c, d). Then the ramification divisor equation (6.3) defines a map

$$\rho^* : \mathbf{A}^{4k+8} \longrightarrow \mathbf{A}^{4k+9}$$

where the latter $\mathbf{A}^{4k+9}$ is the vector space of triples (α, β, γ) with $\deg \alpha, \beta \leq k+2$ and $\deg \gamma \leq 2k+2$. The projection-ramification ρ_{X} map ρ is recovered by composing ρ^* with the projection $\mathbf{A}^{4k+9} \dashrightarrow \mathbf{P}^{4k+8}$.

First, if (a, b, c, d) are general, then one can easily use the fact that $d - bt$ and $at - c$ are relatively prime to conclude that ρ^* is generically injective on tangent spaces, and hence the generic fiber of ρ^* is finite.

We next show that ρ_{X} is dominant. In light of the previous paragraph, it suffices to prove: If (α, β, γ) is a general point in the image of ρ^*, and $\lambda \neq 0, 1$ is a constant, then $\lambda(\alpha, \beta, \gamma)$ is not in the image of ρ^*.

To this end, suppose (a, b, c, d) is a general point in $\mathbf{A}^{4k+8}$. Then $\alpha := d - bt$ and $\beta := at - c$ will be degree $k+2$ polynomials which are relatively prime.

For any polynomial $p(t)$, let p^* denote the highest coefficient of p. Then we observe that $\beta^* = a^*$. Furthermore, the expression for γ is easily seen to be

$$\gamma = (\alpha'\beta - \beta'\alpha) + \alpha a + \beta b \tag{6.4}$$

where $'$ denotes d/dt.

If we scale by λ, we get:

$$\begin{aligned} \lambda\alpha &= \lambda(d - bt) \\ \lambda\beta &= \lambda(at - c) \\ \lambda\gamma &= \lambda(\alpha'\beta - \beta'\alpha) + \lambda\alpha a + \lambda\beta b \end{aligned} \tag{6.5}$$

At the same time, if $\lambda(\alpha, \beta, \gamma)$ is also realized by some quadruple $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ then we get the equations:

$$\begin{aligned} \lambda\alpha &= \tilde{d} - \tilde{b}t \\ \lambda\beta &= \tilde{a}t - \tilde{c} \\ \lambda\gamma &= \lambda^2(\alpha'\beta - \beta'\alpha) + \lambda\alpha\tilde{a} + \lambda\beta\tilde{b} \end{aligned} \tag{6.6}$$

The second equation gives $\tilde{a}^* = \lambda\beta^*$

The last equation gives: $\gamma = \lambda(\alpha'\beta - \beta'\alpha) + \alpha\tilde{a} + \beta\tilde{b}$. Combining with (6.4), we get

$$\alpha(a - \beta') + \beta(b + \alpha') = \alpha(\tilde{a} - \lambda\beta') + \beta(\tilde{b} + \lambda\alpha'). \tag{6.7}$$

Since α and β are relatively prime and have degree greater than $a, b, \tilde{a}, \tilde{b}$, we conclude that

$$\begin{aligned} a - \beta' &= \tilde{a} - \lambda\beta' \\ b + \alpha' &= \tilde{b} + \lambda\alpha' \end{aligned}$$

By examining top coefficients, and using $a^* = \beta^*$, $\tilde{a}^* = \lambda\beta^*$ we get:

$$\beta^* - (k+2)\beta^* = \lambda\beta^* - \lambda(k+2)\beta^*$$

or

$$(1 - \lambda)\beta^* = (1 - \lambda)(k+2)\beta^*$$

Given our assumption on λ, this is only possible if $\beta^* = 0$. However, since (a, b, c, d) were chosen generically, $\beta^* = a^*$ would not be zero, providing our contradiction.

Finally, we conclude that $\deg \rho_{\mathrm{X}} = 1$. It suffices to prove that a general ramification equation $\alpha x_0 + \beta x_1 + \gamma x_2$ of the form Equation 6.3 arises from a unique choice of polynomials (a, b, c, d). The conditions $d - bt = \alpha$ and $at - c = \beta$ produce an affine linear subspace Λ in the vector space of choices (a, b, c, d). With respect to linear coordinates on Λ, the expression for γ is also linear, and hence the available choices of (a, b, c, d) producing Equation 6.3 is an intersection of affine linear spaces. Since we already know ρ^* finite, we immediately conclude $\deg \rho_{\mathrm{X}} = 1$.

□

Since every smooth, three dimensional rational normal scroll specializes isotrivially to the scroll X in Proposition 6.39, we immediately get:

Corollary 6.40. *The projection-ramification map ρ_{X} is dominant for every smooth three dimensional rational normal scroll $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$.*

7. A network of webs, pencils, and points.

Denote by $a^{(b)}$ the bth symmetric power of a vector space of dimension a, by $[a]$ the group PGL_a, by $[a, b]$ the product $\mathrm{PGL}_a \times \mathrm{PGL}_b$, and so on.

$$
\begin{tikzcd}
\mathbf{P}(3^{(2)})/\mathbf{G}_m^3 \times \mathrm{S}_3 \arrow[r, leftrightarrow] & \{(\mathrm{X}, \mathrm{C}) \mid \mathrm{X} \in \mathrm{Hilb}_3 \mathbf{P}(3), \mathrm{C} \in \mathbf{P}(3^{(2)})\}/[3] \arrow[rr, leftrightarrow] \arrow[d, shift right, "\text{Syzygies}"'] & & \mathrm{M}_{0,6}/\mathrm{G} \arrow[dl, red] \arrow[dddd, red] \\
& \mathbf{P}(2 \boxtimes 3 \boxtimes 3) \times \mathbf{P}(3^{(2)})/[2,3,3] \arrow[u, shift right, "\text{Degeneracy locus}"'] \arrow[d, leftrightarrow, "\text{Segre } \mathbf{P}^1 \subset \mathbf{P}^2"] & & \\
& \mathbf{P}(2 \boxtimes 3 \boxtimes 2^{(2)})/[2,3,2] \arrow[d, leftrightarrow, "="] & & \\
\mathbf{Gr}(3, 2^{(4)})/[2] \arrow[r, shift right, "\text{Syzygies}"'] \arrow[d, "\mathrm{D}_4^2"'] \arrow[rr, bend right, "\text{Wronskian}"] & \mathbf{Gr}(3, 2 \boxtimes 2^{(2)})/[2,2] \arrow[l, shift right, "\text{Minors}"'] \arrow[d, "\rho"'] & \mathrm{M}_{0;6}/\mathrm{S}_6 & \\
\mathbf{Gr}(2, 2^{(4)})/[2] \arrow[r, leftrightarrow, "="] \arrow[rrr, bend right, "\text{Wronskian}"'] & \mathbf{P}(2 \boxtimes 2^{(4)})/[2,2] & & \mathrm{M}_{0;6}/\mathrm{S}_6
\end{tikzcd}
$$

Here $\mathrm{G} \subset \mathrm{S}_6$ is the stabilizer of $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$.

The red arrows are not the obvious forgetful maps. Is there an easy description?

The GIT quotient in the top left hand corner $\mathbf{P}(3^{(2)})/\mathbf{G}_m^3 \times \mathrm{S}_3$ is very easy to compute. It is very likely that this threefold is something famous. Actually (before modding out by S_3), it appears to be the cubic hypersurface

$$\mathrm{AB}^2 - \mathrm{CDE} \subset \mathbf{P}^4.$$

Why? Consider the $\mathbf{G}_m^3$ acting on an equation in $(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ of degree 2

$$a\mathrm{X}^2 + b\mathrm{Y}^2 + c\mathrm{Z}^2 + \alpha \mathrm{YZ} + \beta \mathrm{XZ} + \gamma \mathrm{XY}.$$

The quotient will be the Proj of the semi-invariant ring, which is

$$\mathrm{R} = k[abc, \alpha\beta\gamma, a\alpha^2, b\beta^2, c\gamma^2].$$

Denoting these generators by $\mathrm{A}, \ldots, \mathrm{E}$, we see that $\operatorname{Proj} \mathrm{R} \subset \mathbf{P}^4$ is cut out by $\mathrm{AB}^2 - \mathrm{CDE}$. The S_3 action just permutes the $\mathrm{C}, \mathrm{D}, \mathrm{E}$. Is the Richelot involution simply $\mathrm{B} \mapsto -\mathrm{B}$?

8. Some more GIT

8.1. **A moduli space of sequences.** Let us form a moduli space of sequences

$$\mathcal{O}(-2)^2 \xrightarrow{\mathrm{A}} \mathcal{O}^3 \xrightarrow{\mathrm{B}} \mathcal{O}(4). \tag{8.1}$$

We can construct this moduli space using GIT as follows. Let X be the closed subvariety of $\mathbf{P}^{17} \times \mathbf{P}^{14}$ consisting of pairs of matrices (A, B) that satisfy $\mathrm{BA} = 0$. Then our moduli space is the quotient $\mathrm{X} /\!\!/ \mathrm{SL}_2 \times \mathrm{SL}_3$. Note that X has a rank 2 Picard group, so the GIT quotient might depend on a choice of linearization.

For a generic choice of $(\mathrm{A}, \mathrm{B}) \in \mathrm{X}$, the sequence (8.1) will be exact. Therefore, up to the action of $\mathrm{GL}(2)$ and $\mathrm{GL}(3)$, the matrix A can be recovered from B and vice-versa. If the linearization tilts towards B, then the GIT quotient will be $\mathbf{Gr}(3, 2^{(4)})$. If the linearization tilts towards A, then the GIT quotient will be $\mathbf{Gr}(3, 2 \boxtimes 2^{(2)}) /\!\!/ \mathrm{SL}(2)$. For an intermediate choice, we will get something that interpolates between the two.

8.2. **The differential construction.** Given A and B as in (8.1), we get a diagram

$$\begin{tikzcd}
 & \mathcal{O}(-2)^2 \arrow[r, "\mathrm{A}"] \arrow[d, "\mathrm{D}"] & \mathcal{O}^3 \arrow[r, "\mathrm{B}"] \arrow[d, "\mathrm{J}"] & \mathcal{O}(4) \arrow[d] & \\
0 \arrow[r] & \mathcal{O}(2) \arrow[r] & \mathcal{O}(3) \oplus \mathcal{O}(3) \arrow[r] & \mathcal{O}(4) \arrow[r] & 0,
\end{tikzcd}$$

where the sequence on the bottom can be thought of as the jet bundle sequence

$$0 \longrightarrow \mathrm{L} \otimes \Omega \longrightarrow j_2\mathrm{L} \longrightarrow \mathrm{L} \longrightarrow 0,$$

where $\mathrm{L} = \mathcal{O}(4)$. The function

$$(\mathrm{A}, \mathrm{B}) \mapsto \mathrm{D}$$

is the differential construction map.

Let us make the maps above more explicit. Suppose

$$\mathrm{B} = (\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3),$$

where the R_i are quartics. Then the map J is defined by the Jacobian matrix

$$\mathrm{J} = \begin{pmatrix} \partial\mathrm{F}_1/\partial\mathrm{X} & \partial\mathrm{F}_2/\partial\mathrm{X} & \partial\mathrm{F}_3/\partial\mathrm{X} \\ \partial\mathrm{F}_1/\partial\mathrm{Y} & \partial\mathrm{F}_2/\partial\mathrm{Y} & \partial\mathrm{F}_3/\partial\mathrm{Y} \end{pmatrix}.$$

Suppose

$$\mathrm{A} = \begin{pmatrix} \mathrm{P}_1 & \mathrm{P}_2 & \mathrm{P}_3 \\ \mathrm{Q}_1 & \mathrm{Q}_2 & \mathrm{Q}_3 \end{pmatrix},$$

where the P_i and Q_j are quadrics. Then the differential map D is given by

$$\mathrm{D} = \begin{pmatrix} (\mathrm{P}_1 \cdot \partial\mathrm{F}_1/\partial\mathrm{X} + \mathrm{P}_2 \cdot \partial\mathrm{F}_2/\partial\mathrm{X} + \mathrm{P}_3 \cdot \partial\mathrm{F}_3/\partial\mathrm{X})\,/\mathrm{X} \\ (\mathrm{Q}_1 \cdot \partial\mathrm{F}_1/\partial\mathrm{X} + \mathrm{Q}_2 \cdot \partial\mathrm{F}_2/\partial\mathrm{X} + \mathrm{Q}_3 \cdot \partial\mathrm{F}_3/\partial\mathrm{X})\,/\mathrm{X} \end{pmatrix},$$

or equivalently by replacing $\frac{1}{\mathrm{X}} \cdot \partial/\partial\mathrm{X}$ by $-\frac{1}{\mathrm{Y}} \cdot \partial/\partial\mathrm{Y}$.

Set $\mathrm{P} = \mathbf{P}(2 \boxtimes 2^{(4)})$ and let $\mathrm{D}\colon \mathrm{X} \dashrightarrow \mathrm{P}$ be the differential construction map. The explicit equations for D above are homogeneous of degree 1 in the coefficients of $(\mathbf{P}_i, \mathbf{Q}_j)$ and also homogeneous of degree 1 in the coefficients of F_k. In other words, we have

$$\mathrm{D}^* \mathcal{O}(1) = \mathcal{O}(1,1),$$

where $\mathcal{O}(a,b)$ on $\mathrm{X} \subset \mathbf{P}^{17} \times \mathbf{P}^{14}$ represents the restriction of $\mathcal{O}(a,b)$ on the ambient space.

Question 8.1. Does the differential construction preserve semistability? That is, is the following map regular?

$$\mathrm{X} /\!\!/_{(1,1)} \mathrm{SL}_2 \times \mathrm{SL}_3 \dashrightarrow \mathrm{P} /\!\!/ \mathrm{SL}_2$$

Remark 8.2. *If the line bundle* $\mathrm{L} = \mathcal{O}(4)$ *is replaced by* $\mathrm{L} = \mathcal{O}(2)$*, then the sequence* (8.1) *becomes*

$$\mathcal{O}(-1)^2 \xrightarrow{\mathrm{A}} \mathcal{O}^3 \xrightarrow{\mathrm{B}} \mathcal{O}(2). \tag{8.2}$$

In this case, is the moduli space of sequences related to the space of complete conics?

Remark 8.3. *It seems interesting to me to study compactified "moduli spaces of exact sequences" of the form*

$$0 \longrightarrow \mathcal{O}(-1)^2 \xrightarrow{\mathrm{A}} \mathcal{O}^3 \xrightarrow{\mathrm{B}} \mathcal{O}(2) \longrightarrow 0.$$

What makes this interesting is that we get three different compactifications, coming from three different points of view towards such an exact sequence. The first is through A*, the second is through* B*, and the third is through the extension class* $\mathrm{C} \in \mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}(-1)^2)$*. Furthermore, we can try to interpolate between these three views using GIT as follows. Consider*

$$\mathrm{X} \subset \mathbf{P}^{11} \times \mathbf{P}^{8} \times \mathbf{P}^{5}$$

consisting of $(\mathrm{A}, \mathrm{B}, \mathrm{C})$ *such that* $\mathrm{BA} = 0$ *and* $\mathrm{CB} = 0$ *and study the GIT quotients*

$$\mathrm{X} /\!\!/ \mathrm{SL}(2) \times \mathrm{SL}(3).$$

(For simplicity, I have chosen specific numbers for the ranks and degrees of the bundles appearing here, but they can be changed).

9. A Porteous approach

Here is another attempt to find the degrees of the projection-ramification map for surfaces. The idea is to fix a generic ramification divisor, and try to find projections that give exactly that ramification.

For simplicity, let us work on $\mathrm{S} = \mathbf{P}^1 \times \mathbf{P}^1$. Let $\mathrm{L} = (1, n)$. Then we have a projection-ramification map

$$\mathbf{Gr}\left(3, \mathrm{H}^0(\mathrm{S}, \mathrm{L})\right) \dashrightarrow |(1, 3n-2)|$$

Given a net U of sections of L, we interpret the corresponding ramification divisor as the locus of points p in S such that there exists an element of U that is singular at p. If U is generic, then there will be a unique element of U that is singular at p for all p in the ramification divisor.

Let a ramification divisor be given. We view it as a section $\sigma\colon \mathbf{P}^1 \longrightarrow \mathrm{S}$. Let $\Sigma \subset \mathrm{S} \times |1, n|$ be incidence correspondence consisting of (s, u) such that u is singular at s. In this case, the divisor u is $\mathrm{C}_s + \mathrm{F}_s$, where C_s is in the linear series $|1, n-1|$ passing through s, and F_s is the fiber of S through s. The map $\Sigma \longrightarrow \mathrm{S}$ is a $\mathbf{P}^{2n-2}$ bundle. Let $\Sigma|_\sigma \longrightarrow \mathbf{P}^1$ be the pullback of Σ along the section $\sigma\colon \mathbf{P}^1 \longrightarrow \mathrm{S}$. To find projections whose ramification divisor is σ, we attempt to find sections $\mathrm{Z}\colon \mathbf{P}^1 \longrightarrow \Sigma|_\sigma$ such that the composite $\mathrm{Z}\colon \mathbf{P}^1 \longrightarrow |1, n|$ spans a $\mathbf{P}^2$. The net given by this $\mathbf{P}^2$ will have ramification locus (containing) σ by construction. We have to make sure that the ramification locus is not all of S.

Let $\Delta \subset \Sigma$ be the locus consisting of (s, u) where the divisor u is $\mathrm{C}_s + 2\mathrm{F}_s$, where C_s is in the linear series $|1, n-2|$ and F_s is the fiber through s. The sections Z arising from a generic net will never intersect Δ. So we should only find $\mathrm{Z}\colon \mathbf{P}^1 \longrightarrow \Sigma|_\sigma$ satisfying the following two conditions.

(1) The image of Z in $|1, n|$ spans a $\mathbf{P}^2$.
(2) The image of Z is disjoint from Δ.

In this case, I believe the ramification locus of the net given by the $\mathbf{P}^2$ is not all of S. So it must be exactly σ.

Since $\Delta \subset \Sigma$ is a divisor, the second condition pins down the numerical class of Z. The space of all Z in this numerical class satisfying the second condition is a (non-projective) variety. The first condition puts a closed condition on it. We expect the set of points satisfying the condition to be finite, and want to find its size.

Let us first try to identify the bundle $\Sigma \longrightarrow \mathrm{S}$ and its sub-bundle $\Delta \longrightarrow \mathrm{S}$. For Σ, define the bundle K on S by the following exact sequence

$$0 \longrightarrow \mathrm{K} \longrightarrow \mathcal{O}_{\mathrm{S}}^{2n+2} \longrightarrow j_{\mathrm{S}}\mathrm{L} \longrightarrow 0,$$

where $j_{\mathrm{S}}\mathrm{L}$ is the first jet bundle of L on S. Then Σ is the projectivization (space of quotients) of $\mathrm{K}^\vee$. For Δ, define the bundle V on $\mathbf{P}^1$ by the following exact sequence

$$0 \longrightarrow \mathrm{V} \longrightarrow \mathcal{O}_{\mathbf{P}^1}^{2n+2} \longrightarrow j_{\mathbf{P}^1}(\pi_*\mathrm{L}) \longrightarrow 0,$$

where $\pi\colon \mathrm{S} \longrightarrow \mathbf{P}^1$ is the natural projection. The bundle $\Delta \longrightarrow \mathrm{S}$ is the projectivization of $\pi^*\mathrm{V}^\vee$. We have an injection of vector bundles

$$\pi^*\mathrm{V} \longrightarrow \mathrm{K},$$

which induces the inclusion $\Delta \subset \Sigma$.

Proposition 9.1. *We have isomorphisms*

$$\mathrm{V}|_\sigma \cong \mathcal{O}(-2)^{2n-2},$$

and

$$\mathrm{K}|_\sigma \cong \mathrm{V}|_\sigma \oplus \mathcal{O}(-2n).$$

Proof. ToDo. □

Therefore, sections of $\Sigma|_\sigma \longrightarrow \mathbf{P}^1$ disjoint from $\Delta|_\sigma$ are given by quotients

$$\mathrm{K}^* \longrightarrow \mathcal{O}(2n)$$

that restrict to an isomorphism

$$\mathcal{O}(2n) \xrightarrow{\sim} \mathcal{O}(2n).$$

The space of such quotients is an open subset of the projective space

$$\mathbf{P}\,\mathrm{Hom}(\mathrm{K}^*, \mathcal{O}(2n)) \cong \mathbf{P}^{(2n-2)(2n-1)}.$$

Given a quotient $q\colon \mathrm{K}^* \longrightarrow \mathrm{O}(2n)$, the surjection

$$\mathcal{O}^{2n+2} \longrightarrow \mathrm{K}^*$$

induces a map

$$\mathcal{O}^{2n+2} \longrightarrow \mathcal{O}(2n),$$

and hence a map

$$q_f\colon \mathbf{C}^{2n+2} \longrightarrow \mathrm{Sym}^{2n}\,\mathbf{C}^2.$$

The section given by q is planar, if and only if the rank of q_f is at most 3. This is a condition of expected codimension $(2n-2)(2n-1)$.

Proposition 9.2. *The number of projections that have ramification divisor σ is in bijection with quotients $q\colon \mathrm{K}^* \longrightarrow \mathcal{O}(2n)$ satisfying the following two conditions.*

(1) The composition of $\mathcal{O}(2n) \subset \mathrm{K}^*$ with q gives an isomorphism $\mathcal{O}(2n) \longrightarrow \mathcal{O}(2n)$
(2) The rank of q_f is at most 3.

The second condition leads to a porteous calculation on the moduli space of q, but the first condition makes this moduli space quasi-projective. Can we work on the full projective space instead, or does this lead to excess intersection?

9.1. **The "two".** Given a general triple $\{a, b, c\}$ of binary quadratic forms, we can create the three quartic binary forms $a[b, c], b[c, a], c[a, b]$, where $[p, q]$ denotes $p_x q_y - p_y q_x$. As we know, these three forms are actually linearly dependent, yielding a pencil of binary quartics.

In this way, we obtain an *a priori* rational map

$$\mathrm{D} : \mathrm{H} \dashrightarrow \mathrm{G}(1, 4)$$

where H denotes the Hilbert scheme of 3 points on $\mathbf{P}^2$.

The main observation is:

Proposition 9.3. *The rational map* D *extends to a regular map.*

Proof. This is best seen by visualizing D geometrically, and noting that the geometric construction makes sense at every point of H.

If $\{a, b, c\}$ is a general subset of $\mathbf{P}^2$, then the quartic pencil $\mathrm{D}(\{a, b, c\})$ is obtained as follows. Recall that in $\mathbf{P}^2$ we have the canonical diagonal conic C parametrizing square forms. A point $a \in \mathbf{P}^2$ defines a line $\mathrm{P}ol(a) \subset \mathbf{P}^2$ spanned by the two points of C which correspond to the roots of a. Furthermore, a pair of points $b, c \in \mathbf{P}^2$ defines the line $\overline{b, c} \subset \mathbf{P}^2$.

To the triple $\{a, b, c\}$ we attach the triple of pairs of lines $\mathrm{P}ol(a) \cup \overline{b, c}$ (and permutations), which cut the conic C at 3 members of a degree 4 pencil.

This geometric construction works even for non-reduced schemes. For example, if $\mathrm{Z} \subset \mathbf{P}^2$ is a fat point concentrated at a point $a \in \mathbf{P}^2$, we assign the degree 4 pencil on C as: The degree 2 pencil corresponding to $\mathrm{P}ol(a)$ with two base points at $\mathrm{P}ol(a) \cap \mathrm{C}$. □

The map D is only generically finite; the locus of collinear triples is contracted, and has the same image as the locus of fat schemes. However, it is easy to exhibit a point in G over which there are exactly two preimages.

Lemma 9.4. *Let $\Lambda \in \mathrm{G}(1, 4)$ denote the unique pencil with simple base points at $0, 1, \infty$ in $\mathbf{P}^1$. Then the preimage $\mathrm{D}^{-1}(\Lambda)$ consists of two non-reduced points.*

Proof. The two configurations are described as follows: View the three points $0, 1, \infty$ on the diagonal conic C. Then the triple $\{0, 1, \infty\}$ clearly maps to Λ, as does the triangle created by $\mathrm{P}ol(0), \mathrm{P}ol(1), \mathrm{P}ol(\infty)$.

A simple first-order analysis shows that any non-trivial first-order deformation of either of these configurations will have the effect of either removing the base-points, or moving their location.

Furthermore, it is easy to see that these are the only two configurations giving rise to the pencil Λ. □

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