RAMIFICATION DIVISORS OF GENERAL PROJECTIONS

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Abstract. We study the ramification divisors of a general projection of a smooth projective variety onto a linear subspace of the same dimension. We prove that the ramification divisors vary in a maximal dimensional family for a large class of varieties. Going further, we study the map that associates to a linear projection its ramification divisor. We show that this map is dominant for most (but not all!) varieties of minimal degree, using limit linear series of higher rank. We find the degree of this map in some cases, extending the classical appearance of Catalan numbers in the geometry of rational normal curves. We give a geometric explanation of the fibers in terms of torsion points of naturally ocurring elliptic curves in the case of the Veronese surface and the quartic rational suface scroll.

1. Introduction

{sec:intro}

Let $f\colon X \longrightarrow Y$ be a map between smooth algebraic varieties. A fundamental object associated to f is the set $R(f) \subset X$ consisting of critical points of f, namely the points $x \in X$ at which $df\colon T_xX \longrightarrow T_{f(x)}Y$ has less than maximal rank. To what extent can f be recovered from $R(f)$? For example, does every non-trivial perturbation of f induce a non-trivial perturbation of $R(f)$? If this is the case, then how many other $g\colon X \longrightarrow Y$ have $R(g) = R(f)$? The goal of this paper is to explore these questions when X is a smooth projective variety and Y is a projective space of the same dimension as X.

More precisely, let $X \subset \mathbf{P}^n$ be a smooth projective variety of dimension r, not contained in a hyperplane. A general $(n-r-1)$-dimensional linear subspace $L \subset \mathbf{P}^n$ defines a finite surjective map $X \longrightarrow \mathbf{P}^r$. The set of critical points of this map is the ramification divisor $R(L) \subset X$. By the Riemann–Hurwitz formula, $R(L)$ lies in the linear series $|K_X + (r+1)H|$, where K_X is the canonical class, and H is the hyperplane class on X. The association $L \rightsquigarrow R(L)$ gives a rational map

$$\rho\colon \mathbf{Gr}(n-r, n+1) \dashrightarrow |K_X + (r+1)H|.$$

In terms of ρ, we can formulate the two questions raised in the introduction more precisely. Knowing the behavior of the ramification locus under a perturbation is equivalent to knowing whether the map ρ is generically finite, or equivalently, whether the image of ρ has the maximal possible dimension. In the literature, this question is known as the question of maximal variation of the ramification locus. Knowing the number of maps with the same ramification locus is knowing the degree of ρ. Our main goal is to answer these questions.

1.1. **Maximal variation.** The question of maximal variation of ρ first appeared explicitly in the work of Flenner and Manaresi [8] in connection with the transcendence degree of the Stückrad-Vogel cycle in intersection theory. We prove that ρ has maximal variation for a large class of $X \subset \mathbf{P}^n$.

{thm:main}

Theorem A. *Let $X \subset \mathbf{P}^n$ be a non-degenerate, normal, projective variety over a field of characteristic zero. Suppose at least one of the following holds:*

{item:incomp}

{item:dual}

(1) (incompressibility) for every linear subspace $L \subset \mathbf{P}^n$ of dimension $(n-r-1)$, projection from L restrict to a dominant rational map $X \dashrightarrow \mathbf{P}^r$;
(2) (divisorial dual) the dual variety $X^* \subset \mathbf{P}^{n*}$ is a hypersurface.

Then ρ is generically finite onto its image.

In the main text, Theorem A is Corollary 3.15.

Recall that the dual variety $X^* \subset \mathbf{P}^{n*}$ is the closure of the locus of hyperplanes $H \subset \mathbf{P}^n$ such that the intersection of H with the smooth locus of X is singular. We call $X \subset \mathbf{P}^n$ satisfying (1) *incompressible* as it cannot be projected down (compressed) to a smaller dimensional subvariety by a linear projection. The main result of [8] obtained the conclusion of Theorem A assuming incompressibility.

Theorem A substantially increases the class of varieties where we now know maximal variation. Indeed, it is easy to see that if X is a smooth surface over a field of characteristic 0, then the dual variety X^* is a hypersurface. Therefore, maximal variation holds for all surfaces. Note, in contrast, that not all surfaces are incompressible. The smallest counterexample is the cubic surface scroll $X \subset \mathbf{P}^4$—the projection from the directrix $L \subset X$ projects X onto a $\mathbf{P}^1$. Thus, even for surfaces, condition (2) of Theorem A covers new ground. In general, let X be of arbitrary dimension embedded in $\mathbf{P}^n$ by a sufficiently positive line bundle (for example, by a sufficiently high Veronese re-embedding). Then $X \subset \mathbf{P}^n$ may not be incompressible, but the dual variety X^* will be a hypersurface. As a result, $X \subset \mathbf{P}^n$ is covered by condition (2) of Theorem A.

The hypotheses in Theorem A are sufficient, but not necessary. Indeed, consider $X = \mathbf{P}^{r-1} \times \mathbf{P}^1 \subset \mathbf{P}^{2r-1}$, embedded by the Segre embedding, for $r \geq 3$. Then X is neither incompressible nor is X^* a hypersurface, and yet ρ_X is dominant (see Theorem E).

Given that maximal variation holds for a large class of varieties, it is natural to wonder if it always holds. This is not the case.

counterexamples}

Theorem B. *There exist smooth, non-degenerate, rational normal scrolls $X \subset \mathbf{P}^n$ of every dimension $r \geq 4$ and degree $d \geq r+1$ for which the projection-ramification map ρ is not generically finite onto its image.*

In the main text, Theorem B is Corollary 4.6.

The existence varieties for which the projection-ramification map is not generically finite has been alluded to by Zak [14], but we do not know any explicit examples written down in the literature. We describe the rational normal scrolls in Theorem B explicitly; they include scrolls of general moduli.

Having considered the question of maximal variation in general, we turn our attention to cases where the map ρ has a chance to be dominant. Our next result classifies such $X \subset \mathbf{P}^n$. {thm:minimaldegr

Theorem C. *Let $X \subset \mathbf{P}^n$ be a smooth, non-degenerate projective variety of dimension r over a field of characteristic zero. We have the inequality*

$$\dim \mathbf{Gr}(n-r, n+1) \leq \dim |K_X + (r+1)H|.$$

Equality holds if and only if X is a variety of minimal degree, that is $\deg X = n - r + 1$.

In the main text, Theorem C is Theorem 4.2.

Recall the list of smooth varieties of minimal degree: quadric hypersurfaces, the Veronese surface in $\mathbf{P}^5$, and rational normal scrolls. By Theorem A, ρ is dominant for hypersurfaces and surfaces, so what remains are the scrolls. Among the scrolls, the curves (rational normal curves) and surfaces are again covered by Theorem A. For threefold scrolls, show by an explicit calculation and a degeneration argument that ρ is dominant (Corollary 4.10). In higher dimensions, the story is complicated, as evidenced by Theorem B. Nevertheless, we prove the following. {thm:rationalnor

Theorem D. *Let $X = \mathbf{P}E \subset \mathbf{P}^n$ be a rational normal scroll, where E is a ample vector bundle of rank r on $\mathbf{P}^1$, general in its moduli. If* $\deg E = a \cdot (r-1) + b \cdot (2r-1) + 1$ *for non-negative integers a, b, then the projection-ramification map ρ is dominant for X. In particular, the conclusion holds if E is general of degree at least* $(r-1)(2r-1)+1$.

In the main text, Theorem D is Theorem 5.16.

The proof of Theorem D goes by degeneration. We degenerate X to a reducible variety X_0, namely the projectivization of a vector bundle on a two-component nodal rational curve. Suppose we could define a projection-ramification map for X_0 and show that it is dominant, then the same holds X, by the upper semi-continuity of fiber dimensions. Although promising, this line of attack fails with the most näive definition of the projection-ramification map. The right definition requires more sophisticated tools, specifically, the spaces of limit linear series for vector bundles of higher rank developed by Texidor i Bigas [13] and Osserman [11].

1.2. **Enumerative problems.** Theorem C and Theorem D motivate a natural set of enumerative questions: for $X \subset \mathbf{P}^n$ of minimal degree, what is the degree of the projection-ramification map ρ? We make the convention that if ρ is not dominant, then its degree is 0.

For X of dimension 1, namely a rational normal curve, the answer is easy to find—the degree of ρ is the Catalan number $\frac{(2n-2)!}{n!(n-1)!}$. Indeed, in this case, the projection-ramification map

$$\rho \colon \mathbf{Gr}(2, n+1) \longrightarrow \mathbf{P}^{2n-2}$$

is regular, and the pullback of $\mathcal{O}(1)$ is the Plücker line bundle. Therefore, the degree of ρ is the top self-intersection of the Plücker bundle. Schubert calculus gives that this is the Catalan number.

For X of codimension 1, namely a quadric hypersurface, the projection-ramification map

$$\rho\colon \mathbf{Gr}(n, n+1) = \mathbf{P}^n \longrightarrow \mathbf{P}^{n*}$$

is again regular, and is in fact the duality isomorphism induced by the (non-degenerate) quadric X. In particular, it has degree 1.

The cases of the Veronese surface $X \cong \mathbf{P}^2 \subset \mathbf{P}^5$ and the quartic surface scroll $X = \mathbf{P}(\mathcal{O}(2) \oplus \mathcal{O}(2)) \subset \mathbf{P}^5$ are particularly delightful. In these cases, the fibers of ρ have an interpretation in terms of two torsion points of certain elliptic curves, which we now describe. For the Veronese surface, the target of ρ is the linear series of cubics in $\mathbf{P}^2$. The points of fiber of ρ over a cubic $R \subset \mathbf{P}^2$ correspond naturally to the non-trivial two torsion points of $\operatorname{Pic} R$. In particular, the degree of ρ is 3. For the quartic surface scroll, the target of ρ modulo the action of $\operatorname{Aut} X$ is birational to the moduli space of (R, η) where R is a plane cubic and η is a non-trivial two torsion point of $\operatorname{Pic} R$. The points of the fiber of ρ over (R, η) corresponds naturally to the two torsion points of $\operatorname{Pic} R$ other than 0 and η. In particular, the degree of ρ is 2. In this case, the source of ρ modulo the action of $\operatorname{Aut} X$ has several known moduli interpretations. It birational to the moduli of undered triplets of unordered pairs of points on $\mathbf{P}^1$, namely $M_{0,6}/(S_2 \times S_2 \times S_2 \rtimes S_3)$. This space, in turn, is isomorphic to the moduli space of hyperelliptic curves with a maximal isotropic subspace of the $\mathbf{F}_2$-vector space of 2 torsion points, or equivalently, to the moduli of principally polarized abelian surfaces with a maximal isotropic subspace of the $\mathbf{F}_2$-vector space of 2-torsion points [1, Example 4.2]. The involution on this space induced by the 2-to-1 map ρ coincides with the classical Richelot or Fricke involution [1, Remark 4.3].

The following result summarizes our knowledge of the degree of ρ.

{thm:examples}

Theorem E. *Let ρ be the projection-ramification map for $X \subset \mathbf{P}^n$ of minimal degree.*

(1) If $X \subset \mathbf{P}^n$ is a rational normal curve, then ρ is regular and $\deg \rho = \frac{(2n-2)!}{n!(n-1)!}$.
(2) If $X \subset \mathbf{P}^n$ is a quadric hypersurface, then ρ is an isomorphism; in particular, $\deg \rho = 1$.
(3) If $X = \mathbf{P}^{r-1} \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^{2r-1}$ is the Segre embedding, then $\deg \rho = 1$.
(4) If $X \subset \mathbf{P}^5$ is the Veronese surface, then $\deg \rho = 3$.
(5) If $X \subset \mathbf{P}^5$ is a general quartic surface scroll, then $\deg \rho = 2$.
(6) If $X = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(k+1)) \subset \mathbf{P}^{k+3}$ is the surface scroll with the most imbalanced splitting type, then $\deg \rho = 1$.
(7) If $X = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(k+1)) \subset \mathbf{P}^{k+5}$ is the threefold scroll with the most imbalanced splitting type, then $\deg \rho = 1$.

In the main text, the items in Theorem E are § 6.1, § 6.2, Proposition 5.1, Proposition 6.1, Proposition 6.6, Proposition 4.11, and Proposition 4.9, respectively.

1.3. **Further remarks.** There are two natural enumerative problems regarding finite coverings of curves. The first problem, originating in the work of Hurwitz, is to compute the number of branched covers $C \longrightarrow \mathbf{P}^1$ with a specified set of branch points $B \subset \mathbf{P}^1$. These numbers, called the Hurwitz numbers, are difficult to compute, but they exhibit remarkable structure [5, 6]. The second problem is to compute the number of maps $C \longrightarrow \mathbf{P}^1$ with a prescribed set of ramification points $R \subset C$. This problem is much more elementary, and solvable using Schubert calculus. For $C = \mathbf{P}^1$, the numbers are the Catalan numbers, as we have seen.

In higher dimensions, however, the analogue of the Hurwitz problem is expected to be much less interesting, thanks to Chisini's conjecture (proved by Kulikov [9]). A branched cover $S \longrightarrow \mathbf{P}^2$ with generic branching is uniquely determined by its branch divisor $B \subset \mathbf{P}^2$, with finitely many well-understood examples. In contrast, as hinted by Theorem C, the enumerative problem regarding the ramification divisor persists, and poses a significant challenge. In some sense, the enumerative problems regarding branch and ramification divisors trade places, certainly in terms of difficulty, but hopefully also in terms of structure.

1.4. **Further questions.** Our work raises several questions, some of which we hope to return to in a sequel.

{sec:qscroll}

1.4.1. *The enumerative problem for scrolls.* . Recall that every vector bundle on $\mathbf{P}^1$ is isomorphic to a direct sum of line bundles. In particular, an ample vector bundle of rank r and degree d is isomorphic to $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ for positive integers $a_1, \dots, a_r$ satisfying $a_1 \leq \cdots \leq a_r$ and $a_1 + \cdots + a_r = d$, or equivalently, by an r-term partition of d. Let $\Sigma_{r,d}$ be the set of r-term partitions of d. We get a function $\phi \colon \Sigma_{r,d} \longrightarrow \mathbf{Z}_{\geq 0}$ defined by

$$\phi(a_1, \dots, a_r) = \text{Degree of the projection-ramification map for } X \subset \mathbf{P}^{r+d},$$

where $X = \mathbf{P}\left(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)\right)$ is embedded in $\mathbf{P}^{r+d}$ by $\mathcal{O}_X(1)$. The set $\Sigma_{r,d}$ is partially ordered by the dominance order $\prec$. In terms of vector bundles, $\prec$ translates into isotrivial specialization: $(a_1, \dots, a_r) \prec (b_1, \dots, b_r)$ if and only if $\mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_r)$ isotrivially specializes to $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$. In this case, by the lower semi-continuity of degrees of rational maps, we get

$$\phi(a_1, \dots, a_r) \leq \phi(b_1, \dots, b_r).$$

Thus, ϕ is order preserving.

We hope that the enumerative function $\phi \colon \Sigma_{r,d} \longrightarrow \mathbf{Z}$ admits some structure, such as a nice recurrence relation or generating function. Theorem C and Theorem E only scratch the surface as far as ϕ is concerned. Theorem C states that ϕ is not identically zero, at least if d is sufficiently large. Theorem E computes ϕ for the partitions (d), $(1, \dots, 1)$, $(1, k+1)$, $(1, 1, k+1)$, and $(2, 2)$. Some more examples, calculated using randomized trials over finite fields in `Macaulay2` and `MAGMA`, are tabulated in Table 1. We plan to return to a more complete enumerative investigation of ϕ in a future paper.

1.4.2. *Positive characteristics.* The study of the maximal variation of ρ and its degree in positive characteristics will surely bring new surprises and require new techniques. We do not know if Theorem A or Theorem C holds in positive characteristic; our proofs certainly do not. The degrees in Theorem E, and likewise the values of the enumerative function $\phi \colon \Sigma_{r,d} \longrightarrow \mathbf{Z}$ defined in § 1.4.1, depend on the characteristic due to the presence of inseparable covers. Indeed, this is the case even for rational normal curves [10].

TABLE 1. Degrees of the projection-ramification maps for $X = \mathbf{P}(\mathcal{O}(a_1) \oplus \mathcal{O}(a_2))$

{tab:computation

a_1 \ a_2	1	2	3	4
1	1			
2	1	2		
3	1	6	22	
4	1	17	92	422

1.4.3. *Picture over the real numbers.* Consider the projection-ramification map of a rational normal curve of degree n, also called the Wronskian map,

$$\rho\colon \mathbf{Gr}(2, n+1) \longrightarrow \mathbf{P}^{2n-2}.$$

The real algebraic geometry surrounding ρ plays an important role in real enumerative geometry, the theory of real algebraic curves, and control theory, thanks to the B. and M. Shapiro conjecture. Proved by Eremenko and Gabrielov, this conjecture states that if $L \in \mathbf{Gr}(2, n+1)$ is such that the ramification divisor $\rho(L)$ is the sum of $(2n-2)$ real points in $\mathbf{P}^1$, then L is a real point of $\mathbf{Gr}(2, n+1)$ [12, 7]. Theorem C potentially sets the stage for a higher-dimensional generalization of the body of work around the Shapiro conjecture. In particular, it would be interesting to find a uniform topological picture explaining the numbers $\deg \rho$, similar to the "nets" introduced by Eremenko and Gabrielov that explain the appearance of Catalan numbers.

1.5. **Notation and conventions.** We work over an algebraically closed field k of characteristic zero, not simply for convenience, but for necessity—we appeal to Bertini's theorem, generic smoothness, and Kodaira vanishing. All schemes are of finite type over k. A *variety* is an integral scheme. For a scheme X, we let $X^{\mathrm{sm}} \subset X$ be the smooth locus.

We go back and forth without comment between divisors and line bundles, and likewise, locally free sheaves and vector bundles. We follow Grothendieck's convention for projectivization. That is, the projectivization $\mathbf{P}E$ of a vector bundle E is the space of one dimensional quotients of E. For a line bundle L on X, we denote by $|L|$ the projective space $\mathbf{P}H^0(X, L)^*$.

Given a vector bundle F on X, we denote by $P(F)$ the sheaf of principal parts of F. This is defined by the formula

$$P(F) = \pi_{2*}\left({\pi_1}^* F \otimes \mathcal{O}_{X \times X}/I_\Delta^2\right),$$

where the π_i are the projections on the two factors and $\Delta \subset X \times X$ is the diagonal. We remind the reader of the exact sequence

$$0 \longrightarrow F \otimes \Omega_S \longrightarrow P(F) \longrightarrow F \longrightarrow 0$$

and the evaluation map

$$e\colon H^0(X, F) \otimes \mathcal{O}_X \longrightarrow P(F).$$

1.6. **Organization.** In Section 2 we give a precise definition of the projection-ramification map ρ for linear series on normal varieties. In Section 3 we prove Theorem A, first assuming incompressibility (Proposition 3.1), and then assuming non-defectivity (Theorem 3.12). The notion of non-defectivity abstracts the condition of having a divisorial dual variety. Before proving Theorem 3.12, we study this notion in § 3.1. In Section 4, we prove Theorem C as Theorem 4.2. In the same section, we derive explicit formulas for the ramification divisors for scrolls (§ 4.1), give the examples advertised in Theorem B (§ 4.2), and treat the threefold scrolls (§ 4.3). In Section 5, we prove Theorem D as Theorem 5.16, after having treated some low degree examples by hand in § 5.1, introduced limit linear series for vector bundles of higher rank in § 5.2, and defined the projection-ramification map for limit linear series in § 5.3 and § 5.4. In Section 6, we turn to the enumerative problem of finding the degree of ρ. We treat the cases of rational normal curves and quadric hypersurfaces quickly in § 6.1 and § 6.2. We devote § 6.3 to the case of the Veronese surface and § 6.4 to the case of the quartic surface scroll.

2. The projection-ramification map

{sec:prmap}

In this section, we define a projection-ramification map for a pair (X, L) consisting of a proper, normal, variety X and a sufficiently positive line bundle L on X. For $X \subset \mathbf{P}^n$, taking $L = \mathcal{O}(1)$ recovers the projection-ramification map introduced in Section 1. Working with abstract pairs, however, offers more flexibility that is helpful in inductive proofs.

Let X be a proper variety of dimension r over an algebraically closed field k of characteristic zero. A *linear series* on X is a pair (L, W) consisting of a line bundle L on X and a subspace $W \subset H^0(X, L)$. The *complete linear series* associated to L is (L, W) with $W = H^0(X, L)$. A *projection* is a linear series (L, V) with $\dim V = r + 1$. A *projection of* (L, W) is a projection (L, V) with $V \subset W$. As a convention, we use V for projections and W for more general linear series.

{def:properlyram

Definition 2.1 (Properly ramified projection)**.** We say that a projection (L, V) is *properly ramified* if the evaluation homomorphism

$$e \colon V \otimes \mathcal{O}_X \longrightarrow P(L)$$

is an isomorphism over a general point in X. If (L, V) is properly ramified, its *ramification divisor*

$$R(L, V) \subset X$$

is the closure of the scheme defined by the determinant of $e \colon V \otimes \mathcal{O}_{X^{\mathrm{sm}}} \longrightarrow P(L)|_{X^{\mathrm{sm}}}$.

In most cases, L is clear from context, so we drop it from the notation and denote the ramification divisor simply by $R(V)$.

{rem:Jacobian}

Remark 2.2. Suppose for simplicity that V is a base-point free linear series that yields a surjective map $\phi \colon X \longrightarrow \mathbf{P}V$. Then the ramification divisor may be defined as the degeneracy locus of the map

$$d\phi \colon T_X \longrightarrow \phi^* T_{\mathbf{P}V}$$

on tangent spaces. The degeneracy locus is the zero locus of $\det \phi$, which in local coordinates, is given by the determinant of the Jacobian matrix $\left(\frac{\partial \phi_i}{\partial x_j}\right)$. Therefore, the ramification divisor $R(L, V)$ is also often called the *Jacobian* of the linear series (L, V) (see, for example, [2, 1.1.7]).

A projection (L, V) gives the evaluation map

$$e \colon V \otimes \mathcal{O}_X \longrightarrow L.$$

The evaluation map yields a map $p_{V,L} \colon X \dashrightarrow \mathbf{P}V$, regular on the non-empty open set of X where e is surjective. The following is an easy observation, whose proof we skip.

{prop:proj}

Proposition 2.3. *The projection (L, V) is properly ramified if and only if the map on tangent spaces induced by $p_{V,L}$ is generically an isomorphism. In characteristic zero, this is equivalent to the condition that $p_{V,L}$ is dominant.*

For a fixed (L, W), the set of all projections of (L, W) are parametrized by the Grassmannian $\mathbf{Gr}(r+1, W)$. The property of being properly ramified is a Zariski open condition on the Grassmannian.

We now define a map that assigns to a projection its ramification divisor. To do so, we interpret the ramification divisor as an element of a linear series.

Assume, furthermore, that X is normal. Let K_X be the canonical sheaf of X. Denoting by $i \colon X^{\mathrm{sm}} \longrightarrow X$ the inclusion, K_X is given by the push-forward

$$K_X = i_* K_{X^{\mathrm{sm}}}.$$

Note that, since X is normal, the complement of $X^{\mathrm{sm}} \subset X$ has codimension at least 2. The sheaf K_X is coherent, reflexive, and satisfies Serre's S2 condition.

Let L be a line bundle on X. The sheaf $P(L)$ is locally free of rank $(r+1)$ on X^{sm}, and we have a canonical isomorphism

$$\bigwedge^{r+1} P(L)|_{X^{\mathrm{sm}}} \cong K_{X^{\mathrm{sm}}} \otimes L^{r+1}.$$

Given a subspace $V \subset H^0(X, L)$, we apply $\bigwedge^{r+1}$ to the evaluation map

$$e \colon V \otimes \mathcal{O}_{X^{\mathrm{sm}}} \longrightarrow P(L)|_{X^{\mathrm{sm}}},$$

to get

$$\det e \colon \det V \otimes \mathcal{O}_{X^{\mathrm{sm}}} \longrightarrow K_{X^{\mathrm{sm}}} \otimes L^{r+1}.$$

By applying i_* and taking global sections, we get

{eqn:ramsection}

$$r_V \colon \det V \longrightarrow H^0(X, K_X \otimes L^{r+1}). \tag{2.1}$$

If (L, V) is properly ramified, then this map is non-zero, and hence gives a point of the projective space $\mathbf{P}H^0(X, K_X \otimes L^{r+1})^*$. Doing the same construction universally over the Grassmannian $\mathbf{Gr} = \mathbf{Gr}(r+1, W)$ yields a map

{eqn:rammap}

$$r \colon \det \mathcal{V} \longrightarrow H^0(X, K_X \otimes L^{r+1}) \otimes \mathcal{O}_{\mathbf{Gr}}, \tag{2.2}$$

where $\mathcal{V} \subset W \otimes \mathcal{O}_{\mathbf{Gr}}$ is the universal sub-bundle of rank $(r+1)$. Let $U \subset \mathbf{Gr}$ be the open subset of properly ramified projections. Then the map in (2.2) is non-zero at every point of U, and defines a map $U \longrightarrow \mathbf{P}H^0(X, K_X \otimes L^{r+1})^*$ given by the surjection

$$H^0(X, K_X \otimes L^{r+1})^* \otimes \mathcal{O}_U \longrightarrow \det \mathcal{V}|_U^*. \tag{2.3}$$ {eqn:rammapfamil

Note that U is non-empty if and only if W separates tangent vectors at a general point of X.

{def:ProjectionR

Definition 2.4 (Projection-ramification map)**.** Let (L, W) be a linear series that separates tangent vectors at a general point of X. The *projection-ramification* map for (L, W) is the rational map

$$\rho_{(X,L,W)} \colon \mathbf{Gr}(r+1, W) \dashrightarrow \mathbf{P}H^0(X, K_X \otimes L^{r+1})^*$$

defined on the non-empty open subset of properly ramified maps by (2.3).

If any of X, L, or W are clear from context, we drop them from the notation. In particular, for a non-degenerate $X \subset \mathbf{P}^n$, we denote by ρ_X the map $\rho_{X,L,W}$ with $L = \mathcal{O}_X(1)$ and W the image in $H^0(X, L)$ of $H^0(\mathbf{P}^n, \mathcal{O}(1))$.

Note that the map (2.3) factors as

$$\det \mathcal{V} \xrightarrow{a} \bigwedge^{r+1} W \otimes \mathcal{O}_{\mathbf{Gr}} \xrightarrow{b} H^0(X, K_X \otimes L^{r+1}) \otimes \mathcal{O}_{\mathbf{Gr}},$$

where a is $\wedge^{r+1}$ applied to the universal inclusion $\mathcal{V} \subset W \otimes \mathcal{O}_{\mathbf{Gr}}$, and b is induced by $\wedge^{r+1}$ applied to the evaluation map $e \colon W \otimes \mathcal{O}_X \longrightarrow P(L)$. The map a defines the Plücker embedding

$$i \colon \mathbf{Gr}(r+1, W) \longrightarrow \mathbf{P}\left(\bigwedge^{r+1} W^*\right),$$

and the map b defines a linear projection

$$p \colon \mathbf{P}\left(\bigwedge^{r+1} W^*\right) \dashrightarrow \mathbf{P}H^0(X, K_X \otimes L^{r+1}).$$

Thus, $\rho_{X,L,W}$ factors as the Plücker embedding followed by a linear projection.

3. Maximal variation for incompressible and non-defective X

{sec:proof_of_th

The goal of this section is to prove Theorem A. We begin by proving part (1), which is substantially easier.

{prop:incompress

Proposition 3.1 (Theorem A (1))**.** *Let $X \subset \mathbf{P}^n$ be a non-degenerate, normal, incompressible projective variety over a field of characteristic zero. Then ρ_X is a finite map.*

Proof. Set $L = \mathcal{O}(1)$ and let $W \subset H^0(X, L)$ be the image of $H^0(\mathbf{P}^n, \mathcal{O}(1))$. Let $V \subset W$ be an $(r+1)$-dimensional subspace. Since X is incompressible, the projection map $p_{V,L} \colon X \dashrightarrow \mathbf{P}V$ induced by (L, V) is dominant. By Proposition 2.3, this implies that (L, V) is properly ramified. Since V was arbitrary, the projection-ramification map

$$\rho \colon \mathbf{Gr}(r+1, W) \longrightarrow |K_X + (r+1)H|$$

is regular. Since the Picard rank of a Grassmannian is 1, a regular map from a Grassmannian is either constant or finite. It is easy to check that ρ is not constant; so it must be finite. $\square$

For the proof of part (2) of Theorem A, we proceed inductively by showing that a general $(n-r-1)$-dimensional linear subspace which is incident to X is an isolated point in its fiber under ρ. Again, it is more convenient to work with the more abstract set-up of a linear series, allowing for series that are not very ample.

Let X be a proper variety of dimension r, and let (L, W) be a linear series on X. For an ideal sheaf $I \subset \mathcal{O}_X$ we denote by $W \otimes I$ the subspace of W consisting of the sections that vanish modulo I. More precisely, if K is the kernel of the evaluation map

$$W \otimes \mathcal{O}_X \longrightarrow L \otimes \mathcal{O}_X / I,$$

then $W \otimes I = H^0(X, K)$. In particular, for $W = H^0(X, L)$, we have $W \otimes I = H^0(X, L \otimes I)$. For $s \in W \otimes I$, the vanishing locus $v(s)$ refers to the vanishing locus of s as a section of L. We set $|W| = \mathbf{P}W^*$, the space of one-dimensional subspaces of W, and likewise $|W \otimes I| = \mathbf{P}(W \otimes I)^*$. For a complete linear series, we write $|L|$ for $|W|$. Note that $v(s) = v(\lambda s)$ for a non-zero scalar λ, so it causes no ambiguity to talk about $v(s)$ for $s \in |W|$.

non-defectivity}

3.1. **Non-defective linear series.** We study a positivity property of linear series that generalizes the property of having a divisorial dual.

lynon-defective}

Definition 3.2 (Non-defective linear series)**.** We say that a linear series (L, W) is *non-defective* if, for a general point $x \in X$ either $W \otimes \mathfrak{m}_x^2 = 0$, or there exists $s \in W \otimes \mathfrak{m}_x^2$ such that $v(s)$ has an isolated singularity at x.

Note that for $s \in |W|$, the condition that $v(s)$ have an isolated singularity at x is a Zariski open condition on $|W|$. Therefore, if there exists an $s \in |W \otimes \mathfrak{m}_x^2|$ such that $v(s)$ has an isolated singularity at x, then a general $s \in |W \otimes \mathfrak{m}_x^2|$ has the same property.

Remark 3.3. Let x be a point of X. Suppose there exists $s \in |W|$ with an isolated singularity at x. It may be tempting to conclude from this that (L, W) is non-defective. This is not necessarily true! For example, take $X = \mathbf{F}_3$. Denote by E the section of self-intersection -3 and F the fiber of the projection $\mathbf{F}_3 \longrightarrow \mathbf{P}^1$. Let $L = \mathcal{O}_X(E + 2F)$ and $W = H^0(X, L)$. For $x \in E$, the general member of $|W \otimes \mathfrak{m}_x^2|$ has an isolated singularity at x, but the same is not true for a general $x \in X$.

Remark 3.4. Suppose (L, W) is non-defective. Let $x \in X$ be general, and let $s \in |W|$ be such that $v(s)$ has an isolated singularity at x. For all such s, it may be the case $v(s)$ has singularities away from x, even along a positive dimensional locus. For example, let $\pi \colon X \longrightarrow \mathbf{P}^2$ be the blow-up at a point, and E the exceptional divisor. The complete linear series associated to $L = \pi^* \mathcal{O}(2) \otimes \mathcal{O}(2E)$ is non-defective, but for every global section of L, the singular locus of $v(s)$ contains E.

We now define the conormal variety of a linear series, which plays an important role in our analysis of non-defectivity. Let K be the kernel of the evaluation map

$$e\colon W \otimes \mathcal{O}_X \longrightarrow P(L).$$

Let $U \subset X$ be an open subset such that $K|_U$ is locally free and the dual of the inclusion

$$W^* \otimes \mathcal{O}_U \longrightarrow K|_U^*$$

is a surjection. This surjection defines a closed embedding $\mathbf{P}(K|_U) \subset U \times |W|$. The *conormal variety of* (L, W), denoted by $P_{L,W}$, is the closure of $\mathbf{P}(K|_U)$ in $X \times |W|$.

{prop:dimension}

Proposition 3.5. *Suppose* (L, W) *is non-defective. If* $\dim W \geq r + 2$*, then* $P_{L,W}$ *is irreducible of dimension* $\dim W - 2$*. If* $\dim W \leq r + 1$*, then* $P_{L,W}$ *is empty.*

Proof. Set $n = \dim |W| = \dim W - 1$. Let k be the (generic) rank of K, namely the rank of the locally free sheaf $K|_U$. Then $k \geq n - r$. The statement of the proposition is equivalent to showing that if $k > 0$, then $k = n - r$.

For brevity, set $P = P_{L,W}$. Consider the projection $\sigma\colon P \longrightarrow |W|$, obtained by restricting the second projection $X \times |W| \longrightarrow |W|$. For $s \in |W|$, we view $\sigma^{-1}(s)$ as a subscheme of X. We then have

$$\sigma^{-1}(s) \cap U = \operatorname{Sing}(v(s)) \cap U.$$

Suppose $r > 0$. Then P is non-empty and irreducible, since it is the closure of a non-empty and irreducible variety. Since (L, W) is non-defective, a general point $(x, s) \in P$ is such that x is an isolated point of $\operatorname{Sing}(v(s))$. Therefore, $\sigma\colon P \longrightarrow |W|$ is generically finite onto its image. We conclude that $\dim P \leq \dim |W|$, and hence $k \leq n - r + 1$.

To show that $k = n - r$, it suffices to show that $\sigma\colon P \longrightarrow |W|$ is not surjective. We do so using Bertini's theorem. Let $B \subset X$ denote the union of the base locus of $|W|$ and the singular locus of X. Then B is a proper closed subset of X. Let $P^B \subset P$ be the pre-image of B under the projection $\pi\colon P \longrightarrow X$. By the definition of P, the map $\pi\colon P \longrightarrow X$ is surjective, and hence P^B is a proper closed subset of P. Since P is irreducible, we have $\dim P^B < \dim P \leq \dim |W|$, so the projection $P^B \longrightarrow |W|$ cannot be dominant. Let $s \in |W|$ be general, in particular, not in the image of $P^B \longrightarrow |W|$. By Bertini's theorem $v(s)$ is non-singular away from B. Thus, for any $x \in X$, the point $(x, s) \in X \times |W|$ does not lie in P. For $x \in B$, this is because s is not in the image of P^B, and for $x \notin B$, this is because $v(s)$ is non-singular at x. We conclude that s does not lie in the image of $P \longrightarrow |W|$. Hence $P \longrightarrow |W|$ is not surjective. $\square$

{prop:dimensionC

Proposition 3.6. *Let* (L, W) *be a linear series with* $\dim W \geq r + 2$*, and let* $P = P_L$ *be its conormal variety. The projection* $\sigma\colon P \longrightarrow |W|$ *is generically finite onto its image if and only if* (L, W) *is non-defective.*

Proof. Since $\dim W \geq r + 2$, the conormal variety $P = P_{L,W}$ is non-empty. Let $(x, s) \in P$ be a general point. We may assume that $x \in U$. Then x is a singular point of $v(s)$, and it is an isolated singularity of $v(s)$ if and only if (x, s) is an isolated point in the fiber of $\sigma\colon P \longrightarrow |W|$ over s. The conclusion follows. $\square$

The following observation relates non-defectivity with the non-degeneracy of the dual.

op:non-deg-dual}

Proposition 3.7. *Let $X \subset \mathbf{P}^n$ be a non-degenerate projective variety. Let $L = \mathcal{O}_X(1)$ and $W \subset H^0(X, L)$ the image of $H^0(\mathbf{P}^n, \mathcal{O}(1))$. Then (L, W) is non-defective if and only if the dual variety $X^* \subset \mathbf{P}^{n*}$ is a hypersurface.*

Proof. Since $X \subset \mathbf{P}^n$ is not contained in a hyperplane, we have $\dim W = n + 1 \geq r + 1$. Since (L, W) is very ample, it separates tangent vectors on X, so the evaluation map

$$e \colon W \otimes \mathcal{O}_X \longrightarrow P(L)$$

is surjective. It follows that the rank of the kernel is $n - r$, and hence

$$\dim P_{L,W} = (n - r - 1) + r = n - 1.$$

By definition, the dual variety $X^* \subset \mathbf{P}^{n*} = |W|$ is the image of the conormal variety under the projection $P_{L,W} \longrightarrow |W|$. By Proposition 3.6, (L, W) is non-defective if and only if $\dim X^* = n - 1$. □

narydoublepoint}

Proposition 3.8. *Let (L, W) be a non-defective linear series on X with $\dim W \geq r + 2$. Let $x \in X$ be a general point. Then there exists $s \in |W|$ such that $v(s)$ has an ordinary double point singularity at x.*

Proof. By Proposition 3.6, the projection $\sigma \colon P \longrightarrow |W|$ is generically finite onto its image. Let $(x, s) \in P$ be a general point. Since our ground field is of characteristic zero, we may assume that P is smooth at (x, s), that $x \in U \cap X^{\mathrm{sm}}$, and $\sigma \colon P \longrightarrow |W|$ is a local immersion at (x, s). This implies that $x \in \operatorname{Sing}(v(s))$ is isolated, and also that x is a reduced point of the scheme $\operatorname{Sing}(v(s))$. These two properties show that $v(s)$ possesses an ordinary double point at x. To see this, choose local coordinates $(x_1, ..., x_n)$ so that the complete local ring $\widehat{\mathcal{O}}_{X,x}$ is isomorphic to $k[\![x_1, \ldots, x_r]\!]$. After choosing a local trivialization for L around x, the section s corresponds to a power series $s(x_1, \ldots, x_r)$ contained in $\mathfrak{m}_x^2 \widehat{\mathcal{O}}_{X,x}$. The germ of $\operatorname{Sing}(v(s))$ at x is cut out by the power series $\frac{\partial s}{\partial x_1}, \ldots, \frac{\partial s}{\partial x_r}$. Since the germ of $\operatorname{Sing}(v(s))$ at x is the reduced point x, we get that $\frac{\partial s}{\partial x_1}, \ldots, \frac{\partial s}{\partial x_r}$ are linearly independent as elements of $\mathfrak{m}_x / \mathfrak{m}_x^2$. From this, it is easy to check that the tangent cone of $s(x_1, \ldots, x_r)$ at x is a non-degenerate quadric cone. □

eparateTangents}

Proposition 3.9. *If (L, W) is a non-defective linear series with $\dim W \geq r + 1$, then W separates tangent vectors at a general point $x \in X$. That is, the evaluation map*

$$e_x \colon W \otimes \mathcal{O}_X \longrightarrow L / \mathfrak{m}_x^2 L$$

is surjective for general $x \in X$.

Proof. By the definition of $P(L)$, we have a natural isomorphism

$$P(L)|_x = L / \mathfrak{m}_x^2 L,$$

so it suffices to show that the evaluation map

$$e \colon W \otimes \mathcal{O}_X \longrightarrow P(L)$$

is surjective at x. Let k be the generic rank of K, the kernel of e. From the proof of Proposition 3.5, we get

$$k = \dim W - r - 1.$$

Since $(r+1)$ is the generic rank of $P(L)$, we conclude that e is generically surjective. □

{cor:properlyram

Corollary 3.10. *Suppose (L, W) is a non-defective linear series on X with $\dim W \geq r+1$. Then there exists a properly ramified projection (L, V) of (L, W).*

Proof. This follows immediately from Proposition 3.9. □

As a consequence of Corollary 3.10, the projection-ramification rational map $\rho_{X,L,W}$ is defined for a non-defective linear series (L, W) with $\dim W \geq r+1$.

Let $\pi \colon \widetilde{X} \longrightarrow X$ be the blow-up at a point $x \in X$, and $E \subset \widetilde{X}$ the exceptional divisor. A linear series (L, W) on X gives a linear series $(\widetilde{L}, \widetilde{W})$ as follows. Take $\widetilde{L} = \pi^* L \otimes \mathcal{O}_{\widetilde{X}}(-E)$. Note that $H^0(X, L) = H^0(\widetilde{X}, \pi^* L)$, so we may think of W as a subspace of $H^0(\widetilde{X}, \pi^* L)$. Take $\widetilde{W} = W \otimes \mathcal{O}_{\widetilde{X}}(-E)$ with its natural inclusion $\widetilde{W} \subset H^0(\widetilde{X}, \widetilde{L})$.

{prop:blowuppoin

Proposition 3.11. *In the setup above, if (L, W) is non-defective, $\dim W \geq r+2$, and $x \in X$ is general, then $(\widetilde{L}, \widetilde{W})$ is also non-defective.*

Proof. Let y be a general point of $\widetilde{X}$. We have the equality

$$\widetilde{W} \otimes \mathfrak{m}_y^2 = W \otimes \mathfrak{m}_x \cdot \mathfrak{m}_y^2.$$

By Proposition 3.9, for a general $y \in X$, we have

$$\dim(W \otimes \mathfrak{m}_y^2) = \dim W - (r+1).$$

Since $x \in X$ is general, we get

$$\dim(W \otimes \mathfrak{m}_x \cdot \mathfrak{m}_y^2) = \dim W - (r+2).$$

If $\dim W = r+2$, then we get $\widetilde{W} \otimes \mathfrak{m}_y^2 = 0$, so we are done. Assume that $\dim W \geq r+3$. Then $\dim(W \otimes \mathfrak{m}_y^2) \geq 2$. Since (L, W) is non-defective, a general $s \in W \otimes \mathfrak{m}_y^2$ is such that $v(s)$ has an isolated singularity at y. Moreover, since $\dim(W \otimes \mathfrak{m}_y^2) \geq 2$, for every $x \in X$, there exists $s \in V$ such that $v(s)$ passes through x. Hence, as $x \in X$ is general, there exists $s \in W \otimes \mathfrak{m}_y^2$ such that $v(s)$ has an isolated singularity at y and passes through x. That is, there exists $s \in \widetilde{W} \otimes \mathfrak{m}_y^2$ that has an isolated singularity at y. We conclude that $(\widetilde{L}, \widetilde{W})$ is non-defective. □

{sec:nondefectiv

3.2. **Maximal variation for non-defective pairs.** In this section, we prove part (2) of Theorem A. In fact, we prove a more general result (Theorem 3.12).

As before, X is a proper, normal variety of dimension r over an algebraically closed field of characteristic zero.

{thm:mainMain}

Theorem 3.12. *Let (L, W) be a non-defective linear series on X with $\dim W \geq r+2$. Then the projection-ramification map $\rho_{X,L,W}$ is generically finite onto its image.*

For the proof, we need two lemmas, which are essentially local computations. Throughout, X, L, and W are as in the statement of Theorem 3.12.

:tangentconeram}

Lemma 3.13. *Let $x \in X$ be a general point and $V \subset W \otimes \mathfrak{m}_x$ a general $(r+1)$-dimensional subspace. Then V is properly ramified, and the ramification divisor $R(V)$ has an ordinary double point singularity at x.*

Proof. Using Proposition 3.8 and Proposition 3.9, we get a basis $(s_1, ..., s_n, t)$ of V satisfying the following two conditions:

(1) $s_1, \dots, s_n$ generate $L \otimes (\mathfrak{m}_x/\mathfrak{m}_x^2)$, and
(2) $v(t)$ has an ordinary double point singularity at x.

Let $\widehat{\mathcal{O}}_{X,x}$ denote the completion of the local ring at $x \in X$ along its maximal ideal. Upon trivializing L, we may regard s_i and t as elements of $\widehat{\mathcal{O}}_{X,x}$, and can also assume $\widehat{\mathcal{O}}_{X,x} = k[\![s_1, \dots s_n]\!]$. In the bases $(s_1, \dots, s_n, t)$ for V and $(1, s_1, \dots, s_n)$ for $P(L)$, the evaluation map

$$e \colon V \otimes \widehat{\mathcal{O}}_{X,x} \longrightarrow P(L) \otimes \widehat{\mathcal{O}}_{X,x}$$

has the matrix

{matrix}

$$\begin{pmatrix} s_1 & s_2 & \dots & t \\ 1 & 0 & \dots & \partial_1 t \\ 0 & 1 & \dots & \partial_2 t \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \partial_n t \end{pmatrix}, \tag{3.1}$$

where ∂_i denotes $\frac{\partial}{\partial s_i}$. The determinant of the matrix (3.1)

$$t - \sum_i s_i \partial_i t$$

is an analytic local equation for the ramification divisor $R(V)$ near x. Evidently, $R(V)$ shares the same tangent cone as $v(t)$ at x. The proposition follows. $\square$

m:basepointfree}

Lemma 3.14. *Let $x \in X$ be a general point and $V \subset W$ an $(r+1)$-dimensional subspace with a basis $(u, a_1, \dots, a_{r-1}, b)$ where*

(1) u does not vanish at x,
(2) $a_1, \dots, a_{r-1}$ vanish at x, and reduce to linearly independent elements of $L \otimes (\mathfrak{m}_x/\mathfrak{m}_x^2)$, and
(3) $v(b)$ has an ordinary double point at x.

Then $R(V)$ contains x and is smooth at x.

Proof. That $R(V)$ contains x is clear since $V \otimes \mathfrak{m}_x^2 \neq 0$.

For smoothness, we again work in the completion $\widehat{\mathcal{O}}_{X,x}$. After trivializing L, we assume $u, a_1, ..., b$ are elements of $\widehat{\mathcal{O}}_{X,x}$. We choose an element $z \in \widehat{\mathcal{O}}_{X,x}$ such that $(a_1, \dots, a_{r-1}, z)$

forms a system of coordinates, that is $\widehat{\mathcal{O}}_{X,x} \cong k[\![a_1, \dots, a_{r-1}, z]\!]$. With respect to the given basis of V and the basis $1, a_1, \dots, a_{r-1}, z$ for $P(L)$, the evaluation map

$$e \colon V \otimes \widehat{\mathcal{O}}_{X,x} \longrightarrow P(L) \otimes \widehat{\mathcal{O}}_{X,x}$$

has the matrix

$$\begin{pmatrix} u & a_1 & a_2 & \dots & b \\ \partial_1 u & 1 & 0 & \dots & \partial_1 b \\ \partial_2 u & 0 & 1 & \dots & \partial_2 b \\ \vdots & \vdots & \vdots & \vdots & \\ \partial_z u & 0 & 0 & \dots & \partial_z b \end{pmatrix} \tag{3.2}$$ {matrix2}

The determinant of the matrix (3.2) is the analytic local equation for $R(V)$. It is given by

$$\bar{u} \cdot \partial_z b \pm \partial_z u \cdot \bar{b},$$

where, for $r \in \widehat{\mathcal{O}}_{X,x}$ we set

$$\bar{r} = r - a_1 \partial_1 r - a_2 \partial_2 r - \dots - z \partial_z r.$$

Since $b \in \mathfrak{m}_x^2$, we get that $\bar{b} \in \mathfrak{m}_x^2$, and so $\partial_z b \in \mathfrak{m}_x$. Furthermore, since the tangent cone of b is a non-degenerate quadric, we also get that $\partial_z b \notin \mathfrak{m}_x^2$. Since $\overline{u}$ is a unit, we see that the tangent cone of $R(V)$ at x is the hyperplane cut out by $\partial_z b \in \mathfrak{m}_x / \mathfrak{m}_x^2$. So $R(V)$ is smooth at x. □

We now have all the tools for the proof of Theorem 3.12.

Proof of Theorem 3.12. We induct on $\dim W$. The base case $\dim W = r + 1$ is clear.

We now do the induction step. Suppose $\dim W \geq r + 2$. Choose a general point $x \in X$ such that the induced linear series $(\widetilde{L}, \widetilde{W})$ on $\widetilde{X} = \mathrm{Bl}_x X$ is non-defective as in Proposition 3.11. Choose a general $(r+1)$-dimensional subspace $V \subset W \otimes \mathfrak{m}_x = \widetilde{W}$ that satisfies the hypotheses of Lemma 3.13. By the induction hypothesis, V considered as a projection of $(\widetilde{L}, \widetilde{W})$ is an isolated point in the projection-ramification map for $\widetilde{X}$. We now show that it is also an isolated point in the projection-ramification map for X.

Let $(C, 0)$ be a pointed smooth curve and $V \subset W \otimes \mathcal{O}_C$ a sub-bundle of rank $(r+1)$ such that

(1) $V_0 = V$, and
(2) $V_c \neq W_0$ for $c \in C \setminus \{0\}$.

We must show that $R(V_c) \neq R(V)$ for a general $c \in C$.

Suppose $V_c \subset W \otimes \mathfrak{m}_x = \widetilde{W}$ for all $c \in C$. Denote by $\widetilde{R}(V_c)$ the ramification divisor of V_c considered as a projection of $\widetilde{X}$. Since $V = V_0$ is an isolated point in the projection-ramification map for $\widetilde{X}$, we know that $\widetilde{R}(V_c) \neq \widetilde{R}(V_0)$ for a general $c \in C$. Clearly, $R(V_c)$ and $\widetilde{R}(V_c)$ agree away from the exceptional divisor, and hence we conclude that $R(V_c) \neq R(V_0)$ for a general $c \in C$.

On the other hand, suppose $V_c \not\subset W \otimes \mathfrak{m}_x = \widetilde{W}$ for a general $c \in C$. Consider the evaluation maps

$$e_c \colon V_c \longrightarrow L/\mathfrak{m}_x^2 L$$

between an $(r+1)$-dimensional source and $(r+1)$-dimensional target. Since $V = V_0$ satisfies the hypotheses of Lemma 3.13, $\operatorname{rk} e_0 = r$. Therefore, by semi-continuity, $\operatorname{rk} e_c \geq r$ for all $c \in C$. If $\operatorname{rk} e_c = (r+1)$ for a general $c \in C$, then $x \notin R(V_c)$, and hence $R(V_c) \neq R(V)$. Otherwise, by shrinking C if necessary, assume $\operatorname{rk} e_c = r$ for all $c \in C$. In other words, $\dim(V_c \otimes \mathfrak{m}_x^2) = 1$ for all $c \in C$. Let $b_c \in V_C \otimes \mathfrak{m}_x^2$ be a non-zero element. Since $v(b_0)$ has an ordinary double-point singularity at x, so does $v(b_c)$. Also, since $\operatorname{rk}(e_c) = r$ and $V_c \notin W \otimes \mathfrak{m}_x$ for a general c, there exists $u_c \in V_c$ not vanishing at x, and a set of $(r-1)$ other elements that vanish at x but reduce to linearly independent elements modulo $\mathfrak{m}_x^2$. That is, V_c satisfies the hypotheses of Lemma 3.14 for a general $c \in C$. But Lemma 3.14 implies that $R(V_c)$ is smooth at x. Since $R(V_0)$ is singular at x, we conclude that $R(V_0) \neq R(V_c)$. The induction step is now complete. □

We immediately get part (2) of Theorem A.

cor:maintheorem}

Corollary 3.15. *Let $X \subset \mathbf{P}^n$ be a non-degenerate projective variety such that the dual variety $X^* \subset \mathbf{P}^{n*}$ is a hypersurface. Then ρ_X is generically finite onto its image.*

Proof. By Proposition 3.7 the linear series on X that gives the embedding $X \subset \mathbf{P}^n$ is non-defective. Now apply Theorem 3.12. □

{cor:lowdim}

Corollary 3.16. *Let $X \subset \mathbf{P}^n$ be a non-degenerate smooth curve or a surface. Then ρ_X is generically finite onto its image.*

Proof. Curves and surfaces have divisorial duals, so Corollary 3.15 applies. □

c:minimaldegree}

4. Projection-ramification for varieties of minimal degree

In this section, we prove Theorem C, which relates varieties of minimal degree and the projection-ramification map. We then prove Theorem B by constructing examples of rational scrolls where maximal variation fails. Finally, we obtain an alternate and more explicit description of the projection-ramification map for scrolls, which we use repeatedly.

The following is an easy application of the Kodaira vanishing theorem.

{lem:kymh}

Proposition 4.1. *Let $X \subset \mathbf{P}^n$ be a non-degenerate, smooth, projective, variety of dimension $r \geq 1$ over a field of characteristic zero. For all $m \geq r$, we have the inequality*

{eqn:KYmH}

$$\binom{m}{r}(n-r) + \binom{m-1}{r} \leq h^0(X, K_X + mH). \tag{4.1}$$

If equality holds for any $m \geq r$, then X is a variety of minimal degree, that is $\deg X = n - r + 1$. Conversely, for a variety of minimal degree, equality holds for all $m \geq r$.

Proof. Without loss of generality, X is embedded by the complete linear series. Indeed, passing to the complete linear series only increases the left side of the desired inequality, and does not change the right side.

We first prove the inequality (4.1), using a double induction–first on r, and then on m. For the base case $r = 1$, Riemann–Roch gives

{eqn:r1}
$$h^0(X, K_X + mH) = g_X - 1 + mn, \tag{4.2}$$

from which (4.1) follows for all m.

Assume that (4.1) holds for varieties of dimension $(r-1)$ and all $m \geq r-1$. Let $D \subset X$ be a general member of the linear series $|H|$. By Bertini's theorem, D is a smooth variety. The adjunction formula $K_D = (K_X + H)|_D$ yields the exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X + (m-1)H) \longrightarrow \mathcal{O}_X(K_X + mH) \longrightarrow \mathcal{O}_D(K_D + (m-1)H) \longrightarrow 0. \tag{4.3}$$
{eqn:mainexact}

Note that, by the Kodaira vanishing theorem, we have $h^1(K_X + nH) = 0$ for all $n > 1$; we use this repeatedly, without further comment. For $m = r$, the long exact sequence in cohomology associated to (4.3) gives

$$h^0(K_D + (r-1)H) \leq h^0(K_X + rH).$$

By applying the induction hypothesis to D, we have

$$n - r \leq h^0(K_D + (r-1)H) \tag{4.4}$$

Therefore, we conclude that

$$(n - r) \leq h^0(K_D + rH). \tag{4.5}$$

Let $m > r$, and assume that (4.1) holds for X for $m-1$. The long exact sequence in cohomology associated to (4.3) gives

$$h^0(K_X + (m-1)H) + h^0(K_D + (m-1)H) = h^0(K_X + mH). \tag{4.6}$$
{eqn:add}

By applying the induction hypothesis to $m-1$, we get

$$\begin{aligned} h^0(K_X + (m-1)H) &+ h^0(K_D + (m-1)H) \\ &\geq \binom{m-1}{r}(n-r) + \binom{m-2}{r} + \binom{m-1}{r-1}(n-r) + \binom{m-2}{r-1} \\ &= \binom{m}{r}(n-r) + \binom{m-1}{r}. \end{aligned}$$

Together with (4.6), we conclude

$$\binom{m}{r}(n-r) + \binom{m-1}{r} \leq h^0(K_X + mH), \tag{4.7}$$

which is (4.1) for m. The proof of the inequality is thus complete.

We now examine when equality holds in (4.1). For $r = 1$, the equation (4.2) shows that equality holds for some m if and only if $g_X = 0$, that is $X \subset \mathbf{P}^n$ is a rational normal curve, and in this case, equality holds for all m. Furthermore, we observe in the inductive proof that if equality holds for an X of dimension $r > 1$ and some m, then it must hold for the hyperplane slice D and $(m-1)$. Again, by an induction on r, we conclude that $\deg X = n - r + 1$, that is, $X \subset \mathbf{P}^n$ is a variety of minimal degree.

Finally, for $X \subset \mathbf{P}^n$ of minimal degree, induction on r shows that equality holds in (4.1) for all m. □

As a consequence, we immediately deduce Theorem C.

alminimaldegree}

Theorem 4.2 (Theorem C)**.** *Let $X \subset \mathbf{P}^n$ be a smooth, non-degenerate projective variety of dimension $r \geq 1$ over a field of characteristic zero. We have the inequality*

$$\dim \mathbf{Gr}(n-r, n+1) \leq \dim |K_X + (r+1)H|,$$

where equality holds if and only if X is a variety of minimal degree, that is $\deg X = n-r+1$.

Proof. Apply Proposition 4.1 with $m = r+1$. □

{sec:prscrolls}

4.1. **Projection-ramification for scrolls.** Theorem C motivates a deeper investigation of the projection-ramification map for varieties of minimal degree. Indeed, for $X \subset \mathbf{P}^n$ of minimal degree, the projection-ramification map is potentially generically finite. Recall that a large class of varieties of minimal degree are the rational normal scrolls, namely $X = \mathbf{P}E$ for an ample vector bundle E on $\mathbf{P}^1$ embedded by the complete linear series $\mathcal{O}_X(1)$. If $\dim X \geq 3$, then X is neither incompressible nor does it have a divisorial dual variety. Therefore, for such X, Theorem A leaves the question of maximal variation unanswered.

We now examine the projection-ramification map for projectivizations of vector bundles on smooth curves in more detail. Let C be a smooth curve and E an ample vector bundle on C of rank r. Set $X = \mathbf{P}E$, the space of one-dimensional quotients of E, and $L = \mathcal{O}_X(1)$. Denote by $\pi \colon X \longrightarrow C$ the natural map.

Let (L, V) be a projection of X. Recall from (2.1) that such a projection gives a map

$$r_V \colon \det V \longrightarrow H^0(X, K_X \otimes L^{r+1}),$$

whose zero locus is the ramification divisor $R(V) \subset X$. Note that we have an isomorphism $K_X \cong \pi^*(\det E \otimes K_C) \otimes L^{-r}$, and hence, we may view r_v as a map

$$r_V \colon \det V \longrightarrow H^0(C, E \otimes \det E \otimes K_C).$$

We now describe another construction of a section of $E \otimes \det E \otimes K_C$ from V, which we call the *differential construction*. The subspace $V \subset H^0(X, L) = H^0(C, E)$ gives the evaluation map

$$e \colon V \otimes \mathcal{O}_C \longrightarrow E.$$

If V is generic, then e is a surjection, and its kernel is canonically isomorphic to $\det E^* \otimes \det V$. Consider the diagram

al_construction}

$$\begin{tikzcd} 0 \arrow[r] & \det E^* \otimes \det V \arrow[r] \arrow[d, "d_V"] & V \otimes \mathcal{O}_C \arrow[r, "e"] \arrow[d, "e"] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_C \otimes E \arrow[r] & P(E) \arrow[r] & E \arrow[r] & 0, \end{tikzcd} \tag{4.8}$$

where the bottom row is the standard sequence associated to $P(E)$, both maps labeled e are evaluation maps, and the map d_V is the map induced by them. The map d_V gives a map

$$d_V\colon \det V \longrightarrow H^0(C, E \otimes \det E \otimes K_C).$$

{prop:rdv}

Proposition 4.3. *In the setup above, the two maps d_V and r_V are equal.*

Proof. Recall that r_V is induced by the determinant of the evalutation map

$$V \otimes \mathcal{O}_X \longrightarrow P(L).$$

Denote by $P_\pi(L)$ the bundle of principal parts of L along the fibers of π. More explicitly,

$$P_\pi(L) = \pi_{1*}\left(\pi_2^* L \otimes \left(\mathcal{O}_{X\times_\pi X}/I_\Delta^2\right)\right),$$

where $\Delta \subset X \times_\pi X$ is the diagonal and π_i for $i = 1, 2$ are the two projections $X \times_\pi X \longrightarrow X$. It is easy to check that the evaluation map $\pi^* E \longrightarrow L$ induces an isomorphism $\pi^* E \longrightarrow P_\pi(L)$. Furthermore, we have the sequence

$$0 \longrightarrow \pi^* K_C \otimes L \longrightarrow P(L) \longrightarrow P_\pi(L) \longrightarrow 0.$$

By combining this with the identification $\pi^* E = P_\pi(L)$, and the top row of (4.8), we get the diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi^*(\det E^* \otimes \det V) \arrow[r] \arrow[d, "p"] & V \otimes \mathcal{O}_X \arrow[r] \arrow[d, "e"] & \pi^* E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi^* K_C \otimes L \arrow[r] & P(L) \arrow[r] & P_\pi(L) \arrow[r] & 0.
\end{tikzcd} \tag{4.9}$$

{eqn:pxpl}

From the diagram, we see that $\det e = p$, interpreted as elements of the appropriate Hom spaces. By definition, after taking global sections, $\det e$ gives the section r_V. Note that, applying π_* to the bottom row of (4.9) yields the bottom row of (4.8). Hence, after applying π_*, twisting by $\det E$ and taking global sections, p gives the section d_V. We conclude that $r_V = d_V$. $\square$

Let $R = R(V) \subset X$ be the ramification divisor of the projection given by V. Note that R is a divisor of class $\pi^*(\det E \otimes K_C) \otimes \mathcal{O}_X(1)$. Therefore, $R \subset X$ is a sub-scroll, or equivalently, the fibers of $R \longrightarrow C$ are hyperplanes in the corresponding fiber of $X \longrightarrow C$. We can obtain an explicit description of these hyperplanes in two ways, one using the original definition, and one using the differential construction. Fix a point $c \in C$, and a uniformizer t of C at c. Let $X_c \subset X$ and $R_c \subset R$ be the fibers of $X \longrightarrow C$ and $R \longrightarrow C$ over c, respectively.

By definition $R \subset X$ is the set of points $x \in X$ for which there exists $s \in V$ such that $v(s)$ is singular at x. Since s is a section of $L = \mathcal{O}_X(1)$, the hypersurface $v(s)$ is singular at x if and only if it contains the entire fiber of $\pi\colon X \longrightarrow C$ through x. Suppose $\pi(x) = c$. Then, in an open set of X containing X_c, we have $s = ts_1$ for a section s_1 of $\mathcal{O}_X(1)$. Observe that, we have $\operatorname{Sing}(v(s)) \cap F = v(s_1) \cap F$, and therefore, $R_c \subset X_c$ is the hyperplane cut out by s_1.

To obtain the same description using the differential construction, consider the top row of (4.8). Let v be a local section of $V \otimes \mathcal{O}_C$ around c that generates the kernel of $e\colon V \otimes \mathcal{O}_C \longrightarrow E$ at c. The fiber of the evaluation map $V \otimes \mathcal{O}_C \longrightarrow P(L)$ over c sends $v \in V$ to the image of $e(v)$ in $L/\mathfrak{m}_c^2 L$. Since v generates the kernel of $e\colon V \otimes \mathcal{O}_C \longrightarrow L$ at c, we know that image of $e(v)$ in $L/\mathfrak{m}_c L$ is zero. Writing $e(v) = ts_1$ for a section s_1 of E around c, we see that $d_V(v) = s_1 \otimes t \in E \otimes \mathfrak{m}_c/\mathfrak{m}_c^2$. Thus, the fiber of the sub-scroll defined by d_V over c is the hyperplane in X_c cut out by s_1.

Finally, we write an equation of $R(V) \subset X$ over an open subset of C containing c explicitly in coordinates. Choose a trivialization $X_1, \dots, X_r$ for E over an open set $U \subset C$ containing c. Then $X_U \cong \mathbf{P}^{r-1} \times U = \operatorname{Proj} \mathcal{O}_U[X_1, \dots, X_r]$. We have a trivialization of K_C over U given by dt. We then get a trivialization of $P(E)|_U$ by $X_1, \dots, X_r, dt \otimes X_1, \dots, dt \otimes X_r$. Choose a basis $v_0, \dots, v_r$ of V, and suppose the map $e\colon V \otimes \mathcal{O}_U \longrightarrow E_U$ is given by

$$e(v_i) = \sum m_{i,j} X_j,$$

for $m_{i,j} \in \mathcal{O}_U$, where $0 \leq i \leq r$ and $1 \leq j \leq r$. Then the map $\det E^* \otimes \det V \longrightarrow V \otimes \mathcal{O}_U$ defining the kernel of e is given by the $r \times r$ minors of the matrix $(m_{i,j})$. Denote the ℓ-th minor by M_ℓ; that is $M_\ell = (-1)^\ell \det(m_{i,j} \mid i \neq \ell)$. Then the map d_V sends the generator to the element of $E \otimes K_C$ given by

$$\sum_{i,j} M_i \cdot \frac{\partial m_{i,j}}{\partial t} \cdot (dt \otimes X_j).$$

Note that the expression above is the determinant of the $(r+1) \times (r+1)$ matrix

{eqn:Rmatrix}
$$\begin{pmatrix} m_{0,1} & m_{0,2} & \dots & m_{0,r} & \sum_{i=1}^r \frac{\partial m_{0,j}}{\partial t} \cdot dt \otimes X_j \\ m_{1,1} & m_{1,2} & \dots & m_{1,r} & \sum_{i=1}^r \frac{\partial m_{1,j}}{\partial t} \cdot dt \otimes X_j \\ \vdots & \ddots & \dots & \vdots & \vdots \\ m_{r,1} & m_{r,2} & \dots & m_{r,r} & \sum_{i=1}^r \frac{\partial m_{r,j}}{\partial t} \cdot dt \otimes X_j \end{pmatrix}. \tag{4.10}$$

This gives an equation for $R_U \subset X_U = \operatorname{Proj} \mathcal{O}_U[X_1, \dots, X_r]$.

{sec:failure}

4.2. **Failure of maximal variation.** In this section, we show that there exists ample vector bundles E of rank $r \geq 4$ on $\mathbf{P}^1$ such that the projection-ramification map for $X = \mathbf{P}E$ is not generically finite. In other words, a generic projection of X can be deformed in a one-parameter family so that the ramification divisor remains unchanged.

Recall that the projection-ramification map for $X = \mathbf{P}E$ and the complete linear series of $L = \mathcal{O}_X(1)$ is a map

$$\rho\colon \mathbf{Gr}(r+1, H^0(X, L)) \dashrightarrow |K_X \otimes L^{r+1}|,$$

or equivalently a map

$$\rho\colon \mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E)) \dashrightarrow \mathbf{P}H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1})^*.$$

By construction, ρ is equivariant with respect to the action of $\operatorname{Aut}(X)$, and in particular, by the subgroup $\operatorname{Aut}(X/\mathbf{P}^1)$.

We engineer the failure of maximal variation using the following observation.

ivialStabilizer}

Proposition 4.4. *A generic point of* $\mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E))$ *has a trivial stabilizer under the action of* $\mathrm{Aut}(\mathbf{P}E/\mathbf{P}^1)$.

Proof. Fix $(r+1)$ distinct points $p_0, \dots, p_r \in \mathbf{P}^1$. Let $V \subset H^0(\mathbf{P}^1, E)$ be a generic $(r+1)$ dimensional subspace. Let $e \colon V \otimes \mathcal{O}_{\mathbf{P}^1} \longrightarrow E$ be the evaluation map. The points $p_0, \dots, p_r$ give vectors $v_0, \dots, v_r \in V$, unique up to scaling, defined by the property that $e(v_i) = 0$ in the fiber $E|_{p_i}$. Choose a generic point $t \in \mathbf{P}^1$. We get $(r+1)$ points $x_0, \dots, x_r \in \mathbf{P}E^*|_t \cong \mathbf{P}^{r-1}$ given by $e(v_0), \dots, e(v_r)$ evaluated at t. For generic V and t, it is easy to check that these points are in linear general position. Any element of $\mathrm{Aut}(\mathbf{P}E/\mathbf{P}^1)$ that fixes V must fix $x_0, \dots, x_r$. But then it must act as the identity on the projective space $\mathbf{P}E^*|_t$, and hence on the dual projective space $\mathbf{P}E|_t$. Since $t \in \mathbf{P}^1$ is general, it follows that it must be the identity. □

{prop:specialE}

Proposition 4.5. *There exist ample vector bundles E of every rank ≥ 4 such that a general point of* $\mathbf{P}H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1})$ *has a positive-dimensional stabilizer under* $\mathrm{Aut}(\mathbf{P}E/\mathbf{P}^1)$. *In particular, we may take $E = \mathcal{O}(1)^{r-1} \oplus \mathcal{O}(k+1)$ where $k \geq 1$ and $r \geq 4$.*

Proof. It suffices to exhibit an E such that a generic element of $H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1})$ has a positive dimensional stabilizer under the action of $\mathrm{Aut}(E/\mathbf{P}^1)$. Take

$$E = \mathcal{O}(a)^{r-1} \oplus \mathcal{O}(b),$$

where $0 < a < b$ are to be determined. Elements of $\mathrm{Aut}(E/\mathbf{P}^1)$ can be represented by block lower triangular square matrices

$$M = \begin{pmatrix} A & \\ U & B \end{pmatrix},$$

where $A \in \mathrm{GL}_a(k)$, $B \in k^\times$, and $U = (u_i)$ is an $(r-1)$ length row with entries in $H^0(\mathbf{P}^1, \mathcal{O}(b-a))$. Set $d = (r-1)a + b$ so that $\det E = \mathcal{O}(d)$. Suppose a, b, and r, are such that

$$(r-1)(b-a+1) \geq b + d - 1 = (r-1)a + 2b - 1. \tag{4.11}$$ {eqn:requirement

Take a general element of $H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1})$; say it is given by the column vector

$$v = (p_1, \dots, p_{r-1}, q)^T,$$

where the p_i (resp q) are homogeneous polynomials in X, Y of degree $a + d - 2$ (resp $b + d - 2$). We take $A = \mathrm{id}_{r-1}$ and $B = \lambda$ for some $\lambda \in k^\times$, and show that there exists a $U = (u_i)$ such that $Mv = v$. Indeed, we have $Mv = (p_1, \dots, p_r, q')$, where

$$q' = \lambda q + \sum u_i p_i.$$

Let $W \subset H^0(\mathbf{P}^1, \mathcal{O}(a+d-1))$ be the vector space spanned by $p_1, \dots, p_{r-1}$. Consider the multiplication map

$$H^0(\mathbf{P}^1, \mathcal{O}(b-a)) \otimes W \longrightarrow H^0(\mathbf{P}^1, \mathcal{O}(b+d-2)).$$

Thanks to (4.11), the dimension of the source is at least as much as the dimension of the target. It is easy to check that the map is in fact surjective for generic $p_1, \dots, p_{r-1}$. In particular, there exist $u_i \in H^0(\mathbf{P}^1, \mathcal{O}(b-a))$ for $i = 1, \dots, r-1$, such that

$$q(1-\lambda) = \sum u_i p_i.$$

With this choice of $U = (u_i)$, we get M such that $Mv = v$.

Finally, note that the requirement (4.11) is satisfied for $a = 1$ and $b = k+1$ if $k \geq 1$ and $r \geq 4$. ☐

counterexamples}

Corollary 4.6 (Theorem B)**.** *Let $r \geq 3$ and $d \geq r+1$. There exist ample vector bundles E of rank r and degree d on $\mathbf{P}^1$ such that for $X = \mathbf{P}E$ and the complete linear series $L = \mathcal{O}_X(1)$, the projection-ramification map ρ_X is not generically finite onto its image.*

Proof. Take E such that the action of $\operatorname{Aut}(X/\mathbf{P}^1)$ on a generic point of $|K_X \otimes L^{r+1}|$ has a positive-dimensional stabilizer (see Proposition 4.5). Since $\rho_X \colon \mathbf{Gr}(r+1, H^0(X, L)) \dashrightarrow |K_X \otimes L^{r+1}|$ is equivariant with respect to the action of $\operatorname{Aut}(X/\mathbf{P}^1)$, and a generic point of the source does not have a positive-dimensional stabilizer (see Proposition 4.4), it follows that ρ_X cannot be dominant. Since the dimension of the source and target of ρ_X are the same, ρ_X is not generically finite. ☐

Remark 4.7. In all the examples of scrolls where we know that maximal variation fails, the failure can be explained by the presence of generic stabilizers. We do not know, however, if this is the only reason for the failure of maximal variation.

Remark 4.8. If $k = 1$ and $r \geq 4$, then X is the most balanced scroll of its degree and rank, and hence, generic in moduli. Therefore, the non-dominance of projection-ramification is not directly connected to the eccentricity of the splitting type of a scroll.

tric_threefolds}

4.3. **Eccentric threefold scrolls.** Theorem B leaves open the case of threefold scrolls (surface scrolls are covered by Corollary 3.16). We settle this case in this section by showing that the projection-ramification map for threefold scrolls is always generically finite, and thus the statement of Theorem B is sharp in r.

Let $E = \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(k+1)$, for $k \geq 0$. Set $X = \mathbf{P}E$ and $L = \mathcal{O}_X(1)$.

ntric_threefold}

Proposition 4.9. *The map $\rho_X \colon \mathbf{Gr}(4, H^0(X, L)) \dashrightarrow |K_X + 4L|$ is birational.*

Proof. The proof is by direct calculation. Consider the standard open subset $\mathbf{A}^1 = \operatorname{Spec} k[t] \subset \mathbf{P}^1$. Choose trivializations of the four summands of E over $\mathbf{A}^1$ given by sections X_1, X_2, X_3.

Let $W \subset H^0(X, L)$ be a general 4-dimensional subspace. Then the projection map $W \longrightarrow H^0(\mathbf{P}^1, \mathcal{O}(1) \oplus \mathcal{O}(1))$ will be an isomorphism. Therefore, we can choose a basis of W of the form

$$X_1 + aX_3, X_2 + bX_3, tX_1 + cX_3, tX_2 + dX_3,$$

where $a, b, c, d \in k[t]$ have degree at most $k+1$. Using (4.10), we get that the ramification divisor of this W is

qn:threefoldram}

$$\begin{aligned}\rho(W) &= (d-bt)X_1 + (at-c)X_2 + \big((a't-c')(bt-d) + (at-c)(d'-b't)\big)\, X_3 \\ &= \alpha X_1 + \beta X_2 + \gamma X_3, \text{ say.}\end{aligned} \tag{4.12}$$

In this calculation, p' denotes the derivative $\frac{dp}{dt}$. Note that we have

$$\gamma = \alpha'\beta - \beta'\alpha + \alpha a + \beta b.$$

The degrees of α, β, γ are (at most) $k+2$, $k+2$, and $2k+2$, respectively.

Consider the affine space $\mathbf{A}^{4k+8}$ whose coordinates correspond to the coefficients of a, b, c, d, and likewise, the affine space $\mathbf{A}^{4k+9}$ whose coordinates correspond to the coefficients of α, β, γ. The expression in (4.12) defines a map

$$\begin{aligned}\rho^* \colon \mathbf{A}^{4k+8} &\longrightarrow \mathbf{A}^{4k+9} \\ (a,b,c,d) &\mapsto (\alpha, \beta, \gamma).\end{aligned}$$

Note that the choice of basis of W gives a birational isomorphism $\mathbf{Gr}(4, H^0(X,L)) \cong \mathbf{A}^{4k+8}$. Via this isomorphism, the projection-ramification map ρ is simply the composite of ρ^* and the projection $\mathbf{A}^{4k+9} \dashrightarrow \mathbf{P}^{4k+8} = \mathbf{P}H^0(\mathbf{P}^1, E \otimes \det E \otimes \mathcal{O}(-2))^*$.

Let $(a,b,c,d) \in \mathbf{A}^{4k+8}$ be a generic point. We show that the map induced by ρ on tangent spaces is injective at this point. For $\epsilon^2 = 0$, we have

$$\rho^* \colon (a + \hat{a}\epsilon, b + \hat{b}\epsilon, c + \hat{c}\epsilon + d + \hat{d}\epsilon) \mapsto (\alpha + \hat{\alpha}\epsilon, \beta + \hat{\beta}\epsilon, \gamma + \hat{\gamma}\epsilon),$$

where

$$\begin{aligned}\hat{\alpha} &= \hat{d} - \hat{b}t, \\ \hat{\beta} &= \hat{a}t - \hat{c}, \text{ and} \\ \hat{\gamma} &= (bt-d)(\hat{a}'t - \hat{c}') + (at-c)(\hat{d}' - \hat{b}'t) + (\hat{a}t - \hat{c})(d' - b't) + (\hat{b}t - \hat{d})(a't - c).\end{aligned}$$

Suppose $\hat{\alpha} = \hat{\beta} = \hat{\gamma} = 0$. Then $(bt-d)(\hat{a}t - \hat{c}) + (at-c) + (\hat{d}' - \hat{b}'t) = 0$. However, for generic a, b, c, d, the polynomials $(bt-d)$ and $(at-c)$ have degree $(k+2)$ and are relatively prime. So they have no non-trivial syzygy with coefficients of degree at most $k+1$. As a result, we get $\hat{a}t - \hat{c} = 0$ and $\hat{d}' - \hat{b}'t = 0$. Along with $at - c = 0$ and $d - bt = 0$, we get $a = b = c = d = 0$. Thus, $d\rho^*$ is injective. To show that $d\rho$ is injective, it suffices to show that the image of $d\rho^*$ intersects the line joining $(0,0,0)$ an (α, β, γ) transversely. For this, it suffices to show that the three equations $\hat{\alpha} = \alpha, \hat{\beta} = \beta, \hat{\gamma} = \gamma$ give no solutions in (a,b,c,d). Indeed, these three equations imply $(bt-d)(\hat{a}'t - \hat{c}') + (at-c)(\hat{d}' - \hat{b}'t) = 0$, and by the same reason as before, give $\hat{a}'t - \hat{c}' = \hat{d}' - \hat{b}'t = 0$. But it is easy to see that the last two equations combined with $\hat{a}t - \hat{c} = at - c$ and $\hat{b}t - \hat{d} = bt - d$ have no solutions.

Since the map induced by ρ is injective tangent spaces, ρ is generically finite, and hence dominant. To show that ρ is birational, it suffices to show that the generic fiber of ρ is connected. For this, it suffices to show that the generic fiber of ρ^* is connected. However, observe that the fiber of ρ^* over (α, β, γ) is given by the equations

$$\alpha = d - bt, \beta = at - c, \gamma = \alpha'\beta - \beta'\alpha + \alpha a + \beta b,$$

which are affine linear equations in a, b, c, d. Hence, the fibers of ρ^* are affine spaces, which are connected. The proof is now complete. $\square$

riation3scrolls}

Corollary 4.10. *The projection-ramification map ρ_X is dominant for every smooth three dimensional rational normal scroll $X \subset \mathbf{P}^n$.*

Proof. Every such X isotrivially specializes to $\mathbf{P}\left(\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(k+1)\right)$. The statement now follows from the upper semi-continuity of fiber dimension. $\square$

The case of eccentric surface scrolls follows by similar calculations as in the proof of Proposition 4.9; we omit the details.

centric_surface}

Proposition 4.11. *Let $E = \mathcal{O}(1) \oplus \mathcal{O}(k+1)$, for $k \geq 0$. Set $X = \mathbf{P}E$ and $L = \mathcal{O}_X(1)$. Then the projection-ramification map $\rho_X \colon \mathbf{Gr}(3, H^0(X, L)) \dashrightarrow |K_X + 3L|$ is birational.*

{sec:generic}

5. Maximal variation for generic scrolls

In this section, we establish that the projection-ramification map is generically finite (equivalently, dominant) for most scrolls, notwithstanding the examples provided by Theorem B. We begin by treating the cases of some particular scrolls by hand. We then bootstrap these to more general results using degeneration arguments.

{sec:lowdegree}

5.1. **Maximal variation for some particular cases.** Given an ample vector bundle E on $\mathbf{P}^1$, we say that *maximal variation holds for E* if the projection-ramification map is generically finite (equivalently, dominant) for $X = \mathbf{P}E$ embedded by the complete linear series associated to $L = \mathcal{O}_X(1)$.

{prop:segre}

Proposition 5.1. *Maximal variation holds for $E = \mathcal{O}(1)^r$. In fact, the degree of the projection-ramification map in this case is* 1*.*

Proof. We know that the projection-ramification map

$$\rho\colon \mathbf{Gr}(r+1, H^0(\mathbf{P}^1, \mathcal{O}(1)^r)) \dashrightarrow \mathbf{P}H^0(\mathbf{P}^1, \mathcal{O}(r-1)^r)^*$$

is $\operatorname{Aut}\mathbf{P}E$ equivariant. In this case, it is easy to check that the action of $\operatorname{Aut}(\mathbf{P}E/\mathbf{P}^1) = \mathrm{PGL}_r$ has a unique open orbit and trivial generic stabilizers on both the source and the target of ρ. Hence, ρ must be birational. $\square$

{prop:222}

Proposition 5.2. *Maximal variation holds for $E = \mathcal{O}(2)^r$.*

Compared to Proposition 5.1, our proof of Proposition 5.2 is significantly more involved, and does not yield the degree.

Proof. We exhibit a point $\mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E))$ at which ρ is defined, and at which the induced map $d\rho$ on the tangent space is non-singular. It follows that ρ is a local isomorphism at this point, and hence dominant overall.

Our proof is by direct calculation. We calculate on $\mathbf{A}^1 = \operatorname{Spec} k[x] \subset \mathbf{P}^1$ and identify $\mathcal{O}(n)$ with $\mathcal{O}(n \cdot \infty)$. Then the global sections of $\mathcal{O}(n)$ are identified with polynomials in x of degree at most n. Denote the generator of the ith summand of $E(-2)$ by X_i. Consider the point of $\mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E))$ represented by the vector space $V \subset H^0(\mathbf{P}^1, E)$ spanned by the $(r+1)$ sections $v_1, \dots, v_{r+1}$ defined as follows. Set $v_i = (x-a_i)^2 X_i$ for $0 \le i \le r-1$, and $v_r = \sum p_i X_i$, where $a_i \in k$, and $p_j \in H^0(\mathbf{P}^1, \mathcal{O}(2))$ are generic. By (4.10), the ramification divisor associated to V is cut out by the determinant of the matrix

$$M = \begin{pmatrix} (x-a_1)^2 & 0 & \cdots & 0 & 2(x-a_1)X_1 \\ 0 & (x-a_2)^2 & \cdots & 0 & 2(x-a_2)X_2 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & (x-a_r)^2 & 2(x-a_r)X_r \\ p_1 & p_2 & \cdots & p_r & \sum p_i' X_i \end{pmatrix}.$$

We leave it to the reader to check that $R = \det M$ is not identically zero.

To do the tangent space computation, we choose elements $w_i \in H^0(\mathbf{P}^1, E)$, and change v_i to $v_i + \epsilon w_i$, where $\epsilon^2 = 0$. Let R_ϵ be the equation of the discriminant of the projection given by $V_\epsilon \subset H^0(\mathbf{P}^1, E) \otimes k[\epsilon]/\epsilon^2$, where V_ϵ is spanned by $v_1 + \epsilon w_1, \dots, v_{r+1} + \epsilon w_{r+1}$. Concretely, R_ϵ is the determinant of a matrix M_ϵ given by (4.10), which reduces to M modulo ϵ. Note that R_ϵ is an element of $H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2)) \otimes k[\epsilon]/\epsilon^2$, and we have

$$R_\epsilon = R + \epsilon S(w_1, \dots, w_{r+1}),$$

for some $S(w_1, \dots, w_{r+1}) \in H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2))$. Furthermore, the map

$$S\colon H^0(\mathbf{P}^1, E)^{r+1} \longrightarrow H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2)) \tag{5.1}$$ {eqn:mainmap}

is a linear map. To show that $d\rho$ is non-singular at V, it suffices to show that S is surjective. For $1 \le i \le r$ and $1 \le j \le r+1$, let $E_{i,j} \in H^0(\mathbf{P}^1, E)^{r+1}$ be the element corresponding to $(w_1, \dots, w_{r+1})$ where $w_j = X_i$ and $w_\ell = 0$ for all $\ell \ne j$. For $i \ne j$ and $1 \le j \le r$ and $q \in H^0(\mathbf{P}^1, \mathcal{O}(2))$, by direct calculation we get

$$S\left(qE_{i,j}\right) = \frac{(x-a_1)^2 \cdots (x-a_r)^2 p_j}{(x-a_i)^2 (x-a_j)^2} \cdot [q, (x-a_i)^2] \cdot X_i,$$

where the notation $[a, b]$ means $a'b - ab'$. Similarly, we get

$$S\left(qE_{i,r+1}\right) = -\frac{(x-a_1)^2 \cdots (x-a_r)^2}{(x-a_i)^2} \cdot [q, (x-a_i)^2] \cdot X_i,$$

and

$$S\left(qE_{i,i}\right) = \det M_i, \tag{5.2}$$ {eqn:diag}

where M_i is obtained from M by changing the (i,i)-th entry from $(x-a_i)^2$ to q and the $(i, r+1)$-th entry from $2(x-a_i)X_i$ to $q' X_i$.

Fix an i with $1 \leq i \leq r$, and consider the subspace $W_i \subset H^0(\mathbf{P}^1, E)^{r+1}$ spanned by $qE_{i,j}$ for $j \neq i$. By our calculations above, S maps W_i to the subspace of $H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2))$ spanned by $H^0(\mathbf{P}^1, \mathcal{O}(2r)) \otimes X_i$. We begin by identifying $S(W_i)$.

For $1 \leq j \leq r$ and $j \neq i$, set

$$Q_{i,j} = \frac{(x-a_1)^2 \cdots (x-a_r)^2 p_j}{(x-a_i)^2(x-a_j)^2},$$

and

$$Q_{i,r+1} = -\frac{(x-a_1)^2 \cdots (x-a_r)^2}{(x-a_i)^2}.$$

We claim that, there is no non-trivial linear relation among the r polynomials $Q_{i,j}$ for $j \in \{1, \dots, r+1\} \setminus \{i\}$. Indeed, suppose we had a linear relation

$$\sum l_j Q_{i,j} = 0,$$

then dividing throughout by $\frac{(x-a_1)^2 \cdots (x-a_r)^2}{(x-a_i)^2}$ gives the relation

$$\sum_{j=1}^{r} l_j \frac{p_j}{(x-a_j)^2} + l_{r+1} = 0.$$

If $l_j \neq 0$ for some j with $1 \leq j \leq r$, then we have a pole on the left side at $x = a_j$, but not on the right side (note that $(x-a_j)$ does not divide p_j by the genericity of p_j). Therefore, we must have $l_j = 0$ for all j, and hence also $l_{r+1} = 0$. Consider the map

{eqn:big} $$H^0(\mathbf{P}^1, \mathcal{O}(1)) \otimes \langle Q_{i,j} \mid j \in \{1, \dots, r+1\} \setminus \{i\} \rangle \longrightarrow H^0(\mathbf{P}^1, \mathcal{O}(2r-1)). \tag{5.3}$$

We just saw that this map is injective. But both sides have the same dimension, and hence the map must be surjective. Finally, it is easy to see that the image of the map

{eqn:q} $$H^0(\mathbf{P}^1, \mathcal{O}(2)) \longrightarrow H^0(\mathbf{P}^1, \mathcal{O}(2)), \quad q \mapsto [q, (x-a_i)^2] \tag{5.4}$$

is $(x-a_i) \cdot H^0(\mathbf{P}^1, \mathcal{O}(1))$. By (5.3) and (5.4), we conclude that the image of the map

$$S \colon W_i = \langle qE_{i,j} \mid j \in \{1, \dots, r+1\} \setminus \{i\} \longrightarrow H^0(\mathbf{P}^1, \mathcal{O}(2r-1)) \otimes X_i$$

is $(x-a_i)H^0(\mathbf{P}^1, \mathcal{O}(2r-2)) \otimes X_i$. In other words, the cokernel of the map is $X_i \otimes k$ where the map

$$H^0(\mathbf{P}^1, \mathcal{O}(2r)) \otimes X_i \longrightarrow k \otimes X_i$$

is given by evaluation at a_i. Putting together the maps for various i, we see that the cokernel of the map

$$S \colon \bigoplus_i W_i \longrightarrow H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2)) = H^0(\mathbf{P}^1, \mathcal{O}(2r)) \otimes \langle X_1, \dots, X_r \rangle$$

is $k \otimes \langle X_1, \dots, X_r \rangle$, where the map

{eqn:partialsur} $$H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2)) = H^0(\mathbf{P}^1, \mathcal{O}(2r)) \otimes \langle X_1, \dots, X_r \rangle \longrightarrow k \otimes \langle X_1, \dots, X_r \rangle \tag{5.5}$$

on $H^0(\mathbf{P}^1, \mathcal{O}(2r)) \otimes X_i$ is given by evaluation at a_i.

To show that S is surjective, it is now enough to show that the map

{eqn:remainsur} $$H^0(\mathbf{P}^1, \mathcal{O}(2)) \otimes \langle E_{i,i} \mid i \in \{1, \dots, r+1\} \rangle \longrightarrow k \otimes \langle X_1, \dots, X_r \rangle \tag{5.6}$$

obtained by composing (5.1) and (5.5) is surjective. Recall from (5.2) that we have $S(qE_{i,i}) = \det M_i$, where M_i is obtained from M by changing the (i,i)-th entry to q and the $(i, r+1)$-th entry to $q'X_i$. Taking $q = (x - a_i)$ gives

$$S(qE_{i,i}) = \det M_i = \pm \prod_{j \neq i} (a_i - a_j)^2 p_i(a_i) X_i,$$

which is a non-zero multiple of X_i. That is, the images of $(x - a_i)E_{i,i}$ under S span $k \otimes \langle X_1, \dots, X_r \rangle$, and hence the map in (5.6) is surjective. The proof is now complete. □

Our next goal is to bootstrap from Proposition 5.1 and Proposition 5.2 to deduce maximal variation for generic scrolls of sufficiently high degree. We do this by a degeneration argument. We degenerate a vector bundle E to a vector bundle E_0 on the nodal rational curve $P_0 = \mathbf{P}^1 \cup \mathbf{P}^1$, and show that the projection-ramification map for E_0 is dominant. For this to work, we have to define the projection-ramification map for nodal curves. It turns out that with the most näive definition of linear series on scrolls on nodal curves, we do not get a dominant projection-ramification map. We have to work with the limit linear series of higher rank as developed in [13] and[11].

{sec:lls}

5.2. **Limit linear series.** We need limit linear series for the simplest singular curve, namely a (projective, connected) nodal curve C which is the nodal union of two smooth (projective, connected) curves C_1 and C_2, but we need them for vector bundles of rank higher than 1. Let B be the spectrum of a DVR with special point 0, general point η. Let $\pi\colon X \longrightarrow B$ be a smoothing of C with non-singular total space X. That is, π is a flat, proper, family of connected curves, smooth over η, and isomorphic to C over 0. Such a family is a particularly simple example of an almost local smoothing family [11, § 2.1–2.2]. Let g_i be the genus of C_i for $i = 1, 2$, and $g = g_1 + g_2$ the genus of X_η.

Let E be a vector bundle of rank r on C. The *multi-degree* of E is the pair of integers $(\deg E|_{C_1}, \deg E|_{C_2})$. The *degree* or *total degree* of E is the sum $\deg E = \deg E|_{C_1} + \deg E|_{C_2}$.

Once and for all, fix a vector bundle $\mathcal{E}$ of rank r on X, and set $E = \mathcal{E}|_C$. Let E have degree d and multi-degree (w_1, w_2). Fix a positive integer k. Our next task is to recall the definition of the space of limit linear series of dimension k. It will be a B-scheme whose fiber over η is the Grassmannian $\mathbf{Gr}(k, H^0(X_\eta, \mathcal{E}_\eta))$. The key idea is to not only consider the sections of $\mathcal{E}$, but also of its various twists, namely the vector bundles obtained by tensoring with the powers of $\mathcal{O}_X(C_i)$.

Fix maps $\theta_1\colon \mathcal{O}_X \longrightarrow \mathcal{O}_X(C_1)$ and $\theta_2\colon \mathcal{O}_X \longrightarrow \mathcal{O}_X(C_2)$. The choice of these maps is auxilliary, and each one is unique up to multiplication by an element of $\mathcal{O}_B^*$. For $n \in \mathbf{Z}$, set

$$\mathcal{E}_n = \begin{cases} \mathcal{E} \otimes \mathcal{O}_X(C_1)^{\otimes n} & \text{if } n \geq 0, \\ \mathcal{E} \otimes \mathcal{O}_X(C_2)^{\otimes(-n)} & \text{if } n < 0. \end{cases}$$

The maps θ_1 and θ_2 induces maps

$$\theta_n\colon \mathcal{E}_m \longrightarrow \mathcal{E}_{m+n}$$

given by

$$\theta_n = \begin{cases} \theta_1^n & \text{if } n \geq 0, \\ \theta_2^{-n} & \text{if } n < 0. \end{cases}$$

Note that the multi-degree of $\mathcal{E}_n$ is $(w_1 - nr, w_2 + nr)$. In particular, for sufficiently negative n, say for $n \leq n_1$, we have $H^0(C_2, \mathcal{E}_n|_{C_2}) = 0$, and similarly, for sufficiently positive n, say $n \geq n_2$, we have $H^0(C_1, \mathcal{E}_n|_{C_1}) = 0$. Assume, without loss of generality, that $n_2 \geq n_1$. Set

$$d_1 = w_1 - n_1 r, \text{ and } d_2 = w_2 + n_2 r, \text{ and } b = n_2 - n_1.$$

Observe that

$$d_1 + d_2 - rb = d.$$

We say that $\mathcal{E}$ has multi-degree w if for every $s \in S$ mapping to $0 \in B$, the degree of $\mathcal{E}|_s$ on C_v is w_v for $v = 1, 2$. Note that, if $\mathcal{E}$ has multi-degree (w_1, w_2), then $\mathcal{E}_n$ has multi-degree $(w_1 - rn, w_2 + rn)$.

{def:lls}

Definition 5.3 (Limit linear series)**.** Let S be a B-scheme. A *k-dimensional limit linear series* on $\mathcal{E}_S$ consists of sub-bundles $V_n \longrightarrow \pi_*(\mathcal{E}_n)_S$ of rank k for every $n \in \mathbf{Z}$ satisfying the following compatibility condition. For every $m, n \in \mathbf{Z}$, the map

s:compatibility}

$$\pi_* \theta_n \colon \pi_*(\mathcal{E}_m)_S \longrightarrow \pi_*(\mathcal{E}_{m+n})_S \text{ maps } V_m \longrightarrow V_{m+n}. \tag{5.7}$$

Definition 5.3 is a special case of [11, Definition 3.3.2]. From now on, we will talk about the image of an element in V_m in V_{m+n}; this should be understood as the image under the map $\pi_* \theta_n$.

Remark 5.4. The notion of a sub-bundle of a push-forward is a bit subtle; it is treated in depth in [11, Definition B.2.1]. We recall the main points. For a flat proper morphism $X \longrightarrow S$ and a vector bundle $\mathcal{E}$ on S, a *sub-bundle* of $\pi_*\mathcal{E}$ is a vector bundle V on S along with a map $i \colon V \longrightarrow \pi_*\mathcal{E}$ such that for every $T \longrightarrow S$, the pull-back $i_T \colon V_T \longrightarrow \pi_*(\mathcal{E}_T)$ is injective. Note that this is a local condition on S. For Noetherian schemes such as ours, it is enough to check this condition for the $T \longrightarrow S$ that are inclusions of closed points. Alternatively, if $F_0 \longrightarrow F_1 \longrightarrow \cdots$ is a complex of vector bundles on S quasi-isomorphic to $R\pi_*\mathcal{E}$, then a sub-bundle of $\pi_*\mathcal{E}$ is a vector bundle V along with a map $i \colon V \longrightarrow \pi_*\mathcal{E}$ such that the composite $V \longrightarrow F_0$ is an injection of vector bundles (that is, the dual map is surjective).

Remark 5.5. Definition 5.3 defines limit linear series on a particular vector bundle $\mathcal{E}$. We can also vary the choice of the vector bundle, as is done in [11]; in that case, one imposes an additional vanishing condition on the vector bundles to ensure boundedness of the moduli space of limit linear series.

{def:simple_lls}

Definition 5.6 (Simple limit linear series)**.** Let $S = \operatorname{Spec} K$, where K is a field, and let $V = (V_n \mid n \in \mathbf{Z})$ be a limit linear series on S. We say V is *simple* if there exist integers $w_1, \ldots, w_k$, not necessarily distinct, and elements $v_i \in V_{w_i}$ such that for every $w \in \mathbf{Z}$, the images of $v_1, \ldots, v_k$ in V_w form a basis of V_w.

Note that if $S \longrightarrow B$ maps to the generic point η, then the data of a limit linear series $V = (V_n)$ is equivalent to the data of an individual V_n for any $n \in \mathbf{Z}$, and in particular, for $n = 0$. As a result, the functor that associates to $S \longrightarrow \eta$ the set of k-dimensional limit linear series of $\mathcal{E}_S$ is represented by the Grassmannian $\mathbf{Gr}(k, H^0(X_\eta, \mathcal{E}_\eta))$. The main theorem of [11] is the following representability theorem.

{thm:lls}

Theorem 5.7 ([11, Theorem 3.4.7])**.** *The functor that associates to a B-scheme $S \longrightarrow B$ the set of limit linear series on $\mathcal{E}_S$ is representable by a projective B-scheme $\mathcal{G}(k, \mathcal{E})$ isomorphic to the Grassmannian $\mathbf{Gr}(k, H^0(X_\eta, \mathcal{E}_\eta))$ over η. The locus of simple linear series $\mathcal{G}^{\mathrm{simple}}(k, \mathcal{E}) \subset \mathcal{G}(k, \mathcal{E})$ is an open subscheme, and the map $\mathcal{G}^{\mathrm{simple}}(k, \mathcal{E}) \longrightarrow B$ has universal relative dimension at least $k(d - k - r(g-1))$.*

The last stamement implies that if $v \in \mathcal{G}^{\mathrm{simple}}$ is such that $\mathcal{G}^{\mathrm{simple}}$ has relative dimension at most $k(d-k-r(g-1))$ at v, then it has relative dimension exactly $k(d-k-r(g-1))$ at v and, futhermore, it is an open map near v. In particular, v is in the closure of $\mathbf{Gr}(k, H^0(X_\eta, \mathcal{E}_\eta))$.

Remark 5.8. Osserman proves a stronger theorem, namely a relative version of the statement above, over the stack of vector bundles on X. But the statement above is enough for our purposes.

Although the definition of a limit linear series demands that we specify infinitely many vector bundles V_n, one for each $n \in \mathbf{Z}$, this is neither practical nor necessary. In the best case, only specifying the extremal ones, namely V_{n_1} and V_{n_2}, suffices, provided that they satisfy some compatibility conditions. The original definition of limit linear series due to Eisenbud–Harris [4, 3] in the rank 1 case and Teixidor i Bigas [13] in the general case, took this minimalist approach.

Let E_n be the restriction of $\mathcal{E}_n$ to the central fiber $C = X_0$, and set $p = C_1 \cap C_2$.

{def:eht}

Definition 5.9 (EHT limit linear series)**.** A *k-dimensional EHT limit linear series* on E consists of k-dimensional subspaces $W_i \subset H^0(C_i, E_{n_i}|_{C_i})$ for $i = 1, 2$ that satisfy the following two conditions.

{ieq:eht}

(1) If $a_1^i \leq \cdots \leq a_k^i$ is the vanishing sequence for $(\mathcal{E}_{n_i}|_{C_i}, W_i)$ at p for $i = 1, 2$, then for every $v = 1, \ldots, k$ we have
$$a_v^1 + a_{k+1-v}^2 \geq b.$$

{gluing:eht}

(2) There exist bases $s_1^i, \ldots, s_k^i$ for W_i for $i = 1, 2$, such that s_v^i has order of vanishing a_v^i at p, and if we have $a_v^1 + a_{k+1-v}^2 = b$ for some v, then
$$\widetilde{\phi}(s_v^1) = s_{k+1-v}^2,$$
where $\widetilde{\phi} \colon E_{n_1}(-a_v^1 \cdot p)|_p \longrightarrow E_{n_2}(-a_{k+1-v}^2 \cdot p)|_p$ is the isomorphism obtained by taking the appropriate twist of the identity map.

We say that (W_1, W_2) is a *refined EHT limit linear series* if all equality holds in (1) for all $v = 1, \ldots, k$.

This definition is adapted from [11, Definition 4.1.2]. Note that, due to the vanishing condition on the twists of E, the restriction map

$$H^0(C, E_{n_i}) \longrightarrow H^0(C_i, E_{n_i}|_{C_i})$$

is an isomorphism. Via this isomorphism, we sometimes treat W_i as a subspace of $H^0(C_i, \mathcal{E}_{n_i}|_{C_i})$.

Although the notions of a limit linear series and an EHT limit linear series differ in general, they essentially agree when we restrict to the simple limit linear series and the refined EHT limit linear series. More precisely, we have the following statement.

{prop:llseht}

Proposition 5.10. *Let S be a B-scheme, and $V = (V_n \mid n \in \mathbf{Z})$ a limit linear series on $\mathcal{E}_S$. For every $s \in S$ over $0 \in B$, taking $W_i = V_{n_i}|_s$ for $i = 1, 2$ gives an EHT limit linear series. Conversely, assume that S reduced, and let $\mathcal{W}_i \subset \pi_*(\mathcal{E}_{n_i})_S$ be sub-bundles whose restrictions to every $s \in S$ over $\eta \in B$ agree under the isomorphism $(\mathcal{E}_{n_1})_\eta \cong (\mathcal{E}_{n_2})_\eta$, and to eveny $s \in S$ over $0 \in B$ define a refined EHT limit linear series. Then there exists a unique limit linear series $V = (V_n \mid n \in \mathbf{Z})$ on $\mathcal{E}_S$ such that $\mathcal{W}_i = V_{n_i}$. Furthermore, for every $s \in S$ over 0, the series $V|_s$ is simple.*

Proof. Proving that (W_1, W_2) is an EHT limit linear series is straightforward, and left to the reader. It is a special case of [11, Theorem 4.3.4] and the equivalence of type I and type II series in the two component case ([11, Remark 3.4.15].

The converse also follows from the proof of [11, Theorem 4.3.4], but it is not explicitly stated there. So we offer a proof.

First, suppose that S lies over $\eta \in B$. Then $V_n \subset \pi_*(\mathcal{E}_n)_S$ is determined uniquely as the image of $V_{n_i} = \mathcal{W}_{n_i} \subset \pi_*(\mathcal{E}_{n_i})_S$ for either $i = 1$ or $i = 2$.

Next, suppose that $S = \operatorname{Spec} K$, and it lies over $0 \in B$. Denoting $(\mathcal{E}_n)_S$ by E_n, we must construct $V_n \subset H^0(C, E_n)$. By composing $\theta_{n_i - n} \colon E_n \longrightarrow E_{n_i}$ and the restriction $E_{n_i} \longrightarrow E_{n_i}|_{C_i}$, we get a map

$$\iota \colon H^0(C, E_n) \longrightarrow H^0(C_1, E_{n_1}|_{C_1}) \oplus H^0(C_2, E_{n_2}|_{C_2}).$$

The vanishing condition on the twists of E mean that ι is injective. The compatibility condition in Definition 5.3 implies that we must choose V_n so that $\iota(V_n) \subset W_1 \oplus W_2$. We claim that $\dim \iota^{-1}(W_1 \oplus W_2) = k$, so that there is a unique choice of V_n, namely $V_n = \iota^{-1}(W_1 \oplus W_2)$.

Suppose $s \in i^{-1}(W_1 \oplus W_2)$. Then $\iota(s)$ is a linear combination of $(s^1_1, 0), \dots, (s^1_k, 0)$, and $(0, s^2_1), \dots, (0, s^2_k)$. Write $\iota(s) = (s_1, s_2)$. Since s_i is obtained by applying $\theta_{n - n_i}$, and θ on C_i at p corresponds to multiplication by the uniformizer, we see that

{eqn:vanishing}

$$\operatorname{ord}_p(s_1) \geq n - n_1, \text{ and likewise, } \operatorname{ord}_p(s_2) \geq n_2 - n. \tag{5.8}$$

Let $v_1 \in \{1, \dots, k\}$ be the smallest such that $a^1_v \geq n - n_1$, and $v_1 + c$ the smallest such that $a^1_{v_1 + c} > n - n_1$. Since (W_1, W_2) is refined, and $n_2 - n_1 = b$, we see that $v_2 = k + 1 - v_1$ is the largest such that $a^2_{v_2} \leq n_2 - n$, and $v_2 - c$ the smallest such that $a^2_{v_2 + c} < n_2 - n$. The vanishing conditions (5.8) imply that $\iota(s)$ bust be a linear combination of $(s^1_{v_1}, 0), \ldots, (s^1_k, 0)$ and $(0, s^2_{v_2-c}), \ldots, (0, s^2_k)$. Suppose

$$\iota(s) = \sum_{\ell=i}^{k} \alpha_\ell \cdot (s^1_\ell, 0) + \sum_{\ell=j}^{k} \beta_\ell \cdot (0, s^2_\ell),$$

where α_ℓ and β_ℓ are elements of the field K. Since s is a section on the entire nodal curve C, its two restrictions to C_1 and C_2 are equal at p. In terms of the two components of $\iota(s)$, and in light of the gluing condition (2) in Definition 5.9, this equality is equivalent to $\alpha_\ell = \beta_{k+1-\ell}$. That is, $\iota(s)$ is a linear combination of the k elements

$$(s^1_{v_1}, s^2_{v_2}), \ldots, (s^1_{v_1+c-1}, s^2_{v_2-c+1}), (s^1_{v_1+c}, 0), \ldots, (s^1_k, 0), (0, s^2_{v_2+1}), \ldots, (s^2_k, 0).$$

Conversely, it is easy to see that any such linear combination lies in $W_1 \oplus W_2$. Hence the claim that $\dim \iota^{-1}(W_1 \oplus W_2) = k$.

Set $V_n = \iota^{-1}(W_1 \oplus W_2)$. To see that V is simple, we must exhibit appropriate w_i and $v_i \in V_{w_i}$ for $i = 1, \ldots, k$. Take $w_i = n - n_1 - a^1_i$, and let $v_i \in V_{w_i} \subset H^0(C, E_{w_i})$ be such that $\iota(v_i) = (s^1_i, s^2_{k+1-i})$. Then the images of $v_1, \ldots, v_k$ form a basis of V_n for all $n \in \mathbf{Z}$.

For more general S, consider the map

$$\bar\iota \colon \pi_*(\mathcal{E}_n)_S \longrightarrow \pi_*(\mathcal{E}_{n_1})_S/\mathcal{W}_1 \oplus \pi_*(\mathcal{E}_{n_2})_S/\mathcal{W}_2,$$

obtained by composing $\iota = \pi_*(\theta_{n_1-n} \oplus \theta_{n_2-n})$ and the projections $\pi_*(\mathcal{E}_{n_i})_S \longrightarrow \pi_*(\mathcal{E}_{n_i})_S/\mathcal{W}_i$. We proved that, for every $\operatorname{Spec} K \longrightarrow S$, the kernel of $\bar\iota \otimes_{\mathcal{O}_S} K$ is k-dimensional. Since S is reduced, it is easy to prove that $V_n = \ker \iota$ is a sub-bundle of $\pi_*(\mathcal{E}_n)$. It is also easy to check that $V = (V_n \mid n \in \mathbf{Z})$ a limit linear series, the only one that satisfies $V_{n_i} = \mathcal{W}_i$. The proof is now complete. $\square$

Proposition 5.10 allows us to combine the economy of specifying an EHT limit linear series with the convenient functorial definition of a limit linear series. We use this in the definition of the projection-ramification map in terms of limit linear series.

{sec:prnongeneri

5.3. **Projection-ramification with non-generic vanishing sequence.** We consider the projection-ramification map for linear series with a non-generic vanishing sequence. The analysis of such series plays a key role in defining the projection-ramification map for limit linear series.

Let C be a smooth curve and $p \in C$ a point. Let E be a vector bundle on C of rank r. The projective spaces associated to the vector spaces $E(np)|_p$, for $n \in \mathbf{Z}$, are canonically isomorphic to each other, so we identify them. The vanishing sequences considered are at the point p. Choose a uniformizer t of C at p.

Suppose $V \subset H^0(C, E)$ is an $(r+1)$-dimensional subspace with the vanishing sequence

$$(\underbrace{a, \ldots, a}_{i}, \underbrace{a+1, \ldots, a+1}_{r+1-i}), \tag{5.9}$$ {eqn:specialvs}

for some i with $1 \leq i \leq r$, and $a \geq 0$. Let $v_1, \dots, v_{r+1}$ be a basis of V adapted to the vanishing sequence, namely a basis $v_1, \dots, v_{r+1}$ such that in the stalk E_p, we can write

$$v_1 = t^a \widetilde{v}_1, \dots, v_i = t^a \widetilde{v}_i, \quad v_{i+1} = t^{a+1} \widetilde{v}_{i+1}, \dots, v_{r+1} = t^{a+1} \widetilde{v}_{r+1}, \tag{5.10}$$ {eqn:basis}

for some $\widetilde{v}_1, \dots, \widetilde{v}_{r+1} \in E_p$ such that the images of $\widetilde{v}_1, \dots, \widetilde{v}_i$ in the fiber $E|_p$ are linearly independent, and the same holds for the images of $\widetilde{v}_{i+1}, \dots, \widetilde{v}_{r+1}$. Here we are slightly abusing the notation by denoting v_i and its image in E_p under the natural evaluation map by the same letter. Let $V^0 \subset E|_p$ be spanned by the images of $\widetilde{v}_1, \dots, \widetilde{v}_i$, and $V^1 \subset E|_p$ by the images of $\widetilde{v}_{i+1}, \dots, \widetilde{v}_{r+1}$. It is easy to check that a different choice of basis adapted to the vanishing sequence gives the same V^0 and V^1. By construction, $\dim V_0 = i$ and $\dim V^1 = r + 1 - i$, and therefore, $\dim(V^0 \cap V^1) \geq 1$. We say that V has *transverse vanishing* at p if

{eq:genericity}
$$\dim(V^0 \cap V^1) = 1. \tag{5.11}$$

Note that if V is base-point free at p, then $\dim V^0 = r$ and $\dim V^1 = 1$, so V automatically has transverse vanishing.

{prop:agreement}

Proposition 5.11. *Suppose $V \subset H^0(C, E)$ is an $(r+1)$-dimensional subspace with vanishing sequence* (5.9) *and transverse vanishing at p. Then the ramification section r_V of V vanishes to order $(r+1)a + (r-i)$ at p. Furthermore, writing $r_V = t^{(r+1)a+r-i} \cdot \widetilde{r}$, the one-dimensional subspace of $E|_p$ spanned by $\widetilde{r}|_p$ is $V^0 \cap V^1$.*

Proof. Thanks to transverse vanishing, there exists a basis $\{\overline{s}_1, \dots, \overline{s}_r\}$ of $E|_p$ such that

$$V^0 = \langle \overline{s}_1, \dots, \overline{s}_i \rangle \text{ and } V^1 = \langle \overline{s}_{i+1}, \dots, \overline{s}_r, \overline{s}_1 \rangle.$$

Let $v_1, \dots, v_{r+1}$ be a basis of V adapted to the vanishing sequence such that if $\widetilde{v}_i$ are defined as in (5.10) then the images of $\widetilde{v}_1, \dots, \widetilde{v}_r$ in $E|_p$ are $\overline{s}_1, \dots, \overline{s}_r$, respectively, and the image of $\widetilde{v}_{r+1}$ is $\overline{s}_1$. In particular, the r elements $\widetilde{v}_1, \dots, \widetilde{v}_r \in E_p$ give a trivialization of E around p. Write

$$\widetilde{v}_{r+1} = b_1 \widetilde{v}_1 + \cdots + b_r \widetilde{v}_r$$

in E_p, where $b_1, \dots, b_r \in \mathcal{O}_{C,p}$. Since the image of $\widetilde{v}_{r+1}$ in $E|_p$ is $\overline{s}_1$, we get that $b_1 \equiv 1 \pmod{\mathfrak{m}_p}$, and $b_2, \dots, b_r \in \mathfrak{m}_p$. Using the basis $v_1, \dots, v_{r+1}$ of V and the local trivialization $\widetilde{v}_1, \dots, \widetilde{v}_r$ of E, we can write r_V as the determinant (see (4.10)) as follows

$$\begin{aligned} r_V &= \det \begin{pmatrix} t^a & & & & & & a t^{a-1} \widetilde{v}_1 \\ & \ddots & & & & & \vdots \\ & & t^a & & & & a t^{a-1} \widetilde{v}_i \\ & & & t^{a+1} & & & (a+1) t^a \widetilde{v}_{i+1} \\ & & & & \ddots & & \vdots \\ & & & & & t^{a+1} & (a+1) t^a \widetilde{v}_r \\ b_1 t^{a+1} & b_2 t^{a+1} & \cdots & b_{r-1} t^{a+1} & & b_r t^{a+1} & (a+1) t^a \widetilde{v}_1 + t^{a+1}(\cdots) \end{pmatrix} \\ &= t^{(r+1)a+r-i} \widetilde{v}_1 + t^{(r+1)a+r-i+1}(\cdots). \end{aligned}$$

Thus the order of vanishing of r_V is as claimed. Furthermore, $\widetilde{r}$ is given by

$$\widetilde{r} = \widetilde{v}_1 + t(\cdots).$$

Since the image of $\widetilde{v}_1$, namely $\overline{s}_1$, spans $V^0 \cap V^1$, the proof is complete. □

We are primarily interested in generic $(r+1)$-dimensional subspaces $V \subset H^0(C, E)$. A generic such V has the vanishing sequence

$$(0, \dots, 0, 1).$$

For limit linear series, it is important to also study the V with complementary vanishing sequence, namely

$$(0, 1, \dots, 1),$$

which we now do. For simplicity, we restrict to $C = \mathbf{P}^1$.

Let E be an ample vector bundle on $\mathbf{P}^1$ of rank r. Fix a point $p \in \mathbf{P}^1$; all the vanishing sequences are at p. Consider the locally closed subset $U \subset \mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E))$ parametrizing $V \subset H^0(\mathbf{P}^1, E)$ with vanishing sequence

$$(0, \underbrace{1, \dots, 1}_{r}).$$

Given such a V, let $\widetilde{r}_V \in \mathbf{P}H^0(E \otimes \det E \otimes K_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p)^*$ be the reduced ramification section, namely the section obtained by dividing the usual ramification section r_V by the $(r-1)$-th power of a uniformizer at t (see Proposition 5.11). The assignment $V \mapsto \widetilde{r}_V$ gives a variant of the projection-ramification map, which we call the *reduced projection-ramification map*

$$\widetilde{\rho}\colon U \longrightarrow \mathbf{P}H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p))^*. \tag{5.12}$$ {eqn:rrd}

Note that, just as in the case of the usual projection-ramification map, the source and the target of the reduced projection-ramification map are of the same dimension.

Having defined the reduced projection-ramification map, we now relate it back to the usual projection-ramification map, but on a different vector bundle. Given a one-dimensional subspace $\ell \subset E|_p$, define E'_ℓ by the exact sequence

$$0 \longrightarrow E'_\ell \longrightarrow E \longrightarrow E|_p/\ell \longrightarrow 0.$$

There exists a Zariski open subset of the projective space of lines in $E|_p$ such that for all ℓ in this set, the isomorphism class of E'_ℓ remains constant. Denote this isomorphism class by E'_{gen}.

{prop:domred}

Proposition 5.12. *If the usual projection-ramification map*

$$\rho\colon \mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E'_{\text{gen}})) \dashrightarrow \mathbf{P}H^0(\mathbf{P}^1, E'_{\text{gen}} \otimes \det E'_{\text{gen}} \otimes K_{\mathbf{P}^1})^*$$

is dominant, then so is the reduced projection-ramification map

$$\widetilde{\rho}\colon U \longrightarrow \mathbf{P}H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p))^*.$$

Proof. Let $D \in \mathbf{P}H^0(E \otimes \det E \otimes K_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p))^*$ be a generic section. Let $\ell \subset E|_p$ be the one-dimensional subspace defined by $D|_p$, and set $E' = E'_\ell$. Since D is generic, we may assume $E' \cong E'_{\mathrm{gen}}$. The inclusion of sheaves $E' \longrightarrow E$ induces an inclusion of sheaves

$$E' \otimes \det E' \otimes K_{\mathbf{P}^1} \longrightarrow E \otimes \det E \otimes \mathcal{O}(-(r-1)p) \otimes K_{\mathbf{P}^1},$$

and by construction, D is the image of a section $D' \in \mathbf{P}H^0(E' \otimes \det E' \otimes K_{\mathbf{P}^1})^*$. Since ρ is dominant for E', there exists a sequence of subspaces $V'_n \in \mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E'))$ such that the limit of $\rho(V'_n)$ is D'. Let $V_n \subset \mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E))$ be the image of V'_n. Then the limit of $\widetilde{\rho}(V_n)$ is D. Since D was generic, we get that $\widetilde{\rho}$ is dominant. ☐

:domredexamples}

Corollary 5.13. *The reduced projection-ramification map is dominant for the bundles $E = \mathcal{O}(1) \oplus \mathcal{O}(2)^{r-1}$ and $E = \mathcal{O}(2) \oplus \mathcal{O}(3)^{r-1}$.*

Proof. Follows from Proposition 5.12 and that the projection-ramification map is dominant for $E' = \mathcal{O}(1)^r$ and $E' = \mathcal{O}(2)^r$. ☐

{sec:prllls}

5.4. **Projection-ramification for limit linear series.** Recall the setup from § 5.2: $C = C_1 \cup C_2$ is a nodal union of two smooth projective curves of genus g_1 and g_2, and $\pi\colon X \longrightarrow B$ be a smoothing of C. Let $\mathcal{E}$ be a vector bundle of rank r on X whose restriction E to C has multi-degree (w_1, w_2). The integers $n_2 \geq n_1$ are such that we have vanishing $H^0(C_2, E_n|_{C_2}) = 0$ for all $n \leq n_1$ and $H^0(C_1, E_n|_{C_1}) = 0$ for $n \geq n_2$. For convenience, we decrease n_1 and increase n_2 so that the vanishing on C_2 holds for all $n \leq n_1 - (w_1 - 2g_1)$ and on C_1 for all $n \geq n_2 + (w_2 - 2g_2)$. Define

$$d_1 = w_1 - n_1 r, \quad d_2 = w_2 + n_2 r, \text{ and } b = n_2 - n_1,$$

as before.

Set $\mathcal{E}' = \mathcal{E} \otimes \det \mathcal{E} \otimes \omega_{X/B}$. Then $\mathcal{E}'$ is a vector bundle of rank r on X whose restriction E' to C has multi-degree (w'_1, w'_2) where

$$w'_1 = w_1 + r(w_1 - 2g_1 + 1) \text{ and } w'_2 = w_2 + r(w_2 - 2g_2 + 1).$$

We set

$$n'_1 = n_1(1+r) \text{ and } n'_2 = n_2(1+r),$$

and observe that we have vanishings $H^0(C_2, E'_n|_{C_2}) = 0$ for $n \leq n'_1$ and $H^0(C_1, E'_n|_{C_1}) = 0$ for $n \geq n'_2$. We also set

$$b' = n'_2 - n'_1 = b(1+r).$$

Our next goal is to define a rational map

{eq:Rtilde}

$$\rho\colon \mathcal{G}(r+1, \mathcal{E}) \dashrightarrow \mathcal{G}(1, \mathcal{E}') \tag{5.13}$$

that extends the projection-ramification map

$$\rho\colon \mathbf{Gr}(r+1, H^0(X_\eta, \mathcal{E}_\eta)) \dashrightarrow \mathbf{Gr}(1, H^0(X_\eta, \mathcal{E}'_\eta))$$

on X_η. For technical reasons, we define the map in (5.13) only on the reduced scheme underlying $\mathcal{G}(r+1, \mathcal{E})$.

Before defining the map, we identify three conditions on limit linear series on the central fiber that are required for the map to be defined. To do this, consider a limit linear series $(V_n \mid n \in \mathbf{Z})$ on C, and let (W_1, W_2) be the associated EHT limit linear series namely $W_1 = V_{n_1}$ and $W_2 = V_{n_2}$ (see Proposition 5.10). The first condition we want to impose is that (W_1, W_2) be a refined EHT limit linear series; this is an open condition (see [11, Proposition 4.1.5]). The second condition we want to impose is that the vanishing sequence of $W_1 \subset H^0(C_1, E_{n_1}|_{C_1})$ at p is of the form

$$(\underbrace{a, \dots, a}_{i}, \underbrace{a+1, \dots, a+1}_{r+1-i}) \tag{5.14}$$ {eqn:llsvs}

as in (5.9); imposing a particular vanishing sequence is again an open condition (see [11, Proposition 4.2.5]). Since (W_1, W_2) is refined, it follows that the vanishing sequence of $W_2 \subset H^0(C_2, E_{n_2}|_{C_2})$ at p is

$$(\underbrace{b-a-1, \dots, b-a-1}_{r+1-i}, \underbrace{b-a, \dots, b-a}_{i}).$$

Recall from § 5.3 that W_1 yields two vector spaces V^0 and V^1 in the fiber $E_{n_1}|_p$, which we may identify canonically (up to scaling) with the fiber $E|_p$. Likewise, W_2 yields two analogous vector spaces, call them Λ^0 and Λ^1, in $E|_p$. The gluing condition in the definition of EHT limit linear series (Definition 5.9) and the definition of these vector spaces immediately shows that

$$V^0 = \Lambda^1 \text{ and } V^1 = \Lambda^0. \tag{5.15}$$ {eqn:vlambdaswit

The third condition we want to impose is that these two vector spaces be transverse, namely $\dim(V^0 \cap V^1) = 1$.

Let $\mathcal{U} \subset \mathcal{G}(r+1, \mathcal{E})$ be the complement of the union of the following closed sets:

(1) the closure of the subset of $\mathbf{Gr}(r+1, H^0(X_\eta, \mathcal{E}_\eta))$ corresponding to $V \subset H^0(X_\eta, \mathcal{E}_\eta)$ for which the evaluation map $V \otimes \mathcal{O}_{X_\eta} \longrightarrow \mathcal{E}_\eta$ has generic rank less than r.
(2) the set of limit linear series $(V_n \mid n \in \mathbf{Z})$ on C such that the associated EHT limit linear series (W_1, W_2) is not refined, or does not have the vanishing sequence as in (5.14), or does not satisfy the transversality condition $\dim(V^0 \cap V^1) = 1$.

Give $\mathcal{U}$ the reduced scheme structure.

Let S be a reduced B-scheme with a map to $\mathcal{U}$ given by the limit linear series $(V_n \mid n \in \mathbf{Z})$. On X_S, we have a diagram analogous to (4.8), namely

$$\begin{tikzcd} & \det \mathcal{E}_n^* \otimes \det V_n \arrow[r, "j"] \arrow[d, "d"] & V_n \otimes \mathcal{O}_{X_S} \arrow[r, "e"] \arrow[d, "e"] & \mathcal{E}_n \arrow[d, equal] & \\ 0 \arrow[r] & \Omega_{X_S/S} \otimes \mathcal{E}_n \arrow[r] & P(\mathcal{E}_n) \arrow[r] & \mathcal{E}_n \arrow[r] & 0. \end{tikzcd} \tag{5.16}$$ {eq:llspr}

Here $P(\mathcal{E}_n)$ is the sheaf of principal parts of $\mathcal{E}_n$ relative to $X_S \longrightarrow S$, and the bottom row is the natural exact sequence coming from its definition. The top row is a complex, but it may not be exact. The maps labeled e are the evaluation maps. The map j is defined by the maximal minors of $e\colon V_n \otimes \mathcal{O}_{X_S} \longrightarrow \mathcal{E}_n$. The map d is the unique map induced by the other maps in the diagram. By composing d through the inclusion $\Omega_{X_S/S} \longrightarrow \omega_{X_S/S}$, and doing some rearrangement, we obtain a map

{eqn:Rn}
$$r_n\colon \det V_n \longrightarrow \pi_*(\mathcal{E}_n \otimes \det \mathcal{E}_n \otimes \omega^*_{X_S/S}) = \pi_*(\mathcal{E}'_{(r+1)n}). \tag{5.17}$$

Consider the two extremal sections, namely those corresponding to $n = n_1$ and $n = n_2$.

{lem:rameht}

Lemma 5.14. *Over every $s \in S$ over $0 \in \Delta$, the restrictions $r_{n_1}|_s$ and $r_{n_2}|_s$ define a one-dimensional refined EHT limit linear series for E'.*

Proof. Without further comment, we identify $r_{n_i}|_s \in H^0(C, E'_{(r+1)n_i})$ with its image in $H^0(C_i, E'_{(r+1)n_i}|_{C_i})$. We have

$$E'_{(r+1)n_2}|_{C_i} = E_{n_2} \otimes \det E_{n_2} \otimes \omega_C|_{C_1} = E_{n_2} \otimes \det E_{n_2} \otimes \Omega_C|_{C_1} \otimes \mathcal{O}_{C_1}(p),$$

and by construction $r_{n_1}|_s$ is the image of the ramification section of $V_{n_1} \subset H^0(C_1, E_{n_1}|_{C_1})$ under the inclusion map

$$E_{n_1} \otimes \det E_{n_1} \otimes \Omega_C|_{C_1} \longrightarrow E_{n_1} \otimes \det E_{n_1} \otimes \omega_C|_{C_1} = E'_{(r+1)n_1}|_{C_1}.$$

By Proposition 5.11, the ramification section of V_{n_1} has order of vanishing $(r+1)a + (r-i)$ at p, and hence $r_{n_1}|_s$ on C_1 has order of vanishing $(r+1)a + (r-i+1)$ at p. Likewise, $r_{n_2}|_s$ on C_2 has order of vanishing $(r+1)(b-a-1)+i$ at p. Since

$$(r+1)a + (r-i+1) + (r+1)(b-a-1) + i = (r+1)b = b',$$

we see that $r_{n_1}|_s$ and $r_{n_2}|_s$ have complementary orders of vanishing, leading to an equality in condition (1) of Definition 5.9.

We must next ensure that condition (2) of Definition 5.9 holds, that is, the images of $r_{n_i}|_s$ in the appropriate twists of $E_{n_i}|_p$ are equal, at least up to scaling. By Proposition 5.11, the image of $r_{n_1}|_s$ in the appropriate twist of $E_{n_1}|_p$ spans the line $(V^0 \cap V^1)$, and the image of $r_{n_2}|_s$ spans the line $\Lambda^0 \cap \Lambda^1$. But by (5.15), we have $V^1 = \Lambda^0$ and $V^0 = \Lambda^1$, so the two lines are equal. □

Thanks to Lemma 5.14, we apply Proposition 5.10, and conclude that there exists a unique (1-dimensional) limit linear series $(R_n \mid n \in \mathbf{Z})$ of $\mathcal{E}'$ on X_S for which $R_{n'_1} = \det V_{n_1}$ and $R_{n'_2} = \det V_{n_2}$, at least if S is reduced. The transformation

$$(V_n \mid n \in Z) \mapsto (R_n \mid n \in \mathbf{Z})$$

defines a morphism

prop:mapreduced}
$$\rho\colon \mathcal{U} \longrightarrow \mathcal{G}(1, \mathcal{E}'), \tag{5.18}$$

as desired in (5.13). Note that $\mathcal{U}$ has the reduced scheme structure.

op:degeneration} The fruit of our labor is the following corollary.

Corollary 5.15. *Suppose $v \in \mathcal{U}_0$ is such that $\dim_v \mathcal{U}_0 = (r+1)(d-rg-1)$ and v is isolated in the fiber of ρ, then the projection-ramification map $\mathbf{Gr}(r+1, H^0(X_\eta, \mathcal{E}_\eta)) \dashrightarrow \mathbf{P}H^0(X_\eta, \mathcal{E}_\eta \otimes \det E_\eta \otimes K_{X_\eta})$ is generically finite.*

Proof. If $\dim_v U_0 = (r+1)(d-rg-1)$, then v is in the closure of $\mathbf{Gr}(r+1, H^0(X_\eta, \mathcal{E}_\eta))$ by Theorem 5.7. The statement now follows from the upper semi-continuity of fiber dimension. □

5.5. **Maximal variation for generic scrolls of high degree.** We now have all the tools to prove Theorem D

{thm:actualratio

Theorem 5.16 (Theorem D)**.** *Let E be a generic vector bundle on $\mathbf{P}^1$ of rank r and degree $d = a(r-1) + b(2r-1) + 1$, where a, b are positive integers. Then the projection-ramification map is generically finite, and hence dominant, for E. In particular, the projection-ramification map is dominant for generic E of degree $\geq (r-1)(2r-1)+1$.*

Proof. We say that generic dominance holds for rank r and degree d if the projection-ramification map is dominant (equivalently, generically finite) for the generic vector bundle of rank r and degree d. The rank will be fixed througout, so let us drop it from the discussion. Let us prove that if generic dominance holds for degrees d_1 and d_2, then it also holds for degree $d = d_1 + d_2 - 1$. With the base cases $d_1 = r$ (Proposition 5.1) and $d_2 = 2r$ (Proposition 5.2), this proves the theorem.

Take $C_1 = C_2 = \mathbf{P}^1$, and let $C = C_1 \cup C_2$ be their nodal union at one point, which we take to be the point labeled 0 on both $\mathbf{P}^1$s. Let $X \longrightarrow B$ be a smoothing of C. Note that any vector bundle on C is the restriction of a vector bundle on X. Therefore, by Corollary 5.15, it suffices to construct a vector bundle E of degree d on C and a limit linear series $(V_n \mid n \in \mathbf{Z})$ on E such that the following conditions hold for the point v of $\mathcal{G}(r+1, E')$ represented by $(V_n \mid n \in \mathbf{Z})$:

(1) $\dim_v \mathcal{G}(r+1, E) = (r+1)(d-1)$,
(2) ρ is defined at v, and
(3) v is an isolated point in the fiber of ρ.

We construct E as follows. Let E_1 be a generic vector bundle of degree d_1 on C_1, and E_2' a generic vector bundle of degree $d_2 - 1$ on C_2. Choose a generic isomorphism $E_1|_0 \cong E_2'|_0$, and construct the vector bundle E on C by gluing E_1 and E_2' along this isomorphism. Choose $n_1 = a$ and $n_2 = b + a$ for sufficiently negative a and sufficiently positive b. The isomorphism $E_1|_0 \cong E_2'|_0$ yields isomorphisms, canonical up to scaling, of $E_1(m)|_0$ and $E_2'(n)|_0$ for any $m, n \in \mathbf{Z}$.

Having constructed E, we must now construct $(V_n \mid n \in \mathbf{Z})$. By Proposition 5.10, it is enough to construct $V_{n_1} \subset H^0(C_1, E_1 \otimes \mathcal{O}(a))$ and $V_{n_2} \subset H^0(C_2, E_2'(b-a))$, provided they define a refined EHT limit linear series. Let $V \subset H^0(C_1, E_1)$ be a generic $(r+1)$-dimensional vector space. Then it will have the vanishing sequence $(0, \dots, 0, 1)$. Hence, we have $V^0 = E|_0$ and $V^1 \subset E|_0$ is 1-dimensional (see § 5.3 for the definition of these two subspaces). Furthermore, the genericity of V implies that V^1 is a general 1-dimensional subspace. Let $\Lambda \subset H^0(C_2, E_2'(1))$ be the image of a general $(r+1)$ dimensional subspace of $H^0(C_2, E_2)$, where E_2 is the vector bundle of degree d_2 defined by the sequence

$$0 \longrightarrow E_2 \longrightarrow E_2'(1) \longrightarrow E_2'(1)|_0 / V^1 \longrightarrow 0.$$

Then $\Lambda \subset H^0(C_2, E_2'(1))$ has the vanishing sequence $(0, 1, \dots, 1)$, with $\Lambda^0 = V^1$ and $\Lambda^1 = V^0$. Let $V_{n_1} \subset H^0(C_1, E_1 \otimes \mathcal{O}(a))$ be the image of V and $V_{n_2} \subset H^0(C_2, E_2 \otimes \mathcal{O}(b-a))$ the image of Λ. Then V_{n_1} has the vanishing sequence $(a, \dots, a, a+1)$, and Λ the complementary vanishing sequence $(b-a-1, b-a, \dots, b-a)$. By the construction of Λ, there exist bases of V_{n_1} and V_{n_2} that satisfy the gluing condition at 0. In conclusion, V_{n_1} and V_{n_2} form a refined EHT limit linear series, and hence define a limit linear series $v = (V_n \mid n \in \mathbf{Z})$.

It is easy to check that $\dim_v \mathbf{G}(r+1, E) = (r+1)(d-1)$. Indeed, for every limit linear series $w = (W_n \mid n \in \mathbf{Z})$ in an open subset around v, the EHT limit linear series associated to w determines w and has the same vanishing sequence as v. In particular, $W_{n_1} \subset H^0(C_1, E_1(a))$ is the image of an $(r+1)$-dimensional subspace $W \subset H^0(C_1, E_1)$ with vanishing sequence $(0, \dots, 1)$, and $W_{n_2} \subset H^0(C_2, E_2(b-a))$ is the image of an $(r+1)$-dimensional subspace $M \subset H^0(C_2, E_2(1))$ with vanishing sequence $(0, 1, \dots, 1)$. Furthermore, the gluing condition implies that M is in fact the image of an $(r+1)$-dimensional subspace of the kernel of the map

$$E_2'(1) \longrightarrow E_2'(1)/W^1.$$

By the genericity of V, the isomorphism type of the kernel of this map is constant around v; that is, the kernel is isomorphic to E_2. So, a dimension count for $\mathcal{G}(r+1, E)$ around v gives

$$\begin{aligned}
\dim_v \mathcal{G}(r+1, E) &= \dim \mathbf{Gr}(r+1, H^0(C_1, E_1)) + \dim \mathbf{Gr}(r+1, H^0(C_2, E_2)) \\
&= (r+1)(d_1 - 1) + (r+1)(d_2 - 1) \\
&= (r+1)(d_1 + d_2 - 2) \\
&= (r+1)(d-1).
\end{aligned}$$

Finally, we must check that v is an isolated point in the fiber of

$$\rho \colon \mathcal{G}(r+1, E) \dashrightarrow \mathcal{G}(1, E \otimes \det E \otimes \omega_C).$$

For any $w \in \mathcal{G}(r+1, E)$ in an open set around v, either $V \neq W$ or $\Lambda \neq M$, where V, Λ, W, M are as above. By construction, $V \subset H^0(r+1, H^0(C_1, E_1))$ and $\Lambda \subset H^0(r+1, H^0(C_2, E_2'))$ are isolated in their respective projection-ramification maps. Therefore, either $\rho_{C_1}(V) \neq \rho_{C_1}(W)$ or $\rho_{C_2}(\Lambda) \neq \rho_{C_2}(M)$. In either case, we obtain that $\rho(v) \neq \rho(w)$, and hence conclude that v is an isolated point in the fiber of ρ. $\square$

erativeproblems}

6. The Projection-Ramification enumerative problem

In this section, we calculate the degree of the projection-ramification map for as many varieties of minimal degree as we can, leading to a proof of Theorem E. After treating the relatively easy cases by hand, we relate the projection-ramification map for the veronese surface and the quartic normal scroll with classical geometry of cubic plane curves.

{sec:arnc}

6.1. **Rational normal curves.** Let $X \subset \mathbf{P}^n$ be a rational normal curve. Plainly, X is incompressible, and hence the projection-ramification map

$$\rho\colon \mathbf{Gr}(2, n+1) \longrightarrow \mathbf{P}^{2n-2}$$

is a regular map. Therefore, we get

$$\begin{aligned}\deg \rho &= c_1(\rho^* \mathcal{O}(1))^{2n-2} \\ &= c_1(\mathcal{O}_{\mathbf{Gr}(r+1,n+1)}(1))^{2n-2} \\ &= \frac{(2n-2)!}{n!(n-1)!}.\end{aligned}$$

{sec:aquadricsur

6.2. **Quadric hypersurfaces.** A smooth quadric hypersurface $X \subset \mathbf{P}^n$ defined by a homogeneous quadric equation $F(X_0, \dots, X_n) = 0$. An easy calculation shows that the projection-ramification map

$$\rho\colon \mathbf{P}^n \longrightarrow (\mathbf{P}^n)^*$$

is given in coordinates by

$$p = [p_0 : \cdots : p_n] \mapsto \left[\frac{\partial F}{\partial X_0}(p) : \cdots : \frac{\partial F}{\partial X_n}(p)\right].$$

In other words, it is the *polarity isomorphism* induced by F, namely the isomorphism between a projective space and its dual given by the non-degenerate bilinear form associated to F. In particular, we get $\deg \rho = 1$.

{sec:veronese}

6.3. **The Veronese surface.** Let $\mathbf{P}^2 \cong X \subset \mathbf{P}^5$ be the Veronese surface, the image of $\mathbf{P}^2$ under the complete linear series $\mathcal{O}(2)$. In this case, the projection-ramification map

$$\rho\colon \mathbf{Gr}(3, H^0(\mathbf{P}^2, \mathcal{O}(2))) \cong \mathbf{Gr}(3,6) \dashrightarrow \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}(3))^* \cong \mathbf{P}^9$$

can be described as follows. Let $N \subset H^0(\mathbf{P}^2, \mathcal{O}(2))$ be a net of conics. Then $\rho(N)$ corresponds to the cubic curve traced out by the nodes of the singular members of N, called the *Jacobian* of N.

{prop:veronese}

Proposition 6.1. *Let $R \subset \mathbf{P}^2$ be a general cubic. The fiber of ρ over R is in natural bijection with the set of non-trivial* 2*-torsion line bundles on R. In particular, we have* $\deg \rho = 3$.

The rest of § 6.3 is devoted to the proof of this assertion.

For the proof, we recall some classical projective geometry of cubics and nets of conics from [2, § 3]. To distinguish the various copies of $\mathbf{P}^2$ that naturally arise in this story, write $\mathbf{P}^2 = \mathbf{P}V$ for a 3 dimensional vector space V. Let $N \subset H^0(\mathbf{P}V, \mathcal{O}(2)) = \operatorname{Sym}^2 V$ be a general net of conics on $\mathbf{P}V$. Given a point $x \in \mathbf{P}N^*$, we denote the associated conic by Q_x.

Associated to the net N are three important cubic plane curves, namely the Jacobian curve, the discriminant curve, and the Hermite curve. We have already seen the Jacobian curve $R \subset \mathbf{P}V$. The *discriminant curve* $D \subset \mathbf{P}N^*$ is the locus of $x \in \mathbf{P}N^*$ such that Q_x

is singular. Since a pencil of conics contains three singular members, we see that D is a cubic curve. Note that if Q_d is singular, then it is the union of two distinct lines in $\mathbf{P}V$. A component line of Q_x is called a *Reye line*. The *Hermite curve* $E \subset \mathbf{P}V^*$ is the locus of Reye lines. We leave it to the reader to check that it is a cubic curve.

The three cubic curves introduced above are inter-related. First, we have an isomorphism $\tau \colon D \longrightarrow R$ defined by

{eqn:DR}
$$\tau \colon d \mapsto \text{The singular point of } Q_d. \tag{6.1}$$

Second, we have a degree 2 map $E \longrightarrow D$ defined by

$$\ell \mapsto \text{The } d \in D \text{ such that } Q_d \text{ contains } \ell.$$

Evidently, the fiber of this map over a given $x \in D$ corresponds to the two components of Q_d. The (étale) degree 2 map $E \longrightarrow D \cong R$ gives a non-trivial two-torsion element $\eta \in \operatorname{Pic}(R)[2]$ The element η is characterized by the property that it is the unique non-trivial two-torsion element whose pull-back to E is trivial.

Denote by H the hyperplane divisor class on $R \subset \mathbf{P}^2$.

{lem:reye}

Lemma 6.2. *For every $a \in R$, the line joining a and $a + \eta$ is a Reye line. Furthermore, this Reye line is a component of Q_d where $d = \tau^{-1}(H - 2a - \eta)$. Finally, the conjugate Reye line, namely the other component of Q_d, passes through the points b and $b + \eta$ where $b \in R$ differs from a by a non-trivial two-torsion element other than η.*

Proof. Let ℓ be a general Reye line, and let $d \in D$ be such that ℓ is a component of Q_d. Let $x = \tau(d) \in R$ be the singular point of Q_d. Note $\ell \cap R$ consists of three points, one of which is x. It suffices to show that the other two, say y and z, differ by η.

The point y defines a line in $\mathbf{P}V^*$. This line intersects $E \subset \mathbf{P}V^*$ in three points, one of which is ℓ, and the other two are the two components of $Q_{\tau^{-1}(y)}$, namely the two pre-images of $y \in R$ under the double covering $E \longrightarrow R$. Call these two points y_1 and y_2. Define z_1 and z_2 analogously. By construction, the triplets y_1, y_2, ℓ and z_1, z_2, ℓ are collinear triplets on $E \subset \mathbf{P}V^*$, and therefore we have the linear equivalence

$$y_1 + y_2 \sim z_1 + z_2$$

on E. By pushing this forward to R, we get

$$2y \sim 2z.$$

Therefore, $y - z$ is a (non-trivial) two torsion element in $\operatorname{Pic}(R)$. However, the pull-back of $y - z$ is trivial on E, and hence $y - z = \eta$.

Finally, let m be the Reye line conjugate to ℓ. Then it passes contains x, and two other points of R, say y' and z'. By what we just proved, $y' - z' = \eta$. But we also have $y' + z' \sim y + z$. Hence $y - y'$ is a two-torsion element, non-trivial, and distinct from η. The proof is now complete. $\square$

We now have all the tools to prove Proposition 6.1.

Proof of Proposition 6.1. Let $U \subset \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}(3))^*$ be the locus of smooth cubic curves, $J \longrightarrow U$ be the universal Picard scheme, $J[2] \subset J$ the closed subscheme of two-torsion classes, and $J[2]^* \subset J[2]$ the open and closed subscheme of non-trivial two-torsion classes. The projection-ramification map for the Veronese surface factors as

$$\begin{aligned} \rho\colon \mathbf{Gr}(3, H^0(\mathbf{P}^2, \mathcal{O}(2))) \dashrightarrow J[2]^* \dashrightarrow \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}(3))^* \\ N \mapsto (R, \eta) \mapsto R. \end{aligned}$$

We construct $J[2]^* \longrightarrow \mathbf{Gr}(3, H^0(\mathbf{P}^2, \mathcal{O}(2)))$ inverse to the first map. Given $(R, \eta) \in J[2]^*$, we need to construct a net N of conics with Jacobian R. We use Lemma 6.2, which tells us the singular elements of this net in terms of R and η. Let $\{\eta, \eta', \eta''\}$ be the three non-trivial 2 torsion line bundles on R. Define the map $R \longrightarrow \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}(2))^*$ by

$$R \ni a \mapsto (\langle a, a+\eta \rangle) \cdot \left(\langle a+\eta', a+\eta'' \rangle \right),$$

where $\langle p, q \rangle$ denotes the line joining p and q. We leave it to the reader to check that the image of R is a plane cubic curve. The span of the image of R is the desired net N. ☐

{sec:quartic_scr

6.4. **Quartic surface scroll.** Our next objective is to prove that $\deg \rho_X = 2$ for a generic quartic surface scroll $X \subset \mathbf{P}^5$. We begin by recasting ρ_X in terms of nets of conics on $\mathbf{P}^2$, and bring in the projective geometry introduced in § 6.3.

The generic quartic surface scroll $X \subset \mathbf{P}^5$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, embedded by the complete linear system associated to $\mathcal{O}(1,2)$. Say $\mathbf{P}^1 \times \mathbf{P}^1 = \mathbf{P}U \times \mathbf{P}V$, where U and V are two-dimensional vector spaces. Then the projection-ramification map is a $\mathrm{PGL}(U) \times \mathrm{PGL}(V)$-equivariant map

$$\mathbf{Gr}(3, U \otimes \operatorname{Sym}^2 V) \dashrightarrow \mathbf{P}(U \otimes \operatorname{Sym}^4 V)^*.$$

We take the quotient of both sides by the $\mathrm{PGL}(U) \times \mathrm{PGL}(V)$-action. We begin by identifying the two quotients.

Let S be a 3-dimensional quadratic space, that is, a vector space with a non-degenerate quadratic form q. Then we have $\mathrm{Aut}(S) = \mathrm{O}(q) \cong \mathrm{O}(3)$. THe projective space $\mathbf{P}S$ is isomorphic to $\mathbf{P}^2$, and it comes with a distinguished smooth conic $Q \subset \mathbf{P}S$. The automorphism group of the pair $(\mathbf{P}S, Q)$ is $\mathrm{Aut}(Q) \cong \mathrm{PGL}_2$.

{lem:quotgrass}

Lemma 6.3. *The quotient* $\mathbf{Gr}(3, U \otimes \operatorname{Sym}^2 V)/\mathrm{PGL}(U) \times \mathrm{PGL}(V)$ *is birational to the quotient* $\mathrm{Hilb}^3(\mathbf{P}S)/\operatorname{Aut} S$.

Proof. Let W be a 3-dimensional vector space. We have a birational isomorphism

$$\begin{aligned} &\mathbf{Gr}(3, U \otimes \operatorname{Sym}^3 V)/\mathrm{PGL}(U) \times \mathrm{PGL}(V) \\ &\qquad \sim (W^* \otimes U \otimes \operatorname{Sym}^2 V)/\mathrm{GL}(W) \times \mathrm{GL}(U) \times \mathrm{GL}(V). \end{aligned}$$

Interpret the space $(W^* \otimes U \otimes \operatorname{Sym}^2 V)/\mathrm{GL}(W) \times \mathrm{GL}(U)$ as the space of 2×3 matrices with entries in $\operatorname{Sym}^2 V$, modulo row and column transformations. Set $S = \operatorname{Sym}^2 V$; it has a canonical (up to scaling) quadratic form given by the conic $Q \cong \mathbf{P}V \subset \mathbf{P}S$ embedded by

$\mathcal{O}(2)$. We can then interpret $(W^* \otimes U \otimes \operatorname{Sym}^2 V)/\operatorname{GL}(W) \times \operatorname{GL}(U)$ as the space of 2×3 matrices with entries in S. We have a birational isomorphism

$$\begin{aligned}(W^* \otimes U \otimes \operatorname{Sym}^2 V)/\operatorname{GL}(W) \times \operatorname{GL}(U) &\sim \operatorname{Hilb}^3(\mathbf{P}S)\\ 2 \times 3 \text{ matrix } M &\mapsto \text{Vanishing locus of } 2 \times 2 \text{ minors of } M.\end{aligned}$$

By taking the further quotient by $\operatorname{GL}(V)$, we finish the proof. □

{lem:quotram}

Lemma 6.4. *The quotient* $\mathbf{P}(U \otimes \operatorname{Sym}^4 V)^*/\operatorname{PGL}(U) \times \operatorname{PGL}(V)$ *is birational to the quotient* $\mathbf{Gr}(2, (\operatorname{Sym}^2 S)/q)/AutS$.

Proof. We have the birational isomorphism

$$(U \otimes \operatorname{Sym}^4 V)/\operatorname{GL}(U) \sim \mathbf{Gr}(2, \operatorname{Sym}^4 V).$$

Note that $q \in \operatorname{Sym}^2 S$ spans the kernel of the natural surjection $\operatorname{Sym}^2 S \longrightarrow \operatorname{Sym}^4 V$. The assertion follows. □

Via the birational isomorphisms in Lemma 6.3 and Lemma 6.4, the projection-ramification map μ transforms into an $\operatorname{Aut}(S)$-equivariant map

$$\mu\colon \operatorname{Hilb}^3 \mathbf{P}S \dashrightarrow \mathbf{Gr}(2, \operatorname{Sym}^2 S/q).$$

We now describe this map μ. To ease notation, we denote a linear form and its vanishing locus by the same letter. Let $\xi \in \operatorname{Hilb}^3 \mathbf{P}S$ be a general point corresponding to the three vertices of the triangle formed by three lines L_i for $i = 1, 2, 3$. Two lines L_i and L_j define a pencil of quadratic forms on Q. Let R_{ij} be the line whose intersection with Q is the ramification divisor of the pencil $\langle L_i, L_j\rangle$. It is easy to check that the quadrics L_1R_{23}, L_2R_{13}, and L_3R_{12} span a 3-dimensional subspace of $\operatorname{Sym}^2 S$ that contains the quadric q.

{lem:mu}

Lemma 6.5. *In the setup above, the image of* ξ *under* μ *is the image of* $\langle L_1R_{23}, L_2R_{13}, L_3R_{12}\rangle$ *in* $\operatorname{Sym}^2 S/q$.

Proof. The ideal of the point $\xi \in \operatorname{Hilb}^3(S)$ is cut out by 2×3 matrix of linear forms

$$M = \begin{pmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \end{pmatrix}.$$

Let U_0, U_1 be a basis of U. Under the isomorphism in Lemma 6.3, this 2×3 matrix corresponds to the point of $\mathbf{Gr}(3, U \otimes \operatorname{Sym}^2 V)$ given by the subspace of $U \otimes \operatorname{Sym}^3 V$ spanned by $U_0M_{0,i} + U_1M_{1,i}$ for $i = 1, 2, 3$. From (4.10), the ramification divisor of this subspace is given by

$$\begin{aligned}R &= \det \begin{pmatrix} L_1 & 0 & U_0L_1' \\ 0 & L_2 & U_1L_2' \\ L_3 & L_3 & (U_0 + U_1)L_3' \end{pmatrix}\\ &= U_0L_2(L_3'L_1 - L_1L_3') + U_1L_1(L_3'L_2 - L_2L_3')\\ &= U_0L_2R_{13} + U_1L_1R_{23}.\end{aligned}$$

In this calculation, L_i' denotes the derivative $\frac{d}{dt}$ of L_i considered as an element of $k[t]$ by pullback under some parametrization $\operatorname{Spec} k[t] \longrightarrow Q$ and trivialization of $\mathcal{O}(2)|_{\operatorname{Spec} k[t]}$. Although the derivative depends on the choices, the forms $L_i L_j' - L_j L_i'$ do not, and they cut out precisely the ramification divisor of the pencil $\langle L_i, L_j\rangle$. Under the isomorphism in (6.4), the divisor R corresponds to the 2 dimensional subspace of $\operatorname{Sym}^2 S/q$ spanned by L_2R_{13} and L_1R_{23} (The roles of L_1, L_2, L_3 can be changed by linear transformations of M, so we get that L_3R_{12} also lies in this span). The proof is thus complete. $\square$

Recall that the conic $Q \subset \mathbf{P}S$ gives an isomorphism $\mathbf{P}S \cong \mathbf{P}S^*$, called *polarity* with respect to Q. On the vector spaces, it is the isomorphism induced by the bilinear form associated to q. Geometrically, it is characterized by the rule that the polar of a point $p \in Q$ is the tangent line to Q at p. More generally, given a point $p \in \mathbf{P}S$, the pencil of lines through p contains two lines tangent to Q; the polar of p is the line joining the two points of tangency. We denote the polar of a point p (resp. a line L) by $p^\perp$ (resp. $L^\perp$).

Set $M_i = R_{jk}$, and let N be the net spanned by L_iM_i for $i = 1, 2, 3$. By the definition of R_{jk}, we see that M_i is the polar line of the point $L_k \cap L_k$. In other words, the triangles (L_1, L_2, L_3) and (M_1, M_2, M_3) are polar conjugates—lines in one are polars to the vertices of the other.

Recall that $\xi \in \operatorname{Hilb}^3 \mathbf{P}S$ is the point defined by the three vertices of the triangle formed by (L_1, L_2, L_3). Let $\xi' \in \operatorname{Hilb}^3 \mathbf{P}S$ be the point defined by the three vertices of the triangle formed by (M_1, M_2, M_3).

{prop:quarticscr

Proposition 6.6. *In the setup above, ξ and ξ' are the only points of* $\operatorname{Hilb}^3 \mathbf{P}S$ *that map to* $N \in \mathbf{Gr}(2, \operatorname{Sym}^2 S/q)$. *In particular, the degree of* $\mu\colon \operatorname{Hilb}^3 \mathbf{P}S \dashrightarrow \mathbf{Gr}(2, \operatorname{Sym}^2 S/q)$ *is 2.*

Proof. By Lemma 6.5, we see immediately that $\mu(\xi') = \mu(\xi) = N$. To show that no other triangles map to N, consider pairs of triplets $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ and $\nabla = (\nabla_1, \nabla_2, \nabla_3)$ of lines in $\mathbf{P}S$ such that

(1) Δ and ∇ are polar conjugates with respect to Q, and
(2) $\Delta_i \nabla_i$ is an element of N for $i = 1, 2, 3$.

It suffices to show that the only ones satisfying the two conditions are (L_1, L_2, L_3) and (M_1, M_2, M_3), up to permutation.

To show this, we need some observations.

First, suppose A_1B_1 and A_2B_2 are elements of the net N, where A_i and B_j are lines in $\mathbf{P}S$. Then, by definition, A_i and B_j are Reye lines of the net N. Let $p = A_1 \cap A_2$ and $q = B_1 \cap B_2$. We claim that the third Reye line through p, in addition to A_1 and A_2, is the line $\langle p, q\rangle$. Indeed, in the pencil of conics spanned by A_1B_1 and A_2B_2, the third singular conic is $\langle p, q\rangle\langle p', q'\rangle$, where $p' = A_1 \cap B_2$ and $q' = A_2 \cap B_1$.

Second, let $R \subset \mathbf{P}S$ be the Jacobian cubic and $E \subset \mathbf{P}S^*$ be the Hermite cubic of N. Let $E^\perp \subset \mathbf{P}S$ be the image of E under the polarity isomorphism induced by Q. Explicitly, the points of $E^\perp$ are the polars of the Reye lines. We claim that the six points of intersection of R and Q also lie on $E^\perp$. Indeed, to show that $x \in R \cap Q$ also lies on $E^\perp$, it suffices to show that the line T_xQ is a Reye line. Since $x \in R$, there exists an element of N of the form AB where A and B are lines intersecting at x. Note that in the pencil of conics spanned by AB and Q, there is a singular conic containing T_pQ. Therefore, T_pQ is a Reye line.

Third, since $R \cap E^\perp$ contain 6 points on the conic Q, the residual 3 points are collinear. Let them correspond to $x_1, x_2, x_3 \in E$. Denoting by H the hyperplane class of $E \subset \mathbf{P}S^*$, we have the equation in $\operatorname{Pic} E$

$$x_1 + x_2 + x_3 = H.$$

Suppose we have two triangles Δ and ∇ satisfying the two conditions above. Consider the point $p_3 = \Delta_3 \cap \nabla_3$. By the second condition, it lies on R. By the polar conjugacy of Δ and ∇, we have

$$\begin{aligned} p_3^\perp &= \langle \Delta_3^\perp, \nabla_3^\perp \rangle \\ &= \langle \nabla_1 \cap \nabla_2, \Delta_1 \cap \Delta_2 \rangle. \end{aligned}$$

By the first claim, we see that $p_3^\perp$ is a Reye line. Hence p_3 lies on $E^\perp$, and hence on $R \cap E^\perp$. Similarly, $p_1 = \Delta_1 \cap \nabla_1$ and $p_2 = \Delta_2 \cap \nabla_2$ also lie on $E^\perp$. Since N is general, we may assume that the p_i do not lie on Q. Hence, p_1, p_2, p_3 are the points corresponding to the three collinear points in $R \cap E^\perp$. (The fact that p_1, p_2, p_3 are collinear is not surprising—it is because any two polar conjugate triangles are in linear perspective [2, Theorem 2.1.9]). By reordering if necessary, assume that we have $p_i^\perp = x_i$ as elements of E.

Now, observe that the three Reye lines through the vertex $\Delta_1 \cap \Delta_2$ are Δ_1, Δ_2, and $p_3^\perp$, and likewise for the other two vertices. The concurrence of the three lines, along with the equality $p_3^\perp = x_1$, yields the system of equations on $\operatorname{Pic} E$

$$\begin{aligned} \Delta_1 + \Delta_2 + x_3 &= H, \\ \Delta_2 + \Delta_3 + x_1 &= H, \\ \Delta_3 + \Delta_1 + x_2 &= H. \end{aligned}$$

Of course, the same three equations hold if we replace Δ by ∇.

Note that the points $x_1, x_2, x_3 \in E$ are determined by N. Using $x_1 + x_2 + x_3 = H$, a simple calculation gives $2\Delta_1 = 2x_1$. This equation has 4 solutions for Δ_1, namely $x_1 + \epsilon$ for $\epsilon \in \operatorname{Pic} E[2]$. Also, Δ_1 determines Δ_2 and Δ_3 by the equations above, which in turn determine the ∇_i using polarity or the property that ∇_i and Δ_i form a fiber of the map $E \longrightarrow R$. Thus, it suffices to show that only two of the four solutions for Δ_1 can be valid.

Suppose $\Delta_1 = x_1$. Then we get $\Delta_2 = x_2$, and $\Delta_3 = x_3$. However, the lines represented by the x_i are concurrent, whereas the lines Δ_i are not. Therefore, we get that $\Delta_1 \neq x_1$. The same argument shows that $\nabla_1 \neq x_1$. Let the involution of E induced by $E \longrightarrow R$ be given by the addition of $\epsilon_0 \in \operatorname{Pic} E[2]^*$. Since Δ_1 and ∇_1 form a fiber of $E \longrightarrow R$, we have $\nabla_1 = \Delta_1 + \epsilon_0$. So, $\nabla_1 \neq x_1$ translates into $\Delta_1 \neq x_1 + \epsilon_0$. In summary, the only two possible solutions for Δ_1 are $x_1 + \epsilon$ for $\epsilon \in \operatorname{Pic} E[2] \setminus \{0, \epsilon_0\}$. The proof is now complete. $\square$

References

[1] I. Dolgachev and B. Howard, *Configuration spaces of complex and real spheres*, in Recent advances in algebraic geometry, vol. 417 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 2015, pp. 156–179.

[2] I. V. Dolgachev, *Classical algebraic geometry*, Cambridge University Press, Cambridge, 2012. A modern view.

[3] D. Eisenbud and J. Harris, *Limit linear series, the irrationality of M_g, and other applications*, Bull. Amer. Math. Soc. (N.S.), 10 (1984), pp. 277–280.

[4] ———, *Limit linear series: basic theory*, Invent. Math., 85 (1986), pp. 337–371.

[5] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *On Hurwitz numbers and Hodge integrals*, C. R. Acad. Sci. Paris Sér. I Math., 328 (1999), pp. 1175–1180.

[6] ———, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math., 146 (2001), pp. 297–327.

[7] A. Eremenko and A. Gabrielov, *Rational functions with real critical points and the b. and m. shapiro conjecture in real enumerative geometry*, Annals of Mathematics, 155 (2002), pp. 105–129.

[8] H. Flenner and M. Manaresi, *Variation of ramification loci of generic projections*, Mathematische Nachrichten, 194 (1998), pp. 79–92.

[9] V. S. Kulikov, *On Chisini's conjecture. II*, Izv. Ross. Akad. Nauk Ser. Mat., 72 (2008), pp. 63–76.

[10] B. Osserman, *Rational functions with given ramification in characteristic p*, Compos. Math., 142 (2006), pp. 433–450.

[11] B. Osserman, *Limit linear series moduli stacks in higher rank*, May 2014. Preprint.

[12] F. Sottile, *Real schubert calculus: polynomial systems and a conjecture of shapiro and shapiro*, Experiment. Math., 9 (2000), pp. 161–182.

[13] M. Teixidor i Bigas, *Brill-Noether theory for stable vector bundles*, Duke Math. J., 62 (1991), pp. 385–400.

[14] F. Zak, *Review of variation of ramification loci of generic projections by Flenner and Manaresi*. MathSciNet MR1653078.