PROJECTION AND RAMIFICATION

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Abstract. When a projective variety is linearly projected to a projective space of the same dimension, a ramification divisor forms. We study basic properties of this projection-ramification assignment, and uncover enumerative phenomena extending the classical appearance of Catalan numbers in the geometry of rational normal curves.

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{sec:intro}

1. Introduction

Let $X \subset \mathbf{P}^n$ be a smooth projective variety of dimension r, not contained in any hyperplane. Projection from a general $(n-r-1)$-dimensional linear subspace $L \subset \mathbf{P}^n$ defines a finite surjective map

$$p_L \colon X \longrightarrow \mathbf{P}^r.$$

Associated to p_L is its ramification divisor $R_L \subset X$. A simple Riemann–Hurwitz calculation shows that R_L lies in the linear series $|K_X + (r+1)H|$, where K_X is the canonical class, and H is the hyperplane class on X. The association $L \mapsto R_L$ defines a rational map

$$\rho_X \colon \mathbf{Gr}(n-r, n+1) \dashrightarrow |K_X + (r+1)H|,$$

which we call the *projection-ramification* map. The goal of this paper is to explore the relationship between the geometry of X and the properties of ρ_X.

A simple argument shows that ρ_X is itself a linear projection of $\mathbf{Gr}(n-r, n+1)$ in its Plücker embedding. When X is a smooth curve over a field of characteristic 0, the map ρ_X is regular everywhere on $\mathbf{Gr}(n-r, n+1)$. When X is a rational normal curve, the map ρ_X is dominant. In this case, the ramification divisor of a map $\mathbf{P}^1 \longrightarrow \mathbf{P}^1$ of degree n represented by the rational function f/g is cut out by the *Wronskian* expression, namely the degree $(2n-2)$ polynomial $f'g - g'f$. Since ρ_X is regular, its degree is the degree of the Grassmannian, which in this case is the Catalan number $\frac{(2n-2)!}{n!(n-1)!}$. When X has dimension 2 or more, ρ_X may not be regular on the entire Grassmannian, which makes it difficult to understand. Nevertheless, it appears that the geometry of ρ_X is related to some fascinating areas of classical projective geometry, and the enumerative questions surrounding ρ_X hint at a rich underlying structure.

1.1. Maximal variation.

Our focus is the following question.

{q:maxvar}

Question 1.1. Is ρ_X generically finite onto its image? In other words, is the image of ρ_X of maximal possible dimension?

To our knowledge, this question first appeared in the work of Flenner and Manaresi [FM98]. Our first result answers this question affirmatively for a large class of varieties. We say that $X \subset \mathbf{P}^n$ is *incompressible* if for every $(n-r-1)$-dimensional linear subspace $L \subset \mathbf{P}^n$, the projection map $p_L \colon X \dashrightarrow \mathbf{P}^r$ is dominant. Recall that the dual variety $X^* \subset \mathbf{P}^{n*}$ is the locus of hyperplanes in $\mathbf{P}^n$ whose intersection with X is singular.

{theorem:Main}

Theorem A. *Let $X \subset \mathbf{P}^n$ be a non-degenerate, normal, projective variety over a field of characteristic zero. Suppose at least one of the following holds:*

{item:incomp}

(1) X is incompressible, {item:dual}
(2) the dual variety $X^* \subset \mathbf{P}^{n*}$ is a hypersurface.

Then ρ_X is generically finite onto its image.

We do not assume that X is smooth in the statement of Theorem A. This requires defining ρ_X more carefully. To state the conclusion informally, if we move a generic $L \subset \mathbf{P}^n$ of complementary dimension, then the ramification locus $R_L \subset X$ also moves.

The hypotheses in Theorem A are sufficient, but not necessary. Indeed, consider $X = \mathbf{P}^{r-1} \times \mathbf{P}^1 \subset \mathbf{P}^{2r-1}$, embedded by the Segre embedding, for $r \geq 3$. Then X is neither incompressible nor is X^* a hypersurface, and yet ρ_X is dominant (see Theorem E).

To our knowledge, the known results about maximal variation operate under condition (1) in Theorem A. For example, in [FM98], the authors deduce maximal variation under the condition that for every $(n-r-1)$-dimensional linear subspace $L \subset \mathbf{P}^n$, the join $J(L, X)$ equals $\mathbf{P}^n$, or under the condition that X is smooth and the twisted normal bundle $N_{X/\mathbf{P}^n}(-1)$ is ample. Either condition implies that X is incompressible, and hence falls under condition (1). If X is a curve or a hypersurface, then X is incompressible, and covered by condition (1).

Theorem A substantially increases the class of varieties where we now know maximal variation. For example, it is easy to see that if X is a smooth surface over a field of characteristic 0, then X^* is a hypersurface. Therefore, maximal variation holds for all surfaces, although incompressibility may not (The cubic surface scroll $X \subset \mathbf{P}^4$ is the smallest counter example, as the projection from the directrix line of X is not dominant). As another source of new examples, take a sufficiently high degree Veronese re-embedding $X \subset \mathbf{P}^N$ of any smooth X. Then X^* is divisorial, and hence X is covered under Theorem A. But $X \subset \mathbf{P}^N$ will be compressible.

Given that maximal variation holds in such a large class of varieties, it is natural to wonder if it always holds. This is not the case.

{Thm:Counterexam

Theorem B. *There exist smooth, non-degenerate, rational normal scrolls $X^r \subset \mathbf{P}^n$ of every dimension $r \geq 4$ such that the projection-ramification map ρ_X is not generically finite onto its image.*

Theorem B provides the first known examples of varieties with non-maximal variation of ramification divisors. We describe the rational normal scrolls in Theorem B in § 4; they include some of general moduli.

We now turn our attention to cases where the projection-ramification map ρ_X may be dominant. The next result classifies $X \subset \mathbf{P}^n$ for which the source and the target of ρ_X are of the same dimension.

m:minimaldegree}

Theorem C. *Let $X \subset \mathbf{P}^n$ be a smooth, non-degenerate projective variety of dimension r over a field of characteristic zero. We have the inequality*

$$\dim \mathbf{Gr}(n-r, n+1) \leq \dim |K_X + (r+1)H|,$$

where equality holds if and only if X is a variety of minimal degree, namely $\deg X = n-r+1$.

Recall the list of varieties of minimal degree: rational normal curves, quadric hypersurfaces, the Veronese surface in $\mathbf{P}^5$, and rational normal scrolls. By Theorem A, ρ_X is dominant for the first three, so we are led to investigate the scrolls. It came to us as a surprise that ρ_X is *not* dominant for all scrolls (see Theorem B). Nevertheless, it is dominant for most scrolls, which we now make precise.

Recall that if $X \subset \mathbf{P}^n$ is a smooth rational normal scroll, then X is isomorphic to the projectivization of an ample vector bundle E on $\mathbf{P}^1$, and the embedding is given by the complete linear series $|\mathcal{O}_{\mathbf{P}E}(1)|$.

alnormalscrolls}

Theorem D. *Let $X = \mathbf{P}E \subset \mathbf{P}^n$ be a rational normal scroll, where E is a ample vector bundle of rank r on $\mathbf{P}^1$, general in its moduli. If $\deg E = a \cdot (r-1) + b \cdot (2r-1) + 1$ for non-negative integers a, b, then the projection-ramification map ρ_X is dominant for X. In particular, the conclusion holds if E is general of degree at least $(r-1)(2r-1)+1$.*

Thus, at least among the general scrolls, the projection-ramification map is dominant except possibly in small degrees. We prove Theorem D by degeneration, using the theory of limit linear series of higher rank developed by Osserman [Oss14].

1.2. **Enumerative problems.** Theorem C and Theorem D motivate a gamut of enumerative questions.

{q:degree}

Question 1.2. For each variety $X \subset \mathbf{P}^n$ of minimal degree, what is the degree of ρ_X?

{Thm:Examples}

The following result summarizes our knowledge of the answers to Question 1.2.

Theorem E.

(1) If $X \subset \mathbf{P}^n$ is a rational normal curve, then ρ_X is regular and $\deg \rho_X = \frac{(2n-2)!}{n!(n-1)!}$.
(2) If $X \subset \mathbf{P}^n$ is a quadric hypersurface, then ρ_X is an isomorphism.
(3) If $X = \mathbf{P}^{r-1} \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^{2r-1}$ is the Segre embedding, then ρ_X is birational.
(4) If $X \subset \mathbf{P}^5$ is the Veronese surface, then $\deg \rho_X = 3$.
(5) If $X \subset \mathbf{P}^5$ is a general quartic surface scroll, then $\deg \rho_X = 2$.

(6) If $X = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(k+1)) \subset \mathbf{P}^{k+3}$ *is the surface scroll with most imbalanced splitting type, then* ρ_X *is birational.*

(7) If $X = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(k+1)) \subset \mathbf{P}^{k+5}$ *is the threefold scroll with most imbalanced splitting type, then* ρ_X *is birational.*

For X of dimension 1, namely a rational normal curve, the projection-ramification map

$$\rho_X \colon \mathbf{Gr}(2, n+1) \longrightarrow \mathbf{P}^{2n-2}$$

is regular, and defined by the Plücker line bundle on the Grassmannian. Therefore, its degree is the top self-intersection of the Plücker line bundle, which in this case is the Catalan number $\frac{(2n-2)!}{n!(n-1)!}$.

For X of codimension 1, namely a quadric hypersurface, the projection-ramification map

$$\rho_X \colon \mathbf{Gr}(n, n+1) = \mathbf{P}^n \longrightarrow \mathbf{P}^{n*}$$

is again regular, and is in fact the duality isomorphism induced by the (non-degenerate) quadric X.

The case of the Veronese surface and of the quartic surface scroll in Theorem E are particularly delightful; these are treated in § 7. They involve intricate classical projective geometry that intertwines cubic plane curves, Steinerians and Cayleyans, and apolarity.

The cases of the most unbalanced surface and threefold scrolls follow from direct calculation. Note, however, that for the most unbalanced scroll in dimension 4 and higher, the projection-ramification map is not dominant. For scrolls, ρ_X is not regular. Furthermore, the complexity of the base locus of ρ_X effectively blocks any straightforward application of the excess intersection formula.

A smooth rational normal scroll $X \subset \mathbf{P}^n$ of degree d and dimension r is isomorphic to the projectivization of an ample vector bundle E on $\mathbf{P}^1$, which in turn is isomorphic to a direct sum $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ for positive integers $a_1, \dots, a_r$ satisfying $d = a_1 + \cdots + a_r$. Let $\Sigma_{r,d}$ be the set of r-term partitions of d. We get a function $\rho \colon \Sigma_{r,d} \longrightarrow \mathbf{Z}_{\geq 0}$ defined by

$$\rho(a_1, \dots, a_r) = \deg \rho_X,$$

for $X = \mathbf{P}(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r))$. The set $\Sigma_{r,d}$ has a partial ordering $\prec$ given by dominance. If $(a_1, \dots, a_r) \prec (b_1, \dots, b_r)$, then the scroll $\mathbf{P}(\mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_r))$ isotrivially specializes to the scroll $\mathbf{P}(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r))$. By the lower semi-continuity of degrees of rational maps, we get

$$\rho(a_1, \dots, a_r) \leq \rho(b_1, \dots, b_r).$$

Theorem D implies that, at least if d is sufficiently large compared to r, then ρ is not identically zero. Theorem E determines the value of ρ for the partitions (n), $(1, \dots, 1)$, $(1, k+1)$, $(1, 1, k+1)$, and $(2, 2)$. The following table lists some more values of ρ computed using randomized calculations over finite fields using the computer algebra systems `Macaulay2` and `MAGMA`. We plan to return to a more complete enumerative investigation of ρ in a future paper.

$a_1 \backslash a_2$	1	2	3	4
1	1			
2	1	2		
3	1	6	22	
4	1	17	92	422

TABLE 1. Degree of ρ_X for $X = \mathbf{P}(\mathcal{O}(a_1) \oplus \mathcal{O}(a_2))$

{tab:computation

1.3. **Further remarks and questions.** One of the central enumerative problems concerning branch divisors, originating in the work of Hurwitz, is to compute the number of branched covers of the projective line with specified branch set in $\mathbf{P}^1$. This number is called the Hurwitz number. As is well known, the Hurwitz numbers are difficult to compute, but they exhibit remarkable structure. There is a related question of computing the number of rational functions on $\mathbf{P}^1$ with a prescribed ramification set. This question is much more elementary, and yields the Catalan numbers, as we have seen.

In higher dimensions, however, the analogue of the Hurwitz problem is expected to be much less interesting, thanks to Chisini's conjecture (now Kulikov's theorem [Kul99]). Kulikov's theorem asserts that a branched cover $S \longrightarrow P^2$ with generic branching is uniquely determined by its branch divisor $B \subset \mathbf{P}^2$, with finitely many well-understood exceptions. In contrast, the enumerative problem regarding ramification divisors persists in all dimensions, thanks to Theorem C, and poses a significant challenge. In some sense, the "branch" and "ramification" enumerative stories trade places, at least in terms of difficulty, but perhaps also in terms of structure.

The projection-ramification map generalizes the Wronski map

$$\rho\colon \mathbf{Gr}(2, n+1) \longrightarrow \mathbf{P}^{2n-2}.$$

The geometry surrounding the Wronski map has received a lot of attention, thanks to the B. and M. Shapiro conjecture. This conjecture states that the pre-image of any point in $\mathbf{P}^{2n-2}$ defined by a set of $(2n-2)$ real points on $\mathbf{P}^1$ consists entirely of real points in $\mathbf{Gr}(2, n+1)$ [Sot00] (the conjecture has been proved by Eremenko and Gabrielov [EG02]). Theorem C potentially sets the stage for a higher-dimensional generalization of the body of work around the Shapiro conjecture.

The study of ρ_X in positive characteristic is likely to bring new surprises and require different techniques. We do not know if Theorem A or Theorem C holds in positive characteristic; our proof certainly does not. The answers to the enumerative questions Question 1.2 do depend on the characteristic, even in the simplest case of rational normal curves, due to the presence of inseparable covers [Oss06].

1.4. **Notation and conventions.** We work over an algebraically closed field k of characteristic 0 (We use Bertini's theorem and generic smoothness. We also appeal to the Kodaira

Vanishing theorem.) By a *proper variety*, we mean a proper, integral, finite-type k-scheme. For any scheme X, we let X^{sm} denote its smooth locus. If F is a coherent sheaf, we let $P(F)$ denote its sheaf of principal parts. We will let $e\colon H^0(X,F)\longrightarrow P(F)$ denote the natural evaluation morphism – we suppress the dependence on F. If s is a global section of a locally free sheaf, we let $v(s)$ denote the vanishing scheme of s. If L is a line bundle, we let $|L|$ denote the projective space $\mathbf{P}(H^0(L))$. If L is a line bundle on a smooth variety Y, and $s\in H^0(Y,L)$ is a section, then the *singular scheme* $\mathrm{Sing}(v(s))$ of s is the vanishing scheme of $e(s)\in H^0(Y,P(L))$; if K, the kernel sheaf of $e\colon H^0(Y,L)\otimes\mathcal{O}_Y\longrightarrow P(L)$, is locally free, then $\mathrm{Sing}(v(s))$ is the largest closed subscheme $T\subset Y$ such that $s\colon \mathcal{O}_T\longrightarrow H^0(Y,L)\otimes\mathcal{O}_T$ factors through $K|_T$.

2. The projection-ramification map {sec:prmap}

The goal of this section is to define a projection-ramification map for a pair (X,L) consisting of a proper, normal, variety X and a sufficiently positive line bundle L on X. For $X\subset\mathbf{P}^n$, taking $L=\mathcal{O}(1)$ recovers the projection-ramification map introduced in § 1. Working with abstract pairs, however, offers more flexibility that is helpful in inductive proofs. Furthermore, we sometimes pass from X to a blow-up, and so our notion of positivity is broader than (very) ampleness.

Let X be a normal proper variety of dimension r over an algebraically closed field k of characteristic zero. A *projection* of X is a pair (L,V) consisting of a line bundle L on X and a subspace $V\subset H^0(X,L)$ of dimension $r+1$. {definition:prop

Definition 2.1. We say that a projection (L,V) is *properly ramified* if the evaluation homomorphism

$$e\colon V\otimes\mathcal{O}_X\longrightarrow P(L)$$

is an isomorphism over a general point in X. If (L,V) is properly ramified, its *ramification divisor*

$$R(L,V)\subset X$$

is the closure of the scheme defined by the determinant of $e\colon V\otimes\mathcal{O}_{X^{\mathrm{sm}}}\longrightarrow P(L)|_{X^{\mathrm{sm}}}$.

In most cases, L is clear from context, so we drop it from the notation and denote the ramification divisor simply by $R(V)$.

A projection (L,V) gives the evaluation map

$$e\colon V\otimes\mathcal{O}_X\longrightarrow L.$$

The evaluation map yields a map $p_{V,L}\colon X\dashrightarrow\mathbf{P}V$, regular on the non-empty open set of X where e is surjective. The following is an easy observation, whose proof we skip. {prop:proj}

Proposition 2.2. *The projection (L,V) is properly ramified if and only if the map on tangent spaces induced by $p_{V,L}$ is generically an isomorphism. In characteristic zero, this is equivalent to the condition that $p_{V,L}$ is dominant.*

For a fixed L, the set of all projection (L, V) are parametrized by the Grassmannian

$$\mathbf{Gr}(n+1, H^0(X, L)).$$

The property of being properly ramified is a Zariski open condition on the Grassmannian.

We now define a map that assigns to a projection its ramification divisor. To do so, we interpret the ramification divisor as an element of a linear series. Let K_X be the canonical sheaf of X. Denoting by $i\colon X^{\mathrm{sm}} \longrightarrow X$ the inclusion, K_X is given by the push-forward

$$K_X = i_* K_{X^{\mathrm{sm}}}.$$

Note that, since X is normal, the complement of $X^{\mathrm{sm}} \subset X$ has codimension at least 2. The sheaf K_X is coherent, reflexive, and satisfies Serre's S2 condition.

Let L be a line bundle on X. The sheaf $P(L)$ is locally free of rank $(r+1)$ on X^{sm}, and we have a canonical isomorphism

$$\bigwedge^{r+1} P(L)|_{X^{\mathrm{sm}}} \cong K_{X^{\mathrm{sm}}} \otimes L^{r+1}.$$

Given a subspace $V \subset H^0(X, L)$, we apply $\bigwedge^{r+1}$ to the evaluation map

$$e\colon V \otimes \mathcal{O}_{X^{\mathrm{sm}}} \longrightarrow P(L)|_{X^{\mathrm{sm}}},$$

to get

$$\det e\colon \det V \otimes \mathcal{O}_{X^{\mathrm{sm}}} \longrightarrow K_{X^{\mathrm{sm}}} \otimes L^{r+1}.$$

By applying i_* and taking global sections, we get

$$\det V \longrightarrow H^0(X, K_X \otimes L^{r+1}).$$

If (L, V) is properly ramified, then this map is non-zero, and hence gives a point of the projective space $\mathbf{P}H^0(X, K_X \otimes L^{r+1})^*$. Doing the same construction universally over the Grassmannian $\mathbf{Gr} = \mathbf{Gr}(r+1, H^0(X, L))$ yields a map

{eqn:rammap}
$$\det \mathcal{V} \longrightarrow H^0(X, K_X \otimes L^{r+1}) \otimes \mathcal{O}_{\mathbf{Gr}}, \tag{2.1}$$

where $\mathcal{V} \subset H^0(X, L) \otimes \mathcal{O}_{\mathbf{Gr}}$ is the universal sub-bundle of rank $(r+1)$. Let $U \subset \mathbf{Gr}$ be the open subset of properly ramified projections. Then the map in (2.1) is non-zero at every point of U, and defines a map $U \longrightarrow \mathbf{P}H^0(X, K_X \otimes L^{r+1})^*$ given by the surjection

qn:rammapfamily}
$$H^0(X, K_X \otimes L^{r+1})^* \otimes \mathcal{O}_U \longrightarrow \det \mathcal{V}|_U^*. \tag{2.2}$$

Note that U is non-empty if and only if L separates tangent vectors at a general point of X.

ionRamification}

Definition 2.3. Let L be a line bundle that separates tangent vectors at a general point of X. The *projection-ramification* map for L is the rational map

$$\rho_{(X,L)}\colon \mathbf{Gr}(n+1, H^0(X, L)) \dashrightarrow \mathbf{P}H^0(X, K_X \otimes L^{r+1})^*$$

defined on the non-empty open subset of properly ramified maps by (2.2).

Note that the map (2.2) factors as

$$\det \mathcal{V} \xrightarrow{a} \bigwedge^{r+1} H^0(X,L)\otimes \mathcal{O}_{\mathbf{Gr}} \xrightarrow{b} H^0(X, K_X\otimes L^{r+1})\otimes \mathcal{O}_{\mathbf{Gr}},$$

where a is $\wedge^{r+1}$ applied to the universal inclusion $\mathcal{V}\subset H^0(X,L)\otimes\mathcal{O}_{\mathbf{Gr}}$, and b is induced by $\wedge^{r+1}$ applied to the evaluation map $e\colon H^0(X,L)\otimes\mathcal{O}_X \longrightarrow P(L)$. The map a defines the Plücker embedding

$$i\colon \mathbf{Gr}(r+1, H^0(X,L)) \longrightarrow \mathbf{P}\bigwedge^{r+1} H^0(X,L)^*,$$

and the map b defines a linear projection

$$p\colon \mathbf{P}\bigwedge^{r+1} H^0(X,L)^* \dashrightarrow \mathbf{P}H^0(X,K_X\otimes L^{r+1}).$$

Thus, $\rho_{X,L}$ factors as the Plücker embedding followed by a linear projection.

3. Maximal variation for non-defective pairs

{sec:proof_of_th

The goal of this section is to prove Theorem A. We begin by proning part (1), which is substantially easier.

{prop:incompress

Proposition 3.1 (Theorem A (1))**.** *Let $X\subset\mathbf{P}^n$ be a non-degenerate, normal, incompressible projective variety over a field of characteristic zero. Then ρ_X is a finite map.*

Proof. Write $\mathbf{P}^n = \mathbf{P}W$ for an $(n+1)$ dimensional vector space W and set $L=\mathcal{O}(1)$. Since X is non-degenerate, we have an inclusion $W\subset H^0(X,L)$. The projection-ramification map

$$\rho_X\colon \mathbf{Gr}(r+1,W)\dashrightarrow |K_X+(r+1)H|$$

is the composite of the inclusion

$$\mathbf{Gr}(r+1,W)\subset \mathbf{Gr}(r+1,H^0(X,L)),$$

and

$$\rho_{X,L}\colon \mathbf{Gr}(r+1,H^0(X,L))\dashrightarrow \mathbf{P}H^0(X,K_X\otimes L^{r+1}) = |K_X+(r+1)H|.$$

Let $V\subset W$ be an $(r+1)$-dimensional subspace. Since $X\subset\mathbf{P}W$ is incompressible, the projection map $p_{V,L}\colon X\dashrightarrow \mathbf{P}V$ induced by (L,V) is dominant. By **??**, this implies that (L,V) is properly ramified. Hence, the sub-grassmannian $\mathbf{Gr}(r+1,W)\subset\mathbf{Gr}(r+1,H^0(X,L))$ lies entirely in the open subset of properly ramified map, where $\rho_{X,L}$ is regular. Therefore, the projection-ramification map ρ_X extends to a regular map

$$\rho_X\colon \mathbf{Gr}(r+1,W)\longrightarrow |K_X+(r+1)H|.$$

Since the Picard rank of a Grassmannian is 1, a regular map from a Grassmannian is either constant or finite. It is easy to check that ρ_X is not constant; so it must be finite. ☐

The proof of part (2) of Theorem A, we proceed inductively by showing that a general $(n-r-1)$-dimensional linear subspace which is incident to X is an isolated point in its fiber under ρ_X. This strategy relies on a positive property of line bundles, called non-defectivity, which we now study.

non-defectivity}

3.1. **Non-defective line bundles.** Let X be a proper variety of dimension r, and let L be a line bundle on X.

lynon-defective}

Definition 3.2. We say that L is *non-defective* if the following holds: for a general point $x \in X$, either there is no $s \in H^0(X, L)$ such that $v(s)$ is singular at x, or a general s such that $v(s)$ is singular at x is such that $v(s)$ only has an isolated singularity at x.

Note that for $s \in H^0(X, L)$, the vanishing scheme $v(s)$ is singular at x if and only if s lies in $H^0(X, L \otimes \mathfrak{m}_x^2)$. Thus, for a non-defective L, the vector space $H^0(X, L \otimes \mathfrak{m}_x^2)$ is zero, or the vanishing scheme a general element of this vector space only has an isolated singularity at x. Note that for $s \in H^0(X, L \otimes \mathfrak{m}_x^2)$, the condition that $v(s)$ have an isolated singularity at x is a Zariski open condition. Therefore, if there exists an $s \in H^0(X, L \otimes \mathfrak{m}_x^2)$ such that $v(s)$ has an isolated singularity at x, then a general $s \in H^0(X, L \otimes \mathfrak{m}_x^2)$ has the same property.

Remark 3.3. Let x be a point of X. Suppose there exists $s \in H^0(X, L)$ with an isolated singularity at x. It may be tempting to conclude from this that L is non-defective. This is not necessarily true! For example, take $X = \mathbf{F}_3$ and L the line bundle associated to the divisor class $E + 2F$, where F is a fiber of the projection $\mathbf{F}_3 \longrightarrow \mathbf{P}^1$ and E the section of self-intersection -3. Since the dimension of the vector space $H^0(X, L \otimes \mathfrak{m}_x^2)$ may jump at special x, the existence of a nice $s \in H^0(X, L \otimes \mathfrak{m}_x^2)$ for a particular x does not imply the same for a general x.

Remark 3.4. Suppose L is non-defective. Let $x \in X$ be general, and let $s \in H^0(X, L)$ be such that $v(s)$ has an isolated singularity at x. For all such s, it may be the case $v(s)$ has singularities away from x, even along a positive dimensional locus. For example, let $\pi \colon X \longrightarrow \mathbf{P}^2$ be the blow-up at a point, and E the exceptional divisor. Then the line bundle $L = \pi^* \mathcal{O}(2) + 2E$ is non-defective, but every section of L contains E in its singular locus.

We now define the conormal variety of L, which plays an important role in our analysis of non-defectivity. Let K be the kernel of the map

$$e \colon H^0(X, L) \otimes \mathcal{O}_X \longrightarrow P(L).$$

Let $U \subset X$ be an open subset such that $K|_U$ is locally free and the dual of the inclusion

$$H^0(X, L)^* \otimes \mathcal{O}_U \longrightarrow K|_U^*$$

is a surjection. This surjection defines a closed embedding $\mathbf{P}(K|_U) \subset U \times |L|$. The *conormal variety of* L, denoted by P_L, is the closure of $\mathbf{P}(K|_U)$ in $X \times |L|$.

{prop{dimpndimR}

Proposition 3.5. *Suppose L is non-defective. If $h^0(L) \geq r+2$, then P_L is irreducible of dimension* $\dim |L| - 1$*. If $h^0(L) \leq r+1$, then P_L is empty.*

Proof. Set $n = \dim |L|$. Then $h^0(L) = n+1$. Let k be the (generic) rank of K, namely the rank of the locally free sheaf $K|_U$. Then $k \geq n-r$. The statement of the proposition is equivalent to showing that if $k > 0$, then $k = n-r$.

For brevity, set $P = P_L$. Consider the projection $\sigma\colon P \longrightarrow |L|$, obtained by restricting the second projection $X \times |L| \longrightarrow |L|$. For $s \in |L|$, we view $\sigma^{-1}(s)$ as a subscheme of X. We then have

$$\sigma^{-1}(s) \cap U = \mathrm{Sing}(v(s|_U)).$$

Suppose $r > 0$. Then P is non-empty and irreducible, since it is the closure of a non-empty and irreducible variety. Since L is non-defective, a general point $(x,s) \in P$ is such that x is an isolated point of $\mathrm{Sing}(v(s))$. Therefore, $\sigma\colon P \longrightarrow |L|$ is generically finite onto its image. We conclude that $\dim P \leq \dim |L|$, and hence $k \leq n - r + 1$.

To show that $k = n-r$, it suffices to show that $\sigma\colon P \longrightarrow |L|$ is not surjective. We do so using Bertini's theorem. Let $B \subset X$ denote the union of the base locus of the linear series $|L|$ and the singular locus of X. Then B is a proper closed subset of X. Let $P^B \subset P$ be the pre-image of B under the projection $\pi\colon P \longrightarrow X$. By the definition of P, the map $\pi\colon P \longrightarrow X$ is dominant, and hence P^B is a proper closed subset of P. Since P is irreducible, we have $\dim P^B < \dim P \leq \dim |L|$, so the projection $P^B \longrightarrow |L|$ cannot be dominant. Let $s \in |L|$ be general, in particular, not in the image of $P^B \longrightarrow |L|$. By Bertini's theorem $v(s)$ is non-singular away from B. Thus, for any $x \in X$, the point $(x,s) \in X \times |L|$ does not lie in P. For $x \in B$, this is because s is not in the image of P^B, and for $x \notin B$, this is because $v(s)$ is non-singular at x. We conclude that s does not lie in the image of $P \longrightarrow |L|$. Hence $P \longrightarrow |L|$ is not surjective. ☐

{prop:dimensionC

Proposition 3.6. *Let L be a line bundle on X with $h^0(X,L) \geq r+2$, and let $P = P_L$ be its conormal variety. The projection $\sigma\colon P \longrightarrow |L|$ is generically finite onto its image if and only if L is non-defective.*

Proof. Since $h^0(X,L) \geq r+2$, the conormal variety P is non-empty. Let $(x,s) \in P$ be a general point. We may assume that $x \in U$. Then x is a singular point of $v(s)$, and it is an isolated singularity of $v(s)$ if and only if (x,s) is an isolated point in the fiber of $\sigma\colon P \longrightarrow |L|$ over s. The conclusion follows. ☐

{prop:ordinarydo

Proposition 3.7. *Let L be a non-defective line bundle on X with $h^0(X,L) \geq r+2$. Let $x \in X$ be a general point. Then there exists $s \in H^0(X,L)$ such that $v(s)$ has an ordinary double point singularity at x.*

Proof. By Proposition 3.6, the projection $\sigma\colon P \longrightarrow |L|$ is generically finite onto its image. Let $(x,s) \in P$ be a general point. Since our ground field is of characteristic zero, we may assume that P is smooth at (x, s), that $x \in U \cap X^{\mathrm{sm}}$, and $\sigma\colon P \longrightarrow |L|$ is a local immersion at (x, s). This implies that $x \in \mathrm{Sing}(v(s))$ is isolated, and also that x is a reduced point of the scheme $\mathrm{Sing}(v(s))$. These two properties show that $v(s)$ possesses an ordinary double point at x. To see this, choose local coordinates $(x_1, ..., x_n)$ so that the complete local ring $\widehat{\mathcal{O}}_{X,x}$ is isomorphic to $k[\![x_1, \dots, x_r]\!]$. After choosing a local trivialization for L around x, the section s corresponds to a power series $s(x_1, \dots, x_r)$ contained in $\mathfrak{m}_x^2 \widehat{\mathcal{O}}_{X,x}$. The germ of $\mathrm{Sing}(v(s))$ at x is cut out by the power series $\frac{\partial s}{\partial x_1}, \dots, \frac{\partial s}{\partial x_r}$. Since the germ of $\mathrm{Sing}(v(s))$ at x is the reduced point x, we get that $\frac{\partial s}{\partial x_1}, \dots, \frac{\partial s}{\partial x_r}$ are linearly independent as elements of $\mathfrak{m}_x/\mathfrak{m}_x^2$. From this, it is easy to check that the tangent cone of $s(x_1, \dots, x_r)$ at x is a non-degenerate quadric cone. □

eparateTangents}

Proposition 3.8. *If L is a non-defective line bundle on X with $h^0(L) \geq r + 1$, then the global sections of L separate tangent vectors at a general point $x \in X$. That is, the evaluation map*

$$e_x\colon H^0(X, L) \longrightarrow L/\mathfrak{m}_x^2 L$$

is surjective for general $x \in X$.

Proof. By the definition of $P(L)$, we have a natural isomorphism

$$P(L)|_x = L/\mathfrak{m}_x^2 L,$$

so it suffices to show that the evaluation map

$$e\colon H^0(X, L) \longrightarrow P(L)$$

is surjective at x. Let k be the generic rank of K, the kernel of e. From the proof of Proposition 3.5, we get

$$k = h^0(L) - r - 1.$$

Since $(r + 1)$ is the generic rank of $P(L)$, we conclude that e is generically surjective. □

rop:blowuppoint}

Proposition 3.9. *Let L be a non-defective line bundle on X with $h^0(L) \geq r + 2$. Let $x \in X$ be a general point, $\pi\colon \widetilde{X} \longrightarrow X$ the blow up of X at x, and E the exceptional divisor of π. Then $\pi^* L(-E)$ is a non-defective line bundle on $\widetilde{X}$.*

Proof. Let y be a general point of $\widetilde{X}$. We must show that either $H^0(\widetilde{X}, \pi^* L(-E) \otimes \mathfrak{m}_y^2) = 0$ or there exists a global section of $\pi^* L(-E)$ whose vanishing locus has an isolated singularity at y. Note that

$$H^0(\widetilde{X}, \pi^* L(-E)) = H^0(X, L \otimes \mathfrak{m}_x).$$

Hence, we must show that for general $(x, y) \in X \times X$, either $H^0(X, L \otimes \mathfrak{m}_y^2 \cdot \mathfrak{m}_x) = 0$ or there exists a global section of L with an isolated singularity at y and passing through x.

By Proposition 3.8, for a general $y \in X$, we have

$$h^0(X, L \otimes \mathfrak{m}_y^2) = h^0(X, L) - (r + 1).$$

Hence, for a generic $x \in X$, we have

$$h^0(X, L \otimes \mathfrak{m}_y^2 \cdot \mathfrak{m}_x) = h^0(X, L) - (r+2).$$

If $h^0(X, L) = r+2$, then

$$h^0(\widetilde{X}, L \otimes \mathfrak{m}_y^2) = h^0(X, L \otimes \mathfrak{m}_y^2 \cdot \mathfrak{m}_x) = 0,$$

and we are done. Assume that $h^0(X, L) \geq r+3$, and let $y \in X$ be general. Then $h^0(X, L \otimes \mathfrak{m}_y^2) \geq 2$. Since L is non-defective, a general $s \in V = H^0(X, L \otimes \mathfrak{m}_y^2)$ is such that $v(s)$ has an isolated singularity at y. Moreover, since $\dim V \geq 2$, for every $x \in X$, there exists $s \in V$ such that $v(s)$ passes through x. Hence, for a general $x \in X$, there exists $s \in V$ such that $v(s)$ has an isolated singularity at y and passes through x. The proof is now complete. □

{cor:properlyram

Corollary 3.10. *Suppose L is a non-defective line bundle on X with $h^0(X, L) \geq r+1$. Then there exists a properly ramified projection (L, V).*

Proof. This follows immediately from Proposition 3.8. □

By Corollary 3.10, the rational map $\rho_{X,L}$ is defined if L is a non-defective line bundle on X and $h^0(X, L) \geq r+1$.

3.2. **Proof.** We proceed to the proof of Theorem A. More specifically, we prove Theorem 3.12 below, which is more precise and implies Theorem A.

Definition 3.11. A projection (L, V) is *isolated* if it is properly ramified and if $[V] \in \mathbf{Gr}(n+1, H^0(X, L))$ is an isolated point in the corresponding fiber of $\rho_{(X,L)}$.

Our main next objective is to show:

{theorem:MainMai

Theorem 3.12. *If $L \longrightarrow X$ is a non-defective line bundle satisfying $h^0(X, L) \geq n+2$, then there exists an isolated projection (L, V).*

We need two lemmas, which are essentially local computations.

{lemma:tangentco

Lemma 3.13. *Let $L \longrightarrow X$ be a non-defective line bundle with $h^0(L) \geq n+2$, and let $x \in X$ be a general point. If $V \subset H^0(X, L \otimes \mathfrak{m}_x)$ is a general $(n+1)$-dimensional subspace, then the ramification divisor $R(V)$ has an ordinary double point singularity at x.*

Proof. Using Proposition 3.8 and Proposition 3.7, we can assume V has a basis $(s_1, ..., s_n, t)$ satisfying:

(1) $(s_1, ..., s_n)$ generate $L \otimes (\mathfrak{m}_x / \mathfrak{m}_x^2)$ and
(2) $v(t)$ has an ordinary double point singularity at x.

Let $\widehat{\mathcal{O}}_{X,x}$ denote the completion of the local ring at $x \in X$ along its maximal ideal. Upon trivializing L, we may regard s_i and t as elements of $\widehat{\mathcal{O}}_{X,x}$, and can also assume $\widehat{\mathcal{O}}_{X,x} = k[[s_1, ..., s_n]]$. A local matrix representative for the evaluation map

$$e \colon V \otimes \widehat{\mathcal{O}}_{X,x} \longrightarrow P(L) \otimes \widehat{\mathcal{O}}_{X,x}$$

is given by

{matrix}

$$\begin{pmatrix} s_1 & s_2 & \dots & t \\ 1 & 0 & \dots & \partial_1 t \\ 0 & 1 & \dots & \partial_2 t \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \partial_n t \end{pmatrix} \tag{3.1}$$

Here, ∂_i denotes $\frac{\partial}{\partial s_i}$. The determinant of 3.1 is

$$t - \sum_i s_i \partial_i t,$$

and is an analytic local equation for the ramification divisor $R(V)$ near x. It is then evident that $R(V)$ shares the same tangent cone as $v(t)$ at x, and hence the proposition follows. ☐

a:basepointfree}

Lemma 3.14. *Maintain the same setting as in Lemma 3.13, except now assume $V \subset H^0(X, L)$ has a basis of the form $(u, a_1, ..., a_{n-1}, b)$ where*

(1) u does not vanish at x,
(2) $a_1, ..., a_{n-1}$ all vanish at x and induce linearly independent elements in $L \otimes (\mathfrak{m}_x/\mathfrak{m}_x^2)$, and
(3) $v(b)$ has an ordinary double point at x.

Then $x \in R(V)$ and $R(V)$ is smooth at x.

Proof. That $x \in R(V)$ is clear since $V \cap H^0(X, L \otimes \mathfrak{m}_x^2) \neq 0$.

Again we work in the completion $\widehat{\mathcal{O}}_{x,X}$, and after trivializing L, we assume $u, a_1, ..., b$ are elements of $\widehat{\mathcal{O}}_{x,X}$. We choose an element $z \in \widehat{\mathcal{O}}_{x,X}$ in such a way that $(a_1, ..., a_{n-1}, z)$ form a system of coordinates, i.e. $\widehat{\mathcal{O}}_{x,X} \simeq k[[a_1, ..., a_{n-1}, z]]$. With respect to these coordinates, a local matrix representative for

$$e \colon V \otimes \widehat{\mathcal{O}}_{x,X} \longrightarrow P(L) \otimes \widehat{\mathcal{O}}_{x,X}$$

has the form

{matrix2}

$$\begin{pmatrix} u & a_1 & a_2 & \dots & b \\ \partial_1 u & 1 & 0 & \dots & \partial_1 b \\ \partial_2 u & 0 & 1 & \dots & \partial_2 b \\ \vdots & \vdots & \vdots & \vdots & \\ \partial_z u & 0 & 0 & \dots & \partial_z b \end{pmatrix} \tag{3.2}$$

The determinant of 3.2 is the analytic local equation for $R(V)$:

$$\bar{u} \cdot \partial_z b \pm \partial_z u \cdot \bar{b}$$

Here, for any $r \in \widehat{\mathcal{O}}_{x,X}$ we set

$$\bar{r} := r - a_1 \partial_1 r - a_2 \partial_2 r - \ldots - z \partial_z r.$$

Since $b \in \mathfrak{m}_x^2$, we get $\bar{b} \in \mathfrak{m}_x^2$ and so $\partial_z b \in \mathfrak{m}_x$. Furthermore, $\bar{u}$ is a unit. Therefore, $R(V)$ is singular at x if and only if:

$$\partial_z b \in \mathfrak{m}_x^2,$$

but this would render the tangent cone of $v(b)$ degenerate, contrary to our third assumption. So $R(V)$ is smooth at x. □

Now we are ready for the proof of Theorem 3.12.

Proof of Theorem 3.12. We induct on $h^0(L)$. First, assume $h^0(X, L) = n + 2$ and choose $x \in X$ a general point. Then $V = H^0(X, \mathfrak{m}_x \otimes L)$ satisfies the conditions in Lemma 3.13 and in particular V is properly ramified and $R(V)$ has an ordinary double point at x.

Suppose we have a 1-parameter family of $(n + 1)$-dimensional subspaces $V_c \subset H^0(X, L)$ parametrized by points c of a smooth pointed curve $(C, 0)$ satisfying:

(1) $V_0 = V$
(2) $V_c \neq V_0$ for $c \in C$, $c \neq 0$.

We must prove $R(V_c) \neq R(V)$ for a general point $c \in C$. By upper semicontinuity,

$$\dim \left(V_c \cap H^0(X, L \otimes \mathfrak{m}_x^2)\right) \leq 1.$$

If $\dim V_c \cap H^0(X, L \otimes \mathfrak{m}_x^2) = 0$, then $x \notin R(V_c)$ and therefore $R(V_c) \neq R(V)$. Otherwise, if $\dim V_c \cap H^0(X, L \otimes \mathfrak{m}_x^2) = 1$ then a nonzero section $b \in V_c \cap H^0(X, L \otimes \mathfrak{m}_x^2)$ must also possess an ordinary double point singularity at x, as this is an open condition on families of singularities.

Since $V_c \neq V_0 = H^0(X, L \otimes \mathfrak{m})$, there exists a section $u \in V_c$ not vanishing at x. This means V_c obeys the hypotheses in Lemma 3.14. But then the conclusion of Lemma 3.14 implies $R(V_c) \neq R(V_0)$ as the former is smooth while the latter is singular at x, establishing our initial case.

Next suppose $h^0(X, L) > n + 2$, and again choose a general point $x \in X$ where Proposition 3.9 holds. Choose a general $(n + 1)$-dimensional subspace

$$W \subset H^0(X, L \otimes \mathfrak{m}_x)$$

so as to obey the conditions in Lemma 3.13. By induction hypothesis and Proposition 3.9, the projection $(\pi^* L(-E), W)$ is isolated, where $\pi \colon \widetilde{X} \longrightarrow X$ is the blow up at x.

So assume we have a 1-parameter family W_c parameterized by a pointed curve $(C, 0)$ satisfying

(1) $W_0 = W$,
(2) $W_c \neq W_0$ for $c \in C$ general,
(3) W_c is not contained in $H^0(X, L \otimes \mathfrak{m}_x)$ for $c \in C$ general.

We must verify: $R(W_c) \neq R(W_0)$. Again by upper semicontinuity and openness (among singularities) of possession of ordinary double point, we see that W_c meets the conditions of Lemma 3.14. We conclude $R(W_c) \neq R(W_0)$ since the former is smooth while the latter is singular at x, completing the argument. □

Recall that if $X \subset \mathbf{P}^n$ is a projective variety then its dual variety $X^* \subset \mathbf{P}^{n*}$ is the image of the conormal variety P (associated to the line bundle $\mathcal{O}_X(1)$) under the projection $\sigma\colon P \longrightarrow |\mathcal{O}_X(1)| \simeq \mathbf{P}^{n*}$. We immediately get Theorem A:

cor:maintheorem}

Corollary 3.15. *Let $X \subset \mathbf{P}^n$ be a non-degenerate projective variety such that the dual variety $X^* \subset \mathbf{P}^{n*}$ is a hypersurface. Then $\rho_{X,\mathcal{O}_X(1)}$ is generically finite onto its image.*

Proof. Indeed, since $\mathcal{O}_X(1)$ separates tangent vectors at a general point of X, the condition that X^* is a hypersurface implies the projection $\sigma\colon P \longrightarrow |\mathcal{O}_X(1)|$ is generically finite onto X^*, and hence $\mathcal{O}_X(1)$ is non-defective by Proposition 3.6. Thus, Theorem 3.12 applies. □

f_second_result}

4. Proof of Theorem B

Our next objective is to prove Theorem B by exhibiting some examples. Before doing so, we pose the general problem of maximal variation for rational normal scrolls in explicit affine coordinates.

zed_wronski_map}

4.1. **The generalized Wronski map for scrolls in affine coordinates.** Fix variables $x_1, \dots, x_r, t$.

{definition:V}

Definition 4.1. Let $\underline{d} = (d_1, \dots, d_r)$ denote an r-tuple of degrees. We define $V(\underline{d})$ to be the vector space of forms $\sum_{i=1}^r p_i(t) x_i$, where $\deg p_i \leq d_i$.

Remark 4.2. $V(\underline{d})$ is simply the space of global sections of the line bundle $\mathcal{O}_{\mathbf{P}E}(1)$ on the scroll $\mathbf{P}E$ over $\mathbf{P}^1$, where $E = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_r)$.

Next, if $v_1 \wedge \cdots \wedge v_{r+1} \in \bigwedge^{r+1} V(\underline{d})$ is any pure tensor, we set

$$Wr(v_1 \wedge \cdots \wedge v_{r+1}) := \det \begin{pmatrix} -- & v_1 & -- & v_1' \\ -- & v_2 & -- & v_2' \\ \vdots & \vdots & \vdots & \vdots \\ -- & v_{r+1} & -- & v_{r+1}' \end{pmatrix} \in V(\underline{e}) \tag{4.1}$$

where $\underline{e} = (e_1, \dots, e_r)$ is given by $e_i = d_i - 2 + \sum_{j=1}^r d_j$.

ition:Wronskian}

Definition 4.3. The induced map

$$Wr_{\underline{d}}\colon \mathbf{Gr}(r+1, V(\underline{d})) \dashrightarrow \mathbf{P}V(\underline{e}) \tag{4.2}$$

is called the *Wronskian* map.

Remark 4.4. $Wr_{\underline{d}}$ is the projection-ramification map for the scroll $X = \mathbf{P}E$ in coordinates.

The dimensions of source and target of the Wronskian map are equal, hence we may pose the general question:

{problem:Dominan

Problem 4.5. For which degree vectors $\underline{d} = (d_1, ..., d_r)$ is the Wronskian map $Wr_{\underline{d}}$ dominant?

In the next section, we show Problem 4.5 has genuine content by demonstrating that $Wr_{\underline{d}}$ fails to be dominant for degree vectors of the form $(1, 1, 1, \dots, k+1)$, $r \geq 4$.

4.2. **Proof of Theorem B.** Let $E = \mathcal{O}(1)^{r-1} \oplus \mathcal{O}(k+1)$ be the vector bundle over $\mathbf{P}^1$, and set $X = \mathbf{P}E$. We will prove:

Theorem 4.6. *The projection-ramification map for the embedding of X given by $\mathcal{O}_E(1)$ is not dominant once $k(r-3) > 1$.* {NonDominance}

Remark 4.7. The basic phenomenon underlying this example is: a general point in the source Grassmannian has trivial $Aut(X)$-stabilizer, yet every point of $|R|$ has positive dimensional stabilizer.

Remark 4.8. If $k = 1$ and $r \geq 5$, then X is a balanced scroll. Therefore, the non-dominance of projection-ramification is not directly connected to the eccentricity of the splitting type of a scroll. Rather, among balanced scrolls, non-dominance of ρ_X happens only in "low" degree – see Theorem 6.15.

As an immediate corollary, we get a result concerning Grassmannians in their Plucker embeddings. Recall that an r dimensional variety $X \subset \mathbf{P}^n$ is *compressible* if there exists a $(n-r-1)$-dimensional linear space $\Lambda \subset \mathbf{P}^N$ with the property that the projection $p_\Lambda : X \dashrightarrow \mathbf{P}^r$ is not dominant.

Proof of Theorem 4.6. We will show: the general element in the ramification divisor class $|R|$ has a positive dimensional stabilizer under the action of $Aut(X)$. We leave it to the reader to check that the Grassmannian $\mathbf{Gr}$ does not have generic stabilizer under $\mathrm{Aut}(X)$. The theorem then follows by the $Aut(X)$-equivariance of $\rho\colon \mathbf{G} \dashrightarrow |R|$.

In terms of the affine coordinates $(x_1, \dots, x_r, t)$ introduced in the previous section, we find ourselves in the situation corresponding to the degree vector $\underline{d} = (1, ..., 1, k+1)$. The degree vector corresponding to ramification divisors is then $\underline{e} = (r+k+1, r+k+1, \dots, r+2k+1)$.

In these affine coordinates, the substitutions

$$x_1 \mapsto x_1 + p_1(t)x_r \tag{4.3}$$ {substitutions}

$$x_2 \mapsto x_2 + p_2(t)x_r \tag{4.4}$$

$$\vdots \tag{4.5}$$

$$x_n \mapsto x_r \tag{4.6}$$

produce distinct automorphisms in $Aut(X)$ per choice of the p_i, where each $p_i(t)$ has degree $\leq k$.

If $\sum_{j=1}^r a_j x_j$ represents a general element of $V(\underline{e})$, then the substitutions (4.3) have the effect of replacing the coefficient $a_r(t)$ with $a_r + \sum_{j=1}^{r-1} a_j p_j$, and preserving all other coefficients a_i.

Now if $(r-3)k > 1$, then the dimension of the vector space of choices for the polynomials p_i exceeds the dimension of degree $r+2k+1$ polynomials a_r. Hence there exists a particular choice of p_i's (not all zero) such that the above automorphism fixes the equation $\sum_{j=1}^r a_j x_j$. Scaling these p_i's by constants produces the positive dimensional stabilizer mentioned in the theorem.

□

We obtain an immediate corollary:

Corollary 4.9. *The Grassmannian* $\mathbf{Gr}(m,n)$ *is compressible if* $5 \leq m \leq n-m$. ToDo: Is this obvious? Can u

m:minimaldegree}

5. Proof of Theorem C

In this section, we provide the simple proof of Theorem C using the Kodaira vanishing theorem.

{lemma:KYmH}

Lemma 5.1. *Let Y be a smooth projective r-dimensional variety with $r > 0$, and let H be an ample divisor class on Y which induces a morphism*

$$Y \longrightarrow \mathbf{P}^n.$$

Then

$$h^0(K_Y + rH) \geq n - r.$$

Proof. We proceed by induction on $r = \dim Y$, with the case $r = 1$ following directly from the fact that a degree d line bundle on a curve cannot have more than $d+1$ independent sections. So we assume $r > 1$. Let $D \subset Y$ be a general divisor (smooth by Bertini's theorem) in the linear system $|H|$, and consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_Y(K_Y + (r-1)H) \longrightarrow \mathcal{O}_Y(K_Y + rH) \longrightarrow \mathcal{O}_D(K_D + rH) \longrightarrow 0.$$

The Kodaira Vanishing Theorem gives $h^1(\mathcal{O}_Y(K_Y + (r-1)H) = 0$, and hence

$$h^0(Y, \mathcal{O}_Y(K_Y + rH)) \geq h^0(D, \mathcal{O}_D(K_D + (r-1)H)).$$

Notice $h^0(\mathcal{O}_D(H)) \geq n$, and therefore by induction we know:

$$h^0(D, \mathcal{O}_D(K_D + (r-1)H)) \geq (n-1) - (r-1) = n - r.$$

The proposition now follows by the Kodaira vanishing theorem which says that $h^1(K_Y + (r-1)H) = 0$. □

Proof of Theorem C. We first show: if $\dim \mathbf{Gr}(n-r, n+1) \geq \dim |K_X + (r+1)H|$, then X is a variety of minimal degree. Then we argue: in the case of minimal varieties, this inequality is actually an equality.

We proceed by intersecting with a hyperplane: let $X' = X \cap H$ be a general hyperplane section of X. By combining the Kodaira vanishing theorem, adjunction, and Lemma 5.1 we get:

$$h^0(\mathcal{O}_X(K_X + (r+1)H)) - h^0(\mathcal{O}_{X'}(K_{X'} + rH)) \geq n - r.$$

Therefore, the inequality

$$\dim \mathbf{Gr}(n-r, n+1) \geq \dim |K_X + (r+1)H|$$

implies

$$\dim \mathbf{Gr}(n-r, n) \geq \dim |K_{X'} + rH|$$

and hence by the inductive hypothesis, X' is a variety of minimal degree. We deduce X is itself a variety of minimal degree.

Thus we are reduced to considering the case $r = 1$: we leave it to the reader to translate this case into the well-known fact that a non-degenerate degree n smooth curve in $\mathbf{P}^n$ is a rational normal curve.

To complete the proof we observe that if X is a variety of minimal degree, and if $\dim \mathbf{Gr}(n-r, n+1) > \dim |K_X + (r+1)H|$, then by arguing in exactly the same way as above, we would conclude the analogous strict inequality for its iterated hyperplane slices. Again we reduce to the case of X a rational normal curve ($r = 1$), where such an inequality is clearly false. This completes the proof of Theorem C. □

6. Proof of Theorem D

{sec:proof_of_th

In this section, we extend the projection ramification map to vector bundles on nodal curves using limit linear series. We then use degeneration to a nodal curve to prove generic maximal variation for vector bundles on smooth curves.

{sec:lls}

6.1. **Limit linear series.** A linear series on a curve of rank r, degree d, and dimension k consists of a vector bundle E on the curve of rank r and degree d, and a k-dimensional subspace of the vector space $H^0(E)$. A limit linear series is an extension of this idea to singular curves, done in a manner suitable for degeneration techniques.

Let B be a DVR with special point 0 and general point η. Let $\pi\colon X \longrightarrow B$ be a family of connected projective curves of genus g, smooth over η, and at worst nodal over 0, with non-singular total space X. Assume the special fiber $X_0 = C$ is the nodal union of two curves C_1 and C_2 meeting at a unique point p. Then $\pi\colon X \longrightarrow B$ is a particularly simple example of an almost local smoothing family [Oss14, § 2.1–2.2].

We recall the notion of a limit linear series from [Oss14], where it is called linked linear series. In [Oss14], Osserman defines two types of linked linear series. In our setting, where

C has only two components, both notions coincide [Oss14, Remark 3.4.15]. We model our definition on the definition of the type II series.

Fix the following data:

(1) positive integers r, d, and k;
(2) integers d_1, d_2, and b satisfying $d_1 + d_2 - rb = d$;
(3) maps $\theta_v \colon \mathcal{O}_X \longrightarrow \mathcal{O}_X(C_v)$ for $v = 1, 2$ vanishing precisely on C_v;
(4) integers w_1, w_2 satisfying $w_v \equiv d_v \pmod r$ and $w_1 + w_2 = d$.

The integers r, d, and k will denote the rank, the degree, and the dimension of the linear series. The tuple $w = (w_1, w_2)$ will encode the multi-degree of the vector bundle in the limit linear series, and the integers d_1 and d_2 will encode its extremal twists. The maps θ_v are unique up to an element of $\mathcal{O}_B^*$. The choice of w_v, and θ_v is entirely auxilliary; different choices give isomorphisms between the corresponding moduli stacks of limit linear series. The choice of d_v and b is also largely auxilliary; increasing them leads to open inclusions between the corresponding moduli stacks of limit linear series.

Let S be a B-scheme, and let $\mathcal{E}$ be a vector bundle on X_S of rank r and degree d. For every $n \in \mathbf{Z}$, define the vector bundle $\mathcal{E}_n$ by

$$\mathcal{E}_n = \begin{cases} \mathcal{E} \otimes \mathcal{O}_X(C_1)^n & \text{if } n \geq 0, \\ \mathcal{E} \otimes \mathcal{O}_X(C_2)^{-n} & \text{if } n < 0. \end{cases}$$

Define maps

$$\theta_n \colon \mathcal{E}_m \longrightarrow \mathcal{E}_{m+n}$$

by

$$\theta_n = \begin{cases} \theta_1^n & \text{if } n \geq 0, \\ \theta_2^{-n} & \text{if } n < 0. \end{cases}.$$

We say that $\mathcal{E}$ has multi-degree w if for every $s \in S$ mapping to $0 \in B$, the degree of $\mathcal{E}|_s$ on C_v is w_v for $v = 1, 2$. Note that, if $\mathcal{E}$ has multi-degree (w_1, w_2), then $\mathcal{E}_n$ has multi-degree $(w_1 - rn, w_2 + rn)$.

Let $n_1 \in \mathbf{Z}$ be such that

$$(w_1 - n_1 r, w_2 + n_1 r) = (d_1, d_2 - rb),$$

and $n_2 \in \mathbf{Z}$ such that

$$(w_1 - n_2 r, w_2 + n_2 r) = (d_1 - rb, d_2).$$

Observe that $n_2 - n_1 = b$.

{def:lls}

Definition 6.1 (Special case of [Oss14, Definition 3.3.2]). Let S be a B-scheme. A *limit linear series* on X_S consists of $(\mathcal{E}, V_n \mid n \in \mathbf{Z})$, where $\mathcal{E}$ is a vector bundle of rank r, degree d, and multi-degree w on X_S, and V_n is a sub-bundle of $\pi_*\mathcal{E}_n$ of rank k satisfying the following conditions.

{lls:vanishing} (1) (Vanishing) For every $z \in S$ over $0 \in B$ and $v = 1, 2$, we have
$$H^0(C_v, \mathcal{E}_{n_v}|_{C_v}(-(b+1)p)) = 0.$$
(2) (Compatibility) For every $m, n \in \mathbf{Z}$, the map {lls:compatibili
$$\pi_*\theta_n \colon \pi_*\mathcal{E}_m \longrightarrow \pi_*\mathcal{E}_{m+n}$$
maps V_m to V_{m+n}.

The notion of a sub-bundle of a push-forward $\pi_*\mathcal{E}_n$ is as in [Oss14, Definition B.2.1], namely V_n is a vector bundle with a map $V_n \longrightarrow \pi_*\mathcal{E}_n$ which remains injective after arbitrary base-change.

Remark 6.2. In our case, the various twists of $\mathcal{E}$ are indexed by integers n. In general, the twists are indexed by a graph G_{II} that depends on the dual graph of X_0.

Denote by $\mathcal{G}^k_{r,d,d_*,w_*}(X/B)$ the category fibered over the category of B-schemes whose objects over S are the limit linear series on S of rank r, degree d, and multi-degree w, and whose morphisms are isomorphisms over S, defined in the obvious way.

{def:simple_lls}

Definition 6.3. Let $S = \operatorname{Spec} K$, where K is a field, and let $\lambda = (\mathcal{E}, V_n \mid n \in \mathbf{Z})$ be a limit linear series on S. We say λ is *simple* if there exist integers $w_1, \ldots, w_k$, not necessarily distinct, and elements $v_i \in V_{w_i}$ such that for every integer w, the images of $v_1, \ldots, v_k$ in V_w form a basis of V_w. Here the maps $V_{w_i} \longrightarrow V_w$ are as in Definition 6.1 (2) .

{rem:simple_lls}

Remark 6.4. By [Oss14, Lemma 3.4.14], it suffices to check the basis condition for $w = n_1$ and $w = n_2$.

Let $M_{r,d,w}(X/B)$ be the category fibered over the category of B-schemes whose objects over a B-scheme S are vector bundles $\mathcal{E}$ of rank r, degree d, and multi-degree w on X_S, and whose morphisms are isomorphisms over S. Let $M_{r,d,d_*,w_*}(X/B) \subset M_{r,d,w}(X/B)$ be the full-subcategory parametrizing bundles satisfying the vanishing condition in Definition 6.1 (1). Then $M_{r,d,w}(X/B)$ is an Artin stack over B, locally of finite type. By the semi-continuity of cohomology, $M_{r,d,d_*,w_*}(X/B) \subset M_{r,d,w}(X/B)$ is an open substack.

{thm:lls}

Theorem 6.5 ([Oss14, Theorem 3.4.7])**.** *Retain the notation above. Then $\mathcal{G}^k_{r,d,d_*,w_*}(X/B)$ is an Artin stack over B. The natural forgetful map*
$$\beta \colon \mathcal{G}^k_{r,d,d_*,w_*}(X/B) \longrightarrow M_{r,w,d_*}(X/B)$$
is representable by schemes, which are projective locally on the target. The locus of simple limit linear series is an open substack of $\mathcal{G}^k_{r,d,d_*,w_*}(X/B)$; it has universal relative dimension at least $k(d-k-r(g-1))$ over $M_{r,w,d_*}(X/B)$.

The last statement implies: if λ is a simple limit linear series such that the fiber of β through λ has dimension at most $k(d-k-r(g-1))$ at λ, then β is open at λ of relative dimension exactly $k(d-k-r(g-1))$.

Although Definition 6.1 requires specifying infinitely many vector bundles V_n, for $n \in \mathbf{Z}$, specifying finitely many determines the rest. Set $I = [n_1, n_2] \cap \mathbf{Z}$. Define an *$I$-linear series* to be the data of $(\mathcal{E}, V_n \mid n \in I)$ satisfying the conditions (1) and (2) in Definition 6.1 whenever the subscripts lie in I.

:lls_restricted}

Proposition 6.6. *The natural forgetful map from the groupoid of limit linear series to the groupoid of I-linear series is an equivalence.*

It is often enough to specify only the two extremal bundles, for $n = n_1$ and $n = n_2$, provided they satisfy certain compatibility conditions. This approach gives the original incarnation of the notion of limit linear series due to Eisenbud and Harris [**?**] for the rank 1 case and Teixidor i Bigas in the higher rank case [**?**].

Let $\mathcal{E}$ be a vector bundle on C of multi-degree w satisfying the vanishing condition in Definition 6.1.

{def:eht}

Definition 6.7 (Adapted from [Oss14, Definition 4.1.2])**.** Let $W_v \subset H^0(C, \mathcal{E}_{n_v}|_{C_v})$ be a k-dimensional subspace for $v = 1, 2$. We say $(\mathcal{E}, W_1, W_2)$ is an *EHT limit linear series* if the following conditions are satisfied.

{ieq:eht}

(1) If $a_1^v \leq \cdots \leq a_k^v$ is the vanishing sequence for $(\mathcal{E}_{n_v}|_{C_v}, W_v)$ at p for $v = 1, 2$, then for every $i = 1, \ldots, k$ we have
$$a_i^1 + a_{k+1-i}^2 \geq b.$$

{gluing:eht}

(2) There exist bases $s_1^v, \ldots, s_k^v$ for W_v for $v = 1, 2$, such that s_i^v has order of vanishing a_i^v at p, and if we have $a_i^1 + a_{k+1-i}^2 = b$ for some i, then
$$\widetilde{\phi}(s_i^1) = s_{k+1-i}^2,$$
where $\widetilde{\phi}\colon \mathcal{E}_{n_1}(-a_i^1 p)|_p \longrightarrow \mathcal{E}_{n_2}(-a_{k+1-i}^2 p)|_p$ is the isomorphism obtained by taking the appropriate twist of the identity map.

We say $(\mathcal{E}, W_1, W_2)$ is *refined* if all the inequalities in (1) are equalities.

Due to the vanishing condition, the restriction map
$$H^0(C, \mathcal{E}_{n_v}) \longrightarrow H^0(C_v, \mathcal{E}_{n_v}|_{C_v})$$
is an isomorphism. Via this isomorphism, we sometimes treat W_v as a subspace of $H^0(C_v, \mathcal{E}_{n_v}|_{C_v})$.

It is possible to define a stack of EHT limit linear series so that the locus of refined EHT limit linear series forms an open substack [Oss14, § 4].

Let $\lambda = (\mathcal{E}, V_n \mid n \in \mathbf{Z})$ be a limit linear series on C in the sense of Definition 6.1. Set $W_1 = V_{n_1}$ and $W_2 = V_{n_2}$.

{prop:llseht}

Proposition 6.8. *In the notation above, $(\mathcal{E}, W_1, W_2)$ is an EHT limit linear series on C. Conversely, given an EHT limit linear series $\mu = (\mathcal{E}, W_1, W_2)$, there exists a limit linear* *series* $\lambda = (\mathcal{E}, V_n)$ *on* C *such that* $W_1 = V_{n_1}$ *and* $W_2 = V_{n_2}$*. Furthermore, if* μ *is refined, then* λ *is unique and simple.*

Proof. This is a point-wise version of the stack theoretic statement [Oss14, Theorem 4.3.4], plus the equivalence of type I and type II series in the two component case ([Oss14, Remark 3.4.15]).

The assertion about refined series follows from the proof of [Oss14, Theorem 4.3.4], but it is not explicitly stated there. So we offer a proof.

Let μ be a refined EHT limit linear series. We now construct $V_n \subset H^0(\mathcal{E}_n)$. By Proposition 6.6, it suffices to take $n \in [n_1, n_2]$.

By composing the restriction $\mathcal{E}_n \longrightarrow \mathcal{E}_n|_{C_v}$ and the inclusion $\mathcal{E}_n|_{C_v} \longrightarrow \mathcal{E}_{n_v}|_{C_v}$, we get a map

$$\iota\colon H^0(C, \mathcal{E}_n) \longrightarrow H^0(C_1, \mathcal{E}_{n_1}|_{C_1}) \oplus H^0(C_2, \mathcal{E}_{n_2}|_{C_2}).$$

The vanishing condition in Definition 6.1 shows ι is an injection. The compatibility condition in Definition 6.1 implies $\iota(V_n) \subset W_1 \oplus W_2$. Therefore, the subspace $V_n \subset H^0(C, \mathcal{E}_n)$ lies in the kernel of the map

$$\bar{\iota}\colon H^0(C, \mathcal{E}_n) \longrightarrow H^0(C_1, \mathcal{E}_{n_1}|_{C_1})/W_1 \oplus H^0(C_2, \mathcal{E}_{n_2}|_{C_2})/W_2. \tag{6.1}$$ {eqn:iotabar}

Next we show:

$$\dim \ker \bar{\iota} = k. \tag{6.2}$$ {eqn:keriotabar}

Suppose $s \in \ker \bar{\iota}$. Then $\iota(s)$ is a linear combination of $(s_1^1, 0), \ldots, (s_k^1, 0), (0, s_1^2), \ldots, (0, s_k^2)$. Writing $\iota(s) = (s_1, s_2)$, we have

$$\operatorname{ord}_p(s_1) \geq n - n_1 \text{ and } \operatorname{ord}_p(s_2) \geq n_2 - n. \tag{6.3}$$ {eqn:vanishing}

Recall that $a_1^v \leq \cdots \leq a_k^v$ is the vanishing sequence of W_v for $v = 1, 2$. Let i be the smallest such that

$$a_i^1 \geq n - n_1,$$

and let $i + c$ be the smallest such that

$$a_{i+c}^1 > n - n_1.$$

Since μ is a refined series and $n_2 - n_1 = b$, we get $j = k + 1 - i$ is the largest such that

$$a_j^2 \leq n_2 - n,$$

and $j - c$ the largest such that

$$a_{j+c}^2 < n_2 - n.$$

The vanishing conditions (6.3) imply $\iota(s)$ must in fact be a linear combination of $(s_i^1, 0), \ldots, (s_k^1, 0), (0, s_{k-i-c}^2), \ldots$
But since s_v for $v = 1, 2$ are the restriction to C_v of a section on C, they satisfy a gluing condition at p. Write

$$\iota(s) = \sum_{\ell=i}^{k} \alpha_\ell (s_\ell^1, 0) + \sum_{\ell=j}^{k} \beta_\ell (0, s_\ell^2),$$

where α_ℓ, β_ℓ are in the base-field. By the condition (2) in Definition 6.7, the gluing condition for s_1 and s_2 is equivalent to

$$\alpha_\ell = \beta_{k+1-\ell}$$

for $\ell = i, \dots, i+c-1$. Hence $\iota(s)$ must be a linear combination of the k elements

$$(s^1_i, s^2_{k+1-i}), \dots, (s^1_{i+c-1}, s^2_{k+2-i-c}), (s^1_{i+c}, 0), \dots, (s^1_k, 0), (0, s^2_{k+2-i}), \dots, (s^2_k, 0).$$

It follows that $\ker \bar{\iota}$ is k-dimensional.

Since $\ker \bar{\iota}$ is k-dimensional, there is a unique possible choice for V_n, namely $V_n = \ker \bar{\iota}$. It is easy to check that with this choice, the compatibility condition in Definition 6.1 is satisfied. Therefore, we get a limit linear series λ whose associated EHT limit linear series is μ.

It remains to show λ is simple. For $i = 1, \dots, k$, set $n_i = n - n_1 - a^1_i$, and let $s_i \in V_{n_i} \subset H^0(C, \mathcal{E}_{n_i})$ to be the section whose image under ι is (s^1_i, s^2_{k+1-i}). Then the images of $s_1, \dots, s_k$ form a basis of $V_{n_1} = W_1$ and $V_{n_2} = W_2$. By Remark 6.4, we conclude λ is simple. □

{sec:prnodal}

6.2. **Projection-ramification for nodal curves.** Let C be a smooth curve and $p \in C$ a point. Let E be a vector bundle on C of rank r. The projective spaces $\mathbf{P}E(np)|_p$, for $n \in \mathbf{Z}$, are canonically isomorphic to each other, so we identify them.

Suppose $\lambda \subset H^0(C, E)$ is an $(r+1)$-dimensional subspace with the vanishing sequence

{eqn:specialvs}
$$(\underbrace{a, \dots, a}_{i}, \underbrace{a+1, \dots, a+1}_{r+1-i}), \tag{6.4}$$

for some i with $1 \leq i \leq r$, and $a \geq 0$. Let $\Lambda_0 \subset E|_p \cong E(-ap)|_p$ be the image of $\lambda(-ap)$, and $\Lambda_1 \subset E|_p \cong E(-(a+1)p)|_p$ the image of $\lambda(-(a+1)p)$. Then $\dim \Lambda_0 = i$ and $\dim \Lambda_1 = r+1-i$. Assume that Λ_0 and Λ_1 satisfy the following genericity condition

{eq:genericity}
$$\dim(\Lambda_0 \cap \Lambda_1) = 1. \tag{6.5}$$

Recall that the ramification $R(\lambda)$ is a section of $E \otimes \det E \otimes T_C$.

{prop:agreement}

Proposition 6.9. *In the setup above, $R(\lambda)$ vanishes to order $(r+1)a + (r-i)$ at p. Write $\widetilde{R} = R(\lambda)/t^{(r+1)a+r-i}$, where t is a uniformizer at p. Then, the one-dimensional subspace of $E|_p$ spanned by $\widetilde{R}|_p$ is $\Lambda_0 \cap \Lambda_1$.*

Proof. Let $\langle s_1, \dots, s_r \rangle$ be a local trivialization for E in an open set around p such that in these local coordinates, we have

$$\lambda = \{t^a s_1, \dots, t^a s_i, t^{a+1} s_i, t^{a+1} s_{i+1}, \dots, t^{a+1} s_r\}.$$

In these coordinates, $R(\lambda)$ is given by

$$\begin{aligned} R(\lambda) &= \det \begin{pmatrix} t^a & & & & & & at^{a-1}s_1 \\ & t^a & & & & & at^{a-1}s_2 \\ & & \ddots & & & & \vdots \\ & & & t^a & & & at^{a-1}s_i \\ & & & t^{a+1} & & & (a+1)t^a s_i \\ & & & & \ddots & & \vdots \\ & & & & & t^{a+1} & (a+1)t^a s_r \end{pmatrix} \\ &= (-1)^{r-i} t^{(r+1)a+r-i} \cdot s_i. \end{aligned}$$

Since s_i spans $\Lambda_0 \cap \Lambda_1$, the proof is complete. $\square$

Let $\pi\colon X \longrightarrow B$ be a family as in § 6.1 with X_η smooth and $X_0 = C$ a nodal union $C = C_1 \cup_p C_2$, with $g(C_v) = g_v$ for $v = 1, 2$. Fix r, d, d_1, d_2, b, w_1, w_2, θ_1, and θ_2 as in § 6.1, and take $k = r + 1$. Set

$$\begin{aligned} r' &= r, \\ d' &= d + r(d - 2g + 2), \\ d_1' &= d_1 + r(d_1 - 2g_1 + 1), \\ d_2' &= d_2 + r(d_2 - 2g_2 + 1), \\ b' &= b(r+1), \\ w_1' &= w_1 + r(w_1 - 2g_1 + 1), \\ w_2' &= w_2 + r(w_1 - 2g_1 + 1), \\ k' &= 1. \end{aligned}$$

Defining n_1' and n_2' analogously to n_1 and n_2, we get

$$\begin{aligned} n_1' &= n_1(1 + r), \\ n_2' &= n_2(1 + r). \end{aligned}$$

We define a rational map

$$\mathcal{R}\colon \mathcal{G}^{r+1}_{r,d,d_*,w_*}(X/B)^{\mathrm{red}} \dashrightarrow \mathcal{G}^1_{r',d',d'_*,w'_*}(X/B) \tag{6.6}$$ {eq:Rtilde}

that extends the projection-ramification map on the generic fiber. Let $\mathcal{U} \subset \mathcal{G}^{r+1}_{r,d,d_*,w_*}(X/B)$ be the open substack obtained by excluding the following closed loci:

(1) the closure of the locus of linear series $(\mathcal{E}, \lambda)$ on the generic fiber X_η such that $\lambda \otimes \mathcal{O}_{X_\eta} \longrightarrow \mathcal{E}$ has generic rank less than r,
(2) the locus of limit linear series $\lambda = (\mathcal{E}, V_n)$ on the central fiber such that the associated EHT limit linear series $\mu = (\mathcal{E}, W_1, W_2)$ is not refined, or does not have vanishing sequences as in (6.4), or does not satisfy the genericity condition $\dim(\Lambda_0 \cap \Lambda_1) = 1$ as in (6.5).

Let S be a B scheme with a map to $\mathcal{U}$ given by the limit linear series $(\mathcal{E}, V_n)$. On X_S, we have the diagram

$$\begin{tikzcd} & \det \mathcal{E}_n^* \otimes \det V_n \arrow[r, "j"] \arrow[d, "d"] & V_n \otimes \mathcal{O}_{X_S} \arrow[r, "i"] \arrow[d, "\widetilde{i}"] & \mathcal{E}_n \arrow[d, equal] & \\ 0 \arrow[r] & \Omega_{X_S/S} \otimes \mathcal{E}_n \arrow[r] & j_1\mathcal{E}_n \arrow[r] & \mathcal{E}_n \arrow[r] & 0. \end{tikzcd} \tag{6.7}$$ {eq:llspr}

In the top row, the map i is induced by the inclusion $V_n \longrightarrow \pi_*\mathcal{E}_n$ on S, and the map j is given by $j = \wedge^r i^* \otimes \det V_n$. In the bottom row, the sheaf $j_1\mathcal{E}_n$ is the first jet bundle of $\mathcal{E}_n$ along $X_S \longrightarrow S$, and the row is the natural jet bundle exact sequence. The map $\widetilde{i}$ is the canonical lift of i, and the map d is the unique induced map owing to $i \circ j = 0$. By composing d through the inclusion $\Omega_{X_S/S} \longrightarrow \omega_{X_S/S}$, and rearranging the line bundles, we obtain a map

{eqn:Rn}
$$R_n \colon \det V_n \longrightarrow \pi_*(\mathcal{E}_n \otimes \det \mathcal{E}_n \otimes \omega^*_{X_S/S}). \tag{6.8}$$

Set $\mathcal{E}' = \mathcal{E} \otimes \det \mathcal{E} \otimes \omega^*_{X_S/S}$. We want to say that the sections given by R_n of the various twists of $\mathcal{E}'$ define a limit linear series of dimension 1. The catch is that the twists $\mathcal{E}_n \otimes \det \mathcal{E}_n \otimes \omega^*_{X_S/S}$ are only *some* of the twists of $\mathcal{E}'$. However, on the open set $\mathcal{U}$, this is more than enough information—just looking at the extremal twists suffices. Observe that we have $\mathcal{E}_{n_v} \otimes \det \mathcal{E}_{n_v} \otimes \omega_{X_S/S} = \mathcal{E}'_{n'_v}$ for $v = 1, 2$.

{prop:mapreduced}

Proposition 6.10. *In the setup above, assume that S is reduced. Then there is a unique simple limit linear series $(\mathcal{E}', L_n)$ of dimension 1 such that $L_{n'_v} = \det V_{n_v}$ and the map $L_{n'_v} \longrightarrow \pi_*\mathcal{E}'_{n'_v}$ is given by R_n for $v = 1, 2$.*

Proof. First, suppose S is a point mapping to $0 \in B$. The key is that the two extremal sections $L_{n_v} \longrightarrow H^0(\mathcal{E}_{n_v} \otimes \det \mathcal{E}_{n_v} \otimes \omega^*_{X_S/S})$ for $v = 1, 2$ define a refined EHT limit linear series. To see this, suppose $(\mathcal{E}_{n_1}|_{C_1}, V_{n_1})$ has vanishing sequence

$$(\underbrace{a, \dots, a}_{i}, \underbrace{a+1, \dots, a+1}_{r+1-i}),$$

at p for some i with $1 \leq i \leq r$ and $a \geq 0$. Then $(\mathcal{E}_{n_2}|_{C_2}, V_{n_2})$ has the vanishing sequence

$$(\underbrace{b-a-1, \dots, b-a-1}_{r+1-i}, \underbrace{b-a, \dots, b-a}_{i}).$$

By construction, the section R_{n_v} restricted to C_v is the ramification of $(\mathcal{E}_{n_v}|_{C_v}, V_{n_v})$ composed with the inclusion $\omega^*_C|_{C_v} \longrightarrow \Omega^*_{C_v}$. Therefore, by Proposition 6.9, R_{n_1} vanishes at p to order $a'_1 = a(r+1) + (r-i) + 1$ and R_{n_2} to order $a'_2 = (b-a-1)(r+1) + i$. Since $a'_1 + a'_2 = b'$, we have the equality required in condition (1) of Definition 6.7. Note that the spaces Λ_0 and Λ_1 are exchanged when we switch from C_1 to C_2, and so their intersection $\Lambda_0 \cap \Lambda_1$ remains the same. By Proposition 6.9, after dividing by the appropriate power of the uniformizer, the sections R_{n_1} and R_{n_2} at p are proportional; they both span $\Lambda_0 \cap \Lambda_1$. Hence, we also have the gluing condition in required in (2) in Definition 6.7.

For a general S, note that for every point $s \in S$, the map

$$H^0(X_s, \mathcal{E}'_n|_s) \longrightarrow H^0(X_s, \mathcal{E}'_{n_1}|_s)/L_{n_1}|_s \oplus H^0(X_s, \mathcal{E}'_{n_2}|_s)/L_{n_2}|_s$$

has kernel of dimension 1. If s lies over the generic point of Δ, then this is automatic. If s lies over the special point of Δ, then this follows from the fact that $(\mathcal{E}', L_{n_1}, L_{n_2})$ is a refined EHT linear series; see (6.2). Since S is reduced, we conclude that the kernel of the map

$$\pi_*(\mathcal{E}'_n) \longrightarrow \pi_*(\mathcal{E}'_{n_1})/L_{n_1} \oplus \pi_*(\mathcal{E}'_{n_2})/L_{n_2}$$

is a line bundle, say L_n, and the inclusion $L_n \longrightarrow \pi_*(\mathcal{E}'_n)$ is a sub-bundle map. The data $(\mathcal{E}', L_n)$ is the unique simple limit linear series claimed in the statement. □

Remark 6.11. A simple limit linear series of dimension 1 on a vector bundle $\mathcal{E}'$ on C is simply a section of one of its twists that is non-zero on both components of C. From Proposition 6.9, we see that this twist is $\mathcal{E}'_{n_1+m}$ where

$$m = (r+1)a + ar - i.$$

In particular, it is not one of the twists $\mathcal{E}'_{n_1+r(n-n_1)} = \mathcal{E}_n \otimes \det \mathcal{E}_n \otimes \omega^*$ in (6.8) that receive the image of the ramification section R_n.

Thanks to Proposition 6.10, we have a morphism

$$\mathcal{R} \colon \mathcal{U}^{\mathrm{red}} \longrightarrow \mathcal{G}^1_{r',d',d'_*,w'_*(X/B)} \tag{6.9}$$ {eqn:llsR}

defined by

$$(\mathcal{E}, V_n) \mapsto (\mathcal{E}', L_n).$$

6.3. **Maximal variation.** Let E be an ample vector bundle on $\mathbf{P}^1$ of rank r. Fix a point $p \in \mathbf{P}^1$. Consider the locally closed subset $U \subset \mathbf{Gr}(r+1, H^0(E))$ consisting of linear series with vanishing sequence

$$(0, \underbrace{1, \dots, 1}_{r})$$

over p. Given $\lambda \in U$, let $\widetilde{R}(\lambda) \in \mathbf{P}H^0(E \otimes \det E \otimes T_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p)$ be the reduced ramification divisor; see Proposition 6.9. The assignment $\lambda \mapsto \widetilde{R}(\lambda)$ gives a *reduced* projection-ramification map

$$\widetilde{R} \colon U \longrightarrow \mathbf{P}H^0(E \otimes \det E \otimes T_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p)) \tag{6.10}$$ {eqn:rrd}

between varieties of the same dimension.

Given a one-dimensional subspace $\ell \subset E|_p$, define E'_ℓ by the exact sequence

$$0 \longrightarrow E'_\ell \longrightarrow E \longrightarrow E|_p/\ell \longrightarrow 0.$$

There exists a Zariski open subset of $\mathbf{P}_{\mathrm{sub}}(E|_p)$, such that for all ℓ in this set, the isomorphism class of E'_ℓ remains constant. Denote this isomorphism class by E'_{gen}.

{prop:domred}

Proposition 6.12. *If the usual projection-ramification map*

$$R\colon \mathbf{Gr}(r+1, H^0(E'_{\mathrm{gen}})) \dashrightarrow \mathbf{P}H^0(E'_{\mathrm{gen}} \otimes \det E'_{\mathrm{gen}} \otimes T_{\mathbf{P}^1})$$

is dominant, then so is the reduced projection-ramification map

$$\widetilde{R}\colon U \longrightarrow \mathbf{P}H^0(E \otimes \det E \otimes T_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p)).$$

Proof. Let $D \in \mathbf{P}H^0(E \otimes \det E \otimes T_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p))$ be a generic section. Let $\ell \subset E|_p$ be the one-dimensional subspace defined by $D|_p$, and set $E' = E'_\ell$. Since D is generic, we may assume $E' \cong E'_{\mathrm{gen}}$. We have the inclusion

$$E' \otimes \det E' \otimes T_{\mathbf{P}^1} \longrightarrow E \otimes \det E \otimes \mathcal{O}(-(r-1)p) \otimes T_{\mathbf{P}^1},$$

and by construction D is the image of a section $D' \in \mathbf{P}H^0(E' \otimes \det E' \otimes T_{\mathbf{P}^1})$. Since R is dominant for E', there exists a sequence of subspaces $\lambda'_n \in \mathbf{Gr}(r+1, H^0(E'))$ such that $R(\lambda'_n)$ limit to D'. Let $\lambda_n \subset \mathbf{Gr}(r+1, H^0(E))$ be the image of λ'_n. Then $\widetilde{R}(\lambda_n)$ limit to D. Since D was generic, we get that $\widetilde{R}$ is dominant. □

:domredexamples}

Corollary 6.13. *The reduced projection-ramification map is dominant for the bundles $E = \mathcal{O}(1) \oplus \mathcal{O}(2)^{r-1}$ and $E = \mathcal{O}(2) \oplus \mathcal{O}(3)^{r-1}$.*

Proof. Follows from Proposition 6.12 and that the projection-ramification map is dominant for $E' = \mathcal{O}(1)^r$ and $E' = \mathcal{O}(2)^r$. □

For $v = 1, 2$, let C_v be a smooth curve and E_v a vector bundles of rank r on C_v. Let $p_v \in C_v$ be a point. Suppose $\lambda_1 \in \mathbf{Gr}(r+1, H^0(E_1))$ is a linear series with vanishing sequence $(0, \dots, 0, 1)$ at p_1, and $\lambda_2 \in \mathbf{Gr}(r+1, H^0(E_2))$ is a linear series with vanishing sequence $(0, 1, \dots, 1)$ at p_2.

Let C be the nodal union of C_1 and C_2 formed by identifying p_1 and p_2. We construct a simple limit linear series λ on C of rank r and degree $\deg E_1 + \deg E_2 - r$. Choose an isomorphism $\phi\colon E_1(-p)|_{p_1} \longrightarrow E_2|_{p_2}$ that sends the image of $\lambda_1(-p)$ in $E_1(-p)|_{p_1}$ to the image of λ_2 in $E_2|_{p_2}$. Let $\mathcal{E}$ be the vector bundle on C constructed by gluing $E_1(-p)$ and E_2 by ϕ. Let $b = 2m$ be large enough so that $H^0(E_1(-mp)) = 0$ and $H^0(E_2(-mp)) = 0$. Set $d_1 = \deg E_1 + (m-1)r$ and $d_2 = \deg E_2 + mr$. Then $w_1 = d_1 - r$, so $n_1 = m$; and $w_2 = d_2$, so $n_2 = -m$. Let $V_{n_1} \subset H^0(C, \mathcal{E}_{n_1})$ be the subspace that restricts to $\lambda_1((m-1)p) \subset H^0(C_1, E_1((m-1)p))$ and $V_{n_2} \subset H^0(C, \mathcal{E}_{n_2})$ the subspace that restricts to $\lambda_2(mp) \subset H^0(C_2, E_2(mp))$. Then the vanishing sequence of V_{n_1} is $(m-1, \dots, m-1, m)$ and that of V_{n_2} is $(m, m+1, \dots, m+1)$. By the choice of ϕ, we see that the two series are compatible at the node, and hence define a refined EHT limit linear series on C. Let λ be the associated unique simple limit linear series.

Let $X \longrightarrow B$ be a smoothing of C, and $\mathscr{E}$ a vector bundle on X whose restriction to $X_0 = C$ is $\mathcal{E}$. Set $d = d_1 + d_2 - r$ and $g = g(C_1) + g(C_2)$.

prop:attachtail}

Proposition 6.14. *Suppose λ_v is isolated in their respective projection-ramification maps, for $v = 1, 2$. Then λ is isolated in the projection-ramification map $\mathcal{R}$. Suppose, furthermore, that the dimension of the fiber of the forgetful map β at λ is $(r+1)(d-1-rg)$. Then the projection-ramification map is generically finite for the vector bundle $\mathscr{E}_\eta$ on X_η.*

Proof. The projection-ramification map for λ reduces to the projection-ramification map for λ_v (up to twists) on components C_v. So if both λ_v are isolated in the fibers of their projection-ramification maps, so is λ.

If the dimension condition holds, then β is open at λ by Theorem 6.5. In particular, λ is in the closure of $\mathbf{Gr}(r+1, H^0(X_\eta, \mathscr{E}_\eta))$. By the semi-continuity of fiber dimension, it follows that the projection-ramification map

$$R\colon \mathbf{Gr}(r+1, H^0(X_\eta, \mathscr{E}_\eta)) \longrightarrow \mathbf{P} H^0(X_\eta, \mathscr{E}_\eta \otimes \det \mathscr{E}_\eta \otimes \omega^*_{X_\eta})$$

is generically finite. □

{thm:prp1}

Theorem 6.15. *Let E be a generic vector bundle on $\mathbf{P}^1$ of rank r and degree $d = a(r-1)+b(2r-1)+1$, where a, b are positive integers. Then the projection-ramification map is generically finite, and hence dominant, for E. In particular, the projection-ramification map is dominant for generic E of degree $\geq (r-1)(2r-1)+1$.*

Proof. Let E be a generic vector bundle of rank $d \geq 0$ such that the projection-ramification map is dominant for E. Set $C_1 = \mathbf{P}^1$ and $E_1 = E$. Take $C_2 = \mathbf{P}^1$ and $E_2 = \mathcal{O}(1) \oplus \mathcal{O}(2)^{r-1}$ or $E_2 = \mathcal{O}(2) \oplus \mathcal{O}(3)^{r-1}$. Pick points $p_v \in C_v$ for $v = 1, 2$. Let $\lambda_1 \subset H^0(E_1)$ be an $(r+1)$-dimensional subspace with vanishing sequence $(0, \dots, 0, 1)$ at p_1, and $\lambda_2 \subset H^0(E_2)$ an $(r+1)$-dimensional subspace with vanishing sequence $(0, 1, \dots, 1)$. Assume that λ_v are isolated in the respective fibers of their projection-ramification maps. Furthermore, assume that the image ℓ of λ_2 in $E_2|_{p_2}$ is generic in the sense that the kernel of

$$E_2 \longrightarrow E_2|_{p_2}/\ell$$

is the generic vector bundle E_2^{gen} (which will be either $\mathcal{O}(1)^r$ or $\mathcal{O}(2)^r$). Let λ be the limit linear series on $C = C_1 \cup C_2$ constructed from λ_1 and λ_2 as above. Then, we get

$$\begin{aligned}
\dim_\lambda \beta^{-1}(\beta(\lambda)) &= \dim \mathbf{Gr}(r+1, H^0(E_1)) + \dim \mathbf{Gr}(r+1, H^0(E_2^{\text{gen}})) \\
&= (r+1)(\deg E + \deg E_2^{\text{gen}} - 2) \\
&= (r+1)(\deg E + \deg E_2 - r - 1) \\
&= (r+1)(\deg \mathcal{E} - 1).
\end{aligned}$$

Here is how the dimension count goes: it suffices to count dimensions for the refined EHT limit linear series associated to λ, since λ can be recovered uniquely from it. For the EHT limit linear series, we begin by choosing an $(r+1)$-dimensional subspace of $H^0(E_1)$, giving us the first term in the dimension count. This choice gives a one-dimensional subspace $\Lambda_1 \in E_1|_{p_1} = \mathcal{E}|_p = E_2|_{p_2}$. We must choose an $(r+1)$-dimensional subspace of $H^0(E_2)$

with the complementary vanishing sequence and satisfying the compatibility condition over p. These two conditions force it to be a subspace of $H^0(E_2')$, where E_2' is the kernel of

$$E_2 \longrightarrow E_2|_{p_2}/\Lambda_1.$$

Since the kernel is isomorphic to E_2^{gen}, we get the second term in the dimension count.

Let $\pi\colon X \longrightarrow B$ be a smoothing of B. Every vector bundle on C is the restriction of a vector bundle on X. Indeed, this is clearly true for line bundles on C, and hence for vector bundles of arbitrary rank, as these are direct sums of line bundles. Let $\mathscr{E}$ be a vector bundle on X whose restriction to C is $\mathcal{E}$. By Proposition 6.14, the projection-ramification map is generically finite, and hence dominant, for $\mathcal{E}_\eta$. By the semi-continuity of fiber dimension, the same is true for a generic vector bundle of rank r and degree $\deg E + \deg E_2 - r$. The two choices of E_2 give $\deg E_2 - r = r - 1$ and $\deg E_2 - r = 2r - 1$.

In summary, dominance for a generic bundle of rank r and degree d implies the same for a generic bundle of rank r and degree $d + r - 1$ and degree $d + 2r - 1$. Starting with the base case $d = 1$, namely $E = \mathcal{O}^{r-1} \oplus \mathcal{O}(1)$, we obtain the statement by induction. □

6.4. **Maximal variation for $\mathcal{O}(2)^r$.** The goal of this section is to establish dominance of the projection ramification morphism for $E = \mathcal{O}(2)^r$. We do this by a tangent space calculation. For simplicity, we work with inhomogeneous polynomials in $x = X/Y$ instead of homogeneous polynomials in X and Y.

Consider the point λ of $\mathbf{Gr}(r+1, H^0(E))$ represented by the $(r+1) \times r$ matrix

$$\Lambda = \begin{pmatrix} (x-a_1)^2 & 0 & \cdots & 0 \\ 0 & (x-a_2)^2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & (x-a_r)^2 \\ p_1 & p_2 & \cdots & p_r \end{pmatrix},$$

where $a_i \in \mathbf{C}$ and $p_j \in H^0(\mathcal{O}(2))$. We claim that if the a_i and the p_j are generic, then the map on the tangent spaces induced by the projection-ramification construction is surjective at Λ.

Recall that if M is an $(r+1) \times r$ matrix of polynomials in x, then the ramification divisor of the projection map represented by M is given by the formula

$$R(M) = \det\left(M \mid \xi(M)\right),$$

where $\xi(M)$ is the vector given by

$$\xi(M)_i = \sum_{j=1}^{r} \partial_x M_{i,j} \cdot X_j.$$

To do the tangent space computation, we compute the ramification divisor R for the matrix $M = \Lambda + \epsilon\Delta$, where Δ is an $(r+1) \times r$ matrix of elements in $H^0(\mathcal{O}(2))$, assuming

$\epsilon^2 = 0$. The result will be of the form

$$R(\Lambda + \epsilon\Delta) = R(\Lambda) + \epsilon S(\Lambda, \Delta),$$

where $S(R, \Delta)$ is an element of $H^0(2r) \otimes \langle X_1, \dots, X_r \rangle$, linear in the entries of Δ. We must show that the linear map

$$H^0(\mathcal{O}(2)) \otimes M_{r+1,r} \longrightarrow H^0(\mathcal{O}(2r)) \otimes \langle X_1, \dots, X_r \rangle$$

given by

$$\Delta \mapsto S(\Lambda, \Delta)$$

is surjective.

We compute $R(\Lambda + \epsilon\Delta)$ for elementary matrices Δ. Denote by $E_{i,j}$ the elementary matrix with 1 at the (i, j)th place, and 0 everywhere else.

First, suppose

$$\Delta = qE_{j,i},$$

where $i \neq j$, and $1 \leq j \leq r$. By direct calculation, we obtain

$$S(\Lambda, \Delta) = \frac{(x - a_1)^2 \cdots (x - a_r)^2 p_j}{(x - a_i)^2 (x - a_j)^2} \cdot \left[q, (x - a_i)^2\right] \cdot X_i, \tag{6.11}$$

{eq:off_diagonal

where the notation $[a, b]$ means $ab' - a'b$.

Second, suppose

$$\Delta = qE_{r,i},$$

where $1 \leq i \leq r$. Again, by direct calculation, we obtain

$$S(\Lambda, \Delta) = -\frac{(x - a_1)^2 \cdots (x - a_r)^2}{(x - a_i)^2} \cdot \left[q, (x - a_i)^2\right] \cdot X_i. \tag{6.12}$$

{eq:bottom}

Third, suppose

$$\Delta = qE_{i,i},$$

where $1 \leq i \leq r$. As before, by direct calculation, we obtain

$$S(\Lambda, \Delta) = R(\Lambda_i(q)), \tag{6.13}$$

{eq:diagonal}

where $\Lambda_i(q)$ is obtained from Λ by changing the (i, i)th entry from $(x - a_i)^2$ to q.

We want to show that the map

$$H^0(\mathcal{O}(2)) \otimes M_{(r+1)\times r} \longrightarrow H^0(\mathcal{O}(2r)) \otimes \langle X_1, \dots, X_r \rangle \tag{6.14}$$

{eqn:mainmap}

given by

$$\Delta \mapsto S(\Lambda, \Delta)$$

is surjective. Fix a i with $1 \leq i \leq r$ and consider the subspace of the domain given by

$$H^0(\mathcal{O}(2)) \otimes \langle E_{j,i} \mid j \neq i \rangle.$$

By our calculations above, the image of this space lies in $H^0(\mathcal{O}(2r)) \otimes X_i$. We begin by identifying the image. For $1 \leq j \leq r$ and $j \neq i$, set

$$Q_{i,j} = \frac{(x-a_1)^2 \cdots (x-a_r)^2 p_j}{(x-a_i)^2(x-a_j)^2}$$

and set

$$Q_{i,r+1} = \frac{(x-a_1)^2 \cdots (x-a_r)^2}{(x-a_i)^2}.$$

o_linear_syzygy}

Lemma 6.16. *For generic $p_1, \dots, p_r$, there is no non-trivial linear relation among the polynomials $Q_{i,j}$ for $j \in \{1, \dots, r+1\} \setminus \{i\}$.*

Proof. Suppose we had a linear relation

$$\sum l_j Q_{i,j} = 0.$$

Divide throughout by $\frac{(x-a_1)^2 \dots (x-a_r)^2}{(x-a_i)^2}$. Then we get the relation

$$\sum_{j=1}^{r} l_j \frac{p_j}{(x-a_j)^2} + l_{r+1} = 0.$$

If $l_j \neq 0$ for some j with $1 \leq j \leq r$, then we have a pole on the left hand side at $x = a_j$, but not on the right hand side; a contradiction. Therefore, we must have $l_j = 0$ for all j with $1 \leq j \leq r$, and hence also $l_{r+1} = 0$. Thus, the relation was trivial. □

{lem:jacobian}

Lemma 6.17. *The image of the map*

$$H^0(\mathcal{O}(2)) \longrightarrow H^0(\mathcal{O}(2))$$

given by

$$q \mapsto [q, (x-a)^2]$$

is

$$(x-a) \cdot H^0(\mathcal{O}(1)).$$

Proof. Straightforward. □

{lem:imageXi}

Lemma 6.18. *The image of the map*

$$H^0(\mathcal{O}(2)) \otimes \langle E_{j,i} \mid j \in \{1, \dots, r+1\} \setminus \{i\} \rangle \longrightarrow H^0(\mathcal{O}(2r)) \otimes X_i$$

is

$$(x-a) \cdot H^0(\mathcal{O}(2r-1)) \otimes X_i.$$

Proof. By the computation in (6.11) and (6.12) and Lemma 6.17 the image of the map above is the same as the image of the multiplication map

$$\langle Q_{i,j} \mid j \in \{1, \dots, r+1\} \setminus \{i\} \rangle \otimes (x-a) \cdot H^0(\mathcal{O}(1)) \longrightarrow H^0(\mathcal{O}(2r)) \otimes X_i.$$

By Lemma 6.16, the map

$$\langle Q_{i,j} \mid j \in \{1, \dots, r+1\} \setminus \{i\} \rangle \otimes H^0(\mathcal{O}(1)) \longrightarrow H^0(\mathcal{O}(2r-1))$$

is injective. Since both sides have dimension $2r$, the map is an isomorphism. The proof is now complete. $\square$

By Lemma 6.18, the cokernel of the map

$$H^0(\mathcal{O}(2)) \otimes \langle E_{j,i} \mid j \in \{1, \dots, r+1\} \setminus \{i\}\rangle \longrightarrow H^0(\mathcal{O}(2r)) \otimes X_i$$

is $\mathbf{C} \otimes X_i$, where the map $H^0(\mathcal{O}(2r)) \longrightarrow \mathbf{C}$ is the evaluation at a_i. Putting all these maps together for various i, we get that the cokernel of the map

$$H^0(\mathcal{O}(2)) \otimes \langle E_{j,i} \mid j \neq i\rangle \longrightarrow H^0(\mathcal{O}(2r)) \otimes \langle X_1, \dots, X_r\rangle$$

is $\mathbf{C} \otimes \langle X_1, \dots, X_r\rangle$, where the map

$$H^0(\mathcal{O}(2)) \otimes \langle X_1, \dots, X_r\rangle \longrightarrow \mathbf{C} \otimes \langle X_1, \dots, X_r\rangle \tag{6.15}$$ {eqn:remaining}

is given on $H^0(\mathcal{O}(2r)) \otimes X_i$ by the evaluation at a_i.

It remains to show that the map

$$H^0(\mathcal{O}(2)) \otimes \langle E_{i,i}\rangle \longrightarrow \mathbf{C} \otimes \langle X_1, \dots, X_r\rangle, \tag{6.16}$$ {eq:remaining_di

obtained by composing (6.14) and (6.15), is surjective. Recall from (6.13) that the image of $qE_{i,i}$ is given by $R(\Lambda_i(q))$. Suppose $q = (x - a_i)l$, where $l(a_i) \neq 0$. Then, we have

$$R(\Lambda_i(q))_{x=a_j} = 0$$

for $j \neq i$, and

$$R(\Lambda_i(q))_{x=a_i} = \pm l(a_i) \prod_{j \neq i} (a_i - a_j)^2 p_i(a_i) X_i.$$

Thus, up to scaling, $qE_{i,i}$ maps to X_i under (6.16). Therefore, the map (6.16) is surjective.

7. The Projection-Ramification enumerative problem

{sec:enumerative

Our objectives in this section are to prove Theorem E, and also to relate special instances of the projection-ramification enumerative problem with constructions in classical algebraic geometry.

{sub:a_quadric_s

7.1. **Quadric hypersurfaces.** A smooth quadric hypersurface $X \subset \mathbf{P}^n$ defined by an equation $F(x_0, x_1, x_2, ...) = 0$ induces the classical *polarity isomorphism* $\mathbf{P}^n \longleftrightarrow (\mathbf{P}^n)^\vee$ given by

$$p = [p_0 : p_1 : p_2 : ...] \mapsto [\partial_0 F(p) : \partial_1 F(p) : \partial_2 F(p) : ...]$$

where ∂_i denotes derivative with respect to the i-th variable x_i. The duality morphism is equal to ρ_X, and hence $\deg \rho_X = 1$.

7.2. **The Veronese surface.** Let $\mathbf{P}^2 \simeq X \subset \mathbf{P}^5$ be the Veronese surface. Then the projection-ramification morphism

$$\rho_X : \mathbf{Gr}(3,6) \dashrightarrow \mathbf{P}^9$$

assigns to a general net of conics N the cubic curve $C \subset \mathbf{P}^2$ consisting of the nodes of the singular members of N. We will outline the classical algebraic geometry (the relationship between cubic curves and their Hessians) underlying the claim that the degree of ρ_X is 3.

Suppose $N = \langle Q_1, Q_2, Q_3 \rangle$ is a general net of conics in $\mathbf{P}^2$, with Q_i general ternary quadratic forms. For each line $L \subset \mathbf{P}^2$, the net N restricts either to the complete linear series of two points on L or it restricts to a pencil. The latter type of line a *Reye* line.

em:specialLines}

Lemma 7.1. *The set of Reye lines $C' \subset \mathbf{P}^{2*}$ is a smooth cubic equipped with a fixed-point free involution τ with quotient isomorphic to C.*

Proof. Each Reye line L arises from a unique singular conic of the net N, and hence possesses a conjugate line L', which defines the fixed point free involution τ.

Let $S \longrightarrow \mathbf{P}^{2*}$ denote the rank 2 tautological subbundle. The forms Q_1, Q_2, Q_3 define a map of vector bundles

$$\mathcal{O}^3 \longrightarrow \operatorname{Sym}^2 S^*$$

The determinant of this map defines the locus of Reye lines, and a simple Chern class calculation reveals the locus is a cubic $C' \subset \mathbf{P}^{2*}$. The quotient of C' by the involution sending L to L' is clearly identifiable with $C \subset \mathbf{P}^2$. $\square$

If L is a Reye line, then L is a component of a unique singular member of the net N, and therefore has a conjugate line L'. On L, there are now three points of significance: $x = L \cap L'$, which is clearly a point on C, and the residual pair of points $a_L, b_L \in L \cap C$. Similarly for L'. We may view C' as a $2:1$ unramified cover of C. Lying above the points $a_L, b_L \in C$ are points $a'_L, a''_L, b'_L, b''_L \in C'$.

{lem:lineEq}

Lemma 7.2. *Maintain the setting above. Then on C the following linear equivalences hold: $2a_L \sim 2b_L \sim 2a_{L'} \sim 2b_{L'}$.*

Proof. Attached to a_L and b_L are the dual lines $a^*_L, b^*_L \subset \mathbf{P}^{2*}$. The intersections $a^*_L \cap C'$ and $b^*_L \cap C'$ both contain the point $[L] \in \mathbf{P}^{2*}$. The residual intersections of C' with a^*_L and b^*_L are the points a'_L, a''_L and b'_L, b''_L in C'. Hence, on C' we get a linear equivalence $a'_L + a''_L \sim b'_L + b''_L$. Pushing this linear equivalence forward under the quotient map $C' \longrightarrow C$ gives $2a_L \sim 2b_L$.

Proceeding in a similar way, note that the points $[L], [L'] \in C'$ constitute the two points in C' lying above $x \in C$. Further, on C' we get the equivalence $[L] + a'_L + a''_L \sim [L'] + a'_{L'} + a''_{L'}$ since both triads are collinear in $\mathbf{P}^{2*}$. Pushing this equivalence forward to C yields the equivalence $2a_L \sim 2a_{L'}$. $\square$

m:2torsionclass}

Lemma 7.3. *The class $\eta = a_L - b_L \in Jac(C)[2] \setminus \{0\}$ is independent of the point x and the choice of Reye line L.*

Proof. For each $x \in C$ there are two Reye lines L, L' containing x, and two pairs of points a_L, b_L and $a_{L'}, b_{L'}$ respectively.

Now, if four points $p, q, r, s \in C$ satisfy $2p \sim 2q \sim 2r \sim 2s$, then it is always true that $p - q \sim r - s$, as a straightforward divisor calculation shows.

The lemma now follows by the fact that there are only finitely many 2-torsion divisor classes on C, combined with the fact that C' is irreducible. □

The 2-torsion class η defines a translation on C which takes a point $x \in C$ to the unique point denoted $\eta(x) \in C$ which is linearly equivalent to $x + \eta$. Therefore, Lemma 7.3 allows us to describe the set of Reye lines as the lines joining p with $\eta(p)$ for all points $p \in C$.

Thanks to Lemma 7.3, we see the projection-ramification map ρ_X factors as:

$$\rho_X : \mathbf{Gr}(3,6) \dashrightarrow Jac[2] \dashrightarrow \mathbf{P}^9 \qquad (7.1)$$ {eq:compose}

where $Jac[2]$ is the variety parametrizing pairs (C, η) with C a smooth plane cubic and $\eta \in Jac(C)[2]$ a non-trivial 2-torsion element.

To conclude, we argue the first map in (7.1) is birational by constructing its inverse. To this end, suppose C is a smooth plane cubic, and $\eta \in Jac(C)[2]$ a chosen non-trivial 2-torsion element. We will create from this data a net of conics N whose set of nodes is C. Again, we think of η as a translation $C \longrightarrow C$ in the usual way.

For every point $p \in C$, we get a line $L_p \subset \mathbf{P}^2$ joining p and $\eta(p)$. In this way, we obtain a map $f : C \longrightarrow \mathbf{P}^{2*}$ which is $2:1$ onto its image, since $L_p = L_q$ if and only if $p = q$ or $\eta(p) = q$. Since $\eta : C \longrightarrow C$ is fixed-point free, it is easy to see that f is also unramified. Hence, the image of f must be a smooth cubic.

If $\beta \neq \eta \in Jac(C)[2]$ is any other non-trivial 2-torsion element, the pair of points $\beta(p), \beta(\eta(p))$ span a well-defined second line L'_p containing the point p.

The collection of singular conics $L_p \cup L'_p$ parametrized by $p \in C$ induces a map $C \longrightarrow \mathbf{P}^5$, whose degree is 3, since through a general point in $\mathbf{P}^2$ there pass 3 of the lines L_p. Furthermore, a divisor class computation shows that the node point $L_p \cap L_{p'}$ is again on C. Hence the image of $C \longrightarrow \mathbf{P}^5$ spans a plane which by construction is the desired net of conics N whose locus of nodes is C.

7.3. **Quartic surface scroll.** Our next objective is to prove that $\deg \rho_X = 2$ for a generic quartic surface scroll $X \subset \mathbf{P}^5$. Our proof uncovers a rich geometric picture similar to the case of the Veronese surface in the previous subsection.

We begin with the following seemingly unrelated geometric figure: $C \subset \mathbf{P}^2$ is a smooth cubic curve, $a \in \mathbf{P}^2 \setminus X$ a point, and $Q \subset \mathbf{P}^2$ the *polar conic* of a with respect to C – the unique conic which passes through the six points of ramification on C of the projection from a. We assume a is chosen so that Q is a smooth conic.

To set notation moving forward, if $x \in \mathbf{P}^2$ is any point, we let $P_x(C)$ denote the polar conic of x with respect to C. Similarly, we let $P_x(Q)$ denote the polar line of x with respect to the conic Q.

icsaboutHessian}

Lemma 7.4. *The Hessian $Hess(C) \subset \mathbf{P}^2$ consists of the points x such that $P_x(C)$ is singular, and if C is not a Fermat cubic $P_x(C)$ is the union of two distinct lines for every $x \in Hess(C)$. Furthermore, if $x \in Hess(C)$, then the unique singularity $s(x)$ of $P_x(C)$ lies on $Hess(C)$, and the map $x \mapsto s(x)$ is translation by a 2-torsion point $\eta \in Jac(Hess(C))$.*

Proof. Standard. $\boxed{\text{ToDo: Reference, probably Dolgachev}}$ $\square$

ition:polarline}

Proposition 7.5. *Suppose $x \in Hess(C)$. Then the line $P_x(C)$ passes through the point $x + \eta \in Hess(C)$.*

Proof. We have:

$$P_x(Q) = P_x P_a(C) = P_a P_x(C).$$

Since $x \in Hess(C)$, $P_x(C)$ is a singular conic. Hence $P_x(Q)$ must pass through the singularity $\operatorname{Sing} P_x(C)$, which by Lemma 7.4 is the point $x + \eta$. $\square$

Next suppose $\ell_1, m_1, \ell_2, m_2, \ell_3, m_3$ are six distinct lines in $\mathbf{P}^2$ with the properties:

(1) The three singular conics $\ell_i \cup m_i$ are polars of C.
(2) The triangle $\ell_1\ell_2\ell_3$ is *conjugate* to the triangle $m_1m_2m_3$ with respect to Q.

The second condition above simply means that the vertices of one triangle are polar to the lines of the other triangle. By basic projective geometry, the two triangles are then in *linear perspective*, i.e. the three points $x_1 := \ell_1 \cap m_1, x_2 := \ell_2 \cap m_2, x_3 := \ell_3 \cap m_3$ are collinear.

antReyeLineFact}

Proposition 7.6. *Maintain the notation above, and recall the definition of Reye line from the previous subsection. The lines $P_{x_i}(Q)$ are Reye lines of the net of polar conics of C.*

Proof. The triangles $\ell_1\ell_2\ell_3$ and $m_1m_2m_3$ are conjugate with respect to Q. Hence it follows that the polar line $P_{x_3}(Q)$ equals $\ell := \overline{\ell_{12}m_{12}}$, where $\ell_{ij} = \ell_i \cap \ell_j$ and $m_{ij} = m_i \cap m_j$.

We will prove that ℓ is one of the three Reye lines of the net of polars of C which pass through the point ℓ_{12}, the other two Reye lines being ℓ_1 and ℓ_2. From Lemma 7.4, we can find points $y, z \in C$ and write:

$$\ell_1 = \overline{y, y + \eta}$$
$$\ell_2 = \overline{z, z + \eta}$$

Then a divisor class computation shows that the third Reye line through ℓ_{12} must be $\overline{w, w+\eta}$, with w satisfying

$$y + z + w \sim H + \epsilon,$$

where ϵ is any one of the two non-trivial 2-torsion elements on $Hess(C)$ differing from η. We let $s \in Hess(C)$ denote the third point of intersection of the line $\overline{w, w+\eta}$ with C. (Notice that w depends on the choice of ϵ, but the line $\overline{w, w+\eta}$ is independent of this choice.)

From this setup, we get:

$$\begin{aligned} x_1 &\sim H - 2y - \eta \\ x_2 &\sim H - 2z - \eta \\ s &\sim H - 2w - \eta \end{aligned}$$

from which we get:

$$\begin{aligned} s &\sim H - 2w - \eta \\ &\sim H - 2[H + \epsilon - y - z] - \eta \\ &\sim 2y + 2z - H - \eta \\ &\sim H - x_1 - \eta + H - x_2 - \eta - H - \eta \\ &\sim H - x_1 - x_2 - \eta. \end{aligned}$$

Therefore, to prove that ℓ is a Reye line, it suffices to show that the points ℓ_{12}, m_{12}, and $s \sim H - x_1 - x_2 - \eta$ are collinear. But, this is true if and only if their respective polar lines $P_{\ell_{12}}(Q), P_{m_{12}}(Q), P_s(Q)$ are concurrent. The latter is true if and only if the lines $m_3, \ell_3, P_s(Q)$ are concurrent, which in turn translates to the condition that $x_3 \in P_s(Q)$. But, $x_3 \sim H - x_1 - x_2$, and $s \sim H - x_1 - x_2 + \eta$, and so by Proposition 7.6, we conclude that indeed $x_3 \in P_s(Q)$, which is what we needed to show. $\square$

7.3.1. *Returning to the projection ramification problem.* Our next objective is to relate the geometry in the previous subsection to

{sec:rational_cu

7.4. **Rational curves, the differential construction, and the case of Segre varieties.** **??** connects with an old story involving rational curves in projective space.

Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ be a degree d morphism. Its derivative

$$d\gamma : T_{\mathbf{P}^1} \longrightarrow \gamma^*(T_{\mathbf{P}^n})$$

may be viewed as a global section of the rank r vector bundle $\gamma^*(T_{\mathbf{P}^n}) \otimes T^{\vee}_{\mathbf{P}^1}$. The splitting of $\gamma^*(T_{\mathbf{P}^n})$ is known to be balanced for a general morphism γ. In particular, if the divisibility

$$n \mid d$$

holds, and if we set $\ell := d + d/n - 2$, then a general γ satisfies:

$$(\gamma^* T_{\mathbf{P}^n}) \otimes T_{\mathbf{P}^1}^{\vee} \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbf{P}^1}(\ell).$$

The direct sum decomposition is not canonical, it is only defined up to the action of $GL_n(k)$.

Assuming γ is an immersion, the element $d\gamma \in H^0(\mathbf{P}^1, \bigoplus_{i=1}^n \mathcal{O}_{\mathbf{P}^1}(\ell))$ does not vanish anywhere, and hence defines a degree ℓ map

$$D(\gamma) : \mathbf{P}^1 \longrightarrow \mathbf{P}^{n-1},$$

only well-defined up to the action of post-composition by $PGL_n(k)$.

Definition 7.7. Let M_d^n denote the moduli stack parametrizing $PGL_{n+1}(k)$ equivalence classes of degree d maps $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$, and let $U_d^n \subset M_d^n$ denote the open substack parametrizing local immersions with $\gamma^*(T_{\mathbf{P}^n})$ balanced.

Remark 7.8. Notice: $\dim M_d^n = (k+1)(n+1) - (n+1)^2 = (n+1)(k-n) = \dim \mathbf{G}(n,k)$. Furthermore, notice $PGL_2(k)$ acts on U_d^n and M_d^n by pre-composition.

Remark 7.9. Though M_d^n is an Artin stack, the open substack U_d^n is a scheme, provided $n \leq d$, represented by an open subset of $\mathbf{Gr}(n+1, d+1)$.

When $n \mid d$, and $\ell := d + d/n - 2$, we get the morphism of stacks:

$$D_d^n : U_d^n \longrightarrow M_\ell^{n-1}$$
$$\gamma \longmapsto D(\gamma)$$

which we call the *differential construction*. Interestingly, the dimensions of the domain and codomain of the differential construction are equal, and this leads to another collection of enumerative problems:

em:differential}

Problem 7.10. Compute the degrees of the differential constructions $D_d^n : U_d^n \longrightarrow M_\ell^{n-1}$.

Remark 7.11. The maps D_d^n are clearly $PGL_2(k)$ equivariant. The image of the differential construction D_d^n need not be the open set U_ℓ^{n-1}. $\boxed{\text{ToDo: Sure?}}$

n:trivialdegree} The $n = d$ instances of Problem 7.10 are immediate:

Proposition 7.12. *The degree of the differential construction D_d^d is* 1.

pointlinescroll} *Proof.* The space U_d^d is a single $PGL_2(k)$ orbit. □

Definition 7.13. Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ be any map. We define the *point-hyperplane scroll* of γ to be

$$X_\gamma := \{(t, \Lambda) \mid \gamma(t) \in \Lambda\} \subset \mathbf{P}^1 \times (\mathbf{P}^n)^{\vee}$$

We denote by π_1, π_2 the projections of X_γ to $\mathbf{P}^1$ and $(\mathbf{P}^n)^{\vee}$ respectively. Finally, we set $X_\gamma^{\vee} := \mathbf{P}(\gamma^* T_{\mathbf{P}^n})$.

Remark 7.14. The $\mathbf{P}^{n-1}$-bundle X_γ is isomorphic to $\mathbf{P}(\gamma^* T^\vee_{\mathbf{P}^n})$. Hence, for a general map $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$, X_γ and $X^\vee_\gamma$ are balanced scrolls.

{proposition:tra

Proposition 7.15. *Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ be a non constant map.*

(1) The image of $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ is non-degenerate if and only if $\pi_2 : X_\gamma \longrightarrow (\mathbf{P}^n)^\vee$ is finite; in any case, $\deg \pi_2 = \deg \gamma$.
(2) The ramification divisor $R(\pi_2) \subset X_\gamma$ is a smooth, codimension 1 subscroll of X_γ if and only if γ is an immersion.
(3) Assuming γ is an immersion, the dual section $R^\vee(\pi_2) \subset X^\vee_\gamma$ is induced by the inclusion $d\gamma : T_{\mathbf{P}^1} \hookrightarrow \gamma^* T_{\mathbf{P}^n}$.

Proof. $\boxed{\text{ToDo: PROVE}}$ □

Let $X = \mathbf{P}^1 \times \mathbf{P}^{n-1}$, and denote by h and f the divisor classes of the pullback of a hyperplane in $\mathbf{P}^{n-1}$ and a point in $\mathbf{P}^1$, respectively. When $n \mid k$, Proposition 7.15 sets up a commuting diagram:

$$\begin{tikzcd} U^n_k \arrow[r, leftrightarrow, "\text{duality}"] \arrow[d, "D^n_k"] & PGL_{n+1} \Big\backslash \left\{ \begin{matrix} \text{Deg. } k \text{ maps} \\ X \longrightarrow (\mathbf{P}^n)^\vee \\ \text{induced by } |h + \frac{k}{n} f| \end{matrix} \right\} \Big/ PGL_n \arrow[d, "\rho_X"] \\ M^{n-1}_\ell \arrow[r, leftrightarrow, "\text{duality}"] & \left\{ \begin{matrix} \text{Smooth divisors } R \subset X \\ \text{with div. class } |h + \ell f| \end{matrix} \right\} / PGL_n \end{tikzcd}$$

From this, we conclude:

{proposition:equ

Proposition 7.16. *Let $k = nm$, and let $X \subset \mathbf{P}^{n(m+1)-1}$ be the variety $\mathbf{P}^1 \times \mathbf{P}^{n-1}$ embedded by the linear series $|h + mf|$. Then*

$$\deg \rho_X = \deg D^n_k.$$

Corollary 7.17. *If $X \subset \mathbf{P}^{2n-1}$ is a Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^{n-1}$, then $\deg \rho_X = 1$.*

Proof. The corollary follows at once from Proposition 7.16 and Proposition 7.12. □

7.5. **Quartic surface scrolls.**

7.5.1. *The explicit differential construction for trinodal quartics.* A trinodal quartic R can be obtained as an abstract curve by identifying three pairs of points $\{a', a''\}, \{b', b''\}, \{c', c''\}$ on $\mathbf{P}^1$. These pairs can be encoded by the three binary quadratic forms (up to scale) defining them. In terms of these three quadratic forms, we will now describe the differential construction D^2_4.

In what follows, we let $\{q_1, q_2, q_3\}$ denote a point in $\operatorname{Sym}^3 \mathbf{P} H^0(\mathcal{O}_{\mathbf{P}^1}(2))$.

{definition:node

Definition 7.18. Let

$$\nu : \operatorname{Sym}^3 \mathbf{P}H^0(\mathcal{O}_{\mathbf{P}^1}(2)) \dashrightarrow \mathbf{Gr}(3,5)$$

denote the map given by the formula:

$$\nu\left(\{q_1,q_2,q_3\}\right) = \left\{ \begin{array}{c} \text{v. space of meromorphic 1-forms } \omega \text{ on } \mathbf{P}^1 \text{ with at worst} \\ \text{simple poles at the zeros of } q_i \text{ and with } \textit{opposite} \text{ residues} \\ \text{at the pairs of zeros of } q_i\text{, for all } i = 1,2,3 \end{array} \right\}$$

tion:symtwoptwo}

Proposition 7.19. *The map ν is birational.*

Proof. Suppose a general three dimensional space $W \subset H^0(\mathcal{O}_{\mathbf{P}^1}(4))$ is given. Then the induced degree four map $\mathbf{P}^1 \longrightarrow \mathbf{P}W^\vee$ is the normalization of a trinodal quartic R. The vector space W is naturally identified with the sections of the dualizing sheaf of R, which consist of meromorphic 1-forms on $\mathbf{P}^1$ with the properties stated in the proposition. □

{definition:pi}

Definition 7.20. Let

$$\pi : \operatorname{Sym}^3 \mathbf{P}^2 \dashrightarrow \mathbf{Gr}(2,5)$$

be given by the formula

$$\pi\left(\{q_1,q_2,q_3\}\right) = \left\{ \begin{array}{l} \text{v. space of meromorphic 1-forms } \omega \text{ with at worst simple} \\ \text{poles at the zeros of } q_i \text{ and with } \textit{equal} \text{ residues} \\ \text{at the pairs of zeros of } q_i\text{, for all } i = 1,2,3 \end{array} \right\}$$

nterpretTangent}

Proposition 7.21. *The rational map $\pi \circ \nu^{-1} : \mathbf{Gr}(3,5) \dashrightarrow \mathbf{Gr}(2,5)$ is the differential construction D^2_4.*

Proof. Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^2$ be a general map induced by a three dimensional vector space $W \subset H^0(\mathcal{O}_{\mathbf{P}^1}(4))$ having image R, and let (q_1,q_2,q_3) be $\nu^{-1}(\varphi)$. The pencil $D(\gamma)$ is cut out by the perspective conics. $\boxed{\text{ToDo: Why?}}$ According to **??**, the linear series on R cut out by perspective conics is $\mathcal{O}_R(1) \otimes \eta$, where η is the distinguished element $(-1,-1,-1) \in \operatorname{Pic}(R)[2]$. If the space of sections of the line bundle $\mathcal{O}_R(1)$ is identified with $\nu(q_1,q_2,q_3)$, then it follows that the space of sections of the twist $\mathcal{O}_R(1) \otimes \eta$ equals $\pi(q_1,q_2,q_3)$. □

biantwoquadrics}

Definition 7.22. Let $\{a(x,y), b(x,y)\}$ be two homogeneous quadratic polynomials with no common zeros. Their *Jacobian* is

$$J(a,b) := a_x b_y - a_y b_x.$$

Note that the Jacobian vanishes precisely at the two branch points of the map $[x : y] \mapsto [a(x,y) : b(x,y)]$.

rem:onlyapencil}

Theorem 7.23. *Let* $\{q_1, q_2, q_3\} \in \mathrm{Sym}^3 \mathbf{P}^2$ *have six distinct roots. Then the vector space*

$$\langle q_1 J(q_2, q_3), q_2 J(q_1, q_3), q_3 J(q_1, q_2)\rangle$$

is equal to $\pi(q_1, q_2, q_3) \in \mathbf{Gr}(2, 5)$.

Proof. By $SL_2(k)$-equivariance, it suffices to prove the theorem for three quadratic functions $\{xy, q_2, q_3\}$ where q_2 and q_3 are general.

Let α_1, α_2, and β_1, β_2 denote the roots of q_2, q_3. Note that these roots are assumed to be in $\mathbf{A}^1 \subset \mathbf{P}^1$. We let $t = x/y$ denote the affine coordinate.

The vector space $\Pi := \pi(t, q_2(t), q_3(t))$ is equal to the vector space of forms

$$\omega = \frac{f(t)dt}{tq_2(t)q_3(t)},$$

with $\deg(f) \leq 4$, and with the additional constraints

$$\begin{aligned}
\mathrm{Res}_{\alpha_1} \omega &= \mathrm{Res}_{\alpha_2} \omega \\
\mathrm{Res}_{\beta_1} \omega &= \mathrm{Res}_{\beta_2} \omega \\
\mathrm{Res}_{0} \omega &= \mathrm{Res}_{\infty} \omega
\end{aligned}$$

Since we know a priori that the space of such forms is two dimensional, we conclude in particular that there exists a nonzero $\omega \in \Pi$ which is nonzero and vanishing at α_1. However, the first residue condition then forces ω to vanish at α_2 as well. (This is clear from the geometry: an element of the pencil of perspective conics is cut out by a (possibly singular) conic in $\mathbf{P}^2$. If it contains a node, then its pullback to $\mathbf{P}^1$ must vanish at both points above the node.)

Therefore, there exists an $\omega \in \Pi$ of the form

$$\omega = \frac{(t - \alpha_1)(t - \alpha_2)g(t)dt}{tq_2q_3} = \frac{g(t)dt}{tq_3}.$$

The residue conditions at β_i, and $0, \infty$ together imply, up to nonzero scaling,

$$g(t) = t^2 - \beta_1\beta_2.$$

The roots $\pm\sqrt{\beta_1\beta_2}$ are precisely the branch points of the map $[x : y] \longrightarrow [xy : q_3]$. Therefore ω vanishes at the roots of the quartic polynomial $q_1 j(xy, q_3)$. The theorem follows by arguing in the same manner for the two other pairs of roots.

□

Given a general triple $\{a, b, c\}$ of binary quadratic forms, we can create the three quartic binary forms $a[b, c], b[c, a], c[a, b]$, where $[p, q]$ denotes $p_x q_y - p_y q_x$. As we know, these three forms are actually linearly dependent, yielding a pencil of binary quartics.

In this way, we obtain an *a priori* rational map

$$D : \mathrm{Hilb}^3(\mathbf{P}^2) \dashrightarrow \mathbf{Gr}(2,5)$$

where the domain is the Hilbert scheme of 3 points on $\mathbf{P}^2$.

The main observation is:

sition:Dregular}

Proposition 7.24. *The rational map D extends to a regular map.*

Proof. This is best seen by describing D geometrically, and noting that the geometric construction makes sense at every point of H.

If $\{a, b, c\}$ is a general subset of $\mathbf{P}^2$, then the quartic pencil $D(\{a, b, c\})$ is obtained as follows. Recall that in $\mathbf{P}^2$ we have the canonical discriminant conic C parametrizing square forms. A point $a \in \mathbf{P}^2$ defines a line $Pol(a) \subset \mathbf{P}^2$ spanned by the two points of C which correspond to the roots of a. Furthermore, a pair of points $b, c \in \mathbf{P}^2$ defines the line $\overline{b, c} \subset \mathbf{P}^2$.

To the triple $\{a, b, c\}$ we attach the triple of pairs of lines $Pol(a) \cup \overline{b, c}$ (and permutations), which cut the conic C at 3 members of a degree 4 pencil.

This geometric construction works even for non-reduced schemes. For example, if $Z \subset \mathbf{P}^2$ is a fat point concentrated at a point $a \in \mathbf{P}^2$, we assign the degree 4 pencil on C as: The degree 2 pencil corresponding to $Pol(a)$ with two base points at $Pol(a) \cap C$. □

The map D is only generically finite; the locus of collinear triples is contracted, and has the same image as the locus of fat schemes. However, it is easy to exhibit a point in G over which there are exactly two preimages.

ma:TwoPreimages}

Lemma 7.25. *Let $\Lambda \in \mathbf{Gr}(2, 5)$ denote the unique pencil of binary quartics with simple base points at $0, 1, \infty$ in $\mathbf{P}^1$. Then the preimage $D^{-1}(\Lambda)$ consists of two non-reduced points.*

Proof. The two configurations are described as follows: View the three points $0, 1, \infty$ on the diagonal conic C. Then the triple $\{0, 1, \infty\}$ clearly maps to Λ, as does the triangle created by $Pol(0), Pol(1), Pol(\infty)$.

A simple infinitesimal calculation shows any non-trivial first-order deformation of either of these configurations will have the effect of either removing the base-points, or moving their location.

Furthermore, it is clear that these are the only two possible configurations giving rise to the pencil Λ. □

The previous lemma immediately gives:

Theorem 7.26. *Let $X \subset \mathbf{P}^5$ be a balanced quartic surface scroll. Then* $\deg \rho_X = 2$.

{sub:surfaces}

7.6. **Eccentric surface scrolls.** Let $E = \mathcal{O}(1) \oplus \mathcal{O}(k+1)$, $X = \mathbf{P}E$. Choose an affine coordinate t on $\mathbf{P}^1$, and consider the projection-ramification enumerative problem for $X \subset \mathbf{P}^{k+3}$. We claim:

ationalsurfaces}

Proposition 7.27. *Maintaining the setting above, ρ_X is birational.*

Let $A = H^0(\mathcal{O}_X(1))$. This vector space will be identified with the space of expressions of the form $\ell(t)x_1 + q_{k+1}(t)x_2$, where ℓ, q_{k+1} are polynomials of degrees at most 1 and $k+1$ respectively. In what follows, subscripts of polynomials in t represent the degree.

If $W \subset A$ is a general three dimensional vector space, then there will be a unique triple of elements in W of the form

$$\begin{aligned} w_0 &= t(x_1 + q_k(t)x_2) \\ w_\infty &= (x_1 + r_k(t)x_2) \\ w_* &= s_{k+1}(t)x_2 \end{aligned}$$

The Wronski determinant of this triple is:

$$sx_1 + \left[s(qt)' - s'(qt) - t(r's - s'r)\right] x_2 \tag{7.2}$$ {equation:jacobs

Proof of Proposition 7.27. Let $r := \sigma x_1 + \tau x_2 \in H^0(X, \mathcal{O}(R))$ be a general element, we can extract the unique vector space W obeying $\rho_X(W) = [r] \in |R|$ as follows: First, we set $s := \sigma$. Secondly, given s, the equation $[s(qt)' - s'(qt) - t(r's - s'r)] = \tau$ is a system of $2k+2$ linear equations involving the $2k+2$ coefficients of the pair (q, r). We know (from Theorem A) this system has a finite, positive number of solutions. Hence it must have a unique solution, proving the proposition. □

{sub:eccentric_t

7.7. **Eccentric threefold scrolls.** Now let $E = \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(k+1)$, $k \geq 0$, and set $X := \mathbf{P}E$. Embed $X \subset \mathbf{P}^{k+5}$ via the natural $\mathcal{O}(1)$ on X. Again, we choose affine coordinate $t \in \mathbf{P}^1$ and relative coordinates x_1, x_2, x_3 on X corresponding to the three factors of the splitting of E.

{proposition:thr

Proposition 7.28. *Maintain the setting above. Then ρ_X is birational.*

Suppose $W \subset H^0(E)$ is a general 4 dimensional vector space. Then the projection $W \longrightarrow H^0(\mathcal{O}(1) \oplus \mathcal{O}(1))$ will be an isomorphism. Hence, there will be 4 uniquely defined elements of W of the form:

$$\begin{gathered} x_1 + ax_3 \\ x_2 + bx_3 \\ tx_1 + cx_3 \\ tx_2 + dx_3 \end{gathered}$$

where a, b, c, d are degree $\leq k+1$ polynomials in t. The Wronski determinant for this tuple of equations is:

$$\alpha x_1 + \beta x_2 + \gamma x_3 = (d - bt)x_1 + (at - c)x_2 + \left[a' t(bt - d) + b' t(c - at) + c'(d - bt) + d'(at - c)\right] x_3. \tag{7.3}$$
{eq:jacobianthre

Proof of Proposition 7.28. We replace the Grassmannian $\mathbf{Gr}(4, H^0(E))$ with the affine open subset $\mathbf{A}^{4k+8}$ parametrizing quadruples (a, b, c, d). Then the ramification divisor equation (7.3) defines a map

$$\rho^* : \mathbf{A}^{4k+8} \longrightarrow \mathbf{A}^{4k+9}$$

where the latter $\mathbf{A}^{4k+9}$ is the vector space of triples (α, β, γ) with $\deg \alpha, \beta \leq k+2$ and $\deg \gamma \leq 2k+2$. The projection-ramification ρ_X map ρ is recovered by composing ρ^* with the projection $\mathbf{A}^{4k+9} \dashrightarrow \mathbf{P}^{4k+8}$.

First, if (a, b, c, d) are general, then one can directly use the relative primeness of $d - bt$ and $at - c$ (we omit this simple calculation) to conclude that ρ^* is generically injective on tangent spaces, and hence the generic fiber of ρ^* is finite.

We next show ρ_X is dominant. In light of the previous paragraph, it suffices to prove: If (α, β, γ) is a general point in the image of ρ^*, and $\lambda \neq 0, 1$ is a constant, then $\lambda(\alpha, \beta, \gamma)$ is not in the image of ρ^*.

To this end, suppose (a, b, c, d) is a general point in $\mathbf{A}^{4k+8}$. Then $\alpha := d - bt$ and $\beta := at - c$ will be degree $k+2$ polynomials which are relatively prime.

For any polynomial $p(t)$, let p^+ denote the highest degree coefficient of p. Observe that $\beta^+ = a^+$. Furthermore, the expression for γ is easily seen to be

{gammaEq}
$$\gamma = (\alpha' \beta - \beta' \alpha) + \alpha a + \beta b \tag{7.4}$$

where $'$ denotes d/dt.

If we scale by λ, we get:

{firstEquations}
$$\begin{aligned} \lambda\alpha &= \lambda(d - bt) \\ \lambda\beta &= \lambda(at - c) \\ \lambda\gamma &= \lambda(\alpha'\beta - \beta'\alpha) + \lambda\alpha a + \lambda\beta b \end{aligned} \tag{7.5}$$

At the same time, if $\lambda(\alpha, \beta, \gamma)$ is also realized by some quadruple $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ then we get the equations:

{secondEquation}
$$\begin{aligned} \lambda\alpha &= \tilde{d} - \tilde{b}t \\ \lambda\beta &= \tilde{a}t - \tilde{c} \\ \lambda\gamma &= \lambda^2(\alpha'\beta - \beta'\alpha) + \lambda\alpha\tilde{a} + \lambda\beta\tilde{b} \end{aligned} \tag{7.6}$$

The second equation gives $\tilde{a}^+ = \lambda\beta^+$. The last equation gives: $\gamma = \lambda(\alpha'\beta - \beta'\alpha) + \alpha\tilde{a} + \beta\tilde{b}$. Combining with (7.4), we get

$$\alpha(a - \beta') + \beta(b + \alpha') = \alpha(\tilde{a} - \lambda\beta') + \beta(\tilde{b} + \lambda\alpha').$$

Since α and β are relatively prime and have degree greater than $a, b, \tilde{a}, \tilde{b}$, we deduce:

$$a - \beta' = \tilde{a} - \lambda\beta'$$
$$b + \alpha' = \tilde{b} + \lambda\alpha'$$

By examining top coefficients, and using $a^+ = \beta^+$, $\tilde{a}^+ = \lambda\beta^+$ we get:

$$\beta^+ - (k+2)\beta^+ = \lambda\beta^+ - \lambda(k+2)\beta^+$$

or

$$(1-\lambda)\beta^+ = (1-\lambda)(k+2)\beta^+$$

Given our assumption on λ, this is only possible if $\beta^+ = 0$. However, since (a, b, c, d) were chosen generically, $\beta^+ = a^+$ would not be zero, providing our desired contradiction.

Finally, we argue $\deg \rho_X = 1$. It suffices to show that a general ramification equation $\alpha x_1 + \beta x_2 + \gamma x_3$ of the form (7.3) arises from a unique {alphabeta} choice of polynomials (a, b, c, d). The conditions $d - bt = \alpha$ and $at - c = \beta$ produce an affine linear subspace Λ in the vector space of choices (a, b, c, d). With respect to linear coordinates on Λ, the expression for γ is also linear, and hence the available choices of (a, b, c, d) producing Equation 7.3 is an intersection of affine linear spaces. Since we already know ρ^* is generically finite, it follows that $\deg \rho_X = 1$ as desired.

□

Since every smooth three dimensional rational normal scroll specializes isotrivially to the scroll X in Proposition 7.28, we immediately get:

{corollary:maxVa

Corollary 7.29. *The projection-ramification map ρ_X is dominant for every smooth three dimensional rational normal scroll $X \subset \mathbf{P}^n$.*

7.8. **Recasting the projection-ramification map for scrolls.** Let E be a rank r ample vector bundle on $\mathbf{P}^1$, and set $X = \mathbf{P}E$. Then a general $r+1$-dimensional subspace

$$W \subset H^0(X, \mathcal{O}(1)) = H^0(\mathbf{P}^1, E)$$

yields a short exact sequence

$$0 \longrightarrow (\det E)^{-1} \longrightarrow W \otimes \mathcal{O}_{\mathbf{P}^1} \longrightarrow E \longrightarrow 0$$

which corresponds to an element w (up to scalar) of the extension space $\mathrm{Ext}^1(E, (\det E)^{-1})$. The assignment $W \mapsto [w] \in \mathbf{P}(\mathrm{Ext}^1(E, (\det E)^{-1}))$ is easily seen to be a birational map between $\mathbf{G} := \mathbf{Gr}(n+1, H^0(E))$ and $\mathbf{P}(\mathrm{Ext}^1(E, (\det E)^{-1}))$.

The ramification linear series $|R|$ is the projectivization of the vector space $V = H^0(E \otimes \det E \otimes \Omega_{\mathbf{P}^1})$. By Serre duality, V is dual to $\mathrm{Ext}^1(E, (\det E)^{-1})$. Therefore, the projection-ramification map ρ_X may be recast as a map

$$\delta_X : \mathbf{P}(V^*) \dashrightarrow \mathbf{P}(V)$$

8. Further Questions

(1) How many of our theorems are valid in characteristic $p > 0$?
(2) When $\dim \mathbf{Gr} < \dim |R|$ and ρ_X is generically finite onto its image, then is ρ_X birational onto its image?
(3) Is $\mathbf{Gr}(2, 4)$ the only incompressible Grassmannian?
(4) Is it possible to classify the scrolls for which $\deg \rho_X = 1$?
(5) Is there an analogous characterization of varieties of minimal degree using "higher codimension" ramification loci?

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