PROJECTION AND RAMIFICATION

ANAND DEOPURKAR, EDUARD DURYEV, & ANAND PATEL

Abstract. When a projective variety is linearly projected to a projective space of the same dimension, a ramification divisor forms. We study basic properties of this projection-ramification assignment, and uncover enumerative phenomena extending the classical appearance of Catalan numbers in the geometry of rational normal curves.

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1. Introduction

{sec:intro}

Let $X \subset \mathbf{P}^n$ be a smooth projective variety of dimension r, not contained in any hyperplane. Projection from a general $(n-r-1)$-dimensional linear subspace $L \subset \mathbf{P}^n$ defines a finite surjective map $X \longrightarrow \mathbf{P}^r$ and its ramification divisor $R(L) \subset X$. The goal of this paper is to study the relationship between the geometry of X and the map $L \mapsto R(L)$. To make matters more precise, note that by the Riemann–Hurwitz formula, the ramification divisor $R(L)$ lies in the linear series $|K_X + (r+1)H|$, where K_X is the canonical class, and H is the hyperplane class on X. We can then interpret the map $L \mapsto R(L)$ as a (rational) map

$$\rho_X \colon \mathbf{Gr}(n-r, n+1) \dashrightarrow |K_X + (r+1)H|.$$

Our goal is to explore the relationship between the geometry of X and various properties of ρ_X such as generic fiber dimension and degree.

A simple argument shows that ρ_X is itself a linear projection of $\mathbf{Gr}(n-r, n+1)$ in its Plücker embedding. When X is a smooth curve over a field of characteristic 0, the map ρ_X is regular everywhere on $\mathbf{Gr}(n-r, n+1)$. When X is a rational normal curve, the map ρ_X is also finite. In this case, the ramification divisor of a map $\mathbf{P}^1 \longrightarrow \mathbf{P}^1$ of degree n represented by the rational function f/g is cut out by the Jacobian expression, namely the degree $(2n-2)$ polynomial $f'g - g'f$. Since ρ_X is regular, its degree is the degree of the Grassmannian, which in this case is the Catalan number $\frac{(2n-2)!}{n!(n-1)!}$. When X has dimension 2 or more, ρ_X may not be regular on the entire Grassmannian, which makes it difficult to understand. Nevertheless, it appears that the geometry of ρ_X is related to some fascinating areas of classical projective geometry, and the enumerative questions surrounding ρ_X hint at a rich underlying structure.

{q:maxvar}

1.1. **Maximal variation.** Our focus is the following question.

Question 1.1. Is ρ_X generically finite onto its image? In other words, does the image of ρ_X have maximal possible dimension?

To our knowledge, this question first appeared in the work of Flenner and Manaresi [**?**]. Our first result answers this question affirmatively for a large class of varieties. We say that $X \subset \mathbf{P}^n$ is *incompressible* if for every $(n-r-1)$-dimensional linear subspace $L \subset \mathbf{P}^n$, the projection map $p_L\colon X \dashrightarrow \mathbf{P}^r$ is dominant. Recall that the dual variety $X^* \subset \mathbf{P}^{n*}$ is the closure of the locus of hyperplanes in $\mathbf{P}^n$ whose intersection with the smooth part of X is singular.

{thm:main}

Theorem A. *Let $X \subset \mathbf{P}^n$ be a non-degenerate, normal, projective variety over a field of characteristic zero. Suppose at least one of the following holds:*

{item:incomp}

(1) X is incompressible, {item:dual}

(2) the dual variety $X^* \subset \mathbf{P}^{n*}$ is a hypersurface.

Then ρ_X is generically finite onto its image.

We do not assume that X is smooth in the statement of Theorem A. This requires defining ρ_X more carefully. To state the conclusion informally, if we move a generic $L \subset \mathbf{P}^n$ of complementary dimension, then the ramification locus $R_L \subset X$ also moves.

The hypotheses in Theorem A are sufficient, but not necessary. Indeed, consider $X = \mathbf{P}^{r-1} \times \mathbf{P}^1 \subset \mathbf{P}^{2r-1}$, embedded by the Segre embedding, for $r \geq 3$. Then X is neither incompressible nor is X^* a hypersurface, and yet ρ_X is dominant (see Theorem E).

To our knowledge, the known results about maximal variation operate under condition (1) in Theorem A. For example, in [**?**], the authors deduce maximal variation under the condition that for every $(n-r-1)$-dimensional linear subspace $L \subset \mathbf{P}^n$, the join $J(L, X)$ equals $\mathbf{P}^n$, or under the condition that X is smooth and the twisted normal bundle $N_{X/\mathbf{P}^n}(-1)$ is ample. Either condition implies that X is incompressible, and hence falls under condition (1). If X is a curve or a smooth complete intersection, then X is incompressible, and covered by condition (1).

Theorem A substantially increases the class of varieties where we now know maximal variation. For example, it is easy to see that if X is a smooth surface over a field of characteristic 0, then X^* is a hypersurface. Therefore, maximal variation holds for all surfaces, although incompressibility may not (The cubic surface scroll $X \subset \mathbf{P}^4$ is the smallest counter example, as the projection from the directrix line of X is not dominant). As another source of new examples, take a sufficiently high degree Veronese re-embedding $X \subset \mathbf{P}^N$ of any smooth X. Then X^* is divisorial, and hence X is covered under Theorem A. But $X \subset \mathbf{P}^N$ will be compressible.

Given that maximal variation holds in such a large class of varieties, it is natural to wonder if it always holds. This is not the case.

{Thm:Counterexam

Theorem B. *There exist smooth, non-degenerate, rational normal scrolls $X^r \subset \mathbf{P}^n$ of every dimension $r \geq 4$ and degree $d \geq r+1$ such that the projection-ramification map ρ_X is not generically finite onto its image.*

Theorem B provides the first known examples of varieties with non-maximal variation of ramification divisors. We describe the rational normal scrolls in Theorem B in **??**; they include some of general moduli.

We now turn our attention to cases where the projection-ramification map ρ_X may be dominant. The next result classifies $X \subset \mathbf{P}^n$ for which the source and the target of ρ_X are of the same dimension.

m:minimaldegree}

Theorem C. *Let $X \subset \mathbf{P}^n$ be a smooth, non-degenerate projective variety of dimension r over a field of characteristic zero. We have the inequality*

$$\dim \mathbf{Gr}(n-r, n+1) \leq \dim |K_X + (r+1)H|,$$

where equality holds if and only if X is a variety of minimal degree, that is $\deg X = n-r+1$.

Recall the list of smooth varieties of minimal degree: rational normal curves, quadric hypersurfaces, the Veronese surface in $\mathbf{P}^5$, and rational normal scrolls. By Theorem A, ρ_X is dominant for the first three, so we are led to investigate the scrolls. It came to us as a surprise that ρ_X is *not* dominant for all scrolls (see Theorem B). Nevertheless, it is dominant for most scrolls, which we now make precise.

Recall that if $X \subset \mathbf{P}^n$ is a smooth rational normal scroll, then X is isomorphic to the projectivization of an ample vector bundle E on $\mathbf{P}^1$, and the embedding is given by the complete linear series $|\mathcal{O}_{\mathbf{P}E}(1)|$.

alnormalscrolls}

Theorem D. *Let $X = \mathbf{P}E \subset \mathbf{P}^n$ be a rational normal scroll, where E is a ample vector bundle of rank r on $\mathbf{P}^1$, general in its moduli. If* $\deg E = a \cdot (r-1) + b \cdot (2r-1) + 1$ *for non-negative integers a, b, then the projection-ramification map ρ_X is dominant for X. In particular, the conclusion holds if E is general of degree at least* $(r-1)(2r-1)+1$.

Thus, at least among the general scrolls, the projection-ramification map is dominant except possibly in small degrees. We prove Theorem D by degeneration, using the theory of limit linear series of higher rank developed by Teixidor i Bigas [**?**] and Osserman [Oss14]. In the course of proving the theorem, we also demonstrate dominance in the case of $E = \mathcal{O}(1)^r$ and $E = \mathcal{O}(2)^r$.

1.2. **Enumerative problems.** Theorem C and Theorem D motivate a gamut of enumerative questions.

{q:degree}

Question 1.2. When $X \subset \mathbf{P}^n$ is a variety of minimal degree, what is the degree of ρ_X?

{Thm:Examples}

The following result summarizes our knowledge of the answers to Question 1.2.

Theorem E.

(1) If $X \subset \mathbf{P}^n$ is a rational normal curve, then ρ_X is regular and $\deg \rho_X = \frac{(2n-2)!}{n!(n-1)!}$.
(2) If $X \subset \mathbf{P}^n$ is a quadric hypersurface, then ρ_X is an isomorphism.
(3) If $X = \mathbf{P}^{r-1} \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^{2r-1}$ is the Segre embedding, then ρ_X is birational.
(4) If $X \subset \mathbf{P}^5$ is the Veronese surface, then $\deg \rho_X = 3$.
(5) If $X \subset \mathbf{P}^5$ is a general quartic surface scroll, then $\deg \rho_X = 2$.
(6) If $X = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(k+1)) \subset \mathbf{P}^{k+3}$ is the surface scroll with most imbalanced splitting type, then ρ_X is birational.

(7) *If* $X = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(k+1)) \subset \mathbf{P}^{k+5}$ *is the threefold scroll with most imbalanced splitting type, then* ρ_X *is birational.*

For X of dimension 1, namely a rational normal curve, the projection-ramification map

$$\rho_X \colon \mathbf{Gr}(2, n+1) \longrightarrow \mathbf{P}^{2n-2}$$

is regular, and defined by the Plücker line bundle on the Grassmannian. Therefore, its degree is the top self-intersection of the Plücker line bundle, which in this case is the Catalan number $\frac{(2n-2)!}{n!(n-1)!}$.

For X of codimension 1, namely a quadric hypersurface, the projection-ramification map

$$\rho_X \colon \mathbf{Gr}(n, n+1) = \mathbf{P}^n \longrightarrow \mathbf{P}^{n*}$$

is again regular, and is in fact the duality isomorphism induced by the (non-degenerate) quadric X.

The case of the Veronese surface and of the quartic surface scroll in Theorem E are particularly delightful; these are treated in § 6. They involve intricate classical projective geometry that intertwines cubic plane curves, Steinerians and Cayleyans, and apolarity.

The cases of the most unbalanced surface and threefold scrolls follow from direct calculation. Note, however, that for the most unbalanced scroll in dimension 4 and higher, the projection-ramification map is not dominant. For scrolls, ρ_X is not regular. Furthermore, the complexity of the base locus of ρ_X effectively blocks any straightforward application of the excess intersection formula.

A smooth rational normal scroll $X \subset \mathbf{P}^n$ of degree d and dimension r is isomorphic to the projectivization of an ample vector bundle E on $\mathbf{P}^1$, which in turn is isomorphic to a direct sum $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ for positive integers $a_1, \ldots, a_r$ satisfying $d = a_1 + \cdots + a_r$. Let $\Sigma_{r,d}$ be the set of r-term partitions of d. We get a function $\rho \colon \Sigma_{r,d} \longrightarrow \mathbf{Z}_{\geq 0}$ defined by

$$\rho(a_1, \ldots, a_r) = \deg \rho_X,$$

for $X = \mathbf{P}(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r))$. The set $\Sigma_{r,d}$ has a partial ordering $\prec$ given by dominance. If $(a_1, \ldots, a_r) \prec (b_1, \ldots, b_r)$, then the scroll $\mathbf{P}(\mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_r))$ isotrivially specializes to the scroll $\mathbf{P}(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r))$. By the lower semi-continuity of degrees of rational maps, we get

$$\rho(a_1, \ldots, a_r) \leq \rho(b_1, \ldots, b_r).$$

Theorem D implies that, at least if d is sufficiently large compared to r, then ρ is not identically zero. Theorem E determines the value of ρ for the partitions (n), $(1, \ldots, 1)$, $(1, k+1)$, $(1, 1, k+1)$, and $(2, 2)$. The following table lists some more values of ρ computed using randomized calculations over finite fields using the computer algebra systems `Macaulay2` and `MAGMA`. We plan to return to a more complete enumerative investigation of ρ in a future paper.

$a_1 \backslash a_2$	1	2	3	4
1	1			
2	1	2		
3	1	6	22	
4	1	17	92	422

TABLE 1. Degree of ρ_X for $X = \mathbf{P}(\mathcal{O}(a_1) \oplus \mathcal{O}(a_2))$

{tab:computation

1.3. **Further remarks and questions.** One of the central enumerative problems concerning branch divisors, originating in the work of Hurwitz, is to compute the number of branched covers of the projective line with specified branch set in $\mathbf{P}^1$. This number is called the Hurwitz number. As is well known, the Hurwitz numbers are difficult to compute, but they exhibit remarkable structure. There is a related question of computing the number of rational functions on $\mathbf{P}^1$ with a prescribed ramification set. This question is much more elementary, and yields the Catalan numbers, as we have seen.

In higher dimensions, however, the analogue of the Hurwitz problem is expected to be much less interesting, thanks to Chisini's conjecture (now Kulikov's theorem [**?**]). Kulikov's theorem asserts that a branched cover $S \longrightarrow \mathbf{P}^2$ with generic branching is uniquely determined by its branch divisor $B \subset \mathbf{P}^2$, with finitely many well-understood exceptions. In contrast, the enumerative problem regarding ramification divisors persists in all dimensions, thanks to Theorem C, and poses a significant challenge. In some sense, the "branch" and "ramification" enumerative stories trade places, at least in terms of difficulty, but perhaps also in terms of structure.

The projection-ramification map generalizes the Wronski map

$$\rho \colon \mathbf{Gr}(2, n+1) \longrightarrow \mathbf{P}^{2n-2}.$$

The geometry surrounding the Wronski map has received a lot of attention, thanks to the B. and M. Shapiro conjecture. This conjecture states that the pre-image of any point in $\mathbf{P}^{2n-2}$ defined by a set of $(2n-2)$ real points on $\mathbf{P}^1$ consists entirely of real points in $\mathbf{Gr}(2, n+1)$ [**?**] (the conjecture has been proved by Eremenko and Gabrielov [**?**]). Theorem C potentially sets the stage for a higher-dimensional generalization of the body of work around the Shapiro conjecture.

The study of ρ_X in positive characteristic is likely to bring new surprises and require different techniques. We do not know if Theorem A or Theorem C holds in positive characteristic; our proof certainly does not. The answers to the enumerative questions Question 1.2 do depend on the characteristic, even in the simplest case of rational normal curves, due to the presence of inseparable covers [**?**].

1.4. **Notation and conventions.** We work over an algebraically closed field k of characteristic 0 (We use Bertini's theorem and generic smoothness. We also appeal to the Kodaira Vanishing theorem.) By a *proper variety*, we mean a proper, integral, finite-type k-scheme. For any scheme X, we let X^{sm} denote its smooth locus. If F is a coherent sheaf, we let

$P(F)$ denote its sheaf of principal parts. We will let $e\colon H^0(X,F)\longrightarrow P(F)$ denote the natural evaluation morphism – we suppress the dependence on F. If s is a global section of a locally free sheaf, we let $v(s)$ denote the vanishing scheme of s. If L is a line bundle, we let $|L|$ denote the projective space $\mathbf{P}(H^0(L))$. If L is a line bundle on a smooth variety Y, and $s \in H^0(Y,L)$ is a section, then the *singular scheme* $\mathrm{Sing}(v(s))$ of s is the vanishing scheme of $e(s) \in H^0(Y,P(L))$; if K, the kernel sheaf of $e\colon H^0(Y,L)\otimes \mathcal{O}_Y \longrightarrow P(L)$, is locally free, then $\mathrm{Sing}(v(s))$ is the largest closed subscheme $T\subset Y$ such that $s\colon \mathcal{O}_T\longrightarrow H^0(Y,L)\otimes\mathcal{O}_T$ factors through $K|_T$.

2. The projection-ramification map

{sec:prmap}

In this section, we define a projection-ramification map for a pair (X,L) consisting of a proper, normal, variety X and a sufficiently positive line bundle L on X. For $X\subset \mathbf{P}^n$, taking $L=\mathcal{O}(1)$ recovers the projection-ramification map introduced in § 1. Working with abstract pairs, however, offers more flexibility that is helpful in inductive proofs.

Let X be a proper variety of dimension r over an algebraically closed field k of characteristic zero. A *linear series* on X is a pair (L,W) consisting of a line bundle L on X and a subspace $W\subset H^0(X,L)$. The *complete linear series* associated to L is (L,W) with $W=H^0(X,L)$. A *projection* is a linear series (L,V) with $\dim V = r+1$. A *projection of* (L,W) is a projection (L,V) with $V\subset W$. As a convention, we use V for projections and W for more general linear series.

{definition:prop

Definition 2.1. We say that a projection (L,V) is *properly ramified* if the evaluation homomorphism

$$e\colon V\otimes\mathcal{O}_X\longrightarrow P(L)$$

is an isomorphism over a general point in X. If (L,V) is properly ramified, its *ramification divisor*

$$R(L,V)\subset X$$

is the closure of the scheme defined by the determinant of $e\colon V\otimes\mathcal{O}_{X^{\mathrm{sm}}}\longrightarrow P(L)|_{X^{\mathrm{sm}}}$.

In most cases, L is clear from context, so we drop it from the notation and denote the ramification divisor simply by $R(V)$.

{rem:Jacobian}

Remark 2.2. Suppose for simplicity that V is a base-point free linear series that yields a surjective map $\phi\colon X\longrightarrow \mathbf{P}V$. Then the ramification divisor may be defined as the degeneracy locus of the map

$$d\phi\colon T_X\longrightarrow \phi^*T_{\mathbf{P}V}$$

on tangent spaces. The degeneracy locus is the zero locus of $\det\phi$, which in local coordinates, is given by the determinant of the Jacobian matrix $\left(\frac{\partial\phi_i}{\partial x_j}\right)$. Therefore, the ramification divisor $R(L,V)$ is also often called the *Jacobian* of the linear series (L,V) (see, for example, [**?**, 1.1.7]).

A projection (L, V) gives the evaluation map

$$e\colon V \otimes \mathcal{O}_X \longrightarrow L.$$

The evaluation map yields a map $p_{V,L}\colon X \dashrightarrow \mathbf{P}V$, regular on the non-empty open set of X where e is surjective. The following is an easy observation, whose proof we skip.

{prop:proj}

Proposition 2.3. *The projection (L, V) is properly ramified if and only if the map on tangent spaces induced by $p_{V,L}$ is generically an isomorphism. In characteristic zero, this is equivalent to the condition that $p_{V,L}$ is dominant.*

For a fixed (L, W), the set of all projections of (L, W) are parametrized by the Grassmannian $\mathbf{Gr}(r+1, W)$. The property of being properly ramified is a Zariski open condition on the Grassmannian.

We now define a map that assigns to a projection its ramification divisor. To do so, we interpret the ramification divisor as an element of a linear series.

Assume, furthermore, that X is normal. Let K_X be the canonical sheaf of X. Denoting by $i\colon X^{\mathrm{sm}} \longrightarrow X$ the inclusion, K_X is given by the push-forward

$$K_X = i_* K_{X^{\mathrm{sm}}}.$$

Note that, since X is normal, the complement of $X^{\mathrm{sm}} \subset X$ has codimension at least 2. The sheaf K_X is coherent, reflexive, and satisfies Serre's S2 condition.

Let L be a line bundle on X. The sheaf $P(L)$ is locally free of rank $(r+1)$ on X^{sm}, and we have a canonical isomorphism

$$\bigwedge^{r+1} P(L)|_{X^{\mathrm{sm}}} \cong K_{X^{\mathrm{sm}}} \otimes L^{r+1}.$$

Given a subspace $V \subset H^0(X, L)$, we apply $\bigwedge^{r+1}$ to the evaluation map

$$e\colon V \otimes \mathcal{O}_{X^{\mathrm{sm}}} \longrightarrow P(L)|_{X^{\mathrm{sm}}},$$

to get

$$\det e\colon \det V \otimes \mathcal{O}_{X^{\mathrm{sm}}} \longrightarrow K_{X^{\mathrm{sm}}} \otimes L^{r+1}.$$

By applying i_* and taking global sections, we get

{eqn:ramsection}

$$r_V\colon \det V \longrightarrow H^0(X, K_X \otimes L^{r+1}). \tag{2.1}$$

If (L, V) is properly ramified, then this map is non-zero, and hence gives a point of the projective space $\mathbf{P}H^0(X, K_X \otimes L^{r+1})^*$. Doing the same construction universally over the Grassmannian $\mathbf{Gr} = \mathbf{Gr}(r+1, W)$ yields a map

{eqn:rammap}

$$r\colon \det \mathcal{V} \longrightarrow H^0(X, K_X \otimes L^{r+1}) \otimes \mathcal{O}_{\mathbf{Gr}}, \tag{2.2}$$

where $\mathcal{V} \subset W \otimes \mathcal{O}_{\mathbf{Gr}}$ is the universal sub-bundle of rank $(r+1)$. Let $U \subset \mathbf{Gr}$ be the open subset of properly ramified projections. Then the map in (2.2) is non-zero at every point of U, and defines a map $U \longrightarrow \mathbf{P}H^0(X, K_X \otimes L^{r+1})^*$ given by the surjection

qn:rammapfamily}

$$H^0(X, K_X \otimes L^{r+1})^* \otimes \mathcal{O}_U \longrightarrow \det \mathcal{V}|_U^*. \tag{2.3}$$

Note that U is non-empty if and only if W separates tangent vectors at a general point of X.

{def:ProjectionR

Definition 2.4. Let (L, W) be a linear series that separates tangent vectors at a general point of X. The *projection-ramification* map for (L, W) is the rational map

$$\rho_{(X,L,W)} \colon \mathbf{Gr}(r+1, W) \dashrightarrow \mathbf{P}H^0(X, K_X \otimes L^{r+1})^*$$

defined on the non-empty open subset of properly ramified maps by (2.3).

If any of X, L, or W are clear from context, we drop them from the notation. In particular, for a non-degenerate $X \subset \mathbf{P}^n$, we denote by ρ_X the map $\rho_{X,L,W}$ with $L = \mathcal{O}_X(1)$ and W the image in $H^0(X, L)$ of $H^0(\mathbf{P}^n, \mathcal{O}(1))$.

Note that the map (2.3) factors as

$$\det \mathcal{V} \xrightarrow{a} \bigwedge^{r+1} W \otimes \mathcal{O}_{\mathbf{Gr}} \xrightarrow{b} H^0(X, K_X \otimes L^{r+1}) \otimes \mathcal{O}_{\mathbf{Gr}},$$

where a is $\wedge^{r+1}$ applied to the universal inclusion $\mathcal{V} \subset W \otimes \mathcal{O}_{\mathbf{Gr}}$, and b is induced by $\wedge^{r+1}$ applied to the evaluation map $e \colon W \otimes \mathcal{O}_X \longrightarrow P(L)$. The map a defines the Plücker embedding

$$i \colon \mathbf{Gr}(r+1, W) \longrightarrow \mathbf{P}\left(\bigwedge^{r+1} W^*\right),$$

and the map b defines a linear projection

$$p \colon \mathbf{P}\left(\bigwedge^{r+1} W^*\right) \dashrightarrow \mathbf{P}H^0(X, K_X \otimes L^{r+1}).$$

Thus, $\rho_{X,L,W}$ factors as the Plücker embedding followed by a linear projection.

3. Maximal variation for incompressible and non-defective X

{sec:proof_of_th

The goal of this section is to prove Theorem A. We begin by proving part (1), which is substantially easier.

{prop:incompress

Proposition 3.1 (Theorem A (1))**.** *Let $X \subset \mathbf{P}^n$ be a non-degenerate, normal, incompressible projective variety over a field of characteristic zero. Then ρ_X is a finite map.*

Proof. Set $L = \mathcal{O}(1)$ and let $W \subset H^0(X, L)$ be the image of $H^0(\mathbf{P}^n, \mathcal{O}(1))$. Let $V \subset W$ be an $(r+1)$-dimensional subspace. Since X is incompressible, the projection map $p_{V,L} \colon X \dashrightarrow \mathbf{P}V$ induced by (L, V) is dominant. By Proposition 2.3, this implies that (L, V) is properly ramified. Since V was arbitrary, the projection-ramification map

$$\rho \colon \mathbf{Gr}(r+1, W) \longrightarrow |K_X + (r+1)H|$$

is regular. Since the Picard rank of a Grassmannian is 1, a regular map from a Grassmannian is either constant or finite. It is easy to check that ρ is not constant; so it must be finite. $\square$

For the proof of part (2) of Theorem A, we proceed inductively by showing that a general $(n-r-1)$-dimensional linear subspace which is incident to X is an isolated point in its fiber under ρ. Again, it is more convenient to work with the more abstract set-up of a linear series, allowing for series that are not very ample.

Let X be a proper variety of dimension r, and let (L, W) be a linear series on X. For an ideal sheaf $I \subset \mathcal{O}_X$ we denote by $W \otimes I$ the subspace of W consisting of the sections that vanish modulo I. More precisely, if K is the kernel of the evaluation map

$$W \otimes \mathcal{O}_X \longrightarrow L \otimes \mathcal{O}_X/I,$$

then $W \otimes I = H^0(X, K)$. In particular, for $W = H^0(X, L)$, we have $W \otimes I = H^0(X, L \otimes I)$. For $s \in W \otimes I$, the vanishing locus $v(s)$ refers to the vanishing locus of s as a section of L. We set $|W| = \mathbf{P}W^*$, the space of one-dimensional subspaces of W, and likewise $|W \otimes I| = \mathbf{P}(W \otimes I)^*$. For a complete linear series, we write $|L|$ for $|W|$. Note that $v(s) = v(\lambda s)$ for a non-zero scalar λ, so it causes no ambiguity to talk about $v(s)$ for $s \in |W|$.

non-defectivity}

3.1. **Non-defective linear series.** We study a positivity property of linear series that generalizes the property of having a divisorial dual.

lynon-defective}

Definition 3.2. We say that a linear series (L, W) is *non-defective* if, for a general point $x \in X$ either $W \otimes \mathfrak{m}_x^2 = 0$, or there exists $s \in W \otimes \mathfrak{m}_x^2$ such that $v(s)$ has an isolated singularity at x.

Note that for $s \in |W|$, the condition that $v(s)$ have an isolated singularity at x is a Zariski open condition on $|W|$. Therefore, if there exists an $s \in |W \otimes \mathfrak{m}_x^2|$ such that $v(s)$ has an isolated singularity at x, then a general $s \in |W \otimes \mathfrak{m}_x^2|$ has the same property.

Remark 3.3. Let x be a point of X. Suppose there exists $s \in |W|$ with an isolated singularity at x. It may be tempting to conclude from this that (L, W) is non-defective. This is not necessarily true! For example, take $X = \mathbf{F}_3$. Denote by E the section of self-intersection -3 and F the fiber of the projection $\mathbf{F}_3 \longrightarrow \mathbf{P}^1$. Let $L = \mathcal{O}_X(E + 2F)$ and $W = H^0(X, L)$. For $x \in E$, the general member of $|W \otimes \mathfrak{m}_x^2|$ has an isolated singularity at x, but the same is not true for a general $x \in X$.

Remark 3.4. Suppose (L, W) is non-defective. Let $x \in X$ be general, and let $s \in |W|$ be such that $v(s)$ has an isolated singularity at x. For all such s, it may be the case $v(s)$ has singularities away from x, even along a positive dimensional locus. For example, let $\pi \colon X \longrightarrow \mathbf{P}^2$ be the blow-up at a point, and E the exceptional divisor. The complete linear series associated to $L = \pi^*\mathcal{O}(2) \otimes \mathcal{O}(2E)$ is non-defective, but for every global section of L, the singular locus of $v(s)$ contains E.

We now define the conormal variety of a linear series, which plays an important role in our analysis of non-defectivity. Let K be the kernel of the evaluation map

$$e \colon W \otimes \mathcal{O}_X \longrightarrow P(L).$$

Let $U \subset X$ be an open subset such that $K|_U$ is locally free and the dual of the inclusion

$$W^* \otimes \mathcal{O}_U \longrightarrow K|_U^*$$

is a surjection. This surjection defines a closed embedding $\mathbf{P}(K|_U) \subset U \times |W|$. The *conormal variety of* (L, W), denoted by $P_{L,W}$, is the closure of $\mathbf{P}(K|_U)$ in $X \times |W|$.

{prop:dimension}

Proposition 3.5. *Suppose* (L, W) *is non-defective. If* $\dim W \geq r + 2$*, then* $P_{L,W}$ *is irreducible of dimension* $\dim W - 2$*. If* $\dim W \leq r + 1$*, then* $P_{L,W}$ *is empty.*

Proof. Set $n = \dim |W| = \dim W - 1$. Let k be the (generic) rank of K, namely the rank of the locally free sheaf $K|_U$. Then $k \geq n - r$. The statement of the proposition is equivalent to showing that if $k > 0$, then $k = n - r$.

For brevity, set $P = P_{L,W}$. Consider the projection $\sigma \colon P \longrightarrow |W|$, obtained by restricting the second projection $X \times |W| \longrightarrow |W|$. For $s \in |W|$, we view $\sigma^{-1}(s)$ as a subscheme of X. We then have

$$\sigma^{-1}(s) \cap U = \operatorname{Sing}(v(s)) \cap U.$$

Suppose $r > 0$. Then P is non-empty and irreducible, since it is the closure of a non-empty and irreducible variety. Since (L, W) is non-defective, a general point $(x, s) \in P$ is such that x is an isolated point of $\operatorname{Sing}(v(s))$. Therefore, $\sigma \colon P \longrightarrow |W|$ is generically finite onto its image. We conclude that $\dim P \leq \dim |W|$, and hence $k \leq n - r + 1$.

To show that $k = n - r$, it suffices to show that $\sigma \colon P \longrightarrow |W|$ is not surjective. We do so using Bertini's theorem. Let $B \subset X$ denote the union of the base locus of $|W|$ and the singular locus of X. Then B is a proper closed subset of X. Let $P^B \subset P$ be the pre-image of B under the projection $\pi \colon P \longrightarrow X$. By the definition of P, the map $\pi \colon P \longrightarrow X$ is surjective, and hence P^B is a proper closed subset of P. Since P is irreducible, we have $\dim P^B < \dim P \leq \dim |W|$, so the projection $P^B \longrightarrow |W|$ cannot be dominant. Let $s \in |W|$ be general, in particular, not in the image of $P^B \longrightarrow |W|$. By Bertini's theorem $v(s)$ is non-singular away from B. Thus, for any $x \in X$, the point $(x, s) \in X \times |W|$ does not lie in P. For $x \in B$, this is because s is not in the image of P^B, and for $x \notin B$, this is because $v(s)$ is non-singular at x. We conclude that s does not lie in the image of $P \longrightarrow |W|$. Hence $P \longrightarrow |W|$ is not surjective. $\square$

{prop:dimensionC

Proposition 3.6. *Let* (L, W) *be a linear series with* $\dim W \geq r + 2$*, and let* $P = P_L$ *be its conormal variety. The projection* $\sigma \colon P \longrightarrow |W|$ *is generically finite onto its image if and only if* (L, W) *is non-defective.*

Proof. Since $\dim W \geq r + 2$, the conormal variety $P = P_{L,W}$ is non-empty. Let $(x, s) \in P$ be a general point. We may assume that $x \in U$. Then x is a singular point of $v(s)$, and it is an isolated singularity of $v(s)$ if and only if (x, s) is an isolated point in the fiber of $\sigma \colon P \longrightarrow |W|$ over s. The conclusion follows. $\square$

The following observation relates non-defectivity with the non-degeneracy of the dual.

{prop:non-deg-du

Proposition 3.7. *Let $X \subset \mathbf{P}^n$ be a non-degenerate projective variety. Let $L = \mathcal{O}_X(1)$ and $W \subset H^0(X, L)$ the image of $H^0(\mathbf{P}^n, \mathcal{O}(1))$. Then (L, W) is non-defective if and only if the dual variety $X^* \subset \mathbf{P}^{n*}$ is a hypersurface.*

Proof. Since $X \subset \mathbf{P}^n$ is not contained in a hyperplane, we have $\dim W = n + 1 \geq r + 1$. Since (L, W) is very ample, it separates tangent vectors on X, so the evaluation map

$$e \colon W \otimes \mathcal{O}_X \longrightarrow P(L)$$

is surjective. It follows that the rank of the kernel is $n - r$, and hence

$$\dim P_{L,W} = (n - r - 1) + r = n - 1.$$

By definition, the dual variety $X^* \subset \mathbf{P}^{n*} = |W|$ is the image of the conormal variety under the projection $P_{L,W} \longrightarrow |W|$. By Proposition 3.6, (L, W) is non-defective if and only if $\dim X^* = n - 1$. ☐

narydoublepoint}

Proposition 3.8. *Let (L, W) be a non-defective linear series on X with $\dim W \geq r + 2$. Let $x \in X$ be a general point. Then there exists $s \in |W|$ such that $v(s)$ has an ordinary double point singularity at x.*

Proof. By Proposition 3.6, the projection $\sigma \colon P \longrightarrow |W|$ is generically finite onto its image. Let $(x, s) \in P$ be a general point. Since our ground field is of characteristic zero, we may assume that P is smooth at (x, s), that $x \in U \cap X^{\mathrm{sm}}$, and $\sigma \colon P \longrightarrow |W|$ is a local immersion at (x, s). This implies that $x \in \mathrm{Sing}(v(s))$ is isolated, and also that x is a reduced point of the scheme $\mathrm{Sing}(v(s))$. These two properties show that $v(s)$ possesses an ordinary double point at x. To see this, choose local coordinates $(x_1, ..., x_n)$ so that the complete local ring $\widehat{\mathcal{O}}_{X,x}$ is isomorphic to $k[\![x_1, \dots, x_r]\!]$. After choosing a local trivialization for L around x, the section s corresponds to a power series $s(x_1, \dots, x_r)$ contained in $\mathfrak{m}_x^2 \widehat{\mathcal{O}}_{X,x}$. The germ of $\mathrm{Sing}(v(s))$ at x is cut out by the power series $\frac{\partial s}{\partial x_1}, \dots, \frac{\partial s}{\partial x_r}$. Since the germ of $\mathrm{Sing}(v(s))$ at x is the reduced point x, we get that $\frac{\partial s}{\partial x_1}, \dots, \frac{\partial s}{\partial x_r}$ are linearly independent as elements of $\mathfrak{m}_x/\mathfrak{m}_x^2$. From this, it is easy to check that the tangent cone of $s(x_1, \dots, x_r)$ at x is a non-degenerate quadric cone. ☐

eparateTangents}

Proposition 3.9. *If (L, W) is a non-defective linear series with $\dim W \geq r + 1$, then W separates tangent vectors at a general point $x \in X$. That is, the evaluation map*

$$e_x \colon W \otimes \mathcal{O}_X \longrightarrow L/\mathfrak{m}_x^2 L$$

is surjective for general $x \in X$.

Proof. By the definition of $P(L)$, we have a natural isomorphism

$$P(L)|_x = L/\mathfrak{m}_x^2 L,$$

so it suffices to show that the evaluation map

$$e \colon W \otimes \mathcal{O}_X \longrightarrow P(L)$$

is surjective at x. Let k be the generic rank of K, the kernel of e. From the proof of Proposition 3.5, we get

$$k = \dim W - r - 1.$$

Since $(r+1)$ is the generic rank of $P(L)$, we conclude that e is generically surjective. $\square$

{cor:properlyram

Corollary 3.10. *Suppose (L, W) is a non-defective linear series on X with $\dim W \geq r+1$. Then there exists a properly ramified projection (L, V) of (L, W).*

Proof. This follows immediately from Proposition 3.9. $\square$

As a consequence of Corollary 3.10, the projection-ramification rational map $\rho_{X,L,W}$ is defined for a non-defective linear series (L, W) with $\dim W \geq r+1$.

Let $\pi \colon \widetilde{X} \longrightarrow X$ be the blow-up at a point $x \in X$, and $E \subset \widetilde{X}$ the exceptional divisor. A linear series (L, W) on X gives a linear series $(\widetilde{L}, \widetilde{W})$ as follows. Take $\widetilde{L} = \pi^* L \otimes \mathcal{O}_{\widetilde{X}}(-E)$. Note that $H^0(X, L) = H^0(\widetilde{X}, \pi^* L)$, so we may think of W as a subspace of $H^0(\widetilde{X}, \pi^* L)$. Take $\widetilde{W} = W \otimes \mathcal{O}_{\widetilde{X}}(-E)$ with its natural inclusion $\widetilde{W} \subset H^0(\widetilde{X}, \widetilde{L})$.

{prop:blowuppoin

Proposition 3.11. *In the setup above, if (L, W) is non-defective, $\dim W \geq r+2$, and $x \in X$ is general, then $(\widetilde{L}, \widetilde{W})$ is also non-defective.*

Proof. Let y be a general point of $\widetilde{X}$. We have the equality

$$\widetilde{W} \otimes \mathfrak{m}_y^2 = W \otimes \mathfrak{m}_x \cdot \mathfrak{m}_y^2.$$

By Proposition 3.9, for a general $y \in X$, we have

$$\dim(W \otimes \mathfrak{m}_y^2) = \dim W - (r+1).$$

Since $x \in X$ is general, we get

$$\dim(W \otimes \mathfrak{m}_x \cdot \mathfrak{m}_y^2) = \dim W - (r+2).$$

If $\dim W = r+2$, then we get $\widetilde{W} \otimes \mathfrak{m}_y^2 = 0$, so we are done. Assume that $\dim W \geq r+3$. Then $\dim(W \otimes \mathfrak{m}_y^2) \geq 2$. Since (L, W) is non-defective, a general $s \in W \otimes \mathfrak{m}_y^2$ is such that $v(s)$ has an isolated singularity at y. Moreover, since $\dim(W \otimes \mathfrak{m}_y^2) \geq 2$, for every $x \in X$, there exists $s \in V$ such that $v(s)$ passes through x. Hence, as $x \in X$ is general, there exists $s \in W \otimes \mathfrak{m}_y^2$ such that $v(s)$ has an isolated singularity at y and passes through x. That is, there exists $s \in \widetilde{W} \otimes \mathfrak{m}_y^2$ that has an isolated singularity at y. We conclude that $(\widetilde{L}, \widetilde{W})$ is non-defective. $\square$

3.2. **Maximal variation for non-defective pairs.** In this section, we prove part (2) of Theorem A. In fact, we prove a more general result (Theorem 3.12).

As before, X is a proper, normal variety of dimension r over an algebraically closed field of characteristic zero.

{thm:mainMain}

Theorem 3.12. *Let (L, W) be a non-defective linear series on X with $\dim W \geq r+2$. Then the projection-ramification map $\rho_{X,L,W}$ is generically finite onto its image.*

For the proof, we need two lemmas, which are essentially local computations. Throughout, X, L, and W are as in the statement of Theorem 3.12.

:tangentconeRam}

Lemma 3.13. *Let $x \in X$ be a general point and $V \subset W \otimes \mathfrak{m}_x$ a general $(r+1)$-dimensional subspace. Then V is properly ramified, and the ramification divisor $R(V)$ has an ordinary double point singularity at x.*

Proof. Using Proposition 3.8 and Proposition 3.9, we get a basis $(s_1, ..., s_n, t)$ of V satisfying the following two conditions:

(1) $s_1, \dots, s_n$ generate $L \otimes (\mathfrak{m}_x/\mathfrak{m}_x^2)$, and
(2) $v(t)$ has an ordinary double point singularity at x.

Let $\widehat{\mathcal{O}}_{X,x}$ denote the completion of the local ring at $x \in X$ along its maximal ideal. Upon trivializing L, we may regard s_i and t as elements of $\widehat{\mathcal{O}}_{X,x}$, and can also assume $\widehat{\mathcal{O}}_{X,x} = k[\![s_1, \dots s_n]\!]$. In the bases $(s_1, \dots, s_n, t)$ for V and $(1, s_1, \dots, s_n)$ for $P(L)$, the evaluation map

$$e \colon V \otimes \widehat{\mathcal{O}}_{X,x} \longrightarrow P(L) \otimes \widehat{\mathcal{O}}_{X,x}$$

has the matrix

{matrix}

$$\begin{pmatrix} s_1 & s_2 & \dots & t \\ 1 & 0 & \dots & \partial_1 t \\ 0 & 1 & \dots & \partial_2 t \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \partial_n t \end{pmatrix}, \tag{3.1}$$

where ∂_i denotes $\frac{\partial}{\partial s_i}$. The determinant of the matrix (3.1)

$$t - \sum_i s_i \partial_i t$$

is an analytic local equation for the ramification divisor $R(V)$ near x. Evidently, $R(V)$ shares the same tangent cone as $v(t)$ at x. The proposition follows. $\square$

a:basepointfree}

Lemma 3.14. *Let $x \in X$ be a general point and $V \subset W$ an $(r+1)$-dimensional subspace with a basis $(u, a_1, \dots, a_{r-1}, b)$ where*

(1) u does not vanish at x,
(2) $a_1, \dots, a_{r-1}$ vanish at x, and reduce to linearly independent elements of $L \otimes (\mathfrak{m}_x/\mathfrak{m}_x^2)$, and
(3) $v(b)$ has an ordinary double point at x.

Then $R(V)$ contains x and is smooth at x.

Proof. That $R(V)$ contains x is clear since $V \otimes \mathfrak{m}_x^2 \neq 0$.

For smoothness, we again work in the completion $\widehat{\mathcal{O}}_{X,x}$. After trivializing L, we assume $u, a_1, ..., b$ are elements of $\widehat{\mathcal{O}}_{X,x}$. We choose an element $z \in \widehat{\mathcal{O}}_{X,x}$ such that $(a_1, \dots, a_{r-1}, z)$

forms a system of coordinates, that is $\widehat{\mathcal{O}}_{X,x} \cong k[\![a_1, \dots, a_{r-1}, z]\!]$. With respect to the given basis of V and the basis $1, a_1, \dots, a_{r-1}, z$ for $P(L)$, the evaluation map

$$e\colon V \otimes \widehat{\mathcal{O}}_{X,x} \longrightarrow P(L) \otimes \widehat{\mathcal{O}}_{X,x}$$

has the matrix

$$\begin{pmatrix} u & a_1 & a_2 & \dots & b \\ \partial_1 u & 1 & 0 & \dots & \partial_1 b \\ \partial_2 u & 0 & 1 & \dots & \partial_2 b \\ \vdots & \vdots & \vdots & \vdots & \\ \partial_z u & 0 & 0 & \dots & \partial_z b \end{pmatrix} \tag{3.2}$$ {matrix2}

The determinant of the matrix (3.2) is the analytic local equation for $R(V)$. It is given by

$$\bar{u} \cdot \partial_z b \pm \partial_z u \cdot \bar{b},$$

where, for $r \in \widehat{\mathcal{O}}_{X,x}$ we set

$$\bar{r} = r - a_1 \partial_1 r - a_2 \partial_2 r - \dots - z \partial_z r.$$

Since $b \in \mathfrak{m}_x^2$, we get that $\bar{b} \in \mathfrak{m}_x^2$, and so $\partial_z b \in \mathfrak{m}_x$. Furthermore, since the tangent cone of b is a non-degenerate quadric, we also get that $\partial_z b \notin \mathfrak{m}_x^2$. Since $\overline{u}$ is a unit, we see that the tangent cone of $R(V)$ at x is the hyperplane cut out by $\partial_z b \in \mathfrak{m}_x/\mathfrak{m}_x^2$. So $R(V)$ is smooth at x. $\square$

We now have all the tools for the proof of Theorem 3.12.

Proof of Theorem 3.12. We induct on $\dim W$. The base case $\dim W = r + 1$ is clear.

We now do the induction step. Suppose $\dim W \geq r + 2$. Choose a general point $x \in X$ such that the induced linear series $(\widetilde{L}, \widetilde{W})$ on $\widetilde{X} = \mathrm{Bl}_x X$ is non-defective as in Proposition 3.11. Choose a general $(r+1)$-dimensional subspace $V \subset W \otimes \mathfrak{m}_x = \widetilde{W}$ that satisfies the hypotheses of Lemma 3.13. By the induction hypothesis, V considered as a projection of $(\widetilde{L}, \widetilde{W})$ is an isolated point in the projection-ramification map for $\widetilde{X}$. We now show that it is also an isolated point in the projection-ramification map for X.

Let $(C, 0)$ be a pointed smooth curve and $V \subset W \otimes \mathcal{O}_C$ a sub-bundle of rank $(r + 1)$ such that

(1) $V_0 = V$, and
(2) $V_c \neq W_0$ for $c \in C \setminus \{0\}$.

We must show that $R(V_c) \neq R(V)$ for a general $c \in C$.

Suppose $V_c \subset W \otimes \mathfrak{m}_x = \widetilde{W}$ for all $c \in C$. Denote by $\widetilde{R}(V_c)$ the ramification divisor of V_c considered as a projection of $\widetilde{X}$. Since $V = V_0$ is an isolated point in the projection-ramification map for $\widetilde{X}$, we know that $\widetilde{R}(V_c) \neq \widetilde{R}(V_0)$ for a general $c \in C$. Clearly, $R(V_c)$ and $\widetilde{R}(V_c)$ agree away from the exceptional divisor, and hence we conclude that $R(V_c) \neq R(V_0)$ for a general $c \in C$.

On the other hand, suppose $V_c \not\subset W \otimes \mathfrak{m}_x = \widetilde{W}$ for a general $c \in C$. Consider the evaluation maps

$$e_c \colon V_c \longrightarrow L/\mathfrak{m}_x^2 L$$

between an $(r+1)$-dimensional source and $(r+1)$-dimensional target. Since $V = V_0$ satisfies the hypotheses of Lemma 3.13, $\operatorname{rk} e_0 = r$. Therefore, by semi-continuity, $\operatorname{rk} e_c \geq r$ for all $c \in C$. If $\operatorname{rk} e_c = (r+1)$ for a general $c \in C$, then $x \notin R(V_c)$, and hence $R(V_c) \neq R(V)$. Otherwise, by shrinking C if necessary, assume $\operatorname{rk} e_c = r$ for all $c \in C$. In other words, $\dim(V_c \otimes \mathfrak{m}_x^2) = 1$ for all $c \in C$. Let $b_c \in V_C \otimes \mathfrak{m}_x^2$ be a non-zero element. Since $v(b_0)$ has an ordinary double-point singularity at x, so does $v(b_c)$. Also, since $\operatorname{rk}(e_c) = r$ and $V_c \notin W \otimes \mathfrak{m}_x$ for a general c, there exists $u_c \in V_c$ not vanishing at x, and a set of $(r-1)$ other elements that vanish at x but reduce to linearly independent elements modulo $\mathfrak{m}_x^2$. That is, V_c satisfies the hypotheses of Lemma 3.14 for a general $c \in C$. But Lemma 3.14 implies that $R(V_c)$ is smooth at x. Since $R(V_0)$ is singular at x, we conclude that $R(V_0) \neq R(V_c)$. The induction step is now complete. ☐

We immediately get part (2) of Theorem A.

cor:maintheorem}

Corollary 3.15. *Let $X \subset \mathbf{P}^n$ be a non-degenerate projective variety such that the dual variety $X^* \subset \mathbf{P}^{n*}$ is a hypersurface. Then ρ_X is generically finite onto its image.*

Proof. By Proposition 3.7 the linear series on X that gives the embedding $X \subset \mathbf{P}^n$ is non-defective. Now apply Theorem 3.12. ☐

c:minimaldegree}

4. Projection-ramification for varieties of minimal degree

In this section, we prove Theorem C, which relates varieties of minimal degree and the projection-ramification map. We then prove Theorem B by constructing examples of rational scrolls where maximal variation fails. Finally, we obtain an alternate description of the projection-ramification map for scrolls, which is used in **??**.

The following is an easy application of the Kodaira vanishing theorem.

{lemma:KYmH}

Proposition 4.1. *Let $X \subset \mathbf{P}^n$ be a non-degenerate, smooth, projective, variety of dimension $r \geq 1$ over a field of characteristic zero. For all $m \geq r$, we have the inequality*

{eqn:KYmH}

$$\binom{m}{r}(n-r) + \binom{m-1}{r} \leq h^0(X, K_X + mH). \tag{4.1}$$

If equality holds for any $m \geq r$, then X is a variety of minimal degree, that is $\deg X = n - r + 1$. Conversely, for a variety of minimal degree, equality holds for all $m \geq r$.

Proof. Without loss of generality, X is embedded by the complete linear series. Indeed, passing to the complete linear series only increases the left side of the desired inequality, and does not change the right side.

We first prove the inequality (4.1), using a double induction–first on r, and then on m. For the base case $r = 1$, Riemann–Roch gives

{eqn:r1}

$$h^0(X, K_X + mH) = g_X - 1 + mn, \tag{4.2}$$

from which (4.1) follows for all m.

Assume that (4.1) holds for varieties of dimension $(r-1)$ and all $m \geq r-1$. Let $D \subset X$ be a general member of the linear series $|H|$. By Bertini's theorem, D is a smooth variety. The adjunction formula $K_D = (K_X + H)|_D$ yields the exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X + (m-1)H) \longrightarrow \mathcal{O}_X(K_X + mH) \longrightarrow \mathcal{O}_D(K_D + (m-1)H) \longrightarrow 0. \tag{4.3}$$ {eqn:mainexact}

Note that, by the Kodaira vanishing theorem, we have $h^1(K_X + nH) = 0$ for all $n > 1$; we use this repeatedly, without further comment. For $m = r$, the long exact sequence in cohomology associated to (4.3) gives

$$h^0(K_D + (r-1)H) \leq h^0(K_X + rH).$$

By applying the induction hypothesis to D, we have

$$n - r \leq h^0(K_D + (r-1)H) \tag{4.4}$$

Therefore, we conclude that

$$(n-r) \leq h^0(K_D + rH). \tag{4.5}$$

Let $m > r$, and assume that (4.1) holds for X for $m-1$. The long exact sequence in cohomology associated to (4.3) gives

$$h^0(K_X + (m-1)H) + h^0(K_D + (m-1)H) = h^0(K_X + mH). \tag{4.6}$$ {eqn:add}

By applying the induction hypothesis to $m-1$, we get

$$\begin{aligned} h^0(K_X + (m-1)H) &+ h^0(K_D + (m-1)H) \\ &\geq \binom{m-1}{r}(n-r) + \binom{m-2}{r} + \binom{m-1}{r-1}(n-r) + \binom{m-2}{r-1} \\ &= \binom{m}{r}(n-r) + \binom{m-1}{r}. \end{aligned}$$

Together with (4.6), we conclude

$$\binom{m}{r}(n-r) + \binom{m-1}{r} \leq h^0(K_X + mH), \tag{4.7}$$

which is (4.1) for m. The proof of the inequality is thus complete.

We now examine when equality holds in (4.1). For $r = 1$, the equation (4.2) shows that equality holds for some m if and only if $g_X = 0$, that is $X \subset \mathbf{P}^n$ is a rational normal curve, and in this case, equality holds for all m. Furthermore, we observe in the inductive proof that if equality holds for an X of dimension $r > 1$ and some m, then it must hold for the hyperplane slice D and $(m-1)$. Again, by an induction on r, we conclude that $\deg X = n - r + 1$, that is, $X \subset \mathbf{P}^n$ is a variety of minimal degree.

Finally, for $X \subset \mathbf{P}^n$ of minimal degree, induction on r shows that equality holds in (4.1) for all m. $\square$

As a consequence, we immediately deduce Theorem C.

Theorem 4.2 (Theorem C)**.** *Let $X \subset \mathbf{P}^n$ be a smooth, non-degenerate projective variety of dimension $r \geq 1$ over a field of characteristic zero. We have the inequality*

$$\dim \mathbf{Gr}(n-r, n+1) \leq \dim |K_X + (r+1)H|,$$

where equality holds if and only if X is a variety of minimal degree, that is $\deg X = n-r+1$.

Proof. Apply Proposition 4.1 with $m = r+1$. □

4.1. **Projection-ramification for scrolls.** Theorem C motivates a deeper investigation of the projection-ramification map for varieties of minimal degree. Indeed, for $X \subset \mathbf{P}^n$ of minimal degree, the projection-ramification map is potentially generically finite. Recall that a large class of varieties of minimal degree are the rational normal scrolls, namely $X = \mathbf{P}E$ for an ample vector bundle E on $\mathbf{P}^1$ embedded by the complete linear series $\mathcal{O}_X(1)$. If $\dim X \geq 3$, then X is neither incompressible nor does it have a divisorial dual variety. Therefore, for such X, Theorem A leaves the question of maximal variation unanswered.

We now examine the projection-ramification map for projectivizations of vector bundles on smooth curves in more detail. Let C be a smooth curve and E an ample vector bundle on C of rank r. Set $X = \mathbf{P}E$, the space of one-dimensional quotients of E, and $L = \mathcal{O}_X(1)$. Denote by $\pi\colon X \longrightarrow C$ the natural map.

Let (L, V) be a projection of X. Recall from (2.1) that such a projection gives a map

$$r_V\colon \det V \longrightarrow H^0(X, K_X \otimes L^{r+1}),$$

whose zero locus is the ramification divisor $R(V) \subset X$. Note that we have an isomorphism $K_X \cong \pi^*(\det E \otimes K_C) \otimes L^{-r}$, and hence, we may view r_v as a map

$$r_V\colon \det V \longrightarrow H^0(C, E \otimes \det E \otimes K_C).$$

We now describe another construction of a section of $E \otimes \det E \otimes K_C$ from V, which we call the *differential construction*. The subspace $V \subset H^0(X, L) = H^0(C, E)$ gives the evaluation map

$$e\colon V \otimes \mathcal{O}_C \longrightarrow E.$$

If V is generic, then e is a surjection, and its kernel is canonically isomorphic to $\det E^* \otimes \det V$. Consider the diagram

al_construction}

$$\begin{tikzcd}
0 \arrow[r] & \det E^* \otimes \det V \arrow[r] \arrow[d, "d_V"] & V \otimes \mathcal{O}_C \arrow[r, "e"] \arrow[d, "e"] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_C \otimes E \arrow[r] & P(E) \arrow[r] & E \arrow[r] & 0,
\end{tikzcd} \tag{4.8}$$

where the bottom row is the standard sequence associated to $P(E)$, both maps labeled e are evaluation maps, and the map d_V is the map induced by them. The map d_V gives a map

$$d_V\colon \det V \longrightarrow H^0(C, E \otimes \det E \otimes K_C).$$

{prop:rdv}

Proposition 4.3. *In the setup above, the two maps d_V and r_V are equal.*

Proof. Recall that r_V is induced by the determinant of the evalutation map

$$V \otimes \mathcal{O}_X \longrightarrow P(L).$$

Denote by $P_\pi(L)$ the bundle of principal parts of L along the fibers of π. More explicitly,

$$P_\pi(L) = \pi_{1*}\left(\pi_2^* L \otimes \left(\mathcal{O}_{X\times_\pi X}/I_\Delta^2\right)\right),$$

where $\Delta \subset X \times_\pi X$ is the diagonal and π_i for $i = 1, 2$ are the two projections $X \times_\pi X \longrightarrow X$. It is easy to check that the evaluation map $\pi^* E \longrightarrow L$ induces an isomorphism $\pi^* E \longrightarrow P_\pi(L)$. Furthermore, we have the sequence

$$0 \longrightarrow \pi^* K_C \otimes L \longrightarrow P(L) \longrightarrow P_\pi(L) \longrightarrow 0.$$

By combining this with the identification $\pi^* E = P_\pi(L)$, and the top row of (4.8), we get the diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi^*(\det E^* \otimes \det V) \arrow[r] \arrow[d, "p"] & V \otimes \mathcal{O}_X \arrow[r] \arrow[d, "e"] & \pi^* E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \pi^* K_C \otimes L \arrow[r] & P(L) \arrow[r] & P_\pi(L) \arrow[r] & 0. \end{tikzcd} \tag{4.9}$$

{eqn:pxpl}

From the diagram, we see that $\det e = p$, interpreted as elements of the appropriate Hom spaces. By definition, after taking global sections, $\det e$ gives the section r_V. Note that, applying π_* to the bottom row of (4.9) yields the bottom row of (4.8). Hence, after applying π_*, twisting by $\det E$ and taking global sections, p gives the section d_V. We conclude that $r_V = d_V$. $\square$

Let $R = R(V) \subset X$ be the ramification divisor of the projection given by V. Note that R is a divisor of class $\pi^*(\det E \otimes K_C) \otimes \mathcal{O}_X(1)$. Therefore, $R \subset X$ is a sub-scroll, or equivalently, the fibers of $R \longrightarrow C$ are hyperplanes in the corresponding fiber of $X \longrightarrow C$. We can obtain an explicit description of these hyperplanes in two ways, one using the original definition, and one using the differential construction. Fix a point $c \in C$, and a uniformizer t of C at c. Let $X_c \subset X$ and $R_c \subset R$ be the fibers of $X \longrightarrow C$ and $R \longrightarrow C$ over c, respectively.

By definition $R \subset X$ is the set of points $x \in X$ for which there exists $s \in V$ such that $v(s)$ is singular at x. Since s is a section of $L = \mathcal{O}_X(1)$, the hypersurface $v(s)$ is singular at x if and only if it contains the entire fiber of $\pi \colon X \longrightarrow C$ through x. Suppose $\pi(x) = c$. Then, in an open set of X containing X_c, we have $s = ts_1$ for a section s_1 of $\mathcal{O}_X(1)$. Observe that, we have $\operatorname{Sing}(v(s)) \cap F = v(s_1) \cap F$, and therefore, $R_c \subset X_c$ is the hyperplane cut out by s_1.

To obtain the same description using the differential construction, consider the top row of (4.8). Let v be a local section of $V \otimes \mathcal{O}_C$ around c that generates the kernel of $e \colon V \otimes \mathcal{O}_C \longrightarrow E$ at c. The fiber of the evaluation map $V \otimes \mathcal{O}_C \longrightarrow P(L)$ over c sends $v \in V$ to the image of $e(v)$ in $L/\mathfrak{m}_c^2 L$. Since v generates the kernel of $e \colon V \otimes \mathcal{O}_C \longrightarrow L$ at c, we know that image of $e(v)$ in $L/\mathfrak{m}_c L$ is zero. Writing $e(v) = ts_1$ for a section s_1 of E around c, we see that $d_V(v) = s_1 \otimes t \in E \otimes \mathfrak{m}_c/\mathfrak{m}_c^2$. Thus, the fiber of the sub-scroll defined by d_V over c is the hyperplane in X_c cut out by s_1.

Finally, we write an equation of $R(V) \subset X$ over an open subset of C containing c explicitly in coordinates. Choose a trivialization $X_1, \dots, X_r$ for E over an open set $U \subset C$ containing c. Then $X_U \cong \mathbf{P}^{r-1} \times U = \operatorname{Proj} \mathcal{O}_U[X_1, \dots, X_r]$. We have a trivialization of K_C over U given by dt. We then get a trivialization of $P(E)|_U$ by $X_1, \dots, X_r, dt \otimes X_1, \dots, dt \otimes X_r$. Choose a basis $v_0, \dots, v_r$ of V, and suppose the map $e \colon V \otimes \mathcal{O}_U \longrightarrow E_U$ is given by

$$e(v_i) = \sum m_{i,j} X_j,$$

for $m_{i,j} \in \mathcal{O}_U$, where $0 \le i \le r$ and $1 \le j \le r$. Then the map $\det E^* \otimes \det V \longrightarrow V \otimes \mathcal{O}_U$ defining the kernel of e is given by the $r \times r$ minors of the matrix $(m_{i,j})$. Denote the ℓ-th minor by M_ℓ; that is $M_\ell = (-1)^\ell \det(m_{i,j} \mid i \neq \ell)$. Then the map d_V sends the generator to the element of $E \otimes K_C$ given by

$$\sum_{i,j} M_i \cdot \frac{\partial m_{i,j}}{\partial t} \cdot (dt \otimes X_j).$$

Note that the expression above is the determinant of the $(r+1) \times (r+1)$ matrix

{eqn:Rmatrix}
$$\begin{pmatrix} m_{0,1} & m_{0,2} & \dots & m_{0,r} & \sum_{i=1}^r \frac{\partial m_{0,j}}{\partial t} \cdot dt \otimes X_j \\ m_{1,1} & m_{1,2} & \dots & m_{1,r} & \sum_{i=1}^r \frac{\partial m_{1,j}}{\partial t} \cdot dt \otimes X_j \\ \vdots & \ddots & \dots & \vdots & \vdots \\ m_{r,1} & m_{r,2} & \dots & m_{r,r} & \sum_{i=1}^r \frac{\partial m_{r,j}}{\partial t} \cdot dt \otimes X_j \end{pmatrix}. \tag{4.10}$$

This gives an equation for $R_U \subset X_U = \operatorname{Proj} \mathcal{O}_U[X_1, \dots, X_r]$.

4.2. **Failure of maximal variation.** In this section, we show that there exists ample vector bundles E of rank $r \geq 4$ on $\mathbf{P}^1$ such that the projection-ramification map for $X = \mathbf{P}E$ is not generically finite. In other words, a generic projection of X can be deformed in a one-parameter family so that the ramification divisor remains unchanged.

Recall that the projection-ramification map for $X = \mathbf{P}E$ and the complete linear series of $L = \mathcal{O}_X(1)$ is a map

$$\rho \colon \mathbf{Gr}(r+1, H^0(X, L)) \dashrightarrow |K_X \otimes L^{r+1}|,$$

or equivalently a map

$$\rho \colon \mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E)) \dashrightarrow \mathbf{P}H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1})^*.$$

By construction, ρ is equivariant with respect to the action of $\operatorname{Aut}(X)$, and in particular, by the subgroup $\operatorname{Aut}(X/\mathbf{P}^1)$.

We engineer the failure of maximal variation using the following observation.

ivialStabilizer}

Proposition 4.4. *A generic point of* $\mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E))$ *has a trivial stabilizer under the action of* $\operatorname{Aut}(\mathbf{P}E/\mathbf{P}^1)$.

Proof. Fix $(r+1)$ distinct points $p_0, \dots, p_r \in \mathbf{P}^1$. Let $V \subset H^0(\mathbf{P}^1, E)$ be a generic $(r+1)$ dimensional subspace. Let $e \colon V \otimes \mathcal{O}_{\mathbf{P}^1} \longrightarrow E$ be the evaluation map. The points $p_0, \dots, p_r$ give vectors $v_0, \dots, v_r \in V$, unique up to scaling, defined by the property that $e(v_i) = 0$ in the fiber $E|_{p_i}$. Choose a generic point $t \in \mathbf{P}^1$. We get $(r+1)$ points

$x_0, \ldots, x_r \in \mathbf{P}E^*|_t \cong \mathbf{P}^{r-1}$ given by $e(v_0), \ldots, e(v_r)$ evaluated at t. For generic V and t, it is easy to check that these points are in linear general position. Any element of $\mathrm{Aut}(\mathbf{P}E/\mathbf{P}^1)$ that fixes V must fix $x_0, \ldots, x_r$. But then it must act as the identity on the projective space $\mathbf{P}E^*|_t$, and hence on the dual projective space $\mathbf{P}E|_t$. Since $t \in \mathbf{P}^1$ is general, it follows that it must be the identity. $\square$

{prop:specialE}

Proposition 4.5. *There exist ample vector bundles E of every rank ≥ 4 such that a general point of $\mathbf{P}H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1})$ has a positive-dimensional stabilizer under $\mathrm{Aut}(\mathbf{P}E/\mathbf{P}^1)$. In particular, we may take $E = \mathcal{O}(1)^{r-1} \oplus \mathcal{O}(k+1)$ where $k \geq 1$ and $r \geq 4$.*

Proof. It suffices to exhibit an E such that a generic element of $H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1})$ has a positive dimensional stabilizer under the action of $\mathrm{Aut}(E/\mathbf{P}^1)$. Take

$$E = \mathcal{O}(a)^{r-1} \oplus \mathcal{O}(b),$$

where $0 < a < b$ are to be determined. Elements of $\mathrm{Aut}(E/\mathbf{P}^1)$ can be represented by block lower triangular square matrices

$$M = \begin{pmatrix} A & \\ U & B \end{pmatrix},$$

where $A \in \mathrm{GL}_a(k)$, $B \in k^\times$, and $U = (u_i)$ is an $(r-1)$ length row with entries in $H^0(\mathbf{P}^1, \mathcal{O}(b-a))$. Set $d = (r-1)a + b$ so that $\det E = \mathcal{O}(d)$. Suppose a, b, and r, are such that

$$(r-1)(b-a+1) \geq b + d - 1 = (r-1)a + 2b - 1. \tag{4.11}$$ {eqn:requirement

Take a general element of $H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1})$; say it is given by the column vector

$$v = (p_1, \ldots, p_{r-1}, q)^T,$$

where the p_i (resp q) are homogeneous polynomials in X, Y of degree $a + d - 2$ (resp $b + d - 2$). We take $A = \mathrm{id}_{r-1}$ and $B = \lambda$ for some $\lambda \in k^\times$, and show that there exists a $U = (u_i)$ such that $Mv = v$. Indeed, we have $Mv = (p_1, \ldots, p_r, q')$, where

$$q' = \lambda q + \sum u_i p_i.$$

Let $W \subset H^0(\mathbf{P}^1, \mathcal{O}(a + d - 1))$ be the vector space spanned by $p_1, \ldots, p_{r-1}$. Consider the multiplication map

$$H^0(\mathbf{P}^1, \mathcal{O}(b-a)) \otimes W \longrightarrow H^0(\mathbf{P}^1, \mathcal{O}(b + d - 2)).$$

Thanks to (4.11), the dimension of the source is at least as much as the dimension of the target. It is easy to check that the map is in fact surjective for generic $p_1, \ldots, p_{r-1}$. In particular, there exist $u_i \in H^0(\mathbf{P}^1, \mathcal{O}(b-a))$ for $i = 1, \ldots, r-1$, such that

$$q(1-\lambda) = \sum u_i p_i.$$

With this choice of $U = (u_i)$, we get M such that $Mv = v$.

Finally, note that the requirement (4.11) is satisfied for $a = 1$ and $b = k + 1$ if $k \geq 1$ and $r \geq 4$. $\square$

Corollary 4.6 (Theorem B)**.** *Let $r \geq 3$ and $d \geq r+1$. There exist ample vector bundles E of rank r and degree d on $\mathbf{P}^1$ such that for $X = \mathbf{P}E$ and the complete linear series $L = \mathcal{O}_X(1)$, the projection-ramification map ρ_X is not generically finite onto its image.*

Proof. Take E such that the action of $\mathrm{Aut}(X/\mathbf{P}^1)$ on a generic point of $|K_X \otimes L^{r+1}|$ has a positive-dimensional stabilizer (see Proposition 4.5). Since $\rho_X : \mathbf{Gr}(r+1, H^0(X, L)) \dashrightarrow |K_X \otimes L^{r+1}|$ is equivariant with respect to the action of $\mathrm{Aut}(X/\mathbf{P}^1)$, and a generic point of the source does not have a positive-dimensional stabilizer (see Proposition 4.4), it follows that ρ_X cannot be dominant. Since the dimension of the source and target of ρ_X are the same, ρ_X is not generically finite. $\square$

Remark 4.7. In all the examples of scrolls where we know that maximal variation fails, the failure can be explained by the presence of generic stabilizers. We do not know, however, if this is the only reason for the failure of maximal variation.

Remark 4.8. If $k = 1$ and $r \geq 4$, then X is the most balanced scroll of its degree and rank, and hence, generic in moduli. Therefore, the non-dominance of projection-ramification is not directly connected to the eccentricity of the splitting type of a scroll.

5. Maximal variation for generic scrolls

In this section, we establish that the projection-ramification map is generically finite (equivalently, dominant) for most scrolls, notwithstanding the examples provided by Theorem B. We begin by treating the cases of some particular scrolls by hand. We then bootstrap these to more general results using degeneration arguments.

5.1. **Maximal variation for some particular cases.** Given an ample vector bundle E on $\mathbf{P}^1$, we say that *maximal variation holds for E* if the projection-ramification map is generically finite (equivalently, dominant) for $X = \mathbf{P}E$ embedded by the complete linear series associated to $L = \mathcal{O}_X(1)$.

{prop:segre}

Proposition 5.1. *Maximal variation holds for $E = \mathcal{O}(1)^r$. In fact, the degree of the projection-ramification map in this case is* 1*.*

Proof. We know that the projection-ramification map

$$\rho\colon \mathbf{Gr}(r+1, H^0(\mathbf{P}^1, \mathcal{O}(1)^r)) \dashrightarrow \mathbf{P}H^0(\mathbf{P}^1, \mathcal{O}(r-1)^r)^*$$

is $\mathrm{Aut}\,\mathbf{P}E$ equivariant. In this case, it is easy to check that the action of $\mathrm{Aut}(\mathbf{P}E/\mathbf{P}^1) = \mathrm{PGL}_r$ has a unique open orbit and trivial generic stabilizers on both the source and the target of ρ. Hence, ρ must be birational. $\square$

{prop:222}

Proposition 5.2. *Maximal variation holds for $E = \mathcal{O}(2)^r$.*

Compared to Proposition 5.1, our proof of Proposition 5.2 is significantly more involved, and does not yield the degree.

Proof. We exhibit a point $\mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E))$ at which ρ is defined, and at which the induced map $d\rho$ on the tangent space is non-singular. It follows that ρ is a local isomorphism at this point, and hence dominant overall.

Our proof is by direct calculation. We calculate on $\mathbf{A}^1 = \operatorname{Spec} k[x] \subset \mathbf{P}^1$ and identify $\mathcal{O}(n)$ with $\mathcal{O}(n \cdot \infty)$. Then the global sections of $\mathcal{O}(n)$ are identified with polynomials in x of degree at most n. Denote the generator of the ith summand of $E(-2)$ by X_i. Consider the point of $\mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E))$ represented by the vector space $V \subset H^0(\mathbf{P}^1, E)$ spanned by the $(r+1)$ sections $v_1, \ldots, v_{r+1}$ defined as follows. Set $v_i = (x-a_i)^2 X_i$ for $0 \le i \le r-1$, and $v_r = \sum p_i X_i$, where $a_i \in k$, and $p_j \in H^0(\mathbf{P}^1, \mathcal{O}(2))$ are generic. By (4.10), the ramification divisor associated to V is cut out by the determinant of the matrix

$$M = \begin{pmatrix} (x-a_1)^2 & 0 & \cdots & 0 & 2(x-a_1)X_1 \\ 0 & (x-a_2)^2 & \cdots & 0 & 2(x-a_2)X_2 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & (x-a_r)^2 & 2(x-a_r)X_r \\ p_1 & p_2 & \cdots & p_r & \sum p_i' X_i \end{pmatrix}.$$

We leave it to the reader to check that $R = \det M$ is not identically zero.

To do the tangent space computation, we choose elements $w_i \in H^0(\mathbf{P}^1, E)$, and change v_i to $v_i + \epsilon w_i$, where $\epsilon^2 = 0$. Let R_ϵ be the equation of the discriminant of the projection given by $V_\epsilon \subset H^0(\mathbf{P}^1, E) \otimes k[\epsilon]/\epsilon^2$, where V_ϵ is spanned by $v_1 + \epsilon w_1, \ldots, v_{r+1} + \epsilon w_{r+1}$. Concretely, R_ϵ is the determinant of a matrix M_ϵ given by (4.10), which reduces to M modulo ϵ. Note that R_ϵ is an element of $H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2)) \otimes k[\epsilon]/\epsilon^2$, and we have

$$R_\epsilon = R + \epsilon S(w_1, \ldots, w_{r+1}),$$

for some $S(w_1, \ldots, w_{r+1}) \in H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2))$. Furthermore, the map

$$S\colon H^0(\mathbf{P}^1, E)^{r+1} \longrightarrow H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2)) \qquad (5.1) \quad \{\texttt{eqn:mainmap}\}$$

is a linear map. To show that $d\rho$ is non-singular at V, it suffices to show that S is surjective. For $1 \le i \le r$ and $1 \le j \le r+1$, let $E_{i,j} \in H^0(\mathbf{P}^1, E)^{r+1}$ be the element corresponding to $(w_1, \ldots, w_{r+1})$ where $w_j = X_i$ and $w_\ell = 0$ for all $\ell \neq j$. For $i \neq j$ and $1 \le j \le r$ and $q \in H^0(\mathbf{P}^1, \mathcal{O}(2))$, by direct calculation we get

$$S(qE_{i,j}) = \frac{(x-a_1)^2 \cdots (x-a_r)^2 p_j}{(x-a_i)^2 (x-a_j)^2} \cdot [q, (x-a_i)^2] \cdot X_i,$$

where the notation $[a, b]$ means $a'b - ab'$. Similarly, we get

$$S(qE_{i,r+1}) = -\frac{(x-a_1)^2 \cdots (x-a_r)^2}{(x-a_i)^2} \cdot [q, (x-a_i)^2] \cdot X_i,$$

and

$$S(qE_{i,i}) = \det M_i, \qquad (5.2) \quad \{\texttt{eqn:diag}\}$$

where M_i is obtained from M by changing the (i,i)-th entry from $(x-a_i)^2$ to q and the $(i, r+1)$-th entry from $2(x-a_i)X_i$ to $q'X_i$.

Fix an i with $1 \leq i \leq r$, and consider the subspace $W_i \subset H^0(\mathbf{P}^1, E)^{r+1}$ spanned by $qE_{i,j}$ for $j \neq i$. By our calculations above, S maps W_i to the subspace of $H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2))$ spanned by $H^0(\mathbf{P}^1, \mathcal{O}(2r)) \otimes X_i$. We begin by identifying $S(W_i)$.

For $1 \leq j \leq r$ and $j \neq i$, set

$$Q_{i,j} = \frac{(x-a_1)^2 \cdots (x-a_r)^2 p_j}{(x-a_i)^2 (x-a_j)^2},$$

and

$$Q_{i,r+1} = -\frac{(x-a_1)^2 \cdots (x-a_r)^2}{(x-a_i)^2}.$$

We claim that, there is no non-trivial linear relation among the r polynomials $Q_{i,j}$ for $j \in \{1, \dots, r+1\} \setminus \{i\}$. Indeed, suppose we had a linear relation

$$\sum l_j Q_{i,j} = 0,$$

then dividing throughout by $\frac{(x-a_1)^2 \cdots (x-a_r)^2}{(x-a_i)^2}$ gives the relation

$$\sum_{j=1}^{r} l_j \frac{p_j}{(x-a_j)^2} + l_{r+1} = 0.$$

If $l_j \neq 0$ for some j with $1 \leq j \leq r$, then we have a pole on the left side at $x = a_j$, but not on the right side (note that $(x - a_j)$ does not divide p_j by the genericity of p_j). Therefore, we must have $l_j = 0$ for all j, and hence also $l_{r+1} = 0$. Consider the map

{eqn:big}

$$H^0(\mathbf{P}^1, \mathcal{O}(1)) \otimes \langle Q_{i,j} \mid j \in \{1, \dots, r+1\} \setminus \{i\} \rangle \longrightarrow H^0(\mathbf{P}^1, \mathcal{O}(2r-1)). \tag{5.3}$$

We just saw that this map is injective. But both sides have the same dimension, and hence the map must be surjective. Finally, it is easy to see that the image of the map

{eqn:q}

$$H^0(\mathbf{P}^1, \mathcal{O}(2)) \longrightarrow H^0(\mathbf{P}^1, \mathcal{O}(2)), \quad q \mapsto [q, (x-a_i)^2] \tag{5.4}$$

is $(x - a_i) \cdot H^0(\mathbf{P}^1, \mathcal{O}(1))$. By (5.3) and (5.4), we conclude that the image of the map

$$S \colon W_i = \langle qE_{i,j} \mid j \in \{1, \dots, r+1\} \setminus \{i\} \longrightarrow H^0(\mathbf{P}^1, \mathcal{O}(2r-1)) \otimes X_i$$

is $(x-a_i)H^0(\mathbf{P}^1, \mathcal{O}(2r-2)) \otimes X_i$. In other words, the cokernel of the map is $X_i \otimes k$ where the map

$$H^0(\mathbf{P}^1, \mathcal{O}(2r)) \otimes X_i \longrightarrow k \otimes X_i$$

is given by evaluation at a_i. Putting together the maps for various i, we see that the cokernel of the map

$$S \colon \bigoplus_i W_i \longrightarrow H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2)) = H^0(\mathbf{P}^1, \mathcal{O}(2r)) \otimes \langle X_1, \dots, X_r \rangle$$

is $k \otimes \langle X_1, \dots, X_r \rangle$, where the map

{eqn:partialsur}

$$H^0(\mathbf{P}^1, E \otimes \mathcal{O}(2r-2)) = H^0(\mathbf{P}^1, \mathcal{O}(2r)) \otimes \langle X_1, \dots, X_r \rangle \longrightarrow k \otimes \langle X_1, \dots, X_r \rangle \tag{5.5}$$

on $H^0(\mathbf{P}^1, \mathcal{O}(2r)) \otimes X_i$ is given by evaluation at a_i.

To show that S is surjective, it is now enough to show that the map

{eqn:remainsur}

$$H^0(\mathbf{P}^1, \mathcal{O}(2)) \otimes \langle E_{i,i} \mid i \in \{1, \dots, r+1\} \rangle \longrightarrow k \otimes \langle X_1, \dots, X_r \rangle \tag{5.6}$$

obtained by composing (5.1) and (5.5) is surjective. Recall from (5.2) that we have $S(qE_{i,i}) = \det M_i$, where M_i is obtained from M by changing the (i,i)-th entry to q and the $(i, r+1)$-th entry to $q'X_i$. Taking $q = (x - a_i)$ gives

$$S(qE_{i,i}) = \det M_i = \pm \prod_{j \neq i} (a_i - a_j)^2 p_i(a_i) X_i,$$

which is a non-zero multiple of X_i. That is, the images of $(x - a_i)E_{i,i}$ under S span $k \otimes \langle X_1, \dots, X_r \rangle$, and hence the map in (5.6) is surjective. The proof is now complete. □

Our next goal is to bootstrap from Proposition 5.1 and Proposition 5.2 to deduce maximal variation for generic scrolls of sufficiently high degree. We do this by a degeneration argument. We degenerate a vector bundle E to a vector bundle E_0 on the nodal rational curve $P_0 = \mathbf{P}^1 \cup \mathbf{P}^1$, and show that the projection-ramification map for E_0 is dominant. For this to work, we have to define the projection-ramification map for nodal curves. It turns out that with the most näive definition of linear series on scrolls on nodal curves, we do not get a dominant projection-ramification map. We have to work with the limit linear series of higher rank as developed in [**?**] and[Oss14].

{sec:lls}

5.2. **Limit linear series.** We need limit linear series for the simplest singular curve, namely a (projective, connected) nodal curve C which is the nodal union of two smooth (projective, connected) curves C_1 and C_2, but we need them for vector bundles of rank higher than 1. Let B be the spectrum of a DVR with special point 0, general point η. Let $\pi \colon X \longrightarrow B$ be a smoothing of C with non-singular total space X. That is, π is a flat, proper, family of connected curves, smooth over η, and isomorphic to C over 0. Such a family is a particularly simple example of an almost local smoothing family [Oss14, § 2.1–2.2]. Let g_i be the genus of C_i for $i = 1, 2$, and $g = g_1 + g_2$ the genus of X_η.

Let E be a vector bundle of rank r on C. The *multi-degree* of E is the pair of integers $(\deg E|_{C_1}, \deg E|_{C_2})$. The *degree* or *total degree* of E is the sum $\deg E = \deg E|_{C_1} + \deg E|_{C_2}$.

Once and for all, fix a vector bundle $\mathcal{E}$ of rank r on X, and set $E = \mathcal{E}|_C$. Let E have degree d and multi-degree (w_1, w_2). Fix a positive integer k. Our next task is to recall the definition of the space of limit linear series of dimension k. It will be a B-scheme whose fiber over η is the Grassmannian $\mathbf{Gr}(k, H^0(X_\eta, \mathcal{E}_\eta))$. The key idea is to not only consider the sections of $\mathcal{E}$, but also of its various twists, namely the vector bundles obtained by tensoring with the powers of $\mathcal{O}_X(C_i)$.

Fix maps $\theta_1 \colon \mathcal{O}_X \longrightarrow \mathcal{O}_X(C_1)$ and $\theta_2 \colon \mathcal{O}_X \longrightarrow \mathcal{O}_X(C_2)$. The choice of these maps is auxilliary, and each one is unique up to multiplication by an element of $\mathcal{O}_B^*$. For $n \in \mathbf{Z}$, set

$$\mathcal{E}_n = \begin{cases} \mathcal{E} \otimes \mathcal{O}_X(C_1)^{\otimes n} & \text{if } n \geq 0, \\ \mathcal{E} \otimes \mathcal{O}_X(C_2)^{\otimes(-n)} & \text{if } n < 0. \end{cases}$$

The maps θ_1 and θ_2 induces maps

$$\theta_n \colon \mathcal{E}_m \longrightarrow \mathcal{E}_{m+n}$$

given by

$$\theta_n = \begin{cases} \theta_1^n & \text{if } n \geq 0, \\ \theta_2^{-n} & \text{if } n < 0. \end{cases}$$

Note that the multi-degree of $\mathcal{E}_n$ is $(w_1 - nr, w_2 + nr)$. In particular, for sufficiently negative n, say for $n \leq n_1$, we have $H^0(C_2, \mathcal{E}_n|_{C_2}) = 0$, and similarly, for sufficiently positive n, say $n \geq n_2$, we have $H^0(C_1, \mathcal{E}_n|_{C_1}) = 0$. Assume, without loss of generality, that $n_2 \geq n_1$. Set

$$d_1 = w_1 - n_1 r, \text{ and } d_2 = w_2 + n_2 r, \text{ and } b = n_2 - n_1.$$

Observe that

$$d_1 + d_2 - rb = d.$$

We say that $\mathcal{E}$ has multi-degree w if for every $s \in S$ mapping to $0 \in B$, the degree of $\mathcal{E}|_s$ on C_v is w_v for $v = 1, 2$. Note that, if $\mathcal{E}$ has multi-degree (w_1, w_2), then $\mathcal{E}_n$ has multi-degree $(w_1 - rn, w_2 + rn)$.

{def:lls}

Definition 5.3 (Limit linear series)**.** Let S be a B-scheme. A *k-dimensional limit linear series* on $\mathcal{E}_S$ consists of sub-bundles $V_n \longrightarrow \pi_*(\mathcal{E}_n)_S$ of rank k for every $n \in \mathbf{Z}$ satisfying the following compatibility condition. For every $m, n \in \mathbf{Z}$, the map

s:compatibility}

$$\pi_*\theta_n \colon \pi_*(\mathcal{E}_m)_S \longrightarrow \pi_*(\mathcal{E}_{m+n})_S \text{ maps } V_m \longrightarrow V_{m+n}. \tag{5.7}$$

Definition 5.3 is a special case of [Oss14, Definition 3.3.2]. From now on, we will talk about the image of an element in V_m in V_{m+n}; this should be understood as the image under the map $\pi_*\theta_n$.

Remark 5.4. The notion of a sub-bundle of a push-forward is a bit subtle; it is treated in depth in [Oss14, Definition B.2.1]. We recall the main points. For a flat proper morphism $X \longrightarrow S$ and a vector bundle $\mathcal{E}$ on S, a *sub-bundle* of $\pi_*\mathcal{E}$ is a vector bundle V on S along with a map $i \colon V \longrightarrow \pi_*\mathcal{E}$ such that for every $T \longrightarrow S$, the pull-back $i_T \colon V_T \longrightarrow \pi_*(\mathcal{E}_T)$ is injective. Note that this is a local condition on S. For Noetherian schemes such as ours, it is enough to check this condition for the $T \longrightarrow S$ that are inclusions of closed points. Alternatively, if $F_0 \longrightarrow F_1 \longrightarrow \cdots$ is a complex of vector bundles on S quasi-isomorphic to $R\pi_*\mathcal{E}$, then a sub-bundle of $\pi_*\mathcal{E}$ is a vector bundle V along with a map $i \colon V \longrightarrow \pi_*\mathcal{E}$ such that the composite $V \longrightarrow F_0$ is an injection of vector bundles (that is, the dual map is surjective).

Remark 5.5. Definition 5.3 defines limit linear series on a particular vector bundle $\mathcal{E}$. We can also vary the choice of the vector bundle, as is done in [Oss14]; in that case, one imposes an additional vanishing condition on the vector bundles to ensure boundedness of the moduli space of limit linear series.

{def:simple_lls}

Definition 5.6. Let $S = \operatorname{Spec} K$, where K is a field, and let $V = (V_n \mid n \in \mathbf{Z})$ be a limit linear series on S. We say V is *simple* if there exist integers $w_1, \dots, w_k$, not necessarily distinct, and elements $v_i \in V_{w_i}$ such that for every $w \in \mathbf{Z}$, the images of $v_1, \dots, v_k$ in V_w form a basis of V_w.

Note that if $S \longrightarrow B$ maps to the generic point η, then the data of a limit linear series $V = (V_n)$ is equivalent to the data of an individual V_n for any $n \in \mathbf{Z}$, and in particular, for $n = 0$. As a result, the functor that associates to $S \longrightarrow \eta$ the set of k-dimensional limit linear series of $\mathcal{E}_S$ is represented by the Grassmannian $\mathbf{Gr}(k, H^0(X_\eta, \mathcal{E}_\eta))$. The main theorem of [Oss14] is the following representability theorem.

{thm:lls}

Theorem 5.7 ([Oss14, Theorem 3.4.7])**.** *The functor that associates to a B-scheme $S \longrightarrow B$ the set of limit linear series on $\mathcal{E}_S$ is representable by a projective B-scheme $\mathcal{G}(k, \mathcal{E})$ isomorphic to the Grassmannian $\mathbf{Gr}(k, H^0(X_\eta, \mathcal{E}_\eta))$ over η. The locus of simple linear series $\mathcal{G}^{\text{simple}}(k, \mathcal{E}) \subset \mathcal{G}(k, \mathcal{E})$ is an open subscheme, and the map $\mathcal{G}^{\text{simple}}(k, \mathcal{E}) \longrightarrow B$ has universal relative dimension at least $k(d - k - r(g - 1))$.*

The last stamement implies that if $v \in \mathcal{G}^{\text{simple}}$ is such that $\mathcal{G}^{\text{simple}}$ has relative dimension at most $k(d - k - r(g - 1))$ at v, then it has relative dimension exactly $k(d - k - r(g - 1))$ at v and, futhermore, it is an open map near v. In particular, v is in the closure of $\mathbf{Gr}(k, H^0(X_\eta, \mathcal{E}_\eta))$.

Remark 5.8. Osserman proves a stronger theorem, namely a relative version of the statement above, over the stack of vector bundles on X. But the statement above is enough for our purposes.

Although the definition of a limit linear series demands that we specify infinitely many vector bundles V_n, one for each $n \in \mathbf{Z}$, this is neither practical nor necessary. In the best case, only specifying the extremal ones, namely V_{n_1} and V_{n_2}, suffices, provided that they satisfy some compatibility conditions. The original definition of limit linear series due to Eisenbud–Harris [**?**] in the rank 1 case and Teixidor i Bigas [**?**] in the general case, took this minimalist approach.

Let E_n be the restriction of $\mathcal{E}_n$ to the central fiber $C = X_0$, and set $p = C_1 \cap C_2$.

{def:eht}

Definition 5.9. A *k-dimensional EHT limit linear series* on E consists of k-dimensional subspaces $W_i \subset H^0(C_i, E_{n_i}|_{C_i})$ for $i = 1, 2$ that satisfy the following two conditions.

{ieq:eht}

(1) If $a_1^i \leq \cdots \leq a_k^i$ is the vanishing sequence for $(\mathcal{E}_{n_i}|_{C_i}, W_i)$ at p for $i = 1, 2$, then for every $v = 1, \ldots, k$ we have

$$a_v^1 + a_{k+1-v}^2 \geq b.$$

{gluing:eht}

(2) There exist bases $s_1^i, \ldots, s_k^i$ for W_i for $i = 1, 2$, such that s_v^i has order of vanishing a_v^i at p, and if we have $a_v^1 + a_{k+1-v}^2 = b$ for some v, then

$$\widetilde{\phi}(s_v^1) = s_{k+1-v}^2,$$

where $\widetilde{\phi} \colon E_{n_1}(-a_v^1 \cdot p)|_p \longrightarrow E_{n_2}(-a_{k+1-v}^2 \cdot p)|_p$ is the isomorphism obtained by taking the appropriate twist of the identity map.

We say that (W_1, W_2) is a *refined EHT limit linear series* if all equality holds in (1) for all $v = 1, \ldots, k$.

This definition is adapted from [Oss14, Definition 4.1.2]. Note that, due to the vanishing condition on the twists of E, the restriction map

$$H^0(C, E_{n_i}) \longrightarrow H^0(C_i, E_{n_i}|_{C_i})$$

is an isomorphism. Via this isomorphism, we sometimes treat W_i as a subspace of $H^0(C_i, \mathcal{E}_{n_i}|_{C_i})$.

Although the notions of a limit linear series and an EHT limit linear series differ in general, they essentially agree when we restrict to the simple limit linear series and the refined EHT limit linear series. More precisely, we have the following statement.

{prop:llseht}

Proposition 5.10. *Let S be a B-scheme, and $V = (V_n \mid n \in \mathbf{Z})$ a limit linear series on $\mathcal{E}_S$. For every $s \in S$ over $0 \in B$, taking $W_i = V_{n_i}|_s$ for $i = 1, 2$ gives an EHT limit linear series. Conversely, assume that S reduced, and let $\mathcal{W}_i \subset \pi_*(\mathcal{E}_{n_i})_S$ be sub-bundles whose restrictions to every $s \in S$ over $\eta \in B$ agree under the isomorphism $(\mathcal{E}_{n_1})_\eta \cong (\mathcal{E}_{n_2})_\eta$, and to eveny $s \in S$ over $0 \in B$ define a refined EHT limit linear series. Then there exists a unique limit linear series $V = (V_n \mid n \in \mathbf{Z})$ on $\mathcal{E}_S$ such that $\mathcal{W}_i = V_{n_i}$. Furthermore, for every $s \in S$ over 0, the series $V|_s$ is simple.*

Proof. Proving that (W_1, W_2) is an EHT limit linear series is straightforward, and left to the reader. It is a special case of [Oss14, Theorem 4.3.4] and the equivalence of type I and type II series in the two component case ([Oss14, Remark 3.4.15].

The converse also follows from the proof of [Oss14, Theorem 4.3.4], but it is not explicitly stated there. So we offer a proof.

First, suppose that S lies over $\eta \in B$. Then $V_n \subset \pi_*(\mathcal{E}_n)_S$ is determined uniquely as the image of $V_{n_i} = \mathcal{W}_{n_i} \subset \pi_*(\mathcal{E}_{n_i})_S$ for either $i = 1$ or $i = 2$.

Next, suppose that $S = \operatorname{Spec} K$, and it lies over $0 \in B$. Denoting $(\mathcal{E}_n)_S$ by E_n, we must construct $V_n \subset H^0(C, E_n)$. By composing $\theta_{n_i - n} \colon E_n \longrightarrow E_{n_i}$ and the restriction $E_{n_i} \longrightarrow E_{n_i}|_{C_i}$, we get a map

$$\iota \colon H^0(C, E_n) \longrightarrow H^0(C_1, E_{n_1}|_{C_1}) \oplus H^0(C_2, E_{n_2}|_{C_2}).$$

The vanishing condition on the twists of E mean that ι is injective. The compatibility condition in Definition 5.3 implies that we must choose V_n so that $\iota(V_n) \subset W_1 \oplus W_2$. We claim that $\dim \iota^{-1}(W_1 \oplus W_2) = k$, so that there is a unique choice of V_n, namely $V_n = \iota^{-1}(W_1 \oplus W_2)$.

Suppose $s \in i^{-1}(W_1 \oplus W_2)$. Then $\iota(s)$ is a linear combination of $(s^1_1, 0), \dots, (s^1_k, 0)$, and $(0, s^2_1), \dots, (0, s^2_k)$. Write $\iota(s) = (s_1, s_2)$. Since s_i is obtained by applying θ_{n-n_i}, and θ on C_i at p corresponds to multiplication by the uniformizer, we see that

{eqn:vanishing}

$$\operatorname{ord}_p(s_1) \geq n - n_1, \text{ and likewise, } \operatorname{ord}_p(s_2) \geq n_2 - n. \tag{5.8}$$

Let $v_1 \in \{1, \dots, k\}$ be the smallest such that $a^1_v \geq n - n_1$, and $v_1 + c$ the smallest such that $a^1_{v_1+c} > n - n_1$. Since (W_1, W_2) is refined, and $n_2 - n_1 = b$, we see that $v_2 = k + 1 - v_1$ is the largest such that $a^2_{v_2} \leq n_2 - n$, and $v_2 - c$ the smallest such that $a^2_{v_2+c} < n_2 - n$. The vanishing conditions (5.8) imply that $\iota(s)$ bust be a linear combination of $(s^1_{v_1}, 0), \dots, (s^1_k, 0)$ and $(0, s^2_{v_2-c}), \dots, (0, s^2_k)$. Suppose

$$\iota(s) = \sum_{\ell=i}^{k} \alpha_\ell \cdot (s^1_\ell, 0) + \sum_{\ell=j}^{k} \beta_\ell \cdot (0, s^2_\ell),$$

where α_ℓ and β_ℓ are elements of the field K. Since s is a section on the entire nodal curve C, its two restrictions to C_1 and C_2 are equal at p. In terms of the two components of $\iota(s)$, and in light of the gluing condition (2) in Definition 5.9, this equality is equivalent to $\alpha_\ell = \beta_{k+1-\ell}$. That is, $\iota(s)$ is a linear combination of the k elements

$$(s^1_{v_1}, s^2_{v_2}), \dots, (s^1_{v_1+c-1}, s^2_{v_2-c+1}), (s^1_{v_1+c}, 0), \dots, (s^1_k, 0), (0, s^2_{v_2+1}), \dots, (s^2_k, 0).$$

Conversely, it is easy to see that any such linear combination lies in $W_1 \oplus W_2$. Hence the claim that $\dim \iota^{-1}(W_1 \oplus W_2) = k$.

Set $V_n = \iota^{-1}(W_1 \oplus W_2)$. To see that V is simple, we must exhibit appropriate w_i and $v_i \in V_{w_i}$ for $i = 1, \dots, k$. Take $w_i = n - n_1 - a^1_i$, and let $v_i \in V_{w_i} \subset H^0(C, E_{w_i})$ be such that $\iota(v_i) = (s^1_i, s^2_{k+1-i})$. Then the images of $v_1, \dots, v_k$ form a basis of V_n for all $n \in \mathbf{Z}$.

For more general S, consider the map

$$\bar\iota \colon \pi_*(\mathcal{E}_n)_S \longrightarrow \pi_*(\mathcal{E}_{n_1})_S/\mathcal{W}_1 \oplus \pi_*(\mathcal{E}_{n_2})_S/\mathcal{W}_2,$$

obtained by composing $\iota = \pi_*(\theta_{n_1-n} \oplus \theta_{n_2-n})$ and the projections $\pi_*(\mathcal{E}_{n_i})_S \longrightarrow \pi_*(\mathcal{E}_{n_i})_S/\mathcal{W}_i$. We proved that, for every $\operatorname{Spec} K \longrightarrow S$, the kernel of $\bar\iota \otimes_{\mathcal{O}_S} K$ is k-dimensional. Since S is reduced, it is easy to prove that $V_n = \ker \iota$ is a sub-bundle of $\pi_*(\mathcal{E}_n)$. It is also easy to check that $V = (V_n \mid n \in \mathbf{Z})$ a limit linear series, the only one that satisfies $V_{n_i} = \mathcal{W}_i$. The proof is now complete. $\square$

Proposition 5.10 allows us to combine the economy of specifying an EHT limit linear series with the convenient functorial definition of a limit linear series. We use this in the definition of the projection-ramification map in terms of limit linear series.

{sec:prnodal}

5.3. **Projection-ramification with non-generic vanishing sequence.** We consider the projection-ramification map for linear series with a non-generic vanishing sequence. The analysis of such series plays a key role in defining the projection-ramification map for limit linear series.

Let C be a smooth curve and $p \in C$ a point. Let E be a vector bundle on C of rank r. The projective spaces associated to the vector spaces $E(np)|_p$, for $n \in \mathbf{Z}$, are canonically isomorphic to each other, so we identify them. The vanishing sequences considered are at the point p. Choose a uniformizer t of C at p.

Suppose $V \subset H^0(C, E)$ is an $(r+1)$-dimensional subspace with the vanishing sequence

$$(\underbrace{a, \dots, a}_{i}, \underbrace{a+1, \dots, a+1}_{r+1-i}), \tag{5.9}$$ {eqn:specialvs}

for some i with $1 \le i \le r$, and $a \ge 0$. Let $v_1, \dots, v_{r+1}$ be a basis of V adapted to the vanishing sequence, namely a basis $v_1, \dots, v_{r+1}$ such that in the stalk E_p, we can write

$$v_1 = t^a \widetilde{v}_1, \dots, v_i = t^a \widetilde{v}_i, \quad v_{i+1} = t^{a+1}\widetilde{v}_{i+1}, \dots, v_{r+1} = t^{a+1}\widetilde{v}_{r+1}, \tag{5.10}$$ {eqn:basis}

for some $\widetilde{v}_1, \dots, \widetilde{v}_{r+1} \in E_p$ such that the images of $\widetilde{v}_1, \dots, \widetilde{v}_i$ in the fiber $E|_p$ are linearly independent, and the same holds for the images of $\widetilde{v}_{i+1}, \dots, \widetilde{v}_{r+1}$. Here we are slightly abusing the notation by denoting v_i and its image in E_p under the natural evaluation map by the same letter. Let $V^0 \subset E|_p$ be spanned by the images of $\widetilde{v}_1, \dots, \widetilde{v}_i$, and $V^1 \subset E|_p$ by the images of $\widetilde{v}_{i+1}, \dots, \widetilde{v}_{r+1}$. It is easy to check that a different choice of basis adapted to the vanishing sequence gives the same V^0 and V^1. By construction, $\dim V_0 = i$ and $\dim V^1 = r + 1 - i$, and therefore, $\dim(V^0 \cap V^1) \ge 1$. We say that V has *transverse vanishing* at p if

{eq:genericity}
$$\dim(V^0 \cap V^1) = 1. \tag{5.11}$$

Note that if V is base-point free at p, then $\dim V^0 = r$ and $\dim V^1 = 1$, so V automatically has transverse vanishing.

{prop:agreement}

Proposition 5.11. *Suppose $V \subset H^0(C, E)$ is an $(r+1)$-dimensional subspace with vanishing sequence* (5.9) *and transverse vanishing at p. Then the ramification section r_V of V vanishes to order $(r+1)a + (r-i)$ at p. Furthermore, writing $r_V = t^{(r+1)a+r-i} \cdot \widetilde{r}$, the one-dimensional subspace of $E|_p$ spanned by $\widetilde{r}|_p$ is $V^0 \cap V^1$.*

Proof. Thanks to transverse vanishing, there exists a basis $\{\overline{s}_1, \dots, \overline{s}_r\}$ of $E|_p$ such that

$$V^0 = \langle \overline{s}_1, \dots, \overline{s}_i \rangle \text{ and } V^1 = \langle \overline{s}_{i+1}, \dots, \overline{s}_r, \overline{s}_1 \rangle.$$

Let $v_1, \dots, v_{r+1}$ be a basis of V adapted to the vanishing sequence such that if $\widetilde{v}_i$ are defined as in (5.10) then the images of $\widetilde{v}_1, \dots, \widetilde{v}_r$ in $E|_p$ are $\overline{s}_1, \dots, \overline{s}_r$, respectively, and the image of $\widetilde{v}_{r+1}$ is $\overline{s}_1$. In particular, the r elements $\widetilde{v}_1, \dots, \widetilde{v}_r \in E_p$ give a trivialization of E around p. Write

$$\widetilde{v}_{r+1} = b_1 \widetilde{v}_1 + \cdots + b_r \widetilde{v}_r$$

in E_p, where $b_1, \dots, b_r \in \mathcal{O}_{C,p}$. Since the image of $\widetilde{v}_{r+1}$ in $E|_p$ is $\overline{s}_1$, we get that $b_1 \equiv 1 \pmod{\mathfrak{m}_p}$, and $b_2, \dots, b_r \in \mathfrak{m}_p$. Using the basis $v_1, \dots, v_{r+1}$ of V and the local trivialization $\widetilde{v}_1, \dots, \widetilde{v}_r$ of E, we can write r_V as the determinant (see (4.10)) as follows

$$\begin{aligned} r_V &= \det \begin{pmatrix} t^a & & & & & & at^{a-1}\widetilde{v}_1 \\ & \ddots & & & & & \vdots \\ & & t^a & & & & at^{a-1}\widetilde{v}_i \\ & & & t^{a+1} & & & (a+1)t^a \widetilde{v}_{i+1} \\ & & & & \ddots & & \vdots \\ & & & & & t^{a+1} & (a+1)t^a\widetilde{v}_r \\ b_1 t^{a+1} & b_2 t^{a+1} & \cdots & b_{r-1}t^{a+1} & & b_r t^{a+1} & (a+1)t^a \widetilde{v}_1 + t^{a+1}(\cdots) \end{pmatrix} \\ &= t^{(r+1)a+r-i}\widetilde{v}_1 + t^{(r+1)a+r-i+1}(\cdots). \end{aligned}$$

Thus the order of vanishing of r_V is as claimed. Furthermore, $\widetilde{r}$ is given by

$$\widetilde{r} = \widetilde{v}_1 + t(\cdots).$$

Since the image of $\widetilde{v}_1$, namely $\overline{s}_1$, spans $V^0 \cap V^1$, the proof is complete. $\square$

We are primarily interested in generic $(r+1)$-dimensional subspaces $V \subset H^0(C,E)$. A generic such V has the vanishing sequence

$$(0,\ldots,0,1).$$

For limit linear series, it is important to also study the V with complementary vanishing sequence, namely

$$(0,1,\ldots,1),$$

which we now do. For simplicity, we restrict to $C = \mathbf{P}^1$.

Let E be an ample vector bundle on $\mathbf{P}^1$ of rank r. Fix a point $p \in \mathbf{P}^1$; all the vanishing sequences are at p. Consider the locally closed subset $U \subset \mathbf{Gr}(r+1, H^0(\mathbf{P}^1,E))$ parametrizing $V \subset H^0(\mathbf{P}^1,E)$ with vanishing sequence

$$(0,\underbrace{1,\ldots,1}_{r}).$$

Given such a V, let $\widetilde{r}_V \in \mathbf{P}H^0(E \otimes \det E \otimes K_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p)^*$ be the reduced ramification section, namely the section obtained by dividing the usual ramification section r_V by the $(r-1)$-th power of a uniformizer at t (see Proposition 5.11). The assignment $V \mapsto \widetilde{r}_V$ gives a variant of the projection-ramification map, which we call the *reduced projection-ramification map*

$$\widetilde{\rho}\colon U \longrightarrow \mathbf{P}H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p))^*. \tag{5.12}$$ {eqn:rrd}

Note that, just as in the case of the usual projection-ramification map, the source and the target of the reduced projection-ramification map are of the same dimension.

Having defined the reduced projection-ramification map, we now relate it back to the usual projection-ramification map, but on a different vector bundle. Given a one-dimensional subspace $\ell \subset E|_p$, define E'_ℓ by the exact sequence

$$0 \longrightarrow E'_\ell \longrightarrow E \longrightarrow E|_p/\ell \longrightarrow 0.$$

There exists a Zariski open subset of the projective space of lines in $E|_p$ such that for all ℓ in this set, the isomorphism class of E'_ℓ remains constant. Denote this isomorphism class by E'_{gen}.

{prop:domred}

Proposition 5.12. *If the usual projection-ramification map*

$$\rho\colon \mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E'_{\text{gen}})) \dashrightarrow \mathbf{P}H^0(\mathbf{P}^1, E'_{\text{gen}} \otimes \det E'_{\text{gen}} \otimes K_{\mathbf{P}^1})^*$$

is dominant, then so is the reduced projection-ramification map

$$\widetilde{\rho}\colon U \longrightarrow \mathbf{P}H^0(\mathbf{P}^1, E \otimes \det E \otimes K_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p))^*.$$

Proof. Let $D \in \mathbf{P}H^0(E \otimes \det E \otimes K_{\mathbf{P}^1} \otimes \mathcal{O}(-(r-1)p))^*$ be a generic section. Let $\ell \subset E|_p$ be the one-dimensional subspace defined by $D|_p$, and set $E' = E'_\ell$. Since D is generic, we may assume $E' \cong E'_{\mathrm{gen}}$. The inclusion of sheaves $E' \longrightarrow E$ induces an inclusion of sheaves

$$E' \otimes \det E' \otimes K_{\mathbf{P}^1} \longrightarrow E \otimes \det E \otimes \mathcal{O}(-(r-1)p) \otimes K_{\mathbf{P}^1},$$

and by construction, D is the image of a section $D' \in \mathbf{P}H^0(E' \otimes \det E' \otimes K_{\mathbf{P}^1})^*$. Since ρ is dominant for E', there exists a sequence of subspaces $V'_n \in \mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E'))$ such that the limit of $\rho(V'_n)$ is D'. Let $V_n \subset \mathbf{Gr}(r+1, H^0(\mathbf{P}^1, E))$ be the image of V'_n. Then the limit of $\widetilde{\rho}(V_n)$ is D. Since D was generic, we get that $\widetilde{\rho}$ is dominant. ☐

:domredexamples}

Corollary 5.13. *The reduced projection-ramification map is dominant for the bundles $E = \mathcal{O}(1) \oplus \mathcal{O}(2)^{r-1}$ and $E = \mathcal{O}(2) \oplus \mathcal{O}(3)^{r-1}$.*

Proof. Follows from Proposition 5.12 and that the projection-ramification map is dominant for $E' = \mathcal{O}(1)^r$ and $E' = \mathcal{O}(2)^r$. ☐

5.4. **Projection-ramification for limit linear series.** Recall the setup from § 5.2: $C = C_1 \cup C_2$ is a nodal union of two smooth projective curves of genus g_1 and g_2, and $\pi\colon X \longrightarrow B$ be a smoothing of C. Let $\mathcal{E}$ be a vector bundle of rank r on X whose restriction E to C has multi-degree (w_1, w_2). The integers $n_2 \geq n_1$ are such that we have vanishing $H^0(C_2, E_n|_{C_2}) = 0$ for all $n \leq n_1$ and $H^0(C_1, E_n|_{C_1}) = 0$ for $n \geq n_2$. For convenience, we decrease n_1 and increase n_2 so that the vanishing on C_2 holds for all $n \leq n_1 - (w_1 - 2g_1)$ and on C_1 for all $n \geq n_2 + (w_2 - 2g_2)$. Define

$$d_1 = w_1 - n_1 r, \quad d_2 = w_2 + n_2 r, \text{ and } b = n_2 - n_1,$$

as before.

Set $\mathcal{E}' = \mathcal{E} \otimes \det \mathcal{E} \otimes \omega_{X/B}$. Then $\mathcal{E}'$ is a vector bundle of rank r on X whose restriction E' to C has multi-degree (w'_1, w'_2) where

$$w'_1 = w_1 + r(w_1 - 2g_1 + 1) \text{ and } w'_2 = w_2 + r(w_2 - 2g_2 + 1).$$

We set

$$n'_1 = n_1(1+r) \text{ and } n'_2 = n_2(1+r),$$

and observe that we have vanishings $H^0(C_2, E'_n|_{C_2}) = 0$ for $n \leq n'_1$ and $H^0(C_1, E'_n|_{C_1}) = 0$ for $n \geq n'_2$. We also set

$$b' = n'_2 - n'_1 = b(1+r).$$

Our next goal is to define a rational map

{eq:Rtilde}

$$\rho\colon \mathcal{G}(r+1, \mathcal{E}) \dashrightarrow \mathcal{G}(1, \mathcal{E}') \tag{5.13}$$

that extends the projection-ramification map

$$\rho\colon \mathbf{Gr}(r+1, H^0(X_\eta, \mathcal{E}_\eta)) \dashrightarrow \mathbf{Gr}(1, H^0(X_\eta, \mathcal{E}'_\eta))$$

on X_η. For technical reasons, we define the map in (5.13) only on the reduced scheme underlying $\mathcal{G}(r+1, \mathcal{E})$.

Before defining the map, we identify three conditions on limit linear series on the central fiber that are required for the map to be defined. To do this, consider a limit linear series $(V_n \mid n \in \mathbf{Z})$ on C, and let (W_1, W_2) be the associated EHT limit linear series namely $W_1 = V_{n_1}$ and $W_2 = V_{n_2}$ (see Proposition 5.10). The first condition we want to impose is that (W_1, W_2) be a refined EHT limit linear series; this is an open condition (see [Oss14, Proposition 4.1.5]). The second condition we want to impose is that the vanishing sequence of $W_1 \subset H^0(C_1, E_{n_1}|_{C_1})$ at p is of the form

$$(\underbrace{a, \ldots, a}_{i}, \underbrace{a+1, \ldots, a+1}_{r+1-i}) \tag{5.14}$$ {eqn:llsvs}

as in (5.9); imposing a particular vanishing sequence is again an open condition (see [Oss14, Proposition 4.2.5]). Since (W_1, W_2) is refined, it follows that the vanishing sequence of $W_2 \subset H^0(C_2, E_{n_2}|_{C_2})$ at p is

$$(\underbrace{b-a-1, \ldots, b-a-1}_{r+1-i}, \underbrace{b-a, \ldots, b-a}_{i}).$$

Recall from § 5.3 that W_1 yields two vector spaces V^0 and V^1 in the fiber $E_{n_1}|_p$, which we may identify canonically (up to scaling) with the fiber $E|_p$. Likewise, W_2 yields two analogous vector spaces, call them Λ^0 and Λ^1, in $E|_p$. The gluing condition in the definition of EHT limit linear series (Definition 5.9) and the definition of these vector spaces immediately shows that

$$V^0 = \Lambda^1 \text{ and } V^1 = \Lambda^0. \tag{5.15}$$ {eqn:vlambdaswit

The third condition we want to impose is that these two vector spaces be transverse, namely $\dim(V^0 \cap V^1) = 1$.

Let $\mathcal{U} \subset \mathcal{G}(r+1, \mathcal{E})$ be the complement of the union of the following closed sets:

(1) the closure of the subset of $\mathbf{Gr}(r+1, H^0(X_\eta, \mathcal{E}_\eta))$ corresponding to $V \subset H^0(X_\eta, \mathcal{E}_\eta)$ for which the evaluation map $V \otimes \mathcal{O}_{X_\eta} \longrightarrow \mathcal{E}_\eta$ has generic rank less than r.
(2) the set of limit linear series $(V_n \mid n \in \mathbf{Z})$ on C such that the associated EHT limit linear series (W_1, W_2) is not refined, or does not have the vanishing sequence as in (5.14), or does not satisfy the transversality condition $\dim(V^0 \cap V^1) = 1$.

Give $\mathcal{U}$ the reduced scheme structure.

Let S be a reduced B-scheme with a map to $\mathcal{U}$ given by the limit linear series $(V_n \mid n \in \mathbf{Z})$. On X_S, we have a diagram analogous to (4.8), namely

$$\begin{array}{ccccccccc} & & \det \mathcal{E}_n^* \otimes \det V_n & \xrightarrow{j} & V_n \otimes \mathcal{O}_{X_S} & \xrightarrow{e} & \mathcal{E}_n & & \\ & & \downarrow d & & \downarrow e & & \| & & \\ 0 & \longrightarrow & \Omega_{X_S/S} \otimes \mathcal{E}_n & \longrightarrow & P(\mathcal{E}_n) & \longrightarrow & \mathcal{E}_n & \longrightarrow & 0. \end{array} \tag{5.16}$$ {eq:llspr}

Here $P(\mathcal{E}_n)$ is the sheaf of principal parts of $\mathcal{E}_n$ relative to $X_S \longrightarrow S$, and the bottom row is the natural exact sequence coming from its definition. The top row is a complex, but it may not be exact. The maps labeled e are the evaluation maps. The map j is defined by the maximal minors of $e\colon V_n \otimes \mathcal{O}_{X_S} \longrightarrow \mathcal{E}_n$. The map d is the unique map induced by the other maps in the diagram. By composing d through the inclusion $\Omega_{X_S/S} \longrightarrow \omega_{X_S/S}$, and doing some rearrangement, we obtain a map

{eqn:Rn}
$$r_n\colon \det V_n \longrightarrow \pi_*(\mathcal{E}_n \otimes \det \mathcal{E}_n \otimes \omega^*_{X_S/S}) = \pi_*(\mathcal{E}'_{(r+1)n}). \tag{5.17}$$

Consider the two extremal sections, namely those corresponding to $n = n_1$ and $n = n_2$.

{lem:rameht}

Lemma 5.14. *Over every $s \in S$ over $0 \in \Delta$, the restrictions $r_{n_1}|_s$ and $r_{n_2}|_s$ define a one-dimensional refined EHT limit linear series for E'.*

Proof. Without further comment, we identify $r_{n_i}|_s \in H^0(C, E'_{(r+1)n_i})$ with its image in $H^0(C_i, E'_{(r+1)n_i}|_{C_i})$. We have

$$E'_{(r+1)n_2}|_{C_i} = E_{n_2} \otimes \det E_{n_2} \otimes \omega_C|_{C_1} = E_{n_2} \otimes \det E_{n_2} \otimes \Omega_C|_{C_1} \otimes \mathcal{O}_{C_1}(p),$$

and by construction $r_{n_1}|_s$ is the image of the ramification section of $V_{n_1} \subset H^0(C_1, E_{n_1}|_{C_1})$ under the inclusion map

$$E_{n_1} \otimes \det E_{n_1} \otimes \Omega_C|_{C_1} \longrightarrow E_{n_1} \otimes \det E_{n_1} \otimes \omega_C|_{C_1} = E'_{(r+1)n_1}|_{C_1}.$$

By Proposition 5.11, the ramification section of V_{n_1} has order of vanishing $(r+1)a + (r-i)$ at p, and hence $r_{n_1}|_s$ on C_1 has order of vanishing $(r+1)a + (r-i+1)$ at p. Likewise, $r_{n_2}|_s$ on C_2 has order of vanishing $(r+1)(b-a-1) + i$ at p. Since

$$(r+1)a + (r-i+1) + (r+1)(b-a-1) + i = (r+1)b = b',$$

we see that $r_{n_1}|_s$ and $r_{n_2}|_s$ have complementary orders of vanishing, leading to an equality in condition (1) of Definition 5.9.

We must next ensure that condition (2) of Definition 5.9 holds, that is, the images of $r_{n_i}|_s$ in the appropriate twists of $E_{n_i}|_p$ are equal, at least up to scaling. By Proposition 5.11, the image of $r_{n_1}|_s$ in the appropriate twist of $E_{n_1}|_p$ spans the line $(V^0 \cap V^1)$, and the image of $r_{n_2}|_s$ spans the line $\Lambda^0 \cap \Lambda^1$. But by (5.15), we have $V^1 = \Lambda^0$ and $V^0 = \Lambda^1$, so the two lines are equal. $\square$

Thanks to Lemma 5.14, we apply Proposition 5.10, and conclude that there exists a unique (1-dimensional) limit linear series $(R_n \mid n \in \mathbf{Z})$ of $\mathcal{E}'$ on X_S for which $R_{n'_1} = \det V_{n_1}$ and $R_{n'_2} = \det V_{n_2}$, at least if S is reduced. The transformation

$$(V_n \mid n \in Z) \mapsto (R_n \mid n \in \mathbf{Z})$$

defines a morphism

{prop:mapreduced}
$$\rho\colon \mathcal{U} \longrightarrow \mathcal{G}(1, \mathcal{E}'), \tag{5.18}$$

as desired in (5.13). Note that $\mathcal{U}$ has the reduced scheme structure.

The fruit of our labor is the following corollary.

{op:degeneration}

Corollary 5.15. *Suppose $v \in \mathcal{U}_0$ is such that $\dim_v \mathcal{U}_0 = (r+1)(d-rg-1)$ and v is isolated in the fiber of ρ, then the projection-ramification map $\mathbf{Gr}(r+1, H^0(X_\eta, \mathcal{E}_\eta)) \dashrightarrow \mathbf{P}H^0(X_\eta, \mathcal{E}_\eta \otimes \det E_\eta \otimes K_{X_\eta})$ is generically finite.*

Proof. If $\dim_v U_0 = (r+1)(d-rg-1)$, then v is in the closure of $\mathbf{Gr}(r+1, H^0(X_\eta, \mathcal{E}_\eta))$ by Theorem 5.7. The statement now follows from the upper semi-continuity of fiber dimension. ☐

5.5. **Maximal variation for generic scrolls of high degree.** We now have all the tools to prove Theorem D

Theorem 5.16 (Theorem D)**.** *Let E be a generic vector bundle on $\mathbf{P}^1$ of rank r and degree $d = a(r-1) + b(2r-1) + 1$, where a, b are positive integers. Then the projection-ramification map is generically finite, and hence dominant, for E. In particular, the projection-ramification map is dominant for generic E of degree $\geq (r-1)(2r-1)+1$.*

Proof. We say that generic dominance holds for rank r and degree d if the projection-ramification map is dominant (equivalently, generically finite) for the generic vector bundle of rank r and degree d. The rank will be fixed througout, so let us drop it from the discussion. Let us prove that if generic dominance holds for degrees d_1 and d_2, then it also holds for degree $d = d_1 + d_2 - 1$. With the base cases $d_1 = r$ (Proposition 5.1) and $d_2 = 2r$ (Proposition 5.2), this proves the theorem.

Take $C_1 = C_2 = \mathbf{P}^1$, and let $C = C_1 \cup C_2$ be their nodal union at one point, which we take to be the point labeled 0 on both $\mathbf{P}^1$s. Let $X \longrightarrow B$ be a smoothing of C. Note that any vector bundle on C is the restriction of a vector bundle on X. Therefore, by Corollary 5.15, it suffices to construct a vector bundle E of degree d on C and a limit linear series $(V_n \mid n \in \mathbf{Z})$ on E such that the following conditions hold for the point v of $\mathcal{G}(r+1, E')$ represented by $(V_n \mid n \in \mathbf{Z})$:

(1) $\dim_v \mathcal{G}(r+1, E) = (r+1)(d-1)$,
(2) ρ is defined at v, and
(3) v is an isolated point in the fiber of ρ.

We construct E as follows. Let E_1 be a generic vector bundle of degree d_1 on C_1, and E_2' a generic vector bundle of degree $d_2 - 1$ on C_2. Choose a generic isomorphism $E_1|_0 \cong E_2'|_0$, and construct the vector bundle E on C by gluing E_1 and E_2' along this isomorphism. Choose $n_1 = a$ and $n_2 = b + a$ for sufficiently negative a and sufficiently positive b. The isomorphism $E_1|_0 \cong E_2'|_0$ yields isomorphisms, canonical up to scaling, of $E_1(m)|_0$ and $E_2'(n)|_0$ for any $m, n \in \mathbf{Z}$.

Having constructed E, we must now construct $(V_n \mid n \in \mathbf{Z})$. By Proposition 5.10, it is enough to construct $V_{n_1} \subset H^0(C_1, E_1 \otimes \mathcal{O}(a))$ and $V_{n_2} \subset H^0(C_2, E_2'(b-a))$, provided they define a refined EHT limit linear series. Let $V \subset H^0(C_1, E_1)$ be a generic $(r+1)$-dimensional vector space. Then it will have the vanishing sequence $(0, \ldots, 0, 1)$. Hence, we have $V^0 = E|_0$ and $V^1 \subset E|_0$ is 1-dimensional (see § 5.3 for the definition of these two subspaces). Furthermore, the genericity of V implies that V^1 is a general 1-dimensional subspace. Let $\Lambda \subset H^0(C_2, E_2'(1))$ be the image of a general $(r+1)$ dimensional subspace of $H^0(C_2, E_2)$, where E_2 is the vector bundle of degree d_2 defined by the sequence

$$0 \longrightarrow E_2 \longrightarrow E_2'(1) \longrightarrow E_2'(1)|_0 / V^1 \longrightarrow 0.$$

Then $\Lambda \subset H^0(C_2, E_2'(1))$ has the vanishing sequence $(0, 1, \dots, 1)$, with $\Lambda^0 = V^1$ and $\Lambda^1 = V^0$. Let $V_{n_1} \subset H^0(C_1, E_1 \otimes \mathcal{O}(a))$ be the image of V and $V_{n_2} \subset H^0(C_2, E_2 \otimes \mathcal{O}(b-a))$ the image of Λ. Then V_{n_1} has the vanishing sequence $(a, \dots, a, a+1)$, and Λ the complementary vanishing sequence $(b-a-1, b-a, \dots, b-a)$. By the construction of Λ, there exist bases of V_{n_1} and V_{n_2} that satisfy the gluing condition at 0. In conclusion, V_{n_1} and V_{n_2} form a refined EHT limit linear series, and hence define a limit linear series $v = (V_n \mid n \in \mathbf{Z})$.

It is easy to check that $\dim_v \mathbf{G}(r+1, E) = (r+1)(d-1)$. Indeed, for every limit linear series $w = (W_n \mid n \in \mathbf{Z})$ in an open subset around v, the EHT limit linear series associated to w determines w and has the same vanishing sequence as v. In particular, $W_{n_1} \subset H^0(C_1, E_1(a))$ is the image of an $(r+1)$-dimensional subspace $W \subset H^0(C_1, E_1)$ with vanishing sequence $(0, \dots, 1)$, and $W_{n_2} \subset H^0(C_2, E_2(b-a))$ is the image of an $(r+1)$-dimensional subspace $M \subset H^0(C_2, E_2(1))$ with vanishing sequence $(0, 1, \dots, 1)$. Furthermore, the gluing condition implies that M is in fact the image of an $(r+1)$-dimensional subspace of the kernel of the map

$$E_2'(1) \longrightarrow E_2'(1)/W^1.$$

By the genericity of V, the isomorphism type of the kernel of this map is constant around v; that is, the kernel is isomorphic to E_2. So, a dimension count for $\mathcal{G}(r+1, E)$ around v gives

$$\begin{aligned}
\dim_v \mathcal{G}(r+1, E) &= \dim \mathbf{Gr}(r+1, H^0(C_1, E_1)) + \dim \mathbf{Gr}(r+1, H^0(C_2, E_2)) \\
&= (r+1)(d_1 - 1) + (r+1)(d_2 - 1) \\
&= (r+1)(d_1 + d_2 - 2) \\
&= (r+1)(d-1).
\end{aligned}$$

Finally, we must check that v is an isolated point in the fiber of

$$\rho\colon \mathcal{G}(r+1, E) \dashrightarrow \mathcal{G}(1, E \otimes \det E \otimes \omega_C).$$

For any $w \in \mathcal{G}(r+1, E)$ in an open set around v, either $V \neq W$ or $\Lambda \neq M$, where V, Λ, W, M are as above. By construction, $V \subset H^0(r+1, H^0(C_1, E_1))$ and $\Lambda \subset H^0(r+1, H^0(C_2, E_2'))$ are isolated in their respective projection-ramification maps. Therefore, either $\rho_{C_1}(V) \neq \rho_{C_1}(W)$ or $\rho_{C_2}(\Lambda) \neq \rho_{C_2}(M)$. In either case, we obtain that $\rho(v) \neq \rho(w)$, and hence conclude that v is an isolated point in the fiber of ρ. □

erativeproblems}

6. The Projection-Ramification enumerative problem

In this section, we calculate the degree of the projection-ramification map for as many varieties of minimal degree as we can, leading to a proof of Theorem E. After treating the relatively easy cases by hand, we relate the projection-ramification map for the veronese surface and the quartic normal scroll with classical geometry of cubic plane curves.

{sub:a_rnc}

6.1. **Rational normal curves.** Let $X \subset \mathbf{P}^n$ be a rational normal curve. Plainly, X is incompressible, and hence the projection-ramification map

$$\rho \colon \mathbf{Gr}(2, n+1) \longrightarrow \mathbf{P}^{2n-2}$$

is a regular map. Therefore, we get

$$\begin{aligned} \deg \rho &= c_1(\rho^* \mathcal{O}(1))^{2n-2} \\ &= c_1(\mathcal{O}_{\mathbf{Gr}(r+1,n+1)}(1))^{2n-2} \\ &= \frac{(2n-2)!}{n!(n-1)!}. \end{aligned}$$

{sub:a_quadric_s

6.2. **Quadric hypersurfaces.** A smooth quadric hypersurface $X \subset \mathbf{P}^n$ defined by a homogeneous quadric equation $F(X_0, \ldots, X_n) = 0$. An easy calculation shows that the projection-ramification map

$$\rho \colon \mathbf{P}^n \longrightarrow (\mathbf{P}^n)^*$$

is given in coordinates by

$$p = [p_0 : \cdots : p_n] \mapsto \left[\frac{\partial F}{\partial X_0}(p) : \cdots : \frac{\partial F}{\partial X_n}(p) \right].$$

In other words, it is the *polarity isomorphism* induced by F, namely the isomorphism between a projective space and its dual given by the non-degenerate bilinear form associated to F. In particular, we get $\deg \rho = 1$.

{sec:veronese}

6.3. **The Veronese surface.** Let $\mathbf{P}^2 \cong X \subset \mathbf{P}^5$ be the Veronese surface, the image of $\mathbf{P}^2$ under the complete linear series $\mathcal{O}(2)$. In this case, the projection-ramification map

$$\rho \colon \mathbf{Gr}(3, H^0(\mathbf{P}^2, \mathcal{O}(2))) \cong \mathbf{Gr}(3, 6) \dashrightarrow \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}(3))^* \cong \mathbf{P}^9$$

can be described as follows. Let $N \subset H^0(\mathbf{P}^2, \mathcal{O}(2))$ be a net of conics. Then $\rho(N)$ corresponds to the cubic curve traced out by the nodes of the singular members of N, called the *Jacobian* of N.

{prop:veronese}

Proposition 6.1. *Let $R \subset \mathbf{P}^2$ be a general cubic. The fiber of ρ over R is in natural bijection with the set of non-trivial 2-torsion line bundles on R. In particular, we have $\deg \rho = 3$.*

The rest of **??** is devoted to the proof of this assertion.

For the proof, we recall some classical projective geometry of cubics and nets of conics from [**?**, § 3]. To distinguish the various copies of $\mathbf{P}^2$ that naturally arise in this story, write $\mathbf{P}^2 = \mathbf{P}V$ for a 3 dimensional vector space V. Let $N \subset H^0(\mathbf{P}V, \mathcal{O}(2)) = \operatorname{Sym}^2 V$ be a general net of conics on $\mathbf{P}V$. Given a point $x \in \mathbf{P}N^*$, we denote the associated conic by Q_x.

Associated to the net N are three important cubic plane curves, namely the Jacobian curve, the discriminant curve, and the Hermite curve. We have already seen the Jacobian curve $R \subset \mathbf{P}V$. The *discriminant curve* $D \subset \mathbf{P}N^*$ is the locus of $x \in \mathbf{P}N^*$ such that Q_x

is singular. Since a pencil of conics contains three singular members, we see that D is a cubic curve. Note that if Q_d is singular, then it is the union of two distinct lines in $\mathbf{P}V$. A component line of Q_x is called a *Reye line*. The *Hermite curve* $E \subset \mathbf{P}V^*$ is the locus of Reye lines. We leave it to the reader to check that it is a cubic curve.

The three cubic curves introduced above are inter-related. First, we have an isomorphism $\tau\colon D \longrightarrow R$ defined by

{eqn:DR}
$$\tau\colon d \mapsto \text{The singular point of } Q_d. \tag{6.1}$$

Second, we have a degree 2 map $E \longrightarrow D$ defined by

$$\ell \mapsto \text{The } d \in D \text{ such that } Q_d \text{ contains } \ell.$$

Evidently, the fiber of this map over a given $x \in D$ corresponds to the two components of Q_d. The (étale) degree 2 map $E \longrightarrow D \cong R$ gives a non-trivial two-torsion element $\eta \in \operatorname{Pic}(R)[2]$ The element η is characterized by the property that it is the unique non-trivial two-torsion element whose pull-back to E is trivial.

Denote by H the hyperplane divisor class on $R \subset \mathbf{P}^2$.

{lem:reye}
Lemma 6.2. *For every $a \in R$, the line joining a and $a + \eta$ is a Reye line. Furthermore, this Reye line is a component of Q_d where $d = \tau^{-1}(H - 2a - \eta)$. Finally, the conjugate Reye line, namely the other component of Q_d, passes through the points b and $b + \eta$ where $b \in R$ differs from a by a non-trivial two-torsion element other than η.*

Proof. Let ℓ be a general Reye line, and let $d \in D$ be such that ℓ is a component of Q_d. Let $x = \tau(d) \in R$ be the singular point of Q_d. Note $\ell \cap R$ consists of three points, one of which is x. It suffices to show that the other two, say y and z, differ by η.

The point y defines a line in $\mathbf{P}V^*$. This line intersects $E \subset \mathbf{P}V^*$ in three points, one of which is ℓ, and the other two are the two components of $Q_{\tau^{-1}(y)}$, namely the two pre-images of $y \in R$ under the double covering $E \longrightarrow R$. Call these two points y_1 and y_2. Define z_1 and z_2 analogously. By construction, the triplets y_1, y_2, ℓ and z_1, z_2, ℓ are collinear triplets on $E \subset \mathbf{P}V^*$, and therefore we have the linear equivalence

$$y_1 + y_2 \sim z_1 + z_2$$

on E. By pushing this forward to R, we get

$$2y \sim 2z.$$

Therefore, $y - z$ is a (non-trivial) two torsion element in $\operatorname{Pic}(R)$. However, the pull-back of $y - z$ is trivial on E, and hence $y - z = \eta$.

Finally, let m be the Reye line conjugate to ℓ. Then it passes contains x, and two other points of R, say y' and z'. By what we just proved, $y' - z' = \eta$. But we also have $y' + z' \sim y + z$. Hence $y - y'$ is a two-torsion element, non-trivial, and distinct from η. The proof is now complete. □

We now have all the tools to prove Proposition 6.1.

Proof of Proposition 6.1. Let $U \subset \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}(3))^*$ be the locus of smooth cubic curves, $J \longrightarrow U$ be the universal Picard scheme, $J[2] \subset J$ the closed subscheme of two-torsion classes, and $J[2]^* \subset J[2]$ the open and closed subscheme of non-trivial two-torsion classes. The projection-ramification map for the Veronese surface factors as

$$\begin{gathered}\rho\colon \mathbf{Gr}(3, H^0(\mathbf{P}^2, \mathcal{O}(2))) \dashrightarrow J[2]^* \dashrightarrow \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}(3))^* \\ N \mapsto (R, \eta) \mapsto R.\end{gathered}$$

We construct $J[2]^* \longrightarrow \mathbf{Gr}(3, H^0(\mathbf{P}^2, \mathcal{O}(2)))$ inverse to the first map. Given $(R, \eta) \in J[2]^*$, we need to construct a net N of conics with Jacobian R. We use Lemma 6.2, which tells us the singular elements of this net in terms of R and η. Let $\{\eta, \eta', \eta''\}$ be the three non-trivial 2 torsion line bundles on R. Define the map $R \longrightarrow \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}(2))^*$ by

$$R \ni a \mapsto (\langle a, a+\eta \rangle) \cdot \left(\langle a+\eta', a+\eta'' \rangle\right),$$

where $\langle p, q \rangle$ denotes the line joining p and q. We leave it to the reader to check that the image of R is a plane cubic curve. The span of the image of R is the desired net N. ☐

{sec:quartic_scr

6.4. **Quartic surface scroll.** Our next objective is to prove that $\deg \rho_X = 2$ for a generic quartic surface scroll $X \subset \mathbf{P}^5$. We begin by recasting ρ_X in terms of nets of conics on $\mathbf{P}^2$, and bring in the projective geometry introduced in § 6.3.

The generic quartic surface scroll $X \subset \mathbf{P}^5$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, embedded by the complete linear system associated to $\mathcal{O}(1,2)$. Say $\mathbf{P}^1 \times \mathbf{P}^1 = \mathbf{P}U \times \mathbf{P}V$, where U and V are two-dimensional vector spaces. Then the projection-ramification map is a $\mathrm{PGL}(U) \times \mathrm{PGL}(V)$-equivariant map

$$\mathbf{Gr}(3, U \otimes \operatorname{Sym}^2 V) \dashrightarrow \mathbf{P}(U \otimes \operatorname{Sym}^4 V)^*.$$

We take the quotient of both sides by the $\mathrm{PGL}(U) \times \mathrm{PGL}(V)$-action. We begin by identifying the two quotients.

Let S be a 3-dimensional quadratic space, that is, a vector space with a non-degenerate quadratic form q. Then we have $\operatorname{Aut}(S) = \mathrm{O}(q) \cong \mathrm{O}(3)$. THe projective space $\mathbf{P}S$ is isomorphic to $\mathbf{P}^2$, and it comes with a distinguished smooth conic $Q \subset \mathbf{P}S$. The automorphism group of the pair $(\mathbf{P}S, Q)$ is $\operatorname{Aut}(Q) \cong \mathrm{PGL}_2$.

{lem:quotgrass}

Lemma 6.3. *The quotient* $\mathbf{Gr}(3, U \otimes \operatorname{Sym}^2 V)/\mathrm{PGL}(U) \times \mathrm{PGL}(V)$ *is birational to the quotient* $\operatorname{Hilb}^3(\mathbf{P}S)/\operatorname{Aut} S$.

Proof. Let W be a 3-dimensional vector space. We have a birational isomorphism

$$\begin{aligned}\mathbf{Gr}(3, &U \otimes \operatorname{Sym}^3 V)/\mathrm{PGL}(U) \times \mathrm{PGL}(V) \\ &\sim (W^* \otimes U \otimes \operatorname{Sym}^2 V)/\mathrm{GL}(W) \times \mathrm{GL}(U) \times \mathrm{GL}(V).\end{aligned}$$

Interpret the space $(W^* \otimes U \otimes \operatorname{Sym}^2 V)/\mathrm{GL}(W) \times \mathrm{GL}(U)$ as the space of 2×3 matrices with entries in $\operatorname{Sym}^2 V$, modulo row and column transformations. Set $S = \operatorname{Sym}^2 V$; it has a canonical (up to scaling) quadratic form given by the conic $Q \cong \mathbf{P}V \subset \mathbf{P}S$ embedded by

$\mathcal{O}(2)$. We can then interpret $(W^* \otimes U \otimes \operatorname{Sym}^2 V)/\operatorname{GL}(W) \times \operatorname{GL}(U)$ as the space of 2×3 matrices with entries in S. We have a birational isomorphism

$$\begin{aligned}(W^* \otimes U \otimes \operatorname{Sym}^2 V)/\operatorname{GL}(W) \times \operatorname{GL}(U) &\sim \operatorname{Hilb}^3(\mathbf{P}S) \\ 2 \times 3 \text{ matrix } M &\mapsto \text{Vanishing locus of } 2 \times 2 \text{ minors of } M.\end{aligned}$$

By taking the further quotient by $\operatorname{GL}(V)$, we finish the proof. □

{lem:quotram}

Lemma 6.4. *The quotient* $\mathbf{P}(U \otimes \operatorname{Sym}^4 V)^*/\operatorname{PGL}(U) \times \operatorname{PGL}(V)$ *is birational to the quotient* $\mathbf{Gr}(2, (\operatorname{Sym}^2 S)/q)/AutS$.

Proof. We have the birational isomorphism

$$(U \otimes \operatorname{Sym}^4 V)/\operatorname{GL}(U) \sim \mathbf{Gr}(2, \operatorname{Sym}^4 V).$$

Note that $q \in \operatorname{Sym}^2 S$ spans the kernel of the natural surjection $\operatorname{Sym}^2 S \longrightarrow \operatorname{Sym}^4 V$. The assertion follows. □

Via the birational isomorphisms in Lemma 6.3 and Lemma 6.4, the projection-ramification map μ transforms into an $\operatorname{Aut}(S)$-equivariant map

$$\mu \colon \operatorname{Hilb}^3 \mathbf{P}S \dashrightarrow \mathbf{Gr}(2, \operatorname{Sym}^2 S/q).$$

We now describe this map μ. To ease notation, we denote a linear form and its vanishing locus by the same letter. Let $\xi \in \operatorname{Hilb}^3 \mathbf{P}S$ be a general point corresponding to the three vertices of the triangle formed by three lines L_i for $i = 1, 2, 3$. Two lines L_i and L_j define a pencil of quadratic forms on Q. Let R_{ij} be the line whose intersection with Q is the ramification divisor of the pencil $\langle L_i, L_j \rangle$. It is easy to check that the quadrics L_1R_{23}, L_2R_{13}, and L_3R_{12} span a 3-dimensional subspace of $\operatorname{Sym}^2 S$ that contains the quadric q.

{lem:mu}

Lemma 6.5. *In the setup above, the image of* ξ *under* μ *is the image of* $\langle L_1R_{23}, L_2R_{13}, L_3R_{12} \rangle$ *in* $\operatorname{Sym}^2 S/q$.

Proof. The ideal of the point $\xi \in \operatorname{Hilb}^3(S)$ is cut out by 2×3 matrix of linear forms

$$M = \begin{pmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \end{pmatrix}.$$

Let U_0, U_1 be a basis of U. Under the isomorphism in Lemma 6.3, this 2×3 matrix corresponds to the point of $\mathbf{Gr}(3, U \otimes \operatorname{Sym}^2 V)$ given by the subspace of $U \otimes \operatorname{Sym}^3 V$ spanned by $U_0M_{0,i} + U_1M_{1,i}$ for $i = 1, 2, 3$. From (4.10), the ramification divisor of this subspace is given by

$$\begin{aligned} R &= \det \begin{pmatrix} L_1 & 0 & U_0L_1' \\ 0 & L_2 & U_1L_2' \\ L_3 & L_3 & (U_0 + U_1)L_3' \end{pmatrix} \\ &= U_0L_2(L_3'L_1 - L_1L_3') + U_1L_1(L_3'L_2 - L_2L_3') \\ &= U_0L_2R_{13} + U_1L_1R_{23}. \end{aligned}$$

In this calculation, L_i' denotes the derivative $\frac{d}{dt}$ of L_i considered as an element of $k[t]$ by pullback under some parametrization $\operatorname{Spec} k[t] \longrightarrow Q$ and trivialization of $\mathcal{O}(2)|_{\operatorname{Spec} k[t]}$. Although the derivative depends on the choices, the forms $L_iL_j' - L_jL_i'$ do not, and they cut out precisely the ramification divisor of the pencil $\langle L_i, L_j\rangle$. Under the isomorphism in (6.4), the divisor R corresponds to the 2 dimensional subspace of $\operatorname{Sym}^2 S/q$ spanned by L_2R_{13} and L_1R_{23} (The roles of L_1, L_2, L_3 can be changed by linear transformations of M, so we get that L_3R_{12} also lies in this span). The proof is thus complete. □

Recall that the conic $Q \subset \mathbf{P}S$ gives an isomorphism $\mathbf{P}S \cong \mathbf{P}S^*$, called *polarity* with respect to Q. On the vector spaces, it is the isomorphism induced by the bilinear form associated to q. Geometrically, it is characterized by the rule that the polar of a point $p \in Q$ is the tangent line to Q at p. More generally, given a point $p \in \mathbf{P}S$, the pencil of lines through p contains two lines tangent to Q; the polar of p is the line joining the two points of tangency. We denote the polar of a point p (resp. a line L) by $p^\perp$ (resp. $L^\perp$).

Set $M_i = R_{jk}$, and let N be the net spanned by L_iM_i for $i = 1, 2, 3$. By the definition of R_{jk}, we see that M_i is the polar line of the point $L_k \cap L_k$. In other words, the triangles (L_1, L_2, L_3) and (M_1, M_2, M_3) are polar conjugates—lines in one are polars to the vertices of the other.

Recall that $\xi \in \operatorname{Hilb}^3 \mathbf{P}S$ is the point defined by the three vertices of the triangle formed by (L_1, L_2, L_3). Let $\xi' \in \operatorname{Hilb}^3 \mathbf{P}S$ be the point defined by the three vertices of the triangle formed by (M_1, M_2, M_3).

Proposition 6.6. *In the setup above, ξ and ξ' are the only points of* $\operatorname{Hilb}^3 \mathbf{P}S$ *that map to* $N \in \mathbf{Gr}(2, \operatorname{Sym}^2 S/q)$. *In particular, the degree of* $\mu\colon \operatorname{Hilb}^3 \mathbf{P}S \dashrightarrow \mathbf{Gr}(2, \operatorname{Sym}^2 S/q)$ *is 2.*

Proof. By Lemma 6.5, we see immediately that $\mu(\xi') = \mu(\xi) = N$. To show that no other triangles map to N, consider pairs of triplets $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ and $\nabla = (\nabla_1, \nabla_2, \nabla_3)$ of lines in $\mathbf{P}S$ such that

(1) Δ and ∇ are polar conjugates with respect to Q, and
(2) $\Delta_i\nabla_i$ is an element of N for $i = 1, 2, 3$.

It suffices to show that the only ones satisfying the two conditions are (L_1, L_2, L_3) and (M_1, M_2, M_3), up to permutation.

To show this, we need some observations.

First, suppose A_1B_1 and A_2B_2 are elements of the net N, where A_i and B_j are lines in $\mathbf{P}S$. Then, by definition, A_i and B_j are Reye lines of the net N. Let $p = A_1 \cap A_2$ and $q = B_1 \cap B_2$. We claim that the third Reye line through p, in addition to A_1 and A_2, is the line $\langle p, q\rangle$. Indeed, in the pencil of conics spanned by A_1B_1 and A_2B_2, the third singular conic is $\langle p, q\rangle\langle p', q'\rangle$, where $p' = A_1 \cap B_2$ and $q' = A_2 \cap B_1$.

Second, let $R \subset \mathbf{P}S$ be the Jacobian cubic and $E \subset \mathbf{P}S^*$ be the Hermite cubic of N. Let $E^\perp \subset \mathbf{P}S$ be the image of E under the polarity isomorphism induced by Q. Explicitly, the points of $E^\perp$ are the polars of the Reye lines. We claim that the six points of intersection of R and Q also lie on $E^\perp$. Indeed, to show that $x \in R \cap Q$ also lies on $E^\perp$, it suffices to show that the line T_xQ is a Reye line. Since $x \in R$, there exists an element of N of the form AB where A and B are lines intersecting at x. Note that in the pencil of conics spanned by AB and Q, there is a singular conic containing T_pQ. Therefore, T_pQ is a Reye line.

Third, since $R \cap E^\perp$ contain 6 points on the conic Q, the residual 3 points are collinear. Let them correspond to $x_1, x_2, x_3 \in E$. Denoting by H the hyperplane class of $E \subset \mathbf{P}S^*$, we have the equation in $\operatorname{Pic} E$

$$x_1 + x_2 + x_3 = H.$$

Suppose we have two triangles Δ and ∇ satisfying the two conditions above. Consider the point $p_3 = \Delta_3 \cap \nabla_3$. By the second condition, it lies on R. By the polar conjugacy of Δ and ∇, we have

$$\begin{aligned} p_3^\perp &= \langle \Delta_3^\perp, \nabla_3^\perp \rangle \\ &= \langle \nabla_1 \cap \nabla_2, \Delta_1 \cap \Delta_2 \rangle. \end{aligned}$$

By the first claim, we see that $p_3^\perp$ is a Reye line. Hence p_3 lies on $E^\perp$, and hence on $R \cap E^\perp$. Similarly, $p_1 = \Delta_1 \cap \nabla_1$ and $p_2 = \Delta_2 \cap \nabla_2$ also lie on $E^\perp$. Since N is general, we may assume that the p_i do not lie on Q. Hence, p_1, p_2, p_3 are the points corresponding to the three collinear points in $R \cap E^\perp$. (The fact that p_1, p_2, p_3 are collinear is not surprising—it is because any two polar conjugate triangles are in linear perspective [**?**, Theorem 2.1.9]). By reordering if necessary, assume that we have $p_i^\perp = x_i$ as elements of E.

Now, observe that the three Reye lines through the vertex $\Delta_1 \cap \Delta_2$ are Δ_1, Δ_2, and $p_3^\perp$, and likewise for the other two vertices. The concurrence of the three lines, along with the equality $p_3^\perp = x_1$, yields the system of equations on $\operatorname{Pic} E$

$$\begin{aligned} \Delta_1 + \Delta_2 + x_3 &= H, \\ \Delta_2 + \Delta_3 + x_1 &= H, \\ \Delta_3 + \Delta_1 + x_2 &= H. \end{aligned}$$

Of course, the same three equations hold if we replace Δ by ∇.

Note that the points $x_1, x_2, x_3 \in E$ are determined by N. Using $x_1 + x_2 + x_3 = H$, a simple calculation gives $2\Delta_1 = 2x_1$. This equation has 4 solutions for Δ_1, namely $x_1 + \epsilon$ for $\epsilon \in \operatorname{Pic} E[2]$. Also, Δ_1 determines Δ_2 and Δ_3 by the equations above, which in turn determine the ∇_i using polarity or the property that ∇_i and Δ_i form a fiber of the map $E \longrightarrow R$. Thus, it suffices to show that only two of the four solutions for Δ_1 can be valid.

Suppose $\Delta_1 = x_1$. Then we get $\Delta_2 = x_2$, and $\Delta_3 = x_3$. However, the lines represented by the x_i are concurrent, whereas the lines Δ_i are not. Therefore, we get that $\Delta_1 \neq x_1$. The same argument shows that $\nabla_1 \neq x_1$. Let the involution of E induced by $E \longrightarrow R$ be given by the addition of $\epsilon_0 \in \operatorname{Pic} E[2]^*$. Since Δ_1 and ∇_1 form a fiber of $E \longrightarrow R$, we have $\nabla_1 = \Delta_1 + \epsilon_0$. So, $\nabla_1 \neq x_1$ translates into $\Delta_1 \neq x_1 + \epsilon_0$. In summary, the only two possible solutions for Δ_1 are $x_1 + \epsilon$ for $\epsilon \in \operatorname{Pic} E[2] \setminus \{0, \epsilon_0\}$. The proof is now complete. $\square$

We begin with the following seemingly unrelated geometric figure: $C \subset \mathbf{P}^2$ is a smooth cubic curve, $a \in \mathbf{P}^2 \setminus X$ a point, and $Q \subset \mathbf{P}^2$ the *polar conic* of a with respect to C – the unique conic which passes through the six points of ramification on C of the projection from a. We assume a is chosen so that Q is a smooth conic.

To set notation moving forward, if $x \in \mathbf{P}^2$ is any point, we let $P_x(C)$ denote the polar conic of x with respect to C. Similarly, we let $P_x(Q)$ denote the polar line of x with respect to the conic Q.

{lemma:basicsabo

Lemma 6.7. *The Hessian $Hess(C) \subset \mathbf{P}^2$ consists of the points x such that $P_x(C)$ is singular, and if C is not a Fermat cubic $P_x(C)$ is the union of two distinct lines for every $x \in Hess(C)$. Furthermore, if $x \in Hess(C)$, then the unique singularity $s(x)$ of $P_x(C)$ lies on $Hess(C)$, and the map $x \mapsto s(x)$ is translation by a 2-torsion point $\eta \in Jac(Hess(C))$.*

Proof. Standard. $\boxed{\text{ToDo: Reference, probably Dolgachev}}$ □

{proposition:pol

Proposition 6.8. *Suppose $x \in Hess(C)$. Then the line $P_x(C)$ passes through the point $x + \eta \in Hess(C)$.*

Proof. We have:

$$P_x(Q) = P_x P_a(C) = P_a P_x(C).$$

Since $x \in Hess(C)$, $P_x(C)$ is a singular conic. Hence $P_x(Q)$ must pass through the singularity $\operatorname{Sing} P_x(C)$, which by Lemma 6.7 is the point $x + \eta$. □

Next suppose $\ell_1, m_1, \ell_2, m_2, \ell_3, m_3$ are six distinct lines in $\mathbf{P}^2$ with the properties:

(1) The three singular conics $\ell_i \cup m_i$ are polars of C.
(2) The triangle $\ell_1\ell_2\ell_3$ is *conjugate* to the triangle $m_1m_2m_3$ with respect to Q.

The second condition above simply means that the vertices of one triangle are polar to the lines of the other triangle. By basic projective geometry, the two triangles are then in *linear perspective*, i.e. the three points $x_1 := \ell_1 \cap m_1, x_2 := \ell_2 \cap m_2, x_3 := \ell_3 \cap m_3$ are collinear.

{proposition:imp

Proposition 6.9. *Maintain the notation above, and recall the definition of Reye line from the previous subsection. The lines $P_{x_i}(Q)$ are Reye lines of the net of polar conics of C.*

Proof. The triangles $\ell_1\ell_2\ell_3$ and $m_1m_2m_3$ are conjugate with respect to Q. Hence it follows that the polar line $P_{x_3}(Q)$ equals $\ell := \overline{\ell_{12}m_{12}}$, where $\ell_{ij} = \ell_i \cap \ell_j$ and $m_{ij} = m_i \cap m_j$.

We will prove that ℓ is one of the three Reye lines of the net of polars of C which pass through the point ℓ_{12}, the other two Reye lines being ℓ_1 and ℓ_2. From Lemma 6.7, we can find points $y, z \in C$ and write:

$$\ell_1 = \overline{y, y + \eta}$$
$$\ell_2 = \overline{z, z + \eta}$$

Then a divisor class computation shows that the third Reye line through ℓ_{12} must be $\overline{w, w+\eta}$, with w satisfying

$$y + z + w \sim H + \epsilon,$$

where ϵ is any one of the two non-trivial 2-torsion elements on $Hess(C)$ differing from η. We let $s \in Hess(C)$ denote the third point of intersection of the line $\overline{w, w+\eta}$ with C. (Notice that w depends on the choice of ϵ, but the line $\overline{w, w+\eta}$ is independent of this choice.)

From this setup, we get:

$$\begin{aligned} x_1 &\sim H - 2y - \eta \\ x_2 &\sim H - 2z - \eta \\ s &\sim H - 2w - \eta \end{aligned}$$

from which we get:

$$\begin{aligned} s &\sim H - 2w - \eta \\ &\sim H - 2[H + \epsilon - y - z] - \eta \\ &\sim 2y + 2z - H - \eta \\ &\sim H - x_1 - \eta + H - x_2 - \eta - H - \eta \\ &\sim H - x_1 - x_2 - \eta. \end{aligned}$$

Therefore, to prove that ℓ is a Reye line, it suffices to show that the points ℓ_{12}, m_{12}, and $s \sim H - x_1 - x_2 - \eta$ are collinear. But, this is true if and only if their respective polar lines $P_{\ell_{12}}(Q), P_{m_{12}}(Q), P_s(Q)$ are concurrent. The latter is true if and only if the lines $m_3, \ell_3, P_s(Q)$ are concurrent, which in turn translates to the condition that $x_3 \in P_s(Q)$. But, $x_3 \sim H - x_1 - x_2$, and $s \sim H - x_1 - x_2 + \eta$, and so by Proposition 6.9, we conclude that indeed $x_3 \in P_s(Q)$, which is what we needed to show. $\square$

6.4.1. *Returning to the projection ramification problem.* Our next objective is to relate the geometry in the previous subsection to

rojective_space}

6.5. **Rational curves, the differential construction, and the case of Segre varieties.** **??** connects with an old story involving rational curves in projective space.

Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ be a degree d morphism. Its derivative

$$d\gamma : T_{\mathbf{P}^1} \longrightarrow \gamma^*(T_{\mathbf{P}^n})$$

may be viewed as a global section of the rank r vector bundle $\gamma^*(T_{\mathbf{P}^n}) \otimes T_{\mathbf{P}^1}^\vee$. The splitting of $\gamma^*(T_{\mathbf{P}^n})$ is known to be balanced for a general morphism γ. In particular, if the divisibility

$$n \mid d$$

holds, and if we set $\ell := d + d/n - 2$, then a general γ satisfies:

$$(\gamma^* T_{\mathbf{P}^n}) \otimes T_{\mathbf{P}^1}^{\vee} \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbf{P}^1}(\ell).$$

The direct sum decomposition is not canonical, it is only defined up to the action of $GL_n(k)$.

Assuming γ is an immersion, the element $d\gamma \in H^0(\mathbf{P}^1, \bigoplus_{i=1}^n \mathcal{O}_{\mathbf{P}^1}(\ell))$ does not vanish anywhere, and hence defines a degree ℓ map

$$D(\gamma) : \mathbf{P}^1 \longrightarrow \mathbf{P}^{n-1},$$

only well-defined up to the action of post-composition by $PGL_n(k)$.

Definition 6.10. Let M_d^n denote the moduli stack parametrizing $PGL_{n+1}(k)$ equivalence classes of degree d maps $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$, and let $U_d^n \subset M_d^n$ denote the open substack parametrizing local immersions with $\gamma^*(T_{\mathbf{P}^n})$ balanced.

Remark 6.11. Notice: $\dim M_d^n = (k+1)(n+1) - (n+1)^2 = (n+1)(k-n) = \dim \mathbf{G}(n,k)$. Furthermore, notice $PGL_2(k)$ acts on U_d^n and M_d^n by pre-composition.

Remark 6.12. Though M_d^n is an Artin stack, the open substack U_d^n is a scheme, provided $n \leq d$, represented by an open subset of $\mathbf{Gr}(n+1, d+1)$.

When $n \mid d$, and $\ell := d + d/n - 2$, we get the morphism of stacks:

$$D_d^n : U_d^n \longrightarrow M_\ell^{n-1}$$
$$\gamma \longmapsto D(\gamma)$$

which we call the *differential construction*. Interestingly, the dimensions of the domain and codomain of the differential construction are equal, and this leads to another collection of enumerative problems:

{problem:differe

Problem 6.13. Compute the degrees of the differential constructions $D_d^n : U_d^n \longrightarrow M_\ell^{n-1}$.

Remark 6.14. The maps D_d^n are clearly $PGL_2(k)$ equivariant. The image of the differential construction D_d^n need not be the open set U_ℓ^{n-1}. $\boxed{\text{ToDo: Sure?}}$

The $n = d$ instances of Problem 6.13 are immediate:

{proposition:tri

Proposition 6.15. *The degree of the differential construction D_d^d is* 1.

Proof. The space U_d^d is a single $PGL_2(k)$ orbit. $\square$

{definition:poin

Definition 6.16. Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ be any map. We define the *point-hyperplane scroll* of γ to be

$$X_\gamma := \left\{(t, \Lambda) \mid \gamma(t) \in \Lambda\right\} \subset \mathbf{P}^1 \times (\mathbf{P}^n)^{\vee}$$

We denote by π_1, π_2 the projections of X_γ to $\mathbf{P}^1$ and $(\mathbf{P}^n)^{\vee}$ respectively. Finally, we set $X_\gamma^{\vee} := \mathbf{P}(\gamma^* T_{\mathbf{P}^n})$.

Remark 6.17. The $\mathbf{P}^{n-1}$-bundle X_γ is isomorphic to $\mathbf{P}(\gamma^* T^\vee_{\mathbf{P}^n})$. Hence, for a general map $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$, X_γ and $X^\vee_\gamma$ are balanced scrolls.

{proposition:tra

Proposition 6.18. *Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ be a non constant map.*

(1) *The image of $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^n$ is non-degenerate if and only if $\pi_2 : X_\gamma \longrightarrow (\mathbf{P}^n)^\vee$ is finite; in any case, $\deg \pi_2 = \deg \gamma$.*
(2) *The ramification divisor $R(\pi_2) \subset X_\gamma$ is a smooth, codimension 1 subscroll of X_γ if and only if γ is an immersion.*
(3) *Assuming γ is an immersion, the dual section $R^\vee(\pi_2) \subset X^\vee_\gamma$ is induced by the inclusion $d\gamma : T_{\mathbf{P}^1} \hookrightarrow \gamma^* T_{\mathbf{P}^n}$.*

Proof. $\boxed{\text{ToDo: PROVE}}$ □

Let $X = \mathbf{P}^1 \times \mathbf{P}^{n-1}$, and denote by h and f the divisor classes of the pullback of a hyperplane in $\mathbf{P}^{n-1}$ and a point in $\mathbf{P}^1$, respectively. When $n \mid k$, Proposition 6.18 sets up a commuting diagram:

$$\begin{tikzcd}
U^n_k \arrow[r, leftrightarrow, "\text{duality}"] \arrow[d, "D^n_k"] & PGL_{n+1} \Big\backslash \left\{ \begin{array}{c} \text{Deg. } k \text{ maps} \\ X \longrightarrow (\mathbf{P}^n)^\vee \\ \text{induced by } |h + \frac{k}{n} f| \end{array} \right\} \Big/ PGL_n \arrow[d, "\rho_X"] \\
M^{n-1}_\ell \arrow[r, leftrightarrow, "\text{duality}"] & \left\{ \begin{array}{c} \text{Smooth divisors } R \subset X \\ \text{with div. class } |h + \ell f| \end{array} \right\} \Big/ PGL_n
\end{tikzcd}$$

From this, we conclude:

ion:equivalence}

Proposition 6.19. *Let $k = nm$, and let $X \subset \mathbf{P}^{n(m+1)-1}$ be the variety $\mathbf{P}^1 \times \mathbf{P}^{n-1}$ embedded by the linear series $|h + mf|$. Then*

$$\deg \rho_X = \deg D^n_k.$$

Corollary 6.20. *If $X \subset \mathbf{P}^{2n-1}$ is a Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^{n-1}$, then $\deg \rho_X = 1$.*

Proof. The corollary follows at once from Proposition 6.19 and Proposition 6.15. □

6.6. **Quartic surface scrolls.**

6.6.1. *The explicit differential construction for trinodal quartics.* A trinodal quartic R can be obtained as an abstract curve by identifying three pairs of points $\{a', a''\}, \{b', b''\}, \{c', c''\}$ on $\mathbf{P}^1$. These pairs can be encoded by the three binary quadratic forms (up to scale) defining them. In terms of these three quadratic forms, we will now describe the differential construction D^2_4.

In what follows, we let $\{q_1, q_2, q_3\}$ denote a point in $\operatorname{Sym}^3 \mathbf{P} H^0(\mathcal{O}_{\mathbf{P}^1}(2))$.

inition:nodemap}

Definition 6.21. Let

$$\nu : \mathrm{Sym}^3 \mathbf{P}H^0(\mathcal{O}_{\mathbf{P}^1}(2)) \dashrightarrow \mathbf{Gr}(3,5)$$

denote the map given by the formula:

$$\nu(\{q_1,q_2,q_3\}) = \left\{ \begin{array}{c} \text{v. space of meromorphic 1-forms } \omega \text{ on } \mathbf{P}^1 \text{ with at worst} \\ \text{simple poles at the zeros of } q_i \text{ and with } \textit{opposite} \text{ residues} \\ \text{at the pairs of zeros of } q_i \text{, for all } i=1,2,3 \end{array} \right\}$$

{proposition:sym

Proposition 6.22. *The map ν is birational.*

Proof. Suppose a general three dimensional space $W \subset H^0(\mathcal{O}_{\mathbf{P}^1}(4))$ is given. Then the induced degree four map $\mathbf{P}^1 \longrightarrow \mathbf{P}W^\vee$ is the normalization of a trinodal quartic R. The vector space W is naturally identified with the sections of the dualizing sheaf of R, which consist of meromorphic 1-forms on $\mathbf{P}^1$ with the properties stated in the proposition. □

{definition:pi}

Definition 6.23. Let

$$\pi : \mathrm{Sym}^3 \mathbf{P}^2 \dashrightarrow \mathbf{Gr}(2,5)$$

be given by the formula

$$\pi(\{q_1,q_2,q_3\}) = \left\{ \begin{array}{l} \text{v. space of meromorphic 1-forms } \omega \text{ with at worst simple} \\ \text{poles at the zeros of } q_i \text{ and with } \textit{equal} \text{ residues} \\ \text{at the pairs of zeros of } q_i \text{, for all } i=1,2,3 \end{array} \right\}$$

{proposition:rei

Proposition 6.24. *The rational map $\pi \circ \nu^{-1} : \mathbf{Gr}(3,5) \dashrightarrow \mathbf{Gr}(2,5)$ is the differential construction D^2_4.*

Proof. Let $\gamma : \mathbf{P}^1 \longrightarrow \mathbf{P}^2$ be a general map induced by a three dimensional vector space $W \subset H^0(\mathcal{O}_{\mathbf{P}^1}(4))$ having image R, and let (q_1,q_2,q_3) be $\nu^{-1}(\varphi)$. The pencil $D(\gamma)$ is cut out by the perspective conics. $\boxed{\text{ToDo: Why?}}$ According to **??**, the linear series on R cut out by perspective conics is $\mathcal{O}_R(1) \otimes \eta$, where η is the distinguished element $(-1,-1,-1) \in \mathrm{Pic}(R)[2]$. If the space of sections of the line bundle $\mathcal{O}_R(1)$ is identified with $\nu(q_1,q_2,q_3)$, then it follows that the space of sections of the twist $\mathcal{O}_R(1) \otimes \eta$ equals $\pi(q_1,q_2,q_3)$. □

{definition:jaco

Definition 6.25. Let $\{a(x,y), b(x,y)\}$ be two homogeneous quadratic polynomials with no common zeros. Their *Jacobian* is

$$J(a,b) := a_x b_y - a_y b_x.$$

Note that the Jacobian vanishes precisely at the two branch points of the map $[x:y] \mapsto [a(x,y) : b(x,y)]$.

{theorem:onlyape

Theorem 6.26. *Let $\{q_1,q_2,q_3\} \in \mathrm{Sym}^3 \mathbf{P}^2$ have six distinct roots. Then the vector space*

$$\langle q_1 J(q_2,q_3), q_2 J(q_1,q_3), q_3 J(q_1,q_2) \rangle$$

is equal to $\pi(q_1,q_2,q_3) \in \mathbf{Gr}(2,5)$.

Proof. By $SL_2(k)$-equivariance, it suffices to prove the theorem for three quadratic functions $\{xy, q_2, q_3\}$ where q_2 and q_3 are general.

Let α_1, α_2, and β_1, β_2 denote the roots of q_2, q_3. Note that these roots are assumed to be in $\mathbf{A}^1 \subset \mathbf{P}^1$. We let $t = x/y$ denote the affine coordinate.

The vector space $\Pi := \pi(t, q_2(t), q_3(t))$ is equal to the vector space of forms

$$\omega = \frac{f(t)dt}{tq_2(t)q_3(t)},$$

with $\deg(f) \leq 4$, and with the additional constraints

$$\begin{aligned} \operatorname{Res}_{\alpha_1} \omega &= \operatorname{Res}_{\alpha_2} \omega \\ \operatorname{Res}_{\beta_1} \omega &= \operatorname{Res}_{\beta_2} \omega \\ \operatorname{Res}_{0} \omega &= \operatorname{Res}_{\infty} \omega \end{aligned}$$

Since we know a priori that the space of such forms is two dimensional, we conclude in particular that there exists a nonzero $\omega \in \Pi$ which is nonzero and vanishing at α_1. However, the first residue condition then forces ω to vanish at α_2 as well. (This is clear from the geometry: an element of the pencil of perspective conics is cut out by a (possibly singular) conic in $\mathbf{P}^2$. If it contains a node, then its pullback to $\mathbf{P}^1$ must vanish at both points above the node.)

Therefore, there exists an $\omega \in \Pi$ of the form

$$\omega = \frac{(t-\alpha_1)(t-\alpha_2)g(t)dt}{tq_2q_3} = \frac{g(t)dt}{tq_3}.$$

The residue conditions at β_i, and $0, \infty$ together imply, up to nonzero scaling,

$$g(t) = t^2 - \beta_1\beta_2.$$

The roots $\pm\sqrt{\beta_1\beta_2}$ are precisely the branch points of the map $[x : y] \longrightarrow [xy : q_3]$. Therefore ω vanishes at the roots of the quartic polynomial $q_1 j(xy, q_3)$. The theorem follows by arguing in the same manner for the two other pairs of roots.

$\square$

Given a general triple $\{a, b, c\}$ of binary quadratic forms, we can create the three quartic binary forms $a[b, c], b[c, a], c[a, b]$, where $[p, q]$ denotes $p_x q_y - p_y q_x$. As we know, these three forms are actually linearly dependent, yielding a pencil of binary quartics.

In this way, we obtain an *a priori* rational map

$$D : \operatorname{Hilb}^3(\mathbf{P}^2) \dashrightarrow \mathbf{Gr}(2, 5)$$

where the domain is the Hilbert scheme of 3 points on $\mathbf{P}^2$.

sition:Dregular} The main observation is:

Proposition 6.27. *The rational map D extends to a regular map.*

Proof. This is best seen by describing D geometrically, and noting that the geometric construction makes sense at every point of H.

If $\{a,b,c\}$ is a general subset of $\mathbf{P}^2$, then the quartic pencil $D(\{a,b,c\})$ is obtained as follows. Recall that in $\mathbf{P}^2$ we have the canonical discriminant conic C parametrizing square forms. A point $a \in \mathbf{P}^2$ defines a line $Pol(a) \subset \mathbf{P}^2$ spanned by the two points of C which correspond to the roots of a. Furthermore, a pair of points $b,c \in \mathbf{P}^2$ defines the line $\overline{b,c} \subset \mathbf{P}^2$.

To the triple $\{a,b,c\}$ we attach the triple of pairs of lines $Pol(a) \cup \overline{b,c}$ (and permutations), which cut the conic C at 3 members of a degree 4 pencil.

This geometric construction works even for non-reduced schemes. For example, if $Z \subset \mathbf{P}^2$ is a fat point concentrated at a point $a \in \mathbf{P}^2$, we assign the degree 4 pencil on C as: The degree 2 pencil corresponding to $Pol(a)$ with two base points at $Pol(a) \cap C$. □

The map D is only generically finite; the locus of collinear triples is contracted, and has the same image as the locus of fat schemes. However, it is easy to exhibit a point in G over which there are exactly two preimages.

{lemma:TwoPreima

Lemma 6.28. *Let $\Lambda \in \mathbf{Gr}(2,5)$ denote the unique pencil of binary quartics with simple base points at $0,1,\infty$ in $\mathbf{P}^1$. Then the preimage $D^{-1}(\Lambda)$ consists of two non-reduced points.*

Proof. The two configurations are described as follows: View the three points $0,1,\infty$ on the diagonal conic C. Then the triple $\{0,1,\infty\}$ clearly maps to Λ, as does the triangle created by $Pol(0), Pol(1), Pol(\infty)$.

A simple infinitesimal calculation shows any non-trivial first-order deformation of either of these configurations will have the effect of either removing the base-points, or moving their location.

Furthermore, it is clear that these are the only two possible configurations giving rise to the pencil Λ. □

The previous lemma immediately gives:

Theorem 6.29. *Let $X \subset \mathbf{P}^5$ be a balanced quartic surface scroll. Then $\deg \rho_X = 2$.*

{sub:surfaces}

6.7. **Eccentric surface scrolls.** Let $E = \mathcal{O}(1) \oplus \mathcal{O}(k+1)$, $X = \mathbf{P}E$. Choose an affine coordinate t on $\mathbf{P}^1$, and consider the projection-ramification enumerative problem for $X \subset \mathbf{P}^{k+3}$. We claim:

{proposition:rho

Proposition 6.30. *Maintaining the setting above, ρ_X is birational.*

Let $A = H^0(\mathcal{O}_X(1))$. This vector space will be identified with the space of expressions of the form $\ell(t)x_1 + q_{k+1}(t)x_2$, where ℓ, q_{k+1} are polynomials of degrees at most 1 and $k+1$ respectively. In what follows, subscripts of polynomials in t represent the degree.

If $W \subset A$ is a general three dimensional vector space, then there will be a unique triple of elements in W of the form

$$\begin{aligned} w_0 &= t(x_1 + q_k(t)x_2) \\ w_\infty &= (x_1 + r_k(t)x_2) \\ w_* &= s_{k+1}(t)x_2 \end{aligned}$$

The Wronski determinant of this triple is:

on:jacobsurface}

$$sx_1 + \left[s(qt)' - s'(qt) - t(r's - s'r)\right] x_2 \tag{6.2}$$

Proof of Proposition 6.30. Let $r := \sigma x_1 + \tau x_2 \in H^0(X, \mathcal{O}(R))$ be a general element, we can extract the unique vector space W obeying $\rho_X(W) = [r] \in |R|$ as follows: First, we set $s := \sigma$. Secondly, given s, the equation $[s(qt)' - s'(qt) - t(r's - s'r)] = \tau$ is a system of $2k+2$ linear equations involving the $2k+2$ coefficients of the pair (q, r). We know (from Theorem A) this system has a finite, positive number of solutions. Hence it must have a unique solution, proving the proposition. $\square$

tric_threefolds}

6.8. **Eccentric threefold scrolls.** Now let $E = \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(k+1)$, $k \geq 0$, and set $X := \mathbf{P}E$. Embed $X \subset \mathbf{P}^{k+5}$ via the natural $\mathcal{O}(1)$ on X. Again, we choose affine coordinate $t \in \mathbf{P}^1$ and relative coordinates x_1, x_2, x_3 on X corresponding to the three factors of the splitting of E.

ition:threefold}

Proposition 6.31. *Maintain the setting above. Then ρ_X is birational.*

Suppose $W \subset H^0(E)$ is a general 4 dimensional vector space. Then the projection $W \longrightarrow H^0(\mathcal{O}(1) \oplus \mathcal{O}(1))$ will be an isomorphism. Hence, there will be 4 uniquely defined elements of W of the form:

$$\begin{aligned} x_1 + ax_3 \\ x_2 + bx_3 \\ tx_1 + cx_3 \\ tx_2 + dx_3 \end{aligned}$$

where a, b, c, d are degree $\leq k+1$ polynomials in t. The Wronski determinant for this tuple of equations is:

cobianthreefold}

$$\alpha x_1 + \beta x_2 + \gamma x_3 = (d - bt)x_1 + (at - c)x_2 + \left[a't(bt - d) + b't(c - at) + c'(d - bt) + d'(at - c)\right] x_3. \tag{6.3}$$

Proof of Proposition 6.31. We replace the Grassmannian $\mathbf{Gr}(4, H^0(E))$ with the affine open subset $\mathbf{A}^{4k+8}$ parametrizing quadruples (a, b, c, d). Then the ramification divisor equation (6.3) defines a map

$$\rho^* : \mathbf{A}^{4k+8} \longrightarrow \mathbf{A}^{4k+9}$$

where the latter $\mathbf{A}^{4k+9}$ is the vector space of triples (α,β,γ) with $\deg\alpha,\beta\le k+2$ and $\deg\gamma\le 2k+2$. The projection-ramification ρ_X map ρ is recovered by composing ρ^* with the projection $\mathbf{A}^{4k+9}\dashrightarrow\mathbf{P}^{4k+8}$.

First, if (a,b,c,d) are general, then one can directly use the relative primeness of $d-bt$ and $at-c$ (we omit this simple calculation) to conclude that ρ^* is generically injective on tangent spaces, and hence the generic fiber of ρ^* is finite.

We next show ρ_X is dominant. In light of the previous paragraph, it suffices to prove: If (α,β,γ) is a general point in the image of ρ^*, and $\lambda\neq 0,1$ is a constant, then $\lambda(\alpha,\beta,\gamma)$ is not in the image of ρ^*.

To this end, suppose (a,b,c,d) is a general point in $\mathbf{A}^{4k+8}$. Then $\alpha:=d-bt$ and $\beta:=at-c$ will be degree $k+2$ polynomials which are relatively prime.

For any polynomial $p(t)$, let p^+ denote the highest degree coefficient of p. Observe that $\beta^+=a^+$. Furthermore, the expression for γ is easily seen to be

$$\gamma=(\alpha'\beta-\beta'\alpha)+\alpha a+\beta b \tag{6.4}$$ {gammaEq}

where $'$ denotes d/dt.

If we scale by λ, we get:

$$\begin{aligned}\lambda\alpha&=\lambda(d-bt)\\ \lambda\beta&=\lambda(at-c)\\ \lambda\gamma&=\lambda(\alpha'\beta-\beta'\alpha)+\lambda\alpha a+\lambda\beta b\end{aligned} \tag{6.5}$$ {firstEquations}

At the same time, if $\lambda(\alpha,\beta,\gamma)$ is also realized by some quadruple $(\tilde a,\tilde b,\tilde c,\tilde d)$ then we get the equations:

$$\begin{aligned}\lambda\alpha&=\tilde d-\tilde bt\\ \lambda\beta&=\tilde at-\tilde c\\ \lambda\gamma&=\lambda^2(\alpha'\beta-\beta'\alpha)+\lambda\alpha\tilde a+\lambda\beta\tilde b\end{aligned} \tag{6.6}$$ {secondEquation}

The second equation gives $\tilde a^+=\lambda\beta^+$. The last equation gives: $\gamma=\lambda(\alpha'\beta-\beta'\alpha)+\alpha\tilde a+\beta\tilde b$. Combining with (6.4), we get

$$\alpha(a-\beta')+\beta(b+\alpha')=\alpha(\tilde a-\lambda\beta')+\beta(\tilde b+\lambda\alpha').$$

Since α and β are relatively prime and have degree greater than $a,b,\tilde a,\tilde b$, we deduce:

$$\begin{aligned}a-\beta'&=\tilde a-\lambda\beta'\\ b+\alpha'&=\tilde b+\lambda\alpha'\end{aligned}$$

By examining top coefficients, and using $a^+=\beta^+$, $\tilde a^+=\lambda\beta^+$ we get:

$$\beta^+-(k+2)\beta^+=\lambda\beta^+-\lambda(k+2)\beta^+$$

or

$$(1-\lambda)\beta^+=(1-\lambda)(k+2)\beta^+$$

Given our assumption on λ, this is only possible if $\beta^+ = 0$. However, since (a, b, c, d) were chosen generically, $\beta^+ = a^+$ would not be zero, providing our desired contradiction.

Finally, we argue $\deg \rho_X = 1$. It suffices to show that a general ramification equation $\alpha x_1 + \beta x_2 + \gamma x_3$ of the form (6.3) arises from a unique choice of polynomials (a, b, c, d). The conditions $d - bt = \alpha$ and $at - c = \beta$ produce an affine linear subspace Λ in the vector space of choices (a, b, c, d). With respect to linear coordinates on Λ, the expression for γ is also linear, and hence the available choices of (a, b, c, d) producing Equation 6.3 is an intersection of affine linear spaces. Since we already know ρ^* is generically finite, it follows that $\deg \rho_X = 1$ as desired.

□

Since every smooth three dimensional rational normal scroll specializes isotrivially to the scroll X in Proposition 6.31, we immediately get:

riation3Scrolls}

Corollary 6.32. *The projection-ramification map ρ_X is dominant for every smooth three dimensional rational normal scroll $X \subset \mathbf{P}^n$.*

6.9. **Recasting the projection-ramification map for scrolls.** Let E be a rank r ample vector bundle on $\mathbf{P}^1$, and set $X = \mathbf{P}E$. Then a general $r+1$-dimensional subspace

$$W \subset H^0(X, \mathcal{O}(1)) = H^0(\mathbf{P}^1, E)$$

yields a short exact sequence

$$0 \longrightarrow (\det E)^{-1} \longrightarrow W \otimes \mathcal{O}_{\mathbf{P}^1} \longrightarrow E \longrightarrow 0$$

which corresponds to an element w (up to scalar) of the extension space $\operatorname{Ext}^1(E, (\det E)^{-1})$. The assignment $W \mapsto [w] \in \mathbf{P}(\operatorname{Ext}^1(E, (\det E)^{-1}))$ is easily seen to be a birational map between $\mathbf{G} := \mathbf{Gr}(n+1, H^0(E))$ and $\mathbf{P}(\operatorname{Ext}^1(E, (\det E)^{-1}))$.

The ramification linear series $|R|$ is the projectivization of the vector space $V = H^0(E \otimes \det E \otimes K_{\mathbf{P}^1})$. By Serre duality, V is dual to $\operatorname{Ext}^1(E, (\det E)^{-1})$. Therefore, the projection-ramification map ρ_X may be recast as a map

$$\delta_X : \mathbf{P}(V^*) \dashrightarrow \mathbf{P}(V)$$

7. Further Questions

(1) How many of our theorems are valid in characteristic $p > 0$?
(2) When $\dim \mathbf{Gr} < \dim |R|$ and ρ_X is generically finite onto its image, then is ρ_X birational onto its image?
(3) Is $\mathbf{Gr}(2, 4)$ the only incompressible Grassmannian?
(4) Is it possible to classify the scrolls for which $\deg \rho_X = 1$?
(5) Is there an analogous characterization of varieties of minimal degree using "higher codimension" ramification loci?

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