Algebraic geometry (Notes)

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July 24, 2021

1 Affine algebraic sets

1.1 Basic definitions

1.1.1 Affine space

The objects of study in algebraic geometry are called algebraic varieties. The building blocks for general algebraic varieties are certain subsets of the affine space. Let us first recall affine space.

Let k be a field and let n be a non-negative integer. The *affine n-space over k*, denoted by $\mathbb{A}^n_k$ is the set of n-tuples $a_1, \dots, a_n$ whose entries a_i lie in k. Thus, $\mathbb{A}^n_k$ is nothing but the product k^n. The product k^n has quite a bit of extra structure—it is a k-vector space, for example—but we wish to forget it. That is the reason for choosing different notation. In particular, the zero tuple does not play a distinguished role.

1.1.2 Affine algebraic set

Let $k[x_1, \dots, x_n]$ denote the ring of polynomials in variables $x_1, \dots, x_n$ and coefficients in k. An *affine algebraic subset* of the affine space $\mathbb{A}^n_k$ is the common zero locus of a set of polynomials. More precisely, a set $S \subset \mathbb{A}^n_k$ is an affine algebraic subset if there exists a set of polysomials $A \subset k[x_1, \dots, x_n]$ such that

$$S = \{a \in \mathbb{A}^n_k \mid f(a) = 0 \text{ for all } f \in A\}.$$

Given $A \subset k[x_1, \dots, x_n]$, we denote by $V(A)$ the common zero set (V for vanishing) of elements of A:

$$V(A) = \{a \in \mathbb{A}^n_k \mid f(a) = 0 \text{ for all } f \in A\}.$$

Thus, the affine algebraic sets are precisely the sets of the form $V(A)$ for some A.

1.1.3 Examples

The following are affine algebraic sets

1. The empty set
2. Entire affine space
3. Single point

Proof.

1.1.4 Non examples

The following are not affine algebraic sets

1. The unit cube in $\mathbb{A}^n_{\mathbb{R}}$
2. Points with rational coordinates in $\mathbb{A}^n_{\mathbb{C}}$

Proof.

1.1.5 Ideals and their vanishing loci

Let R be a ring. Recall that a subset $I \subset R$ is an *ideal* if it is closed under addition and multiplication by elements of R. Given any subset $A \subset R$ the *ideal generated by* A, denoted by $\langle A \rangle$ is the smallest ideal containing A. This ideal consists of all elements r of R that can be written as a linear combination

$$r = a_1 r_1 + \cdots + a_m r_m,$$

where $a_i \in A$ and $r_i \in R$.

Proposition 1.1. *Let $A \subset k[x_1, \ldots, x_n]$. Then we have $V(A) = V(\langle A \rangle)$.*

1.2 Noetherian rings and the Hilbert basis theorem

In our definition of $V(A)$, the subset A may be infinite. But it turns out that we can replace it by a finite one without changing $V(A)$. This is a consequence of the Hilbert basis theorem, which is ultimately related to a fundamental property of rings, which we now explain.

1.2.1 The increasing chain property

Proposition 1.2. *Let R be a ring. The following are equivalent*

1. *Every ideal of R is finitely generated.*
2. *Every infinite chain of ideals*

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

   stabilises.

Proof.

Definition 1.3. (Noetherian ring) A ring R satisfying the equivalent conditions of 1.2 is called *Noetherian.*

1.2.2 Examples:

The following rings are Noetherian

1. $R = \mathbb{Z}$
2. R a field.

1. Proof SORRY

1.2.3 Non examples:

The following rings are not Noetherian

1. Ring of continuous functions on the interval.

1. Proof SORRY

1.2.4 Theorem (Hilbert basis theorem)

If R is Noetherian, then so is $R[x]$

1. Proof SORRY

   Let $I \subset R[x]$ be an ideal. For every non-negative integer m, define

$$J_m = \{\text{Leading coeff}(f) \mid f \in I, f \neq 0, \quad \deg(f) \leq m\} \cup \{0\}$$

   (a) Claim: J_m is an ideal of R.

(b) Claim: $J_m \subset J_{m+1}$. Since R is Noetherian, the chain $J_1 \subset J_2 \subset \cdots$ stabilises; say $J_m = J_{m+1} = \cdots$. Let S_i be a finite set of generators for J_i, and for $a \in S_i$, let $p_a \in I$ be a non-zero element of degree at most i whose leading coefficient is a.

(c) Claim: The (finite) set $p_a \mid a \in S_1 \cup \cdots \cup S_m$ generates I.
Prove the claims above and finish the proof.

1.2.5 Corollary

$k[x_1, \ldots, x_n]$ is Noetherian.

1. Proof SORRY

1.2.6 Corollary:

Every affine algebraic subset of $\mathbb{A}_k^n$ is the vanishing set of a finite set of polynomials.

1. Proof SORRY

1.3 The Zariski topology

The notion of affine algebraic sets allows us to define a topology on $\mathbb{A}_k^n$. Recall that we can specify a topology on a set by specifying what the open subsets are, or equivalently, what the closed subsets are. In our case, it is more convenient to do the latter. The collection of closed subsets must satisfy the following properties.

1. The empty set and the entire set are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

   We define the *Zariski topology* on $\mathbb{A}_k^n$ by setting the closed subsets to be the affine algebraic sets, namely, the sets of the form $V(A)$ for some $A \subset k[x_1, \ldots, x_n]$.

1.3.1 Proposition (Zariski topology)

The collection of affine algeraic subsets satisfies the three conditions above.

1. Proof SORRY

1.3.2 Example (Affine line)

The Zariski topology on $\mathbb{A}_k^1$ is the *finite complement topology*. The only closed sets are the finite sets (or the whole space). In other words, the only open sets are the complements of finite sets (or the empty set).

1. Proof SORRY

1.3.3 Comparison between Zariski and Euclidean topology over $\mathbb{C}$.

Every Zariski closed (open) subset of $\mathbb{A}^n_{\mathbb{C}}$ is also closed (open) in the usual Euclidean topology. The converse is not true.

1. Proof SORRY

1.3.4 Proposition (Polynomials are continuous)

Let f be a polynomial function on $\mathbb{A}^n_k$. Then f is continuous in the Zariski topology.

1.3.5 Remark (Non Hausdorff)

The Zariski topology has very few open sets, and as a result has terrible separation properties. It is not even Hausdorff (except in very small examples). Nevertheless, we will see that it is extremely useful. For one, it makes sense over every field!

1.4 The Nullstellensatz

We associated a set $V(A)$ to a subset A of the polynomial ring $k[x_1, \ldots, x_n]$. If we think of A as a system of equations $\{f = 0 \mid f \in A\}$, then $V(A)$ is the set of solutions. We can also define a reverse operation. The Nullstellensatz says that if k is algebraically closed, then these two operations are mutually inverse. That is, the data of a system of equations is equivalent to the data of its set of solutions. This pleasant fact allows us go back and forth between algebra (equations) and geometry (the solution set).

1.4.1 Definition (Ideal of a set)

Let $S \subset \mathbb{A}^n_k$ be a set. The *ideal associated to* S, denoted by $I(S)$, is the set

$$I(S) = \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \text{ for all } a \in S\}$$

Recall that an ideal $I \subset k[x_1, \ldots, x_n]$ is *radical* if it has the property that whenever $f^n \in I$ for some $n > 1$, then $f \in I$.

1.4.2 Proposition (Ideal of a set is radical)

The set $I(S)$ is a radical ideal of $k[x_1, \ldots, x_n]$.

1. Proof SORRY

1.4.3 Theorem (Nullstellensatz)

Let k be an algebraically closed field. Then we have a bijection

$$\text{Radical ideals of } k[x_1, \ldots, x_n] \leftrightarrow \text{Zariski closed subsets of } \mathbb{A}^n_k$$

where the map from the left to the right is $I \mapsto V(I)$ and the map from the right to the left is $S \mapsto I(S)$.

1.4.4 Theorem (Maximal ideals)

Let k be an algebraically closed field. Then all the maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $(x_1 - a_1, \ldots, x_2 - a_n)$ for some $(a_1, \ldots, a_n) \in \mathbb{A}^n_k$.

1. Proof (assuming the Nullstellensatz) SORRY
2. Remark SORRY The statement is not true if k is not algebraically closed.

1.4.5 Theorem (Empty set and unit ideal)

Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. If $V(I) = \emptyset$, then $I = (1)$.

1. Proof (assuming the Nullstellensatz) SORRY
2. Remark SORRY The statement is not true if k is not algebraically closed.
3. Remark This theorem says that we have a dichotomy: either a system of equations $f_i = 0$ has a solution, or there exist polynomials g_i such that

$$\sum f_i g_i = 1.$$

1.4.6 Theorem (Vanishing functions)

Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. If f is identically zero on $V(I)$, then $f^n \in I$ for some n.

1. Proof (assuming the Nullstellensatz) SORRY

1.4.7 Proof of the Nullstellensatz

We use the following result from algebra, whose proof we skip.

1. Theorem Let K be any field and let L be a finitely generated K-algebra. If L is a field, then it must be a finite extension of K.

Proof. See `https://web.ma.utexas.edu/users/allcock/expos/nullstellensatz3.pdf` ☐

2. Proof of 1.4.4 Using this theorem, let us first prove the statement about the maximal ideals of $k[x_1, \ldots, x_n]$. Let $m \subset k[x_1, \ldots, x_n]$ be a maximal ideal. Taking $K = k$ and $L = k[x_1, \ldots, x_n]/m$ in 1, and using that k is algebraically closed, we get that the natural map $k \to k[x_1, \ldots, x_n]/m$ is an isomorphism. Let $a_i \in k$ be the pre-image of x_i under this isomorphism. Then we have $m = (x_1 - a_1, \ldots, x_n - a_n)$.

   (a) Prove the last statement SORRY

3. Proof of 1.4.5 Suppose I is not the unit ideal. We show that $V(I)$ is non-empty. To do so, we use that every proper ideal is contained in a maximal ideal.

   (a) Finish the proof SORRY

4. Proof of 1.4.6 We consider the system $g = 0$ for $g \in I$ and $f \neq 0$. Notice that the last one is not an equation, but there is a trick that allows us to convert it into an equation. Let y be a new variable, and consider the polynomial ring $k[x_1, \ldots, x_n, y]$. In the bigger ring, consider the system of equations $g = 0$ for $g \in I$ and $yf - 1 = 0$. By our assumption, this system of equations has no solutions.

   (a) Finish the proof using 1.4.5.

5. Proof of 1.4.3 We show that the maps $I \to V(I)$ and $S \to I(S)$ are mutual inverses. That is, we show that $I(V(I)) = I$ if I is a radical ideal, and $V(I(S)) = S$ if S is a Zariski closed subset of $\mathbb{A}^n_k$.

   Let us first show that $I(V(I)) = I$. Since every element of I vanishes on $V(I)$ by definition, we have $I \subset I(V(I))$. It remains to show that $I(V(I)) \subset I$. Let $f \in I(V(I))$. Then f is identically zero on $V(I)$. By subsubsection 1.4.6, there is some n such that $f^n \in I$. But I is radical, so $f \in I$.

   Let us now show that $V(I(S)) = S$. To do so, we observe a slight strengthening of the result we just proved: for any ideal I, not necessarily radical, we have $I(V(I)) = \sqrt{I}$.

   (a) Prove this strengthening SORRY

6. Proof of 1.4.3 continued Since S is Zariski closed, we know that $S = V(J)$ for some ideal J. So $I(S) = I(V(J)) = \sqrt{J}$. But it is easy to check that $V(J) = V(\sqrt{J})$, and hence $V(I(S)) = S$. The proof of 1.4.3 is then complete.

   (a) Check the "easy to check" fact SORRY

1.5 Affine and quasi-affine varieties

An *affine variety* is a Zariski closed subset of

2 Regular functions and maps

2.1 Definition (Regular function)

Let $S \subset \mathbb{A}_k^n$

2.2 Proposition (Regular function on an affine is a polynomial)

2.3 Example

2.4 Definition (Ring of regular functions)

2.5 Proposition (Ring of regular functions of an affine)

2.6 Definition (Regular map)

2.7 Proposition (Elementary properties of regular maps)

1. The identity map is regular.
2. The composition of two regular maps is regular.

2.8 Proposition (Regular maps preserve regular functions)

2.9 Corollary (Isomorphic varieties have isomorphic rings of functions)

2.10 Proposition (For affines, map between rings induces map between spaces)

2.11 Examples