Algebraic geometry (Notes)

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1 Dimension

WEEK9

The idea of dimension is central to algebraic geometry, but making it rigorous involves some serious algebra. Although we do not have enough time to develop the algebra, it would be a shame to avoid this notion, which is intuitively so clear. As a middle ground, we will take some statements from algebra as given. We will learn three equivalent definitions of dimension, but we will not prove the equivalence.

Let $x \in X$ be a point. We will define an integer $\dim_x X$, the dimension of X near x. At first, the dependence on x seems strange, but it makes sense when you look at some examples. Suppose $X \subset \mathbb{A}^3$ is the union of the xy-plane and the z axis (see 1). Then $\dim_p X = 2$ if p is in the xy-plane (including the origin) but 1 if p is on the z-axis minus the origin.

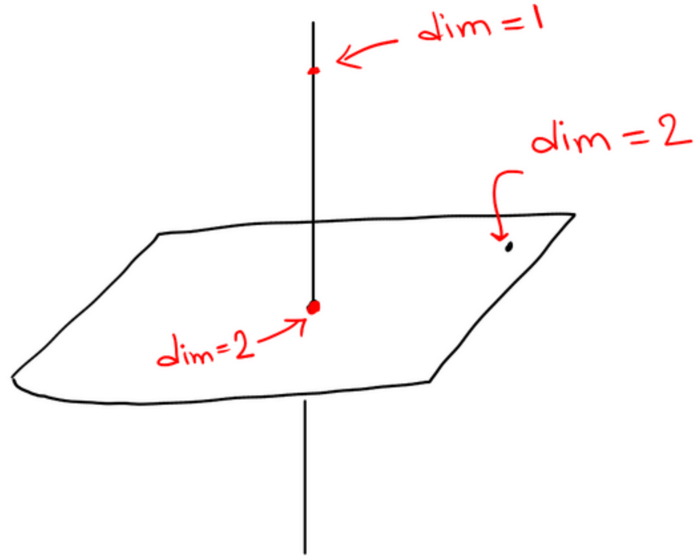

Figure 1: The union of a plane and a line

1.1 Krull dimension

The *Krull dimension* of X at x is the length n of a maximal (strict) chain of *irreducible closed subsets* of X, starting with $\{x\}$:

$$\{x\} \subset X_1 \subset \cdots \subset X_n \subset X.$$

If X is irreducible, then any maximal chain must end with X. (In that case, a non-trivial fact is that all maximal chains have the same length.)

Let us use the temporary notation krdim to denote Krull dimension.

1.1.1 Proposition Let X be irreducible and $Y \subset X$ a proper closed subset. For any $y \in Y$, we have $\mathrm{krdim}_y Y < \mathrm{krdim}_y X$.

1.2 Slicing dimension

The *slicing dimension* of X at x is the smallest number n such that there exists an open subset $U \subset X$ containing x and regular functions $f_1, \ldots, f_n$ on U such that the common vanishig set of $\{f_1, \ldots, f_n\}$ on U is only the point x.

So the dimension is the smallest number of functions we need to slice down X to a single point x. Let us use the temporary notation sldim to denote the slicing dimension.

1.2.1 Proposition (The Principal Ideal Theorem) Let X be any variety, f a regular function on X, and $Y = V(f)$ the zero locus of f. For any $y \in Y$, we have $\mathrm{sldim}_y Y \geq \mathrm{sldim}_y X - 1$.

Slogan: Slicing by 1 function cuts down the dimension by at most 1.

There are instances where the inequality is strict!

1.3 Transcendental dimension

Let X be irreducible. The *transcendental dimension* of X is the transcendence degree of the field of rational functions $k(X)$ over the base-field k. Recall that the transcendence degree of a field extension L/k is the maximal number of elements $f_1, \ldots, f_n \in L$ which are algebraically independent over k; that is, they do not satisfy any polynomial equation with coefficients in k. In Algebra 2, you mostly studied extensions of transcendence degree 0, also called algebraic extensions, in which *every* $f \in L$ satisfies a polynomial equation with coefficients in k. (A non-trivial fact is that all maximal algebraically independent sets have the same size.) Let use the temporary notation trdim to denote the transcendental dimension. Note that this definition does not use the point $x \in X$, but it assumes that X is irreducible.

1.3.1 Proposition Let $f \colon X \to Y$ be a dominant map of irreducible varieties. Then $\mathrm{trdim} Y \leq \mathrm{trdim} X$.

1.4 All definitions are equivalent

All three are reasonable definitions of dimension, so the following is a great relief.

1.4.1 Theorem (krdim = sldim = trdim) Let X be an algebraic variety and $x \in X$ a point. Then we have

$$\mathrm{krdim}_x X = \mathrm{sldim}_x X.$$

Furthermore, if X is irreducible, then both are equal to $\mathrm{trdim} X$.

We denote the dimension by $\dim_x X$. The theorem says that if X is irreducible then this number does not depend on x. If X is reducible, then it is easy to see (using the Krull dimension) that $\dim_x X$ is the maximum of the dimensions of the irreducible components of X that contain x. A variety is *equidimensional* if $\dim_x X$ is the same for all $x \in X$. This is the same as saying that all irreducible components of X have the same dimension.

We will not prove this theorem. Its proper place is a course in commutative algebra. The famous book "Commutative Algebra" by Atiyah and MacDonald has an excellent exposition (in the last chapter), where they give a fourth equivalent definition.

1.5 Applications

1.5.1 Theorem (Dimension of product) For irreducible X and Y, we have

$$\dim(X \times Y) = \dim X + \dim Y.$$

Proof. We first use Krull dimension to get an inequality. Let $m = \dim X$ and let $x \in X$ be arbitrary. We have a (strict) chain of irreducible closed subsets

$$\{x\} \subset X_1 \subset \cdots \subset X_m = X,$$

yielding a chain of irreducible closed subsets

$$\{x\} \times Y \subset X_1 \times Y \cdots \subset X_m \times Y = X \times Y.$$

Let $n = \dim Y$ and let $y \in Y$ be arbitrary. Then we have a (strict) chain of irreducible closed subsets

$$\{y\} \subset Y_1 \subset \cdots \subset Y_n = Y.$$

If we take the product with $\{x\}$ and augment it to the chain above, we get a (strict) chain

$$\{(x, y)\} \subset \{x\} \times Y_1 \cdots \subset \{x\} \times Y_n \subset X_1 \times Y \subset \cdots \subset X_m \times Y.$$

As a result, we have

$$\mathrm{krdim}(X \times Y) \geq m + n.$$

(We don't get equality because we haven't proved that there cannot be a longer chain).

For the opposite inequality, we use slicing dimension. There exist n regular functions in a neighborhood U of x on X whose zero locus is x. There exist m regular functions in a neighborhood V of y on Y whose zero locus is y. In $U \times V$, the $n + m$ functions together have zero locus (x, y). As a result, we have

$$\mathrm{sldim}(X \times Y) \leq m + n.$$

(We don't get equality because we haven't proved that a smaller set of functions does not suffice.)

But since sldim = krdim, we have proved what we wanted. □

1.5.2 Examples The dimension of $\mathbb{A}^1$ is 1 (you should be able to check this using *any* of the definitions). As a result, the dimension of $\mathbb{A}^n$ is n. Consequently, the dimension of $\mathbb{P}^n$ is n and the dimension of $\mathrm{Gr}(m, n)$ is $m(n - m)$.

1.5.3 Theorem (Hypersurfaces in affine space) Let $f \in k[x_1, \ldots, x_n]$ be non-zero. Then $V(f) \subset \mathbb{A}^n$ is equidimensional of dimension $(n-1)$. Conversely, any $X \subset \mathbb{A}^n$ which is equidimensional of dimension $(n - 1)$ has the form $V(f)$ for some $f \in k[x_1, \ldots, x_n]$.

> (1) — Prove this. One direction is easy and applies to any irreducible variety, not just $\mathbb{A}^n$. The converse is specific to $\mathbb{A}^n$, and will use that every irreducible element of $k[x_1, \ldots, x_n]$ defines a prime ideal, which in turn is a consequence of the unique factorisation property for the polynomial ring.

1.5.4 Theorem (Hypersurfaces in projective space) Let $F \in k[X_0, X_1, \ldots, X_n]$ be homogeneous and non-zero. Then $V(F) \subset \mathbb{P}^n$ is equidimensional of dimension $(n - 1)$. Conversely, any $X \subset \mathbb{P}^n$ which is equidimensional of dimension $(n - 1)$ has the form $V(F)$ for some homogeneous $F \in k[X_0, \ldots, X_n]$.

> (2) — Prove this by reducing this to the previous statement using cones.

1.5.5 Theorem (Slicing by hypersurfaces) Let $X \subset \mathbb{P}^n$ be of dimension r and let $F \in k[X_0, \ldots, X_n]$ be homogeneous of positive degree. Then $X \cap V(F)$ is non-empty and of dimension at least $r - 1$.

(3) — Prove this by reducing to the affine cone and applying the principal ideal theorem at the origin.

1.5.6 Corollary In $\mathbb{P}^n$, a collection of at most n homogeneous forms (of positive degree) have a non-empty intersection.

1.5.7 Theorem (No maps from $\mathbb{P}^n$ to $\mathbb{P}^m$ for $n > m$) Suppose $n > m$. Then there are no non-constant regular maps from $\mathbb{P}^n$ to $\mathbb{P}^m$.

The proof relies on the following fact about maps from one projective space to another.

1.5.8 Proposition Let $U \subset \mathbb{P}^n$ be an open subset and $\phi\colon U \to \mathbb{P}^m$ a regular function. Then there exist homogeneous functions $F_0, \dots, F_m \in k[X_0, \dots, X_m]$ of the same degree such that they have no common zero on U and for every $u \in U$, we have

$$\phi(u) = [F_0(u) : \cdots : F_m(u)]$$

Proof. A conceptual proof of this fact uses the classification of line bundles on $\mathbb{P}^n$. Here is more elementary (but clumsy) proof.

Pick some $u \in U$. We first show that ϕ has the required form in *some* open subset containing u. Without loss of generality, assume that u and $\phi(u)$ lie in the charts of the projective spaces here the 0-th coordinate is non-zero. Then $u = [1 : u_1 : \cdots : u_n]$ and $\phi(u) = [1 : v_1 : \cdots : v_m]$. By defintion of a regular map, there exist rational functions $g_i(x_1, \dots, x_n)$ for $i = 1, \dots, m$ such that

$$\phi([1 : x_1 : \cdots : x_n]) = [1 : g_1(x_1, \dots, x_n) : \cdots : g_m(x_1, \dots, x_n)]$$

for all $x = [1 : x_1 : \cdots : x_n]$ in some open subset of U containing u. Multiply this expression for ϕ by a large enough polynomial so that

$$\phi([1 : x_1 : \cdots : x_n]) = [f_0(x_1, \dots, x_n) : \cdots : f_m(x_1, \dots, x_n)],$$

here the f_i are polynomials. Choose $d \geq \deg f_i$ for all i. Homogenise the f_i with respect to x_0 to make them homogeneous of degree d. That is, set $F_i(x_0, \dots, x_n) = x_0^d f(x_1/x_0, \dots, x_n/x_0)$. Then ϕ has the form

$$\phi([x_0, \dots, x_n]) = [F_0(x_0, \dots, x_n) : \cdots : F_m(x_0, \dots, x_n)]$$

for all $x = [x_0 : \cdots : x_n]$ in some open set containing u. We may assume that the F_i do not share a common factor (if they do, cancel it out).

We now show that the F_i cannot have a common zero on U, and therefore, the expression $\phi = [F_i]$ holds on all of U. Suppose $x \in U$ is such that all F_i vanish at x. We show that then the F_i share a common factor. By the argument before, there must be an alternate expression $\phi = [G_i]$ in a neighborhood of x in which some $G_i(x)$ is non-zero. Suppose $G_0(x) \neq 0$. Since we have $[F_i] = [G_i]$ on the open set where both are defined, we have $F_iG_j == G_iF_j$. In particular, we have $F_0G_j = G_0F_j$. Let P be a prime factor of F_0 such that $P(x) = 0$ (all factors of homogeneous polynomials are homogeneous). Then P divides F_0F_j, but P cannot divide G_0, as $G_0(x) \neq 0$. So P divides F_j. Since this is true for all j, we get a common factor P in all F_i. Thus, if the F_i do not share a common factor, then F_i do not have a common zero on U). ☐

1.5.9 Proof of Theorem 1.5.7 Suppose we have a regular map $\phi\colon \mathbb{P}^n \to \mathbb{P}^m$. By Proposition 1.5.8, there exist $F_0, \dots, F_m$ such that they have no common zero and $\phi = [F_0 : \cdots : F_m]$. By Corollary 1.5.6 this is impossible if $m < n$.

1.6 Dimension of fibers and dimension counting

1.6.1 Theorem (Dimensions of fibers) Let $f\colon X \to Y$ be a dominant map between irreducible varieties. Then for every $x \in X$ with $y = f(x)$, we have

$$\dim_x f^{-1}(y) \geq \dim X - \dim Y.$$

Furthermore, there exists a non-empty open $U \subset Y$ such that for every $y \in Y$, the fiber $f^{-1}(y)$ is non-empty and equidimensional of dimension $\dim X - \dim Y$.

That is, for almost all $y \in Y$, the dimension of the fiber is the difference in the dimensions, as expected. But there may be some points in Y whose fiber has a different dimension. But in this case, the dimension can only be bigger, not smaller.

The proof of the theorem uses transcendental dimension. The proof is straightforward, but a bit technical, so I am skipping it. See Chapter 1, Section 6.3 of Shafarevich for the proof.

1.6.2 Example Let us construct an example where the dimension does actually jump. Consider

$$f\colon \mathbb{A}^2 \to \mathbb{A}^2$$

defined by

$$f(x, y) = (xy, y).$$

For all (a, b) such that $b \neq 0$, the fiber is a single point (dimension 0). But over the point $(0, 0)$, the fiber is a copy of $\mathbb{A}^1$ (dimension 1).

1.6.3 Dimension counting Theorem 1.6.1 is used very often in finding dimensions. Here is a typical example.

Let $\mathbb{A}^{n\times n}$ be the affine space of $n \times n$ matrices, and given $r = 0, 1, \dots, n$, let $X_r \subset \mathbb{A}^n$ be the set of matrices of rank at most r. The subset X_r is Zariski closed (it is the vanishing locus of all $(r+1)\times(r+1)$)-minors, and it is not hard to check that it is irreducible. What is its dimension?

Consider $P \subset \mathbb{A}^{n\times n} \times \mathrm{Gr}(n-r, n)$ consisting of (M, V) (where M is an $n \times n$ matrix and $V \subset k^n$ is an $n - r$ dimensional subspace) such that $Mv = 0$ for all $v \in V$. That is, the restriction of the linear map $M\colon k^n \to k^n$ to V is zero.

Claim 1: P is a Zariski closed subset.

(4) — Prove this. But this is less fun than the next two claims, so assume this and do those first.

Claim 2. The dimension of P is $r(2n - r)$.

(5) — Study the fibers of $P \to \mathrm{Gr}(n - r, n)$ to prove this.

Claim 3. The dimension of X_r is $r(2n - r)$.

(6) — Study the image and the fibers of $P \to \mathbb{A}^{n\times n}$ and prove this.