

**MATH 8320: ALGEBRAIC CURVES AND RIEMANN SURFACES —
HOMEWORK 3**

- (1) Compute the cohomology groups $H^i(X, \mathbb{Z})$ using Čech cohomology for $X = \mathbb{P}^1$.
- (2) Let L be a vector bundle on X . Establish a bijection between (a) the set of sections of L , and (b) the set of maps of vector bundles from the trivial bundle $\mathbb{C} \times X$ to L . Conclude that a line bundle is trivial if and only if it has a nowhere vanishing section.
- (3) Let D be a divisor of degree 0 on a compact Riemann surface X . Show that

$$\dim H^0(X, \mathcal{O}(D)) = \begin{cases} 1 & \text{if } D \sim 0 \\ 0 & \text{otherwise.} \end{cases}$$

- (4) Let D be a non-zero effective divisor on a compact Riemann surface X such that

$$\dim H^0(X, \mathcal{O}(D)) = \deg D + 1.$$

Show that $X \cong \mathbb{P}^1$

Hint: Show that there exists $p \in X$ such that $\dim H^0(X, \mathcal{O}(p)) = 2$.

- (5) Let X be a compact Riemann surface and $U = X \setminus S$, where S is a finite set. Show that a every holomorphic map $U \rightarrow \mathbb{P}^n$ extends to a holomorphic map $X \rightarrow \mathbb{P}^n$.
- (6) Let D be a very ample divisor on X . Show that the meromorphic functions in $H^0(X, \mathcal{O}(D))$ separate the points and tangent vectors on X . Conclude that they generate \mathcal{M}_X .
- (7) Let X be a compact Riemann surface and $S \subset X$ a finite set. Use the Riemann-Roch theorem to show that there exists a meromorphic function on X that is holomorphic on $X \setminus S$.
- (8) (Counts as 2 problems) Let U be the Riemann surface in \mathbb{C}^2 defined by $y^3 = x^5 - 1$. We know that there exists a compact Riemann surface X that contains U , and the complement $X \setminus U$ consists of a finite set of points. Let $\pi: X \rightarrow \mathbb{P}^1$ be the extension of the projection $(x, y) \mapsto x$.
- Show that there is a unique point in $X \setminus U$, call it ∞ .
 - Find the ramification divisor of π and hence the genus of X .
 - Write down the canonical series $H^0(X, K)$.
 - Show that the canonical map yields an embedding $X \rightarrow \mathbb{P}^3$.
 - Show that the canonical image of X lies on a singular quadric hypersurface.
- (9) Let D be a divisor on a smooth algebraic curve X of genus g such that $\deg D = 2g - 2$ and $\dim H^0(X, \mathcal{O}(D)) = g$. Show that D is a canonical divisor (that is, $\mathcal{O}_X(D) \cong K_X$).