Algebraic geometry (Notes)

Anand Deopurkar

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1 Affine algebraic sets

1.1 Affine space WEEK1:DONE

The objects of study in algebraic geometry are called algebraic varieties. The building blocks for general algebraic varieties are certain subsets of the affine space. Let us first recall affine space.

Let k be a field and let n be a non-negative integer. The *affine n-space over k*, denoted by $\mathbb{A}^n_k$ is the set of n-tuples $a_1, \dots, a_n$ whose entries a_i lie in k. Thus, $\mathbb{A}^n_k$ is nothing but the product k^n. The product k^n has quite a bit of extra structure—it is a k-vector space, for example—but we wish to forget it. That is the reason for choosing different notation. In particular, the zero tuple does not play a distinguished role.

1.2 Affine algebraic set WEEK1:DONE

Let $k[x_1, \dots, x_n]$ denote the ring of polynomials in variables $x_1, \dots, x_n$ and coefficients in k. An *affine algebraic subset* of the affine space $\mathbb{A}^n_k$ is the common zero locus of a set of polynomials. More precisely, a set $S \subset \mathbb{A}^n_k$ is an affine algebraic subset if there exists a set of polysomials $A \subset k[x_1, \dots, x_n]$ such that

$$S = \{a \in \mathbb{A}^n_k \mid f(a) = 0 \text{ for all } f \in A\}.$$

1.2.1 Definition (Vanishing locus) Given $A \subset k[x_1, \dots, x_n]$, the *vanishing locus of* A, denoted by $V(A)$ is the set

$$V(A) = \{a \in \mathbb{A}^n_k \mid f(a) = 0 \text{ for all } f \in A\}.$$

— Thus the affine algebraic sets are precisely the sets of the form $V(A)$ for some A.

1.2.2 Examples/non-examples

The following are affine algebraic sets

1. The empty set
2. Entire affine space
3. Single point

Proof. Done in class. □

The following are not affine algebraic sets

1. The unit cube in $\mathbb{A}^n_{\mathbb{R}}$
2. Points with rational coordinates in $\mathbb{A}^n_{\mathbb{C}}$

Proof. DIY. □

1.3 Ideals

WEEK1:DONE

Let R be a ring. Recall that a subset $I \subset R$ is an *ideal* if it is closed under addition and multiplication by elements of R. Given any subset $A \subset R$ the *ideal generated by* A, denoted by $\langle A \rangle$ is the smallest ideal containing A. This ideal consists of all elements r of R that can be written as a linear combination

$$r = a_1 r_1 + \cdots + a_m r_m,$$

where $a_i \in A$ and $r_i \in R$.

1.3.1 Proposition

Let $A \subset k[x_1, \ldots, x_n]$. Then we have $V(A) = V(\langle A \rangle)$.

Proof. Done in class. □

1.4 Noetherian rings and the Hilbert basis theorem

WEEK1:DONE

In our definition of $V(A)$, the subset A may be infinite. But it turns out that we can replace it by a finite one without changing $V(A)$. This is a consequence of the Hilbert basis theorem, which, in turn, has to do with a fundamental property of rings.

We begin with a simple observation.

1.4.1 Proposition Let R be a ring. The following are equivalent

1. Every ideal of R is finitely generated.
2. Every infinite chain of ideals
$$I_1 \subset I_2 \subset I_3 \subset \cdots$$
stabilises.

Proof. — 1

1.4.2 Definition (Noetherian ring) A ring R satisfying the equivalent conditions of Proposition 1.4.1 is called *Noetherian*.

1.4.3 Examples/non-examples The following rings are Noetherian

1. $R = \mathbb{Z}$
2. R a field.

Proof. All ideals here can be generated by 1 element. □

The ring of continuous functions on the interval is *not* Noetherian. #+begin$_{\text{proof}}$. Let I_n be the set of functions on $[0,1]$ that vanish on $[0,1/n]$. This forms an increasing chain of ideals that does not stabilise. #+end$_{\text{proof}}$

1.4.4 Proposition (Quotients of Noetherian rings) If R is Noetherian and $I \subset R$ is any ideal, then R/I is Noetherian.

Proof. — 2

1.4.5 Theorem If R is Noetherian, then so is $R[x]$

- Proof Assume R is Noetherian, and let $I \subset R[x]$ be an ideal. We must show that I is finitely generated. The basic idea is to use the division algorithm, while keeping track of the ideals formed by the leading coefficients.

  For every non-negative integer m, define
  $$J_m = \{\text{Leading coeff}(f) \mid f \in I, f \neq 0, \quad \deg(f) \leq m\} \cup \{0\}$$
  We make the following claims.

  1. J_m is an ideal of R.
  2. $J_m \subset J_{m+1}$.

DIY.

Since R is Noetherian, the chain $J_1 \subset J_2 \subset \cdots$ stabilises; say $J_m = J_{m+1} = \cdots$. Let S_i be a finite set of generators for J_i, and for $a \in S_i$, let $p_a \in I$ be a non-zero element of degree at most i whose leading coefficient is a. We claim that the (finite) set $\{p_a \mid a \in S_1 \cup \cdots \cup S_m\}$ generates I.

Proof. Let $G = \{p_a \mid a \in S_1 \cup \cdots \cup S_m\}$. By construction, this is a subset of I, so the ideal it generates is contained in I. We remains to prove that every $f \in I$ is a linear combination of elements of G. It will be convenient to set $S_n = S_m$ for all $n \geq m$.

We induct on the degree of f (leaving the base case to you). Suppose the degree of f is n and the statement is true for elements of degree less than n. By construction, the leading coefficient of f is an R-linear combination of elements of S_n, say

$$\mathrm{LC}(f) = \sum c_i s_i.$$

Let n_i be the degree of p_{s_i}; then by construction $n_i \leq n$. Consider the linear combination $g = \sum c_i p_{s_i} x^{n-n_i}$. See that g lies in I, has degree n, the same leading coefficient as f, and is an $R[x]$-linear combination of elements of G. So $f - g \in I$ has lower degree. By inductive hypothesis, $f - g$ is an $R[x]$-linear combination of elements of G, and hence so is f. □

1.4.6 Corollary (Hilbert basis theorem) $k[x_1, \ldots, x_n]$ is Noetherian.

Proof. Induct on n. □

1.4.7 Corollary Every affine algebraic subset of $\mathbb{A}_k^n$ is the vanishing set of a finite set of polynomials.

Proof. Done in class. □

1.5 The Zariski topology

WEEK2:DONE

The notion of affine algebraic sets allows us to define a topology on $\mathbb{A}_k^n$. Recall that we can specify a topology on a set by specifying what the open subsets are, or equivalently, what the closed subsets are. In our case, it is more convenient to do the latter. The collection of closed subsets must satisfy the following properties.

1. The empty set and the entire set are closed.

2. Arbitrary intersections of closed sets are closed.

3. Finite unions of closed sets are closed.

   We define the *Zariski topology* on $\mathbb{A}_k^n$ by setting the closed subsets to be the affine algebraic sets, namely, the sets of the form $V(A)$ for some $A \subset k[x_1, \ldots, x_n]$.

1.5.1 Proposition The collection of affine algebraic subsets satisfies the three conditions above.

Proof. **The empty set and the entire set are closed.**

$$\begin{aligned}\emptyset &= \{\mathbf{a} \in \mathbb{A}_k^n : 1 = 0\} \\ &= V(\{1\})\end{aligned}$$

So the empty set is closed.

$$\begin{aligned}\mathbb{A}_k^n &= \{\mathbf{a} \in \mathbb{A}_k^n : 0 = 0\} \\ &= V(\{0\})\end{aligned}$$

So the entire set is closed.

Arbitrary intersections of closed sets are closed.

Let $\{V(A_\alpha)\}$ be a collection of closed sets.

$$\begin{aligned}\bigcap_\alpha V(A_\alpha) &= \bigcap_\alpha \{\mathbf{a} \in \mathbb{A}_k^n : p(\mathbf{a}) = 0 \text{ for all } p \in A_\alpha\} \\ &= \{\mathbf{a} \in \mathbb{A}_k^n : p(\mathbf{a}) = 0 \text{ for all } p \in \bigcup_\alpha A_\alpha\} \\ &= V\left(\bigcup_\alpha A_\alpha\right)\end{aligned}$$

So arbitrary intersections of closed sets are closed.

Finite unions of closed sets are closed.

Let $V(A), V(B)$ be closed sets. Let $\mathbf{a} \in V(A) \cup V(B)$. Then $p(\mathbf{a}) = 0$ for all $p \in A$ or $q(\mathbf{a}) = 0$ for all $q \in B$. Without loss of generality, suppose $p(\mathbf{a}) = 0$ for all $p \in A$. Then for all polynomials pq with $p \in A, q \in B$, $pq(\mathbf{a}) = 0$. So $\mathbf{a} \in V(\{pq : p \in A, q \in B\})$ and therefore $V(A) \cup V(B) \subseteq V(\{pq : p \in A, q \in B\})$. Now suppose $\mathbf{a} \notin V(A) \cup V(B)$. Then there exists some $p \in A, q \in B$ such that $pq(\mathbf{a}) \neq 0$. So $\mathbf{a} \notin V(\{pq : p \in A, q \in B\})$ and therefore $V(\{pq : p \in A, q \in B\}) \subseteq V(A) \cup V(B)$.

So $V(A) \cup V(B) = V(\{pq : p \in A, q \in B\})$ and therefore $V(A) \cup V(B)$ is closed. Following this process with an inductive argument, finite unions of closed sets are closed. □

1.5.2 Proposition The Zariski topology on $\mathbb{A}_k^1$ is the *finite complement topology.* The only closed sets are the finite sets (or the whole space). In other words, the only open sets are the complements of finite sets (or the empty set).

Proof. We saw that the subsets $V(A) \subset \mathbb{A}^1_k$ are either the whole $\mathbb{A}^1_k$ or finite sets. $\square$

1.5.3 Comparison between Zariski and Euclidean topology over $\mathbb{C}$. Every Zariski closed (open) subset of $\mathbb{A}^n_{\mathbb{C}}$ is also closed (open) in the usual Euclidean topology. The converse is not true.

Proof. It suffices to prove that $V(A)$ is closed in the usual topology. We have $V(A) = \cap_{f \in A} V(f)$, so it suffices to show that $V(f)$ is closed. But $V(f) = f^{-1}(0)$ is closed, because it is the pre-image of a closed set under a continuous function. $\square$

1.5.4 Proposition (Polynomials are continuous) Let f be a polynomial function on $\mathbb{A}^n_k$, viewed as a map $f \colon \mathbb{A}^n_k \to \mathbb{A}^1_k$. Then f is continuous in the Zariski topology.

Proof. We check that pre-images of closed sets are closed. The only closed sets of $\mathbb{A}^1_k$ is the whole space and finite sets. The pre-image of $\mathbb{A}^1_k$ is $\mathbb{A}^n_k$, which is closed. Since finite unions of closed sets are closed, it suffices to check that the pre-image of a point $a \in \mathbb{A}^1_k$ is closed. But the pre-image of a under f is just $V(f-a)$, which is closed by definition. $\square$

— The Zariski topology has very few open sets, and as a result has terrible separation properties. It is not even Hausdorff (except in very small examples). Nevertheless, we will see that it is extremely useful. For one, it makes sense over every field!

1.6 The Nullstellensatz

WEEK2:DONE

We associated a set $V(A)$ to a subset A of the polynomial ring $k[x_1, \ldots, x_n]$. If we think of A as a system of equations $\{f = 0 \mid f \in A\}$, then $V(A)$ is the set of solutions. We can also define a reverse operation. The Nullstellensatz says that if k is algebraically closed, then these two operations are mutually inverse. That is, the data of a system of equations is equivalent to the data of its set of solutions. This pleasant fact allows us go back and forth between algebra (equations) and geometry (the solution set).

We start with a straightforward definition.

1.6.1 Definition (Ideal vanishing on a set) Let $S \subset \mathbb{A}^n_k$ be a set. The *ideal vanishing on S*, denoted by $I(S)$, is the set

$$I(S) = \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \text{ for all } a \in S\}$$

— Recall that an ideal $I \subset k[x_1, \ldots, x_n]$ is *radical* if it has the property that whenever $f^n \in I$ for some $n > 1$, then $f \in I$.

1.6.2 Proposition The set $I(S)$ is a radical ideal of $k[x_1, \ldots, x_n]$.

Proof. We leave it to you to check that $I(S)$ is an ideal. To see that it is radical, see that if f^n vanishes on S, then so does f. $\square$

1.6.3 Proposition (Easy properties of radical ideals)

1. $I \subset R$ is radical if and only if R/I has no (non-zero) nilpotents.
2. All prime ideals are radical. In particular, all maximal ideals are radical.

Proof. Consider $f \in R$ and its image $\overline{f} \in R/I$. Then $\overline{f}$ is a nilpotent of R/I if and only if $f^n \in I$ and $\overline{f} = 0$ in R/I if and only if $f \in I$. From this, the result follows. If I is prime, then R/I is an integral domain, so it has no nilpotents (it does not even have zero divisors). $\square$

1.6.4 Proposition (Radical of an ideal) Let I be an ideal, and set $\sqrt{I} = \{f \mid f^n \in I \text{ for some } n > 0\}$. Then $\sqrt{I}$ is a radical ideal.

Proof. (Assume a commutative ring) We will first show that $\sqrt{I} \subset R$ is an ideal. Let $f \in \sqrt{I}, r \in R$, and by definition of $\sqrt{I}$, we suppose $f^n \in I$ for some $n > 0$

$$(rf)^n = r^n f^n.$$

Since $r^n \in R, f^n \in I$, by definition of ideal, we have $r^n f^n \in I$. Therefore, $(rf)^n \in I$ for some $n > 0$, and by definition, we have $rf \in \sqrt{I}$. Therefore, $\sqrt{I}$ is closed under multiplication by elements of R.

Let $f, g \in \sqrt{I}$, with $f^n \in I$, $g^m \in I$.

$$\begin{aligned}(f+g)^{m+n} &= c_0 f^{m+n} + c_1 f^{m+n-1} g^1 + \cdots + c_m f^n g^m + \cdots + c_{m+n} g^{m+n} \\ &= c_0 f^m \times f^n + c_1 f^{m-1} g \times f^n + \cdots + c_m f^n g^m \\ &\quad + c_{m+1} f^{n-1} g^1 \times g^m + \cdots + c_{m+n} g^n \times g^m.\end{aligned}$$

(c_i are the corresponding binomial coefficients in I). As shown above, $(f+g)^{m+n}$ can be written as an R-linear combination of f^n and g^m. Since $f^n \in I, g^m \in I$, by definition of ideal, we have $(f+g)^{m+n} \in I$. Therefore, by definition we have $(f+g) \in \sqrt{I}$ and $\sqrt{I}$ is closed under addition. Therefore, $\sqrt{I}$ is an ideal.

Now we need to show that $\sqrt{I}$ is a radical ideal. Suppose $f \in R$ with $f^n \in \sqrt{I}$ for some $n > 0$. Then, by definition of $\sqrt{I}$, we have $(f^n)^m \in I$ for some $m > 0$.

$$(f^n)^m = f^{nm} \in I, nm > 0.$$

Therefore, by definition, we have $f \in \sqrt{I}$. □

1.6.5 Definition (Radical of an ideal) The ideal $\sqrt{I}$ is called the radical of I.

1.6.6 Proposition (V is unchanged by radicals) We have $V(I) = V(\sqrt{I})$.

Proof. $\boxed{\supset}$ Note that $I \subset \sqrt{I}$ and hence $V(\sqrt{I}) \subset V(I)$. More specifically, for any $f \in I$ we have that $f^1 \in I$ and so $f \in \sqrt{I}$. Now suppose $a \in V(\sqrt{I})$. Then $f(a) = 0$ for all $f \in \sqrt{I}$. But since $I \subset \sqrt{I}$, this implies the weaker statement that for all $f \in I$, we have $f(a) = 0$. This is the same as saying that $a \in V(I)$.

$\boxed{\subset}$ Now let $a \in V(I)$. Then let $f \in \sqrt{I}$. By definition of $\sqrt{I}$ there exists some $n > 0$ such that $f^n \in I$ and hence $f^n(a) = 0$ by assumption. We want to show that this implies $f(a) = 0$ which gives us that $a \in V(\sqrt{I})$, completing the proof. This is because f is an arbitrary element of $\sqrt{I}$. We are done if $n = 1$.

Otherwise we use that we are working in a field which has no zero divisors. More specifically, $f^n(a) = f(a)f^{n-1}(a) = 0$ implies that either $f(a) = 0$ or $f^{n-1}(a) = 0$. If $f(a) = 0$ we are done. Otherwise if $f^{n-1}(a) = 0$, we repeat the previous step for $f^{n-1}(a) = f(a)f^{n-2}(a) = 0$ and so on, until we get $f(a) = 0$ or until $n = 2$ in which case we have $f^2(a) = f(a)f(a) = 0$ which implies $f(a) = 0$ as well. □

— We now state a string of important theorems, all called the "Nullstellensatz", starting with the most comprehensive one.

1.6.7 Theorem Let k be an algebraically closed field. Then we have a bijection

$$\text{Radical ideals of } k[x_1, \ldots, x_n] \leftrightarrow \text{Zariski closed subsets of } \mathbb{A}_k^n$$

where the map from the left to the right is $I \mapsto V(I)$ and the map from the right to the left is $S \mapsto I(S)$. The correspondence is inclusion reversing.

1.6.8 Theorem Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. If $V(I) = \emptyset$, then $I = (1)$.

1.6.9 Theorem Let k be an algebraically closed field. Then all the maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $\langle x_1 - a_1, \ldots, x_2 - a_n \rangle$ for some $(a_1, \ldots, a_n) \in \mathbb{A}_k^n$.

— Theorem 1.6.8 says that we have a dichotomy: either a system of equations $f_i = 0$ has a solution, or there exist polynomials g_i such that

$$\sum f_i g_i = 1.$$

1.6.10 Theorem Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. If f is identically zero on $V(I)$, then $f^n \in I$ for some n.

1.7 Proof of the Nullstellensatz

WEEK2:DONE

The proof of Theorem 1.6.7 actually goes via the proofs of the subsequent theorems. We use the following result from algebra, whose proof we skip.

1.7.1 Theorem Let K be any field and let L be a finitely generated K-algebra. If L is a field, then it must be a finite extension of K.

Proof. See `https://web.ma.utexas.edu/users/allcock/expos/nullstellensatz3.pdf` □

1.7.2 Proof of Theorem 1.6.9 Let $m \subset k[x_1, \ldots, x_n]$ be a maximal ideal. Taking $K = k$ and $L = k[x_1, \ldots, x_n]/m$ in Theorem 1.7.1, and using that k is algebraically closed, we get that the natural map $k \to k[x_1, \ldots, x_n]/m$ is an isomorphism. Let $a_i \in k$ be the pre-image of x_i under this isomorphism. Then we have $m = (x_1 - a_1, \ldots, x_n - a_n)$.

Proof of Theorem 1.6.9 . Since m is a maximal ideal, $L := k[x_1, ..., x_n]/m$ is a field. Let $\pi : k[x_1, ..., x_n] \to L$ be the projection map. Consider the inclusion map $\mathfrak{i} : k \to k[x_1, ..., x_n]$. We embed k in L via the map $\phi := \pi \circ \mathfrak{i}$. We now show that ϕ is an ismomorphism.

Surjectivity of ϕ . The existence of this map tells us that L is a k-algebra. Moreover, L is a finitely generated k-algebra, since L is generated by $\{\pi(x_1), ..., \pi(x_n)\}$. Now Theorem 1.7.1 applies, and we deduce that L is a finite extension of k. In particular, L must be an algebraic extension of k. If L were not an algebraic extension of k, then there would exist an element $l \in L$ transcendental over k; but then L could not be a finite extension of K, because the set $\{l^j\}_{j=0,1,2,...}$ would be linearly independent. We conclude that L is an algebraic extension of k.

We know that given any $l \in L$, there is a polynomial $p(y) \in k[y]$, where y is any new variable, such that $p(l) = 0$. Let $p(y)$ be the monic polynomial of least degree satisfying the above. Then $p(y)$ is irreducible, since otherwise it would have a factor of smaller degree also satisfying the above, contradicting the minimality of the degree of $p(y)$. Since k is algebraically closed, the irreducible monic polynomials are all of the form $x - a$, for $a \in k$. As such, we have $p(y) = y - a$ for some $a \in k$. It follows that $l \in k$, since we must have $l = a$. To be precise, what we have really shown is that $l \in \phi(k)$, since k is not itself a subset of L, but can be identified with a subset of L.

We conclude that $L = \phi(k)$. This tells us that ϕ is surjective.

Injectivity of ϕ . Because ϕ is a field homomorphism, ϕ must be injective. Indeed, the kernel of ϕ is an ideal of k. As such, the kernel of ϕ is either the zero ideal or the unit ideal. Since ϕ is not identically zero, the kernel must be the zero ideal. This completes the proof that ϕ is an isomorphism.

Completion of Proof Because $\phi : k \to L$ is an isomorphism, we can define $a_i := \phi^{-1}(\pi(x_i))$, for each $i = 1, ..., n$. We claim that with this choice of $a_1, .., a_n \in k$, the equation in (1) holds. If $p \in (x_1 - a_1, ..., x_n - a_n)$, then there exist $q_i \in k[x_1, ..., x_n]$ such that

$$p = \sum_{i=1}^{n} q_i(x_i - \mathfrak{i}(a_i)). \tag{1}$$

We could remove the $\mathfrak{i}$ in (1); it just serves as a reminder that $k[x_1, ..., x_n]$ contains a copy of k, not k itself. From (1), we obtain

$$\pi(p) = \sum_{i=1}^{n} \pi(q_i)(\pi(x_i) - \phi(a_i)) = 0. \tag{2}$$

The first equality in (1) holds by the fact that π is a ring homomorphism, and the second equality holds because $\phi(a_i) = \pi(x_i)$, for $i = 1, ..., n$. From (2) we conclude that $p \in m$, since the kernel of π is precisely the ideal m. We have shown that

$$(x_1 - a_1, ..., x_n - a_n) \subseteq m. \tag{3}$$

Now suppose that $p \in m$. Then $\pi(p) = 0$. On the other hand, we can write

$$p = \sum_{i=0}^{d} \sum_{j_1+...+j_n=i} \mathfrak{i}(c_{j_1,...,j_n}) x_1^{j_1} ... x_n^{j_n}, \tag{4}$$

where d is the degree of p and the $c_{j_1,...,j_n}$ are elements of k. Equation (4) yields

$$\begin{aligned}
\pi(p) &= \sum_{i=0}^{d} \sum_{j_1+...+j_n=i} \phi(c_{j_1,...,j_n})\pi(x_1)^{j_1}...\pi(x_n)^{j_n} \\
&= \sum_{i=0}^{d} \sum_{j_1+...+j_n=i} \phi(c_{j_1,...,j_n})\phi(a_1)^{j_1}...\phi(a_n)^{j_n} \qquad (5) \\
&= \phi\Bigg(\sum_{i=0}^{d} \sum_{j_1+...+j_n=i} c_{j_1,...,j_n} a_1^{j_1}...a_n^{j_n}\Bigg).
\end{aligned}$$

The second equality in (5) holds by the definition of $a_1, ..., a_n$, and the third equality holds because ϕ is a ring homomorphism. From (5) and the fact that $\pi(p) = 0$, we have that ϕ maps $\sum_{i=0}^{d} \sum_{j_1+...+j_n=i} c_{j_1,...,j_n} a_1^{j_1}...a_n^{j_n}$ to zero. Since ϕ is an isomorphism, it follows that

$$\sum_{i=0}^{d} \sum_{j_1+...+j_n=i} c_{j_1,...,j_n} a_1^{j_1}...a_n^{j_n} = 0, \qquad \text{in the field } k. \tag{6}$$

From (6), we have that the point $(a_1, ..., a_n)$ is a root of the polynomial p. We can write

$$p = \sum_{i=0}^{d} \sum_{j_1+...+j_n=i} \imath(e_{j_1,...,j_n})(x_1 - a_1)^{j_1}...(x_n - a_n)^{j_n}, \tag{7}$$

for suitably chosen $e_{j_1,...,j_n} \in k$. For example, we could define

$$q(x_1, ..., x_n) = p(x_1 + a_1, ..., x_n + a_1). \tag{8}$$

We think of the right-hand side of (8) as a polynomial in $x_1, ..., x_n$. "Evaluating" q at $(x_1 - a_1, ..., x_n - a_n)$ gives back p, by definition, while the right-hand side of (8) becomes a polynomial in the variables $x_1 - a_1, ..., x_n - a_n$. This is one way to show that p can be written in the form (7). Now, the term with $i = 0$ in (7) is the constant term $e_{0,...,0}$. Evaluating p at $(a_1, ..., a_n)$ in (7) shows that $p(a_1, ..., a_n) = e_{0,...,0}$. By (6), we have $p(a_1, ..., a_n) = 0$, so the constant term $e_{0,...,0}$ must also be zero. This means that every term in (7) belongs to $(x_1 - a_1, ..., x_n - a_n)$. It follows that $p \in (x_1 - a_1, ..., x_n - a_n)$. We have shown that

$$m \subseteq (x_1 - a_1, ..., x_n - a_n). \tag{9}$$

From (3) and (9), we conclude that (1) holds. This completes the proof.

1.7.3 Proof of Theorem 1.6.8 Suppose I is not the unit ideal. We show that $V(I)$ is non-empty. To do so, we use that every proper ideal is contained in a maximal ideal.

Finish the proof. — (5)

1.7.4 Proof of Theorem 1.6.10 We consider the system $g = 0$ for $g \in I$ and $f \neq 0$. Notice that the last one is not an equation, but there is a trick that allows us to convert it into an equation. Let y be a new variable, and consider the polynomial ring $k[x_1, \ldots, x_n, y]$. In the bigger ring, consider the system of equations $g = 0$ for $g \in I$ and $yf - 1 = 0$. By our assumption, this system of equations has no solutions.

Solutions to the original and augmented system are in bijection; if $(a_1, ..., a_n)$ satisfies $g = 0$ and $f \neq 0$, then there exists a unique value of y, $\frac{1}{f(a_1,...,a_n)}$, such that the second system is solved. Similarly, a solution to the second system constitutes a solution to the first, by simply ignoring the value of y, because if $yf - 1 = 0$ then f must be non-zero.

Then, by assumption of Theorem 1.6.10, f is identically zero in $V(I)$, so the original system has no solutions. Therefore the augmented system has no solutions, and by Theorem 1.6.8, the ideal generated by $g \in I$ and $yf-1$ is the unit ideal in $k[x_1, ...x_n, y]$. So then we can write

$$1 = \sum c_i(x_1, ..., x_n, y) g_i(x1, ..., x_n, y) + c(x_1, ...x_n, y)(yf - 1)$$

We transform this expression in $k[x_1, ...x_n, y]$ to an expression in the fraction field $k(x_1, ...x_n)$ by setting $y = \frac{1}{f(x_1,...,x_n)}$, and since for this choice of y we have that $yf - 1$ vanishes, we get

$$1 = \sum c_i(x_1, ..., x_n, \frac{1}{f(x_1, ..., x_n)}) g_i(x1, ..., x_n, y) \in k(x_1, ...x_n)$$

Now, since this is a polynomial in $\frac{1}{f(x_1,...,x_n)}$, multiplying through by a sufficiently large power N of f gives

$$f^N = \sum p_i(x_1, ..., x_n) g_i(x1, ..., x_n, y) \in k[x_1, ...x_n]$$

So we can conclude that f^N is in I.

1.7.5 Proof of Theorem 1.6.7. We show that the maps $I \to V(I)$ and $S \to I(S)$ are mutual inverses. That is, we show that $I(V(I)) = I$ if I is a radical ideal, and $V(I(S)) = S$ if S is a Zariski closed subset of $\mathbb{A}^n_k$.

Let us first show that for any ideal I, we have $I(V(I)) = \sqrt{I}$. Suppose $f \in \sqrt{I}$, then $f^n \in I$ for some $n > 0$. But then f^n is identically zero on $V(I)$, and hence so is f; that is, $f \in I(V(I))$. It remains to show that $I(V(I)) \subset \sqrt{I}$. Let $f \in I(V(I))$. Then f is identically zero on $V(I)$. By 1.6.10, there is some n such that $f^n \in I$, and hence $f \in \sqrt{I}$.

Let us now show that $V(I(S)) = S$. Since S is Zariski closed, we know that $S = V(J)$ for some ideal J. So $I(S) = I(V(J)) = \sqrt{J}$. But we know that $V(J) = V(\sqrt{J})$, and hence $V(I(S)) = S$. The proof of Theorem 1.6.7 is then complete.

1.8 Affine and quasi-affine varieties

WEEK2:DONE

An *affine variety* is a subset of the affine space that is closed in the Zariski topology. A *quasi-affine variety* is a subset of the affine space that is locally closed in the Zariski topology. (A locally closed subset of a topological space is a set that can be expressed as an intersection of an open set and a closed set).

2 Regular functions and maps 1

Throughout this section, k is an algebraically closed field.

2.1 Regular functions

WEEK3

Let $S \subset \mathbb{A}^n$ be a set and let $f\colon S \to k$ be a function. Let a be a point of S.

2.1.1 Definition (Regular function) We say that f is *regular* (or *algebraic*) *at* a if there exists a Zariski open set $U \subset \mathbb{A}^n$ and polynomials $p, q \in k[x_1, \dots, x_n]$ with $q(a) \neq 0$ such that

$$f \equiv p/q \text{ on } S \cap U.$$

We say that f is *regular* if it is regular at all points of S.

In other words, f is regular at a point a if locally around a (in the Zariski topology), f can be expessed as a ratio of two polynomials. Although the definition of a regular function makes sense for $S \subset \mathbb{A}^n$, we use it only in the context of quasi-affine varieties.

2.1.2 Examples

1. A constant function is regular.
2. Every polynomial function is regular.
3. Sums and products of regular functions are regular. So, the set of regular functions forms a ring. This ring contains a copy of k, namely the constant functions.

2.1.3 Definition (Ring of regular functions) We denote the ring of regular functions on S by $k[S]$. This ring is a k-algebra.

2.1.4 Proposition (Local nature of regularity) Let f be a function on S, and let $\{U_i\}$ be an open cover of S. If the restriction of f to each U_i is regular, then f is regular.

Proof. — (1)

2.2 Regular functions on an affine variety

WEEK3

It turns out that regular functions on closed subsets of $\mathbb{A}^n$ are just the polynomial functions! So, not only is there a global algebraic expression, we don't even need denominators.

2.2.1 Proposition Let $X \subset \mathbb{A}^n$ be a Zariski closed subset. Let f be a regular function on X. Then there exists a polynomial $P \in k[x_1, \ldots, x_n]$ such that $P(x) = f(x)$ for all $x \in X$.

Proof. By definition, we know that for every $x \in X$, there is a Zariski open set $U \subset X$ and polynomials p, q such that $f = p/q$ on U. The set U and the polynomials p, q may depend on x, so let us denote them by U_x, p_x, and q_x. We need to combine all of these p's and q's and construct a single polynomial P that agrees with f for all x.

This is done by a "partition of unity" argument. First, let us do some preparation. We know that $p_x/q_x = f$ on U_x, but we know nothing about p_x and q_x on the complement of U_x. Our first step is a small trick that lets us assume that both p_x and q_x are identically zero on the complement of U_x.

Since $U_x \subset X$ is open, its complement is closed. By the definition of the Zariski topology, this means that

$$X \setminus U_x = X \cap V(A),$$

for some $A \subset k[x_1, \ldots, x_n]$. Since $x \in U_x$, at least one of the polynomials in A must be non-zero at x. Let g be such a polynomial, and set $U'_x = X \cap \{g \neq 0\}$. Then $U'_x \subset U_x$ is a possibly smaller open set containing x. Set $p'_x = p_x \cdot g$ and $q'_x = q_x \cdot g$. Then we have $f = p'_x/q'_x$ on U'_x, and we also have $p'_x \equiv q'_x \equiv 0$ on $X \setminus U'_x$. So, we may assume from the beginning that both p_x and q_x are identically zero on the complement of U_x..

Now comes the crux of the argument. Suppose $X = V(I)$. Consider the set of "denominators" $\{q_x \mid x \in X\}$. Note that the system of equations

$$g = 0 \text{ for all } g \in I \text{ and } q_x = 0 \text{ for all } x \in X$$

has no solution!

Why is this true? — (2)

By the Nullstellensatz, this means that the ideal $I + \langle q_x \mid q \in X\rangle$ is the unit ideal. That is, we can write

$$1 = g + r_1 q_{x_1} + \cdots + r_m q_{x_m}$$

for some polynomials $r_1, \dots, r_m$. Take $P = r_1 p_{x_1} + \cdots + r_m p_{x_m}$. Then $f = P$ on all of X.

Check the last equality. — (3)

$\square$

—- Let $X \subset \mathbb{A}^n$ be any subset. We have a ring homomorphism

$$\pi\colon k[x_1, \dots, x_n] \to k[X],$$

where a polynomial f is sent to the regular function it defines on X.

2.2.2 Proposition (Ring of regular functions of an affine) Let $X \subset \mathbb{A}^n$ be a closed subset. Then the ring homomorphism $\pi\colon k[x_1, \dots, x_n] \to k[X]$ induces an isomorphism

$$k[x_1, \dots, x_n]/I(X) \xrightarrow{\sim} k[X].$$

Proof. The map π is surjective by Proposition 2.2.1 and its kernel is $I(X)$ by definition. The result follows by the isomorphism theorems. $\square$

2.3 Regular maps

WEEK3

Consider $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ and a function $f\colon X \to Y$. Write f in coordinates as

$$f = (f_1, \dots, f_m).$$

2.3.1 Definition (Regular map) We say that f is *regular at a point* $a \in X$ if all its coordinate functions $f_1, \dots, f_m$ are regular at a. If f is regular at all points of X, then we say that it is *regular*.

2.3.2 Example (Maps to $\mathbb{A}^1$) A regular map to $\mathbb{A}^1$ is the same as a regular function.

2.3.3 Example (An isomorphism) Let $U = \mathbb{A}^1 \setminus \{0\}$ and $V = V(xy - 1) \subset \mathbb{A}^2$. We have a regular function $\phi\colon V \to U$ given by $\phi(x, y) = x$. We have a regular function $\psi\colon U \to V$ given by $\psi(t) = (t, 1/t)$. These functions are mutual inverses, and hence we have a (bi-regular) isomorphism $U \cong V$.

2.4 Properties of regular maps

WEEK3

2.4.1 Proposition (Elementary properties of regular maps)

1. The identity map is regular.
2. The composition of two regular maps is regular.
3. Regular maps are continuous (in the Zariski topology).

Proof. The identity map is given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$; each coordinate is a polynomial, and hence regular. The statement for composition is true because the composition of fractions of polynomials is also a fraction of polynomials. The third statement is left as homework. $\square$

2.4.2 Proposition (Regular maps preserve regular functions) Let $\phi\colon X \to Y$ be a regular map. If f is a regular function on Y, then $f \circ \phi$ is a regular function on X.

Proof. View a regular function as a regular map to $\mathbb{A}^1$. Then this becomes a special case of composition of regular maps. $\square$

— As a result, we get a k-algebra homomorphism $k[Y] \to k[X]$, often denoted by ϕ^*:

$$\phi^*(f) = f \circ \phi.$$

We thus get a (contravariant) functor from the category of (quasi-affine) varieties to k-algebras. On objects, it maps X to $k[X]$. On morphisms, it maps $\phi\colon X \to Y$ to $\phi^*\colon Y \to X$. It is easy to check that this recipe respects composition. That is, if we have maps $\phi\colon X \to Y$ and $\psi\colon Y \to Z$, and if we let $\psi \circ \phi\colon X \to Z$ be the composite, then

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*.$$

2.4.3 Corollary (Isomorphic varieties have isomorphic rings of functions) If $\phi\colon X \to Y$ is an isomorphism of varieties, then $\phi^*\colon k[Y] \to k[X]$ is an isomorphism of k-algebras.

Proof. Let $\psi\colon Y \to X$ be the inverse of ϕ. Then $\psi^*\colon k[X] \to k[Y]$ is the inverse of ϕ^*. $\square$

2.4.4 Proposition (For affines, map between rings induces map between spaces) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be Zariski closed, and let $f\colon k[Y] \to k[X]$ be a homomorphism of k-algebras. Then there is a unique (regular) map $\phi\colon X \to Y$ such that $f = \phi^*$.

Proof. We know that $k[X] = k[x_1, \ldots, x_n]/I(X)$ and $k[Y] = k[y_1, \ldots, y_m]/I(Y)$. Let $\phi_i = f(y_i) \in k[X]$. Consider $\phi\colon X \to \mathbb{A}^m$ given by $\phi = (\phi_1, \ldots, \phi_m)$. Then ϕ sends X to Y and is the unique map satisfying the required properties. $\square$

Prove the last statement. — (4)

2.4.5 Example (Bijection but not an isomorphism) Let $X = \mathbb{A}^1_k$ and $Y = V(y^2 - x^3) \subset \mathbb{A}^2_k$. We have a regular map $f\colon X \to Y$ given by $f(t) = (t^2, t^3)$. It is easy to check that f is a bijection, but not an isomorphism.

Why is this not an isomorphism? — (5)

2.4.6 Example (Distinguished affine opens) Let $U_f \subset \mathbb{A}^n$ be the complement of $V(f)$. Then U_f is isomorphic to an affine variety, namely the variety $V(yf-1) \subset \mathbb{A}^{n+1}$, where y denotes the $(n+1)$-th coordinate.

Prove this. — (6)

2.4.7 Caution (Not all opens are affine) The previous proposition only applies to the complement of $V(f)$ for a single f! The complement of $V(I)$, in general, is not isomorphic to an affine variety. For example, the complement of the origin in $\mathbb{A}^2$ is not isomorphic to an affine variety.

3 Algebraic varieties

3.1 Definition WEEK4

The varieties we have seen so far have been sub-sets of the affine space. Using these as buildig blocks, we can construct general algebraic varieties. The definition is analogous to the definition of a manifold in differential geometry, using open subsets of $\mathbb{R}^n$ as building blocks.

Let X be a topological space. A *quasi-affine chart* on X consists of an open subset $U \subset X$, a quasi-affine variety V and a homeomorphism $\phi_{UV} : U \to V$. Via this isomorphism, we can "transport" the algebraic structure (for example, the notion of a regular function) from V to U.

Let $\phi_1\colon U_1 \to V_1$ and $\phi_2\colon U_2 \to V_2$ be two quasi-affine charts on X (see Figure 1). Set $U_{12} = U_1 \cap U_2$. Consider the open subsets $V_{12} = \phi_1(U_{12}) \subset V_1$ and $V_{21} = \phi_2(U_{12}) \subset V_2$. Being open subsets of quasi-affine varieties, they are themselves quasi-affine varieties. Furthermore, the map

$$\phi_2 \circ \phi_1^{-1}\colon V_{12} \to V_{21}$$

is a homeomorphism. We say that the two charts are *compatible* if this map is a (bi-regular) isomorphism.

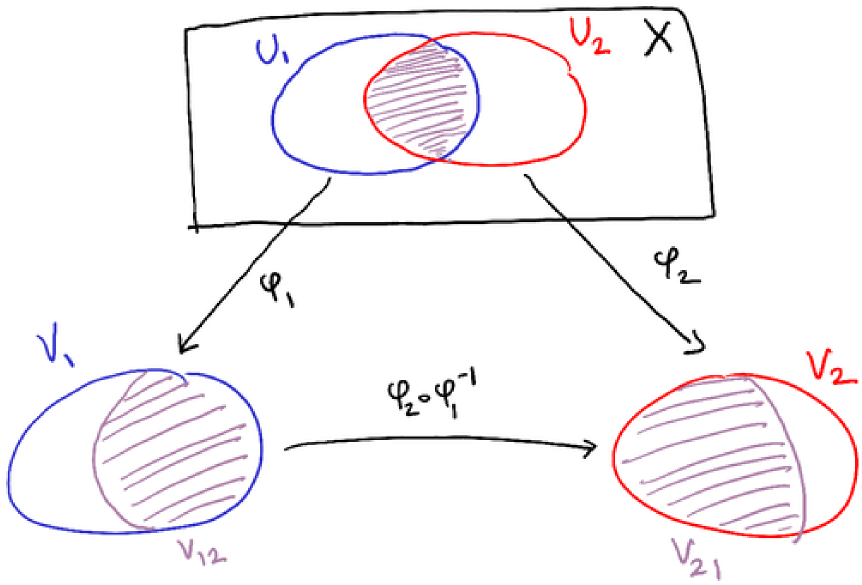

Figure 1: Compatible charts

When we have two charts, one on U_1 and another on U_2, then the intersection $U_1 \cap U_2$ gets two different charts. Compatibility ensures that these two charts are related by a bi-regular isomorphism, so that the algebraic structure coming from one is the same as the one coming from the other.

A *quasi-affine atlas* on X is a collection of compatible charts $\phi_i \colon U_i \to V_i$ such that the U_i cover X.

3.1.1 Definition (Algebraic variety) An *algebraic variety* is a topological space with a quasi-affine atlas.

3.1.2 Example (Quasi-affine varieties) A quasi-affine variety X is itself an algebraic variety. The atlas is the obvious one, consisting of the single chart $\mathrm{id} \colon X \to X$.

3.2 Projective spaces

WEEK4

A fundamental example of an algebraic variety is the projective space.

3.2.1 Definition (Projective space) The *projective n-space over a field k*, denoted by $\mathbb{P}^n_k$, is the set of one-dimensional subspaces of k^{n+1}.

3.2.2 Intuition Before describing how $\mathbb{P}^n_k$ is an algebraic variety, let us build some intuition about projective space. For easy visualisations, it helps to take $k = \mathbb{R}$ or $k = \mathbb{C}$. A one dimensional subspace of k^{n+1} is also called a *line*. Note that, by this definition, a line must contain the origin.

Let us take $n = 0$. Then there is a unique one-dimenional subspace of $k^{n+1} = k$, so $\mathbb{P}^0_k$ is just a single point.

Let us take $n = 1$. Then $\mathbb{P}^1_k$ is the set of lines (through the origin) in k^2. Let us take $k = \mathbb{R}$. Every line through the origin is uniquely determined by its slope, which can be any element of $\mathbb{R}$, so it seems like $\mathbb{P}^1_{\mathbb{R}}$ is just a copy of $\mathbb{R}$. But the vertical line does not have a (finite) slope, so $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. In other words, $\mathbb{P}^1$ contains the usual real line, plus "a point at infinity".

It can be more instructive to see this in a picture. Fix a horizontal line L at, say, $y = -1$. Every line through the origin intersects L at a unique point, except the horizontal line. So if we discard the one point of $\mathbb{P}^1_k$ corresponding to the horizontal line, the rest is just a copy of L. If we had chosen a different reference line L, for example, a vertical one, then we get a similar description of $\mathbb{P}^1$ away from a single point. In fact, we can discard *any* one point of $\mathbb{P}^1$, and the rest will be a copy of $\mathbb{R}$.

Let us take $n = 2$. Then $\mathbb{P}^2_k$ is the set of lines (through the origin) in k^3. We can use the same technique as before: fix a reference plane P at $z = -1$. Then most lines are uniquely characterised by their intersection point with P. The only exceptions are the lines parallel to $z = -1$, that is, the lines lying in the plane $z = 0$, which we miss. But these form a small projective space $\mathbb{P}^1$. So we see that $\mathbb{P}^2 = P \sqcup \mathbb{P}^1$.

3.2.3 Topology A one-dimensional subspace of k^{n+1} is spanned by a non-zero vector $(a_0, \dots, a_n)$. Two vectors $(a_0, \dots, a_n)$ and $(b_0, \dots, b_n)$ span the same subspace if and only if there exists $\lambda \in k^\times$ such that

$$(b_0, \dots, b_n) = (\lambda a_0, \dots, \lambda a_n).$$

So, we can identify $\mathbb{P}^n$ with the equivalence classes of non-zero vectors $(a_0, \dots, a_n)$ where two non-zero vectors are considered equivalent if one is a scalar multiple of the other. In other words, we have

$$\mathbb{P}^n_k = (\mathbb{A}^{n+1} \setminus 0)/\text{scaling}.$$

We denote the equivalence class of $(a_0, \dots, a_n)$ by $[a_0 : \cdots : a_n]$.

We give $\mathbb{P}^n_k$ the quotient topology inherited from $\mathbb{A}^{n+1} \setminus 0$. That is, a set $U \subset \mathbb{P}^n_k$ is open/closed if and only if its pre-image in $\mathbb{A}^{n+1} \setminus 0$ is open/closed.

For example, consider the subset U_n of $\mathbb{P}^n_k$ consisting of $[a_0 : \cdots : a_n]$ with $a_n \neq 0$. Its preimage in the set of $(a_0, \dots, a_n) \in \mathbb{A}^{n+1} \setminus 0$ with $a_n \neq 0$, which is a (Zariski) open set. Hence U_n is open in $\mathbb{P}^n_k$. Likewise, $U_0, U_1, \dots$ are also open. Note that we have

$$\mathbb{P}^n_k = U_0 \cup \cdots \cup U_n;$$

that is, the sets $U_0, \dots, U_n$ form an open cover of $\mathbb{P}^n$.

Consider a point $[a_0 : \cdots : a_n] \in U_0$, so that $a_0 \neq 0$. By scaling by $\lambda = a_0^{-1}$, we have a distinguished representative of this point of the form $[1 : b_1 : \cdots : b_n]$, which we can think of as a point $(b_1, \dots, b_n) \in \mathbb{A}^n$. Thus, we have a bijection $\phi_0 \colon U_0 \to \mathbb{A}^n$, and similarly $\phi_1 U_i \to \mathbb{A}^n$.

3.2.4 Proposition (Charts of the projective space)

1. The bijections $\phi_i \colon U_i \to \mathbb{A}^n$ defined above are homeomorphisms.
2. The charts $\phi_i \colon U_i \to \mathbb{A}^n$ are mutually compatible, and hence give an atlas on $\mathbb{P}^n$.

> 1. This is not obvious, also not hard, but also not very enlightening. Let us skip this.
> 2. Do this! — (1)

3.2.5 Open and closed subvarieties Let X be an algebraic variety, and $Y \subset X$ an open or closed subset. Then Y inherits the structure of an algebraic variety. To get, the atlas for Y, let $\phi_i \colon U_i \to V_i$ be an atlas for X. For Y, we just take $\phi_i \colon U_i \cap Y \to \phi(U_i \cap Y)$.

> Explain why this is an atlas for Y — (2)

3.2.6 Proposition (Closed subvarieties of projective space 1) Let $F \in k[X_0, \dots, X_n]$ be a homogeneous polynomial. Let $V(F) \subset \mathbb{P}^n$ be the set of points $\{[a_0 : \cdots : a_n] \mid F(a_0, \dots, a_n) = 0\}$. Then $V(F)$ is a closed subset.

> Explain why $V(F)$ is well-defined (that is, the condition $F(a_0, \dots, a_n) = 0$ does not depend on the chosen representative of the equivalence class). Then explain why $V(F)$ is closed. — (3)

3.2.7 Proposition (Closed subvarieties of projective space 2) Let $I \subset k[X_0, \dots, X_n]$ be a homogeneous ideal.

> Define $V(I) \subset \mathbb{P}^n$ and show that it is a closed subset. — (4)

3.2.8 Proposition (Closed subvarieties of projective space 3) Conversely, let $X \subset \mathbb{P}^n$ be a closed subset. Then there exists a homogeneous ideal $I \subset k[X_0, \dots, X_n]$ such that $X = V(I)$.

Proof. Assume that X is non-empty. Let $\pi \colon \mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ be the quotient map. Let $C \subset \mathbb{A}^n$ be the closure of $\pi^{-1}(X)$.

Prove that C is conical, that is, if $x \in C$ then $\lambda x \in C$ for every scalar $\lambda \in k$. Conclude using Homework 1 that $C = V(I)$ for a homogeneous ideal I. Prove that $X = V(I)$ in $\mathbb{P}^n$. — (5)

□

3.2.9 Example (Linear subspaces) Suppose $I \subset k[X_0, \ldots, X_n]$ is generated by (homogeneous) linear equations. Then $V(I) \subset \mathbb{A}^{n+1}$ is a sub-vector space $W \subset \mathbb{A}^{n+1}$, and $V(I) \subset \mathbb{P}^n$ is naturally the projective space of W. We call such $V(I) \subset \mathbb{P}^n$ *linear subspaces*, or “lines”, “planes”, etc. See that any two distinct lines in $\mathbb{P}^2$ intersect at a unique point, and through any two distinct points in $\mathbb{P}^2$ passes a unique line.

4 Regular functions and regular maps 2

4.1 Regular functions

WEEK5

4.1.1 Proposition (regularity does not depend on the chart) Let X be an algebraic variety and $f\colon X \to k$ a function. Let $\phi_1\colon U_1 \to V_1$ and $\phi_2\colon U_2 \to V_2$ be two compatible charts such that x lies in both V_1 and V_2. Denote the images of x in the two charts by v_1 and v_2. Consider the functions $f \circ \phi_1^{-1}\colon V_1 \to k$ and $f \circ \phi_2^{-1}\colon V_2 \to k$. Then the first is regular at v_1 if and only if the second is regular at v_2.

Prove this. — (1)

4.1.2 Definition (regular function on a variety) We say that f is regular at x if for some (equivalently, for every) chart $\phi\colon U \to V$ with $x \in U$, the function $f \circ \phi^{-1}\colon V \to k$ is regular at $\phi(x)$. We say that f is regular on X if it is regular at all points $x \in X$.

4.1.3 Example If X is quasi-affine, then we have not done anything new.

4.1.4 Example Let $X = \mathbb{P}^1$. Set $f([X : Y]) = X/Y$. Then f is defined at all points except the point $[1 : 0]$, and is a regular function on $\mathbb{P}^1 \setminus \{[1 : 0]\}$. More generally, let $X = \mathbb{P}^n$ and let $F, G \in k[X_0, \ldots, X_n]$ be homogeneous polynomials of the same degree. The function

$$[X_0 : \cdots : X_n] \mapsto F(X_0, \ldots, X_n)/G(X_0, \ldots, X_n)$$

is regular outside $V(G)$.

Prove this. —- (2)

4.1.5 Example Even more generally, let $X = \mathbb{P}^n$ and let $F_0, \dots, F_m$ be homogeneous polynomials of the same degree. Let $Z \subset \mathbb{P}^n$ be $V(F_0, \dots, F_m)$. Then the formula

$$[X_0 : \dots : X_n] \mapsto [F_0(X_0, \dots, X_n) : \dots : F_m(X_0, \dots, X_n)]$$

defines a regular function from $X \setminus Z$ to $\mathbb{P}^m$.

> Prove this. —- (3)

4.1.6 Example (Automorphisms of $\mathbb{P}^n$) Consider the $n+1$-dimensional k-vector space V spanned by $X_0, \dots, X_n$. Pick any basis $\ell_0, \dots, \ell_n$ of this vector space. Then we have a regular map

$$L\colon \mathbb{P}^n \to \mathbb{P}^n$$
$$[X_0 : \dots : X_n] \mapsto [\ell_0 : \dots : \ell_n].$$

Explicitly, if we write

$$\ell_i = L_{i,0}X_0 + \dots + L_{i,n}X_n$$

and write our homogenous vector as a column vector, then the map is

$$[X] \mapsto [LX].$$

In other words, it is induced by the invertible linear map $L\colon V \to V$. As a result, it has an inverse, induced by the inverse of the matrix M:

$$[X] \mapsto [MX].$$

In this way, we get an action of $GL_n(k)$ on $\mathbb{P}^n$. But notice that a matrix L and a scalar multiple λL induce the same map on $\mathbb{P}^n$. So the action descends to an action of the group $PGL_n(k) = GL_n(k)/\text{scalars}$.

4.1.7 Example (regular functions on $\mathbb{P}^1$) The previous example gave examples of regular functions on (strict) open subsets of the projective space. It turns out that there are *no* regular functions on $\mathbb{P}^n$ other than the constant functions!

> Prove this for $n = 1$. — (4)

4.2 Regular maps WEEK5

4.3 The Veronese embedding WEEK5

Let $n \geq 1$, and consider the k-vector space of degree n homogeneous polynomials in X, Y. This vector space has dimension $n+1$. Choose a basis, for example, let us take

$X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n$. Then we have a regular map

$$v_n \colon \mathbb{P}^1 \to \mathbb{P}^n$$
$$[X : Y] \mapsto [X^n : \cdots : Y^n].$$

4.3.1 Proposition (Veronese curves) The image of v_n is a closed subset of C of $\mathbb{P}^n$. If we denote the homogeneous coordinates on $\mathbb{P}^n$ by $[U_0 : \cdots : U_n]$, then C is cut out by the equations

$$\{U_iU_j - U_kU_\ell \mid 0 \leq i, j, k, l \leq n \text{ and } i + j = k + \ell.$$

Prove this. — (5)

4.3.2 Proposition (Veronese curves continued) The map $v_n \colon \mathbb{P}^1 \to C$ is in fact an isomorphism.

Define the inverse map. — (6)

The proposition above generalises to all dimensions. Consider the k-vector space of degree n homogeneous polynomials in $X_0, \ldots, X_m$. It has dimension $N = \binom{n+m}{m}$. Choosing a basis gives a map $\mathbb{P}^m \to \mathbb{P}^N$. The image of this map is a closed subvariety Z and the map $\mathbb{P}^m \to Z$ is an isomorphism. The equations of Z and the description of the inverse map are analogous to the $m = 1$ case, but (understandably) somewhat more cumbersome.

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