Normalization of Algebraic Varieties

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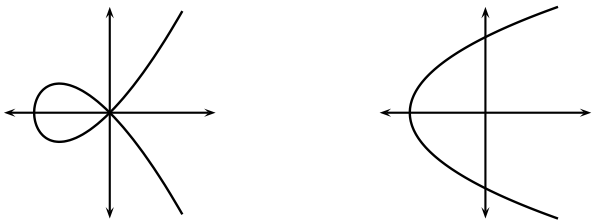
Abstract. We study the process of normalization of affine algebraic varieties and an algorithm to carry it out. We illustrate the application of normalization to the problem of resolution of singularities by proving that normal varieties are regular in codimension 1.

1. Introduction. Consider the plane curve C defined by the equation $y^2 = x^3 + x^2$. It is clear from Figure 1-1(a) that the local behavior of C at the origin O is somehow different from its local behavior at other points. In particular, there is no satisfactory way to define a unique tangent line to C at O. We call such exceptional points *singular* points of the curve.



(a) $y^2 = x^3 + x^2$ (b) $y^2 = x + 1$

Figure 1-1. Removing singularities by a rational transformation.

In general, let S be a d-dimensional variety in the n-dimensional affine space defined by some polynomial equations $f_1(x) = 0, \ldots, f_r(x) = 0$. We can satisfactorily define a unique tangent space at most points of S. However, it is impossible to do so at a few notorious points, called *singular* points. All other points are called *regular* or *simple.* For example, all the points on the surface $z^2 - x^2y = 0$ are regular, except the points on the line $x = 0, z = 0$.

We would like to transform S into a *nonsingular variety*, one with no singular points. The transformation must be reversible, and must involve only rational functions of the coordinates. Such a transformation is called a *birational transformation* and the two varieties are said to be *birational.* For example, if we perform the rational transformation $(x, y) \mapsto (x, yx)$, then the curve $y^2 = x^3 + x^2$ is transformed to the curve $y^2 = x + 1$, which has no singularities. We have thus *resolved* the singularities of C. *Resolving the singularities* of a variety means finding a sequence of birational transformations that transform the variety into a nonsingular variety.

Normalization plays an important role in resolving the singularities of a variety. The normalization process transforms the given variety into one with an integrally closed

algebra of functions. Such varieties are called *normal*; they have "sparse" singularities. In particular, normal curves have no singular points.

In 1882, Dedekind was the first to use the technique of normalization to resolve the singularities of curves. Around 1940, Zariski made major inroads into the problem of resolution by solving it for surfaces and solids in characteristic 0. His student Hironaka settled the problem in characteristic 0 in 1964. Normalization played an important part in their method.

In this paper, we study the process of normalization of affine algebraic varieties. In Section 2, we recall some concepts from algebraic geometry. In Section 3, we consider the local properties of varieties including the notions of regular and singular points. In Section 4, we describe the main construction of the paper: the normalization of affine varieties. In Section 5, we study discrete valuation rings, which play an important role in proving that normal varieties are regular in codimension 1. In Section 6, we show that normal varieties are regular in codimension 1, and prove the converse for curves. Finally, in Section 7, we describe an algorithm to compute the normalization of an affine variety.

2. Preliminaries. This section introduces some fundamental concepts from algebraic geometry. Most of the theorems in this section are not proved. See [**1**] and [**6**] for a full exposition.

Throughout, the letter k stands for an algebraically closed field, and $k[x_1, \ldots, x_n]$ denotes the ring of polynomials in variables $x_1, \ldots, x_n$ with coefficients in k.

Let A be a subset of $k[x_1, \ldots, x_n]$. The *affine algebraic variety*, or just the *affine variety*, defined by A is the set

$$\mathbf{V}(A) = \{(p_1, \ldots, p_n) \in k^n \mid f(p_1, \ldots, p_n) = 0 \text{ for all } f \in A\}.$$

Since we only consider affine varieties in this paper, the term variety always means an affine variety. The following properties of varieties are easy to check:

(1) $\mathbf{V}(\bigcup_\alpha A_\alpha) = \bigcap_\alpha \mathbf{V}(A_\alpha)$,
(2) $\mathbf{V}(A \cdot B) = \mathbf{V}(A) \cup \mathbf{V}(B)$,
(3) $\mathbf{V}(\{0\}) = k^n$,
(4) $\mathbf{V}(k[x_0, \ldots, x_n]) = \varnothing$.

It easy to see that $\mathbf{V}(A) = \mathbf{V}(\langle A \rangle)$ where $\langle A \rangle$ is the ideal generated by A. Since every ideal of $k[x_1, \ldots, x_n]$ is finitely generated, every affine variety is defined by a finite subset of $k[x_1, \ldots, x_n]$.

Let V be a subset of k^n. The ideal of V, denoted by $\mathbf{I}(V)$, is the ideal

$$\mathbf{I}(V) = \{f \in k[x_1, \ldots, x_n] \mid f(p) = 0 \text{ for all } p \in V\}.$$

It is easy to see that $\mathbf{I}(V)$ is indeed an ideal. Moreover, if $f^n \in \mathbf{I}(V)$ for some $n \geq 0$, then $f \in \mathbf{I}(V)$. In other words, $\mathbf{I}(V)$ is a *radical ideal*. The following important theorem relates the operations $\mathbf{V}$ and $\mathbf{I}$.

Theorem 2-1 (Hilbert's Nullstellensatz). *Let I be an ideal of $k[x_1, \ldots, x_n]$. Then $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$.*

Due to the Nullstellensatz, we have an inclusion reversing bijection between radical ideals of $k[x_1, \ldots, x_n]$ and affine algebraic varieties in k^n. The map is given by $I \mapsto \mathbf{V}(I)$, with the inverse $V \mapsto \mathbf{I}(V)$.

We have the following corollary of Theorem 2-1.

Corollary 2-2. *Let $p = (p_1, \ldots, p_n)$ be a point in k^n. Then $\mathbf{I}(p)$ is the maximal ideal $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$. Conversely, all maximal ideals of $k[x_1, \ldots, x_n]$ have this form.*

A polynomial function $f : k^n \to k$ can be restricted to a function on V. These functions are called *polynomial functions* on V, and the algebra $k[V]$ of all polynomial functions on V is called the *algebra of functions* of V. Observe that two polynomials $f, g \in k[x_1, \ldots, x_n]$ give the same function on V if and only if the difference $f - g$ lies in the ideal $\mathbf{I}(V)$. Hence, we have the following proposition.

Proposition 2-3. *The algebra of polynomial functions on a variety $V \subset k^n$ is canonically isomorphic to the quotient $k[x_1, \ldots, x_n]/\mathbf{I}(V)$.*

Note that, since $\mathbf{I}(V)$ is a radical ideal, the algebra $k[V]$ does not have any nilpotents. Thus, $k[V]$ is a finitely generated, reduced k algebra.

For an ideal $I \subset k[V]$, we define the *algebraic subset of V defined by I* to be the set

$$\mathbf{V}_V(I) = \{p \in V \mid f(p) = 0 \text{ for all } f \in I\}.$$

If we let J denote the preimage of I in $k[x_1, \ldots, x_n]$ under the map $k[x_1, \ldots, x_n] \to k[V]$, then it is easy to see that $\mathbf{V}_V(I) = \mathbf{V}(J)$. Hence, algebraic subsets of a variety V are indeed affine algebraic varieties. If the ambient variety V is clear by context, then we often drop the subscript V, and denote $\mathbf{V}_V(I)$ by $\mathbf{V}(I)$.

We can construct an operation analogous to $\mathbf{I}$ on subsets of V. If $W \subset V$, then we let

$$\mathbf{I}_V(W) = \{f \in k[V] \mid f(p) = 0 \text{ for all } p \in W\}.$$

It is easy to see that $\mathbf{I}_V(W)$ is a radical ideal of $k[V]$. In fact, Theorem 2-1 can be extended to $k[V]$ to obtain the following.

Theorem 2-4. *Let V be an affine algebraic variety, and I an ideal of $k[V]$. Then $\mathbf{I}_V(\mathbf{V}_V(I)) = \sqrt{I}$.*

Thus we have an inclusion reversing bijection between the radical ideals of $k[V]$ and algebraic subsets of V given by $I \mapsto \mathbf{V}(I)$. The inverse is $W \mapsto \mathbf{I}(W)$.

Let $V \subset k^n$ be an affine variety. Then the maximal ideals of $k[V]$ are the images of the maximal ideals of $k[x_1, \ldots, x_n]$ containing $\mathbf{I}(V)$ under the map $k[x_1, \ldots, x_n] \to k[V]$. A maximal ideal $\mathbf{I}(p) = \langle x_1 - p_1, \ldots, x_n - p_n \rangle$ contains $\mathbf{I}(V)$ if and only if we have $\mathbf{V}(\mathbf{I}(p)) = \{p\} \subset \mathbf{V}(\mathbf{I}(V)) = V$. Thus we have the following proposition.

Proposition 2-5. *The points of V correspond bijectively to maximal ideals of $k[V]$, where the correspondence is given by $p \mapsto \mathbf{I}_V(p) = \langle x_1 - p_1, \ldots, x_n - p_n \rangle / \mathbf{I}(V)$.*

Under this identification, the closed sets are of the form

$$\mathbf{V}(I) = \{m \mid \text{ is a maximal ideal of } k[V] \text{ containing } I\}.$$

Next, we look at morphisms between affine varieties. Given affine varieties $V \subset k^n$ and $W \subset k^m$, a map $\phi : V \to W$ is called a *polynomial map* or a *morphism of affine varieties* if there exist polynomials $g_1, \ldots, g_n \in k[x_1, \ldots, x_n]$ such that

$$\phi(x_1, \ldots, x_n) = (g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)).$$

Morphisms $\phi : V \to W$ have the following important property: if $f \in k[W]$, then the function on V defined by $x \mapsto f \circ \phi(x)$ is a function in $k[V]$. Hence a morphism

$\phi : V \to W$ induces a homomorphism of k-algebras $\phi^* : k[W] \to k[V]$, which maps $f \mapsto f \circ \phi$.

Conversely, given a homomorphism of k-algebras $\phi^* : k[W] \to k[V]$, we get a morphism $\phi : V \to W$ that induces ϕ^* in the following way. Let $k[V] = k[y_1, \ldots, y_n]/\mathbf{I}(V)$ and $k[W] = k[x_1, \ldots, x_m]/\mathbf{I}(W)$. To simplify notation, let us identify polynomials in $k[x_i]$ and $k[y_j]$ with their images in $k[W]$ and $k[V]$, respectively. Let $f_i = \phi^*(x_i)$ for $i = 1, \ldots, m$. Then the map ϕ is given by

$$\phi(y_1, \ldots, y_n) = (f_1(y_1, \ldots, y_n), \ldots, f_m(y_1, \ldots, y_n)).$$

Two varieties V and W are called *isomorphic* if there exists morphisms $\phi : V \to W$ and $\psi : W \to V$ such that $\psi \circ \phi = \mathrm{id}_W$ and $\phi \circ \psi = \mathrm{id}_V$. This condition is equivalent to the existence of two ring morphisms $f : k[W] \to k[V]$ and $g : k[V] \to k[W]$ such that $f \circ g = \mathrm{id}_{k[W]}$ and $g \circ f = \mathrm{id}_{k[V]}$. In other words, affine varieties are isomorphic if they have isomorphic algebras of functions.

Given a finitely generated reduced k-algebra A, we can construct a variety with A as its algebra of functions. We choose a representation $A \cong k[x_1, \ldots, x_n]/I$ and consider the variety $V = \mathbf{V}(I)$ in k^n. Since A is reduced, I is radical, and hence $\mathbf{I}(\mathbf{V}(I) = I$ by the Nullstellensatz. Therefore, the algebra of V is $k[x_1, \ldots, x_n]/I \cong A$. Since affine varieties are characterized by their algebra of functions, the isomorphism class of V is independent of the representation of A. The points of V can be identified with the set of maximal ideals of A, which is denoted by $\mathrm{Specm}(A)$. We also denote by $\mathrm{Specm}(A)$ the affine variety with algebra of functions A.

Next, we define a topology on a variety V. Observe that the following properties hold for the algebraic subsets of V:

(1) $\mathbf{V}_V(\bigcup_\alpha A_\alpha) = \bigcap_\alpha \mathbf{V}_V(A_\alpha)$,
(2) $\mathbf{V}_V(A \cap B) = \mathbf{V}_V(A) \cup \mathbf{V}_V(B)$,
(3) $\mathbf{V}_V(k[V]) = \varnothing$ and $\mathbf{V}_V(\{0\}) = V$.

Therefore, if we declare algebraic subsets of V to be *closed*, then we get a topology on V. This topology is called the *Zariski topology*.

Given a morphism $\phi : V \to W$, we see that $\phi^{-1}(\mathbf{V}_W(I)) = \mathbf{V}_V(\phi^{*-1}(I))$ for an ideal $I \subset k[W]$. In other words, the inverse images of closed sets are closed. Hence, morphisms of algebraic varieties are continuous maps with respect to the Zariski topology.

Some varieties can be written as a nontrivial union of several varieties. For example, the variety $\mathbf{V}(xy) \subset k^2$ is the union of $\mathbf{V}(x)$ and $\mathbf{V}(y)$. A variety is V called *reducible* if there exist closed subsets V_1 and V_2 of V such that $V = V_1 \cup V_2$, but $V \neq V_1$ and $V \neq V_2$. Varieties that are not reducible are called *irreducible*. It is easy to see that, if V is irreducible, then all open subsets of V are dense in V.

A variety V can be uniquely written as a union of irreducible varieties. In precise words, we have the following proposition.

Proposition 2-6. *Let $V \subset k^n$ be a variety. Then there exist irreducible varieties $V_1, \ldots, V_r$ such that $V = V_1 \cup \cdots \cup V_r$ where $V_i \not\subset V_j$ for $i \neq j$. Moreover, this expression of V is unique up to the order of the V_i.*

As the following proposition shows, we can characterize irreducible affine varieties by their algebra of functions.

Proposition 2-7. *An affine algebraic variety $V \subset k^n$ is irreducible if and only if $k[V]$ in an integral domain.*

Analogous to the algebra of functions, we have the notion of the field of rational functions of an irreducible variety. For an irreducible variety V, the field of fractions $k(V)$ of $k[V]$ is called the *field of rational functions* of V. Since $k[V]$ is a finitely generated k-algebra, $k(V)$ is a finitely generated field extension of k.

In spite of the name "rational function," an element $\phi \in k(V)$ does *not* always define a function on V. Let p be a point in V; then ϕ is said to be *defined* at p if there exist $f, g \in k[V]$ such that $g(p) \neq 0$ and $\phi = f/g$. Notice that, in this case, ϕ is defined on the open set $V - V_V(g)$ containing p. The set of points $\mathrm{dom}(\phi)$ at which ϕ is defined is called the *domain* of ϕ. The previous observation shows that $\mathrm{dom}(\phi)$ is an open subset of V.

We extend the notion of a rational function to rational maps between irreducible varieties.

Let $V \subset k^n$ and $W \subset k^m$ be irreducible affine varieties. A *rational map* from V to W is a function ϕ represented by an m-tuple of rational functions on V

$$\phi = (\phi_1, \ldots, \phi_m) \quad \text{where } \phi_i \in k(V)$$

such that, at every point $p \in V$ where all ϕ_i are defined, $\phi(p) = (\phi_1(p), \ldots, \phi_m(p)) \in W$.

We say that ϕ is *defined* at p if all ϕ_i are defined at p. The set of all points where ϕ is defined is called the domain of ϕ and is denoted by $\mathrm{dom}(\phi)$. Notice that we have $\mathrm{dom}(\phi) = \bigcap \mathrm{dom}(\phi_i)$, and therefore $\mathrm{dom}(\phi)$ is an open subset of V. The morphisms $\phi : V \to W$ are rational maps for which $\mathrm{dom}(\phi) = V$.

To remind us that rational maps from V to W are not defined on all of V, we denote them by dashed arrows $V \dashrightarrow W$.

Like morphisms, rational maps respect the Zariski topology.

Proposition 2-8. *Let $\phi : V \dashrightarrow W$ be a rational map defined on an open subset $U \subset V$. Then ϕ is continuous with respect to the Zariski topology on W and the topology on U induced from the Zariski topology on V.*

For affine varieties, two rational maps that coincide on an open set are equal.

Proposition 2-9. *Let $\phi = (\phi_1, \ldots, \phi_m)$ and $\psi = (\psi_1, \ldots, \psi_m)$ be rational maps from $V \subset k^n$ to $W \subset k^m$. If there exists a nonempty open set $U \subset V$ such that $U \subset \mathrm{dom}(\phi) \cap \mathrm{dom}(\psi)$ and $\phi(p) = \psi(p)$ for all $p \in U$, then $\phi_i = \psi_i$ for all i.*

Proof: The diagonal Δ_W of $W \times W$ is, obviously, closed. So by Proposition 2-8, the preimage $(\phi, \psi)^{-1}\Delta_W \subset V$ is closed and contains U, so is equal to V. Hence $\phi = \psi$. □

Hence, we can specify a rational map by specifying the values it takes on a nonempty open subset of V.

Since rational maps are not defined at all points, we must take care while defining compositions of rational maps. If $\phi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ are rational maps such that $\mathrm{im}(\phi) \cap \mathrm{dom}(\psi)$ is nonempty, then we can consider the map on the nonempty open set $U = \phi^{-1}(\mathrm{dom}(\psi))$ sending $p \mapsto \psi \circ \phi(p)$. It can be seen that the composition $\psi \circ \phi$ on U defines a rational map from X to Z. It is called the composition of ϕ and ψ and is denoted by $\phi \circ \psi$.

A rational map $\phi : V \dashrightarrow W$ is said to be *birational* if there exists a rational map $\psi : W \dashrightarrow V$ such that $\phi \circ \psi = id_W$ and $\psi \circ \phi = id_V$. If there is a birational map $V \dashrightarrow W$, then V and W are said to be *birational* or *birationally equivalent.*

If $\phi : V \dashrightarrow W$ is such that $\phi(V)$ is dense in W, then ϕ is called *dominant.* Such a map induces a map ϕ^* between the fields of rational functions by mapping $f \in k(W)$ to

$f \circ \phi \in k(V)$. Conversely, a map $\phi^* : k(W) \to k(V)$ gives a rational map $\phi : V \dashrightarrow W$. Indeed, we have

$$k[W] = k[x_1, \ldots, x_n]/\mathbf{I}(W) \text{ and } k[V] = k[y_1, \ldots, y_m]/\mathbf{I}(V).$$

Say $\phi^* : x_i \mapsto f_i/g_i$ where $f_i, g_i \in k[V]$. Then we get the rational map

$$y = (y_1, \ldots, y_m) \mapsto \left(\frac{f_1(y)}{g_1(y)}, \ldots, \frac{f_n(y)}{g_n(y)}\right).$$

The field of functions characterizes a variety up to birational equivalence.

Proposition 2-10. *A rational map $\phi : V \dashrightarrow W$ is birational if and only if it is dominant and the induced map $\phi^* : k(W) \to k(V)$ is a field isomorphism. Moreover, two affine varieties V and W are birational if and only if their function fields $k(V)$ and $k(W)$ are isomorphic.*

Let V be an affine variety. The *dimension* of V, denoted by $\dim(V)$, is defined to be the largest n for which we have a chain

$$V_0 \subset \cdots \subset V_n$$

where V_i are distinct nonempty irreducible closed subsets of V. It can be proved that all maximal chains $V_0 \subset \cdots \subset V_m$ of distinct irreducible closed subsets of V have the same length. We see that the dimension of V is the maximum length of chains $P_0 \subset \cdots \subset P_n$ of prime ideals of $k[V]$.

If V is irreducible, then it can be proved [**3**, I.6] that the dimension of V is same as the transcendence degree of the field extension $k(V)/k$.

Example 2-11. Let $V = k^n$. Then $k(V) = k(x_1, \ldots, x_n)$, and hence $\dim(k^n) = n$.

It easy to see that the following propositions hold.

Proposition 2-12. *Let W be a proper irreducible subvariety of an affine variety V. Then* $\dim(W) < \dim(V)$.

Proposition 2-13. *Let $V = V_1 \cup \cdots \cup V_n$, where V_i are affine algebraic varieties. Then* $\dim(V) = \max(\dim(V_1), \ldots, \dim(V_n))$.

3. Local properties of a variety. To study singular and regular points on a variety, we need to look at its local structure. In particular, we need to define tangent spaces, and characterize singular and regular points by the associated tangent spaces. The main algebraic tool for this study is localization.

Let R be a ring. A subset $S \subset R$ is called *multiplicative* if, for every $a, b \in S$, the product ab belongs to S. The localization construction is similar to the construction of the quotient field of an integral domain: it involves adding to R the inverses of elements in S. We define an equivalence relation on $R \times S$ by letting $(a, b) \sim (c, d)$ if $s(ad - bc) = 0$ for some $s \in S$. It is easy to see that $\sim$ is an equivalence relation. The *localization of R with respect to S* is the set of equivalence classes of $R \times S$ under $\sim$. The localization of R with respect to S is denoted by $S^{-1}R$ and the equivalence class of (a, b) is denoted by a/b or $\frac{a}{b}$. We define addition and multiplication on $S^{-1}R$ as follows:

$$a/b + c/d = (ac + bd)/bd \quad \text{and} \quad (a/b) \cdot (c/d) = ac/bd.$$

It is easy to check that these operations make $S^{-1}R$ a ring. Notice that elements of S have inverses in $S^{-1}R$. If $s \in S$, then $s^{-1} = 1/s$.

If R is an integral domain and $0 \notin S$, then the equivalence relation $\sim$ reduces to $(a, b) \sim (c, d)$ if $ad - bc = 0$. In this case, $S^{-1}R$ is the subset $\{a/s \mid a \in R \text{ and } s \in S\}$ of the quotient field of R. In particular, $S^{-1}R$ is also an integral domain.

Example 3-1. Let $P \subset R$ be a prime ideal. Then $S = R - P$ is a multiplicative set. The resulting localization $S^{-1}R$ is denoted by R_P. It has the important property that $PR_p = \{a/b \in R_P \mid a \in P\}$ is the unique maximal ideal of R_P. Indeed, if an ideal I contains an element $x \notin PR_p$, then $x = a/b$ where $a \notin P$ and $b \notin P$. Such an element is invertible with inverse b/a. Therefore, $1 \in I$, and hence $I = R_P$. Thus, all proper ideals of R_P are subsets of PR_P.

A ring with a unique maximal ideal is called a *local ring*. The previous example shows that R_P is a local ring with maximal ideal PR_P.

Example 3-2. As a special case of the previous example, if $m \subset R$ is a maximal ideal, then m is prime, and the localization R_m is a local ring with maximal ideal mR_m.

Example 3-3. For an element $f \in R$, the set $S = \{1, f, f^2, ...\}$ is multiplicative. The localization $S^{-1}R$ is denoted by R_f. If R is a finitely generated k algebra, then so is R_f; it is generated by the generators of R and $1/f$.

It is easy to check the following assertions about ideals of R and ideals of $S^{-1}R$.

Proposition 3-4. *Let R be a ring, $S \subset R$ a multiplicative set, $i : R \to S^{-1}R$ the natural map $x \mapsto x/1$, and $J \subset S^{-1}R$ is an ideal.*

(1) *Then $J = i^{-1}(J)S^{-1}R$, and J is finitely generated if $i^{-1}(J)$ is finitely generated.*
(2) *The correspondence $J \mapsto i^{-1}(J)$ is a bijection between prime ideals of $S^{-1}R$ and prime ideals of R that are disjoint from S*

In particular, (1) implies that, if R is Noetherian, then so is $S^{-1}R$.

The localization construction can be extended to modules M over a ring R. Given a multiplicative subset $S \subset R$, we define an equivalence on $M \times S$ by letting $(m, a) \sim (n, b)$ if $s(an - bm) = 0$ for some $s \in S$. The resulting set of equivalence classes is denoted by $S^{-1}M$, and is an $S^{-1}R$ module with addition and scalar multiplication given by

$$(m, a) + (n, b) = (an + bm, ab) \text{ and } a/s \cdot (m, b) = (am, bs).$$

A map of R-modules $\phi : M \to N$ induces a map of $S^{-1}R$ modules, denoted by $S^{-1}\phi$, from $S^{-1}M \to S^{-1}N$ by sending (m, s) to $(\phi(m), s)$. Observe that, if $\phi : M \to N$ and $\psi : N \to L$ are maps of R-modules, then $S^{-1}(\psi \circ \phi) = S^{-1}\psi \circ S^{-1}\phi$. The following important property of this construction is easy to check.

Proposition 3-5. *Let $\phi : M \to N$ be a map of R-modules and $S \subset R$ a multiplicative subset. Then $\operatorname{im} S^{-1}\phi = S^{-1} \operatorname{im} \phi$ and $\ker S^{-1}\phi = S^{-1} \ker \phi$.*

Proposition 3-5 has the following corollary.

Corollary 3-6. *Let N be a submodule of an R-module M, and let $S \subset R$ be a multiplicative set. Then $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$.*

Proof: Let π be the natural projection from M to M/N and consider the induced map $S^{-1}\pi : S^{-1}M \to S^{-1}(M/N)$. Then

$$\operatorname{im} S^{-1}\pi = S^{-1} \operatorname{im} \pi = S^{-1}(M/N) \text{ and } \ker S^{-1}\pi = S^{-1} \ker \pi = S^{-1}N.$$

So $S^{-1}\pi$ is a surjection with kernel $S^{-1}N$. Thus The assertion holds. □

As the first application of localization, we show that certain open subsets of affine varieties naturally have the structure of an affine variety. Let V be a variety, and $f \in k[V]$ a nonzero function. Let $\mathrm{D}(f)$ denote the complement of the closed set $\mathbf{V}(f)$, called the *distinguished* open set defined by f.

Note that $k[V]_f$ is a finitely generated, reduced k-algebra. The map $k[V] \to k[V]_f$ induces a map

$$i : \operatorname{Specm} k[V]_f \to \operatorname{Specm} k[V] = V.$$

It can be seen that i is a homeomorphism onto $\mathrm{D}(f)$. Thus, $D(f)$ can be identified with the affine algebraic variety $\operatorname{Specm} k[V]_f$. In fact, if f is a non-zero-divisor, then we have $\dim \mathrm{D}(f) = \dim V$. Note that the sets $D(f)$ form a basis of the topology of V.

Note that the various open sets $\mathrm{D}(f)$ form a basis of the topology. Indeed, given any point $p \in V$, and any closed set $W \subset V$ such that $p \notin W$, there is an $f \in \mathbf{I}(W)$ such that $f \notin \mathbf{I}(p)$; clearly, $p \in \mathrm{D}(f) \subset (V - W)$.

Localization also provides us with the algebraic tools to define the local ring of a variety at a point.

Definition 3-7. Let V be an affine variety, and $p \in V$ correspond to the maximal ideal $m = \mathbf{I}(p) \in k[V]$. The *local ring* of V at p is the local ring $k[V]_m$.

To simplify notation, we denote the local ring $k[V]_m$ by $k[V]_p$. The local ring at a point is truly a local object.

Proposition 3-8. *Let V be an affine algebraic variety, and f be a nonzero element of $k[V]$. Let p be a point in $\mathrm{D}(f)$. Then the local ring at p as a point of $\mathrm{D}(f)$ is equal to the local ring at p as a point of V.*

Proof: Let A be the algebra of functions of V. Then we have $k[\mathrm{D}(f)] = A_f$. Let m be the maximal ideal of A corresponding to p. Since $p \in \mathrm{D}(f)$, we see that $f \notin m$. The maximal ideal of p in $k[\mathrm{D}(f)]$ is just the maximal ideal mA_f. Since $f \notin m$, it is easy to see that $A_m = (A_f)_{mA_f}$. □

If V and W are birationally equivalent varieties, then their local structure is the same except at a proper closed set. The following proposition makes this precise.

Proposition 3-9. *Let $\phi : V \dashrightarrow W$ be a birational map. Then there exist nonempty open sets $X \subset V$ and $Y \subset W$ such that the map $\phi : X \to Y$ is bijective, and such that, for all $p \in X$, there are isomorphisms*

$$\phi_p^* : k[Y]_{\phi(p)} \xrightarrow{\sim} k[X]_p.$$

Proof: Let $\psi : W \dashrightarrow V$ be the rational inverse of ϕ. Define open subsets X and Y of V and W as $X = \operatorname{dom}(\phi) \bigcap \phi^{-1}(\operatorname{dom}(\psi))$ and $Y = \operatorname{dom}(\psi) \bigcap \psi^{-1}(\operatorname{dom}(\phi))$. Since nonempty open subsets of irreducible affine varieties are dense, the intersection of two nonempty open sets is nonempty. Therefore, X and Y are nonempty open subsets of V and W respectively. It is easy to see that $\phi : X \to Y$ is a bijection with inverse $\psi : Y \to X$.

If p is a point in X, and if $q = \phi(p) \in Y$, then ϕ induces a map $\phi_p^* : k[W]_q \to k[V]_p$ by $f \mapsto f \circ \phi$. Since ψ and ϕ are inverses, the corresponding map ψ_q^* is inverse to ϕ_p^*. Hence, the map $\phi_p^* : k[W]_{\phi(p)} \to k[V]_p$ is an isomorphism for all $p \in V$. □

If V is irreducible, then the local ring $k[V]_p$ is a subring of the field of rational functions $k(V)$. Observe that the multiplicative set $S = k[V] - m$ used to obtain the localization $k[V]_m$ is precisely the set $\{g \in k[V] \mid g(p) \neq 0\}$. Therefore,

$$k[V]_p = \{f/g \in k(V) \mid g(p) \neq 0\}.$$

In other words, $k[V]_p$ is the subring of rational functions of V that are defined at p.

Having defined the local ring at a point, we can define the tangent space.

Definition 3-10. Let V be an affine algebraic variety, p a point in V, and m the unique maximal ideal of $k[V]_p$. The *Zariski cotangent space* T_p^*V of V at p is defined as the vector space m/m^2. The *tangent space* T_pV of V at p is the dual space $\mathrm{Hom}(m/m^2, k)$.

Since the local ring of a point is a local object, so is the tangent space.

Some justification of the definition is in order. Let γ be an element of T_pV. It gives a map $\delta_\gamma : k[V]_p \to k$ by $f \mapsto \gamma[f - f(p)]$ where $[f - f(p)]$ is the class of $f - f(p)$ in m/m^2. The map δ_γ has the following properties:

(1) $\delta_\gamma(f + g) = \delta_\gamma(f) + \delta_\gamma(g)$.
(2) $\delta_\gamma(c) = 0$ for $c \in k$.
(3) $\delta_\gamma(fg) = f(p)\delta_\gamma(g) + g(p)\delta_\gamma(f)$.

A map $k[V]_p \to k$ satisfying the above three properties is called a *k-derivation* of $k[V]_p$. A k-derivation of $k[V]_p$ is the algebraic analogue of the geometric idea of a directional derivative at p. It is easy to prove that $\gamma \mapsto \delta_\gamma$ is a vector space isomorphism between the space of k-derivations of $k[V]_p$ and $\mathrm{Hom}(m/m^2, k)$. Thus the tangent space T_pV can be thought of as the space of directional derivatives of rational functions defined at p.

Before moving on to the discussion of singular and regular points, we prove a few useful facts about local rings. To do so, we need the following ubiquitous theorem from commutative algebra.

Theorem 3-11 (Cayley–Hamilton theorem). *Let R be a ring, $I \subset R$ an ideal, M a finitely generated R-module, and $\phi : M \to M$ a map of R-modules. If $\phi(M) \subset IM$, then ϕ satisfies an equation*

$$\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_0 = 0$$

in $\mathrm{Hom}(M, M)$ where $a_i \in I^{n-i}$.

Proof: Let x be an indeterminate. Give M the structure of an $R[x]$-module by letting $x \cdot m = \phi(m)$ for $m \in M$. Let $m_1, \ldots, m_n$ be generators of M, and say $xm_i = \sum_j a_{ij}m_j$. Let A be the matrix (a_{ij}), and let $\mathbf{I}$ denote the identity matrix. Let m denote the column vector $[m_1, \ldots, m_n]^T$. Then we have $(A - x\mathbf{I})m = 0$. Multiplying by the matrix of cofactors of $A - x\mathbf{I}$ on the left, we see that $\det(A - x\mathbf{I})\mathbf{I} \cdot m = 0$. Therefore, $\det(A - x\mathbf{I})m_j = 0$ for all $1 \leq j \leq n$, and hence $\det(A - x\mathbf{I})$ acts as 0 on M. Set $p(x) = \det(A - x\mathbf{I})$. Then $p(\phi) = 0$ is the required equation. □

We use the Cayley–Hamilton theorem to prove an important fact about modules over local rings.

Theorem 3-12 (Nakayama's lemma). *Let R be a local ring with maximal ideal m. If M is a finitely generated module over R such that $mM = M$, then $M = 0$.*

Proof: Let $\phi : M \to M$ be the identity map on M. Then $\phi(M) \subset mM$. By Cayley-Hamilton theorem, ϕ satisfies an equation

$$\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_0 = 0,$$

where $a_i \in m$. Therefore, $t = a_{n-1} + \cdots + a_0 \in m$. Therefore, $(1 + t)$ acts as zero on M. Since $t \in m$, we see that $(1 + t) \notin m$, and hence it must be a unit. Since $(1 + t)M = 0$, we conclude that $M = 0$. □

We resume the discussion of tangent spaces with the following proposition about their dimension.

Proposition 3-13. *Let p be a point on a variety V. Then the dimension of T_pV is equal to the minimum number of generators of the maximal ideal of $k[V]_p$.*

Proof: Let m be the maximal ideal of $k[V]_p$. Say m is generated by $t_1, \ldots, t_n$. Then it easy to see that the cotangent space m/m^2 is generated by the classes of the t_i. Hence $\dim(T_pV)$ is at most the minimum number of generators of m.

For the other direction, set $n = \dim(T_pV)$, which is the same as $\dim(m/m^2)$. Let m/m^2 be generated by $\overline{t_1}, \ldots, \overline{t_n}$, where $t_i \in m$ and $\overline{t_i}$ are their images in m/m^2. Let N be the ideal of $k[V]_p$ generated by $t_1, \ldots, t_n$ and let M be the quotient m/N. Then we have

$$M/mM = \frac{m/N}{(m+m^2)/N} = \frac{m/m^2}{N/mN} = 0$$

since N/mN is also generated by $t_1, \ldots, t_n$. It follows that $M = mM$ and by Nakayama's lemma, $M = 0$. We conclude that $m = N$; that is, m is generated by $t_1, \ldots, t_n$. □

Having defined the tangent space at a point, we can define singular and regular points.

Definition 3-14. Let V be an affine algebraic variety. A point $p \in V$ is called *regular* if $\dim(V) = \dim_k(T_pV)$. If a point is not regular, it is called *singular*. A variety is called *regular* if all points $p \in V$ are regular.

Before looking at examples, let us find an alternate and easier way to find the tangent space T_pV. The following proposition is a straightforward consequence of Corollary 3-6.

Proposition 3-15. *Let V be a variety, and p a point in V corresponding to the maximal ideal M of $k[V]$. Let m be the maximal ideal of $k[V]_p$. Then $m/m^2 = M/M^2$.*

Since $m/m^2 = M/M^2$, the tangent space T_pV is isomorphic to $\mathrm{Hom}(M/M^2, k)$. Thus we need not pass to the localization when computing the tangent space.

We now describe an effective criterion to determine whether a point $p \in V$ is singular. Let $V \subset k^n$ be an irreducible variety of dimension d, and $I \subset k[x_1, \ldots, x_n]$ be the ideal of V. Let us compute the cotangent space M/M^2 at a point $p = (p_1, \ldots, p_n)$. Say that M is the maximal ideal $\langle \overline{x_1 - p_1}, \ldots, \overline{x_n - p_n} \rangle$ where the image of a polynomial $f \in k[x_1, \ldots, x_n]$ in $k[V]$ is denoted by $\overline{f}$. The quotient M/M^2 is spanned by the linear polynomials $\overline{x_1 - p_1}, \ldots, \overline{x_n - p_n}$. They satisfy an equation $\sum c_i \overline{(x_i - p_i)} = 0$ if and only if there is an $\overline{f} \in M^2$ such that $\sum c_i \overline{(x_i - p_i)} - \overline{f} = 0$ in $k[V]$. In that case, $\sum c_i(x_i - p_i) - f$ is a polynomial in I where $f \in \langle x_1 - p_1, \ldots, x_n - p_n \rangle^2$.

Given any polynomial $f \in I$, we can express f as $f = \sum (x_i - p_i) \frac{\partial f}{\partial x_i}(p) - g$ where $g \in \langle x_i - p_i \rangle^2$. Therefore, the only linear relations among the $\overline{x_i - p_i}$ in M/M^2 are of the form $\sum (\overline{x_i - p_i}) \overline{\frac{\partial f}{\partial x_i}(p)} = 0$ where $f \in I$. Say $I = \langle f_1, \ldots, f_r \rangle$, and define the *Jacobian* at p to be the matrix

$$\mathrm{Jac}_p(f_1, \ldots, f_r)_{ij} = \frac{\partial f_i}{\partial x_j}(p).$$

Let the standard row vector e_i correspond to basis element $x_i - p_i$. Then the subspace

$$\left\{ \sum (x_i - p_i) \frac{\partial f}{\partial x_i}(p) \;\middle|\; f \in \mathbf{I}(V) \right\} \subset \mathrm{Span}\langle x_i - p_i \rangle$$

is the row space of $\mathrm{Jac}_p(f_1, \ldots, f_r)$. Thus this row space is the kernel of the map

$$\mathrm{Span}\langle x_1 - p_1, \ldots, x_n - p_n\rangle \to M/M^2.$$

Therefore, $\dim(M/M^2) = d$ if and only if $\mathrm{rank}\,(\mathrm{Jac}_p(f_1, \ldots, f_r)) = n - d$. It can be proved [**2**, Corollary 10.7] that $\dim_k T_pV \geq \dim V$ for any point $p \in V$. Hence, we always have the inequality

$$\mathrm{rank}\,(\mathrm{Jac}_p(f_1, \ldots, f_r)) \leq n - d.$$

The singular points p are characterized by strict inequality, which is equivalent to vanishing of all the $(n-d)$ by $(n-d)$ minors of $\mathrm{Jac}_p(f_1, \ldots, f_r)$. Thus we have the following theorem.

Theorem 3-16. *Let $V \subset k^n$ be a d-dimensional irreducible affine variety and the ideal of V be $\langle f_1, \ldots, f_r\rangle \subset k[x_1, \ldots, x_n]$. Let I be the ideal generated by the $(n-d)$ by $(n-d)$ minors of the matrix $Jac_x(f_1, \ldots, f_r)$. Then*

(1) $\dim(T_pV) \geq \dim(V)$ *for all* $p \in V$,
(2) $\dim(T_pV) = \dim(V)$ *iff* $J_x(f_1, \ldots, f_r)$ *has rank* $n - d$.,
(3) *The set of singular points of V is equal to $\mathbf{V}(I)$, and is a proper closed subset.*

Example 3-17. Let $f \in k[x, y]$ be an irreducible polynomial, and $C = \mathbf{V}(f)$ be the associated curve in k^2. Let $p = (p_1, p_2) \in C$. Then the cotangent space at p is spanned by $\{\overline{x_1 - p_1},\ \overline{x_2 - p_2}\}$ modulo the single equation

$$\frac{\partial f}{\partial x_1}(p)\overline{(x_1 - p_1)} + \frac{\partial f}{\partial x_2}(p)\overline{(x_2 - p_2)} = 0.$$

Consequently, the tangent space T_pC is equal to $\left\{(x, y) \mid x\frac{\partial f}{\partial x_1}(p) + y\frac{\partial f}{\partial x_2} = 0\right\}$. Note that T_pC is a line if at least one of $\frac{\partial f}{\partial x_1}(p)$ or $\frac{\partial f}{\partial x_2}(p)$ is nonzero. In that case, the point p is regular. If both $\frac{\partial f}{\partial x_1}(p)$ and $\frac{\partial f}{\partial x_2}(p)$ are zero, then the tangent space is 2-dimensional, and p is singular.

Example 3-18. Let $C = \mathbf{V}(f)$ where $f = y^2 - x^3 - x^2$. Then $\frac{\partial f}{\partial x} = 3x^2 - 2x$ and $\frac{\partial f}{\partial y} = 2y$. We see that the only point of C where both partial derivatives vanish is $(0, 0)$. Therefore, $(0, 0)$ is the only singular point of C.

4. Normalization. This section is about the main construction in this paper: the normalization of an affine variety. Normalization of an affine variety corresponds to taking the integral closure of the algebra of functions. The resulting variety typically has milder singularities than the original variety. All the affine algebraic varieties in this section are assumed to be irreducible.

Let us start by recalling the notions related to integrality.

Definition 4-1. Let A be a subring of a ring B. An element $b \in B$ is said to be *integrally dependent on A*, or *integral over A* if there exists a monic polynomial $p(x)$ with coefficients in A such that $p(b) = 0$. The ring B is said to be *integral over A* if all elements of B are integral over A.

It can be proved [**2**, Thm 4.2] that, if $a, b \in B$ are integral over A, then $a + b$ and ab are integral over A.

Definition 4-2. Let A be a subring of B. Then the set

$$\overline{A} = \{b \in B \mid b \text{ is integral over } B\}$$

is called the *integral closure* of A in B. If A is an integral domain, then the *integral closure of* A is the integral closure of A in its field of fractions $\mathrm{Quot}(A)$. A domain is called *integrally closed* if it is equal to its integral closure.

The integral closure of a domain in its fraction field is called its *normalization*. An integrally closed domain is called *normal*.

The Cayley–Hamilton theorem gives a useful criterion for integrality.

Proposition 4-3. *Let R be a Noetherian domain, $I \subset R$ an ideal, and x an element of $\mathrm{Quot}(R)$. If $xI \subset I$, then x is integral over R.*

Proof: Since R is Noetherian, I is a finitely generated R-module. The element x defines a map $\phi : I \to I$ given by $t \mapsto xt$. Since R is a domain, the map ϕ is not identically zero. By the Cayley–Hamilton theorem, ϕ satisfies a monic polynomial over R. Equivalently, x satisfies a monic polynomial over R. □

We would like to define the normalization of a variety V as the variety having the algebra of functions $\overline{k[V]}$. However, for $\overline{k[V]}$ to be the algebra of functions an affine variety, $k[V]$ must be a finitely generated k-algebra. The following theorem of E. Noether says that it always is.

Theorem 4-4. *Let A be an integral domain that is a finitely generated k-algebra. Then $\overline{A}$ is a finite A-module, and a finitely generated k-algebra.*

The theorem is proved using the Noether normalization theorem and Galois theory. See [**2**, Section 13.3] for a proof.

Definition 4-5. Let V be an affine variety. The variety $\mathrm{Specm}(\overline{k[V]})$ is called the *normalization* of V, and is denoted by $\overline{V}$. The variety V is called *normal* if $k[V]$ is integrally closed.

The normalization is birationally equivalent to the original variety.

Proposition 4-6. *Let V be an affine variety, and $i : \overline{V} \to V$ the map induced by the inclusion $i^* : k[V] \to \overline{k[V]}$. Then i is a birational equivalence.*

Proof: It suffices to prove that the map $i^* : k(V) \to k(\overline{V})$ induced by i is an isomorphism. Clearly

$$k(V) = \mathrm{Quot}(k[V]) = \mathrm{Quot}(\overline{k[V]}) = k(\overline{V}).$$

Since $i^* : k[V] \to k[\overline{V}]$ is the standard inclusion, the map i induces the identity map $i^* : k(V) \to k(\overline{V})$. □

The following theorem shows that being integrally closed is a local property.

Theorem 4-7. *Let A be an integral domain. Then the following are equivalent:*

(1) *A is integrally closed.*
(2) *A_p is integrally closed for all prime ideals p.*
(3) *A_m is integrally closed for all maximal ideals m.*

Proof: Assume (1), and let's prove (2). Let $f/g \in \mathrm{Quot}(A_p)$ be integral over A_p. We must prove that $f/g \in A_p$. Let f/g satisfy the monic polynomial equation

$$\left(\frac{f}{g}\right)^n + a_{n-1}\left(\frac{f}{g}\right)^{n-1} + \cdots + a_0 = 0$$

where $a_i \in A_p$. Say $a_i = b_i/c_i$ where $b_i, c_i \in A$ and $c_i \notin p$. Set $c = c_0 \cdots c_{n-1}$, and consider $cf/g \in \mathrm{Quot}(A)$. Obviously, cf/g satisfies the equation

$$\left(\frac{cf}{g}\right)^n + \left(\frac{cf}{g}\right)^{n-1} ca_{n-1} + \cdots + \left(\frac{cf}{g}\right) c^{n-1}a_1 + c^n a_0 = 0.$$

Observe that the all the coefficients $c^i a_{n-i}$ are elements of A, and therefore, cf/g is integral over A. Since A is integrally closed, cf/g belongs to A. As $c \notin p$, it is a unit of A_p, and therefore, $f/g = (1/c) \cdot (cf/g)$ is an element of A_p. Thus (2) holds.

Since all maximal ideals are prime, (2) trivially implies (3).

To show that (3) implies (1), suppose A_m is integrally closed for all maximal ideals $m \subset A$, and let the element $h \in \mathrm{Quot}(A)$ be integral over A; say

$$h^n + a_{n-1}h^{n-1} + \cdots + a_0 = 0 \tag{4-1}$$

where $a_i \in A$. Let I denote the ideal $\{x \in A \mid xh \in R\}$.

Let us show that I cannot be a proper ideal of A. Suppose $I \neq A$. Then $I \subset m$ for some maximal ideal m. Observe that (4-1) is an equation of integral dependence for h over A_m. Since A_m is integrally closed, $h \in A_m$. So $h = t/s$ for some $t \in A$ and $s \notin m$. Therefore, $sh \in R$ but $s \notin I$, in contradiction to the choice of I. Thus $I = A$, whence $h = 1 \cdot h \in A$. Thus (1) holds, and the proof is complete. □

The following theorem illustrates the importance of normalization in resolution of singularities.

Theorem 4-8. *Let V be a normal variety. Then the set of singular points of V is a closed set of dimension at most* $\dim V - 2$.

Equivalently, any subset of a normal variety of dimension $\dim V - 1$ has a regular point. This result is sometimes expressed by saying that normal varieties are regular in codimension 1.

As the theorem says, the singularities of a normal variety are sparse. In particular, if V is a curve, then $\overline{V}$ has no singular points.

Theorem 4-8 is proved in Section 6, using the theory developed in the next section.

5. Discrete Valuation Rings. To prove Theorem 4-8, we study the local ring of a variety V at an irreducible closed set W. Let P be the ideal of W in $k[V]$. Since $k[V]/P$ is an integral domain, P is prime. In this section, we prove that, if V is normal and W has codimension 1, then the local ring $k[V]_p$ is a discrete valuation ring. We begin by defining the notion of a discrete valuation.

Definition 5-1. Let F be a field. A *discrete valuation* on F is a surjective map $\phi : F \to \mathbb{Z} \cup \{\infty\}$ such that

(1) $\phi(a) = \infty$ if and only if $a = 0$,
(2) $\phi(ab) = \phi(a) + \phi(b)$,

(3) $\phi(a+b) \geq \min(\phi(a), \phi(b))$.

Proposition 5-2. *Let ϕ be a discrete valuation on a field, $R = \{x \in F \mid \phi(x) \geq 0\}$ and $m = \{x \in F \mid \phi(x) > 0\}$. Then*

(1) *R is a local ring with maximal ideal m.*
(2) *m is principal.*
(3) $\text{Quot}(R) = F$.

Proof: Clearly, 0 is an element of R. Note that $\phi(1) = \phi(1) + \phi(1)$, therefore $\phi(1) = 0$. Hence, 1 is an element of R. If $\phi(a) \geq 0$ and $\phi(b) \geq 0$, then

$$\phi(a+b) \geq \min(\phi(a), \phi(b)) \geq 0 \text{ and } \phi(ab) = \phi(a) + \phi(b) \geq 0$$

by Definition 5-1. Thus R is a ring.

It is easy to see that m is a proper ideal of R. To prove that R is local with maximal ideal m, we must prove that there is no proper ideal $I \not\subset m$. It suffices to prove that every $a \in R - m$ is a unit of R. If $a \in R$ and $a \notin m$, then $\phi(a) = 0$, and therefore, $\phi(a^{-1}) = \phi(1) - \phi(a) = 0$. Hence $a^{-1} \in R$, and so a is a unit.

Since ϕ is surjective, there exists $t \in F$ be such that $\phi(t) = 1$. For any $x \in m$, $\phi(x/t) = \phi(x) - \phi(t) = \phi(x) - 1 \geq 0$. Therefore, $x/t \in R$, and hence $x \in \langle t \rangle$. It follows that $m = \langle t \rangle$.

To see that $\text{Quot}(R) = F$, let $a \in F$. Say $\phi(a) = n \in \mathbb{Z}$. Choose $m \geq 0$ such that $m + n \geq 0$. Then $\phi(at^m) = \phi(a) + m\phi(t) \geq 0$. Therefore, $a = x/y$ where $x = at^m \in R$ and $y = t^m \in R$. □

Definition 5-3. Let ϕ be a discrete valuation on a field F. Then the local ring $R = \{x \in F \mid \phi(x) \geq 0\}$ is called a *discrete valuation ring* or a DVR.

To give a simple characterization of DVRs, we need the Krull intersection theorem. First, we prove a lemma.

Lemma 5-4. *Let R be a Noetherian ring, and I and m be two ideals of R. Then there exists an integer s and an ideal $J \supset m^s$ such that $mI = J \cap I$.*

Proof: We use the existence of primary decomposition of an ideal of a Noetherian ring. Let $mI = \cap_i q_i$ be the primary decomposition of mI. Let q_i' be the primary ideals among q_i whose associated prime ideals contain m and q_i'' be those whose associated prime ideals do not contain m.

Since $m \subset \sqrt{q_i'}$, there exists an integer s_i such that $m^{s_i} \subset q_i'$. Set $J = \cap q_i'$ and $s = \max\{s_i\}$. Then $m^s \subset J$. It remains to prove that $J \cap I = mI$.

Set $K = \cap_i q_i''$. Since $mI = K \cap J$, the inclusion $mI \subset J \cap I$ is clear. For the reverse inclusion, we prove that $I \subset K$. Given i, fix $t \in m$ such that $t \notin \sqrt{q_i''}$; such t exists by the choice of q_i''. Then, for all $x \in I$, we have $xt \in mI$, but $t \notin \sqrt{q_i''}$. Since q_i'' is primary, $x \in q_i''$. It follows that $I \subset K$, and hence $J \cap I \subset J \cap K = mI$. □

We use this lemma to prove the Krull intersection theorem.

Theorem 5-5 (Krull Intersection Theorem). *Let R be a Noetherian local ring with maximal ideal m. Then $\cap_i m^i = 0$.*

Proof: Set $I = \cap_i m^i$. By the previous lemma, $mI \supset I \cap m^s$ for some s. But $I \cap m^s = I$ for any s. Hence $mI = I$. Therefore, by Nakayama's lemma, $I = 0$. □

We are now ready to prove the following assertion.

Proposition 5-6. *Let R be a Noetherian local domain. Then R is a* DVR *if and only if the maximal ideal of R is principal.*

Proof: Suppose R is a DVR. Then its maximal ideal is principal by Proposition 5-2.

For the converse, let $\langle t \rangle$ be the maximal ideal of R. We first show that any nonzero element $x \in R$ can be uniquely written as $x = zt^n$ where $z \in R$ is a unit and $n \geq 0$.

First we show uniqueness. If $ut^n = vt^m$ where u, v are units and $m \geq n \geq 0$, then $u = vt^{m-n}$ is a unit. Therefore, t^{m-n} is a unit, and hence $t^{m-n} \notin \langle t \rangle$. It follows that $m - n = 0$, and therefore $u = v$.

To prove existence, let $x \neq 0$ and let $n \geq 0$ be the maximum number such that $x \in \langle t^n \rangle$; the existence of n follows from Theorem 5-5. Say $x = zt^n$ where $z \in R$. Note that $z \notin \langle t \rangle$ as $x \notin \langle t^{n+1} \rangle$. Since $\langle t \rangle$ is the unique maximal ideal of R, it follows that z is a unit.

For $x \in R$, let $x = zt^n$ where z is a unit. Set $\phi(x) = n$ and $\phi(0) = \infty$. By uniqueness, ϕ is well defined. Extend ϕ to $\mathrm{Quot}(R)$ by setting $\phi(a/b) = \phi(a) - \phi(b)$. It is easy to check that ϕ is a discrete valuation on $\mathrm{Quot}(R)$ and $R = \{x \mid \phi(x) \geq 0\}$. □

6. Normalization and Resolution. In this section, we prove Theorem 4-8, that normal varieties are regular in codimension one, and so solve the problem of resolution of singularities for curves. At the end of this section, we prove the converse of the theorem for curves.

In proving Theorem 4-8, the major step is proving that the local ring of a normal variety at a closed set of codimension 1 is a DVR. We need a few intermediate results for this purpose.

First, we prove two results about radical ideals.

Lemma 6-1. *Let R be an integral domain, and $I \subset R$ an ideal. Then the radical of I is intersection of all prime ideals containing I.*

Proof: Let $J = \bigcap_{I \subset P} P$ be the intersection of all prime ideals containing I. If $g \in \sqrt{I}$, then $g^n \in I$ for some n, and hence, $g^n \in P$ for all prime ideals containing P. By the primality of P, we conclude that $g \in P$; hence $g \in J$.

Conversely, let $g \notin \sqrt{I}$. Let $i : R \to R_f$ denote the natural map $x \mapsto x/1$. Since $I \cap S = \varnothing$, the ideal IR_f is a proper ideal in R_f, and hence it is contained in a maximal ideal M. Then $i^{-1}(M) \subset R$ is a prime ideal containing I that does not contain f. Hence $f \notin J$. The proof is complete. □

Lemma 6-2. *Let I be an ideal, and $J = \sqrt{I}$. If J is finitely generated then, there exists $N \geq 0$ such that $J^N \subset I$.*

Proof: Say $J = \langle f_1, \ldots, f_k \rangle$. Since $f_i \in \sqrt{I}$, there exist n_i such that $f_i^{n_i} \in I$. Set $n = \max(n_i)$, so that $f_i^n \in I$ for all $1 \leq i \leq k$. If we set $N = nk$, then every term in the expansion of $(\sum h_i f_i)^N$ is divisible by f_i^n for some i. But $J = \langle f_1, \ldots, f_n \rangle$. So for any $x \in J$, the power x^N belongs to I. Hence $J^N \subset I$. □

We are now ready to prove that, for a normal variety, the local ring at an irreducible closed subset of codimension 1 is a DVR. Recall that the height of a prime ideal P in a ring R is the maximum n such that we have a chain of distinct prime ideals

$$P_0 \subset \cdots \subset P_n = P.$$

Theorem 6-3. *Let A be a Noetherian integrally closed domain, and P a height 1 prime of A. Then the local ring A_P is a* DVR.

Proof: We know from Proposition 3-4 that the map $I \mapsto IA_P$ is a bijection from the prime ideals of A contained in P to the prime ideals of A_P. Since P has height 1 and 0 is a prime ideal, P does not contain any nonzero prime ideal. Hence the only prime ideal of A_P is the maximal ideal PA_P. Denote this maximal ideal by m.

To prove that A is a DVR, choose an element $a \in m$, and set $I = \sqrt{\langle a \rangle}$. Since m is the only nonzero prime ideal of A, we get that $I = m$ by Lemma 6-1. Since A is Noetherian, m is finitely generated. By Lemma 6-2, $m^N \subset \langle a \rangle$ for some integer $N \geq 0$.

Let $k \geq 0$ be smallest such that $m^k \not\subset \langle a \rangle$ and $m^{k+1} \subset \langle a \rangle$. Choose $b \in m^k - \langle a \rangle$, and set $t = a/b$. Since $a \in \langle a \rangle$ and $b \notin \langle a \rangle$, there is no $c \in A$ such that $b = ca$. In other words, t^{-1} is an element of $\mathrm{Quot}(A)$ that is not in A. Since A is integrally closed, t^{-1} is not integral over A. Since m is a finitely generated A module, we conclude by Proposition 4-3 that $t^{-1}m \not\subset m$. On the other hand,

$$t^{-1}m = (1/a) \cdot bm \subset (1/a) \cdot m^{k+1} \subset (1/a) \langle a \rangle \subset A.$$

Since $t^{-1}m$ is an ideal of A and $t^{-1}m \not\subset m$, we conclude that $t^{-1}m = A$. Therefore, $t \in m$, and $m = tA$. Thus A is a Noetherian local integral domain whose maximal ideal is principal. Proposition 5-6 shows that A is a DVR. □

We now have the tools to prove Theorem 4-8.

Theorem 6-4. *Let V be a normal variety. Then the set of singular points of V is a closed set of dimension at most* $\dim V - 2$.

Proof: Set $n = \dim V$. We know from Theorem 3-16 that the set of singular points of V is a proper closed subset of V. To show that it has dimension at most $n - 2$, it in enough to prove that any irreducible closed set of V of dimension $n - 1$ contains a regular point.

Let A be the algebra of functions of V, and $W \subset V$ be an irreducible closed subset of dimension $n - 1$. Let P be the ideal of W in A, which is a prime ideal since W is irreducible. Since W has dimension $n - 1$, there is no irreducible proper closed subset W' of V such that $W \subsetneq W' \subsetneq V$. It follows that P does not properly contain any nonzero prime ideal. We apply Theorem 6-3 to conclude that A_P is a DVR.

Since A_P is a DVR, the maximal ideal PA_P is principal. Let P be generated by u/v where $u \in A$ and $v \notin P$. Let P be generated by $f_1, \ldots, f_r$. Then we can find $u_i \in A$ and $v_i \in A - P$ such that $f_i = uu_i/vv_i$. Let $g = vv_1 \cdots v_r$, and let us restrict our attention to the open set $\mathrm{D}(g)$. The ideal of $W \cap \mathrm{D}(g)$ in A_g is the prime ideal PA_g, which is a proper prime ideal since $g \notin P$. Moreover, see that PA_g is a principal ideal generated by u.

Let R denote the ring A_g, and denote the algebra of functions R/PR of $W \cap \mathrm{D}(g)$ by B. By Theorem 3-16, $W \cap \mathrm{D}(g)$ contains a regular point of $W \cap \mathrm{D}(g)$. Let b be the maximal ideal of q in B. Then the maximal ideal a of q in A_g is the preimage of a under the projection $R \to R/PR$. Note that $\dim W \cap \mathrm{D}(g) = n - 1$. Since q is a regular point of $W \cap \mathrm{D}(g)$, we conclude by Proposition 3-13 that the maximal ideal bB_b is generated by $n - 1$ elements. Note that the maximal ideal aR_a is the preimage of bB_b under the projection map $R_a \to B_b = R_a/PR_a$. Since PR_a is generated by one element u, we conclude that aR_a is generated by the preimages of the generators of bB_b along with u. In conclusion, the maximal ideal the local ring of q as a point of $\mathrm{D}(g)$ is generated by n elements. It follows by Proposition 3-13 that $T_q \mathrm{D}(g)$, which is the same as $T_q V$, has dimension n. Thus q is a regular point of V. □

As a corollary, we have the following statement.

Corollary 6-5. *A normal curve is regular.*

Proof: A point on a curve is an irreducible variety of codimension one. From Theorem 6-4, it follows that any point on a normal curve is regular. □

A weak converse of Theorem 6-4 is also true. Namely, if a variety is regular, then it is normal. Although we do not prove this general statement, we have the tools to prove it for curves.

We begin with an algebraic characterization of regular points on curves.

Corollary 6-6. *A point on a curve is regular iff its local ring is a* DVR*.*

Proof: A point is regular iff the tangent space is 1-dimensional by Theorem 3-16. By Proposition 3-13, it is equivalent to say that the maximal ideal m of the local ring R at that point is principal. By Proposition 5-6, m is principal iff R is a DVR. □

Next, we prove that UFDs are integrally closed.

Proposition 6-7. *A unique factorization domain is integrally closed.*

Proof: Let R be the unique factorization domain, and $f/g \in \mathrm{Quot}(R)$ integral over R. Choose f, g with no common divisor. Then

$$\left(\frac{f}{g}\right)^n + a_{n-1}\left(\frac{f}{g}\right)^{n-1} + \cdots + a_0 = 0$$

for some $a_i \in R$. Therefore, $f^n + a_{n-1}fg^{n-1} + \cdots + a_0 g^n = 0$. It follows that g divides f^n. However, no prime dividing g can divide f^n since f and g have no common divisor. Hence g is a unit, and so $f/g \in R$. Since f/g is arbitrary, R is integrally closed. □

We now prove that all DVRs are UFDs.

Proposition 6-8. *Let R be a discrete valuation ring. Then there exists $t \in R$ such that every nonzero element $x \in R$ can be uniquely written as $x = zt^n$ for some $n \geq 0$. In particular R is a UFD.*

Proof: The assertion is a consequence of the proof of Proposition 5-6. □

Theorem 6-9. *A regular curve is normal.*

Proof: Let C be a regular curve, and p be a point on C. Since C is regular, the local ring of C at p is a DVR by Corollary 6-6. By Proposition 6-8, the local ring is integrally closed. Since p is arbitrary, Theorem 4-7 implies that C is normal. □

Example 6-10. Consider the curve $C = \mathbf{V}(f) \subset k^2$ where $f = y^2 - x^3$. It has a singularity at $(0, 0)$. Consequently, the ring $k[C]$ is not integrally closed. Indeed, the element $y/x \in k(C)$ satisfies the monic equation $t^2 - x = 0$, but is not an element of $k[C]$.

Let us adjoin y/x to $k[C]$ to get $R = k[C][y/x] \subset k(C)$. We claim that $\phi : k[t] \to R$ given by $t \mapsto y/x$ is an isomorphism. Observe that $x = \phi(t^2)$ and $y = \phi(t^3)$; therefore, ϕ is surjective. The kernel of ϕ is a prime ideal in $k[t]$ since R is a domain. Since $k[t]$ is a principal ideal domain and k is algebraically closed, the only prime ideals of $k[t]$ are of the form $\langle t - a \rangle$. Since C is not a point, $R \ncong k[t]/\langle t - a \rangle$. Therefore, $\ker \phi = 0$; hence, ϕ is injective. Thus $\overline{k[C]} \cong k[t]$.

Since $k[t]$ is normal, the normalization of C is k^1. The proof above shows that the parameterization $k^1 \to C$ mapping $t \mapsto (t^2, t^3)$ is a birational equivalence between k^1 and C.

7. Normalization algorithm. Although we have, in principle, solved the problem of resolution of singularities for curves, given a curve C we do not yet have a method to obtain its normalization. What we need is an algorithm to compute the integral closure of a ring that has been given explicitly in terms of generators and relations. This section describes such an algorithm due to Theo De Jong [**7**].

Before introducing the algorithm, we prove a criterion for normality on which the algorithm is based.

Let $\langle f_1, \ldots, f_r \rangle$ be a prime ideal of $k[x_1, \ldots, x_n]$, and $V = \mathbf{V}(f_1, \ldots, f_r)$ be the corresponding affine variety. Let R be the algebra $k[V]$. We define the nonnormal locus of V to be the set

$$V^n = \{p \in V \mid R_p \text{ is not normal.}\}.$$

Observe that V^n is the complement of the largest open set on which the rational map $V \to \overline{V}$ is defined. In particular, V^n is a subset of the set V^s of singular points of V, and hence it is a proper closed subset of V.

Set $I = \mathbf{I}(V^s)$. Then I is a nonzero radical ideal of $k[V]$.

We have the following inclusions of rings:

$$R \subset \mathrm{Hom}_R(I, I) \subset \overline{R}. \tag{7-1}$$

The first inclusion maps $\alpha \in R$ to the map $I \to I$ given by multiplication by α. The second inclusion maps $\phi \in \mathrm{Hom}_R(I, I)$ to $\alpha = \phi(f)/f \in \mathrm{Quot}(R)$ where $f \in I$ is any nonzero element. Then $\phi(g) = \alpha g$ for any $g \in I$ since $\phi(g)f = \phi(fg) = \phi(f)g$; therefore, α does not depend on the choice of f. Since ϕ is a map from I to I, we see that $\alpha I \subset I$. Since R is Noetherian, I is a finitely generated module over R. Proposition 4-3 implies that α belongs to $\overline{R}$. It is clear that both maps in (7-1) are indeed injective.

Moreover, if I, J are nonzero ideals, then we can identify $\mathrm{Hom}_R(I, J)$ with a subset of $\mathrm{Quot}(R)$ by the injection $\phi \mapsto \phi(f)/f$ for a fixed nonzero $f \in I$. Since $\phi(g)f = \phi(f)g$, this definition is independent of the choice of f.

We have the inclusions in (7-1) for an arbitrary ideal I. We choose the particular ideal $I = \mathbf{I}(V^s)$ because of the following lemma.

Lemma 7-1. *In the above setup, $Hom_R(I, I) = Hom_R(I, R) \cap \overline{R}$.*

Proof: Clearly $\mathrm{Hom}_R(I, I) \subset \mathrm{Hom}_R(I, R)$. Hence (7-1) yields

$$\mathrm{Hom}_R(I, I) \subset \mathrm{Hom}_R(I, R) \cap \overline{R}.$$

We need to prove the opposite inclusion.

Let $h \in \overline{R} \cap \mathrm{Hom}_R(I, R)$. We must show that the map $x \mapsto hx$ maps I to I. Since $h \in \overline{R}$, it satisfies an equation

$$h^n = a_0 + a_1 h + \ldots a_{n-1} h^{n-1}$$

where $a_i \in R$ and $n \geq 1$. Therefore, for any $f \in I$, we have

$$(hf)^n = a_0 f^n + a_1 (hf) f^{n-1} + \cdots + a_{n-1} (hf)^{n-1} f \in I,$$

because $hf \in R$ and $f \in I$. But $I = \mathbf{I}(V^s)$ is radical; hence, $hf \in I$. Since $f \in I$ is arbitrary, the proof is complete. □

Using the above theorem, we obtain the following criterion for the normality of R.

Theorem 7-2. *Under the above conditions, $R = Hom_R(I, I)$ iff R is normal.*

Proof: If R is normal, then $R = \overline{R}$, and $R = \mathrm{Hom}_R(I, I)$ owing to the inclusions in (7-1).

Conversely, assume $R = \mathrm{Hom}_R(I, I)$, and let h be integral over R. We need to prove that $h \in R$. Set $P(h) = \{p \in V \mid h \notin R_p\}$. Let $(R : h)$ denote the quotient ideal $\{g \in R \mid gh \in R\}$. Let us prove that

$$P(h) = \mathbf{V}(R : h).$$

It is easy to check that $(R : h)$ is indeed an ideal of R. Note that h is not defined at p if and only if, for every f, g such that $h = f/g$, we have $g(p) = 0$. Moreover, the ideal $(R : h)$ is equal to the set $\{g \in R \mid h = f/g \text{ for some } f\}$. It follows that h is not defined at p iff $g(p) = 0$ for all $g \in (R : h)$.

Since h is integral over R, it is integral over R_m for all maximal ideals m, because the equation of integral dependence of h over R is an equation of integral dependence over R_m for all maximal ideals m. Hence $h \notin R_p$ only if R_p is not integrally closed. Therefore, we have the inclusion $P(h) \subset V^n$.

Let $J = \mathbf{I}(P(h))$ be the ideal of $P(h)$. By Hilbert's Nullstellensatz, $J = \sqrt{R : h}$. Now, Lemma 6-2 implies that there exists $c \geq 0$ such that $J^c \subset (R : h)$; hence, $hJ^c \subset R$. Since $P(h) \subset V^n \subset V^s$, we have $I = \mathbf{I}(V^s) \subset \mathbf{I}(P(h)) = J$. Therefore, $hI^c \subset hJ^c \subset R$. Let $d > 0$ be the minimal number with the property that $hI^d \subset R$. Let us prove that $d = 1$.

Suppose $d > 1$. Then there exists $a \in I^{d-1}$ such that $ha \notin R$. By assumption $R = \mathrm{Hom}_R(I, I)$, and by Lemma 7-1, $\mathrm{Hom}_R(I, I) = \mathrm{Hom}_R(I, R) \cap \overline{R}$. Therefore, we have $R = \{g \in \overline{R} \mid gI \subset R\}$. Since $h \in \overline{R}$ and $a \in R$, we have $ha \in \overline{R}$. Moreover, $haI \subset R$ since $a \in I^{d-1}$ and $hI^d \subset R$. We conclude that $ha \in R$, in contradiction to the choice of a.

Therefore, $d = 1$, and hence $hI \subset R$. As $h \in \overline{R}$, we have $h \in \overline{R} \cap \mathrm{Hom}_R(I, R) = R$. The proof is now complete. □

The theorem yields an algorithm to compute $\overline{R}$, as described in Algorithm 1.

Algorithm 1 Normalization algorithm.

Input: An integral domain $R = k[x_1, \ldots, x_n]/\langle f_1, \ldots, f_r \rangle$.
Output: The integral closure $\overline{R}$.

```
1: Let V = V(f_1, ..., f_r).
2: Determine I such that V_V(I) = V^s.
3: Let I := √I.
4: while R ≠ Hom_R(I, I) do
5:     R := Hom_R(I, I).
6: end while
7: return R.
```

Since R is a finitely generated k-algebra, the algorithm terminates. For, if we denote by R_i the value of R in Step 4 of the ith iteration of the *while* loop, we have an increasing sequence of R-submodules of $\overline{R}$

$$R = R_0 \subset R_1 \subset R_2 \cdots \subset \overline{R}.$$

Since R is Noetherian and $\overline{R}$ is a finitely generated module over R by Theorem 4-4, the sequence stabilizes and the algorithm terminates.

When the algorithm terminates, we have $R = \mathrm{Hom}_R(I, I)$. By Theorem 7-2, R is integrally closed, and hence $R = \overline{R}$. Thus the algorithm is correct.

A few steps in the algorithm need explanation. The generators for the ideal I can be computed from the polynomials f_i using the Jacobian criterion for singularity from Theorem 3-16. The generators for the radical ideal $\sqrt{I}$ in Step 3 can be computed algorithmically, [**5**, Algorithm 4.5.3].

Once we have generators for I, we must calculate the ring $\mathrm{Hom}_R(I, I)$. To be able to do so, we must present I as a finitely generated module over R. In other words, we must find an $m \times k$ matrix M with coefficients R such that $I \cong R^k/MR^m$. Say $I = \langle g_1, \ldots, g_k \rangle$; then we have a surjective map $\phi : R^k \to I$ given by $\phi : e_i \mapsto g_i$ where $e_1, \ldots, e_k$ is the standard basis of R^k. The kernel of this map is

$$\ker \phi = \left\{ (h_1, \ldots, h_k) \mid \sum h_i g_i = 0 \right\} \subset R^k.$$

This submodule of R^k is called the *syzygy module* of $\{g_1, \ldots, g_k\}$.

Expressing $\ker \phi$ as MR^m for some matrix M is equivalent to finding generators for this syzygy module. Finding M can be done algorithmically; see [**5**, Algorithm 2.5.4]. Once we have a presentation $R^k/MR^m \cong I$, the R-module $\mathrm{Hom}_R(I, I)$ can be computed using the algorithm in [**5**, Example 2.1.26]. Once we have a presentation for $\mathrm{Hom}_R(I, I)$, then the injection $i : R \to \mathrm{Hom}_R(I, I)$ can be tested for surjectivity by computing the cokernel of i. Hence the condition in Step 4 can be tested.

To continue the algorithm, we need to compute the ring structure of $\mathrm{Hom}_R(I, I)$. Let $\mathrm{Hom}_R(I, I) \cong R^k/MR^m$ where M is an $m \times k$ matrix, which can be calculated as described before. Let $u_1, \ldots, u_k$ be the generators of $\mathrm{Hom}_R(I, I)$. They satisfy m linear relations given by

$$\sum_{i=1}^{k} M_{ji} u_i = 0 \tag{7-2}$$

for $1 \leq j \leq m$. Next, we compute the compositions $u_i u_j$, which are elements of $\mathrm{Hom}_R(I, I)$, and express them as follows:

$$u_i u_j = \sum_{l=1}^{k} b_{ij}^l u_l. \tag{7-3}$$

Let J be the ideal generated by the linear polynomials $\sum_i M_{ji} y_i$ in (7-2) and the quadratic polynomials $y_i y_j - \sum_l b_{ij}^l y_l$ in (7-3). It is not hard to prove that the map $R[y_1, \ldots, y_k]/J \to \mathrm{Hom}_R(I, I)$ sending $y_i \mapsto u_i$ is an isomorphism. Since we have a presentation $R = k[x_1, \ldots, x_n]/I$, we get the ring structure of $\mathrm{Hom}_R(I, I)$.

$$k[x_1, \ldots, x_n, y_1, \ldots, y_k]/\langle I, J \rangle \cong \mathrm{Hom}_R(I, I).$$

Now that we have the ring structure of $\mathrm{Hom}_R(I, I)$, we can continue the algorithm.

The algorithm is implemented in the computer algebra package Singular in the library "normal.lib."

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