Homework 3

Algebraic Geometry 2021

1. Given 3 distinct points $p, q, r \in \mathbb{P}^1$, prove that there exists a unique element of $A \in \mathrm{PGL}_2(k)$ such that
$$A[0:1] = p, \quad A[1:0] = q, \quad A[1:1] = r.$$
What is the analogous statement for $\mathbb{P}^n$? (Just write the statement, not the proof.)

2. Let Y be a separated variety, and $U \subset X$ a dense subset. If two continuous maps $f, g\colon X \to Y$ agree on U, then show that they must agree on X. In other words, if Y is separated, then a continuous map $U \to Y$ has at most one extension $X \to Y$.

3. Prove that any rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ extends to a regular map.

4. Consider $\mathbb{A}^2$ as an open subset of $\mathbb{P}^2$ in the standard way:
$$\mathbb{A}^2 = \{[x : y : 1] \mid x, y \in k\}.$$
For $f \in k[x, y]$, consider $C = V(f) \subset \mathbb{A}^2$. The closure of C in $\mathbb{P}^2$ has the form $V(F)$ for some homogeneous polynomial $F \in k[X, Y, Z]$. Describe how to obtain F from f and prove that $V(F)$ is indeed the closure of $V(f)$.