(Stable) rationality is not deformation invariant

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Abstract

We prove the existence of a family $\mathcal{X} \to B$ of smooth projective fourfolds, such that the very general fiber $\mathcal{X}_t$ is not stably rational (a fortiori not rational), but some special fibers $\mathcal{X}_t$ are rational (a fortiori stably rational).

0 Introduction

Recall that a variety X is said to be stably rational if $X \times \mathbb{P}^r$ is rational for some r. Artin-Mumford [1] and later on Colliot-Thélène -Ojanguren [7] constructed rather sophisticated examples of unirational nonrational varieties. In contrast, recent work building on [17] (see [8], [3], [4], [15], [14]) established the stable irrationality of many classical types of Fano varieties, thus giving a plethora of negative answers to the stable version of Lüroth problem. A very important open problem concerning stable rationality or rationality is the deformation invariance of the property, which can be stated in the following form.

Question 0.1. *Given a smooth projective morphism $\phi : \mathcal{X} \to B$, is the property of rationality (or stable rationality) of the fiber $\mathcal{X}_t$ open on the base B?*

Remark 0.2. It is clear that disproving deformation invariance of stable rationality also disproves deformation invariance of rationality, but only after taking the product of the original family with $\mathbb{P}^r$, so up to changing the dimension.

Remark 0.3. As we work over $\mathbb{C}$, we can formulate the question for the Euclidean topology or the Zariski topology on B. However both questions are equivalent by the results of [10], where De Fernex and Fusi show that rationality (hence stable rationality) of the fiber is satisfied exactly along a countable union of locally closed algebraic subsets of the base.

Note that if we remove the smoothness assumption, we get a negative answer to Question 0.1 once we allow the simplest singularities: the smooth cubic threefolds are not rational by Clemens-Griffiths [5], but the nodal ones are rational. It is however not clear that this fact gives a strong argument in favour of noninvariance under deformation: An elliptic curve which acquires a node also becomes rational. One can argue that the major issue here is the change in the Kodaira dimension, but it could be as well the change in the topology, which is also present and very important in higher dimension for rationality questions.

The following result provides a negative answer to Question 0.1 for smooth families:

Theorem 0.4. *Let $W \subset \mathbb{P}^4$ be a general quartic threefold singular along a line Δ, and let $\widetilde{W} \subset \widetilde{\mathbb{P}^4}$ be the proper transform of W under the blow-up of Δ. Let $r : Y \to \widetilde{\mathbb{P}^4}$ be the double cover ramified along $\widetilde{W}$. Then*

(i) Y is smooth.

(ii) If there exists a plane $P \subset \mathbb{P}^4$ everywhere tangent to W but not meeting Δ, Y is rational. Such a W exists with $\widetilde{W}$, hence Y, smooth.

(iii) The very general Y as above is not stably rational.

It is also unknown if rationality (or stable rationality) is specialization invariant for families of smooth varieties. This question has been studied in [10] and solved positively in dimension 3 (de Fernex and Fusi show that one can even allow klt singular fibers in dimension 3 but Totaro [16] shows that starting from dimension 5, this becomes wrong even if one only allows fibers with terminal singularities). By the above mentioned result of de Fernex-Fusi, this second question can be stated by asking whether the locally closed subsets are in fact closed. We will say nothing about this last question, but we will prove that countability is necessary:

Theorem 0.5. *The family $\mathcal{Y} \to B$ of smooth projective fourfolds described in Theorem 0.4 has the property that the very general fiber is not stably rational and the set $\{t \in B, \mathcal{Y}_t \text{ is rational}\}$ is dense in B for the usual topology. In particular, rationality of the fibers is not a closed property on the base.*

If we restrict to rationality, this is the picture that is generally conjectured to hold for cubic fourfolds, although nobody up to now has been able to prove that the very general cubic fourfold is not rational.

The family of varieties we will consider for the proof of both theorems is a family of quartic fourfolds admitting a quadric bundle structure over $\mathbb{P}^2$. That some specializations are rational is easy to check since this happens once the family has a rational section. The main problem is to prove the stable irrationality of the very general fiber. We will use the method which has been introduced in [17], improved in [8], and used by many other authors in various contexts, including Beauville [3]. In fact, our examples are obtained by an adaptation of Beauville's construction. This method is a degeneration method and uses the following necessary criterion for stable rationality (see [2], [18]) :

Lemma 0.6. *A stably rational variety admits a Chow decomposition of the diagonal*

$$\Delta_X = X \times x + Z \text{ in } \mathrm{CH}^n(X \times X),$$

where the cycle Z is supported on $D \times X$ for some proper closed algebraic subset $D \subset X$. This implies that the Artin-Mumford invariant $\mathrm{Tors}(H^3(X, \mathbb{Z}))$ vanishes (see [18, Theorem 1.12]).

While the Artin-Mumford invariant $\mathrm{Tors}(H^3(X, \mathbb{Z}))$ is very unstable under smoothification, as one can see in the case of the Artin-Mumford nodal quartic double solids X_0, whose desingularizations $\widetilde{X}_0$ have a nontrivial 2-torsion element in $H^3(\widetilde{X}_0, \mathbb{Z})$ while a general smooth quartic double solid X has no torsion in $H^3(X, \mathbb{Z})$, the criterion given by Lemma 0.6 is robust, as shows Theorem 0.7: Let $\phi : \mathcal{X} \to B$ be a flat projective morphism, where B is smooth. Assume the general fiber is smooth and the special fiber $\mathcal{X}_0$ is irreducible and has a desingularization $\sigma : \widetilde{\mathcal{X}_0} \to \mathcal{X}_0$ with the property that for any subvariety $S \subset \mathcal{X}_0$, the generic fiber of the morphism $\sigma^{-1}(S) \to S$ is smooth and rational over the function field of S.

Theorem 0.7. *With the assumptions above, if $\widetilde{\mathcal{X}_0}$ does not admit a Chow decomposition of the diagonal, (for example if $\widetilde{\mathcal{X}_0}$ has a nontrivial Artin-Mumford invariant), then the very general fiber $\mathcal{X}_t$ does not admit a Chow decomposition of the diagonal, hence is not stably rational.*

This theorem has been first proved in [17] in the case where the special fiber has only ordinary nodes. This allowed us to prove that very general quartic double solids are not stably rational. The theorem in the present form, allowing more general singularities, is due to Colliot-Thélène and Pirutka (see [8], where a more general statement is in fact proved) who used it to prove that very general quartic threefolds are not stably rational. Beauville applied the method to quartic double covers of $\mathbb{P}^4$ or $\mathbb{P}^5$, concluding that the very general such variety is not stably rational. His degeneration involves a reinterpretation of the Artin-Mumford examples : For the Artin-Mumford quartic double solid X_0, the ramification surface of the double cover map $X_0 \to \mathbb{P}^3$ is a 10-nodal quartic surface. Beauville constructs this surface as a discriminant surface associated to a 3-dimensional linear system of quadrics in $\mathbb{P}^3$ parameterized by $\mathbb{P}^3$ and this is the way he gets the higher dimensional generalization.

Thanks. *I thank Arnaud Beauville, Jean-Louis Colliot-Thélène and Burt Totaro for their comments on the original version.*

1 Proof of Theorem 0.4

Proof of (i). Let $\tau : \widetilde{\mathbb{P}^4} \to \mathbb{P}^4$ be the blow-up of Δ and let $E \subset \mathbb{P}^4$ be the exceptional divisor of τ. The linear system $\tau^*\mathcal{O}(2)(-E)$ is very ample on $\widetilde{\mathbb{P}^4}$, hence a fortiori, the line bundle $\tau^*\mathcal{O}(4)(-2E)$ is very ample. In particular, by Bertini, a general member of $|\tau^*\mathcal{O}(4)(-2E)|$ is smooth. But such a variety is the proper transform $\widetilde{W}$ of a quartic hypersurface $W \subset \mathbb{P}^4$ singular along Δ. So $\widetilde{W}$ is smooth, and thus the double cover $r : Y \to \widetilde{\mathbb{P}^4}$ ramified along $\widetilde{W}$ (which exists because $\widetilde{W}$ has its class divisible by 2 in $\mathrm{Pic}\,\widetilde{\mathbb{P}^4}$) is also smooth. $\square$

Proof of (ii). For any plane $R \subset \mathbb{P}^4$ containing Δ, the intersection $R \cap W$ contains Δ with multiplicity 2, so $R \cap W$ is the union of Δ with multiplicity 2 and of a conic $C_R \subset R$. It follows that the induced double cover $r^{-1}(R') \to R'$, where $R' \cong R$ is the proper transform of R in $\widetilde{\mathbb{P}}^4$, ramifies only along C_R. Thus $r^{-1}(R')$ is a 2-dimensional quadric and we proved that denoting $\pi_\Delta : \widetilde{\mathbb{P}^4} \to \mathbb{P}^2$ the linear projection from Δ, $\pi_\Delta \circ r : Y \to \mathbb{P}^2$ makes Y into a quadric bundle over $\mathbb{P}^2$. Assume now that there is a plane $P \subset \mathbb{P}^4$ not intersecting Δ and such that $P \cap W$ is a double conic. Then on the one hand π_Δ restricted to P gives an isomorphism $P \cong \mathbb{P}^2$, so that P can be seen as giving a section of the morphism $\pi_\Delta : \mathbb{P}^4 \to \mathbb{P}^2$ and on the other hand, as P is everywhere tangent to W, $r^{-1}(P) \subset Y$ is a reducible double cover of P. Thus each component P_i of $r^{-1}(P)$ provides a section of the quadric bundle $Y \to \mathbb{P}^2$. Hence Y is rational. It remains to prove that we can construct such a W with Y smooth. We fix in $\mathbb{P}^4$ the line Δ and the conic C such that the plane P generated by C does not intersect Δ. We then have:

Lemma 1.1. *Let $H \subset H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(4))$ be the linear system of quartics f singular along Δ and such that $f_{|P}$ vanishes doubly along C. Then:*

(a) The space H, seen as contained in $H^0(\widetilde{\mathbb{P}^4}, \tau^*\mathcal{O}(4)(-2E))$, has no base-point away from $C \subset \widetilde{\mathbb{P}^4}$.

(b) The general $f \in H$ defines an hypersurface in $\widetilde{\mathbb{P}^4}$ which is smooth along C.

Proof. Let A, B be linear forms generating the ideal of P and let q be a degree 2 homogeneous polynomial on $\mathbb{P}^4$ such that $q_{|P}$ defines $C \subset P$ and q vanishes along Δ. Then H contains $AI_\Delta(1)^2S^1$, $BI_\Delta(1)^2S^1$ and q^2, where $S^1 := H^0(\mathcal{O}_{\mathbb{P}^4}(1))$. It immediately follows that the only base-points of H on $\mathbb{P}^4$ belong to Δ or to C. As the linear system $\langle A, B\rangle$ has no base-points on Δ, $AI_\Delta(1)^2S^1 + BI_\Delta(1)^2S^1$, seen as contained in $H^0(\widetilde{\mathbb{P}^4}, \tau^*(\mathcal{O}(4))(-2E))$, has no base-point along E. This proves (a).

For a given point $c \in C$, the condition that an element f of $AI_\Delta(1)^2S^1 + BI_\Delta(1)^2S^1$ has zero differential at c imposes two independent conditions to f, since the differentials dA and dB are independent along P and $I_\Delta(1)^2S^1$ has no base-point on P. Counting dimensions, we conclude that the general such f has no critical point along C. This proves (b). $\square$

By Bertini's lemma, Lemma 1.1 (a) implies that the singular points of a general $\widetilde{W}$ as above have to lie on C. Hence we conclude by Lemma 1.1 (b) that $\widetilde{W}$ is in fact smooth. $\square$

Proof of (iii). We follow Beauville's paper [3] and specialize our W to a W_0 which is the discriminant hypersurface of a linear system $\mathbb{P}(\Gamma)$ of quadrics in $\mathbb{P}^3$, of (projective) dimension 4. Beauville proves the following:

Theorem 1.2. *Let $P_9 := \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(2)))$, let $\mathcal{D} \subset P_9$ be the discriminant hypersurface and let $Q_2 \subset \mathcal{D}$ be the locus parameterizing quadrics of rank ≤ 2. Let $\mathbb{P}(\Gamma) \subset P_9$ be a generic $\mathbb{P}^4$. Then*

(a) The curve $\Sigma = \mathbb{P}(\Gamma) \cap Q_2$ is smooth, and the quartic hypersurface $W := \mathcal{D} \cap \mathbb{P}(\Gamma)$ has ordinary quadratic singularities of rank 3 along Σ, and is smooth otherwise.

(b) Let $Y \to \mathbb{P}(\Gamma)$ be the double cover ramified along W, and let $\widetilde{Y}$ be its desingularization by blowing-up Σ. Let $\widetilde{X} \subset \widetilde{Y}$ be the inverse image of a general hyperplane in $\mathbb{P}(\Gamma)$. Then $\widetilde{X}$ is the standard desingularization of an Artin-Mumford double solid, and there is a natural Brauer class $\alpha \in \mathrm{Tors}\,(H^3(\widetilde{Y}, \mathbb{Z}))$ satisfying $\alpha_{|\widetilde{X}} \neq 0$.

In (b), the Brauer class is natural on the Zariski open set U of $\widetilde{Y}$ parameterizing quadrics of rank 3 or 4. It is the Brauer class of the following conic bundle on U : U parameterizes quadrics q in $\mathbb{P}^3$ of rank ≥ 3, plus the choice of a ruling ϵ of q. The conic bundle has for fiber over (q, ϵ) the family of lines in the quadric q, in the ruling ϵ.

In our case, we also want W_0 to be singular along a line Δ and this will be obtained as follows: choose a plane $R \subset \mathbb{P}^3$ and consider a pencil $(R'_t)_{t\in\Delta}$ of hyperplanes in $\mathbb{P}^3$ whose base-locus is not contained in R. The line Δ will parameterize quadrics of the form $R \cup R'_t$. These quadrics are all of rank 2. For an adequate choice of homogeneous linear coordinates X, Y, Z, T on $\mathbb{P}^3$, we will thus consider a general vector subspace $\Gamma \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ of dimension 5 containing the linear subsystem of rank 2 generated by XY and XZ. Let $W_0 \subset \mathbb{P}(\Gamma) = \mathbb{P}^4$ be the corresponding discriminant hypersurface. It is singular along the line $\Delta := \langle XY, XZ\rangle \subset \mathbb{P}(\Gamma)$. Let $\widetilde{\mathbb{P}(\Gamma)}$ be the blow-up of $\widetilde{\mathbb{P}(\Gamma)}$ along Δ and let Y_0 be the double cover of $\widetilde{\mathbb{P}(\Gamma)}$ ramified along the proper transform $\widetilde{W}_0$ of W_0. Let $\widetilde{Y}_0$ be any desingularization of Y_0. We only have to show that that we can apply Theorem 0.7.

Proposition 1.3. *The desingularization map $\sigma : \widetilde{Y}_0 \to Y_0$ has the property that for any $S \subset Y_0$, the variety $\sigma^{-1}(S) \to S$ is rational over S.*

Proof. We start with the following lemma:

Lemma 1.4. *Let $\Delta \subset P_9$ be the line $\langle XY, XZ\rangle$, where X, Y, Z are three independent linear forms on $\mathbb{P}^3$. Let $\mathcal{D}_0 \subset \mathcal{D}$ be the locus of quadrics of rank ≥ 2. Then the proper transform $\widetilde{\mathcal{D}_0}$ of $\mathcal{D}_0$ under the blow-up of Δ is singular only along the proper transform $\widetilde{Q_2}$ of Q_2 in $\widetilde{P}_9$. $\widetilde{Q_2}$ is smooth of codimension 3 and the singularities of $\widetilde{\mathcal{D}_0}$ are ordinary quadratic singularities of rank 3 along $\widetilde{Q_2}$.*

Proof. We know that the singularities of $\mathcal{D}$ along the locus $Q_2 \subset P_9$ are exactly of the type described above, namely Q_2 is smooth of codimension 3 in P_9 and the singularities of $\mathcal{D}_0$ along Q_2 are ordinary quadratic singularities of rank 3. The line Δ is contained in Q_2. Locally along Δ, we have analytic coordinates $x_1, \ldots, x_9$ on P_9 such that Q_2 is defined by the equations $x_7 = x_8 = x_9 = 0$, Δ is defined by the equations $x_2 = \ldots = x_9 = 0$ and $\mathcal{D}_0$ is defined by the equations $x_7^2 + x_8^2 + x_9^2 = 0$. The blow-up of Δ gives us local coordinates in Zariski open sets $X_i \neq 0$ of $\widetilde{P}_9$, $i = 2, \ldots, 9$, which are of the form $x_1, y_2, \ldots, \hat{y}_i, \ldots, y_9, x_i$ with blowing-up map τ given by

$$(x_1, y_2, \ldots, \hat{y}_i, \ldots, y_9, x_i) \mapsto (x_1, x_i y_2, \ldots, x_i, \ldots, x_i y_9).$$

If $i \neq 7, 8, 9$, $\tau^*(x_7^2 + x_8^2 + x_9^2) = x_i^2(y_7^2 + y_8^2 + y_9^2)$, so that $\widetilde{\mathcal{D}_0}$ has equation $y_7^2 + y_8^2 + y_9^2 = 0$, which is an ordinary quadratic singularity along a smooth codimension 3 subset, clearly equal to $\widetilde{Q_2}$. If $i = 7$, $\tau^*(x_7^2 + x_8^2 + x_9^2) = x_7^2(1 + y_8^2 + y_9^2)$, and $\widetilde{\mathcal{D}_0}$ has equation $1 + y_8^2 + y_9^2 = 0$, hence is nonsingular in the considered open set. ☐

Corollary 1.5. *Let $\mathbb{P}(\Gamma) \subset P_9$ be a generic linear space of dimension 4 containing Δ and let $\widetilde{\mathbb{P}(\Gamma)}$ be the blow-up of Δ in $\mathbb{P}(\Gamma)$. Then the proper transform $\widetilde{W}_0$ of $W_0 := \mathcal{D} \cap \mathbb{P}(\Gamma)$ in $\widetilde{\mathbb{P}(\Gamma)}$ has ordinary quadratic singularities along a smooth curve $\Sigma_0 \subset \widetilde{\mathbb{P}(\Gamma)}$.*

Furthermore, no component of Σ_0 is contracted by the map $\tau : \widetilde{\mathbb{P}(\Gamma)} \to \mathbb{P}(\Gamma)$.

Proof. Indeed, our 4-dimensional space is only subject to containing Δ. Let $\pi_\Delta : \widetilde{P_9} \to \mathbb{P}^7$ be the linear projection from Δ. The subspaces $\widetilde{\mathbb{P}(\Gamma)}$ are thus inverse images π_Δ^{-1} of planes $P_2 \subset \mathbb{P}^7$. As the quadrics parameterized by Δ have rank 2, the set Q_1 of quadrics of rank < 2 does not intersect Δ and maps under π_Δ to a subvariety of dimension 3 in P_7. Hence a general $P_2 \subset \mathbb{P}^7$ does not meet $\pi_\Delta(Q_1)$ and the corresponding space $\mathbb{P}(\Gamma)$ does not intersect Q_1. Furthermore, its proper transform $\widetilde{\mathbb{P}(\Gamma)} \subset \widetilde{P}_9$ intersects transversally $\widetilde{\mathcal{D}}$ and only along $\widetilde{\mathcal{D}_0}$, so that we can apply Lemma 1.4.

For the last statement, we observe that $\dim \widetilde{Q_2} = 6$ so that the exceptional divisor E_Q of $\widetilde{Q_2}$ has dimension 5. It follows that either E_Q maps via π_Δ to a codimension ≥ 3 subvariety of P_7, or maps in a generically way to a codimension 2 subvariety of P_7. In both cases, for a general plane $\mathbb{P}^2 \subset P_7$, its inverse image $\widetilde{\mathbb{P}(\Gamma)} \subset \widetilde{P_9}$ intersects the exceptional divisor of $\widetilde{Q_2}$ in finitely many points. As Σ_0 is contained in $\widetilde{Q_2}$ by Lemma 1.4, no component of Σ_0 is contained in the exceptional divisor. □

Corollary 1.5 immediately concludes the proof of Proposition 1.3 since it implies that the singularities of Y_0 are ordinary quadratic singularities of rank 4 along a smooth curve $\Sigma_0 \subset Y_0$. We can thus use the desingularization of Y_0 obtained by blowing-up Σ_0. The fibers $\sigma^{-1}(S)$ are then over the generic point of S, either a point, or a smooth 2-dimensional quadric, which is rational over $\mathbb{C}(S)$ since S is a point or a curve. □

We next need to show that $\widetilde{Y}_0$ does not have a Chow decomposition of the diagonal.

Proposition 1.6. *Any desingularization $\widetilde{Y}_0$ of Y_0 satisfies* $\mathrm{Tors}(H^3(\widetilde{Y}_0, \mathbb{Z})) \neq 0$.

We will use below the desingularization $\widetilde{Y}_0$ given by the blow-up of Σ_0.

Proof of Proposition 1.6. Our hypersurface W_0 is a specialization of Beauville's discriminant hypersurface W_t. The degeneration is very simple: in Beauville's case the curve Σ of singularities is smooth and W_t has ordinary quadratic singularities along it. In our case, the curve of singularities of W_0 has two components, namely Δ and a curve Σ'_0 whose proper transform under the blow-up of Δ is Σ_0.

We claim that, denoting by Y'_0 the double cover of $\mathbb{P}(\Gamma)$ ramified along W_0, the natural map $\widetilde{Y}_0 \to Y'_0$ has fibers of dimension at most 2. Indeed, this map factors as $\widetilde{Y}_0 \xrightarrow{\sigma} Y_0 \xrightarrow{\tau} Y'_0$. By Corollary 1.5, the first map σ has 2-dimensional fibers exactly over Σ_0, while the second map τ has 2-dimensional fibers over the curve Δ. The map σ does not contract any component of Σ_0 to a point by Corollary 1.5. This proves the claim.

Corollary 1.7. *There is a surface $T \subset \widetilde{Y}_0$, which maps to a finite set Z of points in Y'_0, such that the curve $(\Delta \cup \Sigma'_0) \setminus Z$ is smooth (not connected), Y'_0 has ordinary quadratic singularities of rank 4 along $(\Delta \cup \Sigma'_0) \setminus Z$ and the map $\widetilde{Y}_0 \setminus T \to Y'_0 \setminus Z$ is the desingularization of $Y'_0 \setminus Z$ by blowing-up $(\Delta \cup \Sigma'_0) \setminus Z$.*

Proof. Corollary 1.5 shows that the singularities of W_0 are ordinary quadratic singularities generically along Σ'_0. It is immediate to check that the same is true generically along Δ. By construction, the map $\widetilde{Y}_0 \to Y_0$ is the blow-up of Σ_0. As Y_0 is generically isomorphic to Y'_0 along Σ_0, the same holds for the natural map $\widetilde{Y}_0 \to Y'_0$. Finally, generically over Δ, $\widetilde{Y}_0$ is obtained by first blowing-up Δ in $\mathbb{P}(\Gamma)$ and then taking the double cover ramified along $\widetilde{W}_0$, but this is equivalent to first taking the double cover Y'_0 of $\mathbb{P}(\Gamma)$ ramified along W_0 and then blowing-up Δ. Thus away from a finite set Z, the situation is as stated in Corollary 1.7. We now conclude using the fact that by the claim proved above, the inverse image of Z is a surface T. □

Corollary 1.8. *The Brauer class α constructed by Beauville on $\widetilde{Y}_0 \setminus (\tau \circ \sigma)^{-1}(Q_2)$ extends to a class $\beta \in \mathrm{Tors}\,(H^3(\widetilde{Y}_0, \mathbb{Z}))$.*

Proof. Recall that the group $\mathrm{Tors}\,(H^3(\widetilde{Y}_0,\mathbb{Z}))$ does not change if one replaces the smooth variety $\widetilde{Y}_0$ by the complement of any codimension ≥ 2 closed algebraic subset. Hence it suffices to show that α extends to an element in $\mathrm{Tors}\,(H^3(\widetilde{Y}_0 \setminus T,\mathbb{Z}))$, where T is as in Corollary 1.7. Let $\mathcal{W} \to B$ be a one parameter family of quartic determinantal hypersurfaces W_t in $\mathbb{P}^4$, such that the central fiber is our W_0 and the general fiber is Beauville's general determinantal quartic W_t. Let $\mathcal{Y} \to B$ be the corresponding family of ramified double covers. Then by Corollary 1.7, the family is equisingular along $\mathcal{Y}_0 \setminus Z$, that is, the central fiber with finitely many points removed, and the blow-up of $\mathcal{Y} \setminus Z$ along the family $\mathcal{S} \setminus Z \to B$ of curves Σ provides a smooth fibration $\widetilde{\mathcal{Y} \setminus Z} \to B$. It follows that the central fiber, which is isomorphic as an algebraic variety to $Y_0 \setminus T$ is homeomorphic as a topological space to an open subset of the general fiber $\widetilde{\mathcal{Y}_t}$. This gives by restriction to this open set the desired extension of Beauville's extended Brauer class (see Theorem 1.2 (b)) on $\widetilde{\mathcal{Y}_t}$. $\square$

We next have the following lemma:

Lemma 1.9. *Let $W_0 \subset \mathbb{P}(\Gamma)$ be as above a general discriminant quartic hypersurface singular along a line Δ. Then a general hyperplane section S of W_0 is a general Artin-Mumford quartic surface.*

Proof. Beauville proves (see Theorem 1.2) that a general discriminant quartic surface is a general Artin-Mumford quartic surface. It thus suffices that for a general linear system of dimension four $\Gamma_1 \subset H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2))$, there is a general five dimensional linear system $\Gamma \subset H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2))$ containing Γ_1 and with the property that there is a 2-dimensional subspace $\Delta \subset \Gamma$ contained in the discriminant hypersurface in $\mathbb{P}(\Gamma)$. This is immediate because the discriminant surface $S \subset \mathbb{P}(\Gamma_1)$ has a node corresponding to a rank 2 quadric $q = P_1 \cup P_2$. It suffices now to take $\Gamma = \langle \Gamma_1, q' \rangle$, where q' is a rank 2 quadric of the form $P_1 \cup P_2'$, P_2' being a general plane. It is clear that the line $\Delta = \langle q, q' \rangle$ is contained in the discriminant hypersurface $W_0 \subset \mathbb{P}(\Gamma)$ and that W_0 constructed above is a general determinantal quartic hypersurface singular along a line. $\square$

We know by Theorem 1.2 that the Brauer class of the conic bundle is nonzero on the desingularization of the Artin-Mumford quartic double solid which by the lemma above is a hypersurface W_0' in $\widetilde{Y}_0$. The 2-torsion class β in $H^3(\widetilde{Y}_0,\mathbb{Z})$ given by Corollary 1.8 thus has its restriction to W_0' is nonzero, so that $\beta \neq 0$. $\square$

The proof of (iii) is now a consequence of Theorem 0.7. Indeed, Proposition 1.3 allows us to apply Theorem 0.7. On the other hand, Proposition 1.6 and Lemma 0.6 show that $\widetilde{Y}_0$ does not admit a Chow (and even cohomological) decomposition of the diagonal. So the very general member Y of the family does not admit a Chow decomposition of the diagonal and hence is not stably rational. $\square$

2 Proof of Theorem 0.5

We will use the following fact from [6, Section 3] (see also [9, Section 8]):

Proposition 2.1. *Let Y be a smooth fourfold fibered in 2-dimensional quadrics over a surface. Then integral Hodge classes of degree 4 on Y are algebraic.*

We also have the following standard lemma due to Springer [13] (it is in fact true in any dimension and over any field).

Lemma 2.2. *Let Q be a smooth quadric surface over a field k of characteristic 0. Then Q has a k-point, hence is rational over k, if and only if Q has a 0-cycle z of odd degree.*

Proof. Indeed, let C be the family of lines in Q. The curve $C_{\overline{k}}$ is the disjoint union of two copies of $\mathbb{P}^1_{\overline{k}}$. Let $k \subset k'$ be the degree 2 (or 1) extension on which the two geometric components of $C_{\overline{k}}$ are defined. Then $C_{k'}$ is the disjoint union of two curves which become isomorphic to $\mathbb{P}^1_{\overline{k'}}$ over $\overline{k'}$. But each of these curves has a divisor of odd degree defined over k', namely the divisor P^*z of lines passing through z, where $P \subset C \times Q$ is the universal correspondence. It follows that each component is isomorphic to $\mathbb{P}^1_{k'}$, and has a k'-point l, providing a line $l \subset Q$ defined over k'. Let i be the Galois involution acting on $C(k')$. Then if $i(l) = l$, (so that in fact $k = k'$ and $i = Id$), l is defined over k and Q has a k-point. Otherwise we get two different conjugate lines l and $i(l)$ in Q which belong to different rulings of Q, and their intersection point is defined over k. □

Corollary 2.3. *Let Y be a fourfold as in Theorem 0.4. Then Y is rational if Y has an integral Hodge class α of degree 4 which has odd intersection number with the fibers Q_t of the morphism $\pi_\Delta \circ r : Y \to \mathbb{P}^2$.*

Proof. Indeed, recall that via $\pi_\Delta \circ r$, Y is fibered into quadric surfaces over $\mathbb{P}^2$. Proposition 2.1 thus applies to Y and α is the class of a codimension algebraic cycle Z on Y. Restricting Z to the generic fiber Y_η of $\pi_\Delta \circ r$, we get a 0-cycle of odd degree on Y_η defined over the function field $\mathbb{C}(\eta)$ of $\mathbb{P}^2$ and Lemma 2.2 then tells that Y_η is rational over $\mathbb{C}(\eta)$. A fortiori, Y is rational. □

Corollary 2.3 reduces the proof of Theorem 0.5 to the following proposition:

Proposition 2.4. *Let B be the family of all smooth fourfolds Y described in Theorem 0.4. Then the set of points $b \in B$ such that Y_b has an integral Hodge class α of degree 4 which has odd intersection number with the fibers $\pi_\Delta \circ r$ is dense in B for the usual topology.*

Proof. This will follow by applying the following infinitesimal criterion (Proposition 2.5) below : Consider our family of fourfolds $\mathcal{Y} \to B$. We have an associated infinitesimal variation of Hodge structures (see [19, 5.1.2]) at any point $t \in B$

$$H^{2,2}(\mathcal{Y}_t) \to \mathrm{Hom}\,(T_{B,t}, H^{1,3}(\mathcal{Y}_t)),$$

$$\alpha \mapsto \overline{\nabla}(\alpha) : T_{B,t} \to H^{1,3}(\mathcal{Y}_t).$$

Using the fact that the Hodge structure on $H^4(\mathcal{Y}_t, \mathbb{Q})$ is of Hodge niveau 2, that is, $H^{4,0}(\mathcal{Y}_t) = 0$, we have (see [19, 5.3.4]):

Proposition 2.5. *If there exist $t_0 \in B$ and $\alpha \in H^{2,2}(\mathcal{Y}_{t_0})$ such that $\overline{\nabla}(\alpha) : T_{B,t_0} \to H^{1,3}(\mathcal{Y}_{t_0})$ is surjective, then for any $t \in B$, and any Euclidean open set $U \subset B$ containing t, the image of the natural map*

$$T_t : \mathcal{H}^{2,2}_{\mathcal{Y}_U,\mathbb{R}} \to H^4(\mathcal{Y}_t, \mathbb{R})$$

defined by composing the inclusion $\mathcal{H}^{2,2}_{\mathcal{Y},\mathbb{R}} \to \mathcal{H}^4_{\mathcal{Y},\mathbb{R}}$ with a local flat trivialization over U of the flat vector bundle $\mathcal{H}^4_{\mathcal{Y},\mathbb{R}}$ with fiber $H^4(\mathcal{Y}_b, \mathbb{R})$ over any $b \in B$, contains an open subset V_U of $H^4(\mathcal{Y}_t, \mathbb{R})$.

Here $\mathcal{H}^{2,2}_{\mathcal{Y},\mathbb{R}}$ is the real vector bundle over B with fiber over $t \in B$ the space $H^{2,2}(\mathcal{Y}_t)_\mathbb{R}$ of real cohomology classes of type $(2,2)$ on $\mathcal{Y}_t$. Note that the image of T_t is the set of real degree 4 cohomology classes on $\mathcal{Y}_t$ which are of type $(2,2)$ at some point $t' \in U$.

Corollary 2.6. *Under the same assumption, for any $t \in B$, and any Euclidean open set $U \subset B$ containing t, there exists $t' \in U$ and $\alpha_{t'} \in H^{2,2}(\mathcal{Y}_{t'}) \cap H^4(\mathcal{Y}_{t'}, \mathbb{Z})$ such that the degree of α on the fibers Q_s of $\mathcal{Y}_{t'} \dashrightarrow \mathbb{P}^2$ is odd.*

Proof. We observe that the open subset V_U of $H^4(\mathcal{Y}_t, \mathbb{R})$ appearing in Proposition 2.5 is in fact a subcone. It is then immediate to prove that a non-empty open subcone of $H^4(\mathcal{Y}_t, \mathbb{R}) = H^4(\mathcal{Y}_t, \mathbb{Z}) \otimes \mathbb{R}$ has to contain an integral class which has odd degree on the fibers Q_s. □

What remains to be done is to check the infinitesimal criterion which is the contents of Lemma 2.8 below. In our case, the infinitesimal variation of Hodge structure of our family can be described by the adequate adaptation of Griffiths theory (see [11]). Recall that our varieties Y are double covers of the blow-up $\widetilde{\mathbb{P}^4}$ of $\mathbb{P}^4$ along a line, ramified along a smooth member $\widetilde{W}$ of $|\mathcal{O}_{\widetilde{\mathbb{P}^4}}(4)(-2E)|$.

Lemma 2.7. *The space $H^{3,1}(Y)$ is naturally isomorphic to $H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}(1)(-E))$. The space $H^{2,2}(Y)^-$ is isomorphic to the quotient $H^0(\widetilde{W}, \mathcal{O}(5)(-3E))/J_f$, where J_f is the Jacobian ideal of the defining equation f of $\widetilde{W}$ in $\widetilde{\mathbb{P}^4}$, that is*

$$J_f = \mathrm{Im}\,(H^0(\widetilde{\mathbb{P}^4}, T_{\widetilde{\mathbb{P}^4}}()) \to H^0(\widetilde{W}, \mathcal{O}(5)(-3E))).$$

Furthermore, the infinitesimal variation of Hodge structure

$$H^{3,1}(Y) \otimes H^0(\widetilde{W}, \mathcal{O}(4)(-2E)) \to H^{2,2}(Y)_{van}$$

gets identified via these isomorphisms with the multiplication

$$H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}(1)(-E)) \otimes H^0(\widetilde{W}, \mathcal{O}(4)(-2E)) \to H^0(\widetilde{W}, \mathcal{O}(5)(-3E))$$

composed with the quotient map $H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}(5)(-3E)) \to H^{2,2}(Y)_{van}$.

We identify here $H^0(\widetilde{W}, \mathcal{O}(4)(-2E))$ with the tangent space to the deformation space of W (as a hypersurface singular along Δ).

Proof of Lemma 2.7. Let $r : Y \to \widetilde{\mathbb{P}^4}$ be the ramified double cover, and let $r : V \to U$ be the restriction of r to the open set $r^{-1}(U)$, where $U = \widetilde{\mathbb{P}^4} \setminus \widetilde{W}$. The long exact sequence of relative cohomology of the pair (Y, V) (combined with the Thom isomorphism) provides

$$\ldots \to H^2(\widetilde{W}, \mathbb{Q}) \to H^4(Y, \mathbb{Q}) \to H^4(V, \mathbb{Q}) \to H^3(\widetilde{W}, \mathbb{Q}) \to \ldots \tag{1}$$

The involution i of Y over $\widetilde{\mathbb{P}}^4$ acts in a compatible way on this long exact sequence, and taking the skew invariant part, we get an isomorphism

$$H^4(Y, \mathbb{Q})^- \cong H^4(V, \mathbb{Q})^-. \tag{2}$$

This isomorphism is an isomorphism of mixed Hodge structures (which are in fact pure), and the right hand side has its $H^{p,q}$-groups computed as $H^q(Y, \Omega_Y^p(\log \widetilde{W}))^-$. As we have a canonical isomorphism

$$\Omega_Y^p(\log \widetilde{W}) \cong r^*(\Omega^p_{\widetilde{\mathbb{P}^4}}(\log \widetilde{W})),$$

we get a decomposition

$$H^q(Y, \Omega_Y^p(\log \widetilde{W})) = H^q(\widetilde{\mathbb{P}^4}, \Omega^p_{\widetilde{\mathbb{P}^4}}(\log \widetilde{W})) \oplus H^q(\widetilde{\mathbb{P}^4}, \Omega^p_{\widetilde{\mathbb{P}^4}}(\log \widetilde{W})(-2+E)), \tag{3}$$

where we use the isomorphism

$$R^0 r_* \mathcal{O}_Y = \mathcal{O}_{\widetilde{\mathbb{P}^4}} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-2+E).$$

The decomposition (3) identifies $H^q(Y, \Omega_Y^p(\log \widetilde{W}))^-$ with $H^q(\widetilde{\mathbb{P}^4}, \Omega^p_{\widetilde{\mathbb{P}^4}}(\log \widetilde{W})(-2+E))$. This way, we are reduced to computing the twisted Dolbeault cohomology groups

$$H^q(\widetilde{\mathbb{P}^4}, \Omega^p_{\widetilde{\mathbb{P}^4}}(\log \widetilde{W})(-2+E)).$$

We now apply Griffiths analysis (see [19, 6.1]), which in our case provides isomorphisms

$$H^1(\widetilde{\mathbb{P}^4}, \Omega^3_{\widetilde{\mathbb{P}^4}}(\log \widetilde{W})(-2+E)) \cong H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1)(-E)),$$

$$H^2(\widetilde{\mathbb{P}^4}, \Omega^2_{\widetilde{\mathbb{P}^4}}(\log \widetilde{W})(-2+E)) \cong H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(5)(-3E))/J_f.$$

The identification of the infinitesimal variation of Hodge structure with the multiplication map is proved exactly as in the case of the Hodge structure of hypersurfaces (see [19, 6.2.1]). □

We finally prove the desired infinitesimal criterion :

Lemma 2.8. *Let W, Y be as above. Then the infinitesimal variation of Hodge structure $\overline{\nabla} : H^{2,2}(Y) \to \mathrm{Hom}\,(H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(4)(-2E)), H^{1,3}(Y))$ has the property that for generic $\alpha \in H^{2,2}(Y)$, $\overline{\nabla}(\alpha) : H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(4)(-2E)) \to H^{1,3}(Y)$ is surjective.*

Proof. Let us consider the map

$$\mu : H^{3,1}(Y) \otimes H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(4)(-2E)) \to H^{2,2}(Y), \tag{4}$$

which is also given by $\overline{\nabla}$. One has (see [19, 5.3.3])

$$\langle \mu(\eta \otimes u), \beta \rangle = -\langle \overline{\nabla}(\beta)(u), \eta \rangle, \tag{5}$$

where the brackets denote the intersection pairing on Y. By Lemma 2.7, the map μ is the multiplication map

$$H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1)(-E)) \otimes H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(4)(-2E)) \to H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(5)(-3E)) \tag{6}$$

composed with the quotient map $H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(5)(-3E)) \to H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(5)(-3E))/J_f$. Suppose to the contrary that for any $\alpha \in H^{2,2}(Y)$, $\overline{\nabla}(\alpha)$ is not surjective. Then by (5), $H^{2,2}(Y)$ is the union over all nonzero $\eta \in H^{3,1}(Y)$ of the vector spaces $\mu(\eta \otimes H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(4)(-2E)))^{\perp}$. As $\dim H^{3,1}(Y) = 3$, this is possible only if one of the two possibilities hold:

- For generic $\eta \in H^{3,1}(Y)$ the rank of

$$\mu(\eta, \cdot) : H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(4)(-2E)) \to H^0(\widetilde{\mathbb{P}^4}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(5)(-3E))/J_f$$

is only 2.

- For a 1-dimensional family of $\eta \in \mathbb{P}(H^{3,1}(Y))$ the rank of $\mu(\eta, \cdot)$ is at most 1.

Both cases are easily excluded. □

The proof of Proposition 2.4 is now complete. □

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