ALGEBRAIC GEOMETRY: HOMEWORK 8

This homework is due on Friday, October 11 by 5pm.

(1) Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree 4 with singularities at $[1:0:0]$, $[0:1:0]$, and $[0:0:1]$. Prove that C is rational.
Hint: Use the Cremona transformation.

(2) Let X and Y be two irreducible varieties that are birationally isomorphic. Prove that there exist non-empty open subsets $U \subset X$ and $V \subset Y$ such that U and V are isomorphic.

(3) Write down a pair of mutually inverse maps between the fields

$$\operatorname{frac}\left(\mathbb{C}[x, y, z]/(x^3 + y^3 + z^3 + 1)\right) \text{ and } \mathbb{C}(s, t).$$

You should describe the maps by writing where each generator goes. But you need not write the calculation to show that they are inverses.

(4) (Food for thought. Not to be turned in.) The isomorphism you wrote above most probably involved some roots of unity. That raises the question: are the fields $\operatorname{frac}\mathbb{Q}[x, y, z]/(x^3 + y^3 + z^3 + 1)$ and $\mathbb{Q}(s, t)$ isomorphic?