ALGEBRAIC GEOMETRY: HOMEWORK 10

This homework is due on Friday October 25 by 5pm.

Let $V = k[X,Y,Z]_d$, the vector space of homogeneous polynomials in three variables of degree d. Think of the projective space $\mathbb{P}V$ as the space of all degree d curves in $\mathbb{P}^2$. Let $\Delta \subset \mathbb{P}V$ be the set of singular curves. In this sequence of exercises, we will prove that Δ is a closed subvariety, irreducible of codimension 1.

(1) Prove the identity ("Euler identity"):

$$d \cdot F = X\frac{\partial F}{\partial X} + Y\frac{\partial F}{\partial Y} + Z\frac{\partial F}{\partial Z}.$$

Thanks to this, a point $p \in \mathbb{P}^2$ is a singular point of F if and only if F and the three partials vanish at p.

(2) Let $D \subset \mathbb{P}V \times \mathbb{P}^2$ be the set of (F, p) such that $V(F)$ is singular at p. Show that D is a closed subvariety. Using the map $D \to \mathbb{P}^2$, prove that D is irreducible and find its dimension.

(3) Using the map $D \to \mathbb{P}V$, prove that Δ is a closed subvariety, and is irreducible of codimension 1.

(4) Does the map $D \to \mathbb{P}V$ have any positive dimensional fibers? Can you describe the subset of $\mathbb{P}V$ over which the fibers are positive dimensional?

(5) (Not to be turned in). Since Δ is irreducible of codimension 1, it is given by the vanishing of *one* homogeneous polynomial on $\mathbb{P}V$. What is the degree of this polynomial?

Remark. The hypersurface $\Delta \subset \mathbb{P}V$ is called the *discriminant* hypersurface.

Remark. Working with three variables X, Y, Z is not important; everything works with n variables.