(1) (Finite k-algebras) Let A be a reduced finitely generated k-algebra with finitely many maximal ideals. Prove that A is isomorphic to $k^{\oplus n}$ for some n.

(In contrast, there are a *ton* of finitely generated non-reduced k-algebras even with just one maximal ideal, for example, $A = k[x, y]/(x^a, y^b)$ for various a, b.)

Proof. By content covered in class $A \cong k[X]$ for some X an affine algebraic set. Now by Hilbert's Nullstellensatz we know that maximal ideals correspond to points of X, and so X must be a finite subset of $\mathbb{A}^m$ for some m. Let $X = \{P_1, \cdots, P_n\}$. Now considering X as the disjoint union of n points and using question 5 on assignment 2 we have:

$$A \cong k[X] \cong k\left[\coprod_{i=1}^{n} P_i\right] \cong \bigoplus_{i=1}^{n} k[P_i] \cong \bigoplus_{i=1}^{n} k[x_1, \cdots x_m]/I(P_i) \cong k^{\oplus n}$$

Where the middle isomorphism is using question 5 on assignment 2, and the last isomorphism comes from the first isomorphism theorem applied to the ring homomorphism "evaluation at P_i". $\square$

(2) (Connectedness and idempotents) Let X be an affine algebraic set and let $f \in k[X]$ an idempotent (that is, $f^2 = f$).

(a) Assume that f is non-trivial (that is, $f \neq 0, 1$). Show that $V(f) \subset X$ is a non-empty proper subset of X that is both open and closed. *Hint:* Consider $1 - f$.

(b) Conversely, suppose X has a non-empty proper subset that is both open and closed. Produce a non-trivial idempotent in $k[X]$.

(As a result, connected affine algebraic sets correspond to k-algebras without non-trivial idempotents).

Part A:

Proof. $V(f)$ is closed by definition. Now note that $(1-f)f = f - f^2 = f - f = 0$ vanishes on X. So consider $x \notin V(f)$, we know from before that $(1 - f(x))f(x) = 0$, therefore $(1 - f)(x) = 0$, hence $x \in V(1 - f)$. This shows us that:

$$V(f) \cup V(1 - f) = X$$

Now $V(f)$ and $V(1 - f)$ are disjoint because if $f(x) = 0$ then $(1 - f)(x) = 1$, so there can be no $x \in V(f) \cap V(1 - f)$. Therefore:

$$V(f)^c = V(1 - f)$$

$V(1 - f)$ is a closed set, so $V(f)$ must also be open. □

Part B:

Proof. Let $V(S)$ be both open and closed, then we can write $V(S)^c = V(R)$, where S and R are without loss of generality radical ideals. Note first that since these are supposed to be proper subsets of X we have that neither R nor $S = (1)$. Now note the following:

$$(0) = I(X) = I(V(S) \cup V(R)) = I(V(S)) \cap I(V(R)) = S \cap R$$

By question 1 from assignment 2 and the Nullstellensatz using the fact that R and S were assumed to be radical. We also have from that fact that $V(S) \cap V(R) = \emptyset$:

$$k[X] = I(\emptyset) = I(V(S) \cap V(R)) = \sqrt{I(V(S)) + I(V(R))} = \sqrt{S + R}$$

Therefore we see that $S + R = (1)$. Now take $f \in S$ and $g \in R$ such that:

$$f + g = 1$$

Consider the product $fg \in S \cap R = (0)$, so we see that $fg = 0$, hence:

$$0 = fg = f(1 - f) = f - f^2$$

Therefore $f^2 = f$. I claim that f is a non-trivial idempotent, this is because $f \in S$ so $f \neq 1$ and $g = 1 - f \in R$, so $1 - f \neq 1$, that is $f \neq 0$. So we have found a non-trivial idempotent element, as required. □

(3) (The nodal cubic) Assume $\operatorname{char} k \neq 2$. Consider $X = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2$ and the map $\phi : \mathbb{A}^1 \to \mathbb{A}^2$ given by

$$\phi : t \mapsto (t^2 - 1, t^3 - t)$$

(a) Show that ϕ maps $\mathbb{A}^1$ to X, is surjective, and is injective except that the pair of points $\{-1, 1\}$ map to the same point $(0, 0)$.

(b) Show that the map $\phi^* : k[X] \to k[\mathbb{A}^1]$ is injective and its image is the subring $\{f | f(1) = f(-1)\}$.

(Thus, X is obtained from $\mathbb{A}^1$ by "gluing the two points 1 and -1". Algebraically, this translates into the fact that functions on X are functions on $\mathbb{A}^1$ that takes the same value at 1 and -1.)

Part A:

Proof. Let $(x, y) \in X$, if $(x, y) \neq (0, 0)$ then let $t = \frac{y}{x}$, which we know exists because $x \neq 0$. Then:

$$\begin{aligned}
\phi(\frac{y}{x}) &= ((\frac{y}{x})^2 - 1, (\frac{y}{x})^3 - \frac{y}{x}) \\
&= (\frac{x^3 + x^2}{x^2} - 1, \frac{y(x^3 + x^2)}{x^3} - \frac{y}{x}) && \text{because } y^2 = x^3 + x^2 \\
&= (x + 1 - 1, y + \frac{y}{x} - \frac{y}{x}) \\
&= (x, y)
\end{aligned}$$

If $x = 0$ then $y = 0$, and note that $\phi(1) = (0, 0)$. Therefore we have shown that ϕ is surjective.

Now suppose $\phi(t) = \phi(s)$ and neither s nor t are ± 1. Then:

$$(t^2 - 1, t^3 - t) = (s^2 - 1, s^3 - s)$$

Equating the last co-ordinate and factoring we get:

$$t(t^2 - 1) = s(s^2 - 1)$$

Since neither s nor t are ± 1, and using the equality from the first co-ordinate we can divide out to get:

$$t = s$$

Hence ϕ is injective except for ± 1, which both map to (0,0) under ϕ. □

Part B:

Proof. First I will show injectivity of ϕ^*. Suppose $f, g \in k[X]$ are such that $\phi^*(f) = \phi^*(g)$. Then by the definition of ϕ^* we have that:

$$f(\phi(t)) = g(\phi(t)) \ , \ \forall \ t \in \mathbb{A}^1$$

Now since ϕ is surjective we have that $g(x) = f(x) \ \forall \ x \in X$. Hence $f = g$, as required for injectivity of ϕ^*.

Now for the image of ϕ^* define $F := \{f \mid f(1) = f(-1)\}$. I will prove that $\operatorname{Image} \phi^* = \{f | f(1) = f(-1)\}$ by showing that:

$$\operatorname{Image} \phi^* \subset F \subset \langle 1, t^2, t^3 - t \rangle \subset \operatorname{Image} \phi^*$$

Where the angle brackets mean "the k-algebra generated by".

Image $\phi^* \subset F$:

Suppose $h = \phi^* f$ for some $f \in k[X]$ then:

$$\begin{aligned} h(1) &= f(\phi(1)) \\ &= f(0,0) \\ &= f(\phi(-1)) \\ &= h(-1) \end{aligned}$$

Therefore $h \in F$, as required.

$F \subset \langle 1, t^2, t^3 - t\rangle$:

Suppose $h \in F$. Therefore $h(1) = h(-1)$. This is equivalent to the fact that the sum of the coefficients of odd powers of t in $h(t)$ is 0.

First note that we have all polynomials with terms of only even degree in $\langle 1, t^2, t^3 - t\rangle$, so we can without loss of generality assume that $h(t)$ has only odd powers of t.

For the sake of brevity call polynomials with only odd powers of t and whose coefficients add to 0 "odd polynomials". We then proceed by induction. Odd polynomials in F of degree ≤ 3 are exactly k-scalar multiples of $t^3 - t$, and so are in $\langle 1, t^2, t^3 - t\rangle$.

Suppose all odd polynomials of degree $\leq 2n$ are in $\langle 1, t^2, t^3 - t\rangle$. Let $h(t) \in F$ be odd and have degree $2n + 1$. Note that:

$$\begin{aligned} h(t) &= a_{2n+1}t^{2n+1} + a_{2n-1}t^{2n-1} + \sum_{i=0}^{n-2} a_{2i+1}t^{2i+1} \\ &= (t^2)^{n-1}a_{2n+1}(t^3 - t) + (a_{2n+1} + a_{2n-1})t^{2n-1} + \sum_{i=0}^{n-2} a_{2i+1}t^{2i+1} \\ &= (t^2)^{n-1}a_{2n+1}(t^3 - t) + g(t) \end{aligned}$$

Now the first term is in $\langle 1, t^2, t^3 - t\rangle$ because it is a product of elements of k and the generators t^2 and $t^3 - t$. The second term $g(x)$ is odd because the sum of its coefficients are the same as the sum of the coefficients of $h(t)$, since it is degree $2n - 1 \leq 2n$ we have by induction that $g(t) \in \langle 1, t^2, t^3 - t\rangle$. Since $\langle 1, t^2, t^3 - t\rangle$ is closed under addition we have that $h(t) \in \langle 1, t^2, t^3 - t\rangle$.

Therefore we have shown by induction that $F \subset \langle 1, t^2, t^3 - t\rangle$.

$\langle 1, t^2, t^3 - t\rangle \subset$ Image ϕ^*:

Since the image of a k-algebra homomorphism is a k-subalgebra it is enough to prove that the generators $1, t^2$ and $t^3 - t$ all lie in the image of ϕ^*. For this we will exhibit explicit polynomials $f \in k[X]$ such that $\phi^*(f) = 1, t^2$ and $t^3 - t$.

$$\begin{aligned} 1 &= (1 \circ \phi)(t) = \phi^*(1)(t) \\ t^2 &= ((x+1) \circ \phi)(t) = \phi^*(x+1)(t) \\ t^3 - t &= (y \circ \phi)(t) = \phi^*(y)(t) \end{aligned}$$

Therefore $\langle 1, t^2, t^3 - t\rangle \subset$ Image ϕ^*.

Hence, via three set inclusions we have shown that:

$$\operatorname{Image} \phi^* = \{h \in k[\mathbb{A}] \mid h(1) = h(-1)\}$$

$\square$

(4) (Affine conics) Assume char $k \neq 2$. Let $f \in k[x, y]$ be an irreducible polynomial of degree 2. Show that $V(f)$ is isomorphic to either $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$.

Hint: Show that f can be brought into the form $xy - 1$ or $y^2 - x$ by a linear change of coordinates on $\mathbb{A}^2$.

Proof. First I will prove that $V(xy - 1) \cong \mathbb{A}^1 \setminus \{0\}$ and $V(y^2 - x) \cong \mathbb{A}^1$, then I will show that $V(f)$ is isomorphic to one of these cases, for f an irreducible degree two polynomial in $k[x, y]$.

First consider the regular function:

$$\varphi : \mathbb{A}^1 \setminus \{0\} \to V(xy - 1), \ \varphi(z) = (z, z^{-1})$$

The function maps into $V(xy - 1)$ because $zz^{-1} - 1 = 0$ and is an isomorphism because the inverse function $(x, y) \mapsto x$ is also regular.

Now consider the regular function:

$$\varphi : \mathbb{A}^1 \to V(y^2 - x), \ \varphi(z) = (z^2, z)$$

We see that φ maps into $V(y^2 - x)$ because $z^2 - z^2 = 0$, and furthermore we see it is an isomorphism because the inverse function $(x, y) \mapsto y$ is also regular.

Now I need to reduce to the two already proved cases. To do this I will note that any invertible affine linear transformation is a regular isomorphism of quasi-affine varieties, so it is enough to reduce to the two cases above via invertible affine linear changes of variables. Let

$$ax^2 + bxy + cy^2 + dx + ey + f$$

be a degree 2, irreducible polynomial in $k[x, y]$. Consider the degree 2 terms. These form a homogeneous polynomial of degree 2. Now since this can really be viewed as a polynomial of 1 variable, we see that it must factor due to the algebraic closure of k. Hence:

$$ax^2 + bxy + cy^2 + dx + ey + f = (gx + hy)(ix + jy) + dx + ey + f$$

for some $g, h, i, j \in k$. Now we have two cases, one where $(gx + hy) = \lambda(ix + jy)$, $\lambda \in k^*$.

Suppose first that the factors do not differ by a scalar. Then we can make the linear change of variables:

$$x_1 := (gx + hy), \ y_1 := (ix + jy)$$

This is invertible because the two factors are not scalar multiples of each other, and because neither factor is 0, because this would imply that the polynomial was in fact degree 1. This change of variables gives us the equality:

$$(gx + hy)(ix + jy) + dx + ey + f = x_1y_1 + \alpha x_1 + \beta y_1 + \gamma$$

For new constants $\alpha, \beta, \gamma \in k$. Now we factorise the result a bit:

$$x_1y_1 + \alpha x_1 + \beta y_1 + \gamma = (x_1 + \beta)(y_1 + \alpha) - (\alpha\beta - \gamma)$$

Now we know $\alpha\beta - \gamma \neq 0$, or the original polynomial would be reducible, hence we can make the next change of co-ordinates:

$$x_2 := \frac{x_1 + \beta}{\alpha\beta - \gamma}, \ y_2 := y_1 + \alpha$$

This is also invertible because it ammounts to scalar multiplication by a non-zero factor and translation. It gives us the equality:

$$(x_1 + \beta)(y_1 + \alpha) - (\alpha\beta - \gamma) = (\alpha\beta - \gamma)(x_2 y_2 - 1)$$

Again since $\alpha\beta - \gamma \neq 0$ we know that the right hand side defines the exact same affine variety as $x_2 y_2 - 1$, so we have successfully reduced to the $V(xy - 1)$ case.

We now must deal with the case where $(gx + hy) = \lambda(ix + jy)$ for $\lambda \in k^*$.

Suppose without loss of generality that $j \neq 0$, since at least one of i and j are non-zero, or the polynomial would have degree 1. Then the following change of variables is invertible:

$$y_1 := \sqrt{\lambda}(ix + jy),\ x_1 := x$$

Note that all square roots exist because we are working in an algebraically closed field. This gives us the equality:

$$(gx + hy)(ix + jy) + dx + ey + f = y_1^2 + \alpha x_1 + \beta y_1 + \gamma$$

for new $\alpha, \beta, \gamma \in k$. Note that $\alpha \neq 0$ because k is algebraically closed, so a degree 2 polynomial in one variable is automatically reducible. We can now use the fact that the characteristic of $k \neq 2$ to write:

$$y_1^2 + \alpha x_1 + \beta y_1 + \gamma = (y_1 + \beta/2)^2 + \alpha x_1 + \gamma - \beta^2/4$$

Now we make another change of variables (this time it is invertible because $\alpha \neq 0$):

$$y_2 := y_1 + \beta/2,\ x_2 := \alpha x_1 + \gamma - \beta^2/4$$

and get the equality:

$$(y_1 + \beta/2)^2 + \alpha x_1 + \gamma - \beta^2/4 = y_2^2 - x_2$$

So we have successfully reduced to the case $y^2 - x$.

Therefore we have shown that all $V(f)$, where f is degree 2 irred in two variables is isomorphic to $\mathbb{A}$ or $\mathbb{A} \setminus \{0\}$. As required. $\square$