RIEMANN–ROCH

1. Ample divisors

Let X be a compact Riemann surface. A divisor A on X is called *ample* if for every coherent $\mathcal{O}_X$-sheaf F on X, we have

$$H^i(X, F \otimes \mathcal{O}_X(nA)) = 0$$

for $i > 0$ and for sufficiently large n.

Theorem 1.1. *There exists an ample divisor on* X.

In the complex analytic world, the proof I know of Theorem 1.1 uses harmonic analysis. The theorem (generalized naturally to all dimensions) is known as the Kodaira vanishing theorem. In the algebraic world, the proof goes by reducing the statement to a similar statement about sheaves on projective space. The theorem (generalized) is known as the Serre vanishing theorem. By GAGA, for projective algebraic varieties the statements in the analytic and the algebraic category are equivalent.

We will not prove Theorem 1.1. It is very likely that you will prove the Kodaira vanishing theorem or the Serre vanishing theorem (or both) in your mathematical life, in their natural settings. In this class, we will just reap the benefits. In the book, Miranda does something similar—he assumes the existence of enough meromorphic functions, which is a consequence of vanishing, and works with custom-defined H^1 spaces, which turn out to be the same as the standard H^1 spaces as a consequence of vanishing.

Recall that if D is a divisor and E is an effective divisor, then we have the exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+E) \to \mathcal{O}_X(D+E)|_E \to 0.$$

By the long exact sequence on cohomology associated to this sequence, the following are easy to check.

(1) A is ample if and only if nA is ample for some $n > 0$.
(2) $H^i(X, D) = 0$ for all D and $i \geq 2$.
(3) $H^i(X, D)$ are finite dimensional vector spaces (this lets us define the Euler characteristic).
(4) If $H^1(X, D) = 0$ and E is effective, then $H^1(X, D+E) = 0$.
(5) If A is ample and E is effective, then $A + E$ is ample.
(6) If A is ample, and D is any divisor, then for all sufficiently large n, the function $n \mapsto h^0(X, D + nA)$ is a linear function of n. More precisely, we have

$$h^0(X, D + nA) = n \deg A + c$$

for some constant c and sufficiently large n.
(7) If A is ample, then sufficiently large multiples nA of A separate points and tangent vectors; that is, they are very ample.

We also get some exciting information about M_X, the field of meromorphic functions on X.

Theorem 1.2. *M_X is a finitely generated field of transcendence degree 1 over $\mathbb{C}$.*

Proof. Let f be a non-constant meromorphic function on X. The function f gives a map $X \to \mathbb{P}^1$, which in turn gives an inclusion of fields

$$\mathbb{C}(t) = M_{\mathbb{P}^1} \to M_X.$$

By the next theorem, we get that $\mathbb{C}(t) \subset M_X$ is a finite extension. □

Theorem 1.3. *Let $\phi\colon X \to Y$ be a non-constant map of degree d. Then $M_Y \subset M_X$ is a field extension of degree d.*

Proof. Let $y \in Y$ be a point such that $\phi^{-1}(y) = \{x_1, \dots, x_d\}$ with $x_i \neq x_j$ if $i \neq j$. It is easy to construct meromorphic functions f_i on X for $i = 1, \dots, d$ such that f_i are holomorphic on $\{x_1, \dots, x_d\}$ and their restriction to $\{x_1, \dots, x_d\}$ gives d-linearly independent functions on $\{x_1, \dots, x_d\}$. For example, we may take f_i to not vanish at x_i and vanish at all other x_j. It is clear that $f_1, \dots, f_d$ are M_Y-linearly independent. Therefore, we have

$$\deg(M_X/M_Y) \geq d.$$

For the opposite inclusion, let $f_1, \dots, f_{d+1}$ be meromorphic functions on X. Let D be a divisor on X such that $f_i \in H^0(X, D)$; for example, take D to be the sum of the divisor of poles of all f_i. Let A be an ample divisor on Y. We have a map

$$\mathbb{C}^{d+1} \otimes H^0(Y, nA) \to H^0(X, D + n\phi^* A)$$

given by

$$e_i \otimes g \mapsto f_i \cdot \phi^* g.$$

We know that

$$\dim\left(\mathbb{C}^{d+1} \otimes H^0(Y, nA)\right) = (d+1) \cdot n \cdot \deg A + O(1),$$

and

$$\dim\left(H^0(X, D + n\phi^* A)\right) \leq d \cdot n \cdot \deg A + O(1).$$

Therefore, the dimension of the source must overtake the dimension of the target for some n, at which point, we have a non-zero kernel. Suppose $\sum e_i \otimes g_i$ lies in the kernel, where $g_i \in H^0(Y, nA) \subset M_Y$. Then we get the equation

$$\sum f_i g_i = 0,$$

which shows that $f_1, \dots, f_{d+1}$ are M_Y-linearly dependent. Therefore, we get

$$\deg(M_X/M_Y) \leq d.$$

□

2. Riemann–Roch

2.1. **Riemann–Roch and Serre duality.** We will elevate the Riemann–Roch formula to the following more precise statement.

Theorem 2.1. *Let X be a compact Riemann surface of genus g and D a divisor on X.*

1. *We have*
$$\chi\left(\mathcal{O}_X(D)\right) - \chi\left(\mathcal{O}_X\right) = \deg D.$$
2. *We have the duality*
$$H^0\left(X, K_X - D\right) \cong H^1\left(X, D\right)^\vee$$
given by the summation of residues.

(3) We have

$$\chi(\mathcal{O}_X) = 1 - g.$$

Of the three statements, the duality is the hardest to prove. The third statement is a numerical consequence of the formula

$$\chi(K_X) - \chi(\mathcal{O}_X) = 2g - 2$$

and the duality for $D = 0$, which implies

$$\chi(K_X) = -\chi(\mathcal{O}_X).$$

The first statement is an easy consequences of the following lemma and induction.

Lemma 2.2. *Let $D' = D + p$. Then $\chi(\mathcal{O}(D')) = \chi(\mathcal{O}(D)) + 1$.*

Proof. We have the exact sequence of sheaves

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D') \to \mathbb{C}_p \to 0.$$

Apply χ and win. □

2.2. **Residues and duality.** The duality statement in the Riemann–Roch theorem rests on the following.

Theorem 2.3 (The residue theorem). *Let ω be a meromorphic differential form on X. Then $\sum_{p\in X} \operatorname{Res}_p \omega = 0$.*

Theorem 2.3 allows us to define a pairing

$$H^1(X,D) \otimes H^0(X, K_X - D) \to \mathbb{C}$$

as follows. Recall that we think of $\mathcal{O}(K_X - D)$ as the sheaf of meromorphic sections ω of Ω_X satisfying the property $(\omega) - D \geq 0$. Informally, we say that ω is a differential with zeros at D (which is literally true only if D is reduced and effective). To define the pairing, we need a better interpretation of $H^1(X,D)$, which comes thanks to the vanishing theorem. For a sufficiently positive (that is, sufficiently large multiple of an ample and effective) divisor A, we have $H^1(X, D+A) = 0$. The exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+A) \to \mathcal{O}_X(D+A)|_A \to 0$$

gives the sequence

$$H^0(X, D+A) \to H^0(A, \mathcal{O}_X(D+A)|_A) \to H^1(X,D) \to 0.$$

Thus, we can think of $H^1(X,D)$ as the quotient of $H^0(A, \mathcal{O}_X(D+A)|_A)$ by $H^0(X, D+A)$.

We now define a pairing

$$H^0(A, \mathcal{O}_X(D+A)|_A) \otimes H^0(K_X - D) \to \mathbb{C}. \tag{1}$$

Let $f \in H^0(A, \mathcal{O}_X(D+A)|_A)$ and $\omega \in H^0(K_X - D)$. The product $f\omega$ lies in $H^0(A, (K_X + A)|_A)$. We have a map

$$H^0(A, (K_X + A)|_A) \to \mathbb{C}$$

given by

$$\eta \mapsto \sum_{a\in A} \operatorname{Res}_e \eta.$$

We thus get the pairing

$$H^0(A, \mathcal{O}_X(D+A)|_A) \otimes H^0(K_X - D) \to \mathbb{C}$$

given by

$$f \otimes \omega \mapsto \sum_{a \in A} \operatorname{Res}_e(f\omega).$$

If f lies in the image of $H^0(X, \mathcal{O}_X(D+A))$, then $f\omega$ lies in $H^0(X, K_X + A)$. By the residue theorem, $\sum_{a\in A} \operatorname{Res}_e(f\omega) = 0$. Therefore, the pairing in (1) induces a pairing

$$H^1(X,D) \otimes H^0(X, K_X - D) \to \mathbb{C}. \tag{2}$$

We can check that the pairing in (2) is independent of A. Indeed, it suffices to show that if $A \le A'$, then A and A' induce the same pairing. We have a map $\mathcal{O}_X(D+A) \to \mathcal{O}_X(D+A')$ given on meromorphic function by multiplication by 1. So we get the two sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(D) \arrow[r] \arrow[d] & \mathcal{O}_X(D+A) \arrow[r] \arrow[d] & \mathcal{O}_X(D+A)|_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X(D) \arrow[r] & \mathcal{O}_X(D+A') \arrow[r] & \mathcal{O}_X(D+A')|_{A'} \arrow[r] & 0,
\end{tikzcd},$$

which in turn give the diagram

$$\begin{tikzcd}
H^0(X, D+A) \arrow[r] \arrow[d] & H^0(A, (D+A)|_A) \arrow[r] \arrow[d] & H^1(X,D) \arrow[r] \arrow[d, equal] & 0 \\
H^0(X, D+A') \arrow[r] & H^0(A', (D+A')|_{A'}) \arrow[r] & H^1(X,D) \arrow[r] & 0.
\end{tikzcd} \tag{3}$$

For $f \in H^0(A, (D+A)|_A)$, we have

$$\operatorname{Res}_{a\in A} f\omega = \operatorname{Res}_{a' \in A'} f\omega,$$

where on the right hand side, f is considered as an element of $H^0(A', (D+A')|_A)$. Indeed, the only extra terms on the right hand side are from $p \in A' - A$. But $f\omega$ has no poles at $p \in A' - A$, so its residue at p is zero.

2.3. **Laurent tail divisors and a concrete interpretation of Serre duality**. To get a more concrete interpretation of $H^1(X,D)$, let the support of D be in $\{p_1, \dots, p_k\}$, and take $A = p_1 + \dots + p_k$. Say $D = \sum n_i p_i$. Choose uniformizers t_i at p_i. Then $H^0((D+nA)|_{nA})$ is the vector space

$$\bigoplus_{i=1}^k t_i^{-n_i-n} \mathbb{C}[t_i]/t^{-n_i}\mathbb{C}[t_i] = \bigoplus_{i=1}^k \mathbb{C}\langle t_i^{-n_i-1}, \dots, t_i^{-n_i-n}\rangle.$$

The map $H^0(X, D+nA) \to H^0((D+nA)|_A)$ sends a meromorphic function f to its "Laurent tail expansion" at each p_i. That is, we write f as a power series in t_i at p_i. Since $(f) + D + nA \ge 0$, the lowest power in the power series is $t_i^{-n_i-n}$. We keep track of the terms in the power series up to $t_i^{-n_i-1}$ and drop the rest. In this sense, the divisor D keeps track of the upper bound n_i of the non-ignored powers in the power series. Thus, we can write

$$H^1(X,D) = \begin{matrix}\text{Laurent tails bounded above by } D, \textit{ modulo} \\ \text{tails of global meromorphic functions.}\end{matrix}$$

In this setup, Serre duality says something amazing. Since the map

$$H^1(X,D) \to H^0(X,K_X - D)^\vee$$

is an isomorphism (in particular, an injection), we get that a Laurent tail τ represents 0 in $H^1(X,D)$ if and only if its image is 0 in $H^0(X,K_X-D)^\vee$ (that was tautological). But the image of τ in $H^0(X,K_X-D)^\vee$ is the functional

$$H^0(X,K_X-D) \to \mathbb{C}$$

given by

$$\omega \mapsto \sum_{i=1}^{k} \operatorname{Res}_{p_i}(\tau\omega).$$

Therefore, τ arises from a global meromorphic function if and only if for all meromorphic forms ω in $K_X - D$, the sum of residues of $\tau\omega$ is zero. This condition is clearly necessary for τ to arise from a global meromorphic function by the residue theorem. Serre duality says that this is sufficient.

Example 2.4. Consider $X = \mathbb{P}^1$ with the standard coordinates x centered at $p = [0:1]$ and y centered at $q = [1:0]$. Let $r = [1:1]$ and $t = (x-1)$ Consider the tail

$$\tau = (2x^{-1} + 3 \text{ at } p, 1 + 3y \text{ at } q, -3t^{-1} \text{ at } r).$$

It represents an element of $H^1(\mathbb{P}^1, D)$ where $D = -1 \cdot p - 2q + 0 \cdot r$. But is $\tau = 0$ in $H^1(\mathbb{P}^1, D)$? That is, does there exist a rational function on $\mathbb{P}^1$ whose tails at p (up to the power x), q (up to the power y^2), and r (up to the power t^0) are as given (and with no poles elsewhere)?

Serre duality lets us find the answer. We have $h^0(\mathbb{P}^1, K_X - D) = 2$, and we can readily write down a basis of this space:

$$H^0(\mathbb{P}^1, K_X - D) = \mathbb{C}\left\langle dx, \frac{1}{x}dx \right\rangle = \mathbb{C}\left\langle \frac{-1}{y^2}dy, \frac{-1}{y}dy \right\rangle = \mathbb{C}\langle \omega_1, \omega_2 \rangle.$$

The sum of residues of $\tau\omega_1$ is $2 - 3 + 1 = 0$. The sum of residues of $\tau\omega_2$ is $3 - 3 + 0 = 0$. Therefore, there exists a rational function with the specified tails!

What is the rational function? As the writer of the example, I know the answer (because I started with the function first and then wrote its tails). The function is

$$\frac{2+x}{x(1-x)}.$$

2.4. **Proof of Serre duality**. Let

$$r_D \colon H^0(X, K_X - D) \to H^1(X,D)^\vee \tag{4}$$

be the map induced by the residue pairing. Our goal is to prove that r is an isomorphism.

Proof of injectivity. Let $\omega \in H^0(X, K_X - D)$ be non-zero. Then there exists $p \in X$ such that ω is holomorphic differential not vanishing at p. Let t be a local coordinate at p and $f = t^{-1}$, and suppose $\omega = (\sum a_i t^i)dt$ with $a_0 \neq 0$.

Since $(\omega) \geq D$, we must have $\operatorname{mult}_p D \leq 0$. Then $f \in H^0(A, (D+A)|_A)$ for any A with $\operatorname{mult}_p A \geq 1 - \operatorname{mult}_p D$ (in particular, a sufficiently positive A). We can thus think of f as an element of $H^1(X, D)$. But $\sum_p \operatorname{Res}_p(f\omega) = a_0 \neq 0$. Therefore, ω maps to a non-zero element of $H^1(X, D)^\vee$. □

Proof of surjectivity. We need four observations. Let P be an effective divisor and $f \in \mathcal{O}(P)$.

First, the map $\mathcal{O}(D) \to \mathcal{O}(D+P)$ by multiplication by f gives a surjection

$$\mu_f \colon H^1(X, D) \to H^1(X, D+P),$$

and hence an injection

$$\mu_f^\vee \colon H^1(X, D+P)^\vee \to H^1(X, D).$$

Second, the residue map r gives a commutative diagram

$$\begin{tikzcd} H^0(X, K_X - D - P) \arrow[r, "r_{D+P}"] \arrow[d, hook, "\mu_f"] & H^1(X, D+P)^\vee \arrow[d, hook, "\mu_f^\vee"] \\ H^0(X, K_X - D) \arrow[r, "r_D"] & H^1(X, D)^\vee. \end{tikzcd}$$

Third, if A is an ample divisor, then $h^1(X, D - nA)$ grows linearly in n, with leading term $n \deg A$.

Fourth, if $\omega \in H^0(X, K_X - D)$ is such that $r_D(\omega) = \mu_f^\vee \lambda$, then there exists $\omega' \in H^0(X, K_X - D - P)$ such that $\omega = \mu_f(\omega')$. (The proof of this is similar to the proof of the injectivity of r_D.)

Now we are in business. Let $\lambda \in H^1(X, D)^\vee$ and $\omega \in H^0(X, K_X - D)$. We have a map

$$\mathbb{C}\langle \omega, \lambda \rangle \otimes H^0(X, \mathcal{O}(nA)) \to H^1(X, D - nA)^\vee$$

given by

$$\omega \otimes f + \lambda \otimes g \mapsto r_{D+nA}(\mu_f(\omega)) + \mu_f^\vee(\lambda).$$

By considering the dimensions of the source and the target for large n, we get that the map must have a kernel. That is, there exists n and $f, g \in H^0(X, \mathcal{O}(nA))$ such that

$$\mu_f^\vee(\lambda) = r_{D+nA}(\mu_g(\omega)).$$

By the fourth property, there exists $\omega' \in H^0(X, D)$ such that $\mu_g(\omega) = \mu_f(\omega')$. Then we get

$$\mu_f^\vee(\lambda) = r_{D+nA}(\mu_f(\omega')) = \mu_f^\vee r_D(\omega').$$

By the injectivity, we get $\lambda = r_D(\omega')$. □