MATH3354 - ASSIGNMENT 6

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Theorem 1. $\mathbb{P}^1 \times \mathbb{A}^1$ *is neither affine nor projective. The ring of regular functions is* $k[t]$.

Proof. Let $\mathbb{P}^1 \times \mathbb{A}^1 = \{([X : Y], t)\}$. First, we will show that the ring of regular functions is $k[t]$, and we will use this information to infer that $\mathbb{P}^1 \times \mathbb{A}^1$ is neither affine nor projective. Let R be the ring of regular functions; we will show $R \subset k[t]$. $\mathbb{P}^1 \times \mathbb{A}^1$ has two charts; $U_0 = \{([X : Y], t) \mid X \neq 0\} \cong \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ and $\{([X : Y], t) \mid Y \neq 0\} \cong \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$. Consider any $f \in R$. By definition,$f : \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is regular if $f|_{U_0} : \mathbb{A}^2 \to \mathbb{A}^1$ and $f|_{U_1} : \mathbb{A}^2 \to \mathbb{A}^1$ are regular, that is $f|_{U_0}$ and $f|_{U_1}$ are globally polynomials. Let $f|_{U_0}([X : Y], t) = g(\frac{Y}{X}, t)$ and $f|_{U_1}([X : Y], t) = h(\frac{X}{Y}, t)$. Moreover, let's write $g(\frac{Y}{X}, t)$ as $\sum_{i=0}^{m} g_i(t)(\frac{Y}{X})^i$ and $h(\frac{X}{Y}, t)$ as $\sum_{i=0}^{l} h_i(t)(\frac{X}{Y})^i$. Since g and h must agree on $U_0 \cap U_1$,

$$\sum_{i=0}^{l} h_i(t)(\frac{X}{Y})^i = \sum_{i=0}^{m} g_i(t)(\frac{Y}{X})^i$$

for all $t \in k$, $X \neq 0$, $Y \neq 0$. Therefore,

$$g_0(t) - h_0(t) = \sum_{i=1}^{l} h_i(t)(\frac{X}{Y})^i - \sum_{i=1}^{m} g_i(t)(\frac{Y}{X})^i$$

for all $t \in k$, $X \neq 0$, $Y \neq 0$. So, setting $Y = 1$,

$$g_0(t) - h_0(t) = \sum_{i=1}^{l} h_i(t)X^i - \sum_{i=1}^{m} g_i(t)(\frac{1}{X})^i$$

for all $t, X \in k$, $X \neq 0$.

But this implies that $h_i = g_i = 0$ for all $i > 0$. This tells us two things. First, that $f|_{U_0}([X : Y], t) = g(\frac{Y}{X}, t) = g_0(t)$ and that $f|_{U_1}([X : Y], t) = h(\frac{X}{Y}, t) = h_0(t)$. Second, that $g_0(t) - h_0(t) = 0$ and therefore $g_0(t) = h_0(t)$. Thus we know that, in fact $f([X : Y], t) = g_0(t) = h_0(t)$ and therefore, $f \in k[t]$.

Showing that $k[t] \subset R$ is straightforward. Consider any polynomial $f(t) \in k[t]$. Then $f(t)$ is globally a polynomial when restricted to the charts U_0, U_1, so $f(t)$ is regular. Thus, $R \subset k[t]$, and $k[t] \subset R$, so the ring of regular functions is $k[t]$.

We can infer immediately that $\mathbb{P}^1 \times \mathbb{A}^1$ is not projective, since every projective variety has k as its ring of regular functions, as shown in class. We can also infer immediately that if $\mathbb{P}^1 \times \mathbb{A}^1$ is affine, then $\mathbb{P}^1 \times \mathbb{A}^1$ is isomorphic to A^1, by the fact that A^1 also has ring of regular functions $k[t]$, and two affine varieties have isomorphic rings of regular functions iff they are isomorphic. We will show $\mathbb{P}^1 \times \mathbb{A}^1$ is not isomorphic to A^1, by contradiction. Suppose, to generate a contradiction, that there exists an isomorphism $\phi : \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. Then, in particular, ϕ is an injective regular map $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. But since the range of ϕ is $\mathbb{A}^1$, ϕ is a regular function, so, as shown above $\phi \in k[t]$, so $\phi([X : Y], t) = f(t)$ for some $f(t) \in k[t]$. But then $\phi([0 : 1], 1) = f(1) = \phi([1 : 0], 1)$ and so ϕ is not injective, giving our desired contradiction. Thus $\mathbb{P}^1 \times \mathbb{A}^1$ is not isomorphic to any affine variety, and not isomorphic to any projective variety either. $\square$

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Theorem 2. *Let $Z \subset \mathbb{P}^n$ be a projective variety, and $X \subset \mathbb{P}^n \times \mathbb{A}^m$ a closed set. For $t \in \mathbb{A}^m$, let $X_t \subset \mathbb{P}^n$ denote the fibre of X over t under the second projection $X \to \mathbb{A}^m$. The set of $t \in \mathbb{A}^m$ such that $Z \subset X_t$ is Zariski closed.*

Proof. We note that the set of $t \in \mathbb{A}^m$ such that $Z \subset X_t$ can be written $\{t \in \mathbb{A}^m \mid Z \subset X_t\}$. Consider any $z \in Z \subset P^n$. The set $\{z\} \times \mathbb{A}^m \cap X$ is Zariski closed, since X is given to be Zariski closed, and $\{z\} \times \mathbb{A}^m$ is Zariski closed. To see that $\{z\} \times \mathbb{A}^m$ is Zariski closed, consider any chart $U_i = \{([X_0 : ... : X_n], (t_1, ..., t_m)) \mid X_1 \neq 0\}$. Let $z = [Z_0 : ... : Z_n]$. If $Z_i = 0$, then $U_i \cap \{z\} \times \mathbb{A}^m = \emptyset$, which is trivially closed. If $Z_i \neq 0$, then $U_i \cap \{z\} \times \mathbb{A}^m = \{(\frac{Z_0}{Z_i}, ..., \frac{Z_n}{Z_i}, t_1, ..., t_n)\} \subset \mathbb{A}^{m+n}$, which is closed, since it is the vanishing set $V(x_1 - \frac{Z_0}{Z_i}, ..., x_n - \frac{Z_n}{Z_i})$. So, as we claimed, $\{z\} \times \mathbb{A}^m \cap X$ is Zariski closed. Therefore, for any $z \in Z$, $\pi(\{z\} \times \mathbb{A}^m \cap X) \subset \mathbb{A}^m$ is closed, because $\mathbb{P}^n$ is universally closed. But we note that $\pi(\{z\} \times \mathbb{A}^m \cap X) = \{t \in \mathbb{A}^m \mid (z, t) \in X\}$. To see this, suppose $t \in \{t \in \mathbb{A}^m \mid (z, t) \in X\}$, then $(z, t) \in X$, so $(z, t) \in \{z\} \times \mathbb{A}^m \cap X$ and therefore, $t \in \pi(\{z\} \times \mathbb{A}^m \cap X)$. Conversely, suppose $t \in \pi(\{z\} \times \mathbb{A}^m \cap X)$, then by the definition of the projection map $(z', t) \in \{z\} \times \mathbb{A}^m \cap X$ for some $z' \in \mathbb{P}^n$. Since $(z', t) \in \{z\} \times \mathbb{A}^m \cap X$, $z' = z$, and so $(z, t) \in \{z\} \times \mathbb{A}^m \cap X$, implying $t \in \{t \in \mathbb{A}^m \mid (z, t) \in X\}$. Thus, for all $z \in Z$, $\{t \in \mathbb{A}^m \mid (z, t) \in X\}$ is Zariski closed. So, since arbitrary intersections of closed sets are closed, $\cap_{z \in Z}\{t \in \mathbb{A}^m \mid (z, t) \in X\}$ is Zariski closed. But we note that

$$\begin{aligned} \cap_{z \in Z}\{t \in \mathbb{A}^m \mid (z, t) \in X\} &= \{t \in \mathbb{A}^m \mid \text{ for all } z \in Z, (z, t) \in X\} \\ &= \{t \in \mathbb{A}^m \mid \text{ for all } z \in Z, z \in \{w \in \mathbb{P}^n \mid (w, t) \in X\}\} \\ &= \{t \in \mathbb{A}^m \mid Z \subset \{w \in \mathbb{P}^n \mid (w, t) \in X\}\} \\ &= \{t \in \mathbb{A}^m \mid Z \subset X_t\} \end{aligned}$$

Therefore $\{t \in \mathbb{A}^m \mid Z \subset X_t\}$ is Zariski closed, completing our proof. $\square$

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Theorem 3. *As usual, let us identify the set of $n \times n$ matrices with $\mathbb{A}^{n^2}$. Let $S \subset \mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$ be the set of pairs of matrices (A, B) such that A and B have a common eigenvector. Prove that S is a Zariski closed subset of $\mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$.*

Proof. First we will show that the set

$$E = \{(v, A, B) \mid \ v \text{ is an eigenvector of } A \text{ and an eigenvector of } B\} \subset \mathbb{P}^{n-1} \times \mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$$

is Zariski closed. We note that E is well-defined; for a vector $v \in \mathbb{A}^n$, v is an eigenvector of A, B iff λv is an eigenvector of A, B for all $\lambda \in k$, $\lambda \neq 0$ (this is true by the definition of an eigenvector), so we can treat v as a vector in $\mathbb{P}^{n-1}$. To show that

E is closed, we must show that for all charts $U_i = \{([X_0 : ... : X_{n-1}], A, B) \mid X_i \neq 0\}$ of $\mathbb{P}^{n-1} \times \mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$, $E \cap U_i \subset \mathbb{A}^{2n^2+n-1}$ is closed. Consider any chart U_i,

$$\begin{aligned} & E \cap U_i \\ &= \{([X_0 : ... : X_{n-1}], A, B) \mid \ [X_0 : ... : X_{n-1}] \text{ is an eigenvector of } A \\ & \qquad \text{and an eigenvector of } B \text{ and } X_i \neq 0\} \\ &\cong \{(\frac{X_0}{X_i}, ..., \hat{X}_i ..., \frac{X_{n-1}}{X_i}, A, B) \mid (\frac{X_0}{X_i}, ..., 1, ..., \frac{X_{n-1}}{X_i}) \text{ is an eigenvector of } A \\ & \qquad \text{and an eigenvector of } B\} \subset \mathbb{A}^{2n^2+n-1} \end{aligned}$$

We will view $E \cap U_i$ as the final set given above, in $\mathbb{A}^{2n^2+n-1}$, and will show that it is a closed subset of $\mathbb{A}^{2n^2+n-1}$. First, we note that $w = (w_1, ..., w_n) = (\frac{X_0}{X_i}, ..., 1, ..., \frac{X_{n-1}}{X_i})$ is an eigenvector of A iff $Aw = (w'_1, ..., w'_n)$ is a scalar multiple of w, and likewise w is an eigenvector of B iff $Bw = (w''_1, ..., w''_n)$ is a scalar multiple of w (noting that since w has a non-zero ith entry, w is not the zero vector, so we don't have to worry about excluding this possibility). These two conditions are satisfied iff for all $j, k \in \{1, ..., n\}$, $w_j w'_k - w_k w'_j = 0$, $w_j w''_k - w_k w''_j = 0$. Let a_{jk} be the entries in A, and b_{jk} be the entries in B. For all $j \in \{1, ..., n\}$, we can write w'_j as a polynomial in $w_1, ..., w_n, a_{11}, ..., a_{nn}$, and likewise we can write w'_j as a polynomial in $w_1, ..., w_n, b_{11}, ..., b_{nn}$ (by the algorithm for matrix multiplication). Let these polynomials be $f_j(w_1, ..., w_n, a_{11}, ..., a_{nn})$, $g_j(w_1, ..., w_n, b_{11}, ..., b_{nn})$ respectively. We conclude that w is an eigenvector of A iff for all $j, k \in \{1, ..., n\}$,

$$w_j f_k(w_1, ..., w_n, a_{11}, ..., a_{nn}) - w_k f_j(w_1, ..., w_n, a_{11}, ..., a_{nn}) = 0$$

and likewise w is an eigenvector of B iff for all $i, j \in \{1, ..., n\}$,

$$w_j g_k(w_1, ..., w_n, b_{11}, ..., b_{nn}) - w_j g_k(w_1, ..., w_n, b_{11}, ..., b_{nn}) = 0$$

So,

$$\begin{aligned} E \cap U_i = V(& w_j f_k(w_1, ..., w_n, a_{11}, ..., a_{nn}) - w_k f_j(w_1, ..., w_n, a_{11}, ..., a_{nn}), \\ & w_j g_k(w_1, ..., w_n, b_{11}, ..., b_{nn}) - w_k g_j(w_1, ..., w_n, b_{11}, ..., b_{nn})) \end{aligned}$$

Thus, $E \cap U_i$ is Zariski closed for all $i \in \{0, ..., n-1\}$, and so $E \subset \mathbb{P}^{n-1} \times \mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$ is a Zariski closed set.
Furthermore, $\pi(E)$ is closed, where π is the natural projection $\mathbb{P}^{n-1} \times \mathbb{A}^{n^2} \times \mathbb{A}^{n^2} \to \mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$ since $\mathbb{P}^{n-1}$ is universally closed. But

$$\begin{aligned} \pi(E) &= \{(A, B) \in \mathbb{A}^{n^2} \times \mathbb{A}^{n^2} \mid \text{there exists some } v \in P^{n-1} \\ & \qquad \text{such that } (v, A, B) \in E\} \\ &= \{(A, B) \in \mathbb{A}^{n^2} \times \mathbb{A}^{n^2} \mid \text{there exists some } v \in P^{n-1} \\ & \qquad \text{such that v is an eigenvector of A and an eigenvector of B}\} \\ &= \{(A, B) \in \mathbb{A}^{n^2} \times \mathbb{A}^{n^2} \mid A \text{ and } B \text{ share an eigenvector}\} \end{aligned}$$

So, we have shown that the set S of pairs of matrices (A, B) such that A and B have a common eigenvector is Zariski closed. □