# VECTOR BUNDLES AND FINITE COVERS

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ABSTRACT. We prove that, up to a twist, every vector bundle on a smooth projective curve arises from the direct image of the structure sheaf of a smooth, connected branched cover.

## 1. INTRODUCTION

Associated to a finite flat morphism $\phi : X \to Y$ is the vector bundle $\phi_* \mathcal{O}_X$ on $Y$. This is the bundle whose fiber over $y \in Y$ is the vector space of functions on $\phi^{-1}(y)$. The goal of this paper is to understand which vector bundles arise in this way, particularly in the context of finite maps between smooth projective varieties. Said differently, we seek to identify the vector bundles on $Y$ that admit the structure of a commutative and associative $\mathcal{O}_Y$-algebra, particularly in the presence of of additional restrictions like regularity.

Our main result is that, up to a twist, every vector bundle on a smooth projective curve $Y$ arises from a smooth branched cover $X \to Y$.

**Theorem 1.1** (Main). *Let $Y$ be a smooth projective curve and let $E$ be a vector bundle on $Y$. There exists an integer $n$ (depending on $E$) such that for any line bundle $L$ on $Y$ of degree at least $n$, there exists a smooth curve $X$ and a finite map $\phi : X \to Y$ such that $\phi_* \mathcal{O}_X$ is isomorphic to $\mathcal{O}_Y \oplus E^\vee \otimes L^\vee$.*

The reason for the $\mathcal{O}_Y$ summand is the following. Pull-back of functions gives a natural map $\mathcal{O}_Y \to \phi_* \mathcal{O}_X$, which admits a splitting by $1/d$ times the trace map. As a result, every bundle of the form $\phi_* \mathcal{O}_X$ contains $\mathcal{O}_Y$ as a direct summand. The dual of the remaining direct summand is called the *Tschirnhausen bundle* and denoted by $E = E_\phi$ (The dual is taken as a convention.) Thus, Theorem 1.1 says that on a smooth projective curve, a twist of every vector bundle is Tschirnhausen.

The reason for having to twist by a line bundle is a bit more subtle. It arises from geometric restrictions on Tschirnhausen bundles of finite maps between smooth projective varieties. For $Y = \mathbf{P}^n$ and a smooth $X$, the Tschirnhausen bundle $E$ is ample by a result of Lazarsfeld [11]. For more general $Y$ and smooth $X$, it enjoys several positivity properties as shown in [14, 15]. The precise necessary and sufficient conditions for being Tschinrhausen (without the twist) are unknown, and seem to be quite delicate even when $Y$ is a curve.

Without any restrictions on $X$, the question of identifying Tschirnhausen bundles is vacuous: every vector bundle $E$ qualifies. Indeed, given $E$, we may take $X$ to be the non-reduced scheme which is the first order neighborhood of the zero section in the total space of $E$.

The simplest non-trivial case of identifying Tschirnhausen bundles, namely the case of $Y = \mathbf{P}^1$, has attracted the attention of several mathematicians; see for example [1, 5, 12, 16]. Historically, it has been called the problem of classifying the *scrollar invariants* of smooth finite covers of $\mathbf{P}^1$. Writing $E_\phi = \mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_{d-1})$, the scrollar invariants are the integers $a_1, \dots, a_{d-1}$. This problem has been completely solved for $d = 2$ and $d = 3$. For $d = 2$, any positive integer is a scrollar invariant. For $d = 3$, a pair of positive integers $(a_1, a_2)$ with $a_1 \le a_2$ arises as scrollar invariants of smooth triple coverings if and only if $a_2 \le 2a_1$. It may be within reach to completely settle the next few values of $d$ using structure theorems for finite covers of low degree, but the problem becomes very hard very fast as $d$ increases. The picture emerging from the collective work of several authors (and visible in the $d = 3$ case) indicates that if the scrollar invariants are too far apart, then they cannot arise as scrollar invariants.

Let us now allow twisting by a line bundle. For $Y = \mathbf{P}^1$, this is equivalent to allowing a simultaneous shift of all the scrollar invariants. In this guise, the work of Ballico [1] comes closest to a characterization. He shows that one can arbitrarily specify the smallest $d/2$ of the $(d-1)$ scrollar invariants. Theorem 1.1 answers the question fully: one can in fact arbitrarily specify *all* the scrollar invariants.

For affine curves, Theorem 1.1 yields the following corollary.

**Corollary 1.2.** *Suppose $Y$ is a smooth affine curve, and $E$ is a vector bundle on $Y$. Then $E$ is the Tschirnhausen bundle for some map $\phi : X \to Y$, with $X$ smooth and connected.*

*Proof.* Extend $E$ to a vector bundle $E'$ on the smooth projective compactification $Y'$ of $Y$. Apply Theorem 1.1 to $E'$, twisting by a sufficiently positive line bundle $L$ on $Y'$ whose divisor class is supported on the complement $Y' \setminus Y$. We obtain a smooth curve $X'$ and a map $\phi : X' \to Y'$ whose Tschirnhausen bundle is $E' \otimes L$; letting $X = \phi^{-1}(Y)$, we obtain the corollary. □

The method of proof of Theorem 1.1 yields a basic result relating the moduli branched covers of $Y$ and the moduli of vector bundles on $Y$. Let $H_{d,g}(Y)$ denote the Hurwitz space of degree $d$ and genus $g$ branched covers of $Y$ and $M_{d-1,d+g-1}(Y)$ the moduli space of semi-stable vector bundles of rank $d-1$ and degree $d+g-1$ on $Y$.

**Theorem 1.3.** *Suppose $g_Y \geq 2$. If $g$ is sufficiently large (depending on $Y$ and $d$), then the Tschirnhausen bundle of a general degree $d$ and genus $g$ branched cover of $Y$ is stable. Moreover, the rational map $H_{d,g}(Y) \dashrightarrow M_{d-1,d+g-1}(Y)$ defined by $\phi \mapsto E_\phi$ is dominant.*

*The same statement holds for $g_Y = 1$, with "stable" replaced with "regular poly-stable."*

Theorem 1.3 is Corollary 3.10 in the main text.

Special cases of Theorem 1.3, namely the cases $d \leq 5$, were proved by Kanev [8,9,10] using the structure theorems of finite covers of low degree [3,4]. The validity of Theorem 1.3 for low $g$ is an interesting open problem. In particular, it would be nice to know whether $\phi \to E_\phi$ is dominant as soon as we have $\dim H_{d,g}(Y) \geq \dim M_{d-1,d+g-1}$.

Our interest in Tschirnhausen bundles for curves originated partly in the study of cycles on $H_{d,g}(Y)$. For a vector bundle $E$ on $Y$, define the *Maroni locus* $M(E) \subset H_{d,g}(Y)$ as the locally closed subset that parametrizes covers with Tschirnhausen bundle isomorphic to $E$. This notion generalizes the Maroni loci for $Y = \mathbf{P}^1$ studied in [6] and [13]. As a consequence of the method of proof of the main theorem, we obtain the following.

**Theorem 1.4.** *Let $E$ be a vector bundle on $Y$ of rank $(d-1)$ and degree $e$. If $g$ is sufficiently large (depending on $Y$ and $E$), then for every line bundle $L$ of degree $d+g-e-1$, the Maroni locus $M(E \otimes L) \subset H_{d,g}(Y)$ contains an irreducible component having the expected codimension $h^1(\operatorname{End} E)$.*

Theorem 1.4 is Corollary 3.11 in the main text.

Going further, it would be valuable to know whether all the components of $M(E \otimes L)$ are of the expected dimension or, even better, if $M(E \otimes L)$ is irreducible. The results of [6] imply irreducibility for $Y = \mathbf{P}^1$ and some vector bundles $E$. But the question remains open in general.

The connection to cycles on $H_{d,g}(Y)$ is through the fundamental class of the closure of $M(E)$. It would be interesting to know if this cycle has any distinguishing properties, such as rigidity or extremality, as is the case for the Maroni divisors for $Y = \mathbf{P}^1$, at least when $d \leq 5$ [13].

We also draw the reader's attention to results, similar in spirit to Theorem 1.3, proved by Beauville, Narasimhan, and Ramanan [2]. The basic problem in their line of inquiry, first posed by Beauville, is to study not the pushforward of $\mathcal{O}_X$ itself but the pushforwards of general line bundles on $X$.

The attempt at extending Theorem 1.1 to higher dimensional varieties $Y$ presents interesting new challenges. We discuss them through some examples in § 4. As it stands, the analogue of Theorem 1.1 for higher dimensional varieties $Y$ is false. We end the paper by posing slight modifications, for which we are still unable to find counterexamples.

1.1. **Strategy of proof.** The proof of Theorem 1.1 proceeds by degeneration. To help the reader, we first outline our approach to a weaker version of Theorem 1.1. In the weaker version, we consider not the vector bundle $E$ itself, but its projectivization $\mathbf{P}E$, which we call the *Tschirnhausen scroll*. Recall that a branched cover with Gorenstein fibers $\phi: X \to Y$ with Tschirnhausen bundle $E$ factors through a *relative canonical embedding* $\iota: X \hookrightarrow \mathbf{P}E$ (see [3]).

**Theorem 1.5.** *Let $E$ be any vector bundle on a smooth projective curve $Y$. Then the scroll $\mathbf{P}E$ is the Tschirnhausen scroll of a finite cover $\phi: X \to Y$ with $X$ smooth.*

The following steps outline a proof of Theorem 1.5 which parallels the proof of the stronger Theorem 1.1. We omit the details, since they are subsumed by the results in the sequel.

(1) First consider the case
$$E = L_1 \oplus \cdots \oplus L_{d-1},$$
where the $L_i$ are line bundles on $Y$ whose degrees satisfy
$$\deg L_i \ll \deg L_{i+1}.$$
For such $E$, construct a nodal cover $\phi: X \to Y$ such that $E_\phi = E$. For example, we may take $X$ to be a nodal union of $d$ copies of $Y$, each mapping isomorphically to $Y$ under $\phi$, where the $i$th copy meets the $(i+1)$th copy along nodes lying in the linear series $|L_i|$.

(2) The next step is to smooth out the nodes of $X \subset \mathbf{P}E$. As it stands, the normal bundle $N_{X/\mathbf{P}E}$ is quite negative; in fact simple examples show that the nodes of $X$ are usually impossible to smooth out. Fixing this negativity is the most crucial step.

(3) Attach several rational normal curves to $X$ as follows. Given a general point $y \in Y$, the $d$ points $\phi^{-1}(y) \subset \mathbf{P}E_y \simeq \mathbf{P}^{d-2}$ are in general linear position, and therefore they lie on many smooth rational normal curves $R_y \subset \mathbf{P}E_y$. Choose a large subset $S \subset Y$, and attach general rational normal curves $R_y$ for each $y \in S$ to $X$, obtaining a new nodal curve $\widetilde{X} \subset \mathbf{P}E$.

(4) Next, see that the new normal bundle $N_{\widetilde{X}/\mathbf{P}E}$ is sufficiently positive. In particular, $\widetilde{X}$ is the flat limit of a family of smooth, relatively-canonically embedded curves $X_t \subset \mathbf{P}E$. Furthermore, the higher cohomology of $N_{X_t/\mathbf{P}E}$ vanishes.

(5) Deduce the result for arbitrary bundles $E$ as follows.
    (a) See that every vector bundle $E$ degenerates isotrivially to a direct sum of line bundles $L_i$ of the form needed above.
    (b) Consider the map $\phi \to \mathbf{P}E_\phi$ be the map from the moduli stack of branched covers of $Y$ to the moduli stack of projective bundles on $Y$. The positivity of $N_{X_t/\mathbf{P}E}$ implies that this map is smooth at $X_t \to Y$.
    (c) Using the openness of smooth maps and the previous two steps, conclude that every projective bundle arises as a Tschirnhausen scroll for a smooth cover.

To have control over the vector bundle $E$ itself, and not just its projectivization, we consider the *canonical affine embedding* of $X$ in the total space of $E$. However, to attach rational normal curves, it is essential to have a projective space bundle. Therefore, we consider the projective closure $P := \mathbf{P}(E^\vee \oplus \mathcal{O}_Y)$ of the total space of $E$. Let $H := \mathbf{P}(E^\vee \oplus \mathcal{O}_Y) \setminus E$ be the divisor of hyperplanes at infinity. The proof of Theorem 1.1 involves carrying out the steps outlined above for the embedding $X \to P$ *relative* to the divisor $H$.

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1.3. **Conventions.** All schemes are finite type over an algebraically closed field $k$ of characteristic 0 (or of characteristic larger than the degree $d$ of the covers we consider). The projectivization $\mathbf{P}V$ of a vector bundle $V$ refers to the space of 1-dimensional quotients of $V$. We identify vector bundles with their sheaves of sections. An injection of vector bundles is understood as an injection of the corresponding locally free sheaves.

## 2. VECTOR BUNDLES, THEIR INFLATIONS, AND DEGENERATIONS

Let $E$ be a vector bundle on $Y$. A *degree $n$ inflation* of $E$ is a vector bundle $\widetilde{E}$ along with an injective map $E \to \widetilde{E}$ whose cokernel is finite of length $n$. If $E \to \widetilde{E}$ is an inflation, then the dual bundle $\widetilde{E}^\vee$ is a sub-bundle of $E^\vee$ and the quotient is finite of length $n$. Thus, a degree $n$ inflation of $E$ is equivalent to a sub-bundle of $E^\vee$ of co-length $n$, which in turn is equivalent to a quotient of $E^\vee$ of length $n$. Therefore, we can identify the moduli space of length $n$ inflations of $E$ with the quot scheme $\operatorname{Quot}(E^\vee, n)$. It is easy to see that $\operatorname{Quot}(E^\vee, n)$ is smooth and connected, in particular irreducible. Therefore, it makes sense to talk about a general length $n$ inflation of $E$.

**Proposition 2.1.** *Let $E$ be a vector bundle on $Y$. For a sufficiently large $n$, a general length $n$ inflation $E \to \widetilde{E}$ satisfies $H^1(\widetilde{E}) = 0$.*

*Proof.* By Serre vanishing, we have $H^1(E \otimes L) = 0$ for a very ample line bundle $L$ on $Y$. Set $N = \operatorname{rk} E \cdot \deg L$. Let $n \geq N$ and let $E \otimes L \to \widetilde{E}$ be a general length $(N-n)$ inflation. Then $E \to \widetilde{E}$ is a length $n$ inflation. We have an exact sequence

$$0 \to E \otimes L \to \widetilde{E} \to Q \to 0. \tag{2.1}$$

Since $H^1(E \otimes L) = 0$, the long exact sequence on cohomology associated to (2.1) implies that $H^1(\widetilde{E}) = 0$. Since the space of length $n$ inflations of $E$ is irreducible, we conclude that $H^1(\widetilde{E}) = 0$ for a general length $n$ inflation $E \to \widetilde{E}$ for any $n \geq N$. □

*Remark* 2.2. Suppose $E \to \widetilde{E}$ is an inflation and $H^1(E) = 0$. Then the long exact sequence associated to

$$0 \to E \to \widetilde{E} \to Q \to 0$$

shows that we also have $H^1(\widetilde{E}) = 0$.

*Remark* 2.3. Consider a degree 1 inflation $E \to \widetilde{E}$. Suppose the cokernel is supported at a point $p$. The dual map $\widetilde{E}^\vee \to E^\vee$ drops rank by 1 at $p$, and therefore the image of $\widetilde{E}^\vee|_p$ in $E^\vee|_p$ is a hyperplane. Conversely, a degree 1 inflation of $E$ is specified by a point $p$ and a hyperplane of $E^\vee|_p$. In this case, $E$ and $\widetilde{E}$ are often said to be related by an elementary transformation.

*Remark* 2.4. The context in which we use inflations is the following. Let $P$ be a smooth variety. Let $R, S \subset P$ be curves that intersect at a point $p$ so that their union $Z$ has a simple node at $p$. Then we have the exact sequence

$$0 \to N_{R/P} \to N_{Z/P}|_R \to k_p \to 0.$$

That is, the bundle $N_{Z/P}|_R$ is a degree 1 inflation of $N_{R/P}$. The hyperplane of $N^\vee_{R/P}|_p$ that specifies this inflation is the kernel of the map

$$N^\vee_{R/P}|_p \to k$$

defined as the composite

$$N^\vee_{R/P}|_p \xrightarrow{d} \Omega_P|_p \to k,$$

where the last map is the contraction with a non-zero vector in $T_pS$. In particular, if the point $p \in R$ and the image of $T_pS$ in $N_{R/P}|_p$ are both general, then $N_{Z/P}|_R$ is a general degree 1 inflation of $N_{R/P}$.

We say that a bundle $E$ *isotrivially degenerates* to a bundle $E_0$ if there exists a pointed smooth (not necessarily projective) curve $(\Delta, 0)$ and a $\Delta$-flat bundle $\mathcal{E}$ on $Y \times \Delta$ such that $\mathcal{E}_{Y\times\{0\}} \cong E_0$ and $\mathcal{E}|_{Y\times\{t\}} \cong E$ for every $t \in \Delta \setminus \{0\}$.

**Proposition 2.5.** *Let $E$ a vector bundle on $Y$, and $N$ a non-negative integer. Then $E$ isotrivially degenerates to a vector bundle $E_0$ of the form*

$$E_0 = L_1 \oplus \cdots \oplus L_r,$$

*where the $L_i$ are line bundles and* $\deg L_i + N \leq \deg L_{i+1}$.

For the proof of Proposition 2.5, we need a lemma.

**Lemma 2.6.** *There exists a filtration*

$$E = F_0 \supset F_1 \supset \cdots \supset F_{r-1} \supset F_r = 0,$$

*satisfying the following properties.*

(1) *For every $i \in \{0, \dots, r-1\}$, the subquotient $F_i/F_{i+1}$ is a line bundle.*
(2) *Set $L_i = F_i/F_{i+1}$ for $i \in \{1, \dots, r-1\}$ and $L_r = F_0/F_1$. For every $i \in \{1, \dots, r-1\}$, we have*

$$\deg L_i + N \leq \deg L_{i+1}.$$

*Proof.* The statement is vacuous for $r = 0$ and 1. So assume $r \geq 2$. Note that if $F_\bullet$ is a filtration of $E$ satisfying the two conditions, and if $L$ is invertible, then $F_\bullet \otimes L$ is such a filtration of $E \otimes L$. Therefore, by twisting by a line bundle of large degree if necessary, we may assume that $\deg E \geq 0$.

Let us construct the filtration from right to left. Let $L_{r-1} \subset E$ be a line bundle with $\deg L_{r-1} \leq -N$ and with a locally free quotient. Set $F_{r-1} = L_{r-1}$. Next, let $L_{r-2} \subset E/F_{r-1}$ be a line bundle with $\deg L_{r-2} \leq \deg L_{r-1} - N$ and with a locally free quotient. Let $F_{r-2} \subset E$ be the preimage of $L_{r-2}$. Continue in this way. More precisely, suppose that we have constructed

$$F_j \supset F_{j+1} \supset \cdots \supset F_{r-1} \supset F_r = 0$$

such that $L_i = F_i/F_{i+1}$ satisfy

$$\deg L_i \leq \deg L_{i+1} - N,$$

and suppose $j \geq 2$. Then let $L_{j-1} \subset E/F_j$ be a line bundle with $\deg L_{j-1} \leq \deg L_j - N$ with a locally free quotient. Let $F_{j-1} \subset E$ be the preimage of $L_{j-1}$. Finally, set $F_0 = E$.

Condition (1) is true by design. Condition (2) is true by design for $i \in \{1, \dots, r-2\}$. For $i = r-1$, note that $\deg L_{r-1} \leq -N$ by construction. On the other hand, we must have $\deg L_r \geq 0$. Indeed, we have $\deg E \geq 0$ but every sub-quotient of $F_\bullet$ except $F_0/F_1$ has negative degree. Therefore, condition (2) holds for $i = r-1$ as well. $\square$

*Proof of Proposition 2.5.* Let $F_\bullet$ be a filtration of $E$ satisfying the conclusions of Lemma 2.6. It is standard that a coherent sheaf degenerates isotrivially to the associated graded sheaf of its filtration. The construction goes as follows. Consider the $\mathcal{O}_Y[t]$-module

$$\bigoplus_{n\in\mathbf{Z}} t^{-n}F_n,$$

where $F_n = 0$ for $n > r$ and $F_n = E$ for $n < 0$. The corresponding sheaf $\mathcal{E}$ on $Y \times \mathbf{A}^1$ is coherent, $k[t]$-flat, satisfies $\mathcal{E}_{Y\times\{t\}} \cong E$ for $t \neq 0$, and $\mathcal{E}_{Y\times\{0\}} \cong L_1 \oplus \cdots \oplus L_r$. □

## 3. Proof of the main theorem

3.1. **The split case.** As a first step, we treat the case of a suitable direct sum of line bundles and allow the source curve to be singular.

**Proposition 3.1.** *Let $E = L_1 \oplus \cdots \oplus L_r$, where the $L_i$ are line bundles on $Y$ with $\deg L_1 \geq 2g_Y - 1$ and $\deg L_{i+1} \geq \deg L_i + (2g_Y - 1)$ for $i \in \{1, \dots, r-1\}$. There exists a nodal curve $X$ and a finite flat map $\phi \colon X \to Y$ of degree $d = r + 1$ such that*

1. *we have $E_\phi \cong E$, and*
2. *the normalization $X^\nu$ is isomorphic to $d$ disjoint copies of $Y$.*

The proof is inductive, based on the following "pinching" construction. Let $\psi \colon Z \to Y$ be a finite cover of degree $d-1$. Let $X$ be the reducible nodal curve $Z \cup Y$, where $Z$ and $Y$ are attached nodally at distinct points. We have a finite flat map $\phi \colon X \to Y$ that restricts to $\psi$ on $Z$ and is identity on $Y$. Let $D \subset Y$ be the preimage of the nodes.

**Lemma 3.2.** *In the setup above, we have an exact sequence*

$$0 \to E_\psi \to E_\phi \to \mathcal{O}_Y(D) \to 0.$$

*Proof.* The closed embedding $Z \to X$ gives a surjection

$$\phi_*\mathcal{O}_X \to \psi_*\mathcal{O}_Z$$

whose kernel is easily seen to be $\mathcal{O}_Y(-D)$. Factoring out the $\mathcal{O}_Y$ summand from both sides and taking duals yields the claimed exact sequence. □

*Proof of Proposition 3.1.* We use induction on $r$, starting with the base case $r = 0$, which is vacuous.

By the inductive hypothesis, we may assume that there exists a nodal curve $Z$ and a finite cover $\psi \colon Z \to Y$ of degree $d-1$ such that $E_\psi \cong L_2 \oplus \cdots \oplus L_r$ and $Z^\nu$ is a disjoint union of $d-1$ copies of $Y$. Let $X = Z \cup Y \to Y$ be a cover of degree $d$ obtained from $Z \to Y$ by a pinching construction such that $\mathcal{O}_Y(D) = L_1$. Then $X^\nu$ is a disjoint union of $d$ copies of $Y$. By Lemma 3.2, we get an exact sequence

$$0 \to L_2 \oplus \cdots \oplus L_r \to E_\phi \to L_1 \to 0. \tag{3.1}$$

But we have $\operatorname{Ext}^1(L_1, L_i) = H^1(L_i \otimes L_1^\vee) = 0$ since $\deg\left(L_i \otimes L_1^\vee\right) \geq 2g_Y - 1$. Therefore, the sequence (3.1) is split, and we get $E_\phi = L_1 \oplus \cdots \oplus L_r$. The induction step is then complete. □

3.2. **Smoothing out.** We now come to the key step of the proof. This step allows us to pass from singular covers to smooth covers and from particular vector bundles to their deformations.

Let $X$ be a nodal curve, $\phi \colon X \to Y$ a finite flat morphism of degree $d$, and $E$ the associated Tschirnhausen bundle. The inclusion $E^\vee \to \phi_*\mathcal{O}_X$ induces a surjection $\operatorname{Sym}^* E^* \to \phi_*\mathcal{O}_X$. Taking the relative spectrum gives an embedding of $X$ in the total space of the vector bundle associated to $E$ (which we also denote by $E$). We call $X \subset E$ the *canonical affine embedding*.

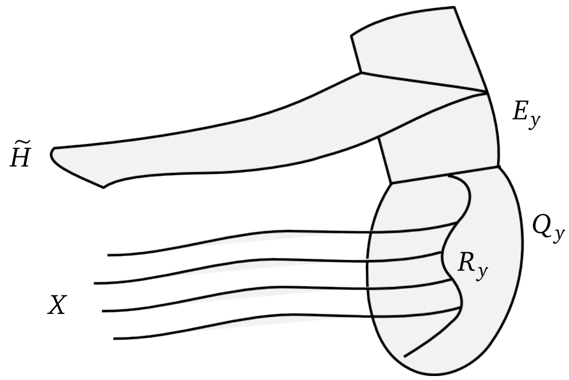


FIGURE 1. Attaching rational normal curves to $X$ to make the normal bundle positive

**Proposition 3.3** (Key)**.** *There exists a line bundle $L$ on $Y$, a smooth curve $X'$, and a finite morphism $\phi' : X' \to Y$ such that the following hold.*

(1) *The Tschirnhausen bundle of $\phi'$ is $E \otimes L$.*
(2) *We have $H^1(X', N_{X'/E'}) = 0$, where $X' \subset E'$ is the canonical affine embedding.*

*Furthermore, there exists an $n$ (depending on $X$), such that the above holds for any $L$ of degree at least $n$.*

The crucial idea in the proof of Proposition 3.3 is the following construction. Let $S \subset Y$ be a finite set such that $X \to Y$ is étale over all points of $S$. Consider the compactification $P = \mathbf{P}(E^\vee \oplus \mathcal{O}_Y)$ of $E$. Let $H \cong \mathbf{P}E^\vee \subset P$ be the family of hyperplanes at infinity, where the embedding $H \subset P$ is defined by the projection $E^\vee \oplus \mathcal{O}_Y \to E^\vee$. The complement of $H \subset P$ is the vector bundle $E$. For $y \in S$, the set $X_y \subset P_y \cong \mathbf{P}^{d-1}$ consists of $d$ points in general position. Therefore, there exists a smooth rational normal curve $R_y$ in $P_y$ containing $X_y$. Let $\widetilde{P} \to P$ be the blow up at $\bigsqcup_{y \in S} H_y$. Denote by the same symbol $R_y$ the proper transform of $R_y$ in $\widetilde{P}$, and by $\widetilde{H}$ the proper transform of $H$ in $\widetilde{P}$.

The fiber of $\widetilde{P} \to Y$ over $y \in S$ consists of two components. One is the exceptional divisor $E_y$ of the blow-up. The second is the proper transform $Q_y$ of $P_y$, which is a copy of $P_y$. The two components intersect transversally along a $\mathbf{P}^{d-2}$. See Figure 1 for a picture of this construction.

Set $Z = X \cup_{y \in S} R_y$. The proof of Proposition 3.3 proceeds through the following result.

**Proposition 3.4.** *For large enough $n$, a general choice of $S \subset Y$ of size $n$, and a general choice of rational normal curves $R_y$ for $y \in S$, the curve $Z$ is unobstructed in $\widetilde{P}$ and is a flat limit of smooth curves in $\widetilde{P}$.*

*Furthermore, if $n$ is large enough, the set $S$ can be chosen so that $\mathcal{O}_Y(S)$ is isomorphic to any prescribed line bundle of degree $n$.*

We need several preparatory lemmas for the proof of Proposition 3.4. First, we set some notation. Denote the normal bundle $N_{Z/\widetilde{P}}$ by $N$ for brevity. Let $\nu \colon Z^\nu \to Z$ be the normalization. Note that $Z^\nu$ is the disjoint union of the components $X_1, \dots, X_l$ of the normalization $X^\nu$ of $X$ and the rational curves $R_y$ for $y \in S$. Denote by $\nu_X : X^\nu \to Z$, $\nu_i : X_i \to Z$, and $\nu_y : R_y \to Z$ the natural maps. Let $\gamma \subset Z$ be the set of nodes of $X$ and $\Gamma \subset X^\nu$ its preimage. Note that every point of $\gamma$ has two preimages in $\Gamma$. Set $\delta_y = R_y \cap X$ and $\delta = \bigcup_{y \in S} \delta_y$. Then the singular set of $Z$ is $\gamma \cup \delta$.

Let $y$ be a point in $S$.

**Lemma 3.5.** *The restriction of $N$ to $R_y$ is isomorphic to $\mathcal{O}(d+1)^{d-2} \oplus \mathcal{O}(1)$, and the $\mathcal{O}(d+1)^{d-2}$ summand is the image of the natural map*

$$N_{R_y/Q_y} \to N|_{R_y}.$$

*Proof.* In the proof, we drop the subscript $y$ from $R_y$ and $Q_y$. First, note that $N|_R$ is a vector bundle of rank $(d-1)$ and degree $(d-2)(d+1)+1$. The map $N_{R/Q} \to N|_R$ is the composite

$$N_{R/Q} \to N_{R/\widetilde{P}} \to N_{Z/\widetilde{P}}|_R = N|_R.$$

Using that $X$ is transverse to $Q$, a simple local computation shows that the injection $N_{R/Q} \to N|_R$ remains an injection when restricted to any point of $R$ (see Remark 2.4). Since $R \subset Q \cong \mathbf{P}^{d-1}$ is a rational normal curve, we know that $N_{R/Q} \cong \mathcal{O}(d+1)^{d-2}$. We thus get an exact sequence

$$0 \to \mathcal{O}(d+1)^{d-2} \to N|_R \to \mathcal{O}(1) \to 0.$$

Since $\operatorname{Ext}^1(\mathcal{O}(1), \mathcal{O}(d+1)) = 0$, this sequence splits. □

We call the $\mathcal{O}(d+1)^{d-2}$ the *vertical* summand of $N|_{R_Y}$. Let $V \subset N|_\delta$ be the image of all the vertical summands of $R_y$ for $y \in S$. Set $F = N|_\delta / V$. Denote by $V_y$ and $F_y$ the restrictions of $V$ and $F$ to $\delta_y$, respectively.

**Lemma 3.6.** *We have an exact sequence*

$$0 \to N_{X/\widetilde{P}} \to N|_X \to F \to 0.$$

*Proof.* We have an injective map of vector bundles $N_{X/\widetilde{P}} \to N|_X$ that drops rank by 1 at every point of $\delta$ (see Remark 2.4). Let $p \in \delta$ lie over $y \in Y$. To show that the cokernel of $N_{X/\widetilde{P}} \to N|_X$ is $F$, we must show that the image of $N_{X/\widetilde{P}}|_p \to N|_p$ is $V|_p$. But this follows from the following commutative diagram

$$\begin{tikzcd}
T_{Q_y}|_p \arrow[r] \arrow[d, equal] & N_{R_y/Q_y}|_p \arrow[r] \arrow[d] & 0 \\
N_{X/\widetilde{P}}|_p \arrow[r] & N|_p. &
\end{tikzcd}$$

□

Let $X_1, \ldots, X_\ell$ be the components of the normalization $X^\nu$ of $X$, and let $\nu_i \colon X_i \to X$ be the induced maps. Let $p \in \delta$ be a point lying on $X_i$. Let $N_i$ be the kernel of the map $N|_X \to F_{\delta\setminus\{p\}}$. Then we have an exact sequence

$$0 \to \nu_i^* N_{X/\widetilde{P}} \to N_i \to F_p \to 0.$$

In other words, $N_i$ is an inflation of $\nu_i^* N_{X/\widetilde{P}}$ at $p$. Furthermore, if the tangent line $T_pR_y \subset T_pQ_y$ is general, then $N_i$ is a general inflation of $\nu_i^* N_{X/\widetilde{P}}$ at $p$. Note that we have an injection $\nu_i^* N_i \to \nu_i^* N$, which is an isomorphism except at the points of $\delta \setminus \{p\}$.

**Lemma 3.7.** *If the size of $S$ is large, its points are general, and the rational normal curves $R_y$ are general, then we have $H^1(X_i, \nu_i^* N) = 0$.*

*Proof.* The preceding analysis show that $\nu_i^* N$ contains a general degree $|S|$ inflation of $\nu_i^* N_{X/\widetilde{P}}$. The lemma follows from Proposition 2.1. □

Thanks to Lemma 3.5 and Lemma 3.7, the restriction of $N$ to the components of the normalization of $Z$ has no higher cohomology. This is necessary, but not sufficient to conclude that $H^1(N) = 0$. To get the latter, we need a more careful analysis.

Recall that $\Gamma \subset X^\nu$ is the preimage of the singular set $\gamma$ of $X$. Set $M = \nu_X^* N_{X/\widetilde{P}}(-\Gamma)$. Note that the natural map $M \to \nu_X^* N_{X/\widetilde{P}}$ is an isomorphism at the points of $\delta$. From now on, we identify $M|_\delta$ and $N|_\delta$.

The surjection $M \to M|_\delta$ gives a surjection

$$\phi_* M \to \phi_*(M|_\delta).$$

Set $R = \cup_{y \in S} R_y$. The surjection $N|_R \to M|_\delta$ gives a map

$$\phi_*(N|_R) \to \phi_*(M|_\delta).$$

Let $W$ be the cokernel; it is supported on $S$. For $y \in S$, we have the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \phi_*\left(\mathcal{O}_{R_y}(d+1)^{d-2}\right) \arrow[r] \arrow[d] & \phi_*\left(N|_{R_y}\right) \arrow[r] \arrow[d] & \phi_*\left(\mathcal{O}_{R_y}(1)\right) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \phi_*\left(V|_{\delta_y}\right) \arrow[r] \arrow[d] & \phi_*\left(M|_{\delta_y}\right) \arrow[r] \arrow[d] & \phi_*\left(F_y\right) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & W_y \arrow[r, equal] & W_y. &
\end{tikzcd} \tag{3.2}$$

In particular, we get $W_y \cong k^{d-2}$. Let $K$ be the kernel of the surjection $\phi_* M \to W$. Then $K$ is a vector bundle of rank $d(d-1)$ on $Y$.

**Proposition 3.8.** *If the size of $S$ is large, its points are general, and the rational normal curves $R_y$ are general, then we have $H^1(Y,K) = 0$.*

*Proof.* The proof is by degeneration. For simplicity, assume first that $S$ consists of a single point $y$. Take a point $p \in \delta_y$. Consider a one parameter degeneration of $R_y$ to a reducible nodal curve $R' = R'_1 \cup R'_2$ contained in $Q_y$ and containing $\delta_y$ of the following form: $R'_1$ is a line containing $p$, and $R'_2$ is a smooth rational curve of degree $(d-2)$ containing $\delta_y \setminus \{p\}$, with $R'_1$ and $R'_2$ meeting nodally at one point away from $\delta_y$. More formally, let $(T, 0)$ be a smooth pointed curve, and $\mathcal{S} \subset Q_y \times T$ a smooth surface containing $\delta_y \times T$ such that $\mathcal{S}_0 = R'$ as described above and $\mathcal{S}_t \subset Q_y$ is a smooth rational normal curve for $t \neq 0$. We may assume that $T_p R' = T_p R'_1$ is a general line in $T_p Q_y$. Set $\mathcal{Z} = (X \times T) \cup \mathcal{S} \subset \widetilde{P} \times T$. Let $\mathcal{N} = N_{\mathcal{Z}/(\widetilde{P} \times T)}$ and $\mathcal{M} = \mathcal{N} \otimes \mathcal{O}_{\mathcal{S}}(-\Gamma \times T)$.

Define $\mathcal{L}$ by the exact sequence

$$0 \to N_{\mathcal{S}/(Q_y \times T)} \to N_{\mathcal{Z}/(\widetilde{P} \times T)}|_{\mathcal{S}} \to \mathcal{L} \to 0.$$

For a general $t \in T$, we have $\mathcal{L}_t \cong \mathcal{O}(1)$. For $t = 0$, we have $\mathcal{L}_0|_{R'_1} \cong \mathcal{O}$ and $\mathcal{L}_0|_{R'_2} \cong \mathcal{O}(1)$. Set $\mathcal{F} = \mathcal{L}|_{\delta \times T}$. We have a surjection $\mathcal{M} \to \mathcal{F}$.

Consider $\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}_{\mathcal{S}}(R'_2)$. Then $\mathcal{L}'$ is isomorphic to $\mathcal{L}$ away from $t = 0$. On $t = 0$, we have $\mathcal{L}'|_{R'_1} \cong \mathcal{O}(1)$ and $\mathcal{L}'|_{R'_2} \cong \mathcal{O}$. Clearly, $\mathcal{L}'|_{\delta \times T}$ is isomorphic to $\mathcal{L}|_{\delta \times T}$. Identify them.

Define $\mathcal{W}$ by the sequence

$$\phi_* \mathcal{L}' \to \phi_* \mathcal{F} \to \mathcal{W} \to 0.$$

Then $\mathcal{W}$ is a $T$-flat $Y \times T$ module supported on $S \times T$. Let $\mathcal{K}$ be the kernel of the surjection $\phi_* \mathcal{M} \to \mathcal{W}$. Over a generic $t \in T$, the bundle $\mathcal{K}|_t$ on $Y$ is simply the bundle $K$. Let us analyze $\mathcal{K}|_0$. Observe that the image of $\phi_* \mathcal{L}'|_0 \to \phi_* \mathcal{F}|_0$ contains the global section of $\mathcal{F}|_0$ that is non-zero at $p$ and zero at the other points of $\delta$. Said differently, $\mathcal{W}_0$ is a further quotient of $\phi_*\left(\mathcal{F}|_{(\delta \setminus \{p\}) \times \{0\}}\right)$.

Define $M^+$ on $X^\nu$ by the sequence

$$0 \to M^+ \to \nu_X^* M \to \mathcal{F}|_{(\delta \setminus \{p\}) \times \{0\}} \to 0.$$

Then $M^+$ is isomorphic to $\nu_X^* N_{X/\widetilde{P}}(-\Gamma)$ on all components of $X^\nu$ except the one containing $p$. On the component containing $p$, it is a degree 1 inflation (see the discussion after Lemma 3.6) of $\nu_i^* N_{X/\widetilde{P}}(-\Gamma)$ at $p$. Furthermore, since $T_p R'$ is a general line in $T_p Q_y$, it is a general such inflation. Since $\mathcal{W}_0$ is a quotient of $\phi_*\left(\mathcal{F}_{(\delta\setminus\{p\})\times\{0\}}\right)$, we get an injection $\phi_* M^+ \to \mathcal{K}|_0$.

Now consider $S \subset Y$ of size $n$, say $S = \{y_1, \dots, y_n\}$. Pick $p_{ij} \in X$ over $y_i$ such that a general inflation of $\nu_X^* N_{X/\widetilde{P}}(-\Gamma)$ at the points $p_{ij}$ has vanishing $H^1$. By considering an $n$-parameter degeneration of the $n$ rational normal curves $R_y$ for $y \in S$, we obtain a family of bundles $\mathcal{K}$ whose generic fiber $\mathcal{K}|_t$ agrees with $K$ and whose special fiber $\mathcal{K}|_0$ contains $\phi_* M^+$, where $M^+$ is a general inflation of $\nu_X^* N_{X/\widetilde{P}}(-\Gamma)$ at the points $p_{ij}$. It follows that $H^1(Y, \mathcal{K}|_0) = 0$, and hence $H^1(Y, K) = 0$ by semicontinuity. □

We now have all the tools to prove Proposition 3.4.

*Proof of Proposition 3.4.* Retain the notation introduced so far in § 3.2.

We have the exact sequence

$$0 \to N \to \nu_* \nu^* N \to N|_{\gamma\cup\delta} \to 0.$$

The long exact sequence in cohomology gives

$$H^0(\nu^* N) \to H^0(N|_{\gamma\cup\delta}) \to H^1(N) \to H^1(\nu^* N) \to 0.$$

By Lemma 3.5, we know that $H^1(\nu_y^* N) = 0$. By Lemma 3.6, we know that $H^1(\nu_X^* N) = 0$. Therefore, we get $H^1(\nu^* N) = 0$.

We now show that $H^0(\nu^* N) \to H^0(N|_{\gamma\cup\delta})$ is surjective. First, by the definition of $W$, we have the exact sequence

$$\bigoplus_{y\in S} H^0(\nu_y^* N) \to H^0(N|_\delta) \to H^0(W) \to 0. \tag{3.3}$$

Second, the sequence

$$0 \to K \to \phi_* \nu_X^* N_{Z/\widetilde{P}} \to \phi_*\left(\nu_X^* N_{Z/\widetilde{P}}|_\Gamma\right) \oplus W \to 0,$$

along with Proposition 3.8, gives a surjecton

$$H^0(\nu_X^* N) \to H^0(\nu^* N|_\Gamma) \oplus H^0(W). \tag{3.4}$$

By combining (3.3) and (3.4), we get a surjection

$$H^0(\nu^* N) \to H^0(\nu^* N|_\Gamma) \oplus H^0(N|_\delta). \tag{3.5}$$

Since $H^0(\nu^* N|_\Gamma) \to H^0(N|_\gamma)$ is surjective, we get that $H^0(\nu^* N) \to H^0(N|_{\gamma\cup\delta})$ is surjective. We conclude that $H^1(N) = 0$.

Consider the effect of enlarging $S$ to $S^+ = S \cup \{y\}$ where $y \in Y \setminus S$ is a point over which $X \to Y$ is étale. Denote by $Z^+$, $N^+$, $F^+$, and $V^+$ the analogues of $Z$, $N$, $F$, and $V$ for $S^+$. Note that $Z^+ = Z \cup R_y$. Recall that $N^+|_{\delta_y} = V_y^+ \bigoplus F_y^+$. The exact sequence

$$0 \to N \to N^+|_Z \to F_y^+ \to 0$$

shows that $H^1(N^+|_Z) = 0$ and $H^0(N^+|_Z) \to H^0(F_y^+)$ is surjective. Since $H^1(N^+|_{R_y}) = 0$ and $H^0(N^+|_{R_y}) \to H^0(V_y^+)$ is surjective, we deduce that $H^1(N^+) = 0$. Therefore, enlarging $S$ retains the vanishing $H^1(N) = 0$.

Since $H^1(N) = 0$, the Hilbert scheme of $\widetilde{P}$ is smooth at $[Z]$. In particular, every first order deformation of $Z \subset \widetilde{P}$ extends. To show that $Z$ is the limit of smooth curves, it suffices to show that for every node $p \in Z$, the natural map $N \to k_p$ is surjective on global sections, where $k_p$ is a skyscraper sheaf at $p$. The surjection $N|_p \to k_p$ is a part of the exact sequence

$$0 \to T_p\widetilde{P}/T_pZ \to N|_p \to k_p \to 0.$$

In particular, for $p \in \delta$ the map $N \to k_p$ is the same as the map $N \to F_p$.

Consider a node $p \in \gamma$. The exact sequence

$$0 \to N \otimes I_\Gamma \to \nu_* \nu^* N \to \nu_*(\nu^* N|_\Gamma) \oplus N|_\delta \to 0$$

and the surjection (3.5) implies $H^1(N \otimes I_\Gamma) = 0$. Therefore, $H^0(N) \to H^0(N|_p)$ and hence $H^0(N) \to H^0(k_p)$ are surjective for all $p \in \gamma$.

Next, consider a node $p \in \delta_y$. Let $S^- = S \setminus \{y\}$. Denote by $Z^-$, $N^-$, $F^-$, and $V^-$ the analogues of $Z$, $N$, $F$, and $V$ for $S^-$. Let $\mu\colon Z^- \sqcup R_y \to Z$ be the natural map, which is the normalization of the nodes $\delta_y$ of $Z$. We have a surjection $H^0(R_y, \mu^* N|_{R_y}) \to H^0(F_p)$. If $S$ is large enough, we may assume that we already have $H^1(N^-) = 0$. Then we have a surjection $H^0(Z^-, \mu^* N|_{Z^-}) \to H^0(F_y)$. Combining the two, we see that we have a surjection $H^0(N) \to H^0(F_p)$.

Finally, assume that $n$ is large enough so that the conclusions above hold for an $S$ of size $n - 2g_Y$. Then we may enlarge $S$ to a set $S^+$ by adding an appropriate set of $2g_Y$ points so that the same conclusions hold and $\mathcal{O}_Y(S^+)$ is isomorphic to a given line bundle of degree $n$. $\square$

We finally prove the key proposition.

*Proof of Proposition 3.3.* By Proposition 3.4, there exists a family of smooth curve in $\widetilde{P}$ whose flat limit is $Z$. Let $X'$ be a general member of such a family. This curve satisfies the following conditions

(1) $\deg(X' \cdot E_y) = d - 1$ for all $y \in S$,
(2) $\deg(X' \cdot Q_y) = 1$ for all $y \in S$,
(3) $X' \cap \widetilde{H} = \varnothing$,
(4) $g(X') = g(X) + n(d-1)$,
(5) $H^1(N_{X'/\widetilde{P}}) = 0$.

Let $\widetilde{P} \to P'$ be the blowing down of all the $Q_y$ for $y \in S$. Then $P' \to Y$ is a $\mathbf{P}^{d-1}$ bundle and the map $X' \to P'$ is an embedding. Similarly, $\widetilde{H} \to P'$ is also an embedding and its complement is the total space of the vector bundle $E' = E \otimes \mathcal{O}_Y(S)$. Note that $X'$ and $\widetilde{H}$ remain disjoint in $P'$, and hence we get an embedding $X' \subset E'$.

We claim that $X' \subset E'$ is the canonical affine embedding. Consider the natural map

$$\phi_* \mathcal{O}_{P'}(\widetilde{H}) = \mathcal{O}_Y \oplus E'^{\vee} \to \phi_* \mathcal{O}_{X'}.$$

The vector bundles $\mathcal{O}_Y \oplus E'^{\vee}$ and $\phi_* \mathcal{O}_{X'}$ have the same degree and rank, and the above map between them is an isomorphism at a generic point of $Y$. Therefore, it is an isomorphism. As a result, we get that $E'$ is the Tschirnhausen bundle of $X'$, and the embedding $X' \to E'$ is the canonical affine embedding.

Next, note that we have an injection of vector bundles

$$N_{X'/\widetilde{P}} \to N_{X'/E'}$$

with finite quotient (supported on $\bigcup_{y \in S} X' \cap E_y$). Since $H^1(N_{X'/\widetilde{P}}) = 0$, we deduce that $H^1(N_{X'/E'}) = 0$.

Finally, by the last assertion of Proposition 3.4, we may take $\mathcal{O}_Y(S)$ to be any prescribed line bundle of degree $n$ if $n$ is large enough. $\square$

3.3. **The general case.** We now use the results of § 3.1 and § 3.2 to deduce the main theorem. Denote by $H_{d,g}(Y)$ the moduli stack of degree $d$ and genus $g$ branched covers of $Y$ and by $\mathrm{Vec}_{d-1,g+d-1}(Y)$ the moduli stack of vector bundles of rank $d-1$ and degree $g+d-1$ on $Y$. The stack $H_{d,g}(Y)$ is Deligne–Mumford and of finite type and the stack $\mathrm{Vec}_{d-1,g+d-1}(Y)$ is Artin and locally of finite type. Both stacks are smooth. The rule $\phi \mapsto E_\phi$ gives a morphism

$$\tau \colon H_{d,g}(Y) \to \mathrm{Vec}_{d-1,g+d-1}(Y).$$

**Theorem 3.9.** *Let $E$ be a vector bundle on $Y$. There exists $n$ (depending on $E$) such that for any line bundle $L$ of degree at least $n$, there exists a smooth curve $X$ and a finite flat morphism $\phi \colon X \to Y$ such that $E_\phi \cong E \otimes L$ and such that the map $\tau$ is smooth at $\phi$.*

*Proof.* We begin by analyzing the map $\tau$ on first order deformations. Let $[\phi \colon X \to Y]$ be a point of $H_{d,g}(Y)$ and set $E = E_\phi$. The space of first order deformations of $\phi$ is given by

$$\mathrm{Def}_\phi = H^0(X, N_\phi),$$

where $N_\phi = \mathrm{coker}(T_X \to \phi^* T_Y)$. The space of first order deformations of $E$ is given by

$$\mathrm{Def}_E = H^1(Y, \mathrm{End}\, E).$$

Consider the canonical affine embedding $X \subset E$. We have an exact sequence

$$0 \to T_{E/Y}|_X \to N_{X/E} \to N_\phi \to 0.$$

Note that $T_{E/Y}|_X = \phi^* E$. The long exact sequence on cohomology gives a map

$$H^0(X, N_\phi) \to H^1(X, \phi^* E) = H^1(Y, E \oplus \mathrm{End}\, E).$$

By composing with the projection $H^1(Y, E \oplus \mathrm{End}\, E) \to H^1(Y, \mathrm{End}\, E)$, we get a map

$$H^0(X, N_\phi) \to H^1(Y, \mathrm{End}\, E).$$

It is straightforward to check that this is the map on the first order deformations induced by $\tau$. Note in particular that if $H^1(X, N_{X/E}) = 0$, then $\tau$ is surjective on first order deformations and hence smooth at $\phi$.

Choose an isotrivial degeneration $E_0$ of $E$ of the form

$$E_0 = L_1 \oplus \cdots \oplus L_{d-1},$$

where the $L_i$'s are line bundles with $\deg L_i + (2g_Y - 1) \leq \deg L_{i+1}$. Such a degeneration exists by Proposition 2.5. After replacing $E$ by $E \otimes \lambda$ for a line bundle $\lambda$ of large degree, we may also assume that $\deg L_1 \geq 2g_Y - 1$. By Proposition 3.1, there exists a nodal curve $X_0$ and a finite flat morphism $\phi_0 \colon X_0 \to Y$ with Tschirnhausen bundle $E_0$. By the key proposition Proposition 3.3, there exists $n$ such that for any line bundle $L$ of degree at least $n$, there exists a smooth curve $X_1$ and a map $\phi_1 \colon X_1 \to Y$ with Tschirnhausen bundle $E_1 = E_0 \otimes L$. Furthermore, we also know that $H^1(N_{X_1/E_1}) = 0$. By our analysis above, we get $\tau$ is smooth at $\phi_1$. Since $E \otimes L$ isotrivially degenerates to $E_0 \otimes L$, we deduce that there exists $\phi \colon X \to Y$ with Tschirnhausen bundle $E \otimes L$ and such that $\tau$ is smooth at $\phi$. □

Denote by $M_{d-1,d+g-1}(Y)$ the moduli space of semistable vector bundles of rank $(d-1)$ and degree $(d+g-1)$ on $Y$. Theorem 3.9 along with the openness of smooth maps gives the following.

**Corollary 3.10.** *If $g$ is sufficiently large (depending on $Y$ and $d$), then the Tschirnhausen bundle of a general degree $d$ and genus $g$ cover of $Y$ is stable. Moreover, the rational map $H_{d,g}(Y) \dashrightarrow M_{d-1,d+g-1}(Y)$ given by $\phi \mapsto E_\phi$ is dominant.*

Recall the Maroni locus $M(E) \subset H_{d,g}(Y)$ defined by

$$M(E) = \{\phi \in H_{d,g}(Y) \mid E_\phi \cong E\}.$$

**Corollary 3.11.** *Let $E$ be a vector bundle on $Y$ of rank $(d-1)$ and degree $e$. If $g$ is sufficiently large (depending on $Y$ and $E$), then for every line bundle $L$ on $Y$ of degree $d+g-1-e$, the Maroni locus $M(E \otimes L)$ contains an irreducible component of the expected codimension $h^1(\operatorname{End} E)$.*

*Proof.* Let $U$ be the open subset of the Hilbert scheme of curves in $P = \mathbf{P}(E^\vee \otimes L^\vee \oplus \mathcal{O}_Y)$ of genus $g$, of degree $d$ over $Y$, that are smooth and disjoint from the hyperplane at infinity $\mathbf{P}(E^\vee)$. Every $[X] \in U$ gives $\phi : X \to Y$ with Tschirnhausen bundle $E \otimes L$. Furthermore, the map

$$U \to M(E \otimes L)$$

is surjective with fibers isomorphic to $\operatorname{Aut}(P/Y)$. The normal bundle $N_{X/P}$ is a vector bundle of rank $(d-1)$ and degree $(d+2)(d+g-1)$.

By the key proposition Proposition 3.3, there exists $[X] \in U$ with $H^1(N_{X/P}) = 0$. Then the dimension of $U$ at $[X]$ is given by

$$\dim_{[X]} U = \chi(N_{X/P}) = d^2 + 2d + 3g - 3.$$

We may assume $\deg L$ to be large enough so that $H^1(E \otimes L) = 0$ and $H^0(E^\vee \otimes L^\vee) = 0$. Then

$$\dim \operatorname{Aut}(P/Y) = d^2 + g - 1 + h^1(\operatorname{End} E).$$

It follows that the component of $M(E \otimes L)$ containing $[\phi : X \to Y]$ has dimension

$$2g + 2d - 2 - h^1(\operatorname{End} E) = \dim H_{d,g} - h^1(\operatorname{End} E).$$

□

## 4. Higher dimensions

In this section, we discuss the possibility of having an analogue Theorem 1.1 for higher dimensional $Y$. Let us begin with the following question.

**Question 4.1.** *Let $Y$ be a smooth projective variety, $L$ and ample line bundle, and $E$ a vector bundle of rank $(d-1)$. Is $E \otimes L^n$ a Tschirnhausen bundle for all sufficiently large $n$.*

It is simple to see that the answer to Question 4.1 is negative, at least without additional hypotheses.

**Example 4.2.** Take $Y = \mathbf{P}^4$, and $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$. Then a sufficiently positive twist $E'$ cannot be the Tschirnhausen bundle of a smooth branched cover $X$: the rank 4 vector bundle $\operatorname{Sym}^3 E' \otimes (\det E')^\vee$ becomes very ample, forcing its fourth chern class to be nonzero. Thus a general section would vanish completely at some points, which in turn would lead to a positive dimensional fiber in the hypothetical branched cover. In fact, this analysis shows that $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$ can be a Tschirnhausen bundle of a smooth triple cover if and only if $b = 2a$.

This example illustrating the failure of Theorem 1.1 can be generalized for all degrees $\geq 3$, provided the base $Y$ is allowed to be very high dimensional. In fact, for each degree $d$ there is a "threshold dimension" $m(d)$ where Question 4.1 has a negative answer for some variety $Y$ of dimension $m(d)$.

The results of Lazarsfeld in [11] show that $m(d) \leq d+1$ for all $d \geq 3$.

**Proposition 4.3.** *Let $E$ be a vector bundle of rank $(d-1)$ on $\mathbf{P}^r$, where $r \geq d+1$. Then $E$ must be contain a line bundle summand. Furthermore, for all sufficiently large $n$, $E(n)$ is not a Tschirnhausen bundle of a smooth, connected cover.*

*Proof.* The proof relies on [11, Proposition 3.1] in [11] which states that for a smooth branched cover $\phi : X \to \mathbf{P}^r$ of degree $d \leq r-1$, the pullback map

$$\phi^* : \operatorname{Pic}(\mathbf{P}^r) \to \operatorname{Pic} X$$

is an isomorphism.

From the arguments in the proof of [11, Lemma 3.4] we deduce an isomorphism

$$\mathcal{O}_{\mathbf{P}^r} \oplus E \simeq \mathcal{O}_{\mathbf{P}^r}(l) \oplus E^\vee(l)$$

where $l = \frac{2 \det E}{d} > 0$. This implies $E = \mathcal{O}_{\mathbf{P}^r}(l) \oplus E'$ for some vector bundle $E'$.

Applying the same reasoning with $E$ replaced by $E(n)$ shows that $E$ must have line bundle summands of infinitely many degrees. □

The following example shows that $m(3) = 4$.

**Example 4.4.** We use the well-known structure theorem on branched covers of degree 3 [3]. If $\phi : X \to Y$ is a degree 3 branched cover with Tschirnhausen bundle $E$, then $X \subset \mathbf{P}E$ is the zero locus of a section $s \in H^0(\mathcal{O}_{\mathbf{P}E}(3) \otimes (\det E)^\vee)$. By pushing forward to $Y$, this is the same as a section of the vector bundle $V := \operatorname{Sym}^3 E \otimes (\det E)^\vee$.

Conversely, starting from an arbitrary rank 2 vector bundle $E$ on $Y$, if we twist by a sufficiently positive line bundle $L$, the vector bundle $V$ becomes very ample. If $\dim Y \leq 3$, then a general section $s \in V$ will be non-vanishing everywhere on $Y$, and therefore the induced divisor $X \subset \mathbf{P}E$ will be smooth, and its projection will be a finite degree 3 cover of $Y$.

If $\dim Y > 3$ (and $Y$ projective), then for sufficiently high twists of $E$ every section of $V$ must vanish at some point $y \in Y$ – the resulting divisor in $\mathbf{P}E$ will fail to have finite projection over $y \in Y$.

Therefore, $m(3) = 4$.

The next example shows that, even if we allow $X$ to be singular but still Gorenstein, $m(d)$ is finite.

**Example 4.5.** First, let $\phi : X \to Y$ be an arbitrary finite, flat, degree $d$ branched cover. Then the sheaf $\phi_* \mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$-algebras, and it splits as $\phi_* = \mathcal{O}_Y \oplus E^\vee$.

Suppose over some point $y \in Y$, the multiplication map

$$m : \operatorname{Sym}^2 E^\vee \to \phi_* \mathcal{O}_X$$

is identically zero. Then, writing $k = k(y)$ as the residue field, it follows that

$$(\phi_* \mathcal{O}_X) \otimes k(y) \simeq k[x_1, ..., x_{d-1}]/(x_1, \dots, x_{d-1})^2,$$

i.e. $\phi^{-1}(y)$ is isomorphic to the length $d$ *fat point*, defined by the square of the maximal ideal of the origin in an affine space. When $d \geq 3$, these fat points are not Gorenstein. Since $Y$ is smooth, this implies $X$ is not Gorenstein.

Now, if $E$ is a vector bundle on $Y$ and $L$ is a sufficiently positive line bundle, then the bundle

$$M := \operatorname{Hom}(\operatorname{Sym}^2(E \otimes L)^\vee, \mathcal{O}_Y \oplus (E \otimes L)^\vee)$$

becomes very ample, and therefore a general section $m \in H^0(Y, M)$ will vanish identically at some points $y \in Y$, provided

$$\dim Y \geq \operatorname{rk} M = d\binom{d}{2}.$$

The conclusion is: Once $\dim Y \geq d\binom{d}{2}$, Question 4.1 has a negative answer even if we relax the smoothness assumption on $X$ and allow arbitrary Gorenstein schemes.

4.1. **Modifications of the original question.** Following the discussion in the previous section, natural modified versions of Question 4.1 emerge.

The first obvious question is

**Question 4.6.** *Is the analogue of Theorem 1.1 true for all $Y$ with* $\dim Y \leq d$*?*

We can relax the finiteness assumption on $\phi$:

**Question 4.7.** *Let $Y$ be a smooth projective variety, $E$ a vector bundle in $Y$. Does there exist a line bundle $L$ on $Y$, a smooth variety $X$, and a generically finite map $\phi : X \to Y$ such that $E \otimes L \simeq (\phi_* \mathcal{O}_X / \mathcal{O}_Y)^\vee$?*

*Remark* 4.8. A similar question is addressed in work of Hirschowitz and Narasimhan [7], where it is shown that any vector bundle on $Y$ is the direct image of *some* line bundle on a smooth variety $X$ under a generically finite morphism.

We can keep the finiteness requirement on $\phi$ in exhange for smoothness of $X$. We end the paper with the following open-ended question:

**Question 4.9.** *What singularity assumptions on $X$ (or the fibers of $\phi$) yield a positive answer to Question 4.1?*

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