Claire Voisin on the question of rationality

February 27, 2019

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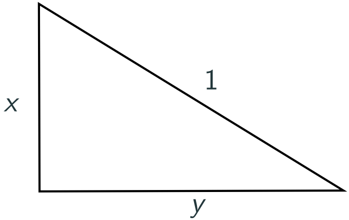

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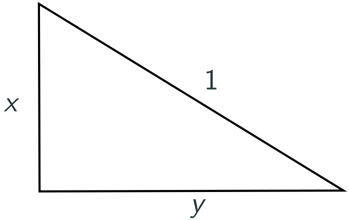

All the solutions:

$$x = \frac{1 - t^2}{1 + t^2} \qquad y = \frac{2t}{1 + t^2}.$$

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Two systems of equations ...

Variables: x, y
Equations: $x^2 + y^2 = 1$.

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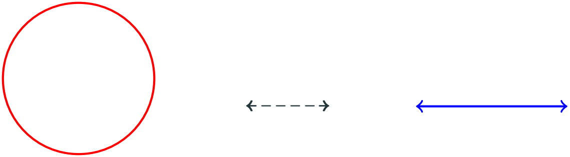

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Algebraic varieties

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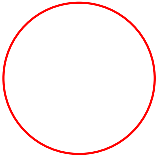

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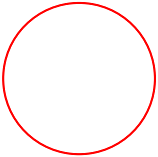

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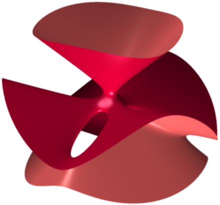

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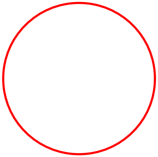

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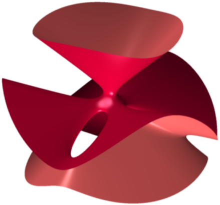

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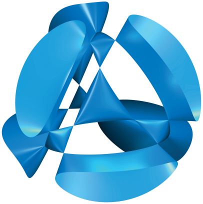

A Kummer K3

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$\mathbf{A}^n =$ The ambient space (no equations)!

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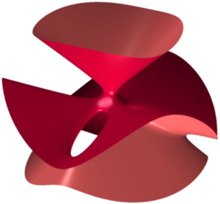

$\dashleftarrow$ rational map $\dashrightarrow$

System of equations $\dashleftarrow$ Coördinate change $\dashrightarrow$ No equations!

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4. Varieties defined by one cubic ?
   4.1 Cubic curves: not rational (ancient)
   4.2 Cubic surfaces: rational (Castelnuovo, Enriques: Early 1900s)
   4.3 Cubic threefolds: not rational (Clemens–Griffiths: 1972)
   4.4 Cubic fourfolds and higher: ???

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So we have the Artin–Mumford invariant

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as a candidate to detect non-rationality.

But $H^3(X, \mathbf{Z})_{\mathrm{tors}} = 0$ for all interesting examples.

Photo credit: CNRS News Article “Claire Voisin, 2016 CNRS Gold Medal”

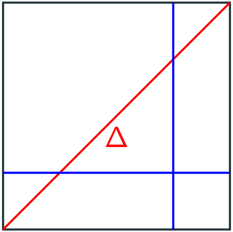
Decomposition of the diagonal

Definition (Voisin, 2015)

X admits a decomposition of the diagonal if in $\mathrm{Chow}(X \times X)$,

$$[\Delta] \sim \{x\} \times X + \alpha$$

for some α supported on $X \times Z$ for $Z \subsetneq X$.



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Theorem (Voisin, 2015)

1. *X rational $\implies$ X admits a decomp. of the diagonal.*
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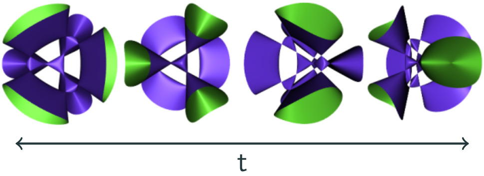
1. *X rational $\implies$ X admits a decomp. of the diagonal.*
2. *X admits decomp. of the diagonal $\implies$ $H^3(X,\mathbf{Z})_{\mathrm{tors}} = 0$.*
3. *If X_t is a family of varieties such that some X_{t_0} does not admit a decomp. of the diagonal, then neither does X_t for almost all t.*

For example, $X_t = \{x^4 + y^4 + z^4 + w^4 - txyzw = 0\}$.



Decomposition of the diagonal

New technique for non-rationality theorems:

1. Consider a family X_t.
2. Find a t_0 such that X_{t_0} does not admit a decomposition of the diagonal (for example, show $H^3(X_{t_0}, \mathbf{Z})_{\mathrm{tors}} \neq 0$).
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New technique for non-rationality theorems:

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- Quartic double solids (Voisin, 2015),
- Rationality is not deformation invariant (Hassett–Pirutka–Tschinkel, 2016).
- Hypersurfaces in $\mathbf{P}^{n+1}$ of degree $d \geq \log_2 n + 2$ (Schreieder, 2018)

There's a lot more...

1. Kodaira problem,
2. Green's conjecture for canonical curves,
3. Chow rings of K3 surfaces,
4. Many questions related to the Hodge conjecture.