Combinatorics and dynamics of Harder-Narasimhan filtrations

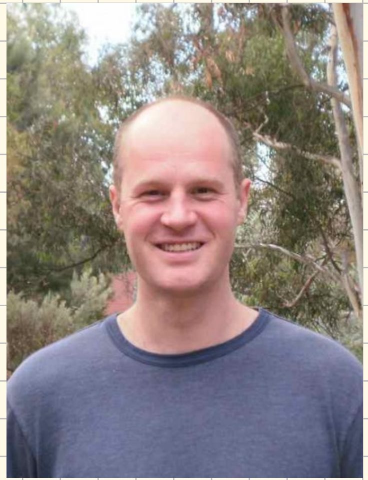
- Anand Deopurkar

with

Asilata Bapat

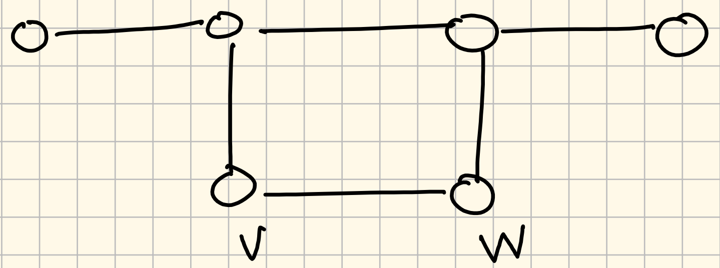
Anthony Licata

(ANU)



Artin-Tits braid groups

Γ



finite simple graph

$\rightsquigarrow$

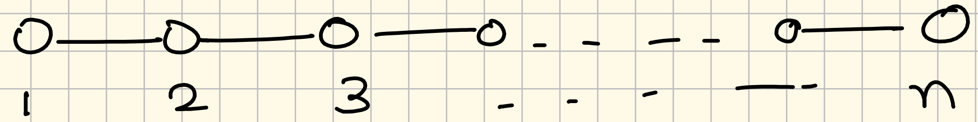
B_Γ = Braid gp associated to Γ

$= \langle \sigma_v \mid v \in \mathrm{Vertex}(\Gamma) \rangle \,/\,$ Relations

$\overset{v}{\circ}\!\!-\!\!-\!\!-\!\!\overset{w}{\circ} \implies \sigma_v \sigma_w \sigma_v = \sigma_w \sigma_v \sigma_w$ (Braid)

$\overset{v}{\circ} \quad \overset{w}{\circ} \implies \sigma_v \sigma_w = \sigma_w \sigma_v$

Example:



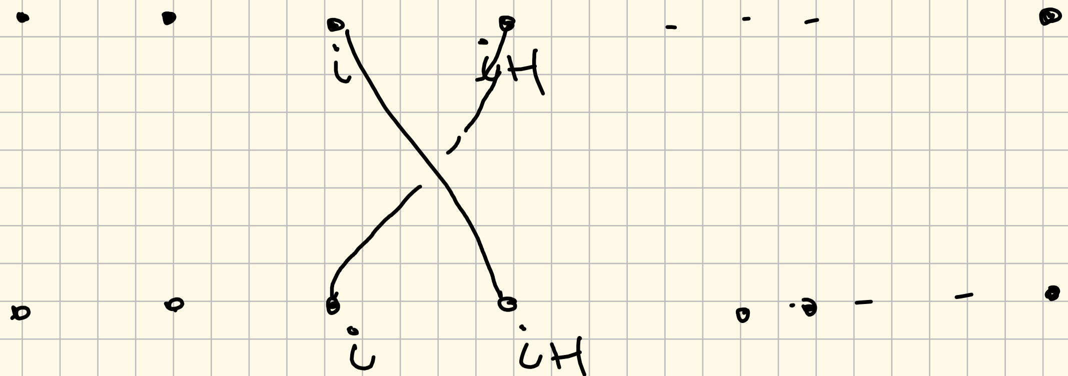
$=$ $(12)(23)(12)$

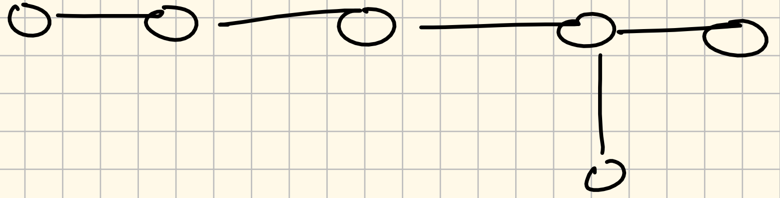
$= (23)(12)(23)$

$$\left\langle \sigma_i \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \leftarrow \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \leftarrow \ |i-j| \geq 2 \end{array} \right\rangle$$

"Usual" braid group on $(n+1)$ strands.

$$W_\Gamma \cong S_n \qquad \sigma_i^2 = 1 \qquad \sigma_i = (i, i+1)$$

$\sigma_i =$ 



$\Gamma \rightsquigarrow B_\Gamma$

$$\langle \sigma_i \mid \text{Relations} \rangle = B_\Gamma \quad \text{Braid group}$$

$$\downarrow$$

Same gens + Relations + extra relations $\quad W_\Gamma \quad$ Coxeter group

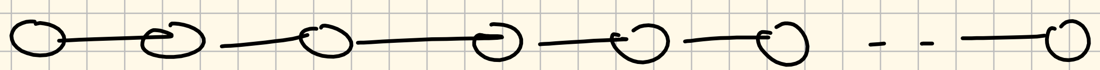
↳ $\sigma_i^2 = 1$

↳ Simpler

Well-understood.

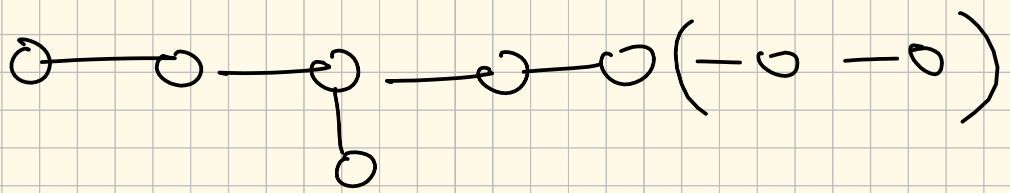
<u>Ex.</u> <u>Thm</u>: W_Γ is finite <u>iff</u> Γ is an A-D-E dynkin diagram

$\underline{W_\Gamma}$ finite $\Longleftrightarrow$



A

D



E.

B_Γ is always infinite.

$\hookrightarrow$ __Not__ very well understood.

__Ex.__ Rep. theory of B_Γ $\leftarrow$ Mystery

__Q:__ Does B_Γ admit a faithful fin. dim. representation?

Does $B_\Gamma \subset \underline{GL_n}$ for some n ??

Representations / actions of B_Γ

How does one study a group? $\quad \mathbb{C}^n$ to somthing else.

$G \circlearrowright X \leftarrow$ object that you like/understand.

$\hookrightarrow$ faithful.

$G \subset \mathrm{Aut}(X) \quad \to$ Tools to understand G.

$X = \mathbb{C}^n$, finite dim. V space.

$G \circlearrowright \mathbb{C}^n \rightsquigarrow G \subset Gl_n$.

faithful

$G = B_\Gamma$ this proves difficult.

There is natural ~~X~~ on which G acts.
(conj. faithful)

Spherical Objects and twists

$G \curvearrowright X$
$\downarrow$
Triang. Category

$\mathcal{C}$ = $\mathbb{C}$-linear triangulated category.
(finite type: for $X, Y \in \mathcal{C}$
the vector space $\bigoplus_n \mathrm{Hom}(X, Y[n])$ is fin. dim)

+ n-Calabi-Yau. i.e.

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(Y, X[n])^*$$

<u>e.g.</u> $\mathcal{C} = D^b \mathrm{Coh}(n\text{-CY manifold})$.

$\mathcal{C}$ = $\mathbb{C}$-linear $\underline{\underline{n}}$-Calabi-Yau category.

$X \in \mathcal{C}$ is called **spherical** if :–

$$\mathrm{Hom}(X, X[k]) = \left.\begin{cases} \mathbb{C} \ \langle id \rangle. & k=0 \\ 0 & \\ 0 & \\ \vdots & \\ 0 & \\ \mathbb{C} & k=n \end{cases}\right\} H^*(S^n; \mathbb{C})$$

The simplest possible End.

$\mathcal{C} \ni X \leftarrow$ spherical. $\qquad \underline{\mathrm{Hom}^*(X,Y)} = \bigoplus \mathrm{Hom}(X, Y[n])$

Then it gives an auto-equivalence

$$\boxed{\begin{aligned} \sigma_X : \mathcal{C} &\longrightarrow \mathcal{C} \\ Y &\longmapsto \underbrace{\sigma_X(Y)} \end{aligned}} \qquad X \xrightarrow{\sigma_X} X[1-n]$$

(Recall:

$$X \underset{\mathbb{C}}{\otimes} \mathrm{Hom}^*(X,Y) \xrightarrow{ev} Y$$

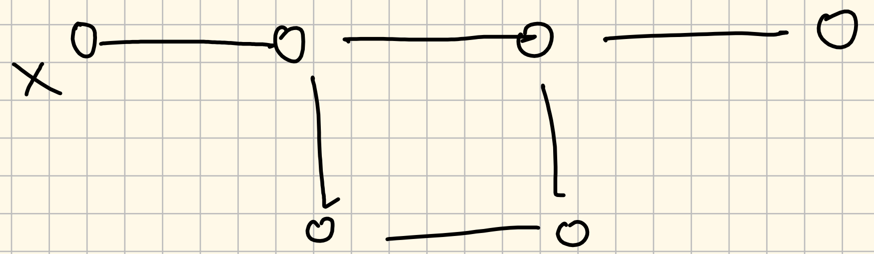
$$\sigma_X(Y) = \mathrm{Cone}(\underline{ev}))$$

From now on, $\mathcal{C}$ will be a 2-CY category

$n=2$.

$\sigma_X : X \longmapsto X[-1]$.

$Br_\Gamma \curvearrowright \mathcal{C}$

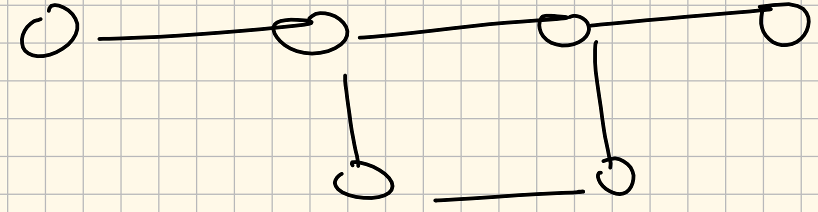


$\circ$ represents a spherical obj of $\mathcal{C}$.

$\mathrm{Hom}^*(X,Y) = 0$ dim

or 1 dim

Def: A Γ-configuration of spherical obj of $\mathcal{C}$ is a collection $\{X_v \mid v \in \mathrm{Vertex}(\Gamma)\}$ such that

$\Gamma =$ 

$$\mathrm{Hom}^*(X,Y) = 0 \quad \text{if} \quad \underset{X}{\circ} \quad \underset{Y}{\circ}$$
$$= \mathbb{C} \quad \text{if} \quad \underset{X}{\circ}\!\!-\!\!\!-\!\!\underset{Y}{\circ}$$

Prop: In this case σ_X, σ_Y satisfy the braid relations. (Khovanov, Seidel, Thomas, ...)

i.e. $\sigma_X \sigma_Y = \sigma_Y \sigma_X$ if $\overset{X}{\circ} \quad \overset{Y}{\circ}$

$\sigma_X \sigma_Y \sigma_X = \sigma_Y \sigma_X \sigma_Y$ if $\underset{X}{\circ}\!\!-\!\!\!-\!\!\underset{Y}{\circ}$

<u>Upshot</u>: A Γ-config of sphericals in $\mathcal{C}$

$$\Downarrow$$

An action of $B_\Gamma \circlearrowright \mathcal{C}$

σ_v acts by σ_{X_v}

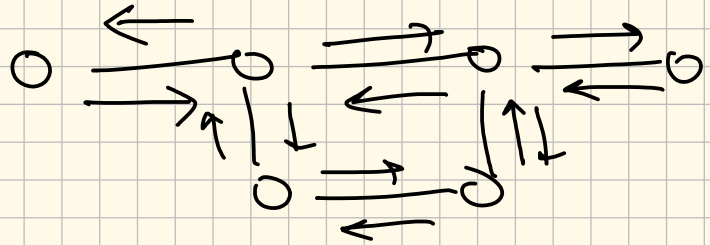
As a <u>result</u>, $\boxed{B_\Gamma \circlearrowright \mathcal{C}}$ ← triang. category

↪ Everywhere in <u>nature</u>.*

* If you can find a Γ-config.

For <u>every</u> Γ, it's possible to construct $\mathcal{C} = \mathcal{C}_\Gamma$ in which we see a Γ-config of sphericals.

<u>Here's how</u>:

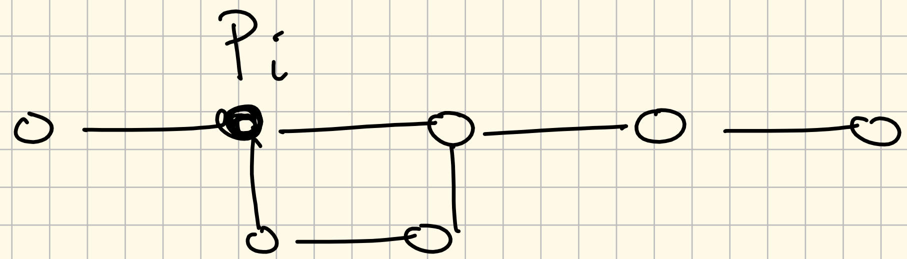


Γ

$$\Gamma \rightsquigarrow Z(\Gamma) = \text{Zig-Zag algebra of } \Gamma$$

$$= \langle \text{Path algebra of } \Gamma^{dbl} \rangle / \text{Rel}$$

<u>Rel</u> :=
- Kill all paths of length 3
- $$\begin{tikzcd} \underset{i}{\circ} \arrow[r, bend right] & \underset{j}{\circ} \arrow[l, bend right] \arrow[r, bend left] & \underset{k}{\circ} \arrow[l, bend left] \end{tikzcd}$$ $(j|i|j) = (j|k|j)$



grading
= path length.

$Z(\Gamma)$ is a graded algebra fin-dim.

$= Z(\Gamma)_0 \oplus Z(\Gamma)_1 \oplus Z(\Gamma)_2$

D^b(gr-mod in $Z(\Gamma)$)

$P_i = Z(\Gamma)\cdot(i)$

$ \| \;\; \langle$ all paths ending at $i \rangle$

Spherical.

Graded proj. module.

Form Γ-config.

$\mathcal{C}_\Gamma = \langle P_i \rangle \subset D^b(\text{gr-mod})$.

$\mathcal{C}_\Gamma = \langle P_i \rangle$ P_i are spherical & form a Γ-configuration.

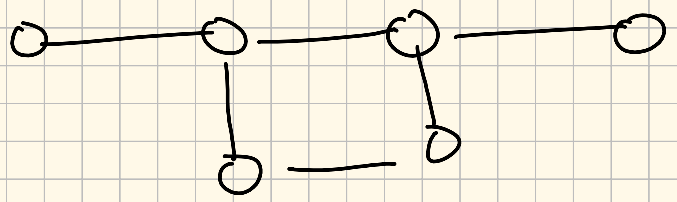
$\hookrightarrow$ simplest / smallest $\mathcal{C}$ in which you see a Γ-config. of sphericals.

By construction $B_\Gamma \circlearrowright \mathcal{C}_\Gamma$

<u>Conj</u>: This is a faithful action.

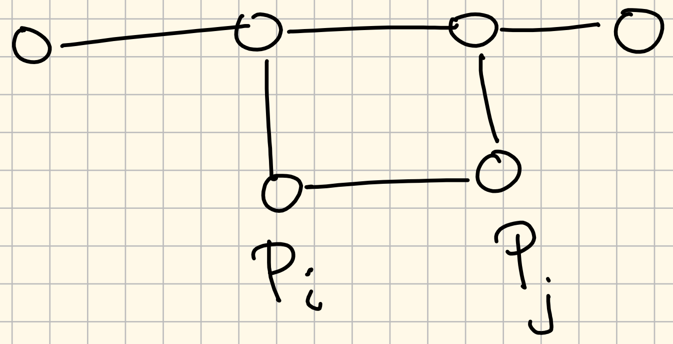
$\hookrightarrow$ You should be able to understand B_Γ through its action on $\mathcal{C}_\Gamma$.

Γ- configuration of sphericals



Category $\mathcal{C}_T$

Category C_I



Example: $\Gamma = A_2$

$$\underset{1}{\circ} \text{———} \underset{2}{\circ}$$

$$\mathcal{C}_\Gamma = \langle P_1, P_2 \rangle \circlearrowleft B_\Gamma = \langle \sigma_1, \sigma_2 \rangle$$

$$\begin{array}{lcccccl} & & 0 & & 1 & & 2 \\ \mathrm{Hom}^*(P_i, P_i) = & & \mathbb{C} & \oplus & 0 & \oplus & \mathbb{C} \\ \mathrm{Hom}^*(P_i, P_j) = & & 0 & \oplus & \mathbb{C} & \oplus & 0 \qquad i \neq j \end{array}$$

$\mathrm{Hom}^1(P_1, P_2) \Rightarrow \exists \quad P_2 \to X \to P_1 \overset{+1}{\dashrightarrow} \qquad X = \sigma_1(P_2)$

$\mathrm{Hom}^1(P_2, P_1) \Rightarrow \exists \quad P_1 \to Y \to P_2 \overset{+1}{\dashrightarrow} \qquad Y = \sigma_2(P_1)$

$$G = B_r \curvearrowright C_r = \mathcal{C}$$

How do you study this?

X

$\cap$

$\mathcal{C}_r$

$g \in G$

Look at $g(X)$

or $g^2(X)$

$g^3(X)$

$\vdots$

"measure the size"

How ~~complicated~~ does this get?

Stability Conditions

$\mathcal{C}$ = triangulated category

Stab cond = Slicing + central charge.

(Semi)-Stable objects.

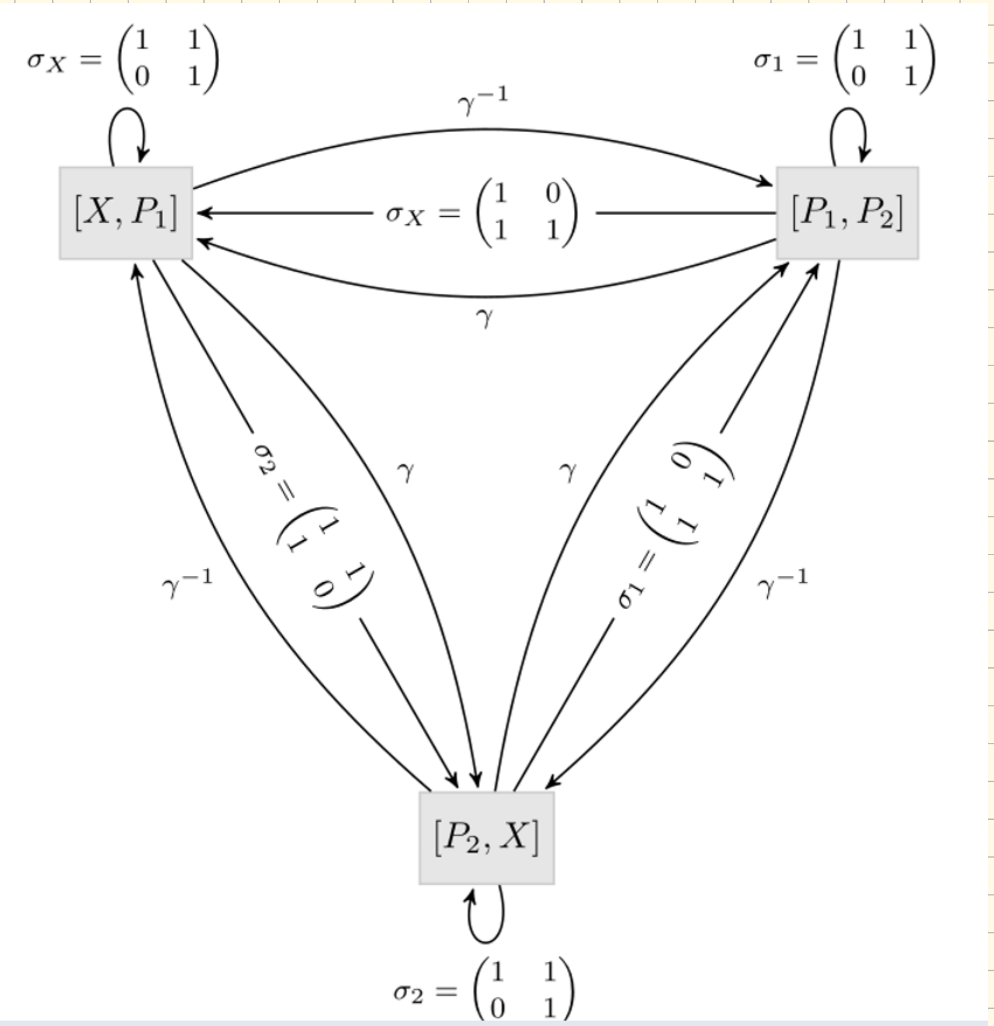
any object $X \rightsquigarrow$ semi-stable factors.
unique.

$$
\begin{tikzcd}
X \arrow[r, rightsquigarrow] \arrow[d] & g(X) \arrow[r, rightsquigarrow] \arrow[d] & g^2(X) \arrow[r] \arrow[d] & g^3(X) \arrow[r, rightsquigarrow] & {} \\
\text{HN factors} \arrow[r, rightsquigarrow] & \text{HN factors} \arrow[r, rightsquigarrow] & \text{HN factors} & &
\end{tikzcd}
$$

↳ understand how this evolves.

Evolution of HN filtrations

HN- Automaton



HN filtration of

$\underbrace{\beta(P_i)}$ for any β.

Stab. cond.

(semi)stables are P_1, P_2, X

HN filtration of a spherical involves only 2 of 3.

Any braid has an expression

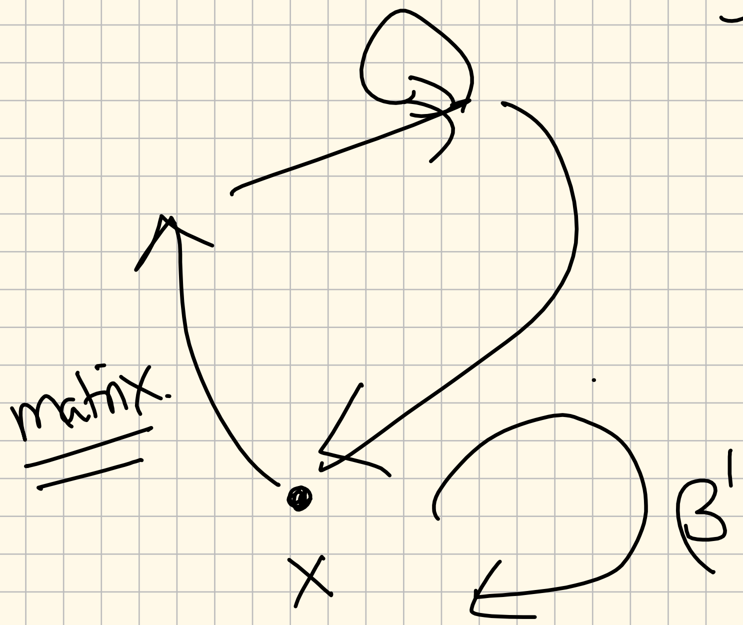
$\beta = \sigma_1 \sigma_2 \sigma_X \gamma \cdots \sigma_2$

↳ "Recognized by this Δ"

More: Every β is conjugate to β'

$\beta' =$ expression . . . . .

loop in the Δ



$(\beta')^n X$ ← easy!

$HN(\beta' X) = \text{Matrix} \cdot HN(X)$.

Cor: Entropy. (ie growth rate)

↳ Eigenvalues of this matrix.

In A_2 case (& other rank 2 categories

$\hat{A}_1$ & also some non-simply laced).

$\boxed{\text{Edmund Heng}}$

There is a linear automaton & "groupoid" that controls the growth of HN mults.

For higher rank : goal ← in progress.

can do : Piecewise linear.

$\Gamma = A_n$: A geometric dictionary

$T = A_n$: Configurations & Stability conditions

$\underline{\Gamma = A_n}$: <u>Supports of Spherical Objects</u>

$\Gamma = A_n$: The sphere of Spherical Objects.