Regular functions and regular maps
<i>_0 </i>
R = Alg closed field
Recall from last time:
X cxx affine algebraic set.
f: X -> k regular if it is the restriction
of a polynomial function.
· · · · · · · · · · · · · · · · · · ·
k[X] = k-algebra of regular functions on X
= K[x1,-1,xn]/I(x).
= Finitely generated nilpotent free
K-algebra.
Observe - Any finitely generalted nilpotent free
Observe - Any finitely generalted nilpotent free k-algebra is of the form k[x]
for some X.
Why? Let A be such an algebra.
Why? Let A be such an aljebra. Let a1,-, an EA be a set of generators.
Then we have a map
q: k[x1,-, xn] -> A
ai Hai

where I = Ken	φ.		
Since A is nilpotent free, I is radical.			
Then take X =	Then take $X = V(I)$.		
By the Null stellen sutz, k[X] = k[X,-,Xn] (I(X)			
	(x) (X)		
= k [x]	11)Xn]/I		
= A	,		
As a result we	have the dictionary.		
Algebra	Geometry		
· Finitely generated	· Alg of regular functions		
· Finitely generated reduced K-aly. A	· Alg of regular functions on affine alg set X.		
· Max ideal of A	. Point of X		
. Given J C A	· Given TCK[X]		
V(J) =	V(J) = \ 2 H(K)=0		
3m moj s.	¥fEJ {		
	1		
In particular V	$(5) = \beta$ if		
-	J = (1)		

Regular Maps
XCA, YCA offine alg sets. f: K-Y is a regular function if
$\exists f_1, \dots, f_m \in k[X]$ such that
$f(x) = (f_i(x), -\cdot\cdot, f_m(x)) + x \in X.$
Equivalently, if there exist Fi,-, Fm
in k(x1,-1xn) such that
Equivalently, if there exist $F_{1,-1}, F_{m}$ in $k(x_{1},-1,x_{n})$ such that $f(x) = (F_{1}(x_{1}),-1, F_{m}(x_{1})) \forall x \in X$.
EX 1: t: 17/2 regular map
EX 1: f: X - /2 regular map (i) f is a regular function.
Ex2: L: ATA linear transformation regular.
is regular.
Ex3: Projections A -1 A
Ex4: Compositions of realer maps

Ex5: XC/A Zaviski closed.
Ex5: XCA Zaniski closed. The inclusion X-1/A is regular.
Def: A regular f: X->Y is an
isomorphism if there exists a regular
inverse map J:Y-X.
1
$E_X 6: X = A$
$E \times 6: X = A$ $Y = \{y^2 - x^3 = 0\} \subset A$
+: X->Y
$f: X \rightarrow Y$ $t \mapsto (t, t)$ is a regular
bijection but not an isomorphism
bijection but not an isomorphism! How does one see that it's not an
iso? Wait and see

Let $\varphi: X \to Y$ be any map.

Then we get an induced map $\varphi^*: Functions on Y \to Functions on X$ $f \mapsto f \circ \varphi$.

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Proposition:	Of is regular if and
only if	φ* sends regular
functions	on Y he regular functions
on X.	

Pf: Suppose of is regular.

If $f: Y \to A$ is a regular function then of the regular because composition of regular maps is regular.

Conversely, suppose f(f) is regular for every regular f. Let $f(x) = (f(x), -1, f_m(x))$.

We want to show each f(x) is regular But f(x) = f(x) = f(x).

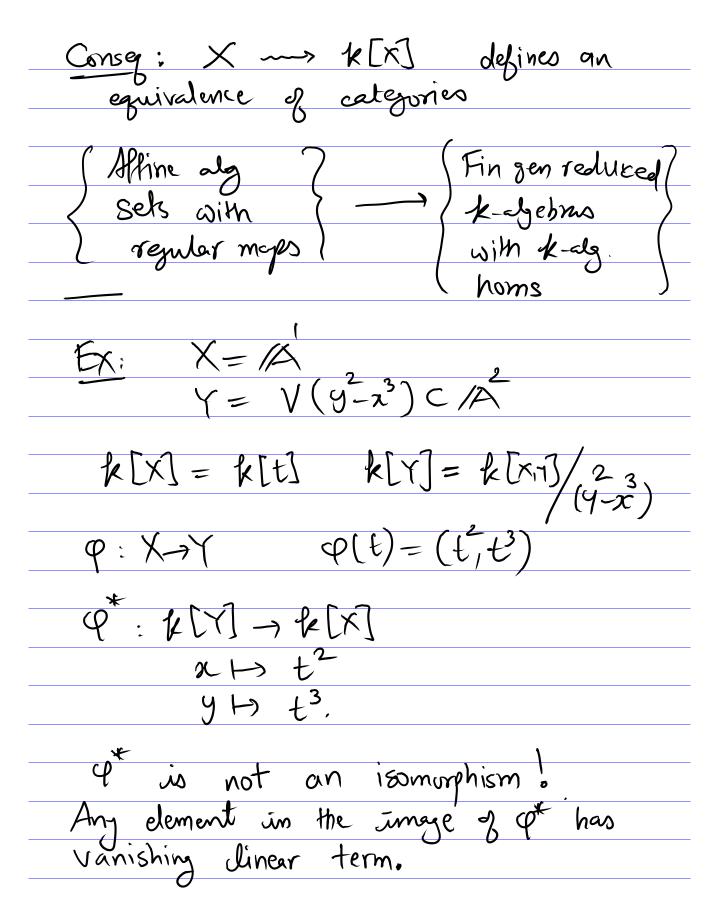
But f(x) = f(x) = f(x) and f(x) = f(x) = f(x).

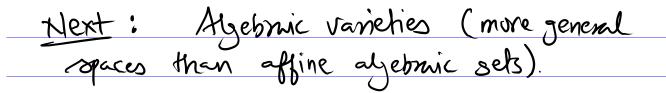
Thus a regular map $\varphi: Y \to X$ induces a k-alg. hom $\varphi^*: k[Y] \to k[X]$.

Prop: Let $\alpha: k[Y] \rightarrow k[X]$ be a k-alg hom. Then there is a unique regular $\varphi: X\rightarrow Y$ much that $\alpha = \varphi^*$.

Pt: suppose Y=V(J) C/Am

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Then k[Y] = k[Y,-,ym] /J
        K[X] = K[X_1, -, X_n] / I.
      \varphi_i = \alpha(y_i) \in k[x]
Consider \varphi := (\varphi_1, -, \varphi_m) : X \rightarrow \mathbb{A}^m
Let us check that I maps X to Y
To see this, we must show that f(P_1(x), -i, P_m(x)) = 0 + 2 \in X
 But f (P,(x), , Pm(x))
        = f(\chi(y_1), --, \chi(y_m))
         = d f(y_1,...,y_m)
            \propto (o)
 So \varphi: X \rightarrow Y. Note \varphi^*(yi) = \lambda(yi)
 so q* = d because { }Yi} generate
 Finally, it Q: X\to Y is such that Q^*=Q, and Q=(P_1,-,P_m), then
 \varphi^*(yi) = \varphi i = \chi(yi), so there is
        one possible P.
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To do that, we want to define the notion of regularity more locally.

Let $X \subset A$ be an affine alg. set, $f: X \to k$ a function, and $x \in X$ a point. We say that f is regular at x if there exist $F, G \in k[X_1, ..., X_n]$ with $G(x) \neq 0$ such that f = F/G on the open set $X \cap \{G \neq 0\}$.

Claim: If f is regular at all xeX, then f is regular (i.e. given by a polynomial).

$f = \frac{F_{i}}{F_{i}}$
Gi
Take $F = h_1F_1 + \cdots + h_kF_k \cdot \in k[X]$ Then $F = f$ on X .
Then $F = f$ on X .
The above motivates the following.
Let XCA be an offine alg. set
and UCX an open set. A function
Let XC/A be an offine alg. set and UCX an open set. A function f. U > 1k is regular on U if it is

 $G(x) \neq 0$ such that $f = E \quad \text{on} \quad U \cap \{G \neq 0\}.$

Similarly $\varphi: U \rightarrow Y$ is regular
if $\varphi = (\varphi_1, -, \varphi_m)$ where each φ :
is a regular function on Q.

Now we have	
(affine algebraic) _	Soffine alg. subsets?
7 vanishes >	of single
\bigcap	\cap
S Quan-affine?	5 open subsets of
7 Varieties 5	offine alg subsets?
	2 Min
	•

Morphisms = Regular maps

Examples: 1

Let $X = /\sqrt{30}$. $Y = \sqrt{(xy-1)} C/\sqrt{2}$.

Then we have an isomorphism

In particular X is (isomorphic to) an affine algebraic variety).

The iso is given by X-Y

and the inverse is	
$Y \rightarrow X$	
(2,4) トラ て.	

(2) More generally, let

$$f \in k[x_1,...,x_n]$$
 and

 $X = \{x \in x^n \mid f(x) \neq 0\}$
 $= x^n - v(f)$.

Let
$$Y \subset A^{nH} = \{(x_1, -, x_n, y)\}$$

 $Y = V (y f(x_1, -, x_n) - 1)$

Then we have an iso X ~ Y
given by

In particular X is an offine alg. variety!

3) Not all quesi-affine varieties are
isomorphic to affine varieties.
To see an example, recall that
affine alg. varieties satisfy the
Null stellen sut 2 — there is a
bijection between max ideals g
k[x] and points of X
given by m H V(m).
2
Take X = 1/4 \ \ \((0,0)\)\}. C/A^2
Claim: The k-algebra of regular
functions on X is the same as
k[x] = k[x,y].
Pf: Deferred.
But now m=(xy) ck[x] is a
non-unit ideal such that $V(m) = \emptyset$
(in X). Therefore, X cannot be
affine.

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