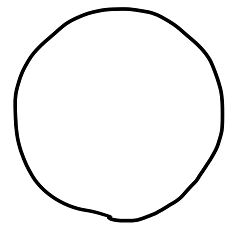
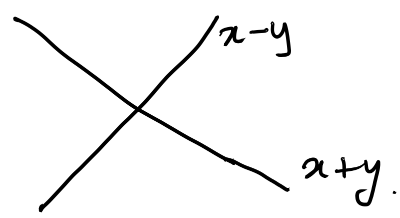
Algebra & geometry

Algebra	Geometry
$x^2+y^2=1$	
$x^2-y^2=0$ $(x+y)(x-y)$	
Factors	components.
system of equations in n-var over k $\longrightarrow$	subset of k^n
$\longleftarrow$	
I $\longrightarrow$	Common zero locus of $I = V(I)$
Poly. vanishing on $S = I(S)$ $\longleftarrow$	S

Obs : ① $f, g \in I(S)$ then $af + bg \in I(S)$ $\forall$ poly. a & b.

② $f^n \in I(S)$ then $f \in I(S)$.

Def: subsets of $k[x_1, \dots, x_n]$ satisfying ① are called ideals & also satisfying ② are called radical ideals.

Def: Subsets of k^n of the form $V(I)$ are called algebraic.

Thm (Hilbert's Nullstellensatz). Let k be algebraically closed (i.e. $\mathbb{C}$, not $\mathbb{R}$). Then

$$\begin{array}{c} \text{Rad. ideals} \\ \text{of } k[X_1, \dots, X_n] \end{array} \longleftrightarrow \begin{array}{c} \text{Alg subsets} \\ \text{of } k^n \end{array}$$

is a bijection.

Sophie Germain: "Algebra is just written geometry; geometry is just drawn algebra."

(if your field is alg. closed.)

Ex. $k = \mathbb{R}$

$$I = k[X] \longrightarrow V(I) = \phi$$

$$I = (x^2+1) = \{(x^2+1) \cdot f \mid f \in k[X]\} \longrightarrow V(I) = \phi.$$

$k = \mathbb{C}$. $\quad \longrightarrow V(I) = \{\pm i\}$

$$\mathbb{P}^n_k = (K^{n+1} \setminus \{0\}) / k^\times$$

$$= \{ (a_0, \ldots, a_n) \mid \text{Not all } a_i = 0 \} / \sim$$

$$(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n).$$

Zero of f only makes sense if f is homogeneous.

N.S.

Homog. radical ideals $\longleftrightarrow$ Alg. subsets of $\mathbb{P}^n_k$

Examples - Points in $\mathbb{P}^2$

Thm: There are only finitely many betti tables of ideals for k points in $\mathbb{P}^n$.

Know the list for $n=1, n=2$.
Hopelessly open for higher n.

Next: One dim. alg. var.. / $\mathbb{C}$.

$X \subset \mathbb{P}^n$ a "curve" over $\mathbb{C}$.

i.e. locally looks like $\mathbb{C}$

i.e. topologically a surface.

Digression - A paradigm shift in pure math. towards abstraction.

Ex: **Pre-modern** | **modern.**

$\boxed{\text{Group}}$ - A set of matrices closed under $*$ & inv. | A set with an operation $*$ ...

e.g. $\left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \right\}$ $\leftarrow$ $\mathbb{Z}/2\mathbb{Z}$

$\left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right\}$ $\leftarrow$

Representations

$\boxed{\text{Manifold}}$ - A subset of $\mathbb{R}^n$ satisfying.... | A set with a topology satisfying $*$

$\xleftarrow{\text{embedding}}$

$\boxed{\text{Alg. var}}$ - A subset of k^n or $\mathbb{P}^n_k$, ----- | A top. space with some extra alg. structure.

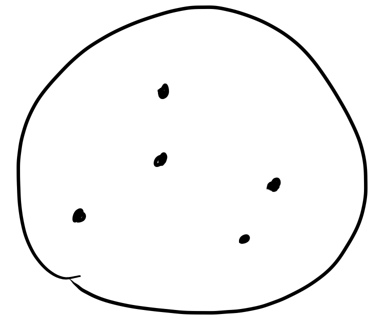
$X \xhookrightarrow{\text{embed}} \mathbb{P}^n_k \leftarrow X$

For dim 1 & 2 & smooth X, same as $\mathbb{C}$-manifold

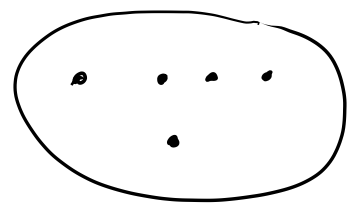
Ex: 5 points in $\mathbb{P}^2$

$X = \{\cdot, \cdot, \cdot, \cdot, \cdot\}$

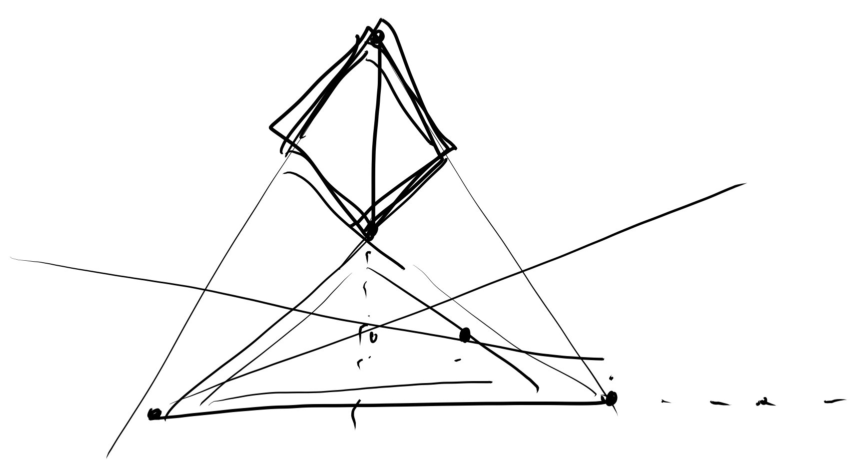
$X \hookrightarrow \mathbb{P}^2$ as

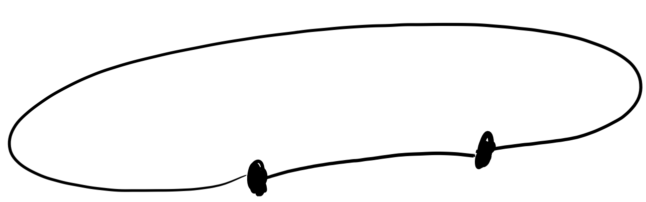


or as



So ideals / betti tables are not "intrinsic" to X but "extrinsic" i.e. a function of how X sits in $\mathbb{P}^n$





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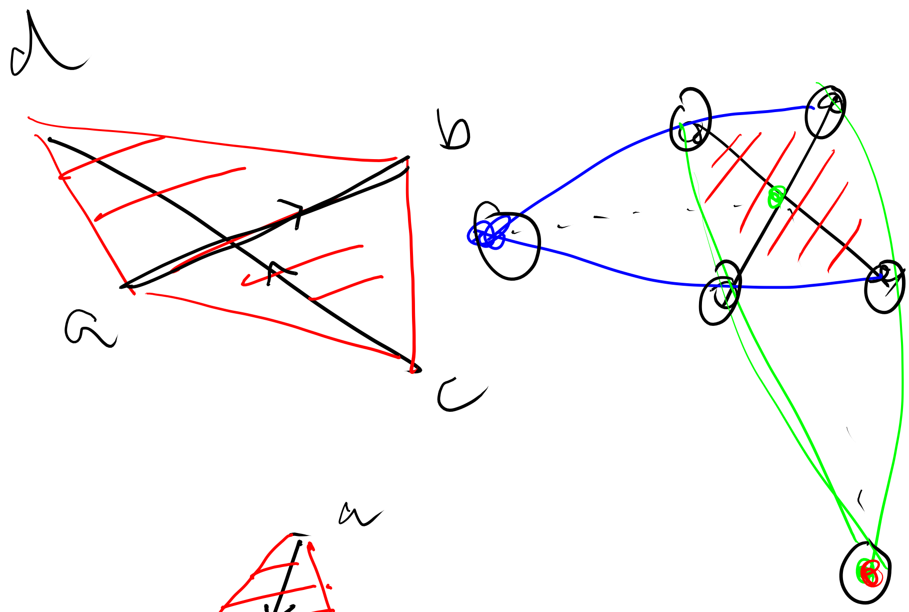

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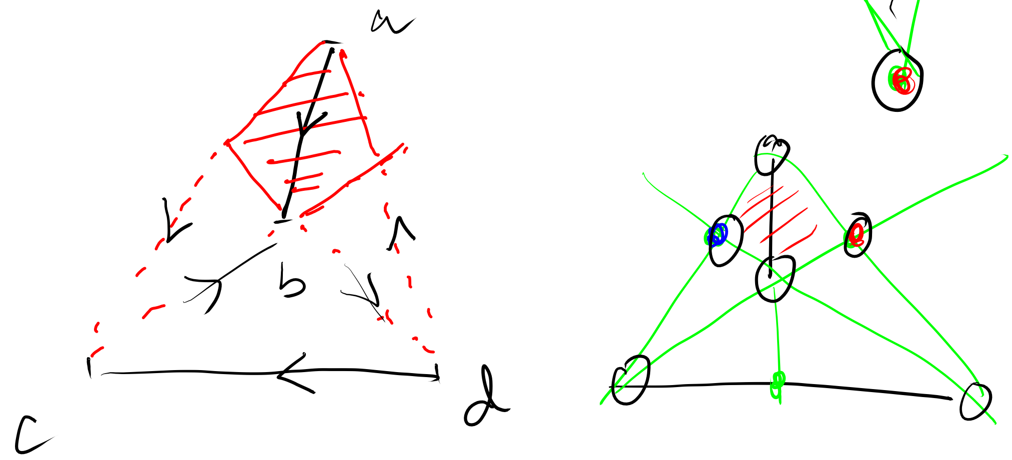

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