ALGEBRAIC GEOMETRY: HOMEWORK 4

This homework is due by 5pm on August 23.

(1) (Affine charts) Let $Q = V(XY - ZW) \subset \mathbb{P}^3$. Write the four affine charts for X and the transition functions between *one* pair of them.

(2) (Projective closure I) Let $I \subset k[x_0, \ldots, x_{n-1}]$ be an ideal and $X = V(I)$. Recall that for $p \in k[x_0, \ldots, x_{n-1}]$,

$$p^{\mathrm{hom}}(X_0, \ldots, X_n) = X_n^{\deg p} \left(\frac{X_0}{X_n}, \ldots, \frac{X_{n-1}}{X_n} \right).$$

Think of $\mathbb{A}^n$ as the open subset of $\mathbb{P}^n$ where the last homogeneous coordinate is non-zero. Show that the closure $\overline{X} \subset \mathbb{P}^n$ of X is given by

$$\overline{X} = V\left(\{ p^{\mathrm{hom}} \mid p \in I \} \right).$$

(3) (Projective closure II) Find the closure in $\mathbb{P}^n$ of the following affine varieties, and identify the points at infinity.
   (a) $V(xy - 1) \subset \mathbb{A}^2$
   (b) $V(y^2 - x) \subset \mathbb{A}^2$
   (c) $V(y - x^2, z - x^3) \subset \mathbb{A}^3$.

(4) (5 points define a conic) Let $X \subset \mathbb{P}^2$ be a set of 5 points, no 3 on a line. Prove that there is a unique conic containing X. (A conic in $\mathbb{P}^2$ is $V(F)$ where F is a homogeneous polynomial of degree 2).

(5) (Pencil of conics) Let F an G be irreducible homogeneous degree 2 polynomials in $k[X, Y, Z]$. For each $[s : t] \in \mathbb{P}^1$, we get a plane conic $Q_{s:t} = V(sF + tG) \subset \mathbb{P}^2$. Such a family of conics is called a *pencil*. Suppose F and G intersect in 4 distinct points. Prove that exactly three members of the pencil are degenerate (reducible), and describe them in terms of the 4 points of intersection of $V(F)$ and $V(G)$.