WHAT'S NEW?

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Here are the new mathematical developments since the last time you wrote a letter.

NEW PAPERS

(1) *Vector bundles and finite covers* (with A. Patel)

We settle a question of Lazarsfeld about vector bundles on curves and finite coverings. Let $\phi : X \to Y$ be a finite flat map of degree d. We can associate to it the vector bundle $\phi_*\mathcal{O}_X$. Roughly, the question is: which vector bundles arise in this way?

The bundle $\phi_*\mathcal{O}_X$ clearly contains $\mathcal{O}_Y$ as a summand (assume characteristic coprime to d). So it suffices to consider the quotient; for convenience let E_ϕ be the dual of the quotient. Our main theorem is the following.

Theorem 1. *Let Y be a smooth projective curve and E be a vector bundle on it. There is an N (depending on E) such that for every $n \geq N$ and line bundle L of degree n, the bundle $E \otimes L$ arises as E_ϕ for some $\phi : X \to Y$ with X smooth.*

Corollaries:

(a) Fix Y and d. If g is large enough, then a generic genus g and degree d cover $\phi : X \to Y$ of Y gives a stable vector bundle E_ϕ (replace "stable" by the correct analogue if $g(Y) = 0$ or 1). This was proved by Kanev for $d \leq 5$. Generally, statements about branched covers for $d \geq 5$ is a different ball-game than $d \leq 5$, because we have explicit structure theorems for $d \leq 5$ that we lack for $d \geq 5$. So, it is cool that we avoid using these structure theorems.

(b) For $Y = \mathbf{P}^1$, the question of characterizing E_ϕ is known as the question of characterizing "scrollar invariants." The theorem above says that anything is possible if you allow for a simultaneous shift.

Technique: It's a very satisfying application of degeneration. We first do the case of direct sums of line bundles using singular X. We then smooth out the X by attaching many rational curves. We smooth things so much that not only is X smooth, but the map from the stack of coverings to the stack of bundles is smooth at $X \to Y$. We can then get all bundles by deformation.

Preprint: `https://deopurkar.github.io/research/papers/ebundle.pdf`.

We were so excited with the result, which is so clean and complete, that we submitted it to *Inventiones*. It got rejected, and is currently waiting to be submitted to the next journal.

(2) *Green's canonical syzygy theorem for ribbons*

I prove Bayer and Eisenbud's conjecture about the minimal free resolutions of ribbons (double structure on $\mathbf{P}^1$).

Theorem 2. *The minimal free resolution of the ideal of a canonically embedded ribbon is linear for p steps if and only if $p \leq$ Clifford index of the ribbon.*

What's unsatisfying is that I have to use Green's conjecture for generic smooth curves (proved by Voisin). An independent proof will be great – but one proof is better than no proof.

As a corollary, I can deduce Green's conjecture for generic curves of every Clifford index very quickly. This was done by Aprodu and Farkas using nodal curves on K3 surfaces, but it was more delicate.

Preprint: `https://deopurkar.github.io/research/papers/RibbonGreen.pdf`.

This is under review at *Math. Zeit.*.

(3) *Syzygy divisors on Hurwitz spaces* (with A. Patel).

In addition to the push-forward bundle E_ϕ mentioned above, we can associate a lot of other vector bundles to a $\phi : X \to Y$ using syzygetic construction. For $Y = \mathbf{P}^1$, these are be generically balanced, and the locus where they are unbalanced forms a divisor. We compute the class of this divisor on the Hurwitz space. This is all in the hope of generating new and interesting effective divisors on Hurwitz spaces, in particular, towards trying to see whether the canonical divisor is effective. The result of our calculation is surprising: the divisor classes we compute are the same (up to scaling), but we suspect that they are not supported on the same locus. This is very reminiscent of the Brill–Noether divisors on $\overline{M}_g$.

Preprint: `https://deopurkar.github.io/research/papers/RibbonGreen.pdf`.

This was written for *Proceedings* of a section of the Joint Math Meetings at which we had spoken.

(4) *Tropical anti-canonical cubics contain 27 lines* (With M. Cueto)

Let X/K be a smooth cubic del Pezzo without Eckhard points (that is, without 3 concurrent lines), where K is a (non-trivially) valued field. The 45 anti-canonical triangles in X give an embedding $X \subset \mathbf{P}^{44}_K$, well defined up to the torus action on $\mathbf{P}^{44}_K$. This is a more natural anti-canonical embedding than the usual $X \subset \mathbf{P}^3$ for tropical geometry. The latter depends on a choice of a basis of $H^0(X, K_X)$ whereas the former doesn't (there's still a choice of scaling, but that does not affect the tropicalization). We show that the only tropical lines on the tropicalization of X are the expected 27. This is in contrast with the picture of the tropicalization of a cubic surface in $\mathbf{P}^3$. Viegeland constructs examples where there are infinitely many tropical lines in this setting.

My (admittedly non-expert) understanding of the broader context: The tropicalization of a sub-variety of a toric variety depends on the embedding; a badly chosen embedding may exhibit pathologies. The inverse limit over all such tropicalizations is the Berkovich analytification. So people view the tropicalizations as "shadows" of the Berkovich space and are interested in finding tropicalizations that are "faithful" (deformation retracts of the analytification). Our result doesn't prove that the tropicalization of X in $\mathbf{P}^{44}$ is faithful. But at least it shows the right number of lines.

Our proof is very computation heavy. We explicitly work with the tropicalization in sage, and rule out lines by pretty much brute force. As an additional benefit, I have a good understanding of moduli of del Pezzos now.

Preprint (in progress): `https://deopurkar.github.io/research/papers/lines_on_tropical_cubics.pdf`

The paper *Covers of stacky curves and limits of plane quintics* is under review at *Trans. AMS*.

New ongoing projects

(1) *Motive of Hurwitz spaces*

Ravi, Anand Patel, and I have been computing the class of the Hurwitz space $H_{d,g}$ in the Grothendieck ring of varieties (localized at $\mathbf{A}^1$ and completed with respect to the dimension filtration). Specifically, we are looking at the generating function

$$F_g(t) = \sum [H_{d,g}] t^d.$$

We expect this to be a rational function whose coefficients involve the motive of M_g and the motive of the universal Jacobian. We can show this for the version of $H_{d,g}$ that parametrizes $\phi : C \to \mathbf{P}^1$ with smooth C and arbitrary branching. We are working on the analogue where we only allow simple branching, or more generally, only allow certain branching profiles. This has turned out to involve complicated inclusion/exclusion combinatorics, which we are currently trying to simplify.

We expect there to be an analogue for higher rank linear series as well, not just pencils. I talk about this in my AMS-Simons proposal.

(2) *Cycles classes on $M_{g,n}$ using cyclic coverings*.

I am pushing Maksym's cyclic covering construction in all directions, hoping to get interesting positive cycle classes on $\overline{M}_{g,n}$ in all codimensions. I talk at length about this in my NSF proposal.

(3) *Degree of irrationality on K3 surfaces*.

The irrationality of X, denoted $\operatorname{irr}(X)$, is the minimum degree of a generically finite map from X to projective space. What can we say about $\operatorname{irr}(X)$ when X is a K3 surface?

With David Stapleton, I did an intriguing dimension count that seemed to say that $\operatorname{irr}(X)$ is *bounded above* as X varies over all K3 surfaces. I talk at length about this in my NSF proposal.

(4) *KSBA compactifications*.

Trigonal curves come with a canonical embedding in a surface scroll. What is the KSBA-style compactification of pairs (S, C) where S is a surface scroll and $C \subset S$ is a canonically embedded trigonal curve? My hunch is that this should have a nice answer analogous to Hassett's result for quartic curves in $\mathbf{P}^2$, and I think I have a pretty good idea of how to proceed. I am (finally!) working out the details with Joe Harris's student Changho Han.

(5) *Self-maps between moduli spaces*.

With Anand Patel, I am exploring many instances of interesting rational self maps on moduli spaces. These are related to many classical constructions (1900-era) studied by Cayley, Coble, etc. We hope we can use modern tools to go further than they could. Here is one example.

Veronese maps and Gale duality: The moduli space of r points in $\mathbf{P}^n$ (modulo $\operatorname{Aut}\mathbf{P}^n$) is birational to the moduli space of r points in $\mathbf{P}^m$ (modulo $\operatorname{Aut}\mathbf{P}^m$) by Gale duality, where $m = r-n-2$. But there is another natural way to map r points in $\mathbf{P}^n$ to r points in $\mathbf{P}^m$—by taking Veronese embeddings:

$$r \text{ points} \subset \mathbf{P}^n \xrightarrow{k\text{th Veronese}} \mathbf{P}^m \rightsquigarrow r \text{ points in } \mathbf{P}^m,$$

where $m = \binom{n+k}{k} - 1$. For (r, n, k) such that $r - n - 2 = \binom{n+k}{k} - 1$, we get a rational self map of the space of r points in $\mathbf{P}^n$ to itself. What is the degree of this map?

For $n = 1$, the numbers are $r = n+3$ and $m = n$, and the degree of the map is 1. This is related to the classical fact that there is a unique rational normal curve in $\mathbf{P}^n$ through $n+3$ points.

For $n = 2$, and $r = 9, k = 2$ the answer is 4 (due to Coble); the 4 comes from the 2 torsion points on the unique cubic curve through 9 general points in $\mathbf{P}^2$. That's the state of the art.

Anand P. and I can handle the next case ($n = 2$ and $r = 13, k = 3$) by ad-hoc methods. These question reduce to interpolation type questions for Veronese maps – how many k Veronese $\mathbf{P}^n$'s can we find through r points in $\mathbf{P}^m$. One non ad-hoc approach would be to build a Kontsevich space of pointed $\mathbf{P}^n$'s mapping to $\mathbf{P}^m$ and do intersection theory on it, but we haven't managed to get this to work.

OTHER THINGS

(1) Got the AMS-Simons Travel Grant.
(2) Co-organized the Summer Workshop in Algebraic Geometry at UGA (a weekend conference) with Angela Gibney, Jesse Kass, and Nicola Tarasca.
(3) Co-organized the Columbia Graduate Student AG Seminar on *Stable rationality and the decomposition of the diagonal* with Johan de Jong.
(4) My AIM workshop proposal on *Stability and Moduli Spaces* with Fedorchuk, Morrison, and Wang got funded. The workshop will happen in January 2017.