MATH3354 - Assignment 9

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Collaboration Statement

I discussed questions 1, 2 and 3 with Angus Mingare, Ben Ellis-Bloor, Gianni Gagliardo and Aymon Wuolanne.

1

Prove that $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ as well as the Fermat cubic S and $\mathbb{P}^2$ are not isomorphic.

We will use the result that there are no non-constant regular maps $\mathbb{P}^n \to \mathbb{P}^m$ for $n > m$.

Suppose we have an isomorphism $g : \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be projection onto the first factor. Projection is regular, so the composition $(\pi \circ g) : \mathbb{P}^2 \to \mathbb{P}^1$ is a surjective regular map. In particular it is a non-constant regular map, a contradiction.

First suppose that we are not in characteristic 2 or 3 and suppose, in order to gain a contradiction, that we have a regular isomorphism $g : \mathbb{P}^2 \to S$. In assignment 5 we proved that when $\operatorname{char} k \neq 2, 3$ there is a non-constant regular map $\pi : S \to \mathbb{P}^1$ from the Fermat Cubic to $\mathbb{P}^1$. Therefore, the map $(\pi \circ g) : \mathbb{P}^2 \to \mathbb{P}^1$ is a non-constant regular map $\mathbb{P}^2 \to \mathbb{P}^1$, a contradiction.

2

The Krull dimension of a ring R is the largest n such that there exists a strictly increasing chain

$$(0) = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n,$$

of prime ideals in R.

(a)

What is the Krull dimension of $\mathbb{Z}$.

First we see that the dimension is at least 1 because we have the chain $(0) \subsetneq (2)$. We note that the prime ideals in $\mathbb{Z}$ are either (0) or of the form (p) for some prime p, since if we have an ideal (n) for $n = ab$ where $1 < a, b < n$, then $a, b \notin (n)$ because $a, b < n$ but $ab \in (n)$, so (n) is not prime.

However, we note that the ideals containing (p) are of the form (d) for $d \mid p$. Since the only divisor of p is 1, this implies that the only ideal which contains (p) is (1), which is not a prime ideal. Similarly, an ideal contained in (p) is of the form (kp) for some $k \in \mathbb{Z}$ which is not a prime ideal unless $k = 0$. Thus the longest possible chain is $(0) \subsetneq (p)$ for some prime p, and so the Krull dimension of $\mathbb{Z}$ is 1.

(b)

Let X be an irreducible affine variety. Prove that the Krull dimension of $k[X]$ is equal to the dimension of X.

If X has Krull dimension n then we have a maximal chain of prime ideals,

$$\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n.$$

We may assume that $\mathfrak{p}_0 = (0)$, because otherwise we could add this ideal to the chain and obtain a strictly longer chain. By the Nullstellensatz this gives us a length n chain of non-empty irreducible closed subsets,

$$V(0) = X \supsetneq V(\mathfrak{p}_1) \supsetneq \dots \supsetneq V(\mathfrak{p}_n).$$

Similarly if we had a maximal length n chain of non-empty irreducible closed subsets, applying I to the chain would give a length n chain of prime ideals. Therefore, the Krull dimension is also equal the largest n such that we have a strictly increasing chain,

$$X_1 \subsetneq \dots \subsetneq X_n \subsetneq X$$

of non-empty irreducible closed sets. Suppose X has Krull dimension n, so we have a chain as above. We know from class that if we have $Y \subsetneq X$ for irreducible closed sets, then $\dim Y < \dim X$. Therefore, with each inclusion in such a chain the dimension goes up by at least 1. Therefore,

$$0 \leq \dim X_1 \leq \dim X_2 - 1 \leq \dots \leq \dim X - n$$

so $n \leq \dim X$. That is, $kr \dim X \leq \dim X$.

Next we want to show that $\dim X \leq kr \dim X$ by induction on the dimension of X. First suppose that $\dim X = 0$, then X is a single point (because X is a non-empty irreducible), so $X = \{x_0\}$. The longest chain of non-empty irreducible closed sets is $\{x_0\}$, so X has Krull dimension 0.

Suppose that $\dim Y \leq kr \dim Y$ whenever $\dim Y \leq n$ and let X be an irreducible dimension $(n+1)$ variety. Since X has positive dimension it is non-empty and not a point, so $k[X]$ is not the trivial ring (by the Nullstellensatz). Therefore, we can choose a non-zero polynomial $f \in k[X]$ which does not vanish identically on X. Then, we know that $V(f) \subset X$ is non-empty, so the Principal Ideal Theorem implies that $V(f)$ has pure dimension $\dim X - 1 = n$. Let Y be an irreducible component of $V(f)$, then Y has dimension n, so by the induction hypothesis $\dim Y \leq kr \dim Y$. Since $kr \dim Y \geq n$ we can choose a length n chain of non-empty irreducible closed subsets of Y, $Y_1 \subsetneq \dots \subsetneq Y$. This gives a length $(n+1)$ chain in X,

$$Y_1 \subsetneq \dots \subsetneq Y \subsetneq X.$$

Hence, $kr \dim X \geq n + 1 = \dim X$. By induction, $\dim X \leq kr \dim X$ for all irreducible affine varieties, so $\dim X = kr \dim X$ since we have already shown the other inequality.

3

Show that $f \in k[X]$ is a zero divisor if and only if f vanishes identically on some irreducible component of X.

($\Longrightarrow$): Suppose $f \in k[X]$ is a zero divisor. Then there is some non-zero $g \in k[X]$ such that $fg = 0$ in $k[X]$. Since g is non-zero in $k[X]$ there is some $y \in X$ such that $g(y) \neq 0$. Say y belongs to the irreducible component Y, so $g \notin I(Y)$. Because Y is irreducible we know that $I(Y)$ is prime, so because $fg \in I(Y)$ and $g \notin I(Y)$, we know that $f \in I(Y)$. Therefore, f vanishes on an irreducible component of X.

($\Longleftarrow$): We can write X in terms of its irreducible components as, $X = Y_1 \cup Y_2 \cup \dots \cup Y_n$. We know that f vanishes identically on some irreducible component of X, say Y_1. Since Y_1 is not contained within $Y_2 \cup \dots \cup Y_n$ we know that $I(Y_2 \cup \dots \cup Y_n)$ is not contained within $I(Y_1)$ by the Nullstellensatz. Hence, we can choose $g \in I(Y_2 \cup \dots \cup Y_n) \setminus I(Y_1)$. Then $g \neq 0$ in $k[X]$ because g does not vanish identically on Y_1. However, $f(x)g(x) = 0$ on all of X because f vanishes on Y_1 and g vanishes on $Y_2 \cup \dots \cup Y_n$. Therefore, $fg = 0$ in $k[X]$, so f is a zero divisor.