Question 1

(1) Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree 4 with singularities at $[1:0:0], [0:1:0]$, and $[0:0:1]$. Prove that C is rational.

Hint:Use the Cremona transformation.

Proof. For a degree 4 curve to have singularities at the three co-ordinate points, its defining homogeneous polynomial must have no monomial terms containing a third power (for example check the coefficient of XZ^3 by taking the partial in X and evaluating at $[0:0:1]$). The combination of 3 partials and 3 points allows us to set 9 coefficients to 0, so a general irreducible degree 4 curve with singularities at the co-ordinate points has defining polynomial:

$$F(X,Y,Z) = aX^2Y^2 + bX^2YZ + cX^2Z^2 + dXY^2Z + eXYZ^2 + fY^2Z^2$$

Now based on this we define the following:

$$G(X,Y,Z) = aZ^2 + bYZ + cY^2 + dXZ + eXY + fX^2$$

Note here that G is an irreducible polynomial, because a factorisation of G would induce a factorisation of F via $Z \mapsto XY$, $Y \mapsto XZ$ and $X \mapsto YZ$, but we know that F is irreducible. Now I claim that $V(F)$ is birationally isomorphic to $V(G)$ via χ, the Cremona transformation.

To to this we will show that χ gives a regular isomorphism:

$$V(XYZ)^c \cap V(F) \to V(XYZ)^c \cap V(G)$$

For this isomorphism to induce a birational isomorphism (and to make any sense really) we need that one of these intersections are non-empty. We know that the first intersection is nonempty, because if $V(F) \subset V(XYZ)$ then since $V(F)$ is irreducible and $V(XYZ)$ is a union of three lines, then $V(F)$ must be contained in a line. But that is not possible because $V(F)$ passes through the three co-ordinate points, which are not colinear.

Now from the last assignment we know that χ gives a regular isomorphism of the set $V(XYZ)^c$, and furthermore is its own inverse. Therefore once we show that χ maps between $V(XYZ)^c \cap V(F)$ and $V(XYZ)^c \cap V(G)$ we will have that χ gives a birational isomorphism of the quasi-projective varieties.

Let $[X:Y:Z] \in V(XYZ)^c \cap V(F)$. Then $F([X:Y:Z]) = 0$. We know already that $\chi([X:Y:Z]) \in V(XYZ)^c$. Now note that:

$$\begin{aligned}
G(\chi([X:Y:Z])) &= G([YZ:XZ:XY]) \\
&= aX^2Y^2 + bX^2YZ + cX^2Z^2 + dXY^2Z + eXYZ^2 + fY^2Z^2 \\
&= F([X:Y:Z]) \\
&= 0
\end{aligned}$$

So $\chi([X:Y:Z]) \in V(G) \cap V(XYZ)^c$.

Let $[X : Y : Z] \in V(XYZ)^c \cap V(G)$. Then $G([X : Y : Z]) = 0$. We know already that $\chi([X : Y : Z]) \in V(XYZ)^c$. Now note that:

$$\begin{aligned}
F(\chi([X : Y : Z])) &= F([YZ : XZ : XY]) \\
&= aX^2Y^2Z^4 + bX^2Y^3Z^3 + cX^2Y^4Z^2 + dX^3Y^2Z^3 + eX^3Y^3Z^2 + fX^4Y^2Z^2 \\
&= (XYZ)^2G([X : Y : Z]) \\
&= 0
\end{aligned}$$

So $\chi([X : Y : Z]) \in V(F) \cap V(XYZ)^c$.

Therefore we conclude that:

$$V(XYZ)^c \cap V(F) \cong V(XYZ)^c \cap V(G)$$

And since these are non-empty open subsets of $V(F)$ and $V(G)$ we must have that they are birational. Now since G is an irreducible quadratic we have from class that $V(G) \cong \mathbb{P}^1$. Therefore $V(F)$ is birationally isomorphic to $\mathbb{P}^1$, as required. $\square$

Question 2

Let X and Y be two irreducible varieties that are birationally isomorphic. Prove that there exist non-empty open subsets $U \subset X$ and $V \subset Y$ such that U and V are isomorphic.

Proof. Suppose X and Y are birational via the pair of dominant maps:

$$(U, \phi : U \to Y),\ (V, \psi : V \to X)$$

Where U and V are open in X and Y respectively. Then since $\phi \circ \psi$ is defined on $\psi^{-1}(U)$, $\psi \circ \phi$ is defined on $\phi^{-1}(V)$ and both compositions are the relevant identity as rational maps, we have that:

$$(\psi^{-1}(U), \phi \circ \psi) \sim (Y, id)$$
$$(\phi^{-1}(V), \psi \circ \phi) \sim (X, id)$$

Therefore:

$$\phi \circ \psi = id \text{ on } \psi^{-1}(U) \text{ and } \psi \circ \phi = id \text{ on } \phi^{-1}(V). \tag{1}$$

Now we define the follwing open sets:

$$\mathcal{O} := \phi^{-1}(\psi^{-1}(U)) \subset \phi^{-1}(V) \subset U \subset X$$
$$\mathcal{P} := \psi^{-1}(\phi^{-1}(V)) \subset \psi^{-1}(U) \subset V \subset Y$$

These sets are non-empty because ϕ and ψ are dominant, and open because ϕ and ψ are continuous.

Now if we have that $\phi(\mathcal{O}) \subset \mathcal{P}$ and $\psi(\mathcal{P}) \subset \mathcal{O}$ then the restrictions of ϕ and ψ to $\mathcal{O}$ and $\mathcal{P}$ respectively will give an isomorphsism $\mathcal{O} \cong \mathcal{P}$ because of line (1).

I will only prove that $\phi(\mathcal{O}) \subset \mathcal{P}$ because the other containment has a symmetric argument. Let $y \in \phi(\mathcal{O})$, so $y = \phi(x)$ for $x \in \mathcal{O} = \phi^{-1}(\psi^{-1}(U))$. Therefore $y \in \psi^{-1}(U)$. Therefore since $\phi \circ \psi = id$ on $\psi^{-1}(U)$ we have that $\phi(\psi(y)) = y \in \psi^{-1}(U) \subset V$. Therefore $y \in \phi^{-1}(\psi^{-1}(V)) = \mathcal{P}$, as required.

Therefore we have found isomorphic open subsets $\mathcal{O} \subset X$ and $\mathcal{P} \subset Y$, as required. □

Question 3

Write down an isomorphism of fields

$$\mathbb{C}(s,t) \;\to\; \operatorname{frac}\left(\mathbb{C}[x,y,z]/(x^3+y^3+z^3+1)\right).$$

You should describe the maps by writing where each generator goes, and describe how you obtained the map. But you need not write down the inverse.

Proof. An isomorphism is defined by:

$$s \mapsto \frac{x+y}{z+1}$$
$$t \mapsto \frac{x+\omega y}{z+\omega}$$

Where $\omega^3 = 1$ but $\omega \neq 1$.

We obtained the map by noticing first that:

$$\mathbb{C}(s,t) \cong \mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{C}(\mathbb{A}^2)$$

And also that:

$$\operatorname{frac}\left(\mathbb{C}[x,y,z]/(x^3+y^3+z^3+1)\right) \cong \mathbb{C}(V(X^3+Y^3+Z^3+W^3)) \cong \mathbb{C}(V(x^3+y^3+z^3+1))$$

We found two skew lines contained in the Fermat cubic:

$$L_1 = V(X+Y, Z+W)$$
$$L_2 = V(X+\omega Y, Z+\omega W)$$

(found in lecture). We then defined two linear projections with each line as the center. These projections had to induce an isomorphism of the line not in the center of projection with a copy of $\mathbb{P}^1$ due to the dimension of the ambient space and the fact that the lines are disjoint. This is a slick way of finding algebraically the description of the birational isomorphism of any smooth cubic surface in $\mathbb{P}^3$ with $\mathbb{P}^2$.

We then used this to define a map:

$$V(X^3+Y^3+Z^3+W^3) \setminus (L_1 \cup L_2) \to \mathbb{P}^1 \times \mathbb{P}^1$$

By applying the relevant projection in each component.
We then restricted this map to make it look like a map:

$$V(x^3+y^3+z^3+1) \setminus [L_1 \cup L_2 \cup V((z+1)(z+\omega))] \to \mathbb{A}^1 \times \mathbb{A}^1$$

Since this was defined an open set of $V(x^3+y^3+z^3+1)$, this allowed us to calculate the induced map on the fields of rational functions:

$$\mathbb{C}(s,t) \;\to\; \operatorname{frac}\left(\mathbb{C}[x,y,z]/(x^3+y^3+z^3+1)\right)$$

This required a choice of affine open subsets of the original projective varieties, different choices of charts would have resulted in potentially different isomorphisms. For example the isomorphism:

$$\begin{aligned} s &\mapsto \frac{z+1}{x+y} \\ t &\mapsto \frac{z+\omega}{x+\omega y} \end{aligned}$$

$\square$