

Claire Voisin on the question of rationality

February 27, 2019

Warm-up

Can you recognise these numbers?

Warm-up

Can you recognise these numbers?

$$\left(\frac{3}{5}, \frac{4}{5}\right),$$

Warm-up

Can you recognise these numbers?

$$\left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{5}{13}, \frac{12}{13}\right),$$

Warm-up

Can you recognise these numbers?

$$\left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{5}{13}, \frac{12}{13}\right), \left(\frac{8}{17}, \frac{15}{17}\right),$$

Warm-up

Can you recognise these numbers?

$$\left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{5}{13}, \frac{12}{13}\right), \left(\frac{8}{17}, \frac{15}{17}\right), \left(\frac{7}{25}, \frac{24}{25}\right),$$

Warm-up

Can you recognise these numbers?

$$\left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{5}{13}, \frac{12}{13}\right), \left(\frac{8}{17}, \frac{15}{17}\right), \left(\frac{7}{25}, \frac{24}{25}\right), \left(\frac{20}{29}, \frac{21}{29}\right), \dots$$

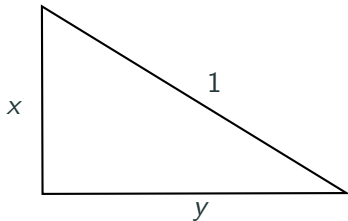
Warm-up

Can you recognise these numbers?

$$\left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{5}{13}, \frac{12}{13}\right), \left(\frac{8}{17}, \frac{15}{17}\right), \left(\frac{7}{25}, \frac{24}{25}\right), \left(\frac{20}{29}, \frac{21}{29}\right), \dots$$

These are solutions (x, y) of

$$x^2 + y^2 = 1.$$



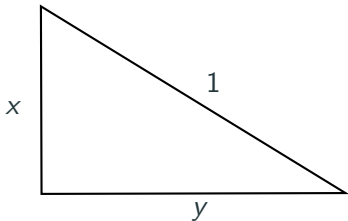
Warm-up

Can you recognise these numbers?

$$\left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{5}{13}, \frac{12}{13}\right), \left(\frac{8}{17}, \frac{15}{17}\right), \left(\frac{7}{25}, \frac{24}{25}\right), \left(\frac{20}{29}, \frac{21}{29}\right), \dots$$

These are solutions (x, y) of

$$x^2 + y^2 = 1.$$



All the solutions:

$$x = \frac{1 - t^2}{1 + t^2} \quad y = \frac{2t}{1 + t^2}.$$

Warm-up

Two systems of equations . . .

Variables: x, y

Equations: $x^2 + y^2 = 1$.

Variables: t

Equations: None.

Warm-up

Two systems of equations ...

Variables: x, y

Equations: $x^2 + y^2 = 1$.

Variables: t

Equations: None.

... are equivalent by

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$$

\longleftarrow

t

x, y

\longrightarrow

$$\frac{y+1-x}{y+1+x}$$

Warm-up

Two systems of equations ...

Variables: x, y

Equations: $x^2 + y^2 = 1$.

Variables: t

Equations: None.

... are equivalent by

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$$

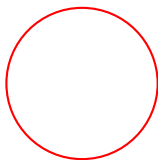
x, y



t



$$\frac{y+1-x}{y+1+x}$$



$$x^2 + y^2 = 1$$



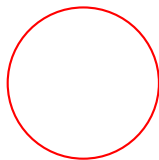
Algebraic varieties

An **algebraic variety** is the set of solutions of a system of polynomial equations.

Algebraic varieties

An **algebraic variety** is the set of solutions of a system of polynomial equations.

Examples

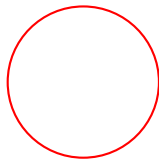


$$x^2 + y^2 = 1$$

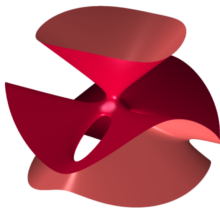
Algebraic varieties

An **algebraic variety** is the set of solutions of a system of polynomial equations.

Examples



$$x^2 + y^2 = 1$$

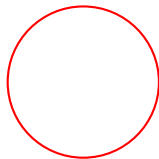


$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$$

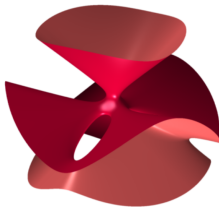
Algebraic varieties

An **algebraic variety** is the set of solutions of a system of polynomial equations.

Examples



$$x^2 + y^2 = 1$$



$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$$



A Kummer K3

Algebraic varieties

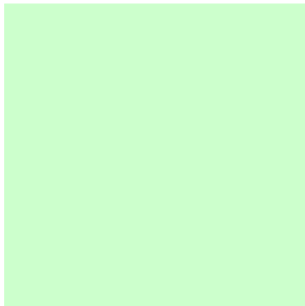
An **algebraic variety** is the set of solutions of a system of polynomial equations.

Example (The best one)

Algebraic varieties

An **algebraic variety** is the set of solutions of a system of polynomial equations.

Example (The best one)



\mathbf{A}^n = The ambient space (no equations)!

Rational varieties

A variety X is **rational** if it is birational to \mathbf{A}^n .

Rational varieties

A variety X is **rational** if it is birational to \mathbf{A}^n .



Rational varieties

A variety X is **rational** if it is birational to \mathbf{A}^n .



System of equations \longleftrightarrow Coördinate change \longleftrightarrow No equations!

Rational varieties

Which varieties are rational?

Rational varieties

Which varieties are rational?

1. The variety defined by $x^2 + y^2 = 1$ is rational.

Rational varieties

Which varieties are rational?

1. The variety defined by $x^2 + y^2 = 1$ is rational.
2. Varieties defined by linear equations are rational.

Rational varieties

Which varieties are rational?

1. The variety defined by $x^2 + y^2 = 1$ is rational.
2. Varieties defined by **linear equations** are rational.
3. Varieties defined by **one quadratic** equation are rational (over **C**).

Rational varieties

Which varieties are rational?

1. The variety defined by $x^2 + y^2 = 1$ is rational.
2. Varieties defined by **linear equations** are rational.
3. Varieties defined by **one quadratic** equation are rational (over **C**).
4. Varieties defined by **one cubic** ?

Rational varieties

Which varieties are rational?

1. The variety defined by $x^2 + y^2 = 1$ is rational.
2. Varieties defined by **linear equations** are rational.
3. Varieties defined by **one quadratic** equation are rational (over **C**).
4. Varieties defined by **one cubic** ?
 - 4.1 Cubic curves: not rational (ancient)
 - 4.2 Cubic surfaces: rational (Castelnuovo, Enriques: Early 1900s)
 - 4.3 Cubic threefolds: not rational (Clemens–Griffiths: 1972)
 - 4.4 Cubic fourfolds and higher: ???

Invariants to detect non-rationality

Artin–Mumford (1971): If X is a rational smooth projective variety, then $H^3(X, \mathbf{Z})$ is torsion-free.

Invariants to detect non-rationality

Artin–Mumford (1971): If X is a rational smooth projective variety, then $H^3(X, \mathbf{Z})$ is torsion-free.

So we have the **Artin–Mumford invariant**

$$H^3(X, \mathbf{Z})_{\text{tors}}$$

as a candidate to detect non-rationality.

Invariants to detect non-rationality

Artin–Mumford (1971): If X is a rational smooth projective variety, then $H^3(X, \mathbf{Z})$ is torsion-free.

So we have the **Artin–Mumford invariant**

$$H^3(X, \mathbf{Z})_{\text{tors}}$$

as a candidate to detect non-rationality.

But $H^3(X, \mathbf{Z})_{\text{tors}} = 0$ for all interesting examples.



Photo credit: CNRS News Article "Claire Voisin, 2016 CNRS Gold Medal"

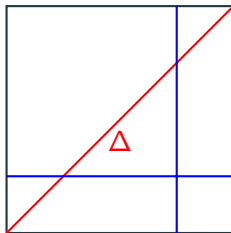
Decomposition of the diagonal

Definition (Voisin, 2015)

X admits a **decomposition of the diagonal** if in $\text{Chow}(X \times X)$,

$$[\Delta] \sim \{x\} \times X + \alpha$$

for some α supported on $X \times Z$ for $Z \subsetneq X$.



Theorem (Voisin, 2015)

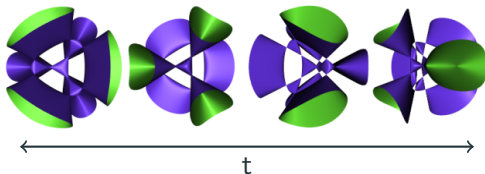
1. X rational $\implies X$ admits a decomp. of the diagonal.
2. X admits decomp. of the diagonal $\implies H^3(X, \mathbf{Z})_{\text{tors}} = 0$.

Decomposition of the diagonal

Theorem (Voisin, 2015)

1. X rational $\implies X$ admits a decomp. of the diagonal.
2. X admits decomp. of the diagonal $\implies H^3(X, \mathbf{Z})_{\text{tors}} = 0$.
3. If X_t is a family of varieties such that some X_{t_0} does not admit a decomp. of the diagonal, then neither does X_t for almost all t .

For example, $X_t = \{x^4 + y^4 + z^4 + w^4 - txyzw = 0\}$.



Decomposition of the diagonal

New technique for non-rationality theorems:

1. Consider a family X_t .
2. Find a t_0 such that X_{t_0} does not admit a decomposition of the diagonal (for example, show $H^3(X_{t_0}, \mathbf{Z})_{\text{tors}} \neq 0$).
3. Theorem: Almost all X_t are not rational!

Decomposition of the diagonal

New technique for non-rationality theorems:

1. Consider a family X_t .
 2. Find a t_0 such that X_{t_0} does not admit a decomposition of the diagonal (for example, show $H^3(X_{t_0}, \mathbf{Z})_{\text{tors}} \neq 0$).
 3. Theorem: Almost all X_t are not rational!
- Quartic double solids (Voisin, 2015),
 - Rationality is not deformation invariant (Hassett–Pirutka–Tschinkel, 2016).
 - Hypersurfaces in \mathbf{P}^{n+1} of degree $d \geq \log_2 n + 2$ (Schreieder, 2018)

There's a lot more...

1. Kodaira problem,
2. Green's conjecture for canonical curves,
3. Chow rings of K3 surfaces,
4. Many questions related to the Hodge conjecture.