ALGEBRAIC GEOMETRY: HOMEWORK 1

The homework is due on Friday, August 2 by 5pm.

(1) Let $U \subset \mathbb{A}^1_{\mathbb{C}}$ be the unit circle; that is, $U = \{z \mid |z| = 1\}$. Is U an affine algebraic subset of $\mathbb{A}^1_{\mathbb{C}}$? Why or why not?

(2) Let I be an ideal of a ring R. The radical of I, denoted $\sqrt{I}$, is defined as the subset of R consisting of elements a such that $a^n \in I$ for some positive integer n. Show that $\sqrt{I}$ is an ideal of R, and $\sqrt{\sqrt{I}} = I$.

(3) Let $k = \mathbb{C}$. Consider the map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$. Is the image of f closed? Open? Dense?

Often, we can identify the points of an affine space with some other objects of interest. With such an identification, we can ask if a subset of the set of objects forms a closed or open set in the Zariski topology. The next problem is an example.

(4) Let n be a positive integer. You may take $k = \mathbb{C}$ if that helps. Identify the set $M_n(k)$ of k-valued $n \times n$ matrices with $\mathbb{A}^{n^2}_k$ by writing the n^2 entries of an $n \times n$ matrix as a n^2-tuple. With this identification, determine whether the following subsets of $\mathbb{A}^{n^2}_k$ are Zariski closed, open, or neither.
   (a) The set of invertible matrices.
   (b) The set of nilpotent matrices.
   (c) For every r, the set of matrices of rank at most r.
   (d) The set of diagonalisable matrices.