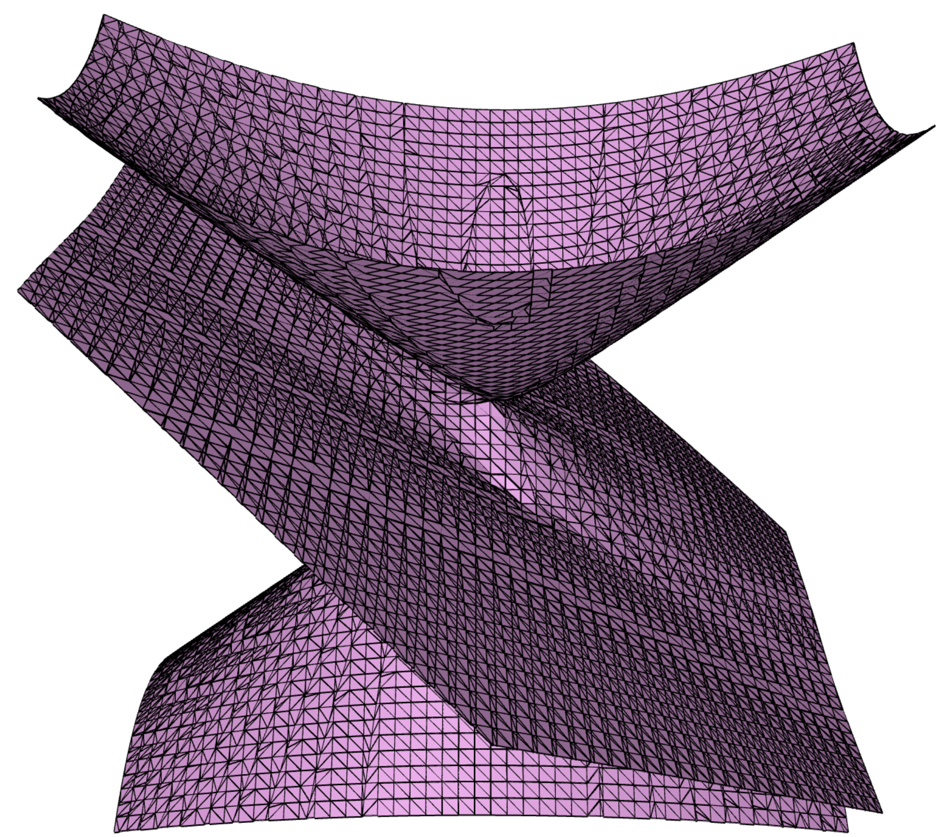
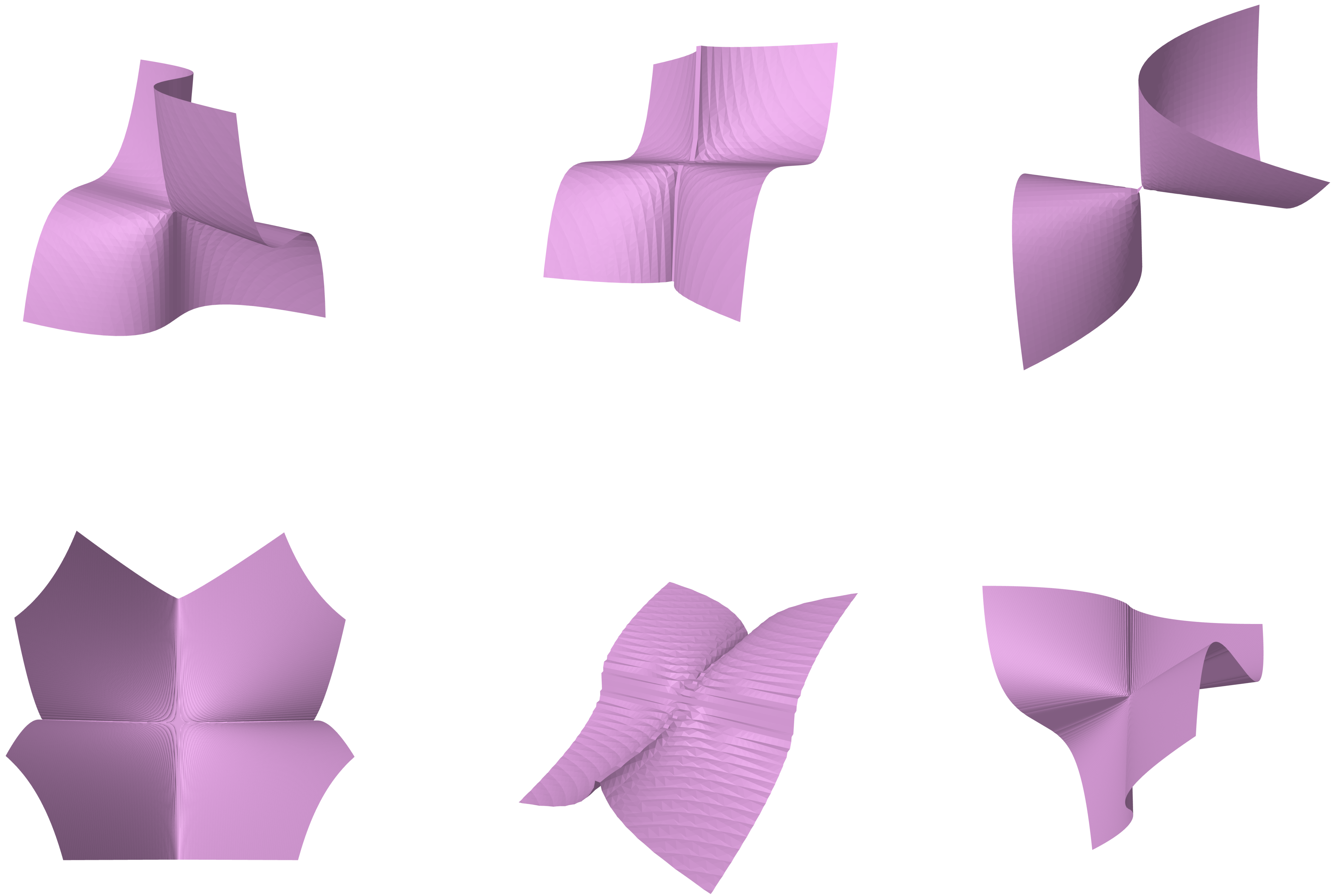
How twisty is that orbit?

Anand Deopurkar

2024

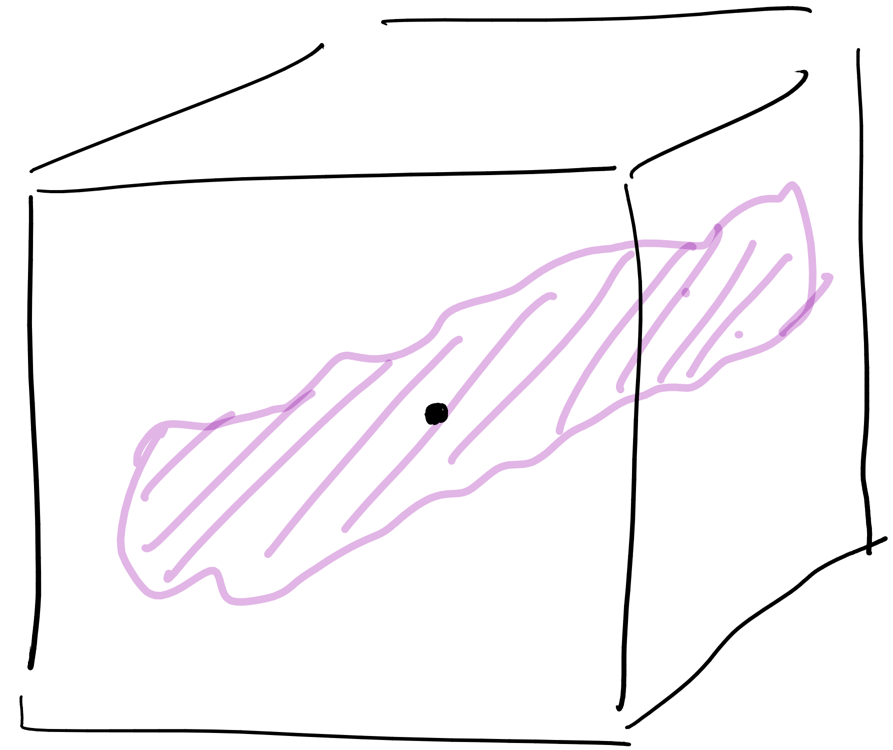
NZMS AustMS AMS



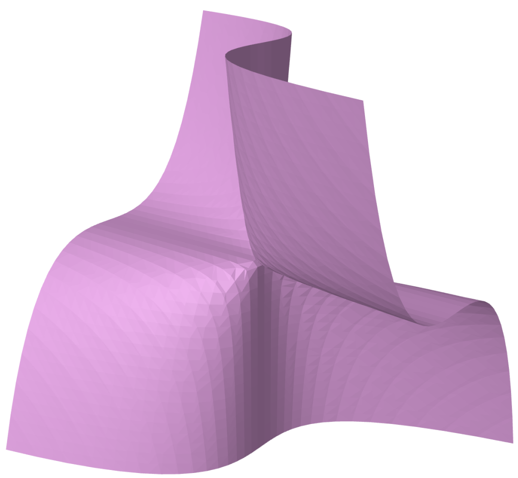


Orbits of group actions

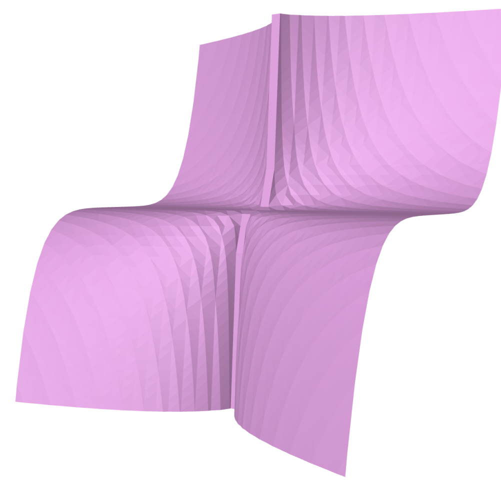
$$\begin{array}{ccc} G & \curvearrowright & V \\ & & \rotatebox{90}{$\in$} \\ \overline{Gv} & \leftsquigarrow & v \end{array}$$



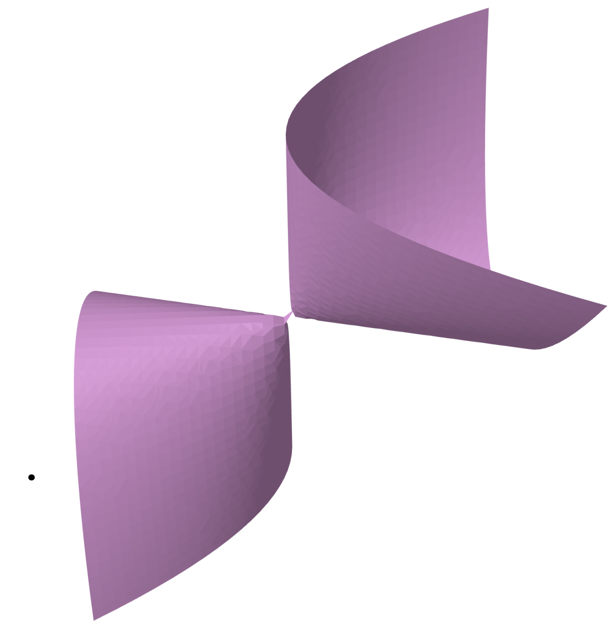
$G = \mathbb{G}_m^2$ $\quad V = \mathbb{A}^3$



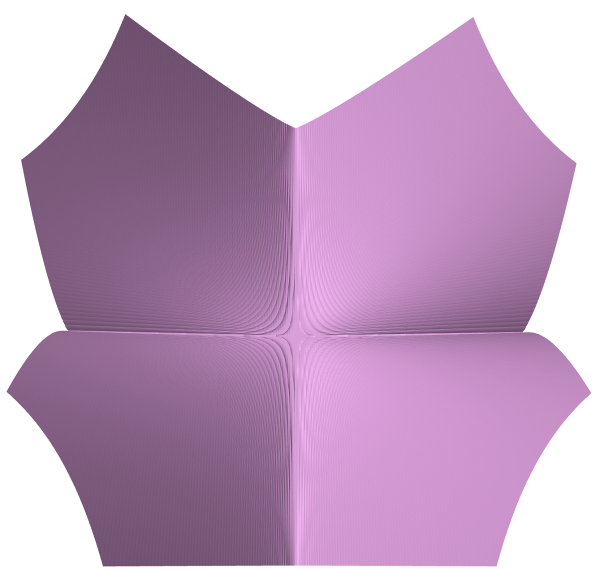
(1,1), (1,2), (2,1)



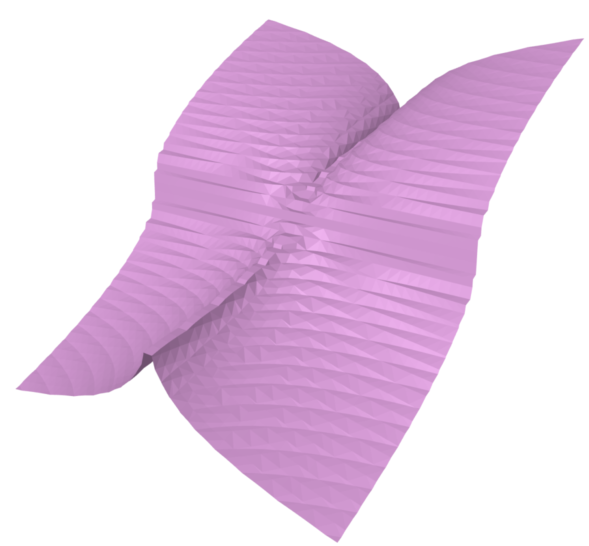
(1,1), (1,2), (3,1)



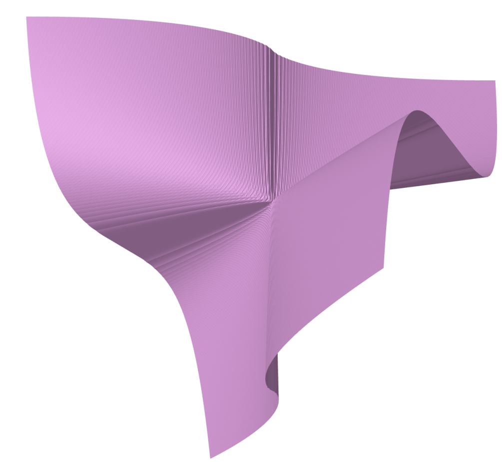
(3,1), (2,2), (1,3)



(3,2), (4,4), (1,3)



(5,1), (1,2), (3,1)

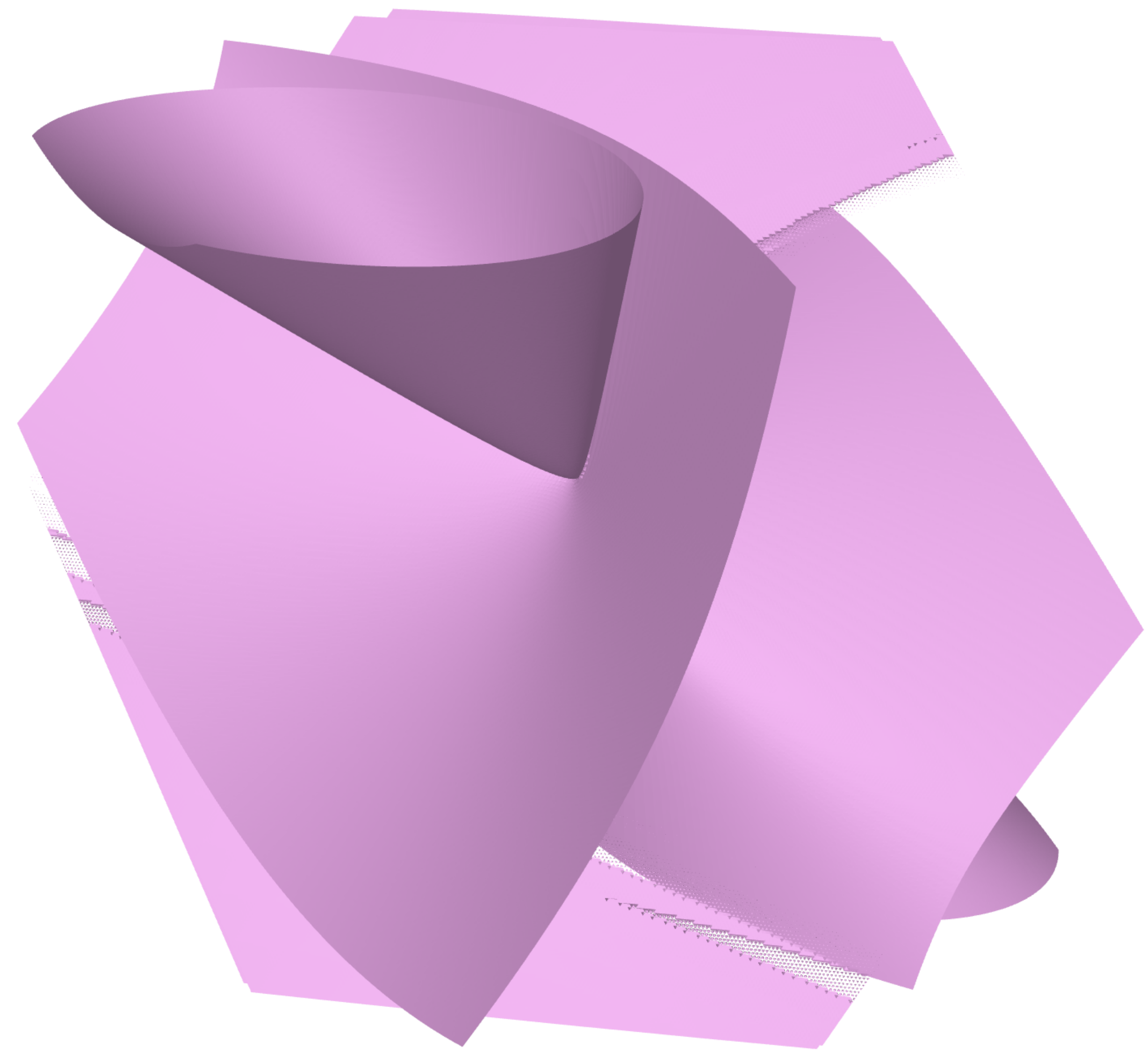


(5,1), (4,2), (3,5)

$G = GL(2)$

$V = \mathrm{Sym}^4(2)$

$v = XY(X-Y)(X-3Y)$

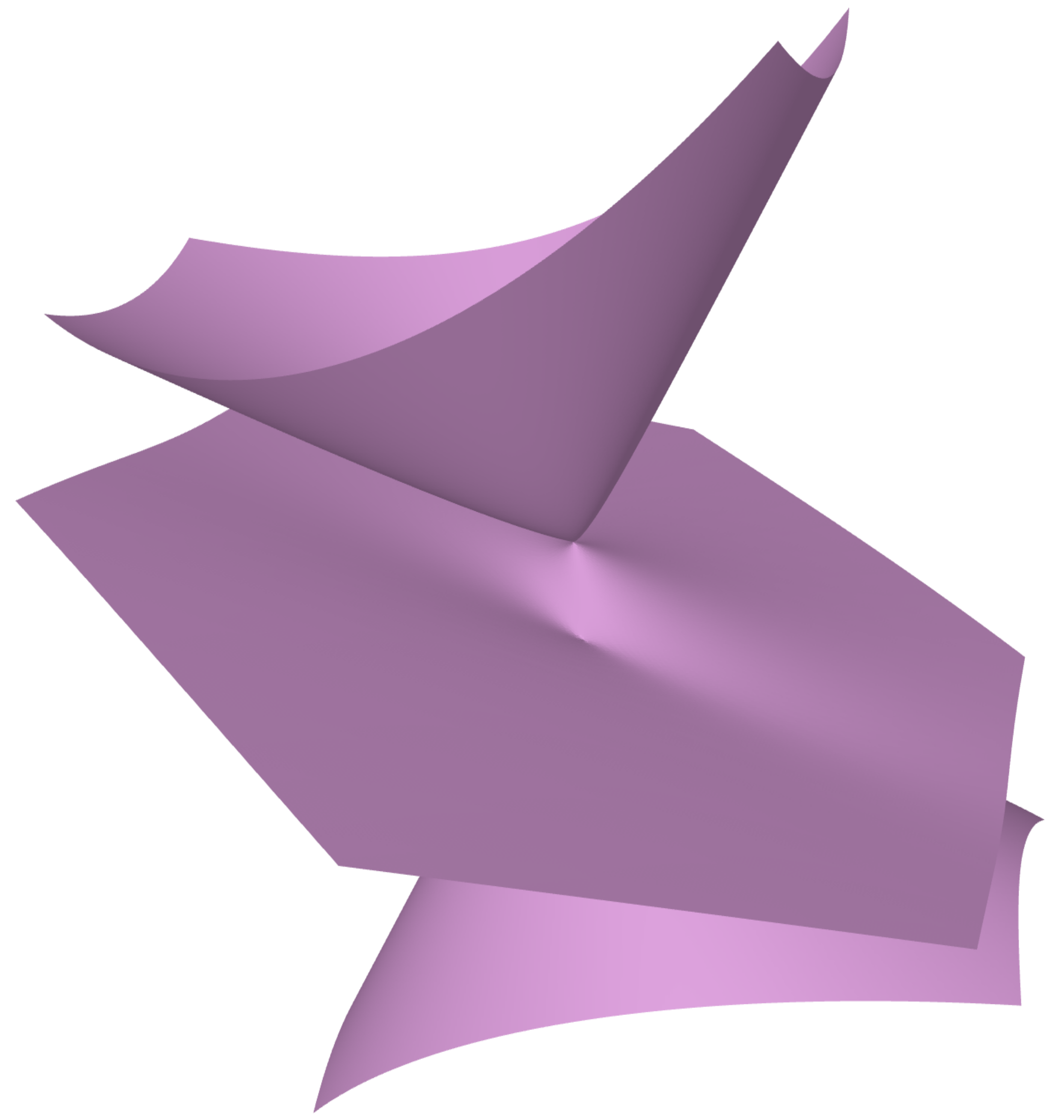


(3d slice)

$G = GL(2)$

$V = \mathrm{Sym}^4(2)$

$v = XY(X-Y)(X+Y)$

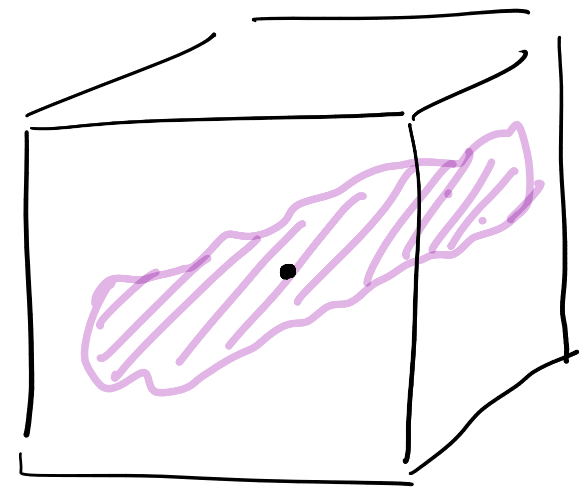


(3d slice)

Orbits of group actions

$G \curvearrowright V$

$X = \overline{Gv} \leftarrow v \in V$

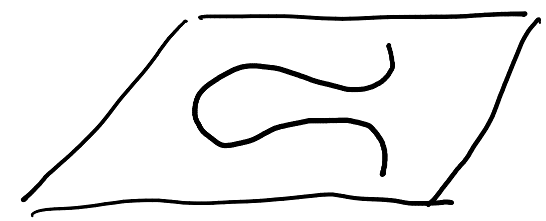


1) $\dim X = ?$
2) $\deg X = ?$
3) $\operatorname{Sing} X = ?$
4) $\partial X = (\overline{Gv} - Gv) = ?$

Geometric ramifications

Ex: $G = GL(2)$ $\quad$ $v =$ 

$V = \mathrm{Sym}^4(2)$ $\quad$ $Gv = \left\{ \begin{array}{l} \text{4-tuples with the} \\ \text{same cross-ratio} \end{array} \right\}$

$G = GL(3)$ $\quad$ $v =$ 

$V = \mathrm{Sym}^3(3)$

$Gv = \left\{ \begin{array}{l} \text{Elliptic curves with the} \\ \text{same } j\text{-invariant} \end{array} \right\}$

Other groups, other representations $\Rightarrow$ Other interpretations.

What is Known?

- $GL(2)$, $\mathrm{Sym}^n(2)$ Enriques–Fano 1890s
- $GL(3)$, $\mathrm{Sym}^n(3)$ Aluffi-Faber 1990s
- $GL(4)$, $\mathrm{Sym}^3(4)$ D-Patel-Tseng 2021

  ↳ "27 questions about cubic surfaces" by Ranestad-Sturmfels

- $\mathbb{G}_m^n$, any rep D. 2024
- $GL(2)$, any rep D. 2024
- Reps for quiver moduli Berget, Buch, Fink, Feher, Fulton, Rimanyi, Weber (2000s)

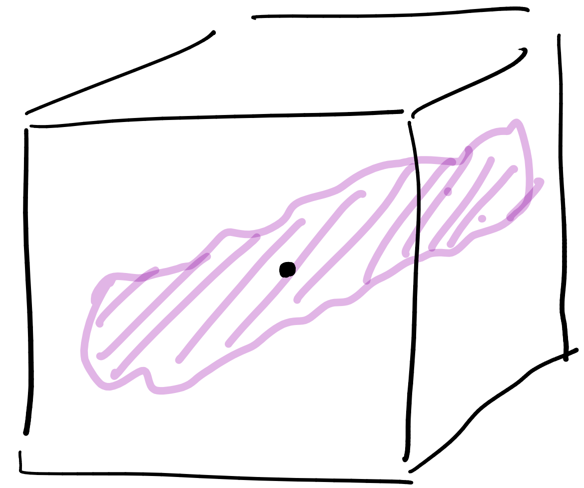
How?


Orbit degrees
Equivariant fundamental classes
imgflip.com

Equivariant Fundamental Class

$$G \curvearrowright V$$

$$X = \overline{G_v} \hookleftarrow v \in V$$



$$\deg X = [X] \in H^*(\mathbb{P}V) \leftarrow \mathbb{Z}$$

$$\mathrm{Efc}(X) = [X] \in H^*_G(V) \leftarrow \text{Polynomial ring}$$

References

1) A universal formula for counting cubic surfaces

DEOPURKAR, PATEL, TSENG,
To appear in Algebraic Geometry

2) Equivariant classes of orbits in GL(2) representations
DEOPURKAR, Pre-print (arxiv)

3) Orbits of linear series on the projective line

DEOPURKAR, PATEL
International Mathematical Research Notices (2024)

From: Anand Deopurkar <anand.deopurkar@anu.edu.au>
Subject: **Exciting! Re: Update on cubic surfaces**
To: **Anand Patel <anandppatel@gmail.com>, Dennis Tseng <ehehheehee@gmail.com>**
Date: Wed, 02 Jun 2021 11:38:30 +1000 (3 years, 26 weeks, 2 days ago)

Hi Anand and Dennis,

I followed yesterday's line of thinking to its conclusion. From the GIT of (X,H), we can write down the following cubic surfaces with Gm actions that are provably not in the closure of a general smooth cubic.

1. $x_0x_2x_3 = x_1^3$
2. $x_3(x_0x_2-x_1^2) = x_0x_1^2$
3. $x_3(x_0x_2-x_1^2) = x_0^2x_1$
4. $x_0^2x_3 = x_1x_2(x_1+x_2)$
5. $x_0x_2^2 = x_1^3 + x_0x_1x_3$

Each one gives a relation (the last one gives the same relation as the first). Together with the relation written on Overleaf for the family B_1, namely $16 a_1^4 + 4 a_1^2a_2 + a_2^2 = 4320$, we can write down the following (integral) relation:

$1080 * a_1^2a_2 - 1080 * a_1*a_3 + 9720 a_4 =$ Cubic surface count.

Amazingly, this agrees with the 96120 count for the family given by a general wel

THANK YOU!