Q1: Identify $\mathbb{A}^n \subset \mathbb{P}^n$ as the open subset U_n of $\mathbb{P}^n$ by $f: \mathbb{A}^n \to U_n$ defined by $f(x_0, \cdots, x_{n-1}) = [x_0 : \cdots : x_{n-1} : 1]$. Since $f(X)$ is Noetherian space, then $f(X) = X_1 \cup \cdots \cup X_m$ where X_i are irreducible components of $f(X)$

Then $\overline{f(X)} = \overline{X_1 \cup \cdots \cup X_m} = \overline{X_1} \cup \cdots \cup \overline{X_m}$

Step 1: Let X be a topological space, $Y \subset X$ be an irreducible subspace, then $\overline{Y}$ is also irreducible

Pf of Lemma 1: Assume $\overline{Y}$ is reducible, then $\overline{Y} = A \cup B$ where $A, B \subsetneq \overline{Y}$ and A, B closed in $\overline{Y}$, so A, B closed in X,

then $Y = Y \cap \overline{Y} = Y \cap (A \cup B) = (Y \cap A) \cup (Y \cap B)$.

then we have $Y = Y \cap A$ (WLOG) since Y irreducible.

Then $A \supset Y \cap A = Y$, then $A \supset \overline{Y} = A \cup B \supset A$, so $A = \overline{Y}$, contradiction

Step 2: $\overline{X_i} \cap U_n = X_i \quad \forall i \in [m]$ Denote closure of A in B by $\overline{A}^B$

Proof of Lemma 2:

$\supset$: $\overline{X_i} \cap U_n = \overline{X_i}^{U_n} \supset X_i$ since $X_i \subset f(X) \subset U_n$

$\subset$: $\overline{X_i} \cap U_n = \overline{X_i} \cap \overline{f(X)} \cap U_n$, so $\overline{X_i} \cap U_n$ is closed in $f(X)$

since $\overline{f(X)} \cap U_n = f(X)$

Note that $\overline{X_i} \cap U_n$ is non-empty open in $\overline{X_i}$ and $\overline{X_i}$ is irreducible.

then $\overline{X_i} \cap U_n$ is irreducible.

Since X_i is an irreducible component, so $\overline{X_i} \cap U_n = X_i$

Step 3: Given Z is irreducible and $\overline{X_i} \subset Z \subset \overline{f(X)}$ closed, then

$Z = \overline{X_i}$

Since $U_n \cap Z$ is open in Z, then $U_n \cap Z$ is irreducible.

Since $Z \subset \overline{f(X)}$ closed, then $Z \cap U_n$ is closed in $f(X)$

And $Z \cap U_n \supset \overline{X_i} \cap U_n = X_i$, then $Z \cap U_n = X_i$ since X_i is an irreducible component of $f(X)$

Then $\overline{Z \cap U_n} = \overline{X_i} \cap Z$. Since $U_n \cap Z$ open in Z, then $U_n \cap Z$ irreducible. Also, $U_n \cap Z$ dense in Z, then $\overline{Z \cap U_n} = Z = \overline{X_i} \cap Z$

so $\overline{X_i} = Z$. Step 3 shows that $\overline{X_i}$ are irreducible components.

Step 4: $\overline{X_i}$ is not in $V(x_n)$

Assume $V(x_n) \supset \overline{X_i}$ for some i. then

$V(x_n) \supset \overline{X_i} \supset X_i$

Given $y \in X_i \subset f(X) \subset U_n$, so $y_n \neq 0$, i.e $y \notin V(x_n)$, contradiction.

Therefore, $X \longrightarrow \overline{f(X)}$ is a well-defined map from closed subset of $\mathbb{A}^n$ to closed subset of $\mathbb{P}^n$ whose irreducible components is not in $V(x_n)$

Now we need to construct an inverse.

Consider $Y \longrightarrow f^{-1}(Y \cap U_n)$ to be the inverse

Since $Y \cap U_n$ closed in U_n, then $f^{-1}(Y \cap U_n)$ closed in $\mathbb{A}^n$, so it is well-defined. To prove it is an actual inverse, suffices to check it is both right inverse and left inverse

① $f^{-1}(\overline{f(X)} \cap U_n) = X$ holds since $\overline{f(X)} \cap U_n = f(X)$ and f is an isomorphism between $\mathbb{A}^n$ and U_n

② $\overline{f(f^{-1}(Y \cap U_n))} = Y$ i.e $\overline{Y \cap U_n} = Y$

Suppose $Y = Y_1 \cup \cdots \cup Y_m$ where Y_i are irreducible component of Y.

then $\overline{Y \cap U_n} = \overline{Y_1 \cap U_n} \cup \cdots \cup \overline{Y_m \cap U_n}$

Since $V(x_n) \not\supset Y_i$, so $Y_i \cap U_n \neq \phi$ and $Y_i \cap U_n$ open in Y_i.

then $\overline{Y_i \cap U_n}^{Y_i} = Y_i$, then $\overline{Y_i \cap U_n} = Y_i$

then $\overline{Y \cap U_n} = \bigcup_{i=1}^{m} \overline{Y_i \cap U_n} = \bigcup_{i=1}^{m} Y_i = Y$

Therefore, the map is a well-defined bijection.

(2) Proof:

Given a rational map (U, φ) where $\varphi: U \to \mathbb{P}^n$ is a regular map, $U \subset \mathbb{P}^1$ open then for $u \in U$, there exists U_1 open subset of U s.t. $u \in U_1$ and for $[x:y] \in U_1$

$\varphi(x,y) = [F_0(x,y) : \cdots : F_n(x,y)]$ where F_i are homogeneous polynomials with the same degree.

Since $F_i(x,y) = y^{\deg F_i} F_i(\frac{x}{y}, 1) = y^{\deg F_i} \prod_{i=1}^{\deg_x F_i} (\frac{x}{y} - c_i) = y^{\deg F_i - \deg_x F_i} \prod_{i=1}^{\deg_x F_i} (x - c_i y)$

then we can define $g(x,y) = \gcd(F_0, \dots, F_n)$, and $g_i = \frac{F_i}{g}$

Define $\psi: \mathbb{P}^1 \to \mathbb{P}^n$ by $\psi = [g_0 ; \cdots ; g_n]$. It is well-defined because if $g_0 = \cdots = g_n = 0$. then $x = y = 0$ since $\{g_i\}$ has no common linear terms.

It is also regular since $g_0, \cdots, g_n$ are homogeneous polynomials with the same degree. Also ψ agree with φ on U_1 since given $[x:y] \in U_1$, then $\psi([x:y]) = [g_0(x,y) : \cdots g_n(x,y)] = [g_0(x,y) g(x,y) : \cdots : g_n(x,y) g(x,y)] = [F_0 : \cdots : F_n] = \varphi([x:y])$

Since $U \subset \mathbb{P}^1$ is irreducible, then $U_1 \subset U$ is dense. Also $U_1 \subset \mathbb{P}^1$ is dense, then we get $\psi|_U = \varphi$ by $\psi|_{U_1} = \varphi|_{U_1}$, which shows that ψ is an extended map of φ.

(3)

(a) Represent the rational map X by $(\mathbb{P}^2 \cap \{X \neq 0\} \cap \{Y \neq 0\} \cap \{Z \neq 0\}, X)$.

Then $X \circ X([X:Y:Z]) = X([YZ:XZ:XY]) = [X^2YZ : XY^2Z : XYZ^2] = [X:Y:Z]$

$= id([X:Y:Z])$

then $X \circ X = id$ on $\mathbb{P}^2 \cap \{X \neq 0\} \cap \{Y \neq 0\} \cap \{Z \neq 0\}$

so $X \circ X = id$ as rational maps

(b) Claim: $U = \mathbb{P}^2 \setminus \{[0:0:1], [0:1:0], [1:0:0]\}$ is the maximal open subset of $\mathbb{P}^2$ on which X is regular.

First, (U, X) is a regular map since X is well-defined on U as $YZ = ZX = XY = 0 \Leftrightarrow$ At least two of X, Y, Z are 0.

Next, U is maximal. Suffices to show that $U \xrightarrow{X} \mathbb{P}^2$ cannot be extended to $V = U \cup \{[0:0:1]\} \xrightarrow{\psi} \mathbb{P}^2$.

Assume so, then for $[0:0:1] \in V$, there exists V_0 open subset of V containing $[0:0:1]$ s.t. $\psi|_{V_0} = [F_0 : F_1 : F_2]$ where F_i are homogeneous polynomials with the same degree. In particular $\psi([0:0:1])$ is well-defined

can suppose $F_0(0,0,1) \neq 0$. Also, $[YZ : XZ : XY] = [F_0(X,Y,Z) : F_1(X,Y,Z) : F_2(X,Y,Z)]$ on V_0, then $XZ F_0(X,Y,Z) = YZ F_1(X,Y,Z)$ on V_0.

Since V_0 is open in $\mathbb{P}^2$, then V_0 is dense in $\mathbb{P}^2$.

Then $XZF_0(X,Y,Z) = YZF_1(X,Y,Z)$ on $\mathbb{P}^2$, so

$XZF_0(X,Y,Z) = YZF_1(X,Y,Z)$ as polynomials.

Then $Y \mid XZF_0(X,Y,Z)$, so $Y \mid F_0(X,Y,Z)$. then $F_0(X,0,Z) = 0$.

but $F_0(0,0,1) \neq 0$, contradiction.

Therefore, U is the maximal open subset of $\mathbb{P}^2$ on which X is regular

(c) Given a line $l: aX+bY+cZ=0$

Let $U=\{[X:Y:Z]\in\mathbb{P}^2 \mid X\neq 0, Y\neq 0, Z\neq 0\}$

By (a), $\chi: U\to U$ is an isomorphism. then

$$V(l)\cap U \cong \underbrace{\chi(V(l)\cap U)}_{U'} \text{ by isomorphism } \chi|_{V(l)\cap U}$$

Then $U'=\chi^{-1}(V(l)\cap U)$

so $U'=\{[X:Y:Z] \mid [YZ:XZ:XY]\in l,\ X\neq 0, Y\neq 0, Z\neq 0\}$

$=\{[X:Y:Z] \mid aYZ+bXZ+cXY=0,\ X\neq 0, Y\neq 0, Z\neq 0\}$

$=V(aYZ+bXZ+cXY)\cap U$

Therefore, $\chi(V(aX+bY+cZ)\cap U)=V(aYZ+bXZ+cXY)\cap U$

Case 1: $a\neq 0, b\neq 0, c\neq 0$, i.e $\{[0:0:1],[1:0:0],[0:1:0]\}\cap V(l)=\phi$.

then $V(aX+bY+cZ)\cap U^c=\{[0:-c:b],[-c:0:a],[-b:a:0]\}$

$V(aYZ+bXZ+cXY)\cap U^c=\{[1:0:0],[0:1:0],[0:0:1]\}$

and $\chi([0:-c:b])=[-cb:0:0]=[1:0:0]$

$\chi([-c:0:a])=[0:-ca:0]=[0:1:0]$ then $\chi: V(aX+bY+cZ) \to V(aYZ+bXZ+cXY)$

$\chi([-b:a:0])=[0:0:-ab]=[0:0:1]$ is an isomorphism.

In this case, χ transforms lines to conics.

Case 2: Only one of a,b,c is 0. Suppose $a=0$, $b\neq 0, c\neq 0$, then

$\{[0:0:1],[1:0:0],[0:1:0]\}\cap V(l)=\{[1:0:0]\}$

$V(bY+cZ)\cap U^c=\{[0:-c:b],[1:0:0]\}$

$\chi(V(bY+cZ))=\chi(V(bY+cZ)\cap U)\cup\chi(\{[0:-c:b],[1:0:0]\})$

$=(V(bXZ+cXY)\cap U)\cup\{[1:0:0]\}$

$=V(bZ+cY)\cap\{X=1\}$

This case shows that χ transforms lines to lines

Case 3: Only one of a, b, c is non-zero. Suppose $a \neq 0$, $b = c = 0$.

then we get $X = 0$

$X(V(X)) = X([0:b:c]) = [1:0:0]$ which means that

X transforms lines to a point.