ALGEBRAIC GEOMETRY: HOMEWORK 5 (SOLUTION SKETCHES)

(1) In this problem, consider $\mathbb{A}^k$ as the open subset of $\mathbb{P}^k$ where the last homogenous coordinate is non-zero.

The following maps from an open subset of $\mathbb{A}^n$ to $\mathbb{A}^m$ extend to regular maps from $\mathbb{P}^n$ to $\mathbb{P}^m$. Write down these extensions using homogeneous polynomials.

(a) $f\colon \mathbb{A}^1 \to \mathbb{A}^2$ defined by $f(t) = (t^2 - 1, t^3 - t)$.

(b) $f\colon \mathbb{A}^2 \setminus V(xy) \to \mathbb{A}^3$ defined by $f(x, y) = (x/y, y/x, 1/xy)$.

Solution. (a) $[X : Y] \mapsto [X^2 - Y^2 : X^3 - XY^2 : Y^3]$

(b) $[X : Y : Z] \mapsto [X^2 : Y^2 : Z^2 : XY]$

□

(2) Show that the natural map

$$\pi\colon \mathbb{A}^2 \setminus \{(0, 0)\} \to \mathbb{P}^1$$

defined by $\pi(x, y) = [x : y]$ does not extend to a regular map $\pi\colon \mathbb{A}^2 \to \mathbb{P}^1$.

Proof. We show that the given map does not even extend to a continuous map (in the Zariski topology). Since regular maps are continuous, this shows in particular that it does not extend to a regular map.

Let $p = (x, y) \in \mathbb{A}^2 \setminus \{(0, 0)\}$. Consider $p_\lambda = (\lambda x, \lambda y)$ for $\lambda \in k^\times$. Then $\pi(p_\lambda) = [x : y]$ for all λ. The origin is in the Zariski closure of the set $\{p_\lambda \mid \lambda \neq 0\}$. Therefore, any continuous extension of π must map $(0, 0)$ to $[x : y]$. But this cannot be true for all (x, y). Therefore, a continuous extension of π does not exist. □

(3) (3-transitivity of PGL_2) Given three distinct points $p_1, p_2, p_3 \in \mathbb{P}^1$, prove that there exists a unique projective linear transformation $\mathbb{P}^1 \to \mathbb{P}^1$ that sends

$$0 = [0 : 1] \mapsto p_1, 1 = [1 : 1] \mapsto p_2, \text{ and } \infty = [1 : 0] \mapsto p_3.$$

Proof. Say $p_i = [x_i : y_i]$. Consider the matrix $M = \begin{pmatrix} x_3 & x_1 \\ y_3 & y_1 \end{pmatrix}$. The projective linear transformation given by M sends $\infty = [1 : 0]$ to p_3 and $0 = [0 : 1]$ to p_1. Suppose M^{-1} sends p_2 to $[a : b] \in \mathbb{P}^1 \setminus \{0, \infty\}$. It suffices to show now that there is a unique projective linear transformation that fixes 0 and ∞ and sends $[1 : 1]$ to $[a : b]$ (why?). Suppose such a transformation is given by a matrix N. Since N fixes $\infty = [1 : 0]$, its first column must be of the form $(\lambda, 0)^T$. Since N fixes $0 = [0 : 1]$, its second column must be of the form $(0, \mu)^T$. Now, if N

sends $[1 : 1]$ to $[a : b]$, we must have $[\lambda : \mu] = [a : b]$. So $\lambda = ta$ and $\mu = tb$ for some $t \in k^\times$. Hence, N is unique up to multiplication by a scalar. $\square$

(4) (A cubic surface as a conic fibration) Suppose $\operatorname{char} k \neq 2, 3$.
Let $S \subset \mathbb{P}^3$ be the Fermat cubic surface

$$S = V(X^3 + Y^3 + Z^3 + W^3).$$

(a) Consider the linear projection $\pi \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ defined by

$$[X : Y : Z : W] \mapsto [X + Y, Z + W].$$

Show that the center L of the linear projection is contained in S.
(b) Show that $\pi \colon S \setminus L \to \mathbb{P}^1$ extends to a regular map $\pi \colon S \to \mathbb{P}^1$.
(c) What is the fiber of $\pi \colon S \to \mathbb{P}^1$ over a point $[a : b] \in \mathbb{P}^1$? (Be careful!)
(d) (Not to be turned in but highly recommended) Draw a (real) picture depicting L, S, a typical fiber of the linear projection $\pi \colon \mathbb{P}^3 \setminus L \to \mathbb{P}^1$, and a typical fiber of $\pi \colon S \to \mathbb{P}^1$.

Solution. The center of projection is $L = V(X + Y, Z + W)$. If both $X + Y$ and $Z + W$ are zero, then $X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2)$ and $(Z^3 + W^3) = (Z + W)(Z^2 - ZW + W^2)$ are 0, so $X^3 + Y^3 + Z^3 + W^3$ is also 0. So L lies in S.

The factorisation above gives a way to define π along L on S. Consider the map $\pi' \colon [X : Y : Z : W] \mapsto [(Z^2 - ZW + W^2) : -(X^2 - XY + Y^2)]$. Note that for every point in L, at least one of the two coordinates defining this map is nonzero. Therefore, π' is defined (and regular) in some Zariski open set containing L. Since $[X + Y : Z + W] = [(Z^2 - ZW + W^2) : -(X^2 - XY + Y^2)]$ on S, this map agrees with π on S, whenever they are both defined. Hence, together with $\pi \colon [X : Y : Z : W] \mapsto [X + Y : Z + W]$ on $S \setminus L$, we get a regular map $S \to \mathbb{P}^1$. By a slight abuse of notation, we denote the extension also by the letter π.

To find the fibers of the (extended) map, we just have to remember that there are *two* expressions defining it. So, $[X : Y : Z : W]$ maps to $[a : b]$ if *both* of the following hold

$$b(X + Y) - a(Z + W) = 0$$
$$b(Z^2 - ZW + W^2) + a(X^2 - XY + Y^2) = 0.$$

(The first is the equation obtained by setting $[X + Y : Z + W] = [a : b]$ and cross-multiplying, and the second similarly using the second expression for the map.) For most points of S, both expressions make sense, but they are equivalent, so the two equations above are also equivalent. For some points of S, one of these formulas is vacuous ($0 = 0$)—in this case, the second formula saves the day by imposing the right condition.

In short, the fiber of π above $[a : b]$ is given by

$$b(X+Y) - a(Z+W) = 0$$
$$b(Z^2 - ZW + W^2) + a(X^2 - XY + Y^2) = 0.$$

This is a plane conic: the first equation cuts out a plane ($\mathbb{P}^2$), and the second equation further cuts out a conic on this plane. To make this even more explicit, using the first expression, we can write one of the variables as a linear combination of the others, and substituting this linear expression in the next equation gives us a homogeneous quadratic in 3 variables. $\square$