Algebraic geometry (Notes)

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1 Regular functions and maps 1

Throughout this section, k is an algebraically closed field.

1.1 Regular functions

WEEK3

Let $S \subset \mathbb{A}^n$ be a set and let $f: S \to k$ be a function. Let a be a point of S.

1.1.1 Definition (Regular function) We say that f is regular (or algebraic) at a if there exists a Zariski open set $U \subset \mathbb{A}^n$ and polynomials $p, q \in k[x_1, \ldots, x_n]$ with $q(a) \neq 0$ such that

$$f \equiv p/q$$
 on $S \cap U$.

We say that f is regular if it is regular at all points of S.

In other words, f is regular at a point a if locally around a (in the Zariski topology), f can be expessed as a ratio of two polynomials. Although the definition of a regular function makes sense for $S \subset \mathbb{A}^n$, we use it only in the context of quasi-affine varieties.

1.1.2 Examples

- 1. A constant function is regular.
- 2. Every polynomial function is regular.
- 3. Sums and products of regular functions are regular. So, the set of regular functions forms a ring. This ring contains a copy of k, namely the constant functions.
- 1.1.3 **Definition (Ring of regular functions)** We denote the ring of regular functions on S by k[S]. This ring is a k-algebra.

1.1.4 Proposition (Local nature of regularity) Let f be a function on S, and let $\{U_i\}$ be an open cover of S. If the restriction of f to each U_i is regular, then f is regular.

Proof.
$$-(1)$$

1.2 Regular functions on an affine variety

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It turns out that regular functions on closed subsets of \mathbb{A}^n are just the polynomial functions! So, not only is there a global algebraic expression, we don't even need denominators.

1.2.1 Proposition Let $X \subset \mathbb{A}^n$ be a Zariski closed subset. Let f be a regular function on X. Then there exists a polynomial $P \in k[x_1, \ldots, x_n]$ such that P(x) = f(x) for all $x \in X$.

Proof. By definition, we know that for every $x \in X$, there is a Zariski open set $U \subset X$ and polynomials p, q such that f = p/q on U. The set U and the polynomials p, q may depend on x, so let us denote them by U_x , p_x , and q_x . We need to combine all of these p's and q's and construct a single polynomial P that agrees with f for all x.

This is done by a "partition of unity" argument. First, let us do some preparation. We know that $p_x/q_x = f$ on U_x , but we know nothing about p_x and q_x on the complement of U_x . Our first step is a small trick that lets us assume that both p_x and q_x are identically zero on the complement of U_x .

Since $U_x \subset X$ is open, its complement is closed. By the definition of the Zariski topology, this means that

$$X \setminus U_r = X \cap V(A),$$

for some $A \subset k[x_1, \ldots, x_n]$. Since $x \in U_x$, at least one of the polynomials in A must be non-zero at x. Let g be such a polynomial, and set $U_x' = X \cap \{g \neq 0\}$. Then $U_x' \subset U_x$ is a possibly smaller open set containing x. Set $p_x' = p_x \cdot g$ and $q_x' = q_x \cdot g$. Then we have $f = p_x'/q_x'$ on U_x' , and we also have $p_x' \equiv q_x' \equiv 0$ on $X \setminus U_x'$. So, we may assume from the beginning that both p_x and q_x are identically zero on the complement of U_x .

Now comes the crux of the argument. Suppose X = V(I). Consider the set of "denominators" $\{q_x \mid x \in X\}$. Note that the system of equations

$$g = 0$$
 for all $g \in I$ and $q_x = 0$ for all $x \in X$

has no solution!

Why is this true? — (2)

By the Nullstellensatz, this means that the ideal $I + \langle q_x \mid q \in X \rangle$ is the unit ideal. That is, we can write

$$1 = g + r_1 q_{x_1} + \dots + r_m q_{x_m}$$

for some polynomials r_1, \ldots, r_m . Take $P = r_1 p_{x_1} + \cdots + r_m p_{x_m}$. Then f = P on all of X.

Check the last equality. — (3)

—- Let $X \subset \mathbb{A}^n$ be any subset. We have a ring homomorphism

$$\pi \colon k[x_1,\ldots,x_n] \to k[X],$$

where a polynomial f is sent to the regular function it defines on X.

1.2.2 Proposition (Ring of regular functions of an affine) Let $X \subset \mathbb{A}^n$ be a closed subset. Then the ring homomorphism $\pi \colon k[x_1, \dots, x_n] \to k[X]$ induces an isomorphism

$$k[x_1, \dots, x_n]/I(X) \xrightarrow{\sim} k[X].$$

Proof. The map π is surjective by Proposition 1.2.1 and its kernel is I(X) by definition. The result follows by the isomorphism theorems.

1.3 Regular maps

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Consider $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ and a function $f: X \to Y$. Write f in coordinates as

$$f=(f_1,\ldots,f_m).$$

- **1.3.1 Definition (Regular map)** We say that f is regular at a point $a \in X$ if all its coordinate functions f_1, \ldots, f_m are regular at a. If f is regular at all points of X, then we say that it is regular.
- **1.3.2** Example (Maps to \mathbb{A}^1) A regular map to \mathbb{A}^1 is the same as a regular function.
- **1.3.3 Example (An isomorphism)** Let $U = \mathbb{A}^1 \setminus \{0\}$ and $V = V(xy 1) \subset \mathbb{A}^2$. We have a regular function $\phi \colon V \to U$ given by $\phi(x,y) = x$. We have a regular function $\psi \colon U \to V$ given by $\psi(t) = (t,1/t)$. These functions are mutual inverses, and hence we have a (bi-regular) isomorphism $U \cong V$.

1.4 Properties of regular maps

WEEK3

1.4.1 Proposition (Elementary properties of regular maps)

- 1. The identity map is regular.
- 2. The composition of two regular maps is regular.
- 3. Regular maps are continuous (in the Zariski topology).

Proof. The identity map is given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$; each coordinate is a polynomial, and hence regular. The statement for composition is true because the composition of fractions of polynomials is also a fraction of polynomials. The third statement is left as homework.

1.4.2 Proposition (Regular maps preserve regular functions) Let $\phi: X \to Y$ be a regular map. If f is a regular function on Y, then $f \circ \phi$ is a regular function on X.

Proof. View a regular function as a regular map to \mathbb{A}^1 . Then this becomes a special case of composition of regular maps.

— As a result, we get a k-algebra homomorphism $k[Y] \to k[X]$, often denoted by ϕ^* :

$$\phi^*(f) = f \circ \phi.$$

We thus get a (contravariant) functor from the category of (quasi-affine) varieties to k-algebras. On objects, it maps X to k[X]. On morphisms, it maps $\phi \colon X \to Y$ to $\phi^* \colon Y \to X$. It is easy to check that this recipe respects composition. That is, if we have maps $\phi \colon X \to Y$ and $\psi \colon Y \to Z$, and if we let $\psi \circ \phi \colon X \to Z$ be the composite, then

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*.$$

1.4.3 Corollary (Isomorphic varieties have isomorphic rings of functions) If $\phi \colon X \to Y$ is an isomorphism of varieties, then $\phi^* \colon k[Y] \to k[X]$ is an isomorphism of k-algebras.

Proof. Let $\psi: Y \to X$ be the inverse of ϕ . Then $\psi^*: k[X] \to k[Y]$ is the inverse of ϕ^* .

1.4.4 Proposition (For affines, map between rings induces map between spaces) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be Zariski closed, and let $f : k[Y] \to k[X]$ be a homomorphism of k-algebras. Then there is a unique (regular) map $\phi : X \to Y$ such that $f = \phi^*$.

Proof. We know that $k[X] = k[x_1, ..., x_n]/I(X)$ and $k[Y] = k[y_1, ..., y_m]/I(Y)$. Let $\phi_i = f(y_i) \in k[X]$. Consider $\phi \colon X \to \mathbb{A}^m$ given by $\phi = (\phi_1, ..., \phi_m)$. Then ϕ sends X to Y and is the unique map satisfying the required properties.

Prove the last statement. — (4)

1.4.5 Example (Bijection but not an isomorphism) Let $X = \mathbb{A}^1_k$ and $Y = V(y^2 - x^3) \subset \mathbb{A}^2_k$. We have a regular map $f \colon X \to Y$ given by $f(t) = (t^2, t^3)$. It is easy to check that f is a bijection, but not an isomorphism.

Why is this not an isomorphism? — (5)

1.4.6 Example (Distinguished affine opens) Let $U_f \subset \mathbb{A}^n$ be the complement of V(f). Then U_f is isomorphic to an affine variety, namely the variety $V(yf-1) \subset \mathbb{A}^{n+1}$, where y denotes the (n+1)-th coordinate.

Prove this. - (6)

1.4.7 Caution (Not all opens are affine) The previous proposition only applies to the complement of V(f) for a single f! The complement of V(I), in general, is not isomorphic to an affine variety. For example, the complement of the origin in \mathbb{A}^2 is not isomorphic to an affine variety.