ANALYSIS AND OPTIMIZATION: MATRIX GAMES

SPRING 2016

1. Introduction

We will use linear programming to analyze simple two-player games, called *matrix games* or *two-player zero sum games*.

Before describing the general setup, let us look at two examples.

Example 1 (Rock, paper, scissors)**.** Player 1 and player 2 simultaneously choose rock, paper, or scissors. The outcome of the game is specified by the following *payoff matrix*

	Rock	Paper	Scissors
Rock	0	−1	1
Paper	1	0	−1
Scissors	−1	1	0.

Player 1's moves correspond to the rows and player 2's moves correspond to the columns. The entry in row i and column j corresponds to the payoff to Player 1 if player 1 chooses move i and player 2 chooses move j. The payoff to player 2 is the negative of this value (hence the name *zero-sum*).

Example 2 (Football)**.** Player 1 is the offense and player 2 is the defense in football. Player 1 can either run or pass, and player 2 can either defend against a run or defend against a pass. The payoff values are the yards gained by the offense.

	defend run	defend pass
run	2	6
pass	8	−6

Example 3. Consider the game given by the following table

$$\begin{pmatrix} -2 & 7 & -5 \\ 1 & 2 & 4 \end{pmatrix}$$

In the last game, row 2 and colum 1 represents an *equilibrium* – neither the row player or the column player has an incentive to change their action. Doing so will only hurt their payoff. In the first two games, there are no such equilibriums.

Let us try to make our games a little more interesting by allowing *mixed strategies*. Instead of playing the game once, suppose the players play the game repeatedly, and instead of making the same moves every time, they switch them up. Let us say that Player 1 makes move i with probability q_i and Player 2 makes move j with probability p_j. Denote by A the payoff matrix. Then the *expected payoff* (for Player 1) is given by

$$\sum_{ij} q_i A_{ij} p_j = q^T A p.$$

(Explain this a bit, if necessary). Player 1 will try to maximize this number and Player 2 will try to minimize this number. Here $p = (p_1, \ldots, p_n)^T$ and $q = (q_1, \ldots, q_m)^T$ (column vectors) are *probability vectors* ($p_j \geq 0$ and $\sum p_j = 1$; similarly $q_i \geq 0$ and $\sum q_i = 1$).

2. Optimal strategies

Definition 4. We say that a pair of strategies (q^0, p^0) is *optimal* (or is an *equilibrium*) if

$$q^T A p^0 \leq (q^0)^T A p^0$$

and

$$(q^0)^T A p^0 \leq q^0 A p^T$$

for any probability vectors p and q. The value $(q^0)^T A p^0$ is called the *value* of the game.

In other words, deviating from q^0 will not help Player 1 and deviating from p^0 will not help Player 2. If the value of the game is positive, then the game is biased towards Player 1. If the value is negative, then it is biased towards Player 2.

Here is another interpretation of the optimum strategies.

Proposition 5. *Assume that (p^0, q^0) is optimal. Then q^0 is the probability vector that maximizes $f(q) = \min_p(q^T A p)$, where the min is taken over all probability vectors p.*

Think of $f(q)$ as the "worst case" payoff function for Player 1. Then the proposition says that q^0 maximizes the worst case payoff.

Proof. Since we have

$$q^T A p^0 \leq (q^0)^T A p^0,$$

we have

$$f(q) \leq (q^0)^T A p^0.$$

But

$$(q^0)^T A p^0 \leq (q^0)^T A p$$

means that

$$f(q^0) = (q^0)^T A p^0.$$

So we get

$$f(q) \leq f(q^0),$$

as required. □

The main theorem is that for every game a pair of optimal strategies always exists! Note that this is only true if we allow probabilistic strategies. We already saw that this fails for pure (i.e. non-probabilistic) strategies. In fact, we will also see how to find the optimal strategies.

Example 6. Show by hand that $p^0 = q^0 = (1/3, 1/3, 1/3)$ is an optimal pair for rock, paper, scissors and the value of the game is 0.

3. Linear programming

We will phrase the problem of finding q^0 as a linear programming problem, and it will turn out that p^0 is the solution of the dual problem. Before we proceed, let us make a simplification. Let us assume that all entries of A are positive. This can be achieved by adding a large positive number k to all the entries. This increases all the payoffs by k but does not change the game in any important way. In particular, an optimal pair for the new game will be an optimal pair for the original game and vice-versa. Note that the value will also go up by k.

Suppose Player 1 chooses strategy q. Then the worst scenario for Player 1 is that player 2 chooses the strategy p that minimizes $q^T Ap$. The minimum value of $q^T Ap$ as p ranges over all probability vectors is the smallest coordinate of $q^T A$ (and it is achieved when p is the corresponding standard basis vector). In particular, if

$$q^T A \geq (a, \dots, a),$$

then Player 1 (in the worst case) is guarranteed to get payoff at least a.

So we can form the problem: maximize a subject to $q^T A \geq (a, \dots, a)$ where $q \geq 0$ is a vector with $\sum q_j = 1$. By replacing q by q/a, we can eliminate the extra variable a and arrive at the problem: maximize $1/(\sum q_j)$ subject to $q^T A \geq (1, \dots, 1)$ where $q \geq 0$. The last problem is equivalent to: minimize $\sum q_j$ subject to $A^T q \geq 1$ and $q_j \geq 0$.

3.1. Review of LP duality.

Let us remember duality. The problem of minimizing $c^T X$ subject to $MX \geq b$ and $X \geq 0$ has a dual problem of maximizing $b^T Y$ subject to $M^T Y \leq c$ and $Y \geq 0$. The main theorem of duality of linear programming is that if both problems have non-empty feasible regions, then there exist optimal solutions X^0 and Y^0 with $b^T Y_0 = c^T X_0$. Also, in the simplex method for finding Y_0, the X_0 appears as "shadow prices" and vice-versa. (The students should know this, and only need to be reminded. A lot of them will not remember this, but just state it and move on.)

Back to existence of optimal strategies. Let us come back to our problem. Here $c = (1, \dots, 1)$ (m times) and $b = (1, \dots, 1)$ (n times). Also, we are using the variable names q and p instead of X and Y. So the dual problem is given by: maximize $\sum p_i$ subject to $Ap \leq 1$ and $p \geq 0$. By the same analysis as before, this is the problem of finding the strategy p^0 for Player 2 that gives them the best (i.e. lowest) worst case payoff.

Since the matrix A has positive entries, it is easy to show that both problems have non-empty feasible regions. (For the original problem choose a *huge* q and for the dual problem, choose a *tiny* p).

So a pair (q^*, p^*) of optimal solutions for the primal/dual give optimal strategies. Note that because of a scaling that we did to arrive at the linear programming problem, the actual probability vector is given by

$$q^0 = q^* / \sum q_j^* \quad p^0 = p^* / \sum p_i^*,$$

and the value of the game (namely the "a" that we eliminated) is the reciprocal of the common optimal value of the primal/dual linear programs.

4. Examples revisited

We already solved the rock, paper, scissors example by hand. Let us look at the football example and find the optimal strategies. First we need to add a large enough k so that the payoffs become positive; let us add $k = 10$. Then the matrix is

$$\begin{pmatrix} 12 & 16 \\ 18 & 4 \end{pmatrix}$$

The linear program that gives the optimal strategy for Player 1 is: minimize $q_1 + q_2$ subject to

$$12q_1 + 18q_2 \geq 1 \quad 16q_1 + 4q_2 \geq 1 \quad q_1, q_2 \geq 0.$$

The simplex method (using a computer) gives the solution

$$q_1 = 1/60, q_2 = 7/120$$

with corresponding dual solution

$$p_1 = 1/20, p_2 = 1/140$$

and the optimal value $3/40$.

Converting these to probability vectors and subtracting 10 from the value gives the strategies $(7/9, 2/9)$ for the offense, $(2/3, 1/3)$ for the defense and value $10/3$. This means that the game is biased towards the offense, and by playing *run* 7 times as many times as *pass*, they will get an *expected* payoff of at least $10/3$ yards, no matter what the defense does. The best course of action for the defense is to defend against *run* twice as many times as defending against *pass*. By playing this way, they will be able to restrict the offense to an *expected* payoff of $10/3$ yards.

5. An economic example

If time permits, here is another "game." (I found it at `http://home.ubalt.edu/ntsbarsh/opre640a/partvi.htm#rinvest`).

Player 1 is an investor who has the choice of buying foreign currency or gold. The payoff depends on how well the country's economy performs. There is no Player 2 as such, but sometimes economists like to think of "fate" as Player 2. We can form a payoff matrix

	High growth	Medium growth	No change	Low growth
Currency	5	4	3	-1
Gold	2	3	4	5.

The investor invests some percent of their money in Currency and the rest in Gold, say (q_1, q_2) where $q_i \geq 0$ and $q_1 + q_2 = 1$. It is reasonable to ask: what is the vector q that will maximize the worst case payoff? Is the worst case payoff positive? What is its value? This is the *guarranteed* returns on the investment, no matter what "fate" plays.

Mathematically, the problem is identical to a two player zero-sum game problem, and we can solve it the same way as before.

After adding 2 to all the entries, we get the problem: minimize $q_1 + q_2$ subject to

$$\begin{aligned} 7q_1 + 4q_2 &\geq 1 \\ 6q_1 + 5q_2 &\geq 1 \\ 5q_1 + 6q_2 &\geq 1 \\ 1q_1 + 7q_2 &\geq 1 \\ q_1, q_2 &\geq 0. \end{aligned}$$

The solution (converted to a probability vector) turns out to be $(1/3, 2/3)$ with a payoff value (of the original problem) 3.

(I use the LP solver at `http://www.zweigmedia.com/RealWorld/simplex.html` to do these linear programming problems.)