VECTOR BUNDLES AND FINITE COVERS

ANAND DEOPURKAR & ANAND PATEL

ABSTRACT. We prove that, up to a twist, every vector bundle on a smooth projective curve arises from the direct image of the structure sheaf of a smooth, connected branched cover.

1. INTRODUCTION

Associated to a finite flat morphism $\phi : X \to Y$ is the vector bundle $\phi_* \mathcal{O}_X$ on Y. This is the bundle whose fiber over $y \in Y$ is the vector space of functions on $\phi^{-1}(y)$. In this paper, we address the following basic question: which vector bundles on a given Y arise in this way? We are particularly interested in cases where X and Y are smooth projective varieties. Formulated differently, the question asks: which vector bundles on a given Y admit the structure of a commutative and associative $\mathcal{O}_Y$-algebra, particularly in the presence of additional restrictions like regularity.

Our main result is that, up to a twist, every vector bundle on a smooth projective curve Y arises from a branched cover $X \to Y$ with smooth projective X. Let d be a positive integer and let k be an algebraically closed field with $\operatorname{char} k = 0$ or $\operatorname{char} k > d$.

Theorem 1.1 (Main). *Let Y be a smooth projective curve over k and let E be a vector bundle of rank $(d-1)$ on Y. There exists an integer n (depending on E) such that for any line bundle L on Y of degree at least n, there exists a smooth curve X and a finite map $\phi : X \to Y$ of degree d such that $\phi_* \mathcal{O}_X$ is isomorphic to $\mathcal{O}_Y \oplus E^\vee \otimes L^\vee$.*

The reason for the $\mathcal{O}_Y$ summand is as follows. Pull-back of functions gives a map $\mathcal{O}_Y \to \phi_* \mathcal{O}_X$, which admits a splitting by $1/d$ times the trace map. Therefore, every bundle of the form $\phi_* \mathcal{O}_X$ contains $\mathcal{O}_Y$ as a direct summand. The dual of the remaining direct summand is called the *Tschirnhausen bundle* and is denoted by $E = E_\phi$ (The dual is taken as a convention.) Theorem 1.1 says that on a smooth projective curve, a sufficiently positive twist of every vector bundle is Tschirnhausen.

The reason for needing the twist is a bit more subtle, and arises from some geometric restrictions on Tschirnhausen bundles. For $Y = \mathbf{P}^n$ and a smooth X, the Tschirnhausen bundle E is ample by a result of Lazarsfeld [18]. For more general Y and smooth X, it enjoys several positivity properties as shown in [23, 24]. The precise necessary and sufficient conditions for being Tschirnhausen (without the twist) are unknown, and seem to be delicate even when Y is a curve.

Without any regularity restrictions on X, identifying Tschirnhausen bundles is trivial: every vector bundle E qualifies. Indeed, given E, we may take X to be the non-reduced scheme which is the first order neighborhood of the zero section in the total space of E.

In number theory, when ϕ is the map corresponding to the extension of rings of integers of number fields, the isomorphism class of E is encoded by the Steinitz invariant. The question of classifying realizable Steinitz invariants is open, with progress under various conditions on the Galois group; see [5] and the references therein.

In algebraic geometry, the simplest non-trivial case of identifying Tschirnhausen bundles, namely the case of $Y = \mathbf{P}^1$, has attracted the attention of several mathematicians; see for example

[1,9,21,26]. Historically, it is known as the problem of classifying *scrollar invariants*. Recall that every vector bundle on $\mathbf{P}^1$ splits as a direct sum of line bundles. Writing $E_\phi = \mathcal{O}(a_1)\oplus\cdots\oplus\mathcal{O}(a_{d-1})$, the scrollar invariants are the integers $a_1, \dots, a_{d-1}$. For $d = 2$, any positive integer a_1 is a scrollar invariant. For $d = 3$, a pair of positive integers (a_1, a_2) with $a_1 \leq a_2$ arises as scrollar invariants of smooth triple coverings if and only if $a_2 \leq 2a_1$ [19, § 9]. It may be within reach to completely settle the next few values of d using structure theorems for finite covers of low degree due to Casnati and Ekedahl [6, 7], but the problem becomes difficult quickly as d increases. The lack of constructive structure theorems for covers of degree ≥ 6 makes any direct attacks unfeasible. Nevertheless, the picture emerging from the collective work of several authors [9,21], and visible in the $d = 3$ case, indicates that if the a_i are too far apart, then they cannot be scrollar invariants.

Our result says that the picture is the cleanest possible if we allow twisting by a line bundle. For $Y = \mathbf{P}^1$, twisting is equivalent to allowing a simultaneous shift. Before our work, the work of Ballico [1] came closest to a characterization of scrollar invariants up to a shift. He showed that one can arbitrarily specify the smallest $d/2$ of the $(d-1)$ scrollar invariants. Our main theorem (Theorem 1.1) applied to $Y = \mathbf{P}^1$ answers the question completely: it says that one can in fact arbitrarily specify *all* of them.

For affine curves, Theorem 1.1 yields the following corollary.

Corollary 1.2. *Suppose Y is a smooth affine curve, and E is a vector bundle on Y. Then E is the Tschirnhausen bundle for some map $\phi : X \to Y$, with X smooth and connected.*

Proof. Extend E to a vector bundle E' on the smooth projective compactification Y' of Y. Apply Theorem 1.1 to E', twisting by a sufficiently positive line bundle L on Y' whose divisor class is supported on the complement $Y' \setminus Y$. We obtain a smooth curve X' and a map $\phi : X' \to Y'$ whose Tschirnhausen bundle is $E' \otimes L$; letting $X = \phi^{-1}(Y)$, we obtain the corollary. □

The method of proof of Theorem 1.1 yields a basic result relating the moduli of branched covers and the moduli of vector bundles. Let $k = \mathbf{C}$. Denote by $H_{d,g}(Y)$ the Hurwitz space, namely the coarse moduli space of primitive branched coverings of Y of degree d and genus g (a branched covering $\phi : X \to Y$ is primitive if $\phi_* : \pi_1(X) \to \pi_1(Y)$ is surjective). The space $H_{d,g}(Y)$ is an irreducible algebraic variety [12, Theorem 9.2]. Denote by $M_{r,k}(Y)$ the moduli space of semi-stable vector bundles of rank r and degree k on Y. It is well-known that $M_{r,k}(Y)$ is an irreducible algebraic variety [28]. Note that the Tschirnhausen bundle of a degree d and genus g cover of Y has rank $d-1$ and degree $g-1-d(g_Y-1)$.

Theorem 1.3. *Suppose $g_Y \geq 2$, and set $b = g-1-d(g_Y-1)$. If g is sufficiently large (depending on Y and d), then the Tschirnhausen bundle of a general degree d and genus g branched cover of Y is stable. Moreover, the rational map $H_{d,g}(Y) \dashrightarrow M_{d-1,b}(Y)$ defined by $\phi \mapsto E_\phi$ is dominant.*

The same statement holds for $g_Y = 1$, with "stable" replaced with "regular poly-stable."

Theorem 1.3 is Theorem 3.15 in the main text.

Special cases of Theorem 1.3, namely the cases $d \leq 5$, were proved by Kanev [15, 16, 17] using structure theorems for finite covers of low degree [6, 7]. The crucial new ingredient in our approach is the use of deformation theory to circumvent such direct attacks, which are unfeasible for $d \geq 6$ for the lack of structure theorems.

The validity of Theorem 1.3 for low g is an interesting open problem. It would be nice to know whether $\phi \mapsto E_\phi$ is dominant as soon as we have $\dim H_{d,g}(Y) \geq \dim M_{d-1,b}(Y)$.

Theorem 1.3 opens the door to studying the geometry of the space of branched covers $H_{d,g}(Y)$ through the geometry of the much better understood space of vector bundles $M_{d-1,b}(Y)$. In fact, our motivation for studying the Tschirnhausen bundles was partly in the study of cycles on

$H_{d,g}(Y)$. For a vector bundle E on Y, define the *Maroni locus* $M(E) \subset H_{d,g}(Y)$ as the locally closed subset that parametrizes covers with Tschirnhausen bundle isomorphic to E. This notion generalizes the Maroni loci for $Y = \mathbf{P}^1$, which play a key role in describing the cones of various cycles classes on $H_{d,g}(Y)$ in [10] and [22]. It would be interesting to know if the cycle of $\overline{M(E)}$ has similar distinguishing properties, such as rigidity or extremality, more generally than for $Y = \mathbf{P}^1$. A first step towards this study is to determine when these cycles are non-empty and of the expected dimension. As a consequence of the method of proof of the main theorem, we obtain the following.

Theorem 1.4. *Set $b = g - 1 - d(g_Y - 1)$. Let E be a vector bundle on Y of rank $(d-1)$ and degree e. If g is sufficiently large (depending on Y and E), then for every line bundle L of degree $b-e$, the Maroni locus $M(E \otimes L) \subset H_{d,g}(Y)$ contains an irreducible component having the expected codimension $h^1(\operatorname{End} E)$.*

Theorem 1.4 is Theorem 3.17 in the main text.

Going further, it would be valuable to know whether all the components of $M(E \otimes L)$ are of the expected dimension or, even better, if $M(E \otimes L)$ is irreducible. The results of [10, § 2] imply irreducibility for $Y = \mathbf{P}^1$ and some vector bundles E. But the question remains open in general.

We also draw the reader's attention to results, similar in spirit to Theorem 1.3, proved by Beauville, Narasimhan, and Ramanan [2]. The basic problem in their line of inquiry is to study not the pushforward of $\mathcal{O}_X$ itself but the pushforwards of general line bundles on X.

The attempt at extending Theorem 1.1 to higher dimensional varieties Y presents interesting new challenges. We discuss them through some examples in § 4. As it stands, the analogue of Theorem 1.1 for higher dimensional varieties Y is false. We end the paper by posing modifications for which we are unable to find counterexamples.

1.1. Strategy of proof.

The proof of Theorem 1.1 proceeds by degeneration. To help the reader, we first outline our approach to a weaker version of Theorem 1.1. In the weaker version, we consider not the vector bundle E itself, but its projectivization $\mathbf{P}E$, which we call the *Tschirnhausen scroll*. A branched cover with Gorenstein fibers $\phi \colon X \to Y$ with Tschirnhausen bundle E factors through a *relative canonical embedding* $\iota \colon X \hookrightarrow \mathbf{P}E$ by the main theorem in [6].

Theorem 1.5. *Let E be any vector bundle on a smooth projective curve Y. Then the scroll $\mathbf{P}E$ is the Tschirnhausen scroll of a finite cover $\phi \colon X \to Y$ with X smooth.*

The following steps outline a proof of Theorem 1.5 which parallels the proof of the stronger Theorem 1.1. We omit the details, since they are subsumed by the results in the paper.

(1) First consider the case
$$E = L_1 \oplus \cdots \oplus L_{d-1},$$
where the L_i are line bundles on Y whose degrees satisfy
$$\deg L_i \ll \deg L_{i+1}.$$
For such E, we construct a nodal cover $\psi \colon X \to Y$ such that $\mathbf{P}E_\psi = \mathbf{P}E$. For example, we may take X to be a nodal union of d copies of Y, each mapping isomorphically to Y under ψ, where the ith copy meets the $(i+1)$th copy along nodes lying in the linear series $|L_i|$.

(2) Consider $X \subset \mathbf{P}E$, where X is the nodal curve constructed above. We now attempt to find a smoothing of X in $\mathbf{P}E$. However, the normal bundle $N_{X/\mathbf{P}E}$ may be quite negative. Fixing this negativity is the most crucial step.

(3) To fix the negativity of $N_{X/\mathbf{P}E}$, we attach several rational normal curves to X as follows. Given a general point $y \in Y$, the d points $\psi^{-1}(y) \subset \mathbf{P}E_y \simeq \mathbf{P}^{d-2}$ are in linear general position, and therefore they lie on many smooth rational normal curves $R_y \subset \mathbf{P}E_y$. Choose a large subset $S \subset Y$, and attach general rational normal curves R_y for each $y \in S$ to X, obtaining a new nodal curve $Z \subset \mathbf{P}E$.
(4) The key technical step is showing that the new normal bundle $N_{Z/\mathbf{P}E}$ is sufficiently positive. Using this positivity, we get that Z is the flat limit of a family of smooth, relatively-canonically embedded curves $X_t \subset \mathbf{P}E$. The generic cover $\phi : X_t \to Y$ in this family satisfies $E_\phi \cong L_1 \oplus \cdots \oplus L_{d-1}$.
(5) We tackle the case of an arbitrary bundle E as follows.
   (a) We note that every vector bundle E degenerates *isotrivially* to a bundle of the form $E_0 = L_1 \oplus \cdots \oplus L_{d-1}$ treated in the previous steps.
   (b) We take a cover $X_0 \to Y$ with Tschirnhausen bundle E_0 constructed above. Using the abundant positivity of $N_{X_0/\mathbf{P}E_0}$, we show that $X_0 \subset \mathbf{P}E_0$ deforms to $X \subset \mathbf{P}E$. The cover $\phi : X \to Y$ satisfies $E_\phi \cong E$.

We need to refine the strategy above to handle the vector bundle E itself, and not just its projectivization. Therefore, we work with the *canonical affine embedding* of X in the total space of E. The proof of Theorem 1.1 involves carrying out the steps outlined above for the embedding $X \subset E$ relative to the divisor of hyperplanes at infinity in a projective completion of E.

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1.3. **Conventions.** We work over an algebraically closed field k. All schemes are of finite type over k. Unless specified otherwise, a point is a k-point. The projectivization $\mathbf{P}V$ of a vector bundle V refers to the space of 1-dimensional *quotients* of V. We identify vector bundles with their sheaves of sections. An injection is understood as an injection of sheaves.

2. Vector bundles, their inflations, and degenerations

This section contains some elementary results on vector bundles on curves. Throughout, Y is a smooth, projective, connected curve over k, an algebraically closed field of arbitrary characteristic.

2.1. **Inflations.** Let E be a vector bundle on Y. A *degree n inflation* of E is a vector bundle $\widetilde{E}$ along with an injective map of sheaves $E \to \widetilde{E}$ whose cokernel is finite of length n. If the cokernel is supported on a subscheme $S \subset Y$, then we say that $E \to \widetilde{E}$ is an inflation of E at S.

Let $E \to \widetilde{E}$ be an inflation of degree n. Then the dual bundle $\widetilde{E}^\vee$ is a sub-sheaf of $E^\vee$ and the quotient is finite of length n. Thus, a degree n inflation of E is equivalent to a sub-sheaf of $E^\vee$ of co-length n, which in turn is equivalent to a quotient of $E^\vee$ of length n. Therefore, we can identify the set of degree n inflations of E with the points of the quot scheme $\operatorname{Quot}(E^\vee, n)$. It is easy to see that $\operatorname{Quot}(E^\vee, n)$ is smooth and connected, hence irreducible. Therefore, it makes sense to talk about a "general" degree n inflation of E.

We wish to study the effect of an inflation on cohomology.

Proposition 2.1. *Let $E \to \widetilde{E}$ be an inflation. Then $h^1(Y, \widetilde{E}) \le h^1(Y, E)$. In particular, if $H^1(Y, E) = 0$, then $H^1(Y, \widetilde{E}) = 0$.*

Proof. Apply the long exact sequence on cohomology to $0 \to E \to \widetilde{E} \to \widetilde{E}/E \to 0$, and use that $\widetilde{E}/E$ has zero-dimensional support. □

Fix a point $y \in Y$. Denote the fiber at y by the subscript y. Consider an inflation $E \to \widetilde{E}$ of degree d whose cokernel is supported (scheme-theoretically) at y. We have an exact sequence

$$0 \to \widetilde{E}^\vee \to E^\vee \to k^d \to 0,$$

where the cokernel is supported at y. Thus, the inflation $E \to \widetilde{E}$ is determined by the surjection

$$E_y^\vee \to k^d. \tag{2.1}$$

We call (2.1) the *defining quotient* of the inflation $E \to \widetilde{E}$.

Let $V \subset E^\vee \otimes \Omega_Y|_y$ be the image of the evaluation map

$$H^0\left(E^\vee \otimes \Omega_Y\right) \to E^\vee \otimes \Omega_Y|_y.$$

Let $q\colon E_y^\vee \to k^d$ be a surjection and denote by $E \to \widetilde{E}_q$ the degree d inflation of E at y corresponding to q.

Proposition 2.2. *With the notation above, let $q_V : V \to k^d \otimes \Omega_Y|_y$ be the restriction of $q \otimes \mathrm{id}$ to V. Then we have*

$$h^0(Y, \widetilde{E}_q) = h^0(Y, E) + d - \operatorname{rk} q_V, \text{ and}$$
$$h^1(Y, \widetilde{E}_q) = h^1(Y, E) - \operatorname{rk} q_V.$$

Proof. We have the exact sequence $0 \to \widetilde{E}_q^\vee \to E^\vee \xrightarrow{q} k^d \to 0$, where the cokernel is supported at y. Tensoring by Ω_Y, taking the long exact sequence in cohomology, and using Serre duality yields the proposition. □

Proposition 2.3. *Suppose E is such that $h^1(Y, E) \neq 0$. Then, for a general degree 1 inflation $E \to \widetilde{E}$, we have*

$$h^1(Y, \widetilde{E}) = h^1(Y, E) - 1 \text{ and } h^0(Y, \widetilde{E}) = h^0(Y, E).$$

Proof. If $h^1(E) = h^0(E^\vee \otimes \Omega_Y) \neq 0$, the space $V \subset E^\vee \otimes \Omega_Y|_y$ defined above is non-zero if $y \in Y$ is general. Then, for a general choice of $q\colon E_q^\vee \to k$, we have $\operatorname{rk} q_V = 1$. The statement now follows from Proposition 2.2. □

We will need a slight strengthening of Proposition 2.3. Suppose $y \in Y$ is such that the image V of the evaluation map

$$H^0(E^\vee \otimes \Omega_Y) \to E^\vee \otimes \Omega_Y|_y$$

is non-zero. Suppose we have a set S of surjections $E_y^\vee \to k^d$. A surjection $q\colon E_y^\vee \to k^d$ gives a $(d-1)$ dimensional linear subspace $\Lambda_q \subset \mathbf{P}E_y^\vee$.

Proposition 2.4. *Suppose the linear span of $\bigcup_{q \in S} \Lambda_q$ is the entire projective space $\mathbf{P}E_y^\vee$. Then for some $q \in S$ we have*

$$h^1(Y, \widetilde{E}_q) \le h^1(Y, E) - 1$$

Proof. By the spanning assumption, we must have $\operatorname{rk} q_V \ge 1$ for some $q \in S$. Then the statement follows from Proposition 2.2. □

Repeated applications of Proposition 2.3 yield the following important consequence.

Corollary 2.5. *Let $n \ge h^1(Y, E)$ be a non-negative integer. Then a general degree n inflation $E \to \widetilde{E}$ satisfies $H^1(Y, \widetilde{E}) = 0$.*

Proposition 2.1 and Corollary 2.5 together imply the following.

Corollary 2.6. *Let E be a vector bundle on Y of rank r. For large enough n, any vector bundle E' of rank r that contains a general degree n inflation of E satisfies $H^1(Y,E')=0$.*

2.2. **Nodal curves and inflations of the normal bundle.** A common setting for inflations in the paper is the following. Let P be a smooth variety. Let R and X be curves in P that intersect at a point p so that their union Z has a node at p. In particular, R and X are smooth at p. Also assume that R and X are local complete intersections elsewhere. In this situation, we get the exact sequence

$$0 \to I_{Z/P}\big|_R \to I_{R/P}\big|_R \to \Omega_X\big|_p \to 0,$$

where the map $I_{R/P}|_R \to \Omega_X|_p$ is induced by the composite of $d\colon I_{R/P} \to \Omega_P$ and the restriction $\Omega_P \to \Omega_X$. The dual sequence

$$0 \to N_{R/P} \to N_{Z/P}\big|_R \to \mathscr{E}\mathrm{xt}^1_{\mathcal{O}_R}\left(\Omega_X\big|_p, \mathcal{O}_R\right) \to 0, \tag{2.2}$$

exhibits $N_{Z/P}|_R$ as a degree 1 inflation of $N_{R/P}$ at p. The defining quotient of this degree 1 inflation

$$q\colon I_{R/P}\big|_p \to k$$

is given by the composite of $d\colon I_{R/P} \to \Omega_P$ and a map $\Omega_P|_p \to k$ obtained by contracting with a generator of T_pX. In particular, if the image of T_pX in $N_{R/P}|_p$ is a general one-dimensional subspace, then $N_{Z/P}|_R$ is a general degree 1 inflation of $N_{R/P}$ at p.

Observe that we have natural identifications

$$\begin{aligned}\mathscr{E}\mathrm{xt}^1_{\mathcal{O}_R}\left(\Omega_X\big|_p, \mathcal{O}_R\right) &= \mathscr{E}\mathrm{xt}^1_{\mathcal{O}_R}\left(\mathcal{O}_p, \mathcal{O}_R\right) \otimes N_{p/X} \\ &= N_{p/R} \otimes N_{p/X}.\end{aligned}$$

Using this identification, we can write the sequence (2.2) and its analogue on X together as

$$\begin{tikzcd} 0 \arrow[r] & N_{R/P} \arrow[r] & N_{Z/P}|_R \arrow[r, "a"] & N_{p/R} \otimes N_{p/X} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N_{X/P} \arrow[r] & N_{Z/P}|_X \arrow[r, "b"] & N_{p/R} \otimes N_{p/X} \arrow[r] & 0. \end{tikzcd} \tag{2.3}$$

The two maps a and b are compatible in the sense that they are induced by a common map

$$N_{Z/P} \to N_{p/R} \otimes N_{p/X}.$$

Finally, note that the discussion above extends naturally to the case of two smooth curves attached nodally at a finite set of points instead of a single point.

2.3. **Isotrivial degenerations.** We say that a bundle E *isotrivially degenerates* to a bundle E_0 if there exists a pointed smooth curve $(\Delta, 0)$ and a bundle $\mathcal{E}$ on $Y \times \Delta$ such that $\mathcal{E}_{Y\times\{0\}} \cong E_0$ and $\mathcal{E}\big|_{Y\times\{t\}} \cong E$ for every $t \in \Delta \setminus \{0\}$.

Proposition 2.7. *Let E a vector bundle on Y, and let N be a non-negative integer. Then E isotrivially degenerates to a vector bundle E_0 of the form*

$$E_0 = L_1 \oplus \cdots \oplus L_r,$$

where the L_i are line bundles and $\deg L_i + N \le \deg L_{i+1}$ for all $i = 1, \ldots, r-1$.

For the proof of Proposition 2.7, we need a lemma.

Lemma 2.8. *There exists a filtration*

$$E = F_0 \supset F_1 \supset \cdots \supset F_{r-1} \supset F_r = 0,$$

satisfying the following properties.

(1) *For every $i \in \{0, \dots, r-1\}$, the sub-quotient F_i/F_{i+1} is a line bundle.*
(2) *Set $L_i = F_i/F_{i+1}$ for $i \in \{1, \dots, r-1\}$ and $L_r = F_0/F_1$. For every $i \in \{1, \dots, r-1\}$, we have*

$$\deg L_i + N \leq \deg L_{i+1}.$$

Proof. The statement is vacuous for $r = 0$ and 1. So assume $r \geq 2$. Note that if $F_\bullet$ is a filtration of E satisfying the two conditions, and if L is a line bundle, then $F_\bullet \otimes L$ is such a filtration of $E \otimes L$. Therefore, by twisting by a line bundle of large degree if necessary, we may assume that $\deg E \geq 0$.

Let us construct the filtration from right to left. Let $L_{r-1} \subset E$ be a line bundle with $\deg L_{r-1} \leq -N$ and with a locally free quotient. Set $F_{r-1} = L_{r-1}$. Next, let $L_{r-2} \subset E/F_{r-1}$ be a line bundle with $\deg L_{r-2} \leq \deg L_{r-1} - N$ and with a locally free quotient. Let $F_{r-2} \subset E$ be the preimage of L_{r-2}. Continue in this way. More precisely, suppose that we have constructed

$$F_j \supset F_{j+1} \supset \cdots \supset F_{r-1} \supset F_r = 0$$

such that $L_i = F_i/F_{i+1}$ satisfy

$$\deg L_i \leq \deg L_{i+1} - N,$$

and suppose $j \geq 2$. Then let $L_{j-1} \subset E/F_j$ be a line bundle with $\deg L_{j-1} \leq \deg L_j - N$ with a locally free quotient. Let $F_{j-1} \subset E$ be the preimage of L_{j-1}. Finally, set $F_0 = E$.

Condition 1 is true by design. Condition 2 is true by design for $i \in \{1, \dots, r-2\}$. For $i = r-1$, note that $\deg L_{r-1} \leq -N$ by construction. On the other hand, we must have $\deg L_r \geq 0$. Indeed, we have $\deg E \geq 0$ but every sub-quotient of $F_\bullet$ except F_0/F_1 has negative degree. Therefore, condition 2 holds for $i = r-1$ as well. □

Proof of Proposition 2.7. Let $F_\bullet$ be a filtration of E satisfying the conclusions of Lemma 2.8. It is standard that a coherent sheaf degenerates isotrivially to the associated graded sheaf of its filtration. The construction goes as follows. Consider the $\mathcal{O}_Y[t]$-module

$$\bigoplus_{n \in \mathbf{Z}} t^{-n} F_n,$$

where $F_n = 0$ for $n > r$ and $F_n = E$ for $n < 0$. The corresponding sheaf $\mathcal{E}$ on $Y \times \mathbf{A}^1$ is coherent, $k[t]$-flat, satisfies $\mathcal{E}_{Y \times \{t\}} \cong E$ for $t \neq 0$, and $\mathcal{E}_{Y \times \{0\}} \cong L_1 \oplus \cdots \oplus L_r$. □

2.4. **The canonical affine embedding.** We end the section with a basic construction that relates finite covers and their Tschirnhausen bundles. Let d be a positive integer and assume that $\operatorname{char} k = 0$ or $\operatorname{char} k > d$.

Let X be a curve of arithmetic genus g_X; let $\phi \colon X \to Y$ be a finite flat morphism of degree d; and let E be the associated Tschirnhausen bundle. Then we have a decomposition $\phi_* \mathcal{O}_X = \mathcal{O}_Y \oplus E^\vee$. The map $E^\vee \to \phi_* \mathcal{O}_X$ induces a surjection $\operatorname{Sym}^* E^\vee \to \phi_* \mathcal{O}_X$. Taking the relative spectrum gives an embedding of X in the total space $\operatorname{Tot}(E)$ of the vector bundle associated to E; we often denote $\operatorname{Tot}(E)$ by E if no confusion is likely. We call $X \subset E$ the *canonical affine embedding*. Note that the degree of E is half of degree of the branch divisor of ϕ, namely

$$\deg E = g_X - 1 - d(g_Y - 1).$$

For all $y \in Y$, the subscheme $X_y \subset E_y$ is in affine general position (not contained in a translate of a strict linear subspace of E_y).

The canonical affine embedding is characterized by the properties above.

Proposition 2.9. *Retain the notation above. Let F be a vector bundle on Y of the same rank and degree as E, and let $\iota\colon X \to F$ be an embedding over Y such that for a general $y \in Y$, the scheme $\iota(X_y) \subset F_y \cong \mathbf{A}^{d-1}$ is in affine general position. Then we have $F \cong E$, and up to an affine linear automorphism of F/Y, the embedding ι is the canonical affine embedding.*

Proof. The restriction map $\operatorname{Sym}^* F^\vee \to \phi_* \mathcal{O}_X = \mathcal{O}_Y \oplus E^\vee$ induces a map

$$\lambda\colon F^\vee \to E^\vee.$$

Since a general fiber $X_y \subset F_y$ is in affine general position, the map λ is an injective map of sheaves. But the source and the target are locally free of the same degree and rank. Therefore, λ is an isomorphism.

Recall that the affine canonical embedding is induced by the map

$$(0, \mathrm{id})\colon E^\vee \to \mathcal{O}_Y \oplus E^\vee = \phi_* \mathcal{O}_X.$$

Suppose ι induces the map

$$(\alpha, \lambda)\colon F^\vee \to \mathcal{O}_Y \oplus E^\vee.$$

Compose ι with the affine linear isomorphism of $T_\alpha\colon \operatorname{Tot}(F) \to \operatorname{Tot}(F)$ over Y defined by the map $\operatorname{Sym}^* F^\vee \to \operatorname{Sym}^* F^\vee$ induced by

$$(-\alpha, \mathrm{id})\colon F^\vee \to \mathcal{O}_Y \oplus F^\vee.$$

Then $T_\alpha \circ \iota\colon X \to F$ is the affine canonical embedding, as desired. □

3. Proof of the main theorem

Let d be a positive integer, and assume that $\operatorname{char} k = 0$ or $\operatorname{char} k > d$. Throughout, Y is a smooth, projective, connected curve over k.

3.1. The split case with singular covers.

As a first step, we treat the case of a suitable direct sum of line bundles and allow the source curve X to be singular.

Proposition 3.1. *Let $E = L_1 \oplus \cdots \oplus L_{d-1}$, where the L_i are line bundles on Y with $\deg L_1 \geq 2g_Y - 1$ and $\deg L_{i+1} \geq \deg L_i + (2g_Y - 1)$ for $i \in \{1, \dots, d-2\}$. There exists a nodal curve X and a finite flat map $\phi\colon X \to Y$ of degree d such that $E_\phi \cong E$.*

The proof is inductive, based on the following "pinching" construction. Let $\psi\colon Z \to Y$ be a finite cover of degree r. Let X be the reducible nodal curve $Z \cup Y$, where Z and Y are attached nodally at distinct points (see Figure 1). More explicitly, let $y_i \in Y$ and $z_i \in Z$ be points such that $\psi(z_i) = y_i$. Define R as the kernel of the map

$$\psi_* \mathcal{O}_Z \oplus \mathcal{O}_Y \to \bigoplus_i k_{y_i},$$

defined around y_i by

$$(f, g) \mapsto f(z_i) - g(y_i).$$

Then $R \subset \psi_* \mathcal{O}_Z \oplus \mathcal{O}_Y$ is an $\mathcal{O}_Y$-subalgebra and $X := \operatorname{Spec}_Y R$ is a nodal curve. Let $\phi\colon X \to Y$ be the natural finite flat map. Set $D = \sum y_i$.

Lemma 3.2. *In the setup above, we have an exact sequence*

$$0 \to E_\psi \to E_\phi \to \mathcal{O}_Y(D) \to 0.$$

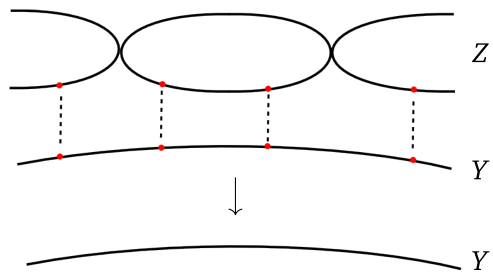

FIGURE 1. The pinching construction, in which pairs of points indicated by dotted lines are identified to form nodes.

Proof. The closed embedding $Z \to X$ gives a surjection

$$\phi_* \mathcal{O}_X \to \psi_* \mathcal{O}_Z$$

whose kernel is $\mathcal{O}_Y(-D)$. Factoring out the $\mathcal{O}_Y$ summand from both sides and taking duals yields the claimed exact sequence. □

Proof of Proposition 3.1. We use induction on d, starting with the base case $d = 1$, which is vacuous.

By the inductive hypothesis, we may assume that there exists a nodal curve Z and a finite cover $\psi \colon Z \to Y$ of degree $(d-1)$ such that $E_\psi \cong L_2 \oplus \cdots \oplus L_{d-1}$. Let $X = Z \cup Y \to Y$ be a cover of degree d obtained from $Z \to Y$ by a pinching construction such that $\mathcal{O}_Y(D) = L_1$. By Lemma 3.2, we get an exact sequence

$$0 \to L_2 \oplus \cdots \oplus L_{d-1} \to E_\phi \to L_1 \to 0. \tag{3.1}$$

But we have $\operatorname{Ext}^1(L_1, L_i) = H^1(L_i \otimes L_1^\vee) = 0$ since $\deg(L_i \otimes L_1^\vee) \geq 2g_Y - 1$. Therefore, the sequence (3.1) is split, and we get $E_\phi = L_1 \oplus \cdots \oplus L_{d-1}$. The induction step is then complete. □

3.2. **Smoothing out.** In this section, we pass from singular covers to smooth covers and from particular vector bundles to their deformations.

Proposition 3.3 (Key)**.** *Let X be a nodal curve and $X \to Y$ a finite flat morphism with Tschirnhausen bundle E. For some line bundle L on Y, the following holds. There exists a smooth curve X' and a finite morphism $X' \to Y$ such that*

(1) *The Tschirnhausen bundle of $X' \to Y$ is $E' = E \otimes L$.*
(2) *We have $H^1(X', N_{X'/E'}) = 0$, where $X' \subset E'$ is the canonical affine embedding.*

Furthermore, there exists an n (depending on $X \to Y$), such that the above holds for any L of degree at least n.

The rest of § 3.2 is devoted to the proof of Proposition 3.3.

Set $\mathbf{P} = \mathbf{P}(E^\vee \oplus \mathcal{O}_Y)$, the space of one-dimensional quotients of $E^\vee \oplus \mathcal{O}_Y$. Let $H \cong \mathbf{P}E^\vee \subset \mathbf{P}$ be the hyperplane at infinity, where the embedding $H \subset \mathbf{P}$ is defined by the projection

$$E^\vee \oplus \mathcal{O}_Y \to E^\vee.$$

The complement of $H \subset \mathbf{P}$ is the total space $\operatorname{Tot}(E)$ of E.

Let $S \subset Y$ be a finite set over which $X \to Y$ is étale. For $y \in S$, the set $X_y \subset \mathbf{P}_y \cong \mathbf{P}^{d-1}$ consists of d points in linear general position, and $H_y \subset \mathbf{P}_y$ is a hyperplane not passing through any of these points.

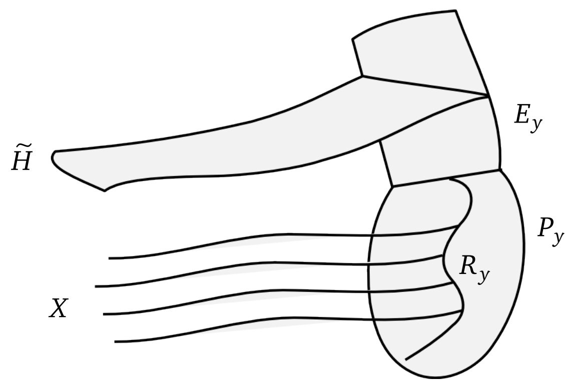

FIGURE 2. Attaching rational normal curves to X to make the normal bundle positive

There exists a smooth rational normal curve $R_y \subset P_y$ which contains X_y and which is transverse to H_y. We can explicitly write down such curves as follows. Pick homogeneous coordinates $[Y_1 : \cdots : Y_d]$ on $P_y \cong \mathbf{P}^{d-1}$ such that the points of X_y are the coordinate points and H_y is the hyperplane $\sum Y_i = 0$. Let $b_1, \dots, b_d \in k^\times$ and $a_1, \dots, a_d \in k$ be arbitrary constants with $a_i \neq a_j$ for $i \neq j$. Let x be a variable and set $F = \prod(x - a_i)$. We can take R_y to be the rational normal curve given parametrically by

$$x \mapsto \left[\frac{b_1 F}{x - a_1} : \cdots : \frac{b_d F}{x - a_d} \right].$$

Note that the parametrization maps the points a_i to the coordinate points. Also, if the b_i are general, then R_y intersects H_y transversely.

Fix $p \in X_y$. Each R_y gives a line $T_{R_y}|_p \subset T_{P_y}|_p$, which we interpret as a point of the corresponding projective space. Using the parametrization, we can check that the set of these points for various R_y is Zariski dense. In other words, a general choice of R_y gives a general tangent line $T_{R_y}|_p \subset T_{P_y}|_p$.

Let $\widetilde{P} \to \mathbf{P}$ be the blow up at $\bigsqcup_{y \in S} H_y$. Denote also by R_y the proper transform of R_y in $\widetilde{P}$. Denote by $\widetilde{H}$ the proper transform of H in $\widetilde{P}$ (see Figure 2).

The fiber of $\widetilde{P} \to Y$ over $y \in S$ consists of two components. One is the exceptional divisor E_y of the blow-up. The second is the proper transform of P_y, which is a copy of P_y; we denote it also by P_y. The two components intersect transversely along a $\mathbf{P}^{d-2}$. Note that $\widetilde{H}$ is disjoint from P_y, and hence also from $R_y \subset P_y$.

Set

$$Z = X \cup_{y \in S} R_y.$$

Our goal is to establish the positivity of $N_{Z/\widetilde{P}}$. First, we set some notation.

$$\begin{aligned}
\nu\colon Z^\nu \to Z &:= \text{The normalization of } Z,\\
\phi : Z^\nu \to Y &:= \text{The composite of } Z^\nu \to Z \text{ and } Z \to Y,\\
X^\nu &= \text{The normalization of } X,\\
\gamma &:= \text{The set of nodes of } X,\\
\Gamma &:= \text{The preimage of } \gamma \text{ in } Z^\nu,\\
\delta_y &:= R_y \cap X,\\
\delta_S &:= \text{The disjoint union of } \delta_y \text{ for } y \in S\\
P_S &:= \text{The disjoint union of } P_y \text{ for } y \in S,\\
R_S &:= \text{The disjoint union of } R_y \text{ for } y \in S.
\end{aligned}$$

Note that Z^ν is the disjoint union of X^ν and R_S. Every point of γ has two preimages in Γ. The singular set of Z is $\gamma \cup \delta_S$.

Let y be a point in S. Denote by $\mathcal{O}(1)$ the line bundle of degree 1 on $R_y \cong \mathbf{P}^1$.

Proposition 3.4. *The restriction of $N_{Z/\widetilde{P}}$ to R_y is isomorphic to $\mathcal{O}(d+1)^{d-2} \oplus \mathcal{O}(1)$, where the sub-bundle $\mathcal{O}(d+1)^{d-2}$ is the image of the natural map*

$$N_{R_y/P_y} \to N_{Z/\widetilde{P}}\big|_{R_y},$$

and the quotient $\mathcal{O}(1)$ is an inflation of $N_{P_y/\widetilde{P}}\big|_{R_y}$ at the points of δ_y.

Proof. For brevity, drop the subscript y from R_y, P_y, and δ_y. First, note that $N_{Z/\widetilde{P}}\big|_R$ is a vector bundle of rank $(d-1)$ and degree $(d-2)(d+1)+1$. The map $N_{R/P} \to N_{Z/\widetilde{P}}\big|_R$ is the composite

$$N_{R/P} \to N_{R/\widetilde{P}} \to N_{Z/\widetilde{P}}\big|_R$$

Using that X is transverse to P, a local computation shows that the injection $N_{R/P} \to N_{Z/\widetilde{P}}\big|_R$ remains an injection when restricted to any point of R. Since $R \subset P \cong \mathbf{P}^{d-1}$ is a rational normal curve, we know that $N_{R/P} \cong \mathcal{O}(d+1)^{d-2}$ (see, for example, [25, II] or [27, Example 4.6.6]). We thus get an exact sequence

$$0 \to \mathcal{O}(d+1)^{d-2} \to N_{Z/\widetilde{P}}\big|_R \to \mathcal{O}(1) \to 0. \tag{3.2}$$

Since $\operatorname{Ext}^1(\mathcal{O}(1), \mathcal{O}(d+1)) = 0$, this sequence splits, and we get the desired isomorphism. The description of the sub and the quotient follows from the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{R/P} & \longrightarrow & N_{R/\widetilde{P}} & \longrightarrow & N_{P/\widetilde{P}}\big|_R & \longrightarrow & 0\\
 & & \downarrow{\simeq} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(d+1)^{d-2} & \longrightarrow & N_{Z/\widetilde{P}}\big|_R & \longrightarrow & \mathcal{O}(1) & \longrightarrow & 0.
\end{array}$$

□

Denote by F the quotient line bundle in the statement of Proposition 3.4, namely

$$F = \operatorname{coker}\left(N_{R_S/\widetilde{P}} \to N_{Z/\widetilde{P}}\big|_{R_S}\right) = N_{P/\widetilde{P}}\big|_{R_S} \otimes \mathcal{O}_{R_S}(\delta_S).$$

Set $D_S = R_S \cap E$, where E is the exceptional divisor of the blow up $\widetilde{P} \to \mathbf{P}$. Then we have

$$\begin{aligned} F &= N_{P/\widetilde{P}}\big|_{R_S} \otimes \mathcal{O}_{R_S}(\delta_S) \\ &= \phi^* N_{S/Y} \otimes \mathcal{O}_{R_S}(\delta_S - D_S). \end{aligned}$$

Combining the diagram in (2.3) and the conclusions of Proposition 3.4 gives the following diagram with exact rows and exact middle column

$$\begin{tikzcd} & N_{R_S/P_S} \arrow[r, hook] \arrow[d, equal] & N_{R_S/\widetilde{P}} \arrow[r, two heads] \arrow[d, hook] & N_{P_S/\widetilde{P}_S}\big|_{R_S} \arrow[d, hook] \\ & N_{R_S/P_S} \arrow[r, hook] & N_{Z/\widetilde{P}}\big|_{R_S} \arrow[r, two heads] \arrow[d, two heads] & F \\ N_{X/\widetilde{P}} \arrow[r, hook] & N_{Z/\widetilde{P}}\big|_X \arrow[r, two heads] & N_{\delta_S/X} \otimes N_{\delta_S/R_S}. & \end{tikzcd} \tag{3.3}$$

In (3.3), the first row is standard, the second row is the definition of F, and the bottom row and the middle column are from (2.3). Let

$$e\colon F \to N_{\delta_S/X} \otimes N_{\delta_S/R_S} \tag{3.4}$$

be the map induced in (3.3). Notice that we have the identifications

$$\begin{aligned} F &= \phi^* N_{S/Y} \otimes \mathcal{O}_{R_S}(\delta_S - D_S), \\ N_{\delta_S/X} &= \phi^* N_{S/Y}\big|_{\delta_S}, \text{ and} \\ N_{\delta_S/R_S} &= \mathcal{O}_{R_S}(\delta_S)\big|_{\delta_S}. \end{aligned}$$

With these identifications, the map

$$e\colon F \to N_{\delta_S/X} \otimes N_{\delta_S/R_S} \tag{3.5}$$

is the composite of the inclusion

$$\phi^* N_{S/Y} \otimes \mathcal{O}_{R_S}(\delta_S - D_S) \to \phi^* N_{S/Y} \otimes \mathcal{O}_{R_S}(\delta_S)$$

and the restriction

$$\phi^* N_{S/Y} \otimes \mathcal{O}_{R_S}(\delta_S) \to \phi^* N_{S/Y}\big|_{\delta_S} \otimes \mathcal{O}_{R_S}(\delta_S)\big|_{\delta_S}.$$

We have the following immediate consequence of the last row in (3.3).

Proposition 3.5. *If the size of S is large, its points are general, and the rational normal curves R_y are general, then we have $H^1(X^\nu, \nu^* N_{Z/\widetilde{P}}|_{X^\nu}) = 0$.*

Proof. Let X_i be a component of X^ν and let $\nu_i : X_i \to Z$ be composite of the inclusion $X_i \subset Z^\nu$ and $\nu\colon Z^\nu \to Z$. Then $\nu^* N_{Z/\widetilde{P}}|_{X_i} = \nu_i^* N_{Z/\widetilde{P}}$. We have the exact sequence

$$0 \to \nu_i^* N_{X/\widetilde{P}} \xrightarrow{\iota} \nu_i^* N_{Z/\widetilde{P}} \to \nu_i^*(N_{\delta_S/X} \otimes N_{\delta_S/R_S}) \to 0.$$

Let p be a point in $\delta_S \cap X_i$ lying over $y \in S$. Since $X_i \to Y$ is étale over y, the natural map

$$T_{P_y}\big|_p \to N_{X/\widetilde{P}}\big|_p$$

is an isomorphism; use it to identify $T_{P_y}|_p$ and $N_{X/\widetilde{P}}|_p$ (and their duals). The defining quotient of the inflation ι at p is the map

$$q\colon \Omega_{P_y}\big|_p \to k,$$

given by contraction with $T_{R_y}|_p \subset T_{P_y}|_p$ (see (2.1)). If $R_y \subset P_y$ is general, then $T_{R_y}|_p \subset T_{P_y}|_p$ is a general line, and hence ι is a general inflation at p. Since the above holds for a point p over every point $y \in S$, we see that $\nu_i^* N_{Z/\widetilde{P}}$ contains a general degree $|S|$ inflation of $\nu_i^* N_{X/\widetilde{P}}$. The proposition now follows from Corollary 2.6. □

Thanks to Proposition 3.4 and Proposition 3.5, the pullbacks of $N_{Z/\widetilde{P}}$ to all the components of Z^ν have no higher cohomology. That is, we have $H^1(\nu^* N_{Z/\widetilde{P}}) = 0$. Our eventual goal is to show that $H^1(N_{Z/\widetilde{P}}) = 0$. For that, we must establish the surjectivity of the map

$$H^0(\nu^* N_{Z/\widetilde{P}}) \to H^0\left(N_{Z/\widetilde{P}}\big|_{\gamma\cup\delta_S}\right)$$

induced by the sequence

$$0 \to N_{Z/\widetilde{P}} \to \nu_* \nu^* N_{Z/\widetilde{P}} \to N_{Z/\widetilde{P}}\big|_{\gamma\cup\delta_S} \to 0.$$

To motivate further constructions, let us describe the key difficulty in showing such a surjection. The nodes γ will not be a big issue, so let us focus on δ_S. By Proposition 3.4, we have the splitting of the normal bundle into "vertical" and "horizontal" components

$$N_{Z/\widetilde{S}}\big|_{R_S} = \mathcal{O}_{R_S}(d+1)^{d-2} \oplus \mathcal{O}_{R_S}(1).$$

The vertical summand is positive enough to have a surjection

$$H^0\left(R_S, \mathcal{O}_{R_S}(d+1)^{d-2}\right) \to H^0\left(\delta_S, \mathcal{O}_{R_S}(d+1)^{d-2}\big|_{\delta_S}\right).$$

It remains to show that we have a surjection

$$H^0\left(X^\nu, \nu^* N_{Z/\widetilde{P}}\big|_{X^\nu}\right) \oplus H^0(R_S, \mathcal{O}_{R_S}(1)) \to H^0\left(\delta_S, \mathcal{O}_{R_S}(1)\big|_{\delta_S}\right).$$

Taking advantage of the first summand (which is clearly necessary) is a global problem. We recast it in terms of a local problem by defining a sheaf K, whose construction depends locally around S. The surjectivity problem will reduce to the vanishing of the higher cohomology of K.

Having explained the motivation, let us construct K. Let $\chi : Z^\chi \to Z$ be the normalization of Z at γ. Abusing notation, also let ϕ denote the map $Z^\chi \to Y$. The bottom part of (3.3) gives the following diagram of sheaves on Z^χ

(3.6)
$$\begin{tikzcd} & & F \arrow[d, "e"] \\ \chi^* N_{X/\widetilde{P}} \arrow[r, hook] & \chi^*\left(N_{Z/\widetilde{P}}\big|_X\right) \arrow[r, two heads] & N_{\delta_S/X} \otimes N_{\delta_S/R_S}. \end{tikzcd}$$

Twist by $\mathcal{O}_{Z^\chi}(-\Gamma)$, which is trivial on F and δ_S, and apply ϕ_* to get

(3.7)
$$\begin{tikzcd} & & \phi_* F \arrow[d, "\phi_* e"] \\ \phi_*\left(\chi^* N_{X/\widetilde{P}}(-\Gamma)\right) \arrow[r, hook] & \phi_*\left(\chi^*\left(N_{Z/\widetilde{P}}\big|_X\right)(-\Gamma)\right) \arrow[r, two heads] & \phi_*\left(N_{\delta_S/X} \otimes N_{\delta_S/R_S}\right). \end{tikzcd}$$

The two sheaves $\chi^* N_{X/\widetilde{P}}(-\Gamma)$ and $\chi^* N_{Z/\widetilde{P}}|_X(-\Gamma)$ are supported on $X^\nu \subset Z^\chi$. On X^ν, the map ϕ restricts to a finite map $X^\nu \to Y$; hence the row remains exact after applying ϕ_*. The sheaf F is supported on $R_S \subset Z^\chi$. On R_S, the map ϕ restricts to a contraction $R_S \to S \subset Y$. As a result, although e is surjective, $\phi_* e$ is not. Since $F \cong \mathcal{O}_{R_S}(1)$ and δ_S consists of d points on each rational curve in R_S, the map $\phi_* e$ is injective.

Let K be the bundle on Y defined by

$$K = \ker\left(\phi_*\left(\chi^*\left(N_{Z/\widetilde{P}}\big|_X\right)(-\Gamma)\right) \to \operatorname{coker}\phi_* e\right). \tag{3.8}$$

The definition of K places it in two important exact sequences. First, we have

$$0 \to K \to \phi_*\chi^*\left(N_{Z/\widetilde{P}}\big|_X\right) \to \phi_*\left(\chi^* N_{Z/\widetilde{P}}\big|_\Gamma\right) \bigoplus \operatorname{coker}\phi_* e \to 0. \tag{3.9}$$

Second, setting

$$M = \chi^* N_{X/\widetilde{P}}(-\Gamma),$$

we have

$$0 \to \phi_* M \to K \to \phi_* F \to 0. \tag{3.10}$$

Note that $\phi_* F$ is supported on S with stalks isomorphic to k^2. Therefore, K is a degree $2|S|$ inflation of $\phi_* M$.

Proposition 3.6. *If the size of S is large, its points are general, and the rational normal curves R_y are general, then we have $H^1(Y, K) = 0$.*

For the proof of Proposition 3.6, we need a non-degeneracy lemma. Let $P = \mathbf{P}^{d-1}$. Let $\delta \subset P$ be a set of d distinct points whose linear span is P, and let $H \subset P$ be a hyperplane disjoint from δ. Let $R \subset P$ be a rational normal curve containing δ and set $D = H \cap R$. Consider the map

$$a \colon H^0\left(\Omega_P\big|_\delta\right) \to H^0(\mathcal{O}_R(\delta - D))^\vee \tag{3.11}$$

obtained by composing the following three maps. First, take the restriction map

$$H^0\left(\Omega_P\big|_\delta\right) \to H^0\left(\Omega_R\big|_\delta\right).$$

Second, take the map

$$H^0(\Omega_R|_\delta) \to H^0(\mathcal{O}_R(\delta))^\vee$$

induced by the natural contraction $\Omega_R|_\delta \otimes \mathcal{O}_R(\delta)|_\delta \to \mathcal{O}_\delta \xrightarrow{\mathrm{tr}} k$ using that $\mathcal{O}_R(\delta)|_\delta = T_R|_\delta$. And third, take the dual of the inclusion

$$H^0(\mathcal{O}_R(\delta))^\vee \to H^0(\mathcal{O}_R(\delta - D))^\vee.$$

It is easy to check that the map a is a surjection. As a result, it gives a line $\Lambda(R) \subset \mathbf{P}H^0(\Omega_P|_\delta)$. (Recall that our projectivizations parametrize quotients.)

Lemma 3.7. *The linear span of the union of the lines $\Lambda(R)$ for all possible choices of R is the entire projective space $\mathbf{P}H^0(\Omega_P|_\delta)$.*

Proof. The proof is by explicit calculation. As done in the beginning of § 3.2, pick homogeneous coordinates $[Y_1 : \cdots : Y_d]$ on $\mathbf{P}^{d-1}$ such that $\delta = \{\delta_1, \ldots, \delta_d\}$ is the set of coordinate points—that is

$$\delta_i = [0 : \cdots : 0 : 1 : 0 : \cdots : 0] \quad \text{(1 in } i\text{th place)}$$

—and such that the hyperplane H is defined by

$$H = \{Y_1 + \cdots + Y_d = 0\}.$$

Let $b_1, \ldots, b_d \in k^\times$ and $a_1, \ldots, a_d \in k$ be arbitrary constants with $a_i \neq a_j$ for $i \neq j$. Consider the rational normal curve $R \subset \mathbf{P}^{d-1}$ given parametrically by

$$\gamma \colon x \mapsto \left[\frac{b_1 F}{x - a_1} : \cdots : \frac{b_d F}{x - a_d}\right],$$

where $F = (x - a_1)\cdots(x - a_d)$.

Let $i, j \in \{1, \dots, d\}$ with $i \neq j$. Define $\omega(i,j) \in H^0(\Omega_P|_\delta)$ by

$$\omega(i,j)\big|_{\delta_\ell} = \begin{cases} d(Y_i/Y_j) & \text{if } \ell = j, \\ 0 & \text{if } \ell \neq j. \end{cases}$$

See that $\{\omega(i,j)\}$ is a basis of $H^0(\Omega_P|_\delta)$. Set $G = \sum b_i F/(x - a_i)$; note that this is the pullback of the defining equation of H to R. Then we have

$$H^0(\mathcal{O}_R(\delta - D)) = \left\{ \frac{(ux+v)G}{F} \mid u, v \in k \right\}.$$

The map in (3.11), viewed as

$$H^0\left(\Omega_P|_\delta\right) \otimes H^0(\mathcal{O}_R(\delta - D)) \to k,$$

takes the following explicit form

$$\omega(i,j) \otimes \frac{(ux+v)G}{F} \mapsto \frac{b_i(ua_j + v)}{a_j - a_i}.$$

In the $d(d-1)$ homogeneous coordinates on $\mathbf{P}H^0(\Omega_P|_\delta)$ corresponding to the basis $\{\omega(i,j)\}$, the line $\Lambda(R)$ is given by

$$\Lambda(R) = \left\{ \left[\frac{b_i(ua_j+v)}{a_j - a_i} \right]_{1 \le i \neq j \le d} \mid [u : v] \in \mathbf{P}^1 \right\}.$$

It is easy to check that the $d(d-1)$ rational functions $\frac{b_i(ua_j+v)}{a_j-a_i}$ in the $(2d+2)$ variables $a_1, \dots, a_d$, $b_1, \dots, b_d$, u, and v are k-linearly independent. Therefore, the linear span of $\bigcup_R \Lambda(R)$ is the entire projective space. $\square$

Proof of Proposition 3.6. We analyze the inflation (3.10) and its dual locally near a point $y \in S$. Let $U \subset Y$ be an open set containing y and set $V := \phi^{-1}U \cap X$. Assume that U contains no other point of S and $V \to U$ is étale.

Let $T_{\widetilde{P}/Y} := \ker(T_{\widetilde{P}} \to T_Y)$ be the vertical tangent bundle of $\widetilde{P}/Y$. Since $V \to U$ is étale, the projection

$$T_{\widetilde{P}/Y}\big|_V \to N_{X/\widetilde{P}}\big|_V$$

is an isomorphism. Since Γ is away from V, we may ignore the twist by Γ for computations on V or U. As a result, on U we can write the sequence (3.10) as

$$0 \to \phi_*\left(T_{\widetilde{P}/Y}\big|_V\right) \to K_U \xrightarrow{b} (\phi_* F)_U \to 0. \tag{3.12}$$

Take the $\mathcal{O}_U$-dual of (3.12), using that the dualizing sheaf of $V \to U$ is trivial, and that

$$F_{R_y} = \phi^* N_{y/Y} \otimes \mathcal{O}_{R_y}(\delta_y - D_y).$$

The dual is

$$0 \to K_U^\vee \to \phi_*\left(\Omega_{\widetilde{P}/Y}\big|_V\right) \xrightarrow{a} H^0(\mathcal{O}_{R_y}(\delta_y - D_y))^\vee \to 0, \tag{3.13}$$

where the last sheaf is supported at y. The map a is determined by its restriction a_y on the fibers at y. To understand the map a, recall that the map b in (3.12) originates in the diagram in (3.7). We have observed in (3.5) that the map $e\colon F \to N_{\delta_S/X} \otimes N_{\delta_S/R_S}$ in the diagram (3.7) is given by the composite of the inclusion $\mathcal{O}_{R_S}(\delta_S - D_S) \to \mathcal{O}_{R_S}(\delta_S)$ and the restriction $\mathcal{O}_{R_S}(\delta_S) \to N_{\delta_S/R_S}$. Using this and by chasing the duals, we see that the map

$$a_y : H^0\left(\Omega_{P_y}\big|_{\delta_y}\right) \to H^0(\mathcal{O}_{R_y}(\delta_y - D_y))^\vee$$

is induced by the inclusion $\mathcal{O}_{R_y}(\delta_y - D_y) \to \mathcal{O}_{R_y}(\delta_y)$, the restriction to δ_y, and the contraction with cotangent vectors, as described in (3.11).

By Lemma 3.7, the lines in $\mathbf{P}H^0\left(\Omega_{P_y}\big|_{\delta_y}\right)$ determined by the maps a_y for various R_y span the entire projective space. If $y \in Y$ is general, then by Proposition 2.4 the inflation of $\phi_* M$ at y given by a_y for a general R_y leads to a non-zero decrease in h^1. As a result, if $|S| \geq h^1(M)$, and $S \subset Y$ and R_y are general, then $H^1(K) = 0$. □

We now have the tools to prove that $H^1(Z, N_{Z/\widetilde{P}}) = 0$.

Proposition 3.8. *If the size of S is large, its points are general, and the rational normal curves R_y are general, then we have $H^1(Z, N_{Z/\widetilde{P}}) = 0$.*

Proof. We have the exact sequence

$$0 \to N_{Z/\widetilde{P}} \to \nu_* \nu^* N_{Z/\widetilde{P}} \to N_{Z/\widetilde{P}}\big|_{\gamma \cup \delta_S} \to 0.$$

The long exact sequence on cohomology gives

$$H^0\left(\nu^* N_{Z/\widetilde{P}}\right) \to H^0\left(N_{Z/\widetilde{P}}\big|_{\gamma\cup\delta_S}\right) \to H^1\left(N_{Z/\widetilde{P}}\right) \to H^1\left(\nu^* N_{Z/\widetilde{P}}\right) \to 0.$$

By Proposition 3.4, we have $H^1(\nu^* N_{Z/\widetilde{P}}|_{R_y}) = 0$. By Proposition 3.5, we have $H^1(\nu^* N_{Z/\widetilde{P}}|_{X^\nu}) = 0$. By combining the two, we get $H^1(\nu^* N_{Z/\widetilde{P}}) = 0$.

We now show that the map

$$\nu_* \nu^* N_{Z/\widetilde{P}} \to N_{Z/\widetilde{P}}\big|_{\gamma\cup\delta_S} \tag{3.14}$$

is surjective on global sections. Note that we have a decomposition

$$\nu^* N_{Z/\widetilde{P}} = \nu^*\left(N_{Z/\widetilde{P}}\big|_X\right) \bigoplus \nu^*\left(N_{Z/\widetilde{P}}\big|_{R_S}\right).$$

Furthermore, by Proposition 3.4, $\nu^* N_{Z/\widetilde{P}}|_{R_S}$ is decomposed by the split exact sequence

$$0 \to N_{R_S/P_S} \to \nu^* N_{Z/\widetilde{P}}\big|_{R_S} \to F \to 0.$$

Consider the diagram of sheaves on Z

$$\begin{tikzcd}
N_{R_S/P_S} \arrow[r, hook] \arrow[d, "q"] & \nu_*\nu^*\left(N_{Z/\widetilde{P}}\big|_X\right) \bigoplus \nu_*\nu^*\left(N_{Z/\widetilde{P}}\big|_{R_S}\right) \arrow[r, two heads] \arrow[d, "r"] & \nu_*\nu^*\left(N_{Z/\widetilde{P}}\big|_X\right) \bigoplus F \arrow[d, "s"] \\
N_{R_S/P_S}\big|_{\delta_S} \arrow[r, hook] & N_{Z/\widetilde{P}}\big|_\gamma \bigoplus N_{Z/\widetilde{P}}\big|_{\delta_S} \arrow[r, two heads] & N_{Z/\widetilde{P}}\big|_\gamma \bigoplus N_{\delta_S/X} \otimes N_{\delta_S/R_S}.
\end{tikzcd}$$

By Proposition 3.4, the bundle N_{R_S/P_S} is positive enough for q to be surjective on global sections. Therefore, to prove that r is surjective on global sections, it suffices to prove the same for s. Recall our notation e for the map $F \to N_{\delta_S/X} \otimes N_{\delta_S/R_S}$. We have the following diagram of sheaves on Y

$$\begin{tikzcd}
\phi_* F \arrow[r, hook] \arrow[d, equal] & \phi_*\left(\nu^*\left(N_{Z/\widetilde{P}}\big|_X\right) \bigoplus F\right) \arrow[r, two heads] \arrow[d] & \phi_*\left(\nu^*\left(N_{Z/\widetilde{P}}\big|_X\right)\right) \arrow[d] \\
\phi_* F \arrow[r, hook] & \phi_*\left(N_{Z/\widetilde{P}}\big|_\gamma \bigoplus N_{\delta_S/X} \otimes N_{\delta_S/R_S}\right) \arrow[r, two heads] & \phi_*\left(N_{Z/\widetilde{P}}\big|_\gamma\right) \bigoplus \operatorname{coker} \phi_* e,
\end{tikzcd}$$

where we have abused notation somewhat to denote both maps $Z^\nu \to Y$ and $Z \to Y$ by the same letter ϕ. From the diagram, we see that it suffices to prove that

$$\phi_*\left(\nu^*\left(N_{Z/\widetilde{P}}\big|_X\right)\right) \to \phi_*\left(N_{Z/\widetilde{P}}\big|_\gamma\right) \oplus \operatorname{coker}\phi_* e \tag{3.15}$$

is surjective on global sections. As a consequence (3.9) of the definition of K, we have the exact sequence

$$0 \to K \to \phi_*\nu^*\left(N_{Z/\widetilde{P}}\big|_X\right) \to \phi_*\left(\nu^* N_{Z/\widetilde{P}}\big|_\Gamma\right) \oplus \operatorname{coker}\phi_* e \to 0.$$

We have replaced χ^* in (3.9) by ν^* above, but this is harmless as the pullbacks are in any case supported on X^ν. By Proposition 3.6, we may assume that $H^1(K) = 0$. Therefore, we get that the map

$$\phi_*\nu^*\left(N_{Z/\widetilde{P}}\big|_X\right) \to \phi_*\left(\nu^* N_{Z/\widetilde{P}}\big|_\Gamma\right) \oplus \operatorname{coker}\phi_* e \tag{3.16}$$

is surjective on global sections. Since

$$H^0\left(\nu^* N_{Z/\widetilde{P}}\big|_\Gamma\right) \to H^0\left(N_{Z/\widetilde{P}}\big|_\gamma\right)$$

is clearly surjective, we conclude that (3.15) is surjective on global sections. The proof of Proposition 3.8 is now complete. □

Remark 3.9. From the surjection (3.16), we observe that the map

$$\nu_*\nu^* N_{Z/\widetilde{P}} \to \nu_*\left(\nu^* N_{Z/\widetilde{P}}\big|_\Gamma\right) \oplus N_{Z/\widetilde{P}}\big|_{\delta_S},$$

is surjective on global sections. This is stronger than what was required for Proposition 3.8; it will be useful later.

The following proposition considers the effect of enlarging S. Let $S^+ = S \cup \{y\}$, where $y \in Y \setminus S$ is any point over which $X \to Y$ is étale. Denote by the superscript + the analogues for S^+ of all the constructions done for S.

Proposition 3.10. *Suppose we have $H^1(Z, N_{Z/\widetilde{P}}) = 0$. Then we also have $H^1(Z^+, N_{Z^+/\widetilde{P}^+}) = 0$.*

Proof. By construction, we have $Z^+ = Z \cup R_y$. Let $\mu\colon Z \sqcup R_y \to Z^+$ be the partial normalization. We have the short exact sequence

$$0 \to N_{Z^+/\widetilde{P}^+} \to \mu_*\mu^* N_{Z^+/\widetilde{P}^+} \to N_{Z^+/\widetilde{P}^+}|_{\delta_y} \to 0.$$

The injection $N_{Z/\widetilde{P}} \to \mu^* N_{Z^+/\widetilde{P}^+}|_Z$ and the hypothesis $H^1(N_{Z/\widetilde{P}}) = 0$ implies that

$$H^1\left(\mu^* N_{Z^+/\widetilde{P}^+}\big|_Z\right) = 0.$$

Proposition 3.4 implies that $H^1(N_{Z^+/\widetilde{P}^+}|_{R_y}) = 0$. By combining the two, we get

$$H^1\left(\mu_*\mu^* N_{Z^+/\widetilde{P}^+}\right) = 0.$$

To finish the proposition, it remains to prove that

$$H^0\left(\mu^* N_{Z^+/\widetilde{P}^+}\right) \to H^0\left(N_{Z^+/\widetilde{P}^+}\big|_{\delta_y}\right)$$

is surjective. Recall that

$$\begin{aligned}\mu^* N_{Z^+/\widetilde{P}^+} &= N_{Z^+/\widetilde{P}^+}\big|_Z \oplus N_{Z^+/\widetilde{P}^+}\big|_{R_y} \\ &= N_{Z^+/\widetilde{P}^+}\big|_Z \oplus N_{R_y/P_y} \oplus F\big|_{R_y}.\end{aligned}$$

We know that $N_{R_y/P_y} \to N_{R_y/P_y}|_{\delta_y}$ is surjective on global sections. Therefore, it suffices to prove that the map

$$N_{Z^+/\widetilde{P}^+}\big|_Z \to N_{\delta_y/X} \otimes N_{\delta_y/R_y} = \operatorname{coker}\left(N_{R_y/P_Y}\big|_{\delta_y} \to N_{Z/\widetilde{P}}\big|_{\delta_y}\right) \tag{3.17}$$

is surjective on global sections. But the exact sequence

$$0 \to N_{Z/\widetilde{P}} \to N_{Z^+/\widetilde{P}^+}\big|_Z \to N_{\delta_y/X} \otimes N_{\delta_y/R_y} \to 0$$

analogous to the bottom row of (3.3) and the vanishing of $H^1(N_{Z/\widetilde{P}})$ imply that (3.17) is indeed surjective on global sections. □

Remark 3.11. In the proof of Proposition 3.10, we showed that

$$N_{Z^+/\widetilde{P}^+}\big|_Z \to N_{\delta_y/X} \otimes N_{\delta_y/R_y}$$

is surjective on global sections. Again, this is stronger than what was required for Proposition 3.10; it will be useful later.

Proposition 3.12. *Suppose the size n of S is large, its points are general, and the rational normal curves R_y are general. Then*

(1) *the Hilbert scheme of subschemes of $\widetilde{P}$ is smooth at $[Z]$;*
(2) *Z is a flat limit of smooth curves in $\widetilde{P}$.*

Furthermore, if n is sufficiently large, then the set S can be chosen so that $\mathcal{O}_Y(S)$ is isomorphic to any prescribed line bundle of degree n on Y.

Proof. Since $H^1(N_{Z/\widetilde{P}}) = 0$, we get that the Hilbert scheme of $\widetilde{P}$ is smooth at $[Z]$, proving (1). As a result, every first order deformation of $Z \subset \widetilde{P}$ extends to a deformation over the germ of a smooth curve. To show that Z is the limit of smooth curves, it suffices to show that for every node $p \in Z$, the natural map $N_{Z/\widetilde{P}} \to \mathscr{E}\mathrm{xt}^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)_p$ is surjective on global sections. Recall that Z has two kinds of nodes: the nodes γ, which are the nodes of X; and the nodes δ_S, which are the nodes introduced because we attached the rational normal curves.

First we deal with the nodes γ. Let $\chi : Z^\chi \to Z$ be the partial normalization at these nodes. Let $I_\gamma \subset \mathcal{O}_Z$ be the ideal sheaf of $\gamma \subset Z$. We have

$$N_{Z/\widetilde{P}} \otimes I_\gamma = \chi_*\left(\chi^* N_{Z/\widetilde{P}}(-\Gamma)\right).$$

Thus, if $\nu \colon Z^\nu \to Z$ is the full normalization, we get the sequence

$$0 \to N_{Z/\widetilde{P}} \otimes I_\gamma \to \nu_*\nu^* N_{Z/\widetilde{P}} \to \nu_*\left(\nu^* N_{Z/\widetilde{P}}\big|_\Gamma\right) \oplus N_{Z/\widetilde{P}}\big|_{\delta_S} \to 0.$$

By the observation in Remark 3.9, we know that

$$\nu_*\nu^* N_{Z/\widetilde{P}} \to \nu_*\left(\nu^* N_{Z/\widetilde{P}}\big|_\Gamma\right) \oplus N_{Z/\widetilde{P}}\big|_{\delta_S}.$$

is surjective on global sections. Therefore, we get that $H^1(N_{Z/\widetilde{P}} \otimes I_\gamma) = 0$. This, in turn, implies that

$$H^0(N_{Z/\widetilde{P}}) \to H^0(N_{Z/\widetilde{P}}|_\gamma)$$

is surjective. By combining with the surjection

$$N_{Z/\widetilde{P}}\big|_\gamma \to \mathscr{E}\mathrm{xt}^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)_\gamma,$$

we conclude that $H^0(N_{Z/\widetilde{P}}) \to H^0(\mathscr{E}\mathrm{xt}^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)_p)$ is surjective for all $p \in \gamma$.

Next, we consider a node $p \in \delta_S$ lying over $y \in S$. We have the equality

$$\mathscr{E}xt^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)_{\delta_y} = N_{\delta_y/R_y} \otimes N_{\delta_y/X}.$$

Set $S^- = S \setminus \{y\}$. Denote by the superscript $-$ the analogous objects for S^-. We may assume that S is big enough to have $H^1(N_{Z^-/\widetilde{P}^-}) = 0$. Let $\mu\colon Z^- \sqcup R_y \to Z$ be the partial normalization at the nodes δ_y. We have the sequence

$$0 \to N_{Z^-/\widetilde{P}^-} \to \mu^* N_{Z/\widetilde{P}}\big|_{Z^-} \to N_{\delta_y/R_y} \otimes N_{\delta_y/X} = \mathscr{E}xt^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)_{\delta_y} \to 0$$

analogous to the bottom row of (3.3). Since $H^1(N_{Z^-/\widetilde{P}^-}) = 0$, the long exact sequence in cohomology implies that

$$\mu^* N_{Z/\widetilde{P}}\big|_{Z^-} \to \mathscr{E}xt^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)_{\delta_y}$$

is surjective on global sections. In particular,

$$\mu^* N_{Z/\widetilde{P}}\big|_{Z^-} \to \mathscr{E}xt^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)_p \tag{3.18}$$

is surjective on global sections.

By Proposition 3.4, the map

$$\mu^* N_{Z/\widetilde{P}}\big|_{R_y} \to N_{Z/\widetilde{P}}\big|_p \tag{3.19}$$

is surjective on global sections.

By combining (3.18) and (3.19), we see that $N_{Z/\widetilde{P}} \to \mathscr{E}xt^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)_p$ is surjective on global sections. We have thus taken care of both types of nodes, proving (2).

It remains to prove the last statement about $\mathcal{O}_Y(S)$. For that, assume that n is large enough so that the conclusions above hold for a generic S of size $n - 2g_Y$. Then we may enlarge S to a set S^+ by adding an appropriate set of $2g_Y$ points so that the same conclusions hold and $\mathcal{O}_Y(S^+)$ is isomorphic to a given line bundle of degree n. $\square$

We now prove the key proposition.

Proof of Proposition 3.3. By Proposition 3.12, there exists a family of smooth curves in $\widetilde{P}$ whose flat limit is Z. Let X' be a general member of such a family. This curve satisfies the following conditions (see Figure 3):

(1) $\deg(X' \cdot E_y) = d - 1$ for all $y \in S$,
(2) $\deg(X' \cdot P_y) = 1$ for all $y \in S$,
(3) $X' \cap \widetilde{H} = \varnothing$,
(4) $g(X') = g(X) + n(d-1)$,
(5) $H^1(N_{X'/\widetilde{P}}) = 0$.

Let $\widetilde{P} \to \mathbf{P}'$ be the blowing down of all the P_y for $y \in S$. Then $\mathbf{P}' \to Y$ is a $\mathbf{P}^{d-1}$ bundle and the map $X' \to \mathbf{P}'$ is an embedding. Similarly, $\widetilde{H} \to \mathbf{P}'$ is also an embedding.

We claim that the complement of $\widetilde{H}$ in $\mathbf{P}'$ is isomorphic to $\operatorname{Tot}(E')$, where $E' = E \otimes \mathcal{O}_Y(S)$.

To see this, let us recall some generalities. Let V be a vector bundle of rank d on Y; set $\mathbf{P} = \mathbf{P}V$; and let $H \subset \mathbf{P}$ be a divisor such that for each $y \in Y$, the fiber H_y is a hyperplane in $\mathbf{P}_y$. In general, the complement $\mathbf{P} \setminus H$ is an affine space bundle over Y. If we have a section $\sigma\colon Y \to \mathbf{P} \setminus H$, then $\mathbf{P} \setminus H \to Y$ is the total space of a vector bundle E. The bundle E can be recovered from σ as

$$E = N_{\sigma(Y)/\mathbf{P}}.$$

Coming back to our situation, let $\sigma\colon Y \to \mathbf{P} = \mathbf{P}(\mathcal{O}_Y \oplus E)$ be a section disjoint from H. Denote by $\sigma'\colon Y \to \mathbf{P}'$ the section obtained from σ by composing with the the blow-up and blow-down rational map $\beta\colon \mathbf{P} \dashrightarrow \mathbf{P}'$ (which is regular in a neighborhood of $\sigma(Y)$). Then $\sigma' \subset \mathbf{P}'$ is disjoint

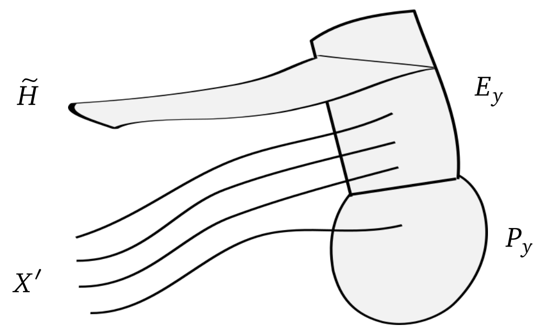

FIGURE 3. A smoothing X' of X after attaching a large number of general rational normal curves

from $\widetilde{H} \subset \mathbf{P}'$. Therefore, $\mathbf{P}' \setminus \widetilde{H}$ is the total space of a vector bundle. To identify this bundle, consider the map

$$N_{\sigma(Y)/\mathbf{P}} \xrightarrow{d\beta} N_{\sigma'(Y)/\mathbf{P}'},$$

which is an isomorphism on $Y \setminus S$ and identically zero when restricted to S. A simple local computation shows that it gives an isomorphism

$$N_{\sigma(Y)/\mathbf{P}} \otimes \mathcal{O}_Y(S) \simeq N_{\sigma'(Y)/\mathbf{P}'}.$$

Therefore, we conclude that the complement of H' in $\mathbf{P}'$ the total space of $E' = E \otimes \mathcal{O}_Y(S)$.

Note that X' and $\widetilde{H}$ remain disjoint in $\mathbf{P}'$, and hence we get an embedding $X' \subset E'$. Proposition 2.9 implies that $X' \subset E'$ is the canonical affine embedding.

Next, note that we have an injection

$$N_{X'/\widetilde{P}} \to N_{X'/E'}$$

with finite quotient, supported on $\bigcup_{y \in S} X' \cap P_y$. Since $H^1(N_{X'/\widetilde{P}}) = 0$, we get $H^1(N_{X'/E'}) = 0$.

Finally, by the last assertion of Proposition 3.12, we may take $\mathcal{O}_Y(S)$ to be any prescribed line bundle of degree n if n is large enough. $\square$

3.3. **The general case.** We now use the results of § 3.1 and § 3.2 to deduce the main theorem. Recall that Y is a connected, projective, and smooth curve over k, an algebraically closed field with $\operatorname{char} k = 0$ or $\operatorname{char} k > d$.

Theorem 3.13. *Let E be a vector bundle on Y of rank $(d-1)$. There exists an n (depending on E) such that for any line bundle L of degree at least n, there exists a smooth curve X and a finite flat morphism $\phi : X \to Y$ of degree d such that $E_\phi \cong E \otimes L$. Furthermore, we have $H^1(X, N_{X/E\otimes L}) = 0$, where $X \subset E \otimes L$ is the canonical affine embedding.*

Proof. Choose an isotrivial degeneration E_0 of E of the form

$$E_0 = L_1 \oplus \cdots \oplus L_{d-1},$$

where the L_i's are line bundles with $\deg L_i + (2g_Y - 1) \le \deg L_{i+1}$. That is, let $(\Delta, 0)$ be a pointed curve and $\mathcal{E}$ a vector bundle on $Y \times \Delta$ such that $\mathcal{E}|_0 = E_0$ and $\mathcal{E}|_t \cong E$ for all $t \in \Delta \setminus \{0\}$. Such a degeneration exists by Proposition 2.7. After replacing $\mathcal{E}$ by $\mathcal{E} \otimes \lambda$ for a line bundle λ on Y of large degree, we may also assume that $\deg L_1 \ge 2g_Y - 1$.

By Proposition 3.1, there exists a nodal curve W and a finite flat morphism $W \to Y$ with Tschirnhausen bundle E_0. By the key proposition (Proposition 3.3), there exists an n such that for any line bundle L of degree at least n, we can find a smooth curve X_0 and a finite map $X_0 \to Y$

with Tschirnhausen bundle $E_0' = E_0 \otimes L$ satisfying $H^1(N_{X_0/E_0'}) = 0$. Set $\mathcal{E}' = \mathcal{E} \otimes L$. Let $\mathcal{H}$ be the component of the relative Hilbert scheme of $\operatorname{Tot}(\mathcal{E}')/\Delta$ containing the point $[X_0 \subset E_0']$. Since $H^1(N_{X_0/E_0'}) = 0$, the map $\mathcal{H} \to \Delta$ is smooth at $[X_0 \subset E_0']$ by [27, Theorem 3.2.12]. In particular, $\mathcal{H} \to \Delta$ is dominant. As a result, there exists a point $[X \subset \mathcal{E}_t'] \in \mathcal{H}$, where X is smooth and $t \in \Delta$ is generic. By the choice of $\mathcal{E}$, we have $\mathcal{E}_t' = E \otimes L$. Since $H^1(N_{X_0/E_0 \otimes L}) = 0$, we can also ensure that $H^1(N_{X/E \otimes L}) = 0$ by semi-continuity. Let $\phi : X \to Y$ be the projection. By Proposition 2.9, we get that $E_\phi \cong E \otimes L$ and $X \subset E \otimes L$ is the canonical affine embedding of ϕ. The proof is now complete. □

Remark 3.14. Theorem 3.13 can be stated in terms of moduli stacks of covers and bundles in the following way. Denote by $\mathcal{H}_d(Y)$ the stack whose S points are finite flat degree d morphisms $\phi : C \to Y \times S$, where $C \to S$ is a smooth curve. Let $\operatorname{Vec}_{d-1}(Y)$ be the stack whose S points are vector bundles of rank $(d-1)$ on $Y \times S$. Both $\mathcal{H}_d(Y)$ and $\operatorname{Vec}_{d-1}(Y)$ are algebraic stacks, locally of finite type, and smooth over k. The rule

$$\tau : \phi \mapsto E_\phi$$

defines a morphism $\tau : H_d(Y) \to \operatorname{Vec}_{d-1}(Y)$. Then Theorem 3.13 says that given $E \in \operatorname{Vec}_{d-1}(Y)$ and given any line bundle L on Y of large enough degree, there exists a point $[\phi : X \to Y]$ of $\mathcal{H}_d(Y)$ such that $\tau(\phi) = E \otimes L$, and furthermore, such that the map τ is smooth at $[\phi]$.

3.4. **Hurwitz spaces and Maroni loci.** We turn to the proof of Theorem 1.3 stated in the introduction. First we establish notation and conventions regarding the various Hurwitz spaces. Throughout § 3.4, take $k = \mathbf{C}$.

Let $\mathcal{H}_{d,g}^{\text{all}}(Y)$ be the stack whose objects over S are S-morphisms $\phi : C \to Y \times S$, where $C \to S$ is a smooth, proper, connected curve of genus g, and ϕ is a finite morphism of degree d. Observe that $\mathcal{H}_{d,g}^{\text{all}}(Y)$ is an open substack of the Kontsevich stack of stable maps $\overline{\mathcal{M}}_g(Y, d[Y])$ constructed, for example, in [11] or in [3]. As a result, $\mathcal{H}_{d,g}^{\text{all}}(Y)$ is a separated Deligne–Mumford stack of finite type over k. Using the deformation theory of maps [27, Example 3.4.14], it follows that $\mathcal{H}_{d,g}^{\text{all}}(Y)$ is smooth and equidimensional of dimension $2b = (2g-2) - d(2g_Y - 2)$. Denote by $\mathcal{H}_{d,g}^{\text{simple}}(Y) \subset \mathcal{H}_{d,g}^{\text{all}}(Y)$ the open substack of simply branched maps, namely the substack whose S-points correspond to maps $\phi : C \to Y \times S$ whose branch divisor $\operatorname{br}\phi \subset Y \times S$ is étale over S (the branch divisor is defined as the vanishing locus of the discriminant [29, Tag 0BVH]). The transformation $\phi \mapsto \operatorname{br}\phi$ gives a morphism

$$\mathcal{H}_{d,g}^{\text{all}}(Y) \to \operatorname{Sym}^{2b} Y$$

with finite fibers. Since the source is equidimensional of the same dimension as the target and the map is quasi-finite, each component of $\mathcal{H}_{d,g}^{\text{all}}(Y)$ maps dominantly on $\operatorname{Sym}^{2b}(Y)$. In particular, $\mathcal{H}_{d,g}^{\text{simple}}(Y)$ is dense in $\mathcal{H}_{d,g}^{\text{all}}(Y)$. By a celebrated theorem of Clebsch [8], if $g_Y = 0$, then $\mathcal{H}_{d,g}^{\text{simple}}(Y)$ is connected (equivalently, irreducible). More generally, by [12, Theorem 9.2], the connected (= irreducible) components of $\mathcal{H}_{d,g}^{\text{all}}(Y)$ are classified by the subgroup $\phi_*\pi_1(C)$ of $\pi_1(Y)$. Recall that ϕ is called *primitive* if $\phi_*\pi_1(C) = \pi_1(Y)$, or equivalently, if ϕ does not factor through an étale covering $\widetilde{Y} \to Y$. Denote by $\mathcal{H}_{d,g}^{\text{primitive}}(Y) \subset \mathcal{H}_{d,g}^{\text{all}}(Y)$ the connected (= irreducible) component whose points correspond to primitive covers.

The connection between primitive and simply branched covers is the following. By [4, Proposition 2.5], if $\phi : C \to Y$ is a simply branched covering, then ϕ is primitive if and only if the monodromy map

$$\pi_1(Y \setminus \operatorname{br}\phi) \to S_d$$

is surjective. Therefore, we can view $\mathcal{H}^{\text{primitive}}_{d,g}(Y)$ as a partial compactification of the stack of simply branched covers of Y with full monodromy group S_d. By convention, $\mathcal{H}_{d,g}(Y)$ (without any superscript) denotes the component $\mathcal{H}^{\text{primitive}}_{d,g}(Y)$ of $\mathcal{H}^{\text{all}}_{d,g}(Y)$.

Being open substacks of the Kontsevich stack, the Hurwitz stacks described above admit quasi-projective coarse moduli spaces, which we denote by the roman equivalent $H_{d,g}$ of $\mathcal{H}_{d,g}$. Denote by $M_{r,k}(Y)$ the moduli space of vector bundles of rank r and degree k on Y. Let $\mathcal{U} \subset \mathcal{H}_{d,g}(Y)$ be the (possibly empty) open substack consisting of points $[\phi] \in \mathcal{H}_{d,g}(Y)$ such that E_ϕ is semi-stable. We have a morphism $\mathcal{U} \to M_{d-1,b}(Y)$ defined functorially as follows. An object $\phi \colon C \to Y \times S$ of $\mathcal{U}$ maps to the unique morphism $S \to M_{d-1,b}(Y)$ induced by the bundle E_ϕ on $Y \times S$. Let $U \subset H_{d,g}(Y)$ be the coarse space of $\mathcal{U}$. By the universal property of coarse spaces, the morphism $\mathcal{U} \to M_{d-1,b}(Y)$ descends to a morphism $U \to M_{d-1,b}(Y)$. If U is non-empty, then we can think of $U \to M_{d-1,b}(Y)$ as a rational map $H_{d,g}(Y) \dashrightarrow M_{d-1,b}(Y)$.

Recall that Y is a smooth, projective, connected curve over $\mathbf{C}$.

Theorem 3.15. *Let $g_Y \geq 2$. If g is sufficiently large (depending on Y and d), then the Tschirnhausen bundle associated to a general point of $H_{d,g}(Y)$ is stable. Moreover, the rational map*

$$H_{d,g}(Y) \dashrightarrow M_{d-1,b}(Y)$$

given by $[\phi] \mapsto E_\phi$ is dominant.

The same statement holds for $g_Y = 1$ with "stable" replaced by "regular poly-stable."

Proof. Let $g_Y \geq 2$; the proof for $g_Y = 1$ is identical with "stable" replaced by "regular poly-stable."

Let $\phi_0 \colon X_0 \to Y$ be an element of the primitive Hurwitz space $H_{d,g_0}(Y)$ with Tschirnhausen bundle E_0. For some line bundle L of sufficiently large degree, there exists $\phi \colon X \to Y$ with Tschirnhausen bundle $E = E_0 \otimes L$ with $H^1(N_{X/E}) = 0$ by Proposition 3.3. From the proof of Proposition 3.3, we know that $X \to Y$ is obtained as a deformation of the singular curve formed by attaching vertical rational curves to X_0. Recall that in a deformation, the π_1 of a general fiber surjects on to the π_1 of the special fiber. Hence, since $\pi_1(X_0) \to \pi_1(Y)$ is surjective, so is $\pi_1(X) \to \pi_1(Y)$. That is, $X \to Y$ is primitive.

We know that the moduli stack of vector bundles on Y is irreducible [14, Appendix A] and therefore, the locus of stable bundles forms a dense open substack. So, we can find a vector bundle $\mathcal{E}$ on $Y \times \Delta$ such that $\mathcal{E}_{Y \times \{0\}} = E$ and $\mathcal{E}_{Y \times \{t\}}$ is stable for $t \in \Delta \setminus \{0\}$. As $H^1(N_{X/E}) = 0$, the curve $X \subset E$ deforms to the generic fiber of $\mathcal{E} \to \Delta$, by the same relative Hilbert scheme argument as used in the proof of Theorem 3.13. Let $X_t \subset \mathcal{E}_t$ be such a deformation. Then $X_t \to Y$ is a primitive cover with a stable Tschirnhausen bundle. We conclude that for sufficiently large g, the Tschirnhausen bundle of a general element of $H_{d,g}(Y)$ is stable.

Let $\phi : X \to Y$ be an element of $H_{d,g}(Y)$ with stable Tschirnhausen bundle E such that $H^1(N_{X/E}) = 0$. The above argument shows that such coverings exist if g is sufficiently large. Let S be a versal deformation space for E and $\mathcal{E}$ a versal vector bundle on $Y \times S$. See [20, Lemma 2.1] for a construction of S in the analytic category. In the algebraic category, we can take S to be a suitable Quot scheme (see, for example, [14, Proposition A.1]). Let $\mathcal{H}$ be the component of the relative Hilbert scheme of $\operatorname{Tot}(\mathcal{E})/S$ containing the point $[X \subset E]$, and let $\mathcal{H}^{\text{sm}} \subset \mathcal{H}$ be the open subset parametrizing $[X_t \subset \mathcal{E}_t]$ with smooth X_t. Since $H^1(N_{X/E}) = 0$, the map $\mathcal{H}^{\text{sm}} \to S$ is smooth at $[X \subset E]$ by [27, Theorem 3.2.12]. In particular, it is dominant. By Proposition 2.9, we know that for $[X_t \subset \mathcal{E}_t] \in \mathcal{H}^{\text{sm}}$, the bundle $\mathcal{E}_t$ is indeed the Tschirnhausen bundle of $X_t \to Y$. We conclude that the map $H_{d,g}(Y) \dashrightarrow M_{d-1,b}(Y)$ is dominant. □

Remark 3.16. It is natural to ask for an effective lower bound on g in Theorem 3.15. The best result is obtained by taking X_0 to be the disjoint union of d copies of Y; then $g_0 = dg_Y - d + 1$.

That X_0 is not connected does not pose any obstacle—the curve X obtained by attaching vertical rational curves and smoothing out is connected and gives a primitive covering of Y.

How many rational curves do we need to attach? The crucial requirement is the vanishing of $H^1(K)$, where K is defined in (3.8). Note that, in this case, we have $M = \mathcal{O}_{X_0}^{d-1}$. The proof of Proposition 3.6 and Proposition 3.12 show that $h^1(M)+1$ many rational curves suffice. Attaching each rational curve raises the genus by $(d-1)$. Since $h^1(M) = d(d-1)g_Y$, we can thus produce an X of genus g where

$$g \geq d g_Y + d(d-1)^2 g_Y,$$

and $g - g_Y \equiv 0 \pmod{d-1}$. By slightly changing the initial curve X_0, we get similar bounds of order $d^3 g_Y$ for other congruence classes of $g \pmod{d-1}$. By studying the extension (3.10) more closely, it may be possible to sharpen these bounds, but we do not pursue this further.

Recall that the Maroni locus $M(E)$ is the locally closed subset of $H_{d,g}(Y)$ defined by

$$M(E) = \left\{[\phi] \in H_{d,g}(Y) \mid E_\phi \cong E\right\}.$$

Theorem 3.17. *Let E be a vector bundle on Y of rank $(d-1)$ and degree e. If g is sufficiently large (depending on Y and E), then for every line bundle L on Y of degree $b-e$, the Maroni locus $M(E\otimes L)$ contains an irreducible component of the expected codimension $h^1(Y, \operatorname{End} E)$.*

Proof. Set $E' = E \otimes L$. Let H^{sm} be the open subset of the Hilbert scheme of curves in $\operatorname{Tot}(E')$ parametrizing $[X \subset E']$ with X smooth of genus g embedded so that for all $y \in Y$, the scheme $X_y \subset E'_y$ is in affine general position. By Proposition 2.9, the Tschirnhausen bundle map

$$\tau \colon H^{\mathrm{sm}} \to M(E')$$

is a surjection. Furthermore, the fibers of τ are orbits under the group A of affine linear transformations of E' over Y. Plainly, the action of the group is faithful.

By Proposition 3.3, there exists $[X \subset E'] \in H^{\mathrm{sm}}$ with $H^1(N_{X/E'}) = 0$. We can now do a dimension count. Note that $N_{X/E'}$ is a vector bundle on X of rank $(d-1)$ and degree $(d+2)b$, where $b = g_X - 1 - d(g_Y - 1)$. Then the dimension of H^{sm} at $[X \subset E']$ is given by

$$\begin{aligned}\dim_{[X]} H^{\mathrm{sm}} &= \chi(N_{X/E'}) \\ &= (d+2)b - (g_X - 1)(d-1) \\ &= 3b - d(d-1)(g_Y - 1)\end{aligned}$$

The dimension of the fiber of τ is given by

$$\begin{aligned}\dim A &= \hom(E'^{\vee}, \mathcal{O}_Y \oplus E'^{\vee}) \\ &= b - d(d-1)(g_Y - 1) + h^1(\operatorname{End} E).\end{aligned}$$

As a result, the dimension of $M(E')$ at $[\phi]$ is given by

$$\begin{aligned}\dim_{[\phi]} M(E') &= \dim_{[X]} H^{\mathrm{sm}} - \dim A \\ &= 2b - h^1(\operatorname{End} E).\end{aligned}$$

Since $\dim H_{d,g}(Y) = 2b$, the proof is complete. □

4. Higher dimensions

In this section, we discuss the possibility of having an analogue of Theorem 1.1 for higher dimensional Y. For simplicity, take $k = \mathbf{C}$.

Let us begin with the following question.

Question 4.1. *Let Y be a smooth projective variety, L an ample line bundle on Y, and E a vector bundle of rank $(d-1)$ on Y. Is $E \otimes L^n$ a Tschirnhausen bundle for all sufficiently large n?*

The answer to Question 4.1 is "No", at least without additional hypotheses.

Example 4.2. Take $Y = \mathbf{P}^4$, and $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$. Then a sufficiently positive twist E' of E cannot be the Tschirnhausen bundle of a smooth branched cover X.

To see this, recall that the data of a Gorenstein triple cover $X \to Y$ with Tschirnhausen bundle E' is equivalent to the data of a nowhere vanishing global section of $\operatorname{Sym}^3 E' \otimes (\det E')^\vee$ ([19] or [7]). For $E' = E \otimes L^n$ with large n, the rank 4 vector bundle $\operatorname{Sym}^3 E' \otimes (\det E')^\vee$ is very ample. Thus, its fourth Chern class is nonzero. Therefore, a general global section must vanish at some points.

In fact, it is easy to see by direct calculation that the fourth Chern class of $\operatorname{Sym}^3 E \otimes (\det E)^\vee$ can vanish if and only if $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$ where $b = 2a$. Conversely, $E = \mathcal{O}(a) \oplus \mathcal{O}(2a)$ is the Tschirnhausen bundle of a cyclic triple cover of $\mathbf{P}^4$. Thus, $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$ can be a Tschirnhausen bundle of a smooth triple cover of $\mathbf{P}^4$ if and only if $b = 2a$.

Example 4.2 illustrating the failure of Theorem 1.1 can be generalized to all degrees ≥ 3, provided the base Y is allowed to be high dimensional.

Proposition 4.3. *Let $d \geq 3$. The answer to Question 4.1 is "No" for all Y of dimension at least $d\binom{d}{2}$.*

Proof. Let $\phi : X \to Y$ be a finite, flat, degree d map. Then the sheaf $\phi_* \mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$-algebras, and it splits as $\phi_* = \mathcal{O}_Y \oplus E^\vee$.

Suppose over some point $y \in Y$, the multiplication map

$$m : \operatorname{Sym}^2 E^\vee \to \phi_* \mathcal{O}_X$$

is identically zero. Then, we have a k-algebra isomorphism

$$(\phi_* \mathcal{O}_X)|_y \cong k[x_1, ..., x_{d-1}]/(x_1, \dots, x_{d-1})^2.$$

That is, $\phi^{-1}(y)$ is isomorphic to the length d "fat point", defined by the square of the maximal ideal of the origin in an affine space. When $d \geq 3$, these fat points are not Gorenstein. Since Y is smooth, this implies X can not even be Gorenstein, let alone smooth.

Now, if E is a vector bundle on Y and L is a sufficiently positive line bundle, then the bundle

$$M := \operatorname{Hom}(\operatorname{Sym}^2(E \otimes L)^\vee, \mathcal{O}_Y \oplus (E \otimes L)^\vee)$$

is very ample. A general global section $m \in H^0(Y, M)$ will vanish identically at some points $y \in Y$ provided

$$\dim Y \geq \operatorname{rk} M = d\binom{d}{2}.$$

We conclude that if $\dim Y \geq d\binom{d}{2}$, then Question 4.1 has a negative answer. ☐

Observe that Proposition 4.3 remains true even if we relax the requirement that X be smooth to X be Gorenstein.

The following result due to Lazarsfeld suggests the possibility that Proposition 4.3 may be true with a much better lower bound than $d\binom{d}{2}$.

Proposition 4.4. *Let E be a vector bundle of rank $(d-1)$ on $\mathbf{P}^r$, where $r \geq d+1$. Then $E(n)$ is not a Tschirnhausen bundle of a smooth, connected cover for sufficiently large n.*

Proof. The proof relies on [18, Proposition 3.1] which states that for a branched cover $\phi : X \to \mathbf{P}^r$ of degree $d \leq r-1$ with X smooth, the pullback map

$$\phi^* \colon \operatorname{Pic}(\mathbf{P}^r) \to \operatorname{Pic} X$$

is an isomorphism. In particular, the dualizing sheaf ω_ϕ is isomorphic to $\phi^*\mathcal{O}(l)$ for some l. Therefore, we get

$$\mathcal{O}_{\mathbf{P}^r} \oplus E = \phi_*\omega_\phi = \phi_*\mathcal{O}(l) = \mathcal{O}_{\mathbf{P}^r}(l) \oplus E^\vee(l).$$

This implies that $\mathcal{O}_{\mathbf{P}^r}(l)$ is a summand of E.

Suppose $E(n)$ is a Tschirnhausen bundle of a smooth connected cover for infinitely many n. Applying the reasoning above with E replaced by $E(n)$ shows that E must have line bundle summands of infinitely many degrees. Since this is impossible, the proposition follows. □

The reasoning in Example 4.2 implies the following.

Proposition 4.5. *For degree 3, Question 4.1 has an affirmative answer if and only if* $\dim Y < 4$*.*

Proof. Let $\phi : X \to Y$ be a Gorenstein finite covering of degree 3 with Tschirnhausen bundle E. Then by the structure theorem of triple covers in [19] or [7], we get an embedding $X \subset \mathbf{P}E$ as a divisor of class $\mathcal{O}_{\mathbf{P}E}(3)$. Thus, X is given by a global section on $\mathbf{P}E$ of $\mathcal{O}_{\mathbf{P}E}(3)$, or equivalently a global section on Y of $\operatorname{Sym}^3 E \otimes \det E^\vee$. Note that since $X \to Y$ is flat, the global section of $\operatorname{Sym}^3 E \otimes \det E^\vee$ is nowhere vanishing.

Suppose we are given an arbitrary rank 2 vector bundle E on Y. Set $D = \mathcal{O}_{\mathbf{P}E}(3)$ and $V = \operatorname{Sym}^3 E \otimes \det E^\vee$. If we twist E by L^n, then $\mathbf{P}E$ is unchanged but D changes to $D + 3nL$ and V changes to $V \otimes L^n$. For sufficiently large n, the bundle $V \otimes L^n$ is ample. If $\dim Y < 4$, then a general section of $V \otimes L^n$ is nowhere zero on Y. Furthermore, the divisor $X \subset \mathbf{P}E$ cut out by the corresponding section of $\mathcal{O}(D+3nL)$ is smooth by Bertini's theorem. By construction, the resulting $X \to Y$ has Tschirnhausen bundle $E \otimes L^n$.

On the other hand, if $\dim Y \geq 4$, then every global section of $V \otimes L^n$ must vanish at some point in Y. Thus, $E \otimes L^n$ cannot arise as a Tschirnhausen bundle. □

4.1. **Modifications of the original question.** Following the discussion in the previous section, natural modified versions of Question 4.1 emerge. The first obvious question is the following.

Question 4.6. *Is the analogue of Theorem 1.1 true for all Y with* $\dim Y \leq d$*?*

We can also relax the finiteness assumption on ϕ.

Question 4.7. *Let Y be a smooth projective variety, E a vector bundle in Y. Is E isomorphic to $(\phi_*\mathcal{O}_X/\mathcal{O}_Y)^\vee$, up to a twist, for a* generically *finite map $\phi : X \to Y$ with smooth X?*

Remark 4.8. A similar question is addressed in work of Hirschowitz and Narasimhan [13], where it is shown that any vector bundle on Y is the direct image of *some* line bundle on a smooth variety X under a generically finite morphism.

Alternatively, we can keep the finiteness requirement on ϕ in exchange for the smoothness of X. We end the paper with the following open-ended question.

Question 4.9. *What singularity assumptions on X (or the fibers of ϕ) yield a positive answer to Question 4.1?*

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF GEORGIA, ATHENS, GA
E-mail address: `deopurkar@uga.edu`

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA
E-mail address: apatel@math.harvard.edu