STATEMENT OF CURRENT RESEARCH

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SUMMARY

A fundamental question in mathematics is the question of classification. Given a collection of objects, like groups or manifolds, and a notion of equivalence, like isomorphism or homeomorphism, this is the question of describing the set of equivalence classes. A fascinating feature of algebraic geometry is that (very often) the set of equivalence classes itself has the structure of an algebraic variety, called a *moduli space*. Studying this space is tantamount to studying all the objects in the classification at once!

My research explores many features of the geometry of moduli spaces, such as their cohomology rings, their divisor class groups, their canonical and minimal birational models, and more recently, their motivic zeta functions. The moduli spaces I study include the moduli space $\overline{M}_{g,n}$ of n-pointed algebraic curves of genus g, the Hurwitz space $\overline{H}_{d,g/h}$ of branched covers of curves, the Kontsevich space $\mathrm{Maps}_g(X)$ of maps of genus g curves to a target X, and more recently, the moduli spaces of some higher dimensional varieties. I use a variety of techniques, such as deformation theory, geometric invariant theory, algebraic stacks, and enumerative geometry.

The rest of this statement contains an outline of three specific lines of research. Roughly speaking, the goals of these projects are as follows. The first project is about the relationship between various moduli spaces on the cohomological and motivic level. It has connections to certain stabilization phenomena in algebraic topology and certain statistical heuristics in number theory. The second project is about a geometric construction of positive cycle classes on moduli spaces of curves. These are related to conformal blocks, which originally arose in mathematical physics, and address long-standing conjectures to describe the cones of positive cycle classes on moduli spaces of curves. The third project is about a notion of stability that uses syzygies of projective varieties, motivated by the log minimal model program for $\overline{M}_g$.

1. COHOMOLOGICAL AND MOTIVIC RELATIONSHIPS BETWEEN MODULI SPACES OF CURVES

Of central importance in the modern study of curves is the moduli space M_g. We study M_g through many associated moduli spaces. A classical such moduli space is the Hurwitz space $H_{d,g}$ of finite covers $\phi : C \to \mathbf{P}^1$, where C is a smooth curve of genus g and ϕ is a simply branched finite map of degree d (See [11] for a survey on Hurwitz spaces). The space $H_{d,g}$ admits a map to M_g given by C, and also a map to $M_{0,n}$ given by the configuration of the branch points. These maps extend to standard compactifications, namely the Deligne–Mumford compactification $\overline{M}_g$, the Mumford–Knudsen compactification $\overline{M}_{0,n}$, and the admissible cover compactification $\overline{H}_{d,g}$. Joint with Anand Patel and Ravi Vakil, the goal of this project is to precisely formulate the relationship between these spaces. We can first ask about the relationship on the level of cohomology.

Question 1.1. *What is the relationship between the cohomology/Chow rings of $\overline{H}_{d,g}$, $\overline{M}_{0,n}$, and $\overline{M}_g$?*

It will be more fruitful to concentrate on a subring of the cohomology/Chow ring generated by certain 'tautological' classes. Such a tautological ring for $\overline{M}_{g,n}$ is a vigorous and fascinating area of current research [7,9,13].

Goal 1.2. *Define a tautological ring of $\overline{H}_{d,g}$ and relate it to the tautological rings of $\overline{M}_g$ and $\overline{M}_{0,n}$.*

We have a provisional definition of the tautological ring, and we show that it is generated by the fundamental classes of the boundary strata of $\overline{H}_{d,g}$. We expect the structure of this tautological ring to stabilize as d approaches infinity, and this stable limit to reflect the tautological ring of $\overline{M}_g$. Likewise, we expect a stabilization to a much simpler ring as both d and g approach infinity, similar to that for $\overline{M}_g$ [8].

Our next goal is to find relationships between $\overline{H}_{d,g}$ and $\overline{M}_g$ on the level of motives. Such a relationship will be in the form of an equation involving the classes of these varieties in the Grothendieck ring of varieties (localized and completed in an appropriate sense). More precisely, we conjecture the following.

Conjecture 1.3. *The power series*

$$H_g(u) = \sum_{d \geq 0} [H_{d,g}] u^d$$

is a rational function in u with coefficients in $[M_g]$ and $[J]$, where $J \to M_g$ is the universal Jacobian variety. Furthermore, the limit

$$\lim_{d \to \infty} \frac{[H_{d,g}]}{[\mathbf{A}^{2d+2g-5}]}$$

exists and is expressible in terms of the motivic zeta function of the universal curve over M_g.

We can prove both statements if we replace $H_{d,g}$ with a partial compactification $\widetilde{H}_{d,g}$ that parametrizes finite covers that are not necessarily simply branched.

A relationship between M_g and $H_{d,g}$ in the Grothendieck ring of varieties will be interesting not only to geometers and topologists but also to number theorists. It will relate, for example, the number of $\mathbf{F}_q$ points of these two varieties. Since branched covers of curves are the geometric analogue of extensions of number fields, such statistical information will reflect heuristics about number fields, such as those due to Cohen–Lenstra.

Question 1.4. *Let k be a finite field of size q. What is the relationship between the number of k-points on $H_{d,g}$ and M_g? In particular, is the number of $\mathbf{F}_q$-points of $H_{d,0}$ a polynomial function of q?*

Experimental evidence for $d = 2, 3, 4$ suggests that the answer to the last question is "Yes".

Although our current focus is on $\overline{H}_{d,g}$, the questions above can be asked for moduli spaces of curves equipped with a higher dimensional linear series. We expect our results and techniques to extend to the more general setting.

2. Cycle classes on $\overline{M}_{g,n}$ via cyclic covers and conformal blocks

A fundamental open question about the geometry of $\overline{M}_{g,n}$ is a description of its cone of nef divisors. Thanks to the work of Gibney–Keel–Morrison [6], the answers for $g = 0$ determine the answers for all g. For $g = 0$, the famous F-conjecture predicts that the obvious necessary conditions for a divisor on $\overline{M}_{0,n}$ to be nef are also sufficient. The challenge is to show that the conditions are indeed sufficient, for example, by constructing enough nef divisors.

The goal of this project is to study a family of positive cycle classes on $\overline{M}_{g,n}$ arising from cyclic covering constructions, generalizing [3]. In codimension one, we hope to use them to make progress towards the F-conjecture. In higher codimensions, we hope to use them to understand the mostly unexplored territory of positivity of higher codimension cycles on $\overline{M}_{0,n}$ following the framework of Fulger and Lehmann [5].

The idea is the following. Fix positive integers p, k, $m_1, \dots, m_n$, and m. Given a point $t = [\mathbf{P}^1, p_1, \dots, p_n]$ of $M_{0,n}$, construct the smooth projective variety X_t as an appropriate desingularization of the $\mathbf{Z}/p\mathbf{Z}$ cyclic-cover of $(\mathbf{P}^1)^k$ branched over the divisors

$$D_i = \{(x_1, \dots, x_k) \in (\mathbf{P}^1)^k \mid x_j = p_i \text{ for some } j\}$$

with multiplicity m_i, and over the divisor

$$D = \{(x_1, \dots, x_k) \in (\mathbf{P}^1)^k \mid x_i = x_j \text{ for some } i \neq j\}$$

with multiplicity m. For $k = 1$, the cyclic covers are curves, and give precisely the construction of Fedorchuk [3]. The varieties X_t fit together to give a family of smooth projective varieties $\mathscr{X} \to M_{0,n}$.

Theorem 2.1 (In progress). *The family $\pi\colon \mathscr{X} \to M_{0,n}$ extends to a family $\pi\colon \overline{\mathscr{X}} \to \widetilde{M}_{0,n}$, where $\widetilde{M}_{0,n}$ is an orbifold obtained by taking pth roots of all the boundary divisors of $\overline{M}_{0,n}$. The total space $\overline{\mathscr{X}}$ is smooth, and its fibers over the boundary have normal crossings singularities.*

For $i \geq 0$, consider the vector bundle $\Lambda_i = \pi_* \omega_\pi^i$ on $\widetilde{M}_{0,n}$. Due to the $\mathbf{Z}/p\mathbf{Z}$ action, it splits into a direct sum of eigen-bundles $\Lambda_i = \bigoplus_j \Lambda_{i,j}$. By results of Viehweg [14], the bundles Λ_i, and hence the bundles $\Lambda_{i,j}$, are weakly semi-positive (in particular nef). In particular, their Chern classes give positive cycle classes.

Problem 2.2. *Compute the Chern class $c(\pi_* \Lambda_{i,j})$.*

Cyclic covering constructions as above are intimately related to spaces of conformal blocks. By the work of Ramadas [10], Schechtman–Varchenko [12], Belkale [1] (and others), some of the above vector bundles $\Lambda_{i,j}$ ought to be isomorphic to the vector bundles of conformal blocks on $M_{0,n}$. It would be interesting to understand their relationship on the boundary.

Question 2.3. *How are the bundles $\Lambda_{i,j}$ on $\overline{M}_{0,n}$ related to bundles of conformal blocks?*

Understanding the connection will give a geometric approach to the conformal blocks bundles and their Chern classes.

The cyclic covering construction extends to higher genera, which suggests the following.

Problem 2.4. *Explore the divisorial and higher codimension classes that arise by cyclic coverings in higher genera.*

3. Stability of varieties and the birational geometry of their moduli spaces

To construct compact moduli spaces in algebraic geometry, we cannot usually restrict to smooth varieties. We must allow a carefully chosen class of singular varieties—a class that is big enough to admit limits in one parameter families, but small enough to not have multiple possible limits. The choice of this class, often called the class of stable varieties, is usually not unique. This choice can be used effectively to study a moduli space from the point of view of the log minimal model program in birational geometry [4].

The specific goal of this project is to systematically study a notion of stability that arises in the log minimal model program for $\overline{M}_g$ [2]. Let $X \subset \mathbf{P}V$ be a projective variety. If X satisfies property N_p, we can associate to it a point $\operatorname{Syz}_p(X)$ in a Grassmannian variety. This point encodes the pth syzygies among the generators of the homogeneous ideal of X (See [2] for precise definitions). The loci of syzygy points of canonically embedded curves modulo the natural action of $\mathrm{SL}\,V$ are expected to give the final steps in the log minimal model program for $\overline{M}_g$. The first step in this direction is to show that these quotients are birational to $\overline{M}_g$. This is equivalent to the following.

Problem 3.1. *Show that for a general curve X, the syzygy points $\operatorname{Syz}_p(X)$ are GIT stable.*

It would be valuable to study syzygy stability in a broader context, going beyond curves.

Question 3.2. *What geometric properties of X are reflected in the GIT stability of $\operatorname{Syz}_p(X)$ with respect to the natural $\mathrm{SL}\,V$ action on the Grassmannian? What is the relationship between syzygy stability and other notions of stability, such as K-stability or KSBA-stability?*

Hoping for an answer in complete generality is perhaps too optimistic. But many cases of geometric interest in addition to canonical curves should yield insightful answers. These include polarized K3 surfaces, polarized abelian varieties, determinantal varieties, and homogeneous spaces.

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