

HOMEWORK 1

ALGEBRAIC GEOMETRY 2021

- (1) A *homogeneous* polynomial in variables X_1, \dots, X_n is one whose every (non-zero) term has the same total degree. For example, $X_1^2 + X_2X_3$ is homogeneous (of degree 2), whereas $X_1 + X_2X_3$ is not homogeneous. Every polynomial p can be written uniquely as $p = p_0 + p_1 + \dots + p_d$, where p_i is homogeneous of degree i . The p_i 's are called the *homogeneous components* of p .

Let k be a field and let $I \subset k[X_1, \dots, X_n]$ be an ideal. Prove that the following are equivalent:

- (a) I is generated by homogeneous polynomials.
- (b) For every p in I , all the homogeneous components of p are also in I .

Definition: An ideal satisfying the above conditions is called a *homogeneous ideal*.

- (2) Let k be an infinite field. Let $X \subset \mathbb{A}_k^n$ be a Zariski closed subset. Prove that the following are equivalent:
- (a) $I(X)$ is a homogeneous ideal.
 - (b) For every x in X and λ in k , the scalar multiple $\lambda \cdot x$ is also in X .
- (3) Let k be an algebraically closed field and let $X \subset \mathbb{A}_k^2$ be the subset defined by the (infinite) system of equations

$$\begin{aligned}x + y &= 0 \\x^2 + y^2 &= 0 \\x^3 + y^3 &= 0 \\&\dots\end{aligned}$$

Find out a finite system of polynomial equations that defines X , or prove that such a system does not exist.

Caution: The answer may depend on the characteristic of k .

- (4) Let P_n denote the set of monic polynomials in one variable T with coefficients in \mathbb{C} . Identify the set P_n with the affine space $\mathbb{A}_{\mathbb{C}}^n$ by the rule

$$T^n + a_{n-1}T^{n-1} + \dots + a_1T + a_0 \leftrightarrow (a_0, \dots, a_{n-1}).$$

Prove that the set of polynomials with n distinct roots is a Zariski open subset of $\mathbb{A}_{\mathbb{C}}^n$. Describe the equations that define its complement when $n = 2$.

Hint: Remember 'the discriminant' from Algebra 2.

- (5) Describe all maximal ideals of the following rings
- (a) $\mathbb{C}[x, y, z]/(xy, yz, xz)$
 - (b) $\mathbb{R}[x, y]$