HOMEWORK 1

ALGEBRAIC GEOMETRY 2021

(1) A *homogeneous* polynomial in variables $X_1, \dots, X_n$ is one whose every (non-zero) term has the same total degree. For example, $X_1^2 + X_2X_3$ is homogeneous (of degree 2), whereas $X_1 + X_2X_3$ is not homogeneous. Every polynomial p can be written uniquely as $p = p_0 + p_1 + \cdots + p_d$, where p_i is homogeneous of degree i. The p_i's are called the *homogeneous components* of p.

Let k be a field and let $I \subset k[X_1, \dots, X_n]$ be an ideal. Prove that the following are equivalent:

(a) I is generated by homogeneous polynomials.

(b) For every p in I, all the homogeneous components of p are also in I.

Definition: An ideal satisfying the above conditions is called a *homogeneous ideal*.

(2) Let k be an infinite field. Let $I \subset k[X_1, \dots, X_n]$ be an ideal, and let $X \subset \mathbb{A}_k^n$ be the vanishing set of I, namely

$$X = \{x \in \mathbb{A}_k^n \mid p(x) = 0 \text{ for all } p \in I\}.$$

Prove that the following are equivalent:

(a) I is a homogeneous ideal.

(b) For every x in X and λ in k, the scalar multiple $\lambda \cdot x$ is also in X.

(3) Let k be an algebraically closed field and let $X \subset \mathbb{A}_k^2$ be the subset defined by the (infinite) system of equations

$$\begin{aligned} x + y &= 0 \\ x^2 + y^2 &= 0 \\ x^3 + y^3 &= 0 \\ &\cdots \end{aligned}$$

Find out a finite system of polynomial equations that defines X, or prove that such a system does not exist.

Caution: The answer may depend on the characteristic of k.

(4) Let P_n denote the set of monic polynomials in one variable T with coefficients in $\mathbb{C}$. Identify the set P_n with the affine space $\mathbb{A}_{\mathbb{C}}^n$ by the rule

$$T^n + a_{n-1}T^{n-1} + \cdots + a_1T + a_0 \leftrightarrow (a_0, \dots, a_{n-1}).$$

Prove that the set of polynomials with n distinct roots is a Zariski open subset of $\mathbb{A}_{\mathbb{C}}^n$. Describe the equations that define its complement when $n = 2$.

Hint: Remember 'the discriminant' from Algebra 2.

(5) Describe all maximal ideals of the following rings

(a) $\mathbb{C}[x, y, z]/(xy, yz, xz)$

(b) $\mathbb{R}[x, y]$