RESEARCH STATEMENT

ANAND DEOPURKAR

I am an algebraic geometer with broader interests in algebra, number theory, combinatorics, and topology. A common theme in my work is the geometry of moduli spaces. My work is rooted in classical questions, but is modern in its techniques, which include geometric invariant theory, deformation theory, algebraic stacks, and birational geometry.

An important insight in mathematics is that it is wiser to study a collection of related objects at once rather than to study them in isolation. A classic application of this philosophy is my favorite proof of the Cayley–Hamilton theorem in linear algebra, which states that a matrix satisfies its own characteristic polynomial. Instead of focusing on a single matrix, consider the space of all matrices, a dense subset of which consists of diagonalizable ones. The theorem is immediate for diagonalizable matrices, and it follows by continuity for all matrices! Similar strategies, namely judicious uses of deformations and specializations, play a key role in my work. For example, a careful degeneration argument recently allowed me to answer an old question of Lazarsfeld about branched covers and vector bundles [17].

In algebraic geometry, a collection of related objects is often parametrized by an algebraic variety, called a *moduli space*. In most cases, moduli spaces are non-compact, which severely limits their use in applications. A central theme in my work is finding ways to construct *modular compactifications*—compact spaces that parametrize a larger collection of objects, which often includes a carefully selected set of degenerate objects. Finding a class of suitable degenerations is a delicate problem, which often benefits from a change of perspective. Recently, a new perspective on plane curves using algebraic stacks allowed me to answer a classical question about their limits [11].

Modular compactifications have proven to be a powerful tool for solving problems by reducing them to easier problems about degenerations. Furthermore, we have realized that *many* modular compactifications can arise from a single moduli space, and it is fruitful to study all of these at once. It allows us to transport information from one space to another, leading to wall-crossing formulas. Studying how the different compactifications relate to each other and what they tell us about the original moduli space is a central theme in my work.

My current projects are aimed at understanding the cones of positive cycles classes on moduli spaces of pointed curves, developing a systematic understanding of linear series on surfaces using the geometry of Hilbert schemes of points, and using minimal free resolutions and Geometric Invariant Theory to construct new birational models of moduli spaces of polarized varieties. In the following sections, I discuss highlights from my past work and my plans for future research.

1. Birational geometry of moduli spaces

Let X be a projective variety. A fundamental question in algebraic geometry is to classify all maps from X to other projective varieties. A systematic study of this question is called the *Mori program*. One of the broader goals of my research is to carry out the Mori program for moduli spaces. A more precise statement of the goal is the following.

Goal 1.1. *Let X be a projective moduli space. Describe the decomposition of $\overline{\mathrm{Eff}}(X)$, the cone of pseudo-effective divisors, into Mori chambers. Describe the corresponding birational models as modular compactifications.*

1.1. **Mori program for Hurwitz spaces.** The Hurwitz space $H_{d,g}$ is the moduli space of simply branched d-sheeted covers $\phi\colon C \to \mathbf{P}^1$, where C is a curve of genus g. Hurwitz spaces have been studied for over a century, not only by algebraic geometers, but also by number theorists, topologists, and representation theorists. I study them from the standpoint of the Mori program. I construct a range of modular compactifications of $H_{d,g}$, described in the following theorem, which simultaneously generalizes previous constructions of Harris–Mumford [25], Abramovich–Corti–Vistoli [1], and Fedorchuk [21]. This theorem begins a Mori program for the Hurwitz spaces.

Theorem 1.2 (Deopurkar [9, Theorem B])**.** *Let ϵ be a rational number. There is a projective compactification $\overline{H}_{d,g}(\epsilon)$ of $H_{d,g}$ parametrizing ϵ-admissible covers. Roughly, these are branched covers $\phi\colon C \to P$ where P is a nodal rational curve on which up to $1/\epsilon$ of the branch points may coincide.*

Using the ideas of Theorem 1.2, I explicitly describe the Mori chamber decomposition in the case of $d = 3$. Let T be the moduli space of pairs (C, D), where C is a curve of genus g and D a reduced divisor of degree 3 with $h^0(C, D) = 2$. The curves C appearing here are known as *trigonal curves*—they are degree 3 covers of $\mathbf{P}^1$.

Theorem 1.3 (Deopurkar [10, Theorem B+C])**.** *The space T of marked trigonal curves admits a sequence of modular compactifications*

$$\overline{T}^{g} \dashrightarrow \cdots \dashrightarrow \overline{T}^{\mu} \dashrightarrow \cdots \dashrightarrow \overline{T}^{0 \text{ or } 1},$$

indexed by an integer $\mu \equiv g \pmod 2$ between 0 and g. The sequence begins with the contraction of the hyperelliptic locus, continues through flips of the Maroni loci, and culminates in a Fano fibration. We describe the Mori chambers corresponding to $\overline{T}^{\mu}$.

I expect the spaces of ϵ-admissible covers to play a role in a very different context, namely in Teichmüller dynamics, which I discuss in more detail in § 2.

1.2. **Vector bundles and finite covers.** At the crux of the proof of Theorem 1.2 is the connection between branched coverings and vector bundles. A branched covering $f : X \to Y$ yields a vector bundle E on Y defined by

$$f_*\mathcal{O}_X = \mathcal{O}_X \oplus E^\vee.$$

A foundational question is to characterize the vector bundles E that arise in this way. In my recent work with Patel, I completely settle an asymptotic version of the characterization problem, answering a question of Lazarsfeld arising in his 1980 paper [34].

Theorem 1.4 (Deopurkar–Patel [17])**.** *Let Y be a smooth curve and E a vector bundle on Y. Then every sufficiently positive twist of E arises from a branched cover.*

The theorem generalizes analogous results of Kanev [28, 29, 30, 31] for bundles of rank at most 4. Our techniques are completely different, which let us surmount the obstacles in Kanev's method.

Theorem 1.4 opens two exciting lines of further investigation. First, it gives us tools to relate two important moduli spaces: the moduli space of branched covers of Y, and the moduli space of vector bundles on Y. Secondly, it invites a similar investigation in higher dimensions. We are pursuing both directions.

1.3. **Varieties of stable limits.** Let X be a very singular algebraic variety. In a one-parameter smoothing of X, we can replace X by another variety with milder singularities by a process called stable reduction. My work addresses the following goal, which lies at the crux of defining modular compactifications.

Goal 1.5. *Describe the possible stable replacements of a singular X.*

Using Theorem 1.3, I get the answer for all double and triple point curve singularities, extending the work of Fedorchuk [21]. One of my long-term aims is to develop tools to answer Goal 1.5 in higher dimensions.

1.4. **KSBA compactification of log surfaces.** Analogous to the Deligne–Mumford compactification of pointed curves, there is a compactification of divisorially marked surfaces of log-general type due to Kollár–Shepherd-Barron and Alexeev (KSBA) [3, 33]. Unlike the case of curves, however, even the coarse geometric properties of these spaces remain unknown. Moduli of trigonal curves provide a bridge to this little-understood area. A trigonal curve C embeds canonically in a Hirzebruch surface S such that the pair (S, C) is of log general type.

Goal 1.6. *Describe the KSBA compactification of pairs (S, C), where C is a trigonal curve canonically embedded in S.*

My work on compactifications of Hurwitz spaces provides the tools necessary to completely answer this question. Together with Changho Han, I am working out the details. Our results will give explicit examples of KSBA compactifications, very few of which are currently known.

2. Cycle classes and their positivity

An important invariant of a variety X is its cohomology or Chow ring. We know that many features of the geometry of X are governed by the Chow groups, and the cones of positive cycle classes in the Chow groups. In this regard, my research addresses the following broad objectives.

Goal 2.1. *Describe the cohomology ring of a moduli space X and the subring generated by the tautological classes. Describe which classes can be represented by effective or nef cycles.*

The analysis of the cone of effective codimension one cycles has important implications for the birational geometry of the space.

Goal 2.2. *Determine the place of X in the birational classification of varieties. For example, determine if it is rational, rationally connected, Calabi–Yau, or of general type.*

The geometric questions in Goal 2.2 have deep arithmetic implications, thanks to the conjectures of Bombieri and Lang.

2.1. **Cycle classes on Hurwitz spaces.** Despite enormous efforts, the questions in Goal 2.1 and Goal 2.2 remain open for Hurwitz spaces. Even the description of the Chow group $A^1(H_{d,g})$ is still conjectural.

Conjecture 2.3 (See [18]). *We have $A^1(H_{d,g}) \otimes \mathbf{Q} = 0$.*

Patel and I prove Conjecture 2.3 for d up to 5.

Theorem 2.4 (Deopurkar–Patel [16, Theorem A]). *We have $A^1(H_{d,g}) \otimes \mathbf{Q} = 0$ for $d \leq 5$.*

For $d > 2g-2$, this conjecture is equivalent to a similar statement about M_g, which was proved by Harer using transcendental techniques [24]. A purely algebraic proof of Conjecture 2.3 will give the first algebraic proof of a fundamental fact about M_g.

We also study the question of effectivity mentioned in Goal 2.1. Denote by λ and δ the pullbacks to $\overline{H}_{d,g}$ of the Hodge class and the total boundary class of $\overline{M}_g$. For $d = 3$, we describe which linear combinations of λ and δ are effective, extending prior work of Cornalba–Harris and Stankova-Frenkel [7, 44].

Theorem 2.5 (Deopurkar–Patel [15, Theorem 1.2]). *The smallest s such that a multiple of $s\lambda - \delta$ is effective on $\overline{H}_{3,g}$ is given by*

$$s = \begin{cases} 7 + 6/g & \text{for even } g, \\ 7 + 20/(3g+1) & \text{for odd } g. \end{cases}$$

My future research in this direction will be guided by the following fundamental open questions about the birational geometry of $\overline{H}_{d,g}$.

Goal 2.6. *Determine the cones of effective and nef divisors on $\overline{H}_{d,g}$. In particular, analyze the positivity of the canonical divisor and find the birational type of $\overline{H}_{d,g}$.*

Patel and I are currently studying effective cycles on $\overline{H}_{d,g}$ that arise from the relative minimal free resolution of a branched cover. The first cycle in this series plays a crucial role in the proof of Theorem 2.5. I believe the later ones will be equally important.

2.2. **Hurwitz correspondences and Teichmüller dynamics.** Consider a more general Hurwitz space $\overline{H}_{d,g/h}$ that parametrizes genus g covers of genus h curves. It admits a map to $\overline{M}_{h,b}$ given by the configuration of the branch points. It also admits a map to $\overline{M}_{g,b}$ given by the configuration of the ramification points. We thus get a correspondence between $\overline{M}_{h,b}$ and $\overline{M}_{g,b}$, which we call a *Hurwitz correspondence*. It induces a map between the cohomology groups of $\overline{M}_{h,b}$ and $\overline{M}_{g,b}$. These are important maps to study.

Hurwitz correspondences for $g = h = 0$ arise in studying the dynamics of post-critically finite maps in Teichmüller theory [32, § 2]. Consider an orientation-preserving (topological) branched cover of the sphere, say $f : (S^2, P) \to (S^2, P)$, where P is the post-critical set of f. Assume that P is finite. A theorem of Thurston characterizes those f that arise from algebraic maps, using the induced map on the Teichmüller space of (S^2, P). The map on Teichmüller space rarely induces a map on the moduli space $M_{0,P}$, but it does induce a Hurwitz correspondence [32]. The eigenvalues of the induced map on cohomology are important dynamical invariants.

Rohini Ramadas has developed tools to compute the induced maps on cohomology in a range of degrees. We are working together on a complete solution using alternate compactifications of Hurwitz spaces.

2.3. **Tautological classes on Hilbert schemes and the existence of linear series.** A fundamental question in algebraic geometry is to determine whether a given variety X admits a map to a given projective space $\mathbf{P}^r$ of a given degree d. As a culmination of work spanning decades, we have a good understanding of this question for curves. The answers are collectively known as Brill–Noether theory. The situation in higher dimensions is much murkier. In a joint project with David Stapleton, I am exploring the basic question of Brill–Noether theory for surfaces, namely, for which r and d should there exist a *rational* map $X \dashrightarrow \mathbf{P}^r$ of degree d.

A particularly important case is when $r = \dim X$. The minimum d for which there exists a generically finite degree d rational map $X \dashrightarrow \mathbf{P}^r$ is called the *degree of irrationality* of X, denoted by $\mathrm{irr}(X)$. This generalization of the gonality of a curve measures how far X is from being rational, and has been studied by many authors [19, 36, 45, 51]. There has been exciting new progress in determining $\mathrm{irr}(X)$ for very general hypersurfaces of large degree by Ein, Lazarsfeld, and Ullery [19]. But the question remains open for a large class of varieties.

The concrete objectives of our project are the following.

Goal 2.7. *Determine when we expect a surface X to possess a linear series of rank r and degree d. In particular, determine how the irrationality of a very general K3 surface of genus g grows as a function of g.*

Our approach involves analyzing a particular cohomology class arising from tautological vector bundles on Hilbert schemes of points on surfaces. Either outcome—vanishing or non-vanishing—will lead to interesting mathematics. Vanishing will give non-trivial relations in the Chow ring; non-vanishing will give upper bounds on the degree of irrationality.

2.4. **Positive cycle classes on $\overline{M}_{g,n}$ from variations of Hodge structures.** Denote by $\overline{M}_{g,n}$ the moduli space of Deligne–Mumford stable n-pointed curves of genus g. An important open problem about $\overline{M}_{g,n}$ is to determine the cones of positive cycle classes in its Chow groups. The answer is unknown even in the case of divisors, let alone in higher codimensions, and is a subject of several open conjectures. The problem is important for two reasons. First, it is crucial for understanding the birational geometry of $\overline{M}_{g,n}$, which is of intrinsic interest in algebraic geometry. Second, exploring this question for $\overline{M}_{g,n}$ will reveal fundamental structural properties of cones of positive cycle classes on projective varieties, especially in higher codimension. This is a fascinating new area of research with vigorous recent activity.

The main challenge in this area is the construction of new positive cycle classes. In an ongoing project, I address this challenge using Chern classes of bundles arising from variations of Hodge structures (VHS) of highly non-trivial families of projective varieties over $\overline{M}_{g,n}$. I construct these families as cyclic coverings of hyperplane arrangements associated to pointed curves. The Chern classes of vector bundles associated to the resulting VHS gives an infinite class of nef cycle classes in all codimensions. I am exploring the structure of these cycle classes and investigating their role in the geometry of $\overline{M}_{g,n}$.

3. Syzygies, free resolutions, and GIT

Let $X \subset \mathbf{P}^n$ be a projective variety. A time-honored theme in algebraic geometry is the connection between the algebraic information encoded in the homogeneous coordinate ring of X and the intrinsic geometry of X. I explore this connection from the point of view of moduli.

For suitable positive integers m and p, there are points in various Grassmannians associated to X, called the *mth Hilbert point* or the *pth syzygy point*. Let I be the homogeneous ideal of X. The mth Hilbert point corresponds to the vector space of degree m polynomials in I, and the pth syzygy point to the vector space of pth syzygies among the generators of I. As X varies, the Hilbert and syzygy points for various m and p trace out loci in the respective Grassmannians. For sufficiently large m, the locus of mth Hilbert points is precisely the Hilbert scheme constructed by Grothendieck. For smaller m, however, we get variations that are more sensitive to the geometry of X. Already for twisted cubics in $\mathbf{P}^3$, these variations and their relationship with the Hilbert scheme have been a fascinating topic of study [20, 41].

My work addresses the geometry of the loci of syzygy points, an area which is completely unexplored. I expect that the geometry of these loci and their interaction with the natural action of SL_{n+1} will reflect subtle geometric properties of X. For example, I show the following for canonical curves in the first non-trivial case.

Theorem 3.1 (Deopurkar [13])**.** *Denote by* $\mathrm{Syz}_1(C)$ *the first syzygy point of a canonically embedded non-trigonal curve of genus 7. We have the following implications regarding the geometry of C and the GIT stability of* $\mathrm{Syz}_1(C)$*:*

(1) *C is generic* $\implies$ $\mathrm{Syz}_1(C)$ *is stable.*
(2) *C is generic with a g^1_4* $\implies$ $\mathrm{Syz}_1(C)$ *is stable.*
(3) *C is generic with a g^2_6* $\implies$ $\mathrm{Syz}_1(C)$ *is strictly semi-stable.*
(4) *C is bi-elliptic* $\implies$ $\mathrm{Syz}_1(C)$ *is unstable.*

For curves, the connection between the geometry and the GIT stability of syzygies is expected to play a key role in the Mori program for $\overline{M}_g$ (see § 3.1). I will explore such a connection systematically for curves as well as higher-dimensional varieties.

Goal 3.2. *Understand the locus of syzygy points of schemes $X \subset \mathbf{P}^n$ and its connection to the Hilbert scheme. Relate the SL_{n+1} stability/instability of the syzygy point of X to intrinsic geometric properties of X.*

My immediate focus will be on low genus curves, polarized K3 surfaces, and certain Fano 3-folds. Thanks to the work of Mukai, we can describe such varieties as linear sections of homogeneous spaces [37, 38].

Goal 3.3. *Describe the compact moduli spaces of low genus curves, polarized K3 surfaces, and Fano 3-folds using Mukai's descriptions. Relate these models to those constructed using Hilbert or syzygy points.*

With Han-Bom Moon, I am working out the Mukai model of the moduli space of genus 7 curves, which is the quotient of the Grassmannian $\mathbf{Gr}(7, 16)$ by an action of Spin_{10}.

3.1. Application to the log minimal model program (MMP) for $\overline{M}_g$.

Denote by δ the total boundary class and by K the canonical class of $\overline{M}_g$. Pioneered by Hassett and Keel, the log MMP for $\overline{M}_g$ aims to give a geometric description of the spaces

$$\overline{M}_g(\alpha) = \operatorname{Proj} \bigoplus_{n \geq 0} H^0\left(\overline{M}_g, n(K + \alpha\delta)\right)$$

for $\alpha \in [0, 1]$. As α goes from 1 to 0, the above sequence goes from the Deligne–Mumford model to the canonical model, realizing the minimal model program for $\overline{M}_g$. For α close to 1, the space $\overline{M}_g(\alpha)$ is known to be a GIT quotient of a locus of Hilbert points [26, 27, 43]. For α close to 0, it is expected to be a GIT quotient of a locus of syzygy points.

Conjecture 3.4. *Denote by Syz_p the closure of the locus of pth syzygy points of canonically embedded curves of genus g. The GIT quotients $\mathrm{Syz}_p /\!\!/ \mathrm{SL}_g$ are log canonical models of $\overline{M}_g$ for (explicitly computable) α-values that range from $(g+6)/(7g+6)$ to 0 as p varies.*

The construction of $\overline{M}_g(\alpha)$ as a GIT quotient of Syz_p was proposed by Keel and Farkas a couple of decades ago, but there was essentially no progress on this proposal. My coauthors and I prove the following generic semistability result, which provides an important step in this direction.

Theorem 3.5 (Deopurkar–Fedorchuk–Swinarski [14])**.** *A general curve of odd genus has a semistable first syzygy point.*

For small genera, we have verified that generic curves have semistable pth syzygy points for all p. Proving this will be a significant advance towards Conjecture 3.4.

Goal 3.6. *Show that a general curve of genus g has semistable pth syzygy points for all g and p. Relate instability with intrinsic geometric properties of the curve.*

We have an approach to Goal 3.6 using non-reduced curves with rich automorphism groups. We conjecture that a particular double structure on $\mathbf{P}^1$, namely the *balanced ribbon* will have a semistable syzygy point for all p. Going further, we would like to interpret $\overline{M}_g(\alpha)$ as a moduli space in its own right. In particular, the moduli interpretation of the canonical model of $\overline{M}_g$ will be extremely interesting.

3.2. **Relationship with Green's conjecture.** A necessary step for Goal 3.6 is related to a fascinating conjecture of Mark Green. Let R be the homogeneous coordinate ring of a canonically embedded curve $C \subset \mathbf{P}^{g-1}$. In 1984, Green made the following conjecture, which relates the shape of the minimal free resolution of R and the presence of linear series on C.

Conjecture 3.7 (Green [22])**.** *The Koszul cohomology group $K_{p,2}$ of R vanishes if and only if p is smaller than the Clifford index of C.*

In the early 2000s, Voisin proved the conjecture for a generic C using K3 surfaces [48, 49]. Subsequently, Farkas and Aprodu have settled it for a large class of C [4, 5].

In 1995, Bayer and Eisenbud made an analogous conjecture where C is replaced by a *ribbon*—a non-reduced curve obtained by appropriately gluing $\operatorname{Spec} \mathbf{C}[x,\epsilon]/\epsilon^2$ and $\operatorname{Spec} \mathbf{C}[y,\eta]/\eta^2$ [6]. In a recent paper, I prove their conjecture using Voisin's theorem.

Theorem 3.8 (Deopurkar [12])**.** *Green's conjecture (Conjecture 3.7) holds for all ribbons.*

As a corollary, I get a short proof that Green's conjecture holds for a non-empty open subset of curves of every genus and Clifford index. Theorem 3.8 also reinforces the viability of our approach for proving generic stability of syzygy points using ribbons.

4. Moduli of fibrations and maps to stacks

A powerful technique to study a complicated geometric object is to express it as a fibration where the fiber and the base are simpler. Consider a variety S fibered over a curve C by a map $\phi : S \to C$. We can interpret the data of $(\phi : S \to C)$ as a map from C to the moduli stack X of the fibers of ϕ. Seminal work Abramovich and Vistoli has made this a feasible approach [2].

Using insights from compactifications of Hurwitz spaces, I construct explicit and well-behaved compactifications of maps to X in the one-dimensional case.

Theorem 4.1 (Deopurkar [11, Theorem 2.5])**.** *Let X be a one-dimensional, smooth, and proper Deligne–Mumford stack of finite type. The space of finite maps from curves to X admits a smooth modular compactification with a normal crossings boundary divisor. The boundary points in this compactification parametrize admissible cover degenerations of finite covers of X.*

The theorem has immediate classical applications. First, taking $X = \overline{M}_{1,1}$ yields a compactification of elliptically fibered surfaces. Second, taking $X = [\overline{M}_{0,4}/S_4]$ yields a compactification of tetragonal curves on Hirzebruch surfaces. Using this compactification, I solve the first non-trivial case of a long-standing problem.

Problem 4.2. *Describe the stable limits of smooth plane curves of degree d. In other words, describe the closure in $\overline{M}_g$ of the locus of smooth plane curves.*

Theorem 4.3 (Deopurkar, [11, Theorem 1.1])**.** *Let $Q \subset \overline{M}_6$ be the locus of plane quintic curves and $\overline{Q}$ its closure. The boundary $\overline{Q} \setminus Q$ consists of 14 irreducible components, which we describe explicitly.*

In the proof of Theorem 4.3, I generalize the picture of tetragonal curves, trigonal curves, and theta characteristics described by Vakil [46].

Theorem 4.1 opens new directions for further research, many of which are approachable. Here is an attractive one.

Goal 4.4. *Understand the compactification of the space of elliptic fibrations, and more generally the space of genus 1 fibrations, using Theorem 4.1.*

The question is particularly interesting for the space of maps of degree 24, where the elliptically fibered surface is a K3 surface.

Another direction is to address higher-dimensional targets X. Abramovich and Vistoli have constructed a compactification $\overline{M}_g(X)$ of the space of maps from curves of genus g to X in great generality. Although the spaces $\overline{M}_g(X)$ have been studied from the point of view of Gromov–Witten theory, their geometric properties have been less explored.

Goal 4.5. *Study the geometric properties such as smoothness, connectedness, and irreducibility of the Abramovich–Vistoli compactifications $\overline{M}_g(X)$.*

Answers to these questions will give tools to study moduli spaces of higher-dimensional varieties. They are also likely to unify and illuminate many classical results as they did for $X = [\overline{M}_{0,4}/S_4]$ in my work [11]. Furthermore, it would be wonderful to construct compactifications with nice geometric properties for higher-dimensional X in the style of Theorem 4.1. This is a challenging question even when X is a scheme.

5. Moduli of surfaces and tropical geometry

Tropical and non-Archimedean geometry has had marvelous applications in many areas of algebraic geometry, such as Brill–Noether theory, Gromov–Witten theory, mirror symmetry, birational geometry, and real algebraic geometry. It has led to remarkable connections between algebraic geometry and combinatorics, leading to progress in both fields.

Let X be an algebraic variety embedded in a toric variety T over a complete valued field K. We can associate to the data of $X \subset T$ a finite polyhedral complex X^{trop} [35]. This tropicalization procedure replaces questions about algebraic functions on X by questions about piecewise linear functions on X^{trop}. We can view X^{trop} as a polyhedral approximation of the Berkovich analytic space X^{an} associated to X [40].

In a joint project with Maria Angelica Cueto, I am studying moduli spaces of del Pezzo surfaces from the tropical perspective. Our current focus is on del Pezzo surfaces of degree 3, or cubic surfaces. We consider the smooth projective model Y of the moduli space of cubic surfaces constructed by Naruki using cross-ratios associated to root systems [39]. The space Y has a GIT interpretation and, conjecturally, a functorial interpretation as the moduli space of weighted stable log pairs [23].

Ren, Shaw, and Sturmfels use a $W(E_6)$-equivariant description of Y due to Yoshida to construct a tropicalization Y^{trop} along with a family of tropical surfaces $G^{\text{trop}} \to Y^{\text{trop}}$ [42, 50]. Our goal is to describe the tropical anti-canonical embeddings of the surfaces in this family.

Let X be a cubic surface over a complete non-trivially valued field K. We consider the generators for $H^0(X, -K_X)$ given by the 45 tri-tangent planes and the resulting embedding $X \subset \mathbf{P}^{44}$. Our main result is the following analogue of a classical enumerative result.

Theorem 5.1 (Cueto–Deopurkar [8]). *The tropical variety $X^{\text{trop}} \subset T\mathbf{P}^{44}$ contains exactly 27 tropical lines.*

Consequently, the pathology of infinitely many lines exhibited by tropical cubic surfaces in $T\mathbf{P}^3$ goes away in the more canonical embedding of a cubic del Pezzo surface given by the anti-canonical triangles [47].

Our next goal is to answer a question of Sturmfels [42].

Goal 5.2. *Describe $X^{\text{trop}} \subset T\mathbf{P}^{44}$ and $P^{\text{trop}} \subset T\mathbf{P}^{44}$, where $P \cong \mathbf{P}^3 \subset \mathbf{P}^{44}$ is the span of X.*

We show that X^{trop} is isomorphic to the corresponding fiber of $G^{\text{trop}} \to Y^{\text{trop}}$. The problem of describing the tropical linear space TP is much more challenging, and is currently under investigation. These tropical investigations will give us insight into the classical geometry of degenerations of divisorially marked threefolds.

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