VECTOR BUNDLES AND FINITE COVERS

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ABSTRACT. We prove that, up to a twist, every vector bundle on a smooth projective curve arises from the direct image of the structure sheaf of a smooth, connected branched cover.

1. INTRODUCTION

Associated to a finite flat morphism $\phi : X \to Y$ is the vector bundle $\phi_* \mathcal{O}_X$ on Y. This is the bundle whose fiber over $y \in Y$ is the vector space of functions on $\phi^{-1}(y)$. In this paper, we address the following basic question: which vector bundles on a given Y arise in this way? We are particularly interested in cases where X and Y are smooth projective varieties. Formulated differently, the question asks: which vector bundles on a given Y admit the structure of a commutative and associative $\mathcal{O}_Y$-algebra, particularly in the presence of of additional restrictions like regularity.

Our main result is that, up to a twist, every vector bundle on a smooth projective curve Y arises from a smooth branched cover $X \to Y$.

Theorem 1.1 (Main). *Let Y be a smooth projective curve and let E be a vector bundle on Y. There exists an integer n (depending on E) such that for any line bundle L on Y of degree at least n, there exists a smooth curve X and a finite map $\phi : X \to Y$ such that $\phi_* \mathcal{O}_X$ is isomorphic to $\mathcal{O}_Y \oplus E^\vee \otimes L^\vee$.*

The reason for the $\mathcal{O}_Y$ summand is the following. Pull-back of functions gives a natural map $\mathcal{O}_Y \to \phi_* \mathcal{O}_X$, which admits a splitting by $1/d$ times the trace map (assume d is invertible). As a result, every bundle of the form $\phi_* \mathcal{O}_X$ contains $\mathcal{O}_Y$ as a direct summand. The dual of the remaining direct summand is called the *Tschirnhausen bundle* and is denoted by $E = E_\phi$ (The dual is taken as a convention.) Thus, Theorem 1.1 says that on a smooth projective curve, a sufficiently positive twist of every vector bundle is Tschirnhausen.

The reason for needing the twist is a bit more subtle. It arises from geometric restrictions on Tschirnhausen bundles of finite maps between smooth projective varieties. For $Y = \mathbf{P}^n$ and a smooth X, the Tschirnhausen bundle E is ample by a result of Lazarsfeld [11]. For more general Y and smooth X, it enjoys several positivity properties as shown in [14, 15]. The precise necessary and sufficient conditions for being Tschirnhausen (without the twist) are unknown, and seem to be quite delicate even when Y is a curve.

Without any regularity restrictions on X, the question of identifying Tschirnhausen bundles is vacuous: every vector bundle E qualifies. Indeed, given E, we may take X to be the non-reduced scheme which is the first order neighborhood of the zero section in the total space of E.

The simplest non-trivial case of identifying Tschirnhausen bundles, namely the case of $Y = \mathbf{P}^1$, has attracted the attention of several mathematicians; see for example [1, 5, 12, 16]. Historically, it has been called the problem of classifying the *scrollar invariants* of smooth finite covers of $\mathbf{P}^1$. Recall that every vector bundle on $\mathbf{P}^1$ splits as a direct sum of line bundles. Writing $E_\phi = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{d-1})$, the scrollar invariants are the integers $a_1, \ldots, a_{d-1}$. This problem has been completely solved for $d = 2$ and $d = 3$. For $d = 2$, any positive integer is a scrollar invariant. For $d = 3$, a pair of positive integers (a_1, a_2) with $a_1 \leq a_2$ arises as scrollar invariants of smooth triple coverings if and only if $a_2 \leq 2a_1$. It may be within reach to completely settle the next few values of d using structure theorems for finite covers of low degree (see [3, 4]), but the problem becomes very hard very fast as d increases. Indeed, the lack of constructive structure theorems for finite covers of degree d for $d \geq 6$ makes any direct attacks unfeasible. Nevertheless, the picture emerging from the collective work of several authors [5, 12], and visible in the $d = 3$ case, indicates that if the a_i are too far apart, then they cannot arise as scrollar invariants.

Our result says that the picture is the cleanest possible if we allow twisting by a line bundle. Note that for $Y = \mathbf{P}^1$, twisting is equivalent to allowing a simultaneous shift of all the scrollar invariants. Before our work, the work of Ballico [1] came closest to a characterization of scrollar invariants up to a shift. He showed that one can arbitrarily specify the smallest $d/2$ of the $(d-1)$ scrollar invariants. Our main theorem (Theorem 1.1) applied to $Y = \mathbf{P}^1$ answers the question completely: it says that one can in fact arbitrarily specify *all* the scrollar invariants.

For affine curves, Theorem 1.1 yields the following corollary.

Corollary 1.2. *Suppose Y is a smooth affine curve, and E is a vector bundle on Y. Then E is the Tschirnhausen bundle for some map $\phi : X \to Y$, with X smooth and connected.*

Proof. Extend E to a vector bundle E' on the smooth projective compactification Y' of Y. Apply Theorem 1.1 to E', twisting by a sufficiently positive line bundle L on Y' whose divisor class is supported on the complement $Y' \setminus Y$. We obtain a smooth curve X' and a map $\phi : X' \to Y'$ whose Tschirnhausen bundle is $E' \otimes L$; letting $X = \phi^{-1}(Y)$, we obtain the corollary. □

The method of proof of Theorem 1.1 yields a basic result relating the moduli of branched covers of Y and the moduli of vector bundles on Y. Let $H_{d,g}(Y)$ denote the Hurwitz space of degree d and genus g branched covers of Y and $M_{r,k}(Y)$ the (coarse) moduli space of semi-stable vector bundles of rank r and degree k on Y. Note that the Tschirnhausen bundle of a degree d and genus g cover of Y has rank $d-1$ and degree $g - 1 - d(g_Y - 1)$.

Theorem 1.3. *Suppose $g_Y \geq 2$, and set $b = g - 1 - d(g_Y - 1)$. If g is sufficiently large (depending on Y and d), then the Tschirnhausen bundle of a general degree d and genus g branched cover of Y is stable. Moreover, the rational map $H_{d,g}(Y) \dashrightarrow M_{d-1,b}(Y)$ defined by $\phi \mapsto E_\phi$ is dominant.*

The same statement holds for $g_Y = 1$, with "stable" replaced with "regular poly-stable."

Theorem 1.3 is Corollary 3.12 in the main text.

Special cases of Theorem 1.3, namely the cases $d \leq 5$, were proved by Kanev [8, 9, 10] using structure theorems for finite covers of low degree [3, 4]. The crucial new ingredient in our approach is the use of deformation theory to circumvent such direct attacks, which are unfeasible for $d \geq 6$ for the lack of structure theorems.

The validity of Theorem 1.3 for low g is an interesting open problem. It would be nice to know whether $\phi \to E_\phi$ is dominant as soon as we have $\dim H_{d,g}(Y) \geq \dim M_{d-1,b}$.

Our interest in Tschirnhausen bundles for curves originated partly in the study of cycles on $H_{d,g}(Y)$. For a vector bundle E on Y, define the *Maroni locus* $M(E) \subset H_{d,g}(Y)$ as the locally closed subset that parametrizes covers with Tschirnhausen bundle isomorphic to E. This notion generalizes the Maroni loci for $Y = \mathbf{P}^1$ studied in [6] and [13]. As a consequence of the method of proof of the main theorem, we obtain the following.

Theorem 1.4. *Let E be a vector bundle on Y of rank $(d-1)$ and degree e. If g is sufficiently large (depending on Y and E), then for every line bundle L of degree $b - e$, the Maroni locus $M(E \otimes L) \subset H_{d,g}(Y)$ contains an irreducible component having the expected codimension $h^1(\operatorname{End} E)$.*

Theorem 1.4 is Corollary 3.13 in the main text.

Going further, it would be valuable to know whether all the components of $M(E \otimes L)$ are of the expected dimension or, even better, if $M(E \otimes L)$ is irreducible. The results of [6] imply irreducibility for $Y = \mathbf{P}^1$ and some vector bundles E. But the question remains open in general.

The connection to cycles on $H_{d,g}(Y)$ is through the fundamental class of the closure of $M(E)$. It would be interesting to know if this cycle has any distinguishing properties, such as rigidity or extremality, as is the case for the Maroni divisors for $Y = \mathbf{P}^1$, at least when $d \leq 5$ [13].

We also draw the reader's attention to results, similar in spirit to Theorem 1.3, proved by Beauville, Narasimhan, and Ramanan [2]. The basic problem in their line of inquiry is to study not the pushforward of $\mathcal{O}_X$ itself but the pushforwards of general line bundles on X.

The attempt at extending Theorem 1.1 to higher dimensional varieties Y presents interesting new challenges. We discuss them through some examples in § 4. As it stands, the analogue of Theorem 1.1 for higher dimensional varieties Y is false. We end the paper by posing modifications for which we are unable to find counterexamples.

1.1. **Strategy of proof.** The proof of Theorem 1.1 proceeds by degeneration. To help the reader, we first outline our approach to a weaker version of Theorem 1.1. In the weaker version, we consider not the vector bundle E itself, but its projectivization $\mathbf{P}E$, which we call the *Tschirnhausen scroll*. Recall that a branched cover with Gorenstein fibers $\phi \colon X \to Y$ with Tschirnhausen bundle E factors through a *relative canonical embedding* $\iota \colon X \hookrightarrow \mathbf{P}E$; see [3].

Theorem 1.5. *Let E be any vector bundle on a smooth projective curve Y. Then the scroll $\mathbf{P}E$ is the Tschirnhausen scroll of a finite cover $\phi \colon X \to Y$ with X smooth.*

The following steps outline a proof of Theorem 1.5 which parallels the proof of the stronger Theorem 1.1. We omit the details, since they are subsumed by the results in the paper.

(1) First consider the case
$$E = L_1 \oplus \cdots \oplus L_{d-1},$$
where the L_i are line bundles on Y whose degrees satisfy
$$\deg L_i \ll \deg L_{i+1}.$$
For such E, we construct a nodal cover $\phi \colon X \to Y$ such that $\mathbf{P}E_\phi = \mathbf{P}E$. For example, we may take X to be a nodal union of d copies of Y, each mapping isomorphically to Y under ϕ, where the ith copy meets the $(i+1)$th copy along nodes lying in the linear series $|L_i|$.

(2) Consider $X \subset \mathbf{P}E$, where X is the nodal curve constructed above. We now attempt to find a smoothing of X in $\mathbf{P}E$. However, the normal bundle $N_{X/\mathbf{P}E}$ may be quite negative. Fixing this negativity is the most crucial step.

(3) To fix the negativity of $N_{X/\mathbf{P}E}$, we attach several rational normal curves to X as follows. Given a general point $y \in Y$, the d points $\phi^{-1}(y) \subset \mathbf{P}E_y \simeq \mathbf{P}^{d-2}$ are in general linear position, and therefore they lie on many smooth rational normal curves $R_y \subset \mathbf{P}E_y$. Choose a large subset $S \subset Y$, and attach general rational normal curves R_y for each $y \in S$ to X, obtaining a new nodal curve $Z \subset \mathbf{P}E$.

(4) The key technical step is showing that the new normal bundle $N_{Z/\mathbf{P}E}$ is sufficiently positive. Using this positivity, we get that Z is the flat limit of a family of smooth, relatively-canonically embedded curves $X_t \subset \mathbf{P}E$. The special case of $E = L_1 \oplus \cdots \oplus L_{d-1}$ is then complete.

(5) We now tackle the case of an arbitrary bundle E as follows.
   (a) We note that every vector bundle E degenerates *isotrivially* to a bundle of the form $L_1 \oplus \cdots \oplus L_{d-1}$ treated in the previous steps.

(b) We then consider the map τ from the moduli stack of branched covers of Y to the moduli stack of projective bundles on Y given by the rule
$$\tau : \phi \rightsquigarrow \mathbf{P}E_{\phi}.$$
Consider a point $[\phi : X_t \to Y]$ of the moduli stack of branched covers of Y constructed in the previous step. The abundant positivity of $N_{X_t/\mathbf{P}E}$ established above implies that τ is smooth at $[\phi : X_t \to Y]$. Said differently, the map τ yields a smooth map from a miniversal deformation space of $[\phi]$ to a miniversal deformation space of E_{ϕ}.

(c) Using the openness of smooth maps and the isotrivial degeneration property, we conclude that every projective bundle lies in the image of τ. That is, every projective bundle arises from a branched cover $\phi : X \to Y$.

We need to refine the strategy above to be able to handle the vector bundle E itself, and not just its projectivization. For this purpose, we begin with the *canonical affine embedding* of X in the total space of E. Ultimately, we work with embedding of X in the projective closure of this total space, namely in $P := \mathbf{P}(E^{\vee} \oplus \mathcal{O}_Y)$. Let $H := P \setminus E$ be the divisor of hyperplanes at infinity. The proof of Theorem 1.1 involves carrying out the steps outlined above for the embedding $X \to P$ *relative* to the divisor H.

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1.3. **Conventions.** All schemes are finite type over an algebraically closed field k of characteristic 0 (or of characteristic larger than the degree d of the covers we consider). The projectivization $\mathbf{P}V$ of a vector bundle V refers to the space of 1-dimensional quotients of V. We identify vector bundles with their sheaves of sections. An injection is understood as an injection of sheaves.

2. VECTOR BUNDLES, THEIR INFLATIONS, AND DEGENERATIONS

This section contains some preliminary results on vector bundles on curves. None of them are difficult; some are well-known. But we have included them for completeness.

Throughout, Y is a smooth connected curve. Let E be a vector bundle on Y. A *degree n inflation* of E is a vector bundle $\widetilde{E}$ along with an injective map $E \to \widetilde{E}$ whose cokernel is finite of length n. If $E \to \widetilde{E}$ is an inflation, then the dual bundle $\widetilde{E}^{\vee}$ is a sub-sheaf of $E^{\vee}$ and the quotient is finite of length n. Thus, a degree n inflation of E is equivalent to a sub-sheaf of $E^{\vee}$ of co-length n, which in turn is equivalent to a quotient of $E^{\vee}$ of length n. Therefore, we can identify the moduli space of length n inflations of E with the quot scheme $\operatorname{Quot}(E^{\vee}, n)$. It is easy to see that $\operatorname{Quot}(E^{\vee}, n)$ is smooth and connected, in particular irreducible. Therefore, it makes sense to talk about a general length n inflation of E.

Proposition 2.1. *Let E be a vector bundle on Y. For a sufficiently large n, a general length n inflation $E \to \widetilde{E}$ satisfies $H^1(\widetilde{E}) = 0$.*

Proof. By Serre vanishing, we have $H^1(E \otimes L) = 0$ for a very ample line bundle L on Y. Set $N = \operatorname{rk} E \cdot \deg L$. Let $n \geq N$ and let $E \otimes L \to \widetilde{E}$ be a general length $(N - n)$ inflation. Then $E \to \widetilde{E}$ is a length n inflation. We have an exact sequence
$$0 \to E \otimes L \to \widetilde{E} \to Q \to 0. \tag{2.1}$$

Since $H^1(E \otimes L) = 0$, the long exact sequence on cohomology associated to (2.1) implies that $H^1(\widetilde{E}) = 0$. Since the space of length n inflations of E is irreducible, we conclude that $H^1(\widetilde{E}) = 0$ for a general length n inflation $E \to \widetilde{E}$ for any $n \geq N$. $\square$

Remark 2.2. Suppose $E \to \widetilde{E}$ is an inflation and $H^1(E) = 0$. Then the long exact sequence associated to

$$0 \to E \to \widetilde{E} \to Q \to 0$$

shows that we also have $H^1(\widetilde{E}) = 0$.

Remark 2.2 and Proposition 2.1 together imply the following.

Corollary 2.3. *Let E be a vector bundle on Y of rank r. For large enough n, any vector bundle E' of rank r that contains a general degree n inflation of E satisfies $H^1(E') = 0$.*

Remark 2.4. Consider a degree 1 inflation $E \to \widetilde{E}$. Suppose the cokernel is supported at a point p. The dual map $\widetilde{E}^\vee \to E^\vee$ drops rank by 1 at p, and therefore the image of $\widetilde{E}^\vee|_p$ in $E^\vee|_p$ is a hyperplane. Conversely, a degree 1 inflation of E is specified by a point p and a hyperplane of $E^\vee|_p$. In this case, E and $\widetilde{E}$ are often said to be related by an elementary transformation.

Remark 2.5. A common setting for inflations in the paper is the following. Let P be a smooth variety. Let $R, S \subset P$ be curves that intersect at a point p so that their union Z has a simple node at p. Then we have the exact sequence

$$0 \to N_{R/P} \to N_{Z/P}|_R \to k_p \to 0.$$

That is, the bundle $N_{Z/P}\big|_R$ is a degree 1 inflation of $N_{R/P}$. The hyperplane of $N_{R/P}^\vee|_p$ that specifies this inflation is the kernel of the map

$$N_{R/P}^\vee|_p \to k$$

defined as the composite

$$N_{R/P}^\vee|_p \xrightarrow{d} \Omega_P\big|_p \to k,$$

where the last map is the contraction with a non-zero vector in T_pS.

We say that a bundle E *isotrivially degenerates* to a bundle E_0 if there exists a pointed smooth (not necessarily projective) curve $(\Delta, 0)$ and a Δ-flat bundle $\mathcal{E}$ on $Y \times \Delta$ such that $\mathcal{E}_{Y \times \{0\}} \cong E_0$ and $\mathcal{E}\big|_{Y \times \{t\}} \cong E$ for every $t \in \Delta \setminus \{0\}$.

Proposition 2.6. *Let E a vector bundle on Y, and N a non-negative integer. Then E isotrivially degenerates to a vector bundle E_0 of the form*

$$E_0 = L_1 \oplus \cdots \oplus L_r,$$

where the L_i are line bundles and $\deg L_i + N \leq \deg L_{i+1}$ *for all* $i = 1, \dots, r-1$.

For the proof of Proposition 2.6, we need a lemma.

Lemma 2.7. *There exists a filtration*

$$E = F_0 \supset F_1 \supset \cdots \supset F_{r-1} \supset F_r = 0,$$

satisfying the following properties.

(1) *For every $i \in \{0, \dots, r-1\}$, the sub-quotient F_i/F_{i+1} is a line bundle.*
(2) *Set $L_i = F_i/F_{i+1}$ for $i \in \{1, \dots, r-1\}$ and $L_r = F_0/F_1$. For every $i \in \{1, \dots, r-1\}$, we have*

$$\deg L_i + N \leq \deg L_{i+1}.$$

Proof. The statement is vacuous for $r = 0$ and 1. So assume $r \geq 2$. Note that if $F_\bullet$ is a filtration of E satisfying the two conditions, and if L is invertible, then $F_\bullet \otimes L$ is such a filtration of $E \otimes L$. Therefore, by twisting by a line bundle of large degree if necessary, we may assume that $\deg E \geq 0$.

Let us construct the filtration from right to left. Let $L_{r-1} \subset E$ be a line bundle with $\deg L_{r-1} \leq -N$ and with a locally free quotient. Set $F_{r-1} = L_{r-1}$. Next, let $L_{r-2} \subset E/F_{r-1}$ be a line bundle with $\deg L_{r-2} \leq \deg L_{r-1} - N$ and with a locally free quotient. Let $F_{r-2} \subset E$ be the preimage of L_{r-2}. Continue in this way. More precisely, suppose that we have constructed

$$F_j \supset F_{j+1} \supset \cdots \supset F_{r-1} \supset F_r = 0$$

such that $L_i = F_i/F_{i+1}$ satisfy

$$\deg L_i \leq \deg L_{i+1} - N,$$

and suppose $j \geq 2$. Then let $L_{j-1} \subset E/F_j$ be a line bundle with $\deg L_{j-1} \leq \deg L_j - N$ with a locally free quotient. Let $F_{j-1} \subset E$ be the preimage of L_{j-1}. Finally, set $F_0 = E$.

Condition 1 is true by design. Condition 2 is true by design for $i \in \{1, \ldots, r-2\}$. For $i = r-1$, note that $\deg L_{r-1} \leq -N$ by construction. On the other hand, we must have $\deg L_r \geq 0$. Indeed, we have $\deg E \geq 0$ but every sub-quotient of $F_\bullet$ except F_0/F_1 has negative degree. Therefore, condition 2 holds for $i = r-1$ as well. □

Proof of Proposition 2.6. Let $F_\bullet$ be a filtration of E satisfying the conclusions of Lemma 2.7. It is standard that a coherent sheaf degenerates isotrivially to the associated graded sheaf of its filtration. The construction goes as follows. Consider the $\mathcal{O}_Y[t]$-module

$$\bigoplus_{n \in \mathbf{Z}} t^{-n} F_n,$$

where $F_n = 0$ for $n > r$ and $F_n = E$ for $n < 0$. The corresponding sheaf $\mathcal{E}$ on $Y \times \mathbf{A}^1$ is coherent, $k[t]$-flat, satisfies $\mathcal{E}_{Y \times \{t\}} \cong E$ for $t \neq 0$, and $\mathcal{E}_{Y \times \{0\}} \cong L_1 \oplus \cdots \oplus L_r$. □

3. Proof of the main theorem

3.1. **The split case.** As a first step, we treat the case of a suitable direct sum of line bundles and allow the source curve X to be singular.

Proposition 3.1. *Let $E = L_1 \oplus \cdots \oplus L_r$, where the L_i are line bundles on Y with $\deg L_1 \geq 2g_Y - 1$ and $\deg L_{i+1} \geq \deg L_i + (2g_Y - 1)$ for $i \in \{1, \ldots, r-1\}$. There exists a nodal curve X and a finite flat map $\phi \colon X \to Y$ of degree $d = r + 1$ such that $E_\phi \cong E$.*

The proof is inductive, based on the following "pinching" construction. Let $\psi \colon Z \to Y$ be a finite cover of degree $d-1$. Let X be the reducible nodal curve $Z \cup Y$, where Z and Y are attached nodally at distinct points (see Figure 1). We have a finite flat map $\phi \colon X \to Y$ that restricts to ψ on Z and is identity on Y. Let $D \subset Y$ be the preimage of the nodes.

Lemma 3.2. *In the setup above, we have an exact sequence*

$$0 \to E_\psi \to E_\phi \to \mathcal{O}_Y(D) \to 0.$$

Proof. The closed embedding $Z \to X$ gives a surjection

$$\phi_* \mathcal{O}_X \to \psi_* \mathcal{O}_Z$$

whose kernel is easily seen to be $\mathcal{O}_Y(-D)$. Factoring out the $\mathcal{O}_Y$ summand from both sides and taking duals yields the claimed exact sequence. □

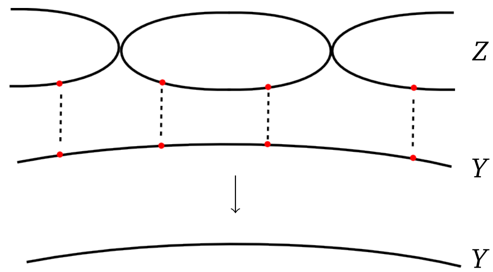

FIGURE 1. The pinching construction, in which pairs of points indicated by dotted lines are identified to form nodes.

Proof of Proposition 3.1. We use induction on r, starting with the base case $r = 0$, which is vacuous.

By the inductive hypothesis, we may assume that there exists a nodal curve Z and a finite cover $\psi: Z \to Y$ of degree $d-1$ such that $E_\psi \cong L_2 \oplus \cdots \oplus L_r$. Let $X = Z \cup Y \to Y$ be a cover of degree d obtained from $Z \to Y$ by a pinching construction such that $\mathcal{O}_Y(D) = L_1$. By Lemma 3.2, we get an exact sequence

$$0 \to L_2 \oplus \cdots \oplus L_r \to E_\phi \to L_1 \to 0. \tag{3.1}$$

But we have $\mathrm{Ext}^1(L_1, L_i) = H^1(L_i \otimes L_1^\vee) = 0$ since $\deg\left(L_i \otimes L_1^\vee\right) \geq 2g_Y - 1$. Therefore, the sequence (3.1) is split, and we get $E_\phi = L_1 \oplus \cdots \oplus L_r$. The induction step is then complete. □

3.2. **Smoothing out.** We now come to the key step of the proof. This step allows us to pass from singular covers to smooth covers and from particular vector bundles to their deformations.

Let X be a nodal curve, $\phi: X \to Y$ a finite flat morphism of degree d, and E the associated Tschirnhausen bundle. The map $E^\vee \to \phi_*\mathcal{O}_X$ induces a surjection $\mathrm{Sym}^* E^\vee \to \phi_*\mathcal{O}_X$. Taking the relative spectrum gives an embedding of X in the total space of the vector bundle associated to E (which we also denote by E). We call $X \subset E$ the *canonical affine embedding*.

Proposition 3.3 (Key)**.** *There exists a line bundle L on Y, a smooth curve X', and a finite morphism $\phi': X' \to Y$ such that the following hold.*

(1) *The Tschirnhausen bundle E' of ϕ' is $E \otimes L$.*
(2) *We have $H^1(X', N_{X'/E'}) = 0$, where $X' \subset E'$ is the canonical affine embedding.*

Furthermore, there exists an n (depending on X and E), such that the above holds for any L of degree at least n.

The rest of § 3.2 is devoted to the proof of Proposition 3.3.

We now describe the main construction used for the proof of Proposition 3.3. To do so, we must first compactify the total space of the vector bundle E, which we do in the standard way. Namely, we take the compactification $\mathbf{P} = \mathbf{P}(E^\vee \oplus \mathcal{O}_Y)$ of E. [1] Let $H \cong \mathbf{P}E^\vee \subset \mathbf{P}$ be the family of hyperplanes at infinity, where the embedding $H \subset \mathbf{P}$ is defined by the projection $E^\vee \oplus \mathcal{O}_Y \to E^\vee$. The complement of $H \subset \mathbf{P}$ is the total space of the vector bundle E.

Let $S \subset Y$ be a finite set such that $X \to Y$ is étale over all points of S. For $y \in S$, the set $X_y \subset P_y \cong \mathbf{P}^{d-1}$ consists of d points in linear general position. Therefore, there exists a smooth rational normal curve R_y in P_y containing X_y. Let $\widetilde{P} \to \mathbf{P}$ be the blow up at $\bigsqcup_{y \in S} H_y$. Denote by

[1] We remind the reader that projectivization refers to the space of 1-dimensional *quotients*.

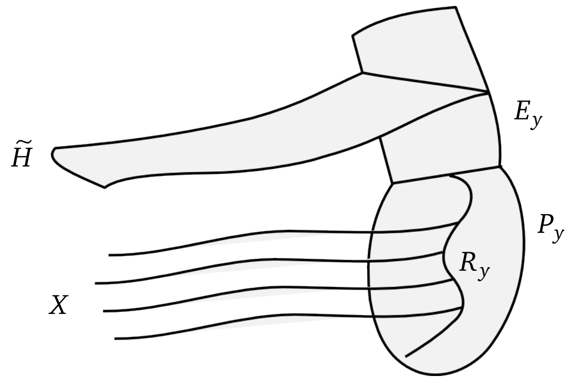

FIGURE 2. Attaching rational normal curves to X to make the normal bundle positive

the same symbol R_y the proper transform of R_y in $\widetilde{P}$ (this will not cause confusion). Denote by $\widetilde{H}$ the proper transform of H in $\widetilde{P}$ (see Figure 2).

The fiber of $\widetilde{P} \to Y$ over $y \in S$ consists of two components. One is the exceptional divisor E_y of the blow-up. The second is the proper transform of P_y, which is a copy of P_y; we denote it also by P_y. The two components intersect transversely along a $\mathbf{P}^{d-2}$. Set $Z = X \cup_{y \in S} R_y$. Establishing the positivity of $N_{Z/\widetilde{P}}$ is at the heart of the proof of Proposition 3.3.

Proposition 3.4. *If the size of S is large, its points are general, and the rational normal curves R_y are general, then we have $H^1\left(N_{Z/\widetilde{P}}\right) = 0$.*

For the proof, we need some preparatory lemmas. First, we set some notation.

$$\begin{aligned}
\nu\colon Z^\nu \to Z &:= \text{The normalization of } Z,\\
X_1, \dots, X_l &:= \text{The components of the normalization } X^\nu \text{ of } X,\\
\gamma &:= \text{The set of nodes of } X,\\
\Gamma &:= \text{The preimage of } \gamma \text{ in } X^\nu,\\
\delta_y &:= R_y \cap X,\\
P_S &:= \text{The disjoint union of } P_y \text{ for } y \in S,\\
R_S &:= \text{The disjoint union of } R_y \text{ for } y \in S,\\
\delta_S &:= \text{The disjoint union of } \delta_y \text{ for } y \in S.
\end{aligned}$$

Furthermore, denote by $\nu_X : X^\nu \to Z$, $\nu_i : X_i \to Z$, and $\nu_y : R_y \to Z$ the natural maps. Denote the map to Y from X, Z, X^ν, and Z^ν by the same letter ϕ. Note that Z^ν is the disjoint union of $X_1, \dots, X_l$ and R_S. Every point of γ has two preimages in Γ. Lastly, note that the singular set of Z is $\gamma \cup \delta_S$.

Let y be a point in S. Denote by $\mathcal{O}(1)$ the line bundle of degree 1 on $R_y \cong \mathbf{P}^1$.

Lemma 3.5. *The restriction of $N_{Z/\widetilde{P}}$ to R_y is isomorphic to $\mathcal{O}(d+1)^{d-2} \oplus \mathcal{O}(1)$, and the $\mathcal{O}(d+1)^{d-2}$ summand is the image of the natural map*

$$N_{R_y/P_y} \to N_{Z/\widetilde{P}}\big|_{R_y}.$$

Proof. In the proof, we drop the subscript y from R_y and P_y. First, note that $N_{Z/\widetilde{P}}\big|_R$ is a vector bundle of rank $(d-1)$ and degree $(d-2)(d+1)+1$. The map $N_{R/P} \to N_{Z/\widetilde{P}}\big|_R$ is the composite

$$N_{R/P} \to N_{R/\widetilde{P}} \to N_{Z/\widetilde{P}}\big|_R$$

Using that X is transverse to P, a simple local computation shows that the injection $N_{R/P} \to N_{Z/\widetilde{P}}\big|_R$ remains an injection when restricted to any point of R. Since $R \subset P \cong \mathbf{P}^{d-1}$ is a rational normal curve, we know that $N_{R/P} \cong \mathcal{O}(d+1)^{d-2}$. We thus get an exact sequence

$$0 \to \mathcal{O}(d+1)^{d-2} \to N_{Z/\widetilde{P}}\big|_R \to \mathcal{O}(1) \to 0. \tag{3.2}$$

Since $\operatorname{Ext}^1(\mathcal{O}(1), \mathcal{O}(d+1)) = 0$, this sequence splits. □

We call the $\mathcal{O}(d+1)^{d-2}$ the *vertical* summand of $N_{Z/\widetilde{P}}\big|_{R_y}$. Set $V = N_{R_S/P_S}$, and think of it as a sub-bundle of $N_{Z/\widetilde{P}}\big|_{R_S}$. Let $F = N_{Z/\widetilde{P}}|_{R_S}/V$ be the quotient. On R_S, we then have the sequence

$$0 \to V \to N_{Z/\widetilde{P}}\big|_{R_S} \to F \to 0. \tag{3.3}$$

Having analyzed $N_{Z/\widetilde{P}}$ on R_S, the next two lemmas analyze it on X and X^ν.

Lemma 3.6. *We have an exact sequence*

$$0 \to N_{X/\widetilde{P}} \to N_{Z/\widetilde{P}}\big|_X \to F|_{\delta_S} \to 0.$$

Proof. We have an injective map of vector bundles $N_{X/\widetilde{P}} \to N_{Z/\widetilde{P}}\big|_X$ that drops rank by 1 at every point of δ_S (see Remark 2.5). Let $p \in \delta_S$ lie over $y \in Y$. To show that the cokernel of $N_{X/\widetilde{P}} \to N_{Z/\widetilde{P}}\big|_X$ is $F|_{\delta_S}$, we must show that the image of $N_{X/\widetilde{P}}|_p \to N_{Z/\widetilde{P}}\big|_p$ is $V|_p$. But this follows from the following commutative diagram

$$\begin{tikzcd} T_{P_y}|_p \arrow[r] \arrow[d, equal] & N_{R_y/P_y}\big|_p \arrow[r] \arrow[d] & 0 \\ N_{X/\widetilde{P}}\big|_p \arrow[r] & N_{Z/\widetilde{P}}\big|_p. & \end{tikzcd}$$

□

Lemma 3.7. *If the size of S is large, its points are general, and the rational normal curves R_y are general, then we have $H^1(X_i, \nu_i^* N_{Z/\widetilde{P}}) = 0$.*

Proof. Let $p \in \delta_S$ be a point lying on X_i. Let N be the kernel of the map $N_{Z/\widetilde{P}}\big|_X \to F|_{\delta_S \setminus \{p\}}$. Then we have an exact sequence

$$0 \to \nu_i^* N_{X/\widetilde{P}} \to \nu_i^* N \to F|_p \to 0.$$

In other words, $\nu_i^* N$ is an inflation of $\nu_i^* N_{X/\widetilde{P}}$ at p. If the tangent line $T_p R_y \subset T_p P_y$ is general, then it is a general such inflation. Note that we have an injection $\nu_i^* N \to \nu_i^* N_{Z/\widetilde{P}}$. Thus, under the given genericity assumptions, $\nu_i^* N_{Z/\widetilde{P}}$ contains a general degree $|S|$ inflation of $\nu_i^* N_{X/\widetilde{P}}$. The lemma now follows from Corollary 2.3. □

Thanks to Lemma 3.5 and Lemma 3.7, the pullbacks of $N_{Z/\widetilde{P}}$ to all the components of Z^ν have no higher cohomology. That is, we have $H^1\left(\nu^* N_{Z/\widetilde{P}}\right) = 0$. This is necessary, but not sufficient, for $H^1\left(N_{Z/\widetilde{P}}\right) = 0$. What remains is the surjectivity of the map

$$H^0(\nu^* N_{Z/\widetilde{P}}) \to H^0\left(N_{Z/\widetilde{P}}\big|_{\gamma \cup \delta_S}\right).$$

Let us try to convey the difficulty in showing such a surjection. The nodes γ will not be a big issue, so let us focus on δ_S. Here, we have the splitting of the normal bundle into vertical and horizontal components

$$N_{Z/\widetilde{S}}\big|_{\delta_S} = V|_{\delta_S} \oplus F|_{\delta_S}.$$

We can easily take care of the vertical component. Indeed, the summand V of $N_{Z/\widetilde{P}}|_{R_S}$ is positive enough so that its global sections surject onto $H^0\left(V|_{\delta_S}\right)$ by Lemma 3.5. However, the other summand F is *not* so positive. As a result, we need the help of the global sections from $\nu_X^* N_{Z/\widetilde{P}}$ in order to get a surjection onto $H^0(F|_{\delta_S})$. In Lemma 3.7, we showed that $\nu_X^* N_{Z/\widetilde{P}}$ is inflated enough from $\nu_X^* N_{X/\widetilde{P}}$ so that it has no H^1. But to get a surjection onto $H^0\left(F|_{\delta_S}\right)$ we need something stronger. We need the vanishing of H^1 not just for $\nu_X^* N_{Z/\widetilde{P}}$ but for a certain sub-sheaf, arising as the kernel of a map to a certain quotient of $F|_{\delta_S}$. Precisely, this is the sheaf K that will be defined soon.

Recall that $\Gamma \subset X^\nu$ is the preimage of the singular set γ of X. Set $M = \nu_X^* N_{X/\widetilde{P}}(-\Gamma)$ and $N = \nu_X^* N_{Z/\widetilde{P}}(-\Gamma)$. Note that the natural map $N \to \nu_X^* N_{Z/\widetilde{P}}$ is an isomorphism at the points of δ_S. From now on, we identify $N|_{\delta_S}$ and $N_{Z/\widetilde{P}}|_{\delta_S}$.

Since $\phi \colon X^\nu \to Y$ is a finite map, the surjection $N \to N|_{\delta_S}$ gives a surjection

$$\phi_* N \to \phi_*\left(N|_{\delta_S}\right).$$

The surjection $N_{Z/\widetilde{P}}\big|_{R_S} \to N_{Z/\widetilde{P}}\big|_{\delta_S} = N\big|_{\delta_S}$ gives a map (not a surjection!)

$$\phi_*\left(N_{Z/\widetilde{P}}\big|_{R_S}\right) \to \phi_*\left(N|_{\delta_S}\right). \tag{3.4}$$

Let W be the cokernel of the map in (3.4). Since both terms in (3.4) are supported on S, so is W.

We have the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \phi_* V \arrow[r] \arrow[d] & \phi_*\left(N_{Z/\widetilde{P}}\big|_{R_S}\right) \arrow[r] \arrow[d] & \phi_* F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \phi_*\left(V|_{\delta_S}\right) \arrow[r] & \phi_*\left(N\big|_{\delta_S}\right) \arrow[r] & \phi_*\left(F|_{\delta_S}\right) \arrow[r] & 0.
\end{tikzcd} \tag{3.5}$$

Since $V \cong \mathcal{O}(d+1)^{d-2}$ (see Lemma 3.5), and δ_S consists of only d points, the left vertical map is surjective. By the snake lemma, we get an isomorphism between the cokernel of the middle vertical map and the cokernel of the right vertical map. That is, we get the sequence

$$0 \to \phi_* F \to \phi_*(F|_{\delta_S}) \to W \to 0. \tag{3.6}$$

In particular, we see that $W_y \cong k^{d-2}$ for $y \in S$. Let K be the vector bundle on Y defined by

$$0 \to K \to \phi_* N \to W \to 0. \tag{3.7}$$

The map $\phi_* N \to W$ is the composite of the surjections $\phi_* N \to \phi_*\left(N\big|_{\delta_S}\right)$ and $\phi_*\left(N\big|_{\delta_S}\right) \to W$. Since $\phi \colon X^\nu \to Y$ is finite of degree d, and N is a vector bundle of rank $(d-1)$ on X^ν, we see that K is a vector bundle of rank $d(d-1)$ on Y.

Lemma 3.8. *If the size of S is large, its points are general, and the rational normal curves R_y are general, then we have $H^1(Y, K) = 0$.*

Proof. The proof is by degeneration. Assume that S consists of a single point y. This is mainly to ease notation, as it allows us to drop the subscripts y and S, and write δ for $\delta_y = \delta_S$, and so on.

Let p be a point in δ. Let $(T, 0)$ be (the germ of) a smooth pointed curve; this will be the base of our degeneration. Let $\mathcal{R} \subset P \times T$ be a closed sub-scheme, flat of relative dimension 1 over T satisfying the following properties. First, $\mathcal{R}_t$ must contain δ for all $t \in T$. Second, $\mathcal{R}_t$ for $t \neq 0$

must be a smooth rational normal curve in $P \cong \mathbf{P}^{d-1}$. Third, $\mathcal{R}_0$ must be a reducible nodal rational curve, say $R_0 = L \cup C$, where L is a line containing p and C is a smooth rational curve of degree $(d-2)$ containing $\delta \setminus \{p\}$. The two components L and C must meet nodally at a point away from δ (see Figure 3). Assume that $T_p\mathcal{R}_0 = T_pL$ is a generic line in T_pP, and furthermore that the total space $\mathcal{R}$ is a smooth surface. Denote by R a generic fiber of $\mathcal{R} \to T$. More generally, adopt the notational convention that roman analogues of calligraphic letters denote generic fibers.

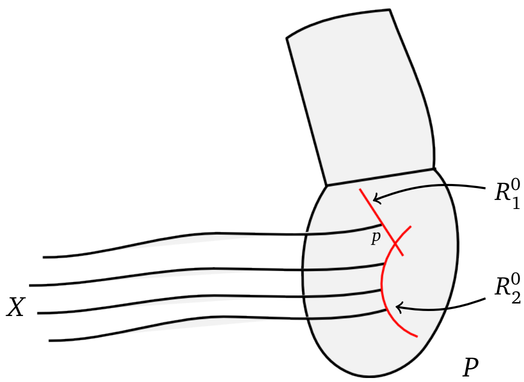

FIGURE 3. A degeneration of the rational curve R_y used in the proof of Lemma 3.8.

The family $\mathcal{R} \subset P \times T$ naturally gives a family $\mathcal{Z} \subset \widetilde{P} \times T$, where $\mathcal{Z}_t$ is identical to Z except the rational curve R is replaced by $\mathcal{R}_t$. We can now make all our previous constructions in this family. Indeed, set $\mathcal{V} = N_{\mathcal{R}/P\times T}\big|_{\mathcal{R}}$ and define $\mathcal{F}$ by the sequence

$$0 \to \mathcal{V} \to N_{\mathcal{Z}/\widetilde{P}\times T}\big|_{\mathcal{R}} \to \mathcal{F} \to 0$$

analogous to (3.3). Then $\mathcal{F}$ is a line bundle on $\mathcal{R}$. For a generic t, we have $\mathcal{F}_t \cong \mathcal{O}(1)$, but for $t = 0$, we have $\mathcal{F}_0|_L = \mathcal{O}$ and $\mathcal{F}_0|_C = \mathcal{O}(1)$.

On $X^\nu \times T$, consider $\mathcal{M} = \nu_1^* N_{X\times T/\widetilde{P}\times T}(-\Gamma \times T)$ and $\mathcal{N} = \nu_2^* N_{\mathcal{Z}/\widetilde{P}\times T}(-\Gamma\times T)$, where $\nu_1 : X^\nu \times T \to X \times T$ and $\nu_2 : X^\nu \times T \to \mathcal{Z}$ are the obvious maps. Identify $\mathcal{N}|_{\delta\times T}$ and $N_{\mathcal{Z}/\widetilde{P}\times T}|_{\delta\times T}$, as before. The analogue of (3.5) continues to hold with $\mathcal{V}$, $\mathcal{N}$, $\mathcal{Z}$, and $\mathcal{F}$ in place of V, N, Z, and F, respectively. Let $\mathcal{W}$ be the cokernel of the middle (or equivalently the right) vertical map of (3.5) and define $\mathcal{K}$, as before, by the sequence

$$0 \to \mathcal{K} \to \phi_*\mathcal{N} \to \mathcal{W} \to 0.$$

It is easy to see that $\mathcal{M}$, $\mathcal{N}$, $\mathcal{V}$, $\mathcal{Z}$, $\mathcal{F}$, $\mathcal{W}$, and $\mathcal{K}$ are flat over T.

Suppose we prove that $H^1(Y, \mathcal{K}_0) = 0$. Then it follows that $H^1(Y, K) = 0$ by semi-continuity.

Unfortunately, we have to take a more twisted approach. Set $\mathcal{F}' = \mathcal{F} \otimes \mathcal{O}_{\mathcal{R}}(C)$. Plainly, $\mathcal{F}'$ is isomorphic to $\mathcal{F}$ away from the central fiber $\mathcal{R}_0$. But note that $\mathcal{F}'_0|_L = \mathcal{O}(1)$ and $\mathcal{F}'_0|_C = \mathcal{O}$. Also $\mathcal{F}'|_{\delta\times T}$ is (non-canonically) isomorphic to $\mathcal{F}|_{\delta\times T}$. Fix such an isomorphism. Define $\mathcal{W}'$ by the sequence

$$0 \to \phi_*\mathcal{F}' \to \phi_*(\mathcal{F}|_{\delta\times T}) \to \mathcal{W}' \to 0.$$

Define $\mathcal{K}'$ as the kernel of $\phi_*\mathcal{N} \to \mathcal{W}'$.

Observe that $\mathcal{K}$ and $\mathcal{K}'$ are isomorphic away from $t = 0$. Therefore, proving $H^1(Y, \mathcal{K}'_0) = 0$ also implies that $H^1(Y, K) = 0$ by semi-continuity.

Let us analyze $\mathcal{K}'_0$. Since $\mathcal{R}_0$ is reducible, it is easy to get information about the image of

$$\phi_*(\mathcal{F}'_0) \to \phi_*(\mathcal{F}_0|_\delta). \tag{3.8}$$

Particularly, consider the unique nonzero (up to scaling) global section of $\mathcal{F}_0'$ that vanishes entirely along C. This section is non-zero at p and zero at all the points of $\delta \setminus \{p\}$. Therefore, the support of the cokernel $\mathcal{W}_0'$ of (3.8) does not contain p. Said differently, we have a surjection

$$\phi_*\left(\mathcal{F}_0\big|_{\delta\setminus\{p\}}\right) \to \mathcal{W}_0'. \tag{3.9}$$

Define M^+ on X^ν by the sequence

$$0 \to M^+ \to \mathcal{N}_0 \to \mathcal{F}_0\big|_{\delta\setminus\{p\}} \to 0. \tag{3.10}$$

The exact sequence in Lemma 3.6 twisted by $\mathcal{O}_X(-\Gamma)$ shows that we have the sequence

$$0 \to \nu_X^* N_{X/\widetilde{\mathbb{P}}}(-\Gamma) = M \to M^+ \to k_p \to 0.$$

In other words, M^+ is isomorphic to M on all components of X^ν except the one containing p. On the component X_i of X^ν containing p, it is a degree 1 inflation of $M|_{X_i}$. The data of this inflation is given by the tangent line $T_p\mathcal{R}_0$ (see Lemma 3.6 and Remark 2.5). Since this line is general, it is a general such inflation.

The definition (3.10) of M^+ and the surjection (3.9) imply that we have an injection

$$\phi_*(M^+) \to \mathcal{K}_0'. \tag{3.11}$$

To summarize, we have degenerated the bundle K to a bundle $\mathcal{K}_0'$, which contains $\phi_* M^+$, where M^+ is a general degree 1 inflation of M on a component of X^ν.

We now take $S \subset Y$ of size n, with n large, and carry out the above degeneration over all $y \in S$. More precisely, pick $p_y \in X$ over $y \in S$ such that a general inflation of M at the points p_y has vanishing higher cohomology. By considering degenerations of the n rational normal curves R_y for $y \in S$ of the form described above, we degenerate the bundle K to a bundle $\mathcal{K}_0'$ that contains $\phi_* M^+$, where M^+ is a general inflation of M at the points p_y. Corollary 2.3 and the finiteness of $\phi \colon X^\nu \to Y$ implies that $H^1(Y, \mathcal{K}_0') = 0$. Then we get $H^1(Y, K) = 0$ by semi-continuity. □

We now have the tools to prove that $H^1(N_{Z/\widetilde{\mathbb{P}}}) = 0$.

Proof of Proposition 3.4. We have the exact sequence

$$0 \to N_{Z/\widetilde{\mathbb{P}}} \to \nu_* \nu^* N_{Z/\widetilde{\mathbb{P}}} \to N_{Z/\widetilde{\mathbb{P}}}\big|_{\gamma\cup\delta_S} \to 0.$$

The long exact sequence on cohomology gives

$$H^0\left(\nu^* N_{Z/\widetilde{\mathbb{P}}}\right) \to H^0\left(N_{Z/\widetilde{\mathbb{P}}}\big|_{\gamma\cup\delta_S}\right) \to H^1\left(N_{Z/\widetilde{\mathbb{P}}}\right) \to H^1\left(\nu^* N_{Z/\widetilde{\mathbb{P}}}\right) \to 0.$$

By Lemma 3.5, we know that $H^1\left(\nu_y^* N_{Z/\widetilde{\mathbb{P}}}\right) = 0$. By Lemma 3.6, we know that $H^1\left(\nu_X^* N_{Z/\widetilde{\mathbb{P}}}\right) = 0$. Therefore, we get $H^1\left(\nu^* N_{Z/\widetilde{\mathbb{P}}}\right) = 0$.

We must now show that $H^0\left(\nu^* N_{Z/\widetilde{\mathbb{P}}}\right) \to H^0\left(N_{Z/\widetilde{\mathbb{P}}}\big|_{\gamma\cup\delta_S}\right)$ is surjective. First, by the definition of W as the cokernel in (3.4), we have the exact sequence

$$\bigoplus_{y\in S} H^0\left(\nu_y^* N_{Z/\widetilde{\mathbb{P}}}\right) \to H^0\left(N_{Z/\widetilde{\mathbb{P}}}\big|_{\delta_S}\right) \to H^0(W) \to 0. \tag{3.12}$$

Second, we have the sequence

$$0 \to \nu_X^* N_{Z/\widetilde{\mathbb{P}}}(-\Gamma) \to \nu_X^* N_{Z/\widetilde{\mathbb{P}}} \to \nu_X^* N_{Z/\widetilde{\mathbb{P}}}\big|_\Gamma \to 0.$$

Third, by the definition of K in (3.7), we have the sequence

$$0 \to K \to \phi_* \nu_X^* N_{Z/\widetilde{\mathbb{P}}}(-\Gamma) \to W \to 0.$$

By Lemma 3.8, we have $H^1(K)=0$. Therefore, we also have $H^1\left(\nu_X^* N_{Z/\widetilde{\mathbb{P}}}(-\Gamma)\right)=0$. By considering the surjections on global sections induced by the two previous exact sequences, we get the surjection

$$H^0\left(\nu_X^* N_{Z/\widetilde{\mathbb{P}}}\right) \to H^0\left(\nu_X^* N_{Z/\widetilde{\mathbb{P}}}\big|_{\Gamma}\right) \oplus H^0(W). \tag{3.13}$$

By combining (3.12) and (3.13), we get a surjection

$$H^0\left(\nu^* N_{Z/\widetilde{\mathbb{P}}}\right) \to H^0\left(\nu^* N_{Z/\widetilde{\mathbb{P}}}\big|_{\Gamma}\right) \oplus H^0\left(N_{Z/\widetilde{\mathbb{P}}}\big|_{\delta_S}\right). \tag{3.14}$$

Since $H^0\left(\nu^* N_{Z/\widetilde{\mathbb{P}}}\big|_{\Gamma}\right) \to H^0\left(N_{Z/\widetilde{\mathbb{P}}}|_{\gamma}\right)$ is surjective, we get that

$$H^0\left(\nu^* N_{Z/\widetilde{\mathbb{P}}}\right) \to H^0\left(N_{Z/\widetilde{\mathbb{P}}}\big|_{\gamma\cup\delta_S}\right)$$

is surjective. We conclude that $H^1\left(N_{Z/\widetilde{\mathbb{P}}}\right)=0$. □

We now prove that enlarging S does not create any problems. Let $S^+ = S\cup\{y\}$, where $y\in Y\setminus S$ is a point over which $X\to Y$ is étale. Denote by the superscript + the analogues for S^+ of all the constructions done for S.

Proposition 3.9. *Suppose we have $H^1(N_{Z/\widetilde{\mathbb{P}}})=0$. Then we also have $H^1(N_{Z^+/\widetilde{\mathbb{P}}^+})=0$.*

Proof. By construction, we have $Z^+ = Z\cup R_y$. We have the exact sequence

$$0\to N_{Z/\widetilde{\mathbb{P}}} \to N_{Z^+/\widetilde{\mathbb{P}}^+}\big|_Z \to F^+|_{\delta_y} \to 0.$$

The long exact sequence on cohomology implies that

$$H^1\left(N_{Z^+/\widetilde{\mathbb{P}}^+}\big|_Z\right)=0, \tag{3.15}$$

and

$$H^0\left(N_{Z^+/\widetilde{\mathbb{P}}^+}\big|_Z\right) \to H^0\left(F^+\big|_{\delta_y}\right) \tag{3.16}$$

is surjective. By Lemma 3.5, the map

$$H^0\left(V^+\big|_{R_y}\right) \to H^0\left(V^+\big|_{\delta_y}\right) \tag{3.17}$$

is surjective. Let $\mu\colon Z\sqcup R_y \to Z^+$ denote the partial normalization. Recall that $V^+\big|_{R_y}$ is a summand of $N_{Z^+/\widetilde{\mathbb{P}}^+}\big|_{R_y}$ and the map $V^+\to F^+$ is zero. Thus, the surjections (3.16) and (3.17) together give a surjection

$$H^0\left(\mu^* N_{Z^+/\widetilde{\mathbb{P}}^+}\right) \to H^0\left(N_{Z^+/\widetilde{\mathbb{P}}^+}\big|_{\delta_y}\right).$$

Lemma 3.5 and (3.15) together imply that

$$H^1\left(\mu^* N_{Z^+/\widetilde{\mathbb{P}}^+}\right)=0.$$

The proposition now follows from the long exact sequence on cohomology associated to

$$0\to N_{Z^+/\widetilde{\mathbb{P}}^+} \to \mu_*\mu^* N_{Z^+/\widetilde{\mathbb{P}}^+} \to N_{Z^+/\widetilde{\mathbb{P}}^+}\big|_{\delta_y} \to 0.$$

□

Proposition 3.10. *Suppose the size n of S is large, its points are general, and the rational normal curves R_y are general. Then*

(1) *the Hilbert scheme of $\widetilde{\mathbb{P}}$ is smooth at $[Z]$;*
(2) *Z is a flat limit of smooth curves in $\widetilde{\mathbb{P}}$.*

Furthermore, if n is sufficiently large, then the set S can be chosen so that $\mathcal{O}_Y(S)$ is isomorphic to any prescribed line bundle of degree n on Y.

Proof. Since $H^1(N_{Z/\widetilde{P}}) = 0$, we conclude that the Hilbert scheme of $\widetilde{P}$ is smooth at $[Z]$, proving (1). In particular, every first order deformation of $Z \subset \widetilde{P}$ extends to a deformation over the germ of a smooth curve. To show that Z is the limit of smooth curves, it suffices to show that for every node $p \in Z$, the natural map $N_{Z/\widetilde{P}} \to k_p$ is surjective on global sections, where k_p is a skyscraper sheaf at p. Recall that Z has two kinds of nodes: the nodes γ, which are the nodes of X; and the nodes δ_S, which are the nodes introduced because we attached the rational normal curves.

Consider first the nodes γ. Let $\mu \colon Z^\mu \to Z$ be the partial normalization at these nodes. Let $I_\gamma \subset \mathcal{O}_Z$ be the ideal sheaf of $\gamma \subset Z$. It is easy to see that we have the equality

$$N_{Z/\widetilde{P}} \otimes I_\gamma = \mu_* \left(\mu^* N_{Z/\widetilde{P}}(-\Gamma)\right).$$

Thus, if $\nu \colon Z^\nu \to Z$ is the full normalization, we get the sequence

$$0 \to N_{Z/\widetilde{P}} \otimes I_\gamma \to \nu_* \nu^* N_{Z/\widetilde{P}} \to \nu_* \left(\nu^* N_{Z/\widetilde{P}}\big|_\Gamma \right) \oplus N_{Z/\widetilde{P}}\big|_{\delta_S} \to 0.$$

In the proof of Proposition 3.4, we showed that the map

$$\nu_* \nu^* N_{Z/\widetilde{P}} \to \nu_* \left(\nu^* N_{Z/\widetilde{P}}\big|_\Gamma \right) \oplus N_{Z/\widetilde{P}}\big|_{\delta_S}.$$

is surjective; see (3.14). Therefore, we get that $H^1(N_{Z/\widetilde{P}} \otimes I_\gamma) = 0$. This, in turn, implies that $H^0(N_{Z/\widetilde{P}}) \to H^0(N_{Z/\widetilde{P}}|_\gamma)$ is surjective. In particular, we get the much weaker statement that $H^0(N_{Z/\widetilde{P}}) \to H^0(k_p)$ is surjective for all $p \in \gamma$.

Consider next a node $p \in \delta_S$. The surjection $N_{Z/\widetilde{P}}|_p \to k_p$ is a part of the exact sequence

$$0 \to T_p \widetilde{P} / T_p Z \to N_{Z/\widetilde{P}}|_p \to k_p \to 0.$$

Note that the kernel here is exactly $V|_p$. Therefore, the map $N_{Z/\widetilde{P}} \to k_p$ is the same as the map $N_{Z/\widetilde{P}} \to F_p$. Set $S^- = S \setminus \{y\}$. Denote by the superscript $-$ the analogous objects for S^-. By Lemma 3.5, we have a surjection

$$H^0\left(N_{Z/\widetilde{P}}\big|_{R_y}\right) \to H^0(F_p). \tag{3.18}$$

Let $\mu \colon Z^- \sqcup R_y \to Z$ be the partial normalization of the nodes δ_y. We may assume that S^- is already large and generic enough so that $H^1\left(N_{Z^-/\widetilde{P}^-}\right) = 0$. In the proof of Proposition 3.9, we showed that the map

$$H^0\left(\mu^* N_{Z/\widetilde{P}}\big|_{Z^-}\right) \to H^0\left(F|_{\delta_y}\right)$$

is surjective; see (3.16). In particular, we get a surjection

$$H^0\left(\mu^* N_{Z/\widetilde{P}}\big|_{Z^-}\right) \to H^0\left(F_p\right). \tag{3.19}$$

Combining (3.18) and (3.19), we see that $H^0\left(N_{Z/\widetilde{P}}\right) \to H^0(F_p)$ is surjective. We have thus taken care of both types of nodes, proving (2).

It remains to prove the last statement about $\mathcal{O}_Y(S)$. For that, assume that n is large enough so that the conclusions above hold for a generic S of size $n - 2g_Y$. Then we may enlarge S to a set S^+ by adding an appropriate set of $2g_Y$ points so that the same conclusions hold and $\mathcal{O}_Y(S^+)$ is isomorphic to a given line bundle of degree n. □

We now prove the key proposition.

Proof of Proposition 3.3. By Proposition 3.10, there exists a family of smooth curves in $\widetilde{P}$ whose flat limit is Z. Let X' be a general member of such a family. This curve satisfies the following conditions (see Figure 4):

(1) $\deg(X' \cdot E_y) = d-1$ for all $y \in S$,
(2) $\deg(X' \cdot P_y) = 1$ for all $y \in S$,
(3) $X' \cap \widetilde{H} = \varnothing$,
(4) $g(X') = g(X) + n(d-1)$,
(5) $H^1\left(N_{X'/\widetilde{P}}\right) = 0$.

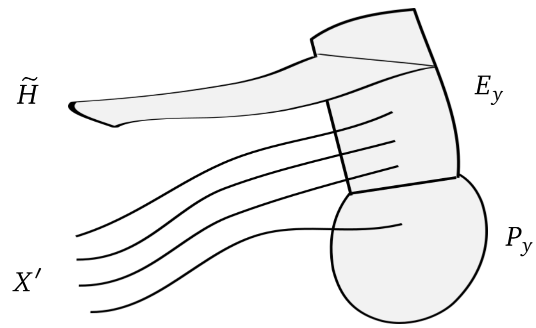

FIGURE 4. A smoothing X' of X union a large number of general rational normal curves

Let $\widetilde{P} \to \mathbf{P}'$ be the blowing down of all the P_y for $y \in S$. Then $\mathbf{P}' \to Y$ is a $\mathbf{P}^{d-1}$ bundle and the map $X' \to \mathbf{P}'$ is an embedding. Similarly, $\widetilde{H} \to \mathbf{P}'$ is also an embedding and its complement is the total space of the vector bundle $E' = E \otimes \mathcal{O}_Y(S)$. Note that X' and $\widetilde{H}$ remain disjoint in $\mathbf{P}'$, and hence we get an embedding $X' \subset E'$.

We claim that $X' \subset E'$ is the canonical affine embedding. To see this, consider the natural map

$$\phi_* \mathcal{O}_{\mathbf{P}'}(\widetilde{H}) = \mathcal{O}_Y \oplus E'^{\vee} \to \phi_* \mathcal{O}_{X'}.$$

For a generic $y \in Y$, the fiber $X_y \subset \mathbf{P}'_y$ is non-degenerate, and so the map above is an isomorphism. But the vector bundles $\mathcal{O}_Y \oplus E'^{\vee}$ and $\phi_* \mathcal{O}_{X'}$ have the same degree and rank. Therefore, the map must be an isomorphism for all $y \in Y$. As a result, we get that E' is the Tschirnhausen bundle of X', and the embedding $X' \to E'$ is the canonical affine embedding.

Next, note that we have an injection

$$N_{X'/\widetilde{P}} \to N_{X'/E'}$$

with finite quotient, supported on $\bigcup_{y \in S} X' \cap P_y$. Since $H^1\left(N_{X'/\widetilde{P}}\right) = 0$, we get $H^1\left(N_{X'/E'}\right) = 0$.

Finally, by the last assertion of Proposition 3.10, we may take $\mathcal{O}_Y(S)$ to be any prescribed line bundle of degree n if n is large enough. □

3.3. **The general case.** We now use the results of § 3.1 and § 3.2 to deduce the main theorem. Denote by $H_{d,g}(Y)$ the moduli stack of degree d and genus g branched covers of Y. Set $b = g - 1 - d(g_Y - 1)$. Denote by $\mathrm{Vec}_{d-1,b}(Y)$ the moduli stack of vector bundles of rank $d-1$ and degree b on Y. The stack $H_{d,g}(Y)$ is Deligne–Mumford and of finite type and the stack $\mathrm{Vec}_{d-1,b}(Y)$ is Artin and locally of finite type. Both stacks are smooth. The rule $\phi \mapsto E_\phi$ gives a morphism

$$\tau : H_{d,g}(Y) \to \mathrm{Vec}_{d-1,b}(Y).$$

We will only need a local analysis of this morphism. So the use of stacks can be avoided by phrasing everything in terms of miniversal deformation spaces, if so desired.

Theorem 3.11. *Let E be a vector bundle on Y. There exists n (depending on E) such that for any line bundle L of degree at least n, there exists a smooth curve X and a finite flat morphism $\phi: X \to Y$ such that $E_\phi \cong E \otimes L$ and such that the map τ is smooth at ϕ.*

Proof. We begin by analyzing the map τ on first order deformations. Let $[\phi: X \to Y]$ be a point of $H_{d,g}(Y)$ and set $E = E_\phi$. The space of first order deformations of ϕ is given by

$$\operatorname{Def}_\phi = H^0(X, N_\phi),$$

where $N_\phi = \operatorname{coker}(T_X \to \phi^* T_Y)$. The space of first order deformations of E is given by

$$\operatorname{Def}_E = H^1(Y, \operatorname{End} E).$$

Consider the canonical affine embedding $X \subset E$. We have an exact sequence

$$0 \to T_{E/Y}\big|_X \to N_{X/E} \to N_\phi \to 0.$$

Note that $T_{E/Y}\big|_X = \phi^* E$. The long exact sequence on cohomology gives a map

$$H^0(X, N_\phi) \to H^1(X, \phi^* E) = H^1(Y, E \oplus \operatorname{End} E).$$

By composing with the projection $H^1(Y, E \oplus \operatorname{End} E) \to H^1(Y, \operatorname{End} E)$, we get a map

$$H^0(X, N_\phi) \to H^1(Y, \operatorname{End} E).$$

It is straightforward to check that this is the map on the first order deformations induced by τ. In particular, if $H^1(X, N_{X/E}) = 0$, then τ is surjective on first order deformations and hence smooth at ϕ.

Choose an isotrivial degeneration E_0 of E of the form

$$E_0 = L_1 \oplus \cdots \oplus L_{d-1},$$

where the L_i's are line bundles with $\deg L_i + (2g_Y - 1) \leq \deg L_{i+1}$. Such a degeneration exists by Proposition 2.6. After replacing E by $E \otimes \lambda$ for a line bundle λ of large degree, we may also assume that $\deg L_1 \geq 2g_Y - 1$. By Proposition 3.1, there exists a nodal curve X_0 and a finite flat morphism $\phi_0 : X_0 \to Y$ with Tschirnhausen bundle E_0. By the key proposition (Proposition 3.3), there exists n such that for any line bundle L of degree at least n, there exists a smooth curve X' and a map $\phi' : X' \to Y$ with Tschirnhausen bundle $E' = E_0 \otimes L$. Furthermore, we also know that $H^1(N_{X'/E'}) = 0$. By our analysis above, we conclude that τ is smooth at ϕ'.

We know that $E \otimes L$ isotrivially degenerates to $E' = E_0 \otimes L$. We now use the openness of smooth maps to deduce that there exists $\phi : X \to Y$ with Tschirnhausen bundle $E \otimes L$ and such that τ is smooth at ϕ. □

Set $b = g - 1 - d(g_Y - 1)$. Denote by $M_{d-1,b}(Y)$ the (coarse) moduli space of semi-stable vector bundles of rank $(d-1)$ and degree b on Y. Theorem 3.11 yields the following.

Corollary 3.12. *Let $g(Y) \geq 2$. If g is sufficiently large (depending on Y and d), then the Tschirnhausen bundle of a general degree d and genus g cover of Y is stable. Moreover, the rational map $H_{d,g}(Y) \dashrightarrow M_{d-1,b}(Y)$ given by $\phi \mapsto E_\phi$ is dominant.*

The same statement holds for $g(Y) = 1$ with "stable" replaced by "regular poly-stable."

Proof. Let $g(Y) \geq 2$; the proof for $g(Y) = 1$ is identical with "stable" replaced by "regular poly-stable." By Theorem 3.11, there exists a cover ϕ such that $\tau : H_{d,g}(Y) \to \operatorname{Vec}_{d-1,b}(Y)$ is smooth, and hence open at ϕ. Since stability is an open condition and $\operatorname{Vec}_{d-1,b}(Y)$ is irreducible, the image of τ contains a stable bundle. The dominance of $H_{d,g}(Y) \dashrightarrow M_{d-1,b}(Y)$ also follows from the openness of τ at ϕ. $\square$

Recall that the Maroni locus $M(E) \subset H_{d,g}(Y)$ is defined by

$$M(E) = \{\phi \in H_{d,g}(Y) \mid E_\phi \cong E\}.$$

Corollary 3.13. *Let E be a vector bundle on Y of rank $(d-1)$ and degree e. If g is sufficiently large (depending on Y and E), then for every line bundle L on Y of degree $b-e$, the Maroni locus $M(E\otimes L)$ contains an irreducible component of the expected codimension $h^1(\operatorname{End} E)$.*

Proof. Let U be the open subset of the Hilbert scheme of curves in $P = \mathbf{P}(E^\vee \otimes L^\vee \oplus \mathcal{O}_Y)$ of genus g, of degree d over Y, that are smooth and disjoint from the hyperplane at infinity $\mathbf{P}(E^\vee)$. Every $[X] \in U$ gives $\phi : X \to Y$ with Tschirnhausen bundle $E \otimes L$. Furthermore, the map

$$U \to M(E \otimes L)$$

is surjective with fibers isomorphic to $\operatorname{Aut}(P/Y)$. The normal bundle $N_{X/P}$ is a vector bundle of rank $(d-1)$ and degree $(d+2)b$.

By the key proposition Proposition 3.3, there exists $[X] \in U$ with $H^1(N_{X/P}) = 0$. Then the dimension of U at $[X]$ is given by

$$\dim_{[X]} U = \chi(N_{X/P}) = d^2 + 2d + 3g - 3.$$

We may assume $\deg L$ to be large enough so that $H^1(E \otimes L) = 0$ and $H^0(E^\vee \otimes L^\vee) = 0$. Then

$$\dim \operatorname{Aut}(P/Y) = d^2 + g - 1 + h^1(\operatorname{End} E).$$

It follows that the component of $M(E \otimes L)$ containing $[\phi : X \to Y]$ has dimension

$$2g + 2d - 2 - h^1(\operatorname{End} E) = \dim H_{d,g} - h^1(\operatorname{End} E).$$

$\square$

4. Higher dimensions

In this section, we discuss the possibility of having an analogue Theorem 1.1 for higher dimensional Y. Let us begin with the following question.

Question 4.1. *Let Y be a smooth projective variety, L an ample line bundle on Y, and E a vector bundle of rank $(d-1)$ on Y. Is $E \otimes L^n$ a Tschirnhausen bundle for all sufficiently large n?*

It is simple to see that the answer to Question 4.1 is “No”, at least without additional hypotheses.

Example 4.2. Take $Y = \mathbf{P}^4$, and $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$. Then a sufficiently positive twist E' of E cannot be the Tschirnhausen bundle of a smooth branched cover X.

To see this, recall that the data of a Gorenstein triple cover $X \to Y$ with Tschirnhausen bundle E' is equivalent to the data of a nowhere vanishing global section of $\operatorname{Sym}^3 E' \otimes (\det E')^\vee$; see [4]. For $E' = E \otimes L^n$ with large n, the rank 4 vector bundle $\operatorname{Sym}^3 E' \otimes (\det E')^\vee$ is very ample. Thus, its fourth Chern class is nonzero. Therefore, a general global section must vanish at some points.

In fact, it is easy to see by direct calculation that the fourth Chern class of $\operatorname{Sym}^3 E \otimes (\det E)^\vee$ can vanish if and only if $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$ where $b = 2a$. Conversely, $E = \mathcal{O}(a) \oplus \mathcal{O}(2a)$ is the Tschirnhausen bundle of a cyclic triple cover of $\mathbf{P}^4$. Thus, $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$ can be a Tschirnhausen bundle of a smooth triple cover of $\mathbf{P}^4$ if and only if $b = 2a$.

Example 4.2 illustrating the failure of Theorem 1.1 can be generalized to all degrees ≥ 3, provided the base Y is allowed to be high dimensional.

Proposition 4.3. *Let $d \geq 3$. The answer to Question 4.1 is "No" for all Y of dimension at least $d\binom{d}{2}$.*

Proof. Let $\phi : X \to Y$ be a finite, flat, degree d map. Then the sheaf $\phi_* \mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$-algebras, and it splits as $\phi_* = \mathcal{O}_Y \oplus E^\vee$.

Suppose over some point $y \in Y$, the multiplication map

$$m : \operatorname{Sym}^2 E^\vee \to \phi_* \mathcal{O}_X$$

is identically zero. Then, we have a k-algebra isomorphism

$$(\phi_* \mathcal{O}_X)|_y \cong k[x_1, ..., x_{d-1}]/(x_1, \ldots, x_{d-1})^2.$$

That is, $\phi^{-1}(y)$ is isomorphic to the length d "fat point", defined by the square of the maximal ideal of the origin in an affine space. When $d \geq 3$, these fat points are not Gorenstein. Since Y is smooth, this implies X can not even be Gorenstein, let alone smooth.

Now, if E is a vector bundle on Y and L is a sufficiently positive line bundle, then the bundle

$$M := \operatorname{Hom}(\operatorname{Sym}^2(E \otimes L)^\vee, \mathcal{O}_Y \oplus (E \otimes L)^\vee)$$

is very ample. A general global section $m \in H^0(Y, M)$ will vanish identically at some points $y \in Y$ provided

$$\dim Y \geq \operatorname{rk} M = d\binom{d}{2}.$$

We conclude that if $\dim Y \geq d\binom{d}{2}$, then Question 4.1 has a negative answer. □

Observe that Proposition 4.3 remains true even if we relax the requirement that X be smooth to X be Gorenstein.

The following result due to Lazarsfeld suggests the possibility that Proposition 4.3 may be true with a much better lower bound than $d\binom{d}{2}$.

Proposition 4.4. *Let E be a vector bundle of rank $(d-1)$ on $\mathbf{P}^r$, where $r \geq d+1$. Then $E(n)$ is not a Tschirnhausen bundle of a smooth, connected cover for sufficiently large n.*

Proof. The proof relies on [11, Proposition 3.1] which states that for a smooth branched cover $\phi : X \to \mathbf{P}^r$ of degree $d \leq r-1$, the pullback map

$$\phi^* : \operatorname{Pic}(\mathbf{P}^r) \to \operatorname{Pic} X$$

is an isomorphism. In particular, the dualizing sheaf ω_ϕ is isomorphic to $\phi^* \mathcal{O}(l)$ for some l. Therefore, we get

$$\mathcal{O}_{\mathbf{P}^r} \oplus E = \phi_* \omega_\phi = \phi_* \mathcal{O}(l) = \mathcal{O}_{\mathbf{P}^r}(l) \oplus E^\vee(l).$$

This implies that $\mathcal{O}_{\mathbf{P}^r}(l)$ is a summand of E.

Suppose $E(n)$ is a Tschirnhausen bundle of a smooth connected cover for infinitely many n. Applying the reasoning above with E replaced by $E(n)$ shows that E must have line bundle summands of infinitely many degrees. Since this is impossible, the proposition follows. □

The reasoning in Example 4.2 implies the following.

Proposition 4.5. *For degree 3, Question 4.1 has an affirmative answer if and only if* $\dim Y < 4$.

Proof. Let $\phi : X \to Y$ be a Gorenstein finite covering of degree 3 with Tschirnhausen bundle E. Then $X \subset \mathbf{P}E$ is a divisor of class $\mathcal{O}_{\mathbf{P}E}(3)$; see [4]. Thus, X is given by a global section on $\mathbf{P}E$ of $\mathcal{O}_{\mathbf{P}E}(3)$, or equivalently a global section on Y of $\operatorname{Sym}^3 E \otimes \det E^\vee$. Note that since $X \to Y$ is flat, the global section of $\operatorname{Sym}^3 E \otimes \det E^\vee$ is nowhere vanishing.

Suppose we are given an arbitrary rank 2 vector bundle E on Y. Set $D = \mathcal{O}_{\mathbf{P}E}(3)$ and $V = \operatorname{Sym}^3 E \otimes \det E^\vee$. If we twist E by L^n, then $\mathbf{P}E$ is unchanged but D changes to $D + 3nL$ and V

changes to $V \otimes L^n$. For sufficiently large n, the bundle $V \otimes L^n$ is ample. If $\dim Y < 4$, then a general section of $V \otimes L^n$ is nowhere zero on Y. Furthermore, the divisor $X \subset \mathbf{P}E$ cut out by the corresponding section of $\mathcal{O}(D+3nL)$ is smooth by Bertini's theorem. By construction, the resulting $X \to Y$ has Tschirnhausen bundle $E \otimes L^n$.

On the other hand, if $\dim Y \geq 4$, then every global section of $V \otimes L^n$ must vanish at some point in Y. Thus, $E \otimes L^n$ cannot arise as a Tschirnhausen bundle. □

4.1. **Modifications of the original question.** Following the discussion in the previous section, natural modified versions of Question 4.1 emerge. The first obvious question is the following.

Question 4.6. *Is the analogue of Theorem 1.1 true for all Y with $\dim Y \leq d$?*

We can also relax the finiteness assumption on ϕ.

Question 4.7. *Let Y be a smooth projective variety, E a vector bundle in Y. Is E isomorphic to $(\phi_* \mathcal{O}_X / \mathcal{O}_Y)^\vee$, up to a twist, for a* generically *finite map $\phi : X \to Y$ with smooth X?*

Remark 4.8. A similar question is addressed in work of Hirschowitz and Narasimhan [7], where it is shown that any vector bundle on Y is the direct image of *some* line bundle on a smooth variety X under a generically finite morphism.

Alternatively, we can keep the finiteness requirement on ϕ in exchange for the smoothness of X. We end the paper with the following open-ended question.

Question 4.9. *What singularity assumptions on X (or the fibers of ϕ) yield a positive answer to Question 4.1?*

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