Theorem on dimensions of fibers

$\varphi : X \to Y$ map between irreducible quasi proj. varieties.

f surjective

$\dim Y = m$ $\qquad X_y := \varphi^{-1}(y)$

$\dim X = n$

Thm (Dim of fibers)

① $m \leq n$

② For every $y \in Y$,

$$\dim X_y \geq n-m.$$

③ $\exists$ open $U \subset Y$ s.t. $\forall\, y \in U$

$$\dim X_y = n-m.$$

For the proof we need some preparation.

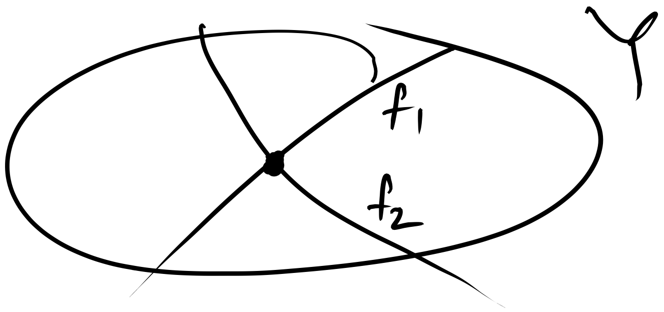
1. <u>Slicing by hyperplanes</u>.

Let Y be affine of dim m & $y \in Y$.

Then there exist $f_1, \dots, f_m \in k[Y]$

s.t. $V(f_1, \dots, f_m) \subset Y$ is finite &

$y \in V(f_1, \dots, f_m)$.



2. <u>Chevalley's thm</u>

$f: X \to Y$ dominant map of irred. var.

$\Rightarrow$ $f(X)$ contains an open subset of Y

(<u>Pf</u> <u>skipped</u> – Black box.)

(BUT we won't even use this!

I am keeping it just as useful general knowledge.

Pf of theorem : 1 We have seen already.

2 Suffices to take Y affine. Take $y \in Y$

Now $\exists\ f_1, \dots, f_m \in k[Y]$ s.t.

$q \in V(f_1, \dots, f_m)$ & $V(f_1, \dots, f_m)$ is finite.

By shrinking Y, assume

$$V(f_1, \dots, f_m) = \{y\}.$$

Then $$X_y = X \cap \left\{ \begin{array}{c} \phi^*(f_1) = 0 \\ \vdots \\ \phi^*(f_m) = 0 \end{array} \right\}$$

By the principal ideal thm,

$$\dim X_y \geq n - m.$$

3 Take $V \subset X$ open affine

We'll show that $\exists$ non ϕ open $U_V \subset X$

s.t. $\forall\ y \in U_V$, the fiber

V_y is (either ϕ or) $n-m$ dim.

By taking $U = \cap U_V$ as V ranges over a finite open cover of X, we get the theorem.

Now, consider $V \to Y$.
The idea of the proof is easy.
Write $k[V] = k[Y][t_1, \dots, t_N] / I$

Since $\operatorname{trdeg}_k k(V) = m$

$\operatorname{trdeg}_k k(Y) = n$

$\operatorname{trdeg}_{k(V)} k(Y) = n-m$.

So, wlog, assume $t_1, \dots, t_r \in k[V]$ are alg. indep over $k(Y)$ and $\exists$
$P_i \in k[Y][t_1, \dots, t_{n-m}, t_i]$
not identically 0 such that
$P_i(t_1, \dots, t_{n-m}, t_i) \equiv 0$ on V.
(so $P_i(t_1, \dots, t_{n-m}, t_i) \in I$).

Think of P_i as a polynomial in the t-variables with coeff in $k[Y]$.
Given $y \in Y$, we can evaluate

these coeff. at y & get a polynomial

$$ev_y(P_i) \in k[t_1, \dots, t_{n-m}, t_i].$$

Let $U \subset Y$ be the open set where $ev_y(P_i)$ is not the zero polynomial $\forall i$.

For $y \in Y$, consider $k[Z]$ where Z is a component of V_y.

Then $k[Z]$ is a quotient of $k[V_y]$

And $k[V_y]$ is

$$k[V_y] = k[Y][t_1, \dots, t_N] \Big/ \sqrt{(I + I(y))}$$

$$\cong k[t_1, \dots, t_N] \Big/ \sqrt{ev_y(f) \text{ for } f \in I}$$

$$= \text{A quotient of } k[t_1, \dots, t_N] \Big/ (ev_y(P_i) \text{ for } i = n-m+1, \dots, N)$$

So if $V_y \neq \emptyset$ & if $Z \subset V_y$ is an irr. comp., then $k(Z)$ is gen. over k by the images of $t_1, \dots, t_N$ and $t_{n-m+1}, \dots, t_N$ are algebraic over the subfield gen by the images of $t_1, \dots, t_{n-m}$. Indeed, t_i for $i > 0$ satisfies the <u>non-zero</u> polynomial

$$ev_y(P_i) \in k[t_1, \dots, t_{n-m}, t_i]$$

on Z.

so $\dim Z \leq n-m$.

$\square$

What if X or Y is not irreducible. Then the previous theorem does not apply, but we can work on individual irred. components as a way out. But to guarantee that the image of a component of X is a closed subvar of Y, we assume that $\varphi: X \to Y$ is closed.

This assumption is omitted in Shafarevich, so its corollary to Thm ...... is wrong.

Setup: X, Y quasi-proj. $\varphi: X \to Y$ surjective & closed. Let

$$X = X_1 \cup \cdots \cup X_e$$

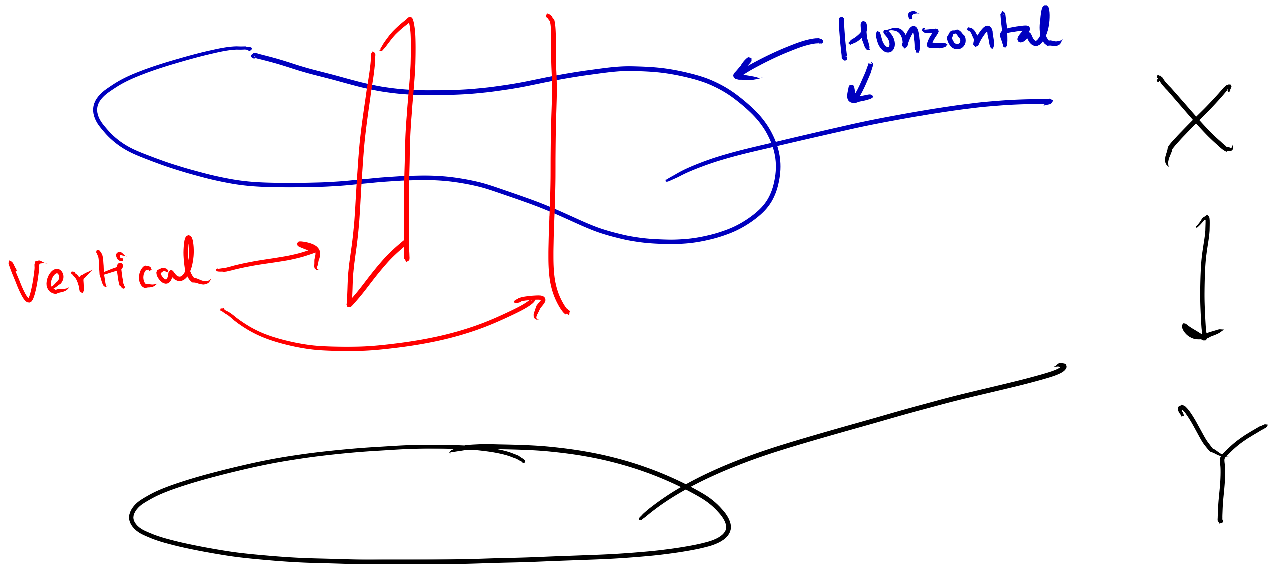
be the decomposition of X into irreducible components. Let $Y_i = \varphi(X_i)$. Then Y_i is a closed and irreducible subset of Y.

There are two cases –

1. $Y_i \subset Y$ is an irred. component
2. Y_i is strictly contained in an irred component.

In case ①, say X_i is "horizontal"

② —— " —— "vertical"



Set relative dim (X_i/Y_i)
$$= \dim X_i - \dim Y_i$$

Let $r = \min_{X_i \text{ horizontal}} (\text{rel dim}(X_i/Y_i))$

Applying the theorem on dim of fibers, we get —

Thm: ① $\forall\, y \in Y$, $\dim \varphi^{-1}(y) \geq r$.

② $\exists$ non ϕ open $U \subset Y$ s.t. $\forall y \in U$
$$\dim \varphi^{-1}(y) = r$$

Note: U may not be dense.

<u>Thm</u> (Semicontinuity). Let $l \in \mathbb{Z}$.

The set

$$Z_l = \{y \in Y \mid \dim X_y \geq l\} \subset Y$$

is closed.

<u>Pf</u>: We induct on $\dim Y$. $\dim Y = 0$ is easy.

We may assume Y is irreducible.

(If not, work on $\varphi^{-1}(Z) \to Z$ for every irred comp. $Z \subset Y$).

If $l \leq r$, then $Z_l = Y$.

Otherwise, by the previous theorem,

$\exists$ $Z \subsetneq Y$ closed s.t. $Z_l \subset Z$.

But then $\dim Z < \dim Y$, so the statement is true for the map

$$\varphi^{-1}(Z) \to Z.$$

But Z_l is the same for this map & the original map. So we get

$\underset{\text{closed}}{Z_l \subset Z}$ & hence $Z_l \subset Y$ closed

$\square$

Another useful fact:

<u>Thm</u>: $\varphi: X \to Y$ surj & closed map of q.proj var. If Y is irred & X_y is irred of the same dim $\forall y \in Y$, then X is irred of dim $= \dim Y + \dim X_y$.

<u>Pf</u>: We'll prove the contrapositive. Suppose X is reducible.

Suppose X has two horizontal comp. X_1 and X_2. Suppose $U \subset Y$ is a nonempty open over which there are no vertical components. Also suppose $\forall y \in U$ that $X_{1,y}$ & $X_{2,y}$ are disjoint. (To get such U, take the complement of $\varphi(X_1 \cap X_2) \cup \varphi(\text{vertical comps})$ in Y). But then $\forall y \in U$, the fiber $X_y = X_{1,y} \sqcup X_{2,y} \cup \ldots$ is reducible.

Suppose X has a vertical component X_1 and a horizontal component X_2.

Since $X_1 \subsetneq X_2$, $\exists\, y \in Y$ such that $X_{1,y} \subsetneq X_{2,y}$. But then

$$X_y = X_{2,y} \cup X_{1,y} \cup \dots$$

Now $\dim X_{2,y} \geq \text{rel dim}(X_2/Y)$.

So either $\dim X_y > \text{rel dim}(X_2/Y)$ or $\dim X_y = \text{rel dim}(X_2, Y)$ & $X_{2,y}$ is an irred. comp of X_y, but not all of X_y. So X_y is reducible.

$\square$.

As an application, consider

$$\mathbb{P} = \mathbb{P}\mathrm{Mat}_{(m \times n)} \qquad \text{for } m \leq n$$

& $\Delta_r \subset \mathbb{P}$ the set of matrices of rank $\leq m-r$. Let's show Δ_r is irreducible & find its dimension.

Observe: $M \in \Delta_r$ iff $\exists$ subspace $V \subset K^m$ of dim r s.t. $MV = 0$.

So consider

$$D_r = \{ ([M],[v]) \mid Mv = 0 \}$$
$$\underset{\text{closed}}{\subset} \mathbb{P}\mathrm{Mat}_{m \times n} \times Gr(r,m).$$

[$Gr(r,m)$ = Set of r. dim subspaces of K^m

↪ irred proj var of dim $r(m-r)$

... coming soon]

Consider $D_r \to Gr(r,m)$.

Fiber over $[V] = \{ M \mid MV = 0 \}$

$= \{ M : K^m \to K^n \quad V \mapsto 0 \} / \text{scaling}$

$= \{ M : K^m/V \to K^n \} / \text{scaling}$

$\cong \mathbb{P}^{n(m-r)}$

So D_r is irred of dim $n(m-r) + r(m-r)$

$\Rightarrow \Delta_r$ is irred.

Observe $D_r \to \Delta_r$ is of rel dim 0.

To show this, it suffices to show <u>one</u> $M \in \Delta_r$ s.t. fiber of $D_r \to \Delta_r$

Over $[M]$ is 0-dim. This is easy.
Take any M of rank $= m-r$.

$\Rightarrow$ Δ_r is irred of dim $= \dim D_r$
$= (n+r)(m-r)$.

$\boxed{\text{The Grassmannian}}$

$Gr(r,m) =$ Set of r-dim sub. of K^m.

Let $U \subset \underbrace{K^m \times \dots \times K^m}_{r \text{ times}}$ be the set

$U = \{ (v_1, \dots, v_r) \mid v_i \text{ lin indep} \}$.

Then GL_r acts on U by linear combination & $Gr(r,m) = U/GL_r$ is the orbit space.

Topology of $Gr(r,m) =$ Quotient top.
i.e. $Z \subset Gr(r,m)$ is closed iff its preimage in U is closed in the Zariski topology on U.

More explicitly

$$U = \left\{ M = \underbrace{\begin{bmatrix} \times & \cdots & \\ & \ddots & \\ & \cdots & \times \end{bmatrix}}_{r} \Big\} m \text{ of rank } r \right\}$$

$\circlearrowright$

GL_r by right mult.

Since U is irred., $Gr(r,m)$ is also.

<u>Affine Charts</u>: Given $I \subset \{1, \cdots, m\}$ of size r, let

$$U_I = \{ M_{I \times r} \text{ is invertible} \} \subset U$$

let $W_I \subset Gr$ be the image of U_I.

Then $\{U_I\}$ forms an open cover of Gr.

For each point in W_I, we can choose a canonical M to represent it, namely the unique M such that

$$M_{I \times r} = id_{r \times r}.$$

(Given any other representative M', one can get M by $M = M' \cdot (M'_{I\times r})^{-1}$).

For simplicity of notation, let

$$I = \{1, 2, \dots, r\}.$$

Then the unique representatives of pts in W_I are

$$\begin{array}{cc} & \\ m-r \Big\{ & \left[\begin{array}{c} I_{r\times r} \\ \hline N \end{array}\right] \\ & \underbrace{\hphantom{I_{r\times r}}}_{r} \end{array}.$$

So we get a bijection

$$W_I \longleftrightarrow \mathbb{A}^{(m-r)\times r}$$

$$\left[\begin{array}{c} I_{r\times r} \\ \hline N \end{array}\right] \longleftrightarrow N$$

Claim: This is a homeomorphism with the Zariski top on $\mathbb{A}^{(m-r)\times r}$.

These homeomorphisms provide the charts for $Gr(r, m)$.

Exercise: Verify that the transition maps are regular.

Projectivity

Want an iso $Gr \to$ Closed subset of $\mathbb{P}^n$

Let $n = \binom{m}{r} - 1$; think of the homog. coord. X_I as indexed by r-elt subsets $I \subset \{1, \dots, m\}$.

Define

$$p : Gr(r,m) \to \mathbb{P}^n$$
$$[M] \mapsto [\det(M_{I \times r})]_I .$$

This map is well-defined.
i.e. does not depend on the representative M
& not all coordinates are 0.

Let us check that $Im(p)$ is closed and p is an iso. onto it. To check that $Im(p)$ is closed it suffices to check $Im(p) \cap \{\text{chart of } \mathbb{P}^n\}$ closed. Take the chart $X_I \neq 0 . \cong \mathbb{A}^n$.

Its preimage is

$$p^{-1}(X_I) = W_I .$$

For simplicity $I = \{1, \dots, r\}$ Using the special rep.
$\begin{bmatrix} I_{r \times r} \\ N \end{bmatrix}$, the map $p : W_I \to \mathbb{A}^n$ is

$$N \mapsto (N_{ij} ; \alpha(N_{ij}))$$

where $\alpha(N_{ij})$ is a tuple of some poly. functions on N_{ij}. To see this note that when we take $r \times r$ minors, the very first one is 1, which should be omitted in the chart $\{x_I \neq 0\}$.
The next ones, corresponding to $(r-1)$ rows from first r & one from below give the entries (N_{ij}). The rest are some polynomials in the N_{ij}.

$$\Rightarrow \quad Im(p) = \text{Graph of } \alpha$$
$$= \text{Closed} \subset \mathbb{A}^N$$

& $p : W_I \longrightarrow Im(p)$ is an iso.

$\square$