# HOW TO THINK ABOUT O(1) AND AMPLENESS?

NB: For simplicity ,all schemes here are separated and finite type over a field $k$.

Fix a hyperplane $H \subset \mathbb{P}^n$. It is easier to think about $O(H)$ than $O(1)$. The sections of $O(H)$ are rational functions with possible (simple) poles along $H$. For example, $O(H)$ has global sections, namely the rational functions $X_0/H, \ldots, X_n/H$. A different choice of $H$ gives an isomorphic sheaf $O(H)$; by $O(1)$, we mean an (unspecified) element in this isomorphism class.

It is likewise easy to understand $O(dH)$. Its sections are rational functions with at worst $d$-fold poles along $H$. For example, $O(dH)$ has global sections, namely the rational functions $F/H^d$, where $F$ is a homogeneous polynomial of degree $d$.

For a coherent sheaf $F$, we understand $F(d) = F \otimes O(d)$ as the sheaf whose sections are the sections of $F$ with at worst $d$-fold poles along $H$. That is, sections which can be expressed locally as $sf$, where $s$ is a section of $F$ and $f$ is a rational function with at worst $d$-fold poles along $H$.

A key property of this twisting is: for every $F$, there exists $d$ such that $F(d)$ is globally generated ("plenty of global sections"). This is not surprising at all. On an affine, every module is globally generated, and a finitely generated module is generated by a finite number of global sections. If we take the standard affine cover of $\mathbb{P}^n$, then each of these global sections may have a pole along the hyperplane at infinity. That is, when you express it in one of the other charts, it will have denominators. This means that, although it may not extend to a global section of $F$, it does extend to a global section of $F(d)$ for some $d$. By choosing a large enough $d$, we can make sure that there are plenty of global sections, and $F(d)$ is indeed globally generated.

This key property suggests a general definition. A line bundle (invertible sheaf) on a scheme $X$ is ample if for every coherent sheaf $F$, there exists a $d > 0$ such that $F \otimes L^d$ is globally generated. For example, $O(1)$ on $\mathbb{P}^n$ is ample. More generally, if $X \subset \mathbb{P}^n$ is a closed subscheme, then the restriction of $O(1)$ to $X$ is ample. This follows immediately from the ampleness of $O(1)$ on $\mathbb{P}^n$. Given a sheaf $F$ on $X$, you simply "regard it as a sheaf on $\mathbb{P}^n$", that is, consider the sheaf $i_*F$ where $i$ is the inclusion. Even more generally, if $X \subset \mathbb{P}^n$ is a locally closed subscheme, then also the restriction of $O(1)$ to $X$ is ample.

It turns out that, up to taking powers, these are the only examples. More precisely, if $L$ is an ample line bundle on $X$, then there exists an embedding of $X$ in $\mathbb{P}^n$ such that the restriction of $O(1)$ is $L^m$ for some $m > 0$. In particular, if $X$ admits an ample line bundle, then $X$ is quasi-projective.

We use twisting by an ample line bundle very frequently in algebraic geometry to "make things positive." This could mean many things, for example, getting lots of global sections,

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or for killing higher cohomology: given a projective scheme $X$ and an ample line bundle $L$, for every coherent sheaf $F$, we have $H^i(X, F \otimes L^n) = 0$ for $i > 0$ and $n$ large enough.

Let $X$ be projective, and $L$ be ample. Then $F \otimes L^n$ is globally generated if and only if we have a surjection $O_X^N \to F \otimes L^n$ for some $N$. (Choose global sections $s_1, \ldots, s_N$ that generate, and use them to construct the map $O_X^N \to F \otimes L^n$). Equivalently, we have a surjection from $L^{-n}$ direct sum $N$ times to $F$.