Algebraic geometry (Notes)

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1 Affine algebraic sets

1.1 Affine space WEEK1:

The objects of study in algebraic geometry are called algebraic varieties. The building blocks for general algebraic varieties are certain subsets of the affine space. Let us first recall affine space.

Let k be a field and let n be a non-negative integer. The *affine n-space over k*, denoted by $\mathbb{A}^n_k$ is the set of n-tuples $a_1, \dots, a_n$ whose entries a_i lie in k. Thus, $\mathbb{A}^n_k$ is nothing but the product k^n. The product k^n has quite a bit of extra structure—it is a k-vector space, for example—but we wish to forget it. That is the reason for choosing different notation. In particular, the zero tuple does not play a distinguished role.

1.2 Affine algebraic set WEEK1:

Let $k[x_1, \dots, x_n]$ denote the ring of polynomials in variables $x_1, \dots, x_n$ and coefficients in k. An *affine algebraic subset* of the affine space $\mathbb{A}^n_k$ is the common zero locus of a set of polynomials. More precisely, a set $S \subset \mathbb{A}^n_k$ is an affine algebraic subset if there exists a set of polysomials $A \subset k[x_1, \dots, x_n]$ such that

$$S = \{a \in \mathbb{A}^n_k \mid f(a) = 0 \text{ for all } f \in A\}.$$

1.2.1 Definition (Vanishing locus) Given $A \subset k[x_1, \dots, x_n]$, the *vanishing locus of A*, denoted by $V(A)$ is the set

$$V(A) = \{a \in \mathbb{A}^n_k \mid f(a) = 0 \text{ for all } f \in A\}.$$

— Thus the affine algebraic sets are precisely the sets of the form $V(A)$ for some A.

1.2.2 Examples/non-examples The following are affine algebraic sets

1. The empty set
2. Entire affine space
3. Single point

Proof. Done in class. □

The following are not affine algebraic sets

1. The unit cube in $\mathbb{A}^n_{\mathbb{R}}$
2. Points with rational coordinates in $\mathbb{A}^n_{\mathbb{C}}$

Proof. DIY. □

1.3 Ideals

WEEK1:

Let R be a ring. Recall that a subset $I \subset R$ is an *ideal* if it is closed under addition and multiplication by elements of R. Given any subset $A \subset R$ the *ideal generated by* A, denoted by $\langle A \rangle$ is the smallest ideal containing A. This ideal consists of all elements r of R that can be written as a linear combination

$$r = a_1 r_1 + \cdots + a_m r_m,$$

where $a_i \in A$ and $r_i \in R$.

1.3.1 Proposition Let $A \subset k[x_1, \ldots, x_n]$. Then we have $V(A) = V(\langle A \rangle)$.

Proof. Done in class. □

1.4 Noetherian rings and the Hilbert basis theorem

WEEK1:

In our definition of $V(A)$, the subset A may be infinite. But it turns out that we can replace it by a finite one without changing $V(A)$. This is a consequence of the Hilbert basis theorem, which, in turn, has to do with a fundamental property of rings.

We begin with a simple observation.

1.4.1 Proposition Let R be a ring. The following are equivalent

1. Every ideal of R is finitely generated.
2. Every infinite chain of ideals
$$I_1 \subset I_2 \subset I_3 \subset \cdots$$
stabilises.

Proof. — 1

1.4.2 Definition (Noetherian ring) A ring R satisfying the equivalent conditions of Proposition 1.4.1 is called *Noetherian*.

1.4.3 Examples/non-examples The following rings are Noetherian

1. $R = \mathbb{Z}$
2. R a field.

Proof. All ideals here can be generated by 1 element. □

The ring of continuous functions on the interval is *not* Noetherian. #+begin$_{\text{proof}}$. Let I_n be the set of functions on $[0,1]$ that vanish on $[0,1/n]$. This forms an increasing chain of ideals that does not stabilise. #+end$_{\text{proof}}$

1.4.4 Proposition (Quotients of Noetherian rings) If R is Noetherian and $I \subset R$ is any ideal, then R/I is Noetherian.

Proof. — 2

1.4.5 Theorem If R is Noetherian, then so is $R[x]$

- Proof Assume R is Noetherian, and let $I \subset R[x]$ be an ideal. We must show that I is finitely generated. The basic idea is to use the division algorithm, while keeping track of the ideals formed by the leading coefficients.

  For every non-negative integer m, define
  $$J_m = \{\text{Leading coeff}(f) \mid f \in I, f \neq 0, \quad \deg(f) \leq m\} \cup \{0\}$$
  We make the following claims.

  1. J_m is an ideal of R.
  2. $J_m \subset J_{m+1}$.

DIY.

Since R is Noetherian, the chain $J_1 \subset J_2 \subset \cdots$ stabilises; say $J_m = J_{m+1} = \cdots$. Let S_i be a finite set of generators for J_i, and for $a \in S_i$, let $p_a \in I$ be a non-zero element of degree at most i whose leading coefficient is a. We claim that the (finite) set $\{p_a \mid a \in S_1 \cup \cdots \cup S_m\}$ generates I.

Proof. Let $G = \{p_a \mid a \in S_1 \cup \cdots \cup S_m\}$. By construction, this is a subset of I, so the ideal it generates is contained in I. We remains to prove that every $f \in I$ is a linear combination of elements of G. It will be convenient to set $S_n = S_m$ for all $n \geq m$.

We induct on the degree of f (leaving the base case to you). Suppose the degree of f is n and the statement is true for elements of degree less than n. By construction, the leading coefficient of f is an R-linear combination of elements of S_n, say

$$\mathrm{LC}(f) = \sum c_i s_i.$$

Let n_i be the degree of p_{s_i}; then by construction $n_i \leq n$. Consider the linear combination $g = \sum c_i p_{s_i} x^{n-n_i}$. See that g lies in I, has degree n, the same leading coefficient as f, and is an $R[x]$-linear combination of elements of G. So $f - g \in I$ has lower degree. By inductive hypothesis, $f - g$ is an $R[x]$-linear combination of elements of G, and hence so is f. □

1.4.6 Corollary (Hilbert basis theorem) $k[x_1, \ldots, x_n]$ is Noetherian.

Proof. Induct on n. □

1.4.7 Corollary Every affine algebraic subset of $\mathbb{A}^n_k$ is the vanishing set of a finite set of polynomials.

Proof. Done in class. □

1.5 The Zariski topology

WEEK2:

The notion of affine algebraic sets allows us to define a topology on $\mathbb{A}^n_k$. Recall that we can specify a topology on a set by specifying what the open subsets are, or equivalently, what the closed subsets are. In our case, it is more convenient to do the latter. The collection of closed subsets must satisfy the following properties.

1. The empty set and the entire set are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

We define the *Zariski topology* on $\mathbb{A}^n_k$ by setting the closed subsets to be the affine algebraic sets, namely, the sets of the form $V(A)$ for some $A \subset k[x_1, \ldots, x_n]$.

1.5.1 Proposition The collection of affine algebraic subsets satisfies the three conditions above.

Proof. **The empty set and the entire set are closed.**

$$\begin{aligned}\emptyset &= \{\mathbf{a} \in \mathbb{A}_k^n : 1 = 0\} \\ &= V(\{1\})\end{aligned}$$

So the empty set is closed.

$$\begin{aligned}\mathbb{A}_k^n &= \{\mathbf{a} \in \mathbb{A}_k^n : 0 = 0\} \\ &= V(\{0\})\end{aligned}$$

So the entire set is closed.

Arbitrary intersections of closed sets are closed.

Let $\{V(A_\alpha)\}$ be a collection of closed sets.

$$\begin{aligned}\bigcap_\alpha V(A_\alpha) &= \bigcap_\alpha \{\mathbf{a} \in \mathbb{A}_k^n : p(\mathbf{a}) = 0 \text{ for all } p \in A_\alpha\} \\ &= \{\mathbf{a} \in \mathbb{A}_k^n : p(\mathbf{a}) = 0 \text{ for all } p \in \bigcup_\alpha A_\alpha\} \\ &= V\left(\bigcup_\alpha A_\alpha\right)\end{aligned}$$

So arbitrary intersections of closed sets are closed.

Finite unions of closed sets are closed.

Let $V(A), V(B)$ be closed sets. Let $\mathbf{a} \in V(A) \cup V(B)$. Then $p(\mathbf{a}) = 0$ for all $p \in A$ or $q(\mathbf{a}) = 0$ for all $q \in B$. Without loss of generality, suppose $p(\mathbf{a}) = 0$ for all $p \in A$. Then for all polynomials pq with $p \in A, q \in B$, $pq(\mathbf{a}) = 0$. So $\mathbf{a} \in V(\{pq : p \in A, q \in B\})$ and therefore $V(A) \cup V(B) \subseteq V(\{pq : p \in A, q \in B\})$. Now suppose $\mathbf{a} \notin V(A) \cup V(B)$. Then there exists some $p \in A, q \in B$ such that $pq(\mathbf{a}) \neq 0$. So $\mathbf{a} \notin V(\{pq : p \in A, q \in B\})$ and therefore $V(\{pq : p \in A, q \in B\}) \subseteq V(A) \cup V(B)$.

So $V(A) \cup V(B) = V(\{pq : p \in A, q \in B\})$ and therefore $V(A) \cup V(B)$ is closed. Following this process with an inductive argument, finite unions of closed sets are closed. □

1.5.2 Proposition The Zariski topology on $\mathbb{A}_k^1$ is the *finite complement topology.* The only closed sets are the finite sets (or the whole space). In other words, the only open sets are the complements of finite sets (or the empty set).

Proof. We saw that the subsets $V(A) \subset \mathbb{A}^1_k$ are either the whole $\mathbb{A}^1_k$ or finite sets. $\square$

1.5.3 Comparison between Zariski and Euclidean topology over $\mathbb{C}$. Every Zariski closed (open) subset of $\mathbb{A}^n_{\mathbb{C}}$ is also closed (open) in the usual Euclidean topology. The converse is not true.

Proof. It suffices to prove that $V(A)$ is closed in the usual topology. We have $V(A) = \cap_{f\in A} V(f)$, so it suffices to show that $V(f)$ is closed. But $V(f) = f^{-1}(0)$ is closed, because it is the pre-image of a closed set under a continuous function. $\square$

1.5.4 Proposition (Polynomials are continuous) Let f be a polynomial function on $\mathbb{A}^n_k$, viewed as a map $f\colon \mathbb{A}^n_k \to \mathbb{A}^1_k$. Then f is continuous in the Zariski topology.

Proof. We check that pre-images of closed sets are closed. The only closed sets of $\mathbb{A}^1_k$ is the whole space and finite sets. The pre-image of $\mathbb{A}^1_k$ is $\mathbb{A}^n_k$, which is closed. Since finite unions of closed sets are closed, it suffices to check that the pre-image of a point $a \in \mathbb{A}^1_k$ is closed. But the pre-image of a under f is just $V(f - a)$, which is closed by definition. $\square$

— The Zariski topology has very few open sets, and as a result has terrible separation properties. It is not even Hausdorff (except in very small examples). Nevertheless, we will see that it is extremely useful. For one, it makes sense over every field!

1.6 The Nullstellensatz

WEEK2:

We associated a set $V(A)$ to a subset A of the polynomial ring $k[x_1, \dots, x_n]$. If we think of A as a system of equations $\{f = 0 \mid f \in A\}$, then $V(A)$ is the set of solutions. We can also define a reverse operation. The Nullstellensatz says that if k is algebraically closed, then these two operations are mutually inverse. That is, the data of a system of equations is equivalent to the data of its set of solutions. This pleasant fact allows us go back and forth between algebra (equations) and geometry (the solution set).

We start with a straightforward definition.

1.6.1 Definition (Ideal vanishing on a set) Let $S \subset \mathbb{A}^n_k$ be a set. The *ideal vanishing on S*, denoted by $I(S)$, is the set

$$I(S) = \{f \in k[x_1, \dots, x_n] \mid f(a) = 0 \text{ for all } a \in S\}$$

— Recall that an ideal $I \subset k[x_1, \dots, x_n]$ is *radical* if it has the property that whenever $f^n \in I$ for some $n > 1$, then $f \in I$.

1.6.2 Proposition The set $I(S)$ is a radical ideal of $k[x_1, \ldots, x_n]$.

Proof. We leave it to you to check that $I(S)$ is an ideal. To see that it is radical, see that if f^n vanishes on S, then so does f. $\square$

1.6.3 Proposition (Easy properties of radical ideals)

1. $I \subset R$ is radical if and only if R/I has no (non-zero) nilpotents.
2. All prime ideals are radical. In particular, all maximal ideals are radical.

Proof. Consider $f \in R$ and its image $\overline{f} \in R/I$. Then $\overline{f}$ is a nilpotent of R/I if and only if $f^n \in I$ and $\overline{f} = 0$ in R/I if and only if $f \in I$. From this, the result follows. If I is prime, then R/I is an integral domain, so it has no nilpotents (it does not even have zero divisors). $\square$

1.6.4 Proposition (Radical of an ideal) Let I be an ideal, and set $\sqrt{I} = \{f \mid f^n \in I \text{ for some } n > 0\}$. Then $\sqrt{I}$ is a radical ideal.

Proof. (Assume a commutative ring) We will first show that $\sqrt{I} \subset R$ is an ideal. Let $f \in \sqrt{I}, r \in R$, and by definition of $\sqrt{I}$, we suppose $f^n \in I$ for some $n > 0$

$$(rf)^n = r^n f^n.$$

Since $r^n \in R, f^n \in I$, by definition of ideal, we have $r^n f^n \in I$. Therefore, $(rf)^n \in I$ for some $n > 0$, and by definition, we have $rf \in \sqrt{I}$. Therefore, $\sqrt{I}$ is closed under multiplication by elements of R.

Let $f, g \in \sqrt{I}$, with $f^n \in I$, $g^m \in I$.

$$\begin{aligned}(f+g)^{m+n} &= c_0 f^{m+n} + c_1 f^{m+n-1} g^1 + \cdots + c_m f^n g^m + \cdots + c_{m+n} g^{m+n} \\ &= c_0 f^m \times f^n + c_1 f^{m-1} g \times f^n + \cdots + c_m f^n g^m \\ &\quad + c_{m+1} f^{n-1} g^1 \times g^m + \cdots + c_{m+n} g^n \times g^m.\end{aligned}$$

(c_i are the corresponding binomial coefficients in I). As shown above, $(f+g)^{m+n}$ can be written as an R-linear combination of f^n and g^m. Since $f^n \in I, g^m \in I$, by definition of ideal, we have $(f+g)^{m+n} \in I$. Therefore, by definition we have $(f+g) \in \sqrt{I}$ and $\sqrt{I}$ is closed under addition. Therefore, $\sqrt{I}$ is an ideal.

Now we need to show that $\sqrt{I}$ is a radical ideal. Suppose $f \in R$ with $f^n \in \sqrt{I}$ for some $n > 0$. Then, by definition of $\sqrt{I}$, we have $(f^n)^m \in I$ for some $m > 0$.

$$(f^n)^m = f^{nm} \in I, nm > 0.$$

Therefore, by definition, we have $f \in \sqrt{I}$. $\square$

1.6.5 Definition (Radical of an ideal) The ideal $\sqrt{I}$ is called the radical of I.

1.6.6 Proposition (V is unchanged by radicals) We have $V(I) = V(\sqrt{I})$.

Proof. $\boxed{\supset}$ Note that $I \subset \sqrt{I}$ and hence $V(\sqrt{I}) \subset V(I)$. More specifically, for any $f \in I$ we have that $f^1 \in I$ and so $f \in \sqrt{I}$. Now suppose $a \in V(\sqrt{I})$. Then $f(a) = 0$ for all $f \in \sqrt{I}$. But since $I \subset \sqrt{I}$, this implies the weaker statement that for all $f \in I$, we have $f(a) = 0$. This is the same as saying that $a \in V(I)$.

$\boxed{\subset}$ Now let $a \in V(I)$. Then let $f \in \sqrt{I}$. By definition of $\sqrt{I}$ there exists some $n > 0$ such that $f^n \in I$ and hence $f^n(a) = 0$ by assumption. We want to show that this implies $f(a) = 0$ which gives us that $a \in V(\sqrt{I})$, completing the proof. This is because f is an arbitrary element of $\sqrt{I}$. We are done if $n = 1$.

Otherwise we use that we are working in a field which has no zero divisors. More specifically, $f^n(a) = f(a)f^{n-1}(a) = 0$ implies that either $f(a) = 0$ or $f^{n-1}(a) = 0$. If $f(a) = 0$ we are done. Otherwise if $f^{n-1}(a) = 0$, we repeat the previous step for $f^{n-1}(a) = f(a)f^{n-2}(a) = 0$ and so on, until we get $f(a) = 0$ or until $n = 2$ in which case we have $f^2(a) = f(a)f(a) = 0$ which implies $f(a) = 0$ as well. $\square$

— We now state a string of important theorems, all called the "Nullstellensatz", starting with the most comprehensive one.

1.6.7 Theorem Let k be an algebraically closed field. Then we have a bijection

$$\text{Radical ideals of } k[x_1, \ldots, x_n] \leftrightarrow \text{Zariski closed subsets of } \mathbb{A}^n_k$$

where the map from the left to the right is $I \mapsto V(I)$ and the map from the right to the left is $S \mapsto I(S)$. The correspondence is inclusion reversing.

1.6.8 Theorem Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. If $V(I) = \emptyset$, then $I = (1)$.

1.6.9 Theorem Let k be an algebraically closed field. Then all the maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $\langle x_1 - a_1, \ldots, x_2 - a_n \rangle$ for some $(a_1, \ldots, a_n) \in \mathbb{A}^n_k$.

— Theorem 1.6.8 says that we have a dichotomy: either a system of equations $f_i = 0$ has a solution, or there exist polynomials g_i such that

$$\sum f_i g_i = 1.$$

1.6.10 Theorem Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. If f is identically zero on $V(I)$, then $f^n \in I$ for some n.

1.7 Proof of the Nullstellensatz

WEEK2:

The proof of Theorem 1.6.7 actually goes via the proofs of the subsequent theorems. We use the following result from algebra, whose proof we skip.

1.7.1 Theorem Let K be any field and let L be a finitely generated K-algebra. If L is a field, then it must be a finite extension of K.

Proof. See `https://web.ma.utexas.edu/users/allcock/expos/nullstellensatz3.pdf` □

1.7.2 Proof of Theorem 1.6.9 Let $m \subset k[x_1, \ldots, x_n]$ be a maximal ideal. Taking $K = k$ and $L = k[x_1, \ldots, x_n]/m$ in Theorem 1.7.1, and using that k is algebraically closed, we get that the natural map $k \to k[x_1, \ldots, x_n]/m$ is an isomorphism. Let $a_i \in k$ be the pre-image of x_i under this isomorphism. Then we have $m = (x_1 - a_1, \ldots, x_n - a_n)$.

Proof. Since m is a maximal ideal, $L := k[x_1, ..., x_n]/m$ is a field. Let $\pi : k[x_1, ..., x_n] \to L$ be the projection map. Consider the inclusion map $\mathfrak{i} : k \to k[x_1, ..., x_n]$. We embed k in L via the map $\phi := \pi \circ \mathfrak{i}$. We now show that ϕ is an ismomorphism.

Surjectivity of ϕ. The existence of this map tells us that L is a k algebra. Moreover, L is a finitely generated k algebra, since L is generated by $\{\pi(x_1), ..., \pi(x_n)\}$. Now Theorem 1.7.1 applies, and we deduce that L is a finite extension of k. In particular, L must be an algebraic extension of k. If L were not an algebraic extension of k, then there would exist an element $l \in L$ transcendental over k; but then L could not be a finite extension of K, because the set $\{l^j\}_{j=0,1,2,...}$ would be linearly independent. We conclude that L is an algebraic extension of k.

We know that given any $l \in L$, there is a polynomial $p(y) \in k[y]$, where y is any new variable, such that $p(l) = 0$. Let $p(y)$ be the monic polynomial of least degree satisfying the above. Then $p(y)$ is irreducible, since otherwise it would have a factor of smaller degree also satisfying the above, contradicting the minimality of the degree of $p(y)$. Since k is algebraically closed, the irreducible monic polynomials are all of the form $x - a$, for $a \in k$. As such, we have $p(y) = y - a$ for some $a \in k$. It follows that $l \in k$, since we must have $l = a$. To be precise, what we have really shown is that $l \in \phi(k)$, since k is not itself a subset of L, but can be identified with a subset of L. We conclude that $L = \phi(k)$. This tells us that ϕ is surjective.

Injectivity of ϕ. Because ϕ is a field homomorphism, ϕ must be injective. Indeed, the kernel of ϕ is an ideal of k. As such, the kernel of ϕ is either the zero ideal or the unit ideal. Since ϕ is not identically zero, the kernel must be the zero ideal. This completes the proof that ϕ is an isomorphism.

Completion of Proof Because $\phi : k \to L$ is an isomorphism, we can define $a_i := \phi^{-1}(\pi(x_i))$, for each $i = 1, ..., n$. We claim that with this choice of $a_1, .., a_n \in k$, the equation in (11) holds. If $p \in (x_1 - a_1, ..., x_n - a_n)$, then there exist $q_i \in k[x_1, ..., x_n]$ such that

$$p = \sum_{i=1}^{n} q_i(x_i - \mathfrak{i}(a_i)). \tag{1}$$

We could remove the $\mathfrak{i}$ in (11); it just serves as a reminder that $k[x_1, ..., x_n]$ contains a copy of k, not k itself. From (11), we obtain

$$\pi(p) = \sum_{i=1}^{n} \pi(q_i)(\pi(x_i) - \phi(a_i)) = 0. \tag{2}$$

The first equality in (11) holds by the fact that π is a ring homomorphism, and the second equality holds because $\phi(a_i) = \pi(x_i)$, for $i = 1, ..., n$. From (2) we conclude that $p \in m$, since the kernel of π is precisely the ideal m. We have shown that

$$(x_1 - a_1, ..., x_n - a_n) \subseteq m. \tag{3}$$

Now suppose that $p \in m$. Then $\pi(p) = 0$. On the other hand, we can write

$$p = \sum_{i=0}^{d} \sum_{j_1+...+j_n=i} \mathfrak{i}(c_{j_1,...,j_n}) x_1^{j_1} ... x_n^{j_n}, \tag{4}$$

where d is the degree of p and the $c_{j_1,...,j_n}$ are elements of k. Equation (4) yields

$$\begin{aligned} \pi(p) &= \sum_{i=0}^{d} \sum_{j_1+...+j_n=i} \phi(c_{j_1,...,j_n}) \pi(x_1)^{j_1} ... \pi(x_n)^{j_n} \\ &= \sum_{i=0}^{d} \sum_{j_1+...+j_n=i} \phi(c_{j_1,...,j_n}) \phi(a_1)^{j_1} ... \phi(a_n)^{j_n} \\ &= \phi\left(\sum_{i=0}^{d} \sum_{j_1+...+j_n=i} c_{j_1,...,j_n} a_1^{j_1} ... a_n^{j_n} \right). \end{aligned} \tag{5}$$

The second equality in (5) holds by the definition of $a_1, ..., a_n$, and the third equality holds because ϕ is a ring homomorphism. From (5) and the fact that $\pi(p) = 0$, we have that ϕ maps $\sum_{i=0}^{d}\sum_{j_1+...+j_n=i} c_{j_1,...,j_n} a_1^{j_1}...a_n^{j_n}$ to zero. Since ϕ is an isomorphism, it follows that

$$\sum_{i=0}^{d}\sum_{j_1+...+j_n=i} c_{j_1,...,j_n} a_1^{j_1}...a_n^{j_n} = 0, \quad \text{in the field } k. \tag{6}$$

From (6), we have that the point $(a_1, ..., a_n)$ is a root of the polynomial p. We can write

$$p = \sum_{i=0}^{d}\sum_{j_1+...+j_n=i} \mathfrak{i}(e_{j_1,...,j_n})(x_1-a_1)^{j_1}...(x_n-a_n)^{j_n}, \tag{7}$$

for suitably chosen $e_{j_1,...,j_n} \in k$. For example, we could define

$$q(x_1, ..., x_n) = p(x_1+a_1, ..., x_n+a_1). \tag{8}$$

We think of the right-hand side of (8) as a polynomial in $x_1, ..., x_n$. "Evaluating" q at $(x_1-a_1, ..., x_n-a_n)$ gives back p, by definition, while the right-hand side of (8) becomes a polynomial in the variables $x_1-a_1, ..., x_n-a_n$. This is one way to show that p can be written in the form (7). Now, the term with $i=0$ in (7) is the constant term $e_{0,...,0}$. Evaluating p at $(a_1, ..., a_n)$ in (7) shows that $p(a_1, ..., a_n) = e_{0,...,0}$. By (6), we have $p(a_1, ..., a_n) = 0$, so the constant term $e_{0,...,0}$ must also be zero. This means that every term in (7) belongs to $(x_1-a_1, ..., x_n-a_n)$. It follows that $p \in (x_1-a_1, ..., x_n-a_n)$. We have shown that

$$m \subseteq (x_1-a_1, ..., x_n-a_n). \tag{9}$$

From 3 and 9, we conclude that 11 holds. This completes the proof. □

1.7.3 Proof of Theorem 1.6.8 Suppose I is not the unit ideal. We show that $V(I)$ is non-empty. To do so, we use that every proper ideal is contained in a maximal ideal.

Suppose I is not the unit ideal.

We show that $V(I)$ is non-empty.

To do so, we use that every proper ideal is contained in a maximal ideal.

So, as I is proper, it is contained in some maximal ideal M.

But

$$I \subset M \implies V(M) \subset V(I).$$

But by theorem 1.6.9,

$$M = \langle x_1 - a_1, \ldots, x_n - a_n \rangle,$$

where $a_i \in l$ is the preimage of x_i under the isomoprhism of the natural map $k \to k[x_1, \ldots, x_n]/M$, for each i.

So $V(M) = \{(a_1, \ldots, a_n)\}$.

Thus, $\neq V(M) \subset V(I)$, i.e. $V(I)$ is non-empty. The contrapositive completes the proof.

1.7.4 Proof of Theorem 1.6.10 We consider the system $g = 0$ for $g \in I$ and $f \neq 0$. Notice that the last one is not an equation, but there is a trick that allows us to convert it into an equation. Let y be a new variable, and consider the polynomial ring $k[x_1, \ldots, x_n, y]$. In the bigger ring, consider the system of equations $g = 0$ for $g \in I$ and $yf - 1 = 0$. By our assumption, this system of equations has no solutions.

Why is this? Solutions to the original and augmented system are in bijection; if $(a_1, ..., a_n)$ satisfies $g = 0$ and $f \neq 0$, then there exists a unique value of y, $\frac{1}{f(a_1,...,a_n)}$, such that the second system is solved. Similarly, a solution to the second system constitutes a solution to the first, by simply ignoring the value of y, because if $yf - 1 = 0$ then f must be non-zero. Then, by assumption of Theorem 1.6.10, f is identically zero in $V(I)$, so the original system has no solutions. Therefore the augmented system has no solutions, and by Theorem 1.6.8, the ideal generated by $g \in I$ and $yf - 1$ is the unit ideal in $k[x_1, ...x_n, y]$. So then we can write

$$1 = \sum c_i(x_1, ..., x_n, y)g_i(x1, ..., x_n, y) + c(x_1, ...x_n, y)(yf - 1)$$

We transform this expression in $k[x_1, ...x_n, y]$ to an expression in the fraction field $k(x_1, ...x_n)$ by setting $y = \frac{1}{f(x_1,...,x_n)}$, and since for this choice of y we have that $yf - 1$ vanishes, we get

$$1 = \sum c_i(x_1, ..., x_n, \frac{1}{f(x_1, ..., x_n)})g_i(x1, ..., x_n, y) \in k(x_1, ...x_n)$$

Now, since this is a polynomial in $\frac{1}{f(x_1,...,x_n)}$, multiplying through by a sufficiently large power N of f gives

$$f^N = \sum p_i(x_1, ..., x_n)g_i(x1, ..., x_n, y) \in k[x_1, ...x_n]$$

So we can conclude that f^N is in I.

1.7.5 Proof of Theorem 1.6.7. We show that the maps $I \to V(I)$ and $S \to I(S)$ are mutual inverses. That is, we show that $I(V(I)) = I$ if I is a radical ideal, and $V(I(S)) = S$ if S is a Zariski closed subset of $\mathbb{A}^n_k$.

Let us first show that for any ideal I, we have $I(V(I)) = \sqrt{I}$. Suppose $f \in \sqrt{I}$, then $f^n \in I$ for some $n > 0$. But then f^n is identically zero on $V(I)$, and hence so is f; that is, $f \in I(V(I))$. It remains to show that $I(V(I)) \subset \sqrt{I}$. Let $f \in I(V(I))$. Then f is identically zero on $V(I)$. By 1.6.10, there is some n such that $f^n \in I$, and hence $f \in \sqrt{I}$.

Let us now show that $V(I(S)) = S$. Since S is Zariski closed, we know that $S = V(J)$ for some ideal J. So $I(S) = I(V(J)) = \sqrt{J}$. But we know that $V(J) = V(\sqrt{J})$, and hence $V(I(S)) = S$. The proof of Theorem 1.6.7 is then complete.

1.8 Affine and quasi-affine varieties

WEEK2:

An *affine variety* is a subset of the affine space that is closed in the Zariski topology. A *quasi-affine variety* is a subset of the affine space that is locally closed in the Zariski topology. (A locally closed subset of a topological space is a set that can be expressed as an intersection of an open set and a closed set).

2 Regular functions and maps 1

Throughout this section, k is an algebraically closed field.

2.1 Regular functions

WEEK3:

Let $S \subset \mathbb{A}^n$ be a set and let $f\colon S \to k$ be a function. Let a be a point of S.

2.1.1 Definition (Regular function) We say that f is *regular* (or *algebraic*) *at* a if there exists a Zariski open set $U \subset \mathbb{A}^n$ and polynomials $p, q \in k[x_1, \ldots, x_n]$ with $q(a) \neq 0$ such that

$$f \equiv p/q \text{ on } S \cap U.$$

We say that f is *regular* if it is regular at all points of S.

In other words, f is regular at a point a if locally around a (in the Zariski topology), f can be expessed as a ratio of two polynomials. Although the definition of a regular function makes sense for $S \subset \mathbb{A}^n$, we use it only in the context of quasi-affine varieties.

2.1.2 Examples

1. A constant function is regular.
2. Every polynomial function is regular.
3. Sums and products of regular functions are regular. So, the set of regular functions forms a ring. This ring contains a copy of k, namely the constant functions.

2.1.3 Definition (Ring of regular functions) We denote the ring of regular functions on S by $k[S]$. This ring is a k-algebra.

2.1.4 Proposition (Local nature of regularity) Let f be a function on S, and let $\{U_i\}$ be an open cover of S. If the restriction of f to each U_i is regular, then f is regular.

Proof. Let $a \in S$. Then, since $\{U_i\}$ is an open cover of S, there exists an open set $U \in \{U_i\}$ such that $a \in U$. Since the restriction of f to U is regular, it must in particular be regular at a. Thus, there exists an open set V containing the point a such that

$$f \equiv p/q \text{ on } V \cap U$$

for some polynomials $p, q \in k[x_1, \ldots, x_n]$. Then, taking $V' = V \cap U$, which is an open set in S, we have that

$$f \equiv p/q \text{ on } V' \cap S$$

Therefore, f is regular at a. Since a was chosen arbitrarily in S, it follows that f is regular. □

2.2 Regular functions on an affine variety

WEEK3:

It turns out that regular functions on closed subsets of $\mathbb{A}^n$ are just the polynomial functions! So, not only is there a global algebraic expression, we don't even need denominators.

2.2.1 Proposition Let $X \subset \mathbb{A}^n$ be a Zariski closed subset. Let f be a regular function on X. Then there exists a polynomial $P \in k[x_1, \ldots, x_n]$ such that $P(x) = f(x)$ for all $x \in X$.

Proof. By definition, we know that for every $x \in X$, there is a Zariski open set $U \subset X$ and polynomials p, q such that $f = p/q$ on U. The set U and the polynomials p, q may depend on x, so let us denote them by U_x, p_x, and q_x. We need to combine all of these p's and q's and construct a single polynomial P that agrees with f for all x.

This is done by a "partition of unity" argument. First, let us do some preparation. We know that $p_x/q_x = f$ on U_x, but we know nothing about p_x and q_x on the complement of U_x. Our first step is a small trick that lets us assume that both p_x and q_x are identically zero on the complement of U_x.

Since $U_x \subset X$ is open, its complement is closed. By the definition of the Zariski topology, this means that

$$X \setminus U_x = X \cap V(A),$$

for some $A \subset k[x_1, \ldots, x_n]$. Since $x \in U_x$, at least one of the polynomials in A must be non-zero at x. Let g be such a polynomial, and set $U'_x = X \cap \{g \neq 0\}$. Then $U'_x \subset U_x$ is a possibly smaller open set containing x. Set $p'_x = p_x \cdot g$ and $q'_x = q_x \cdot g$. Then we have $f = p'_x/q'_x$ on U'_x, and we also have $p'_x \equiv q'_x \equiv 0$ on $X \setminus U'_x$. So, we may assume from the beginning that both p_x and q_x are identically zero on the complement of U_x..

Now comes the crux of the argument. Suppose $X = V(I)$. Consider the set of "denominators" $\{q_x \mid x \in X\}$. Note that the system of equations

$$g = 0 \text{ for all } g \in I \text{ and } q_x = 0 \text{ for all } x \in X$$

has no solution!

> Why is this the case? $\{q_x = 0 \text{ for all } x \in X\} \subseteq X^c$ because for any $x \in X$, there exists a q_x such that $q_x(x) \neq 0$, by definition of the q_x's. Since $\{g = 0 \text{ for all } g \in I\} = V(I) = X$, the system of equations has no solutions.

By the Nullstellensatz, this means that the ideal $I + \langle q_x \mid q \in X\rangle$ is the unit ideal. That is, we can write

$$1 = g + r_1 q_{x_1} + \cdots + r_m q_{x_m}$$

for some polynomials $r_1, \ldots, r_m$. Take $P = r_1 p_{x_1} + \cdots + r_m p_{x_m}$. Then $f = P$ on all of X.

> Why is this the case? We have that $X = U_{x_1} \cup \cdots \cup U_{x_m}$, i.e. X is the union of finitely many U_{x_i}'s. Let $x \in X$ and assume x is in only some of these U_{x_i}'s. Without loss of generality, assume $x \in U_{x_1}, \ldots, U_{x_j}$ and $x \notin U_{x_{j+1}}, \ldots, U_{x_m}$. Then on $U_{x_1} \cap \cdots \cap U_{x_j}$, we have $f(x) = \dfrac{p_{x_1}(x)}{q_{x_1}(x)} = \cdots = \dfrac{p_{x_j}(x)}{q_{x_j}(x)}$. Also, $1 = r_1(x) q_{x_1}(x) + \ldots r_j(x) q_{x_j}(x)$ and $P(x) = r_1(x) p_{x_1}(x) + \cdots + r_j(x) p_{x_j}(x)$.
>
> But for all $i \in \{1, \ldots, j\}$ and $\lambda_i \in k[x_1, \ldots, x_n]$ with at least one $\lambda_i \neq 0$
>
> $$\frac{\sum_{i=1}^{j} \lambda_i(x) p_{x_i}(x)}{\sum_{i=1}^{j} \lambda_i(x) q_{x_i}(x)} = \frac{p_{x_i}(x)}{q_{x_i}(x)} = f(x).$$
>
> More specifically, $P(x) = \dfrac{P(x)}{1} = \dfrac{\sum_{i=1}^{j} r_i(x) p_{x_i}(x)}{\sum_{i=1}^{j} r_i(x) q_{x_i}(x)} = f(x)$. Therefore, $f = P$ on all of X.

□

—- Let $X \subset \mathbb{A}^n$ be any subset. We have a ring homomorphism

$$\pi \colon k[x_1, \ldots, x_n] \to k[X],$$

where a polynomial f is sent to the regular function it defines on X.

2.2.2 Proposition (Ring of regular functions of an affine) Let $X \subset \mathbb{A}^n$ be a closed subset. Then the ring homomorphism $\pi\colon k[x_1, \ldots, x_n] \to k[X]$ induces an isomorphism

$$k[x_1, \ldots, x_n]/I(X) \xrightarrow{\sim} k[X].$$

Proof. The map π is surjective by Proposition 2.2.1 and its kernel is $I(X)$ by definition. The result follows by the isomorphism theorems. □

2.3 Regular maps

WEEK3:

Consider $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ and a function $f\colon X \to Y$. Write f in coordinates as

$$f = (f_1, \ldots, f_m).$$

2.3.1 Definition (Regular map) We say that f is *regular at a point* $a \in X$ if all its coordinate functions $f_1, \ldots, f_m$ are regular at a. If f is regular at all points of X, then we say that it is *regular*.

2.3.2 Example (Maps to $\mathbb{A}^1$) A regular map to $\mathbb{A}^1$ is the same as a regular function.

2.3.3 Example (An isomorphism) Let $U = \mathbb{A}^1 \setminus \{0\}$ and $V = V(xy - 1) \subset \mathbb{A}^2$. We have a regular function $\phi\colon V \to U$ given by $\phi(x, y) = x$. We have a regular function $\psi\colon U \to V$ given by $\psi(t) = (t, 1/t)$. These functions are mutual inverses, and hence we have a (bi-regular) isomorphism $U \cong V$.

2.4 Properties of regular maps

WEEK3:

2.4.1 Proposition (Elementary properties of regular maps)

1. The identity map is regular.
2. The composition of two regular maps is regular.
3. Regular maps are continuous (in the Zariski topology).

Proof. The identity map is given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$; each coordinate is a polynomial, and hence regular. The statement for composition is true because the composition of fractions of polynomials is also a fraction of polynomials. The third statement is left as homework. □

2.4.2 Proposition (Regular maps preserve regular functions) Let $\phi\colon X \to Y$ be a regular map. If f is a regular function on Y, then $f \circ \phi$ is a regular function on X.

Proof. View a regular function as a regular map to $\mathbb{A}^1$. Then this becomes a special case of composition of regular maps. □

— As a result, we get a k-algebra homomorphism $k[Y] \to k[X]$, often denoted by ϕ^*:

$$\phi^*(f) = f \circ \phi.$$

We thus get a (contravariant) functor from the category of (quasi-affine) varieties to k-algebras. On objects, it maps X to $k[X]$. On morphisms, it maps $\phi\colon X \to Y$ to $\phi^*\colon Y \to X$. It is easy to check that this recipe respects composition. That is, if we have maps $\phi\colon X \to Y$ and $\psi\colon Y \to Z$, and if we let $\psi \circ \phi\colon X \to Z$ be the composite, then

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*.$$

2.4.3 Corollary (Isomorphic varieties have isomorphic rings of functions) If $\phi\colon X \to Y$ is an isomorphism of varieties, then $\phi^*\colon k[Y] \to k[X]$ is an isomorphism of k-algebras.

Proof. Let $\psi\colon Y \to X$ be the inverse of ϕ. Then $\psi^*\colon k[X] \to k[Y]$ is the inverse of ϕ^*. □

2.4.4 Proposition (For affines, map between rings induces map between spaces) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be Zariski closed, and let $f\colon k[Y] \to k[X]$ be a homomorphism of k-algebras. Then there is a unique (regular) map $\phi\colon X \to Y$ such that $f = \phi^*$.

Proof. We know that $k[X] = k[x_1, \dots, x_n]/I(X)$ and $k[Y] = k[y_1, \dots, y_m]/I(Y)$. Let $\phi_i = f(y_i) \in k[X]$. Consider $\phi\colon X \to \mathbb{A}^m$ given by $\phi = (\phi_1, \dots, \phi_m)$. Then ϕ sends X to Y and is the unique map satisfying the required properties. □

Let us justify the last part of the proof. For each i we have that $\phi_i = f(y_i)$ is a regular function, so ϕ is a regular map. Let $g \in I(Y)$. Then

$$\begin{aligned} g \circ \phi &= g \circ (f(y_1), \dots, f(y_m)) \\ &= f(g(y_1, \dots, y_m)) \\ &= 0, \end{aligned}$$

since f is a \$k\$-algebra homomorphism. Thus $\phi(X) \subset Y$. For $i \in \{1, \dots, m\}$ we have

$$\phi^*(y_i) = y_i \circ \phi = \phi_i = f(y_i),$$

so that $\phi^* = f$. Finally, let $\psi : X \to Y$ satisfy $\psi^* = f$. Then, for each i, we have

$$\psi_i = y_i \circ \psi = \psi^*(y_i) = f(y_i) = \phi_i,$$

so $\psi = \phi$.

2.4.5 Example (Bijection but not an isomorphism) Let $X = \mathbb{A}^1_k$ and $Y = V(y^2 - x^3) \subset \mathbb{A}^2_k$. We have a regular map $f\colon X \to Y$ given by $f(t) = (t^2, t^3)$. It is easy to check that f is a bijection, but not an isomorphism.

Here is the argument.

Isomorphic varieties have isomorphic rings of functions. From 1.4.3 we know that $f : X \to Y$ induces the map $f^* : k[Y] \to k[X]$.

Claim: f^* is not surjective.

$x \mapsto t^2$
$y \mapsto t^3$

t is not in the image of f_*. Monomials in $\mathrm{Im}(f_*)$ have degrees that are $2\alpha + 3\beta$ where α and β are non-zero integers. We can only add and subtract monomial terms with equal powers. Thus we only need to consider whether we can get a monomial in t by multiplying t^2 and t^3 by other polynomials in t^2 and t^3. We cannot. Thus it is shown that f_* is not a surjective map.

This implies f is not an isomorphism, if it were, f would have an inverse, f^{-1}. f^{-1} would then induce the inverse of f_*^{-1}. Which as we have seen, does not exist.

2.4.6 Example (Distinguished affine opens) Let $U_f \subset \mathbb{A}^n$ be the complement of $V(f)$. Then U_f is isomorphic to an affine variety, namely the variety $V(yf - 1) \subset \mathbb{A}^{n+1}$, where y denotes the $(n+1)$-th coordinate.

Proof. We have that $U_f = V(f)^c = \{(x_1, \ldots, x_n) | f(x_1, \ldots, x_n) \neq 0\}$.

Also, $V(yf - 1) = \{(x_1, \ldots, x_n, y) | y \cdot f(x_1, \ldots, x_n) - 1 = 0\}$.

So we can define a map $\phi : V(yf - 1) \to U_f$, where

$$\phi(x_1, \ldots, x_n, y) = (x_1, \ldots, x_n)$$

% This is a clear homomorphism. (Fixed by Anand) This is clearly a regular map.

We can define another map $\psi : U_f \to V(yf - 1)$, where

$$\phi(x_1, \ldots, x_n) = \left(x_1, \ldots, x_n, \frac{1}{f(x_1, \ldots, x_n)}\right)$$

This is well-defined, since $f(x_1, \ldots, x_n) \neq 0$, and this is also a regular map. % clearly also a homomorphism. (Fixed by Anand)

Then

$$\begin{aligned}\psi \circ \phi(x_1, \ldots, x_n, y) &= \psi(x_1, \ldots, x_n) \\ &= \left(x_1, \ldots, x_n, \frac{1}{f(x_1, \ldots, x_n)}\right)\end{aligned}$$

But y must satisfy $yf(x_1, \ldots, x_n) - 1$, so $y = \frac{1}{f(x_1,\ldots,x_n)}$, and thus $\psi \circ \phi = id_{V(yf-1)}$. Also,

$$\begin{aligned}\phi \circ \psi(x_1, \ldots, x_n) &= \phi\left(x_1, \ldots, x_n, \frac{1}{f(x_1, \ldots, x_n)}\right) \\ &= (x_1, \ldots, x_n)\end{aligned}$$

So $\phi \circ \psi = id_{U_f}$, and therefore U_f and $V(yf-1)$ are isomorphic. □

2.4.7 Caution (Not all opens are affine) The previous proposition only applies to the complement of $V(f)$ for a single f! The complement of $V(I)$, in general, is not isomorphic to an affine variety. For example, the complement of the origin in $\mathbb{A}^2$ is not isomorphic to an affine variety.

3 Algebraic varieties

3.1 Definition

WEEK4:

The varieties we have seen so far have been sub-sets of the affine space. Using these as buildig blocks, we can construct general algebraic varieties. The definition is analogous to the definition of a manifold in differential geometry, using open subsets of $\mathbb{R}^n$ as building blocks.

Let X be a topological space. A *quasi-affine chart* on X consists of an open subset $U \subset X$, a quasi-affine variety V and a homeomorphism $\phi_{UV} : U \to V$. Via this isomorphism, we can "transport" the algebraic structure (for example, the notion of a regular function) from V to U.

Let $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ be two quasi-affine charts on X (see Figure 1). Set $U_{12} = U_1 \cap U_2$. Consider the open subsets $V_{12} = \phi_1(U_{12}) \subset V_1$ and $V_{21} = \phi_2(U_{12}) \subset V_2$. Being open subsets of quasi-affine varieties, they are themselves quasi-affine varieties. Furthermore, the map

$$\phi_2 \circ \phi_1^{-1} : V_{12} \to V_{21}$$

is a homeomorphism. We say that the two charts are *compatible* if this map is a (bi-regular) isomorphism.

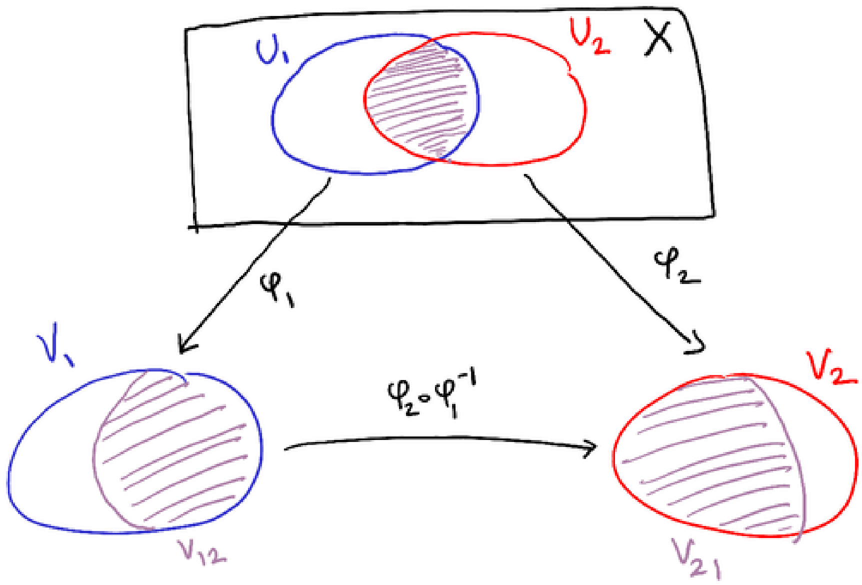

Figure 1: Compatible charts

When we have two charts, one on U_1 and another on U_2, then the intersection $U_1 \cap U_2$ gets two different charts. Compatibility ensures that these two charts are related by a bi-regular isomorphism, so that the algebraic structure coming from one is the same as the one coming from the other.

A *quasi-affine atlas* on X is a collection of compatible charts $\phi_i \colon U_i \to V_i$ such that the U_i cover X.

3.1.1 Definition (Algebraic variety) An *algebraic variety* is a topological space with a quasi-affine atlas.

3.1.2 Example (Quasi-affine varieties) A quasi-affine variety X is itself an algebraic variety. The atlas is the obvious one, consisting of the single chart $\mathrm{id} \colon X \to X$.

3.2 Projective spaces

WEEK4:

A fundamental example of an algebraic variety is the projective space.

3.2.1 Definition (Projective space) The *projective n-space over a field k*, denoted by $\mathbb{P}^n_k$, is the set of one-dimensional subspaces of k^{n+1}.

3.2.2 Intuition Before describing how $\mathbb{P}^n_k$ is an algebraic variety, let us build some intuition about projective space. For easy visualisations, it helps to take $k = \mathbb{R}$ or $k = \mathbb{C}$. A one dimensional subspace of k^{n+1} is also called a *line*. Note that, by this definition, a line must contain the origin.

Let us take $n = 0$. Then there is a unique one-dimenional subspace of $k^{n+1} = k$, so $\mathbb{P}^0_k$ is just a single point.

Let us take $n = 1$. Then $\mathbb{P}^1_k$ is the set of lines (through the origin) in k^2. Let us take $k = \mathbb{R}$. Every line through the origin is uniquely determined by its slope, which can be any element of $\mathbb{R}$, so it seems like $\mathbb{P}^1_{\mathbb{R}}$ is just a copy of $\mathbb{R}$. But the vertical line does not have a (finite) slope, so $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. In other words, $\mathbb{P}^1$ contains the usual real line, plus "a point at infinity".

It can be more instructive to see this in a picture. Fix a horizontal line L at, say, $y = -1$. Every line through the origin intersects L at a unique point, except the horizontal line. So if we discard the one point of $\mathbb{P}^1_k$ corresponding to the horizontal line, the rest is just a copy of L. If we had chosen a different reference line L, for example, a vertical one, then we get a similar description of $\mathbb{P}^1$ away from a single point. In fact, we can discard *any* one point of $\mathbb{P}^1$, and the rest will be a copy of $\mathbb{R}$.

Let us take $n = 2$. Then $\mathbb{P}^2_k$ is the set of lines (through the origin) in k^3. We can use the same technique as before: fix a reference plane P at $z = -1$. Then most lines are uniquely characterised by their intersection point with P. The only exceptions are the lines parallel to $z = -1$, that is, the lines lying in the plane $z = 0$, which we miss. But these form a small projective space $\mathbb{P}^1$. So we see that $\mathbb{P}^2 = P \sqcup \mathbb{P}^1$.

3.2.3 Topology A one-dimensional subspace of k^{n+1} is spanned by a non-zero vector $(a_0, \dots, a_n)$. Two vectors $(a_0, \dots, a_n)$ and $(b_0, \dots, b_n)$ span the same subspace if and only if there exists $\lambda \in k^\times$ such that

$$(b_0, \dots, b_n) = (\lambda a_0, \dots, \lambda a_n).$$

So, we can identify $\mathbb{P}^n$ with the equivalence classes of non-zero vectors $(a_0, \dots, a_n)$ where two non-zero vectors are considered equivalent if one is a scalar multiple of the other. In other words, we have

$$\mathbb{P}^n_k = (\mathbb{A}^{n+1} \setminus 0)/\text{scaling}.$$

We denote the equivalence class of $(a_0, \dots, a_n)$ by $[a_0 : \cdots : a_n]$.

We give $\mathbb{P}^n_k$ the quotient topology inherited from $\mathbb{A}^{n+1} \setminus 0$. That is, a set $U \subset \mathbb{P}^n_k$ is open/closed if and only if its pre-image in $\mathbb{A}^{n+1} \setminus 0$ is open/closed.

For example, consider the subset U_n of $\mathbb{P}^n_k$ consisting of $[a_0 : \cdots : a_n]$ with $a_n \neq 0$. Its preimage in the set of $(a_0, \dots, a_n) \in \mathbb{A}^{n+1} \setminus 0$ with $a_n \neq 0$, which is a (Zariski) open set. Hence U_n is open in $\mathbb{P}^n_k$. Likewise, $U_0, U_1, \dots$ are also open. Note that we have

$$\mathbb{P}^n_k = U_0 \cup \cdots \cup U_n;$$

that is, the sets $U_0, \dots, U_n$ form an open cover of $\mathbb{P}^n$.

Consider a point $[a_0 : \cdots : a_n] \in U_0$, so that $a_0 \neq 0$. By scaling by $\lambda = a_0^{-1}$, we have a distinguished representative of this point of the form $[1 : b_1 : \cdots : b_n]$, which we can think of as a point $(b_1, \dots, b_n) \in \mathbb{A}^n$. Thus, we have a bijection $\phi_0 \colon U_0 \to \mathbb{A}^n$, and similarly $\phi_1 U_i \to \mathbb{A}^n$.

3.2.4 Proposition (Charts of the projective space)

1. The bijections $\phi_i \colon U_i \to \mathbb{A}^n$ defined above are homeomorphisms.

2. The charts $\phi_i \colon U_i \to \mathbb{A}^n$ are mutually compatible, and hence give an atlas on $\mathbb{P}^n$.

1. This is not obvious, also not hard, but also not very enlightening. Let us skip this.

2. *Proof.* For the charts $\varphi_i : U_i \to \mathbb{A}^n$ and $\varphi_j : U_j \to \mathbb{A}^n$, $0 < i < j < n$, for φ_i and φ_j we have

$$[X_0 : ... : X_i : ... : X_n] \mapsto (X_0/X_i, ..., X_n/X_i) = (a_1, ..., a_n)$$

$$[X_0 : ... : X_j : ... : X_n] \mapsto (X_0/X_j, ..., X_n/X_j) = (b_1, ..., b_n)$$

In $U_i \cap U_j$ we have $X_i, X_j \neq 0$, this corresponds to $\{a_j \neq 0\} \subset \mathbb{A}^n$ and $\{b_{i+1} \neq 0\} \subset \mathbb{A}^n$ under φ_i and φ_j,

$$\begin{aligned} (a_1, ..., a_n) &\overset{\varphi_i^{-1}}{\mapsto} [a_1 : ... : a_i : 1 : a_{i+1} : ... : a_n] \\ [a_1 : ... : a_i : 1 : a_{i+1} : ... : a_n] &\overset{\varphi_j}{\mapsto} (a_1/a_j, ..., a_i/a_j, 1/a_j, a_{i+1}/a_j, ..., a_n/a_j) \\ (a_1, ..., a_n) &\overset{\varphi_j \circ \varphi_i^{-1}}{\longrightarrow} (a_1/a_j, ..., a_i/a_j, 1/a_j, a_{i+1}/a_j, ..., a_n/a_j) \end{aligned}$$

Let $\varphi_j \circ \varphi_i^{-1} = (f_{ij}^1, ..., f_{ij}^n)$, by considering all cases $0 \leq i < j \leq n$ and $0 \leq j < i \leq n$ with a similar method we find that

$$f_{ij}^k = \begin{cases} a_k/a_j, (k \leq i < j) \vee (i < j < k) \\ 1/a_j, (i < j) \wedge (k = i + 1) \\ a_{k-1}/a_j, i + 1 < k \leq j \\ a_k/a_{j+1}, (k \leq j < i) \vee (j < i < k) \\ 1/a_{j+1}, j < i = k \\ a_{k+1}/a_{j+1}, j < k < i \end{cases}$$

Thus $\varphi_j \circ \varphi_i^{-1}$ is regular for all i and j and since $(\varphi_j \circ \varphi_i^{-1})^{-1} = \varphi_i \circ \varphi_j^{-1} = (f_{ji}^1, ..., f_{ji}^n)$ is also regular, therefore all $\varphi_j \circ \varphi_i^{-1}$ are biregular. $\square$

3.2.5 Open and closed subvarieties Let X be an algebraic variety, and $Y \subset X$ an open or closed subset. Then Y inherits the structure of an algebraic variety. To get, the atlas for Y, let $\phi_i \colon U_i \to V_i$ be an atlas for X. For Y, we just take $\phi_i \colon U_i \cap Y \to \phi(U_i \cap Y)$.

Explain why this is an atlas for Y.

Proof. Suppose Y is a closed subset of X. First, we need to show that $\{U_i \cap Y\}$ is an open covering of Y: Since $\bigcup U_i = X$, $\bigcup (U_i \cap Y) = Y$ and $\{U_i \cap Y\}$ covers Y. Also, Y is a subspace of X implies $U_i \cap Y$ is open in Y. [By the definition of topological subspace] Then we need to prove $\phi_i(U_i \cap Y)$ is a quasi-affine variety: Since $U_i \cap Y \subset U_i$ and U_i is a subspace of X, $U_i \cap Y$ is closed in U_i. Given that ϕ_i is a homeomorphism, $\phi_i(U_i \cap Y)$ is also closed in V_i. Since a closed subset of quasi-affine varieties is also a quasi-affine variety, $\phi_i(U_i \cap Y)$ is a quasi-affine variety. Thus, $\phi_i : U_i \to \phi_i(Ui \cap Y)$ is a chart for Y. And if we restrict the original transition maps on $U_i \cap Y$, the new transition maps are still bi-regular. Hence $\{\phi_i : U_i \cap Y \to \phi_i(U_i \cap Y)\}$ is a quasi-affine atlas for Y and Y is also an algebraic variety with inherited structure from X. The case when Y is an open subset of X is similar. $\square$

3.2.6 Proposition (Closed subvarieties of projective space 1) Let $F \in k[X_0, \dots, X_n]$ be a homogeneous polynomial. Let $V(F) \subset \mathbb{P}^n$ be the set of points $\{[a_0 : \cdots : a_n] \mid F(a_0, \dots, a_n) = 0\}$. Then $V(F)$ is a closed subset.

Explain why $V(F)$ is well-defined (that is, the condition $F(a_0, \dots, a_n) = 0$ does not depend on the chosen representative of the equivalence class). Then explain why $V(F)$ is closed.

Proof. The fact $V(F)$ is well defined follows from $F(x) = 0$ implies $F(\lambda x) = 0$ for all $\lambda \in k$ in the case of F homogeneous, as all representatives of the equivalence class are related by scaling.

Let E be the set of exponents, such that $F(x) = \sum\limits_{a \in E} c_a x^a$. Noting that $|a|$ is the same for all \$n\$-tuples of exponents as F is homogeneous, denote this degree as m.

$$F(x) = \sum_{a \in E} c_a x^a$$

$$F(\lambda x) = \sum_{a \in E} c_a (\lambda x)^a = \sum_{a \in E} c_a \lambda^{|a|} x^a = \sum_{a \in E} c_a \lambda^m x^a = \lambda^m \sum_{a \in E} c_a x^a = \lambda^m f(x) = \lambda^m \cdot 0 = 0$$

Thus $V(F)$ is well defined. □\$\

$V(F)$ closed in $\mathbb{P}^n$ if its pre-image in $\mathbb{A}^{n+1} \backslash 0$ is closed. Due to our definitions, the pre-image is given by the Zariski closed set $V(F) \subset \mathbb{A}^{n+1} \backslash 0$. □ □

3.2.7 Proposition (Closed subvarieties of projective space 2) Let $I \subset k[X_0, \ldots, X_n]$ be a homogeneous ideal.

Define $V(I) \subset \mathbb{P}^n$ and show that it is a closed subset.

Proof. Let $I \subset k[X_0, \ldots, X_n]$.

We have two equivalent definitions of $V(I) \subset \mathbb{P}^n$:

1. Take $V(I) \subset \mathbb{A}^{n+1}/\{0\}$.

   Set $V(I) \subset \mathbb{P}^n$ as the image of $V(I) \subset \mathbb{A}^{n+1}/\{0\}$.

2. $$V(I) := \{[x_0 : \ldots : x_n] | F(x_0, \ldots, x_n) = 0 \forall \text{ homogeneous } F \in I\}$$

We have that

$$V(I) = \cap V(F),$$

where the intersection is taken over all homogeneous $F \in I$. But by Proposition 3.2.6, $V(F)$ is closed, and thus the arbitrary union of closed sets is closed, i.e. $V(I)$ is closed. □

3.2.8 Proposition (Closed subvarieties of projective space 3) Conversely, let $X \subset \mathbb{P}^n$ be a closed subset. Then there exists a homogeneous ideal $I \subset k[X_0, \ldots, X_n]$ such that $X = V(I)$.

Proof. Assume that X is non-empty. Let $\pi \colon \mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ be the quotient map. Let $C \subset \mathbb{A}^n$ be the closure of $\pi^{-1}(X)$.

#+begin$_{\text{skipped}}$ Prove that C is conical, that is, if $x \in C$ then $\lambda x \in C$ for every scalar $\lambda \in k$. Conclude using Homework 1 that $C = V(I)$ for a homogeneous ideal I. Prove that $X = V(I)$ in $\mathbb{P}^n$.

#+begin$_{\text{proof}}$ Suppose that X is non-empty. Let $\pi : \mathbb{A}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{P}^n$ be the quotient map. Then $\pi^{-1}(X)$ is closed in $\mathbb{A}^{n+1} \setminus \{0\}$. Let $C \subseteq \mathbb{A}^{n+1}$ be the closure of $\pi^{-1}(X)$ in

$\mathbb{A}^{n+1}$. Let $p \in k[X_0, \ldots, X_n]$ with $p(\mathbf{y}) = 0$ for all $\mathbf{y} \in \pi^{-1}(X)$. Let $[\mathbf{x}] \in X$ for some $\mathbf{x} \in \mathbb{A}^{n+1} \setminus \{\mathbf{0}\}$. Then $\lambda\mathbf{x} \in \pi^{-1}(X)$ for all $\lambda \in k$ with $\lambda \neq 0$. Let $p = p_d + \cdots + p_0$ be the decomposition of p into its homogeneous components. Define $q \in k[Y]$ by $q(Y) = Y^d p_d(\mathbf{x}) + \cdots + Y p_1(\mathbf{x}) + p_0(\mathbf{x})$. Let $\lambda \in k$ with $\lambda \neq 0$.

$$\begin{aligned} q(\lambda) &= \lambda^d p_d(\mathbf{x}) + \cdots + \lambda p_1(\mathbf{x}) + p_0(\mathbf{x}) \\ &= p_d(\lambda\mathbf{x}) + \cdots + p_1(\lambda\mathbf{x}) + p_0(\lambda\mathbf{x}) \\ &= p(\lambda\mathbf{x}) \\ &= 0 \end{aligned}$$

So q has infinitely many roots and therefore q is the zero polynomial. This gives that $p_0(\mathbf{x}) = 0$ and so p_0 is the zero constant.

$$\begin{aligned} p(\mathbf{0}) &= p_d(\mathbf{0}) + \cdots + p_0(\mathbf{0}) \\ &= p_0(\mathbf{0}) \\ &= 0 \end{aligned}$$

So $\mathbf{0}$ is a root of p. Therefore $\mathbf{0}$ is an element of C, so $C = \pi^{-1}(X) \cup \{0\}$. So for all $\lambda \in k$ we have that $\lambda\mathbf{x} \in C$. Then by Homework 1 we have that $C = V(I)$ where $I \subseteq k[X_0, \ldots, X_n]$ is a homogeneous ideal.

$$\begin{aligned} \pi(V(I) \setminus \{0\}) &= \pi(\pi^{-1}(X)) \\ &= X \end{aligned}$$

Therefore $X = V(I)$ where $V(I)$ is identified as a subset of $\mathbb{P}^n$.

Now suppose that X is empty. Then X is the image of the empty set under π. The empty set is the vanishing set of the unit ideal, which is homogeneous.

Therefore there exists a homogeneous ideal $I \subseteq k[X_0, \ldots, X_n]$ such that $X = V(I)$. $\square$

$\#{+}\mathrm{end}_{\mathrm{skipped}}$
$\#{+}\mathrm{end}_{\mathrm{proof}}$

3.2.9 Example (Linear subspaces) Suppose $I \subset k[X_0, \ldots, X_n]$ is generated by (homogeneous) linear equations. Then $V(I) \subset \mathbb{A}^{n+1}$ is a sub-vector space $W \subset \mathbb{A}^{n+1}$, and $V(I) \subset \mathbb{P}^n$ is naturally the projective space of W. We call such $V(I) \subset \mathbb{P}^n$ *linear subspaces*, or “lines”, “planes”, etc. See that any two distinct lines in $\mathbb{P}^2$ intersect at a unique point, and through any two distinct points in $\mathbb{P}^2$ passes a unique line.

4 Regular functions and regular maps 2

4.1 Regular functions and maps

4.1.1 Proposition (regularity does not depend on the chart) Let X be an algebraic variety and $f\colon X \to k$ a function. Let $\phi_1\colon U_1 \to V_1$ and $\phi_2\colon U_2 \to V_2$ be two compatible charts such that x lies in both U_1 and U_2. Denote the images of x in the two charts by v_1 and v_2. Consider the functions $f \circ \phi_1^{-1}\colon V_1 \to k$ and $f \circ \phi_2^{-1}\colon V_2 \to k$. Then the first is regular at v_1 if and only if the second is regular at v_2.

Prove this.

Proof. Suppose X is an algebraic variety and that $f : X \to k$ is a function. Suppose $x \in X$ lies in the domains U_1 and U_2 of two compatible charts $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$. Let $v_1 = \phi_1(x)$ and $v_2 = \phi_2(x)$. We prove that $f \circ \phi_1^{-1} : V_1 \to K$ is regular at v_1 if and only if $f \circ \phi_2^{-1} : V_2 \to K$ is regular at v_2. \ \ Suppose that $f \circ \phi_1^{-1} : V_1 \to K$ is regular at v_1. We write

$$f \circ \phi_2^{-1} = (f \circ \phi_1^{-1}) \circ (\phi_1 \circ \phi_2^{-1}). \tag{10}$$

Note that the right-hand side of (10) makes sense as a map from $\phi_2(U_1 \cap U_2)$ not from all of V_2 to all of V_1. This is no cause for concern though, since regularity is a local property. Note that $(\phi_1 \circ \phi_2^{-1})(v_2) = v_1$. By compatibility of the charts, we know that $\phi_1 \circ \phi_2^{-1}$ is regular. Using this and the assumption that $f \circ \phi_1^{-1}$ is regular at v_1, we find that the composition on the right-hand side of (10) is regular at v_2. That is, the restriction of $f \circ \phi_2^{-1}$ is regular at v_2. Since regularity is a local property, we have that $f \circ \phi_2^{-1}$ is regular at v_2. \ \ The proof of the converse implication is exactly the same, with equation (10) replaced by

$$f \circ \phi_1^{-1} = (f \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1}). \tag{11}$$

For the converse implication, we again use compatibility of the maps ϕ_1 and ϕ_2. We establish that the composition on the right-hand side of (11) is regular at v_1. Accordingly, $f \circ \phi_1^{-1}$ is regular at v_1. This completes the proof. $\square$

4.1.2 Definition (regular function on a variety) Let $f\colon X \to k$ be a continuous function. We say that f is regular at x if for some (equivalently, for every) chart $\phi\colon U \to V$ with $x \in U$, the function $f \circ \phi^{-1}\colon V \to k$ is regular at $\phi(x)$. We say that f is regular on X if it is regular at all points $x \in X$.

4.1.3 Definition (regular map between varieties) Let X and Y be algebraic varieties and $f\colon X \to Y$ a continuous map. We say that f is regular at a point $x \in X$ if for any (equivalently, for every) chart $\phi\colon U \to V$ with $x \in U$ and $\psi\colon U' \to V'$ with $f(x) \in U'$, the composite map

$$\psi \circ f \circ \phi^{-1} \colon V \dashrightarrow V'$$

is regular at $\phi(x)$.T The reason for the dashed arrow is that the domain of $\psi \circ f \circ \phi^{-1}$ may not be all of V, but only an open subset of V. To be precise, the domain is $\phi(U \cap f^{-1}(U'))$. But the domain contains $\phi(x)$, so it makes sense to talk about the regularity at $\phi(x)$.

See Figure 2 for a picture (the bottom arrow should be dashed).

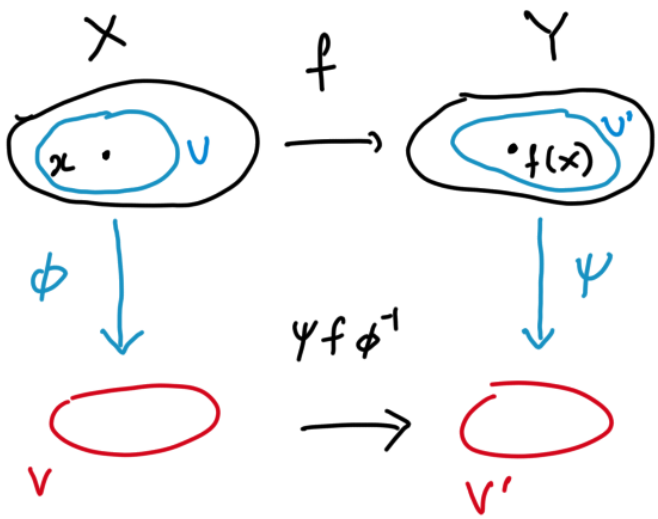

Figure 2: A map is regular if it is regular with respect to the charts.

4.2 Examples

WEEK5

For quasi-affine varieties, these definitions do not add anything new.

4.2.1 Example Let $X = \mathbb{P}^1$. Set $f([X : Y]) = X/Y$. Then f is defined at all points except the point $[1 : 0]$, and is a regular function on $\mathbb{P}^1 \setminus \{[1 : 0]\}$. More generally, let $X = \mathbb{P}^n$ and let $F, G \in k[X_0, \ldots, X_n]$ be homogeneous polynomials of the same degree. The function

$$[X_0 : \cdots : X_n] \mapsto F(X_0, \ldots, X_n)/G(X_0, \ldots, X_n)$$

is regular outside $V(G)$.

Prove this.

Proof. Call the function f.
Note that f is well defined on $\mathbf{P}^n \backslash V(G)$ since it is a ratio of homogeneous polynomials

of the same degree, so \$f(λ x) = F$(\lambda x)\frac{}{G(\lambda x)=\frac{\lambda^d F(x)}{\lambda^d G(x)}=f(x)\$.Considerthestandardatlasfor\mathbf{P}^n}$,
$\phi_i : U_i \to \mathbf{A}^n$, where $U_i = \{x \in \mathbf{P}^n \mid x_i \neq 0\}$.
Let $x \in \mathbf{P}^n \backslash V(G)$; say x is nonzero in its k^{th} coordinate.
Consider the open set W_k of $\mathbf{A}^n$ defined as the complement of the zero locus of the polynomial on $\mathbf{A}^n$ defined by

$$G_k := G(x_1, x_2, ..., x_{k-1}, 1, x_{k+1}, ...x_n)$$

Since $x \in \mathbf{P}^n \backslash V(G)$, $\phi_k(x) \in W_k$.
Now, we show that $f \circ \phi_k^{-1}$ is regular on W_k. Suppose $a = (a_1, ..., a_n) \in W_k$; then

$$f \circ \phi_k^{-1}(a) = f[a_0 : ... : 1 : ... : a_n] = \frac{F(a_0, ..., 1, ..., a_n)}{G(a_0, ..., 1, ..., a_n)}, \ \forall a \in W_k$$

Which is well defined since $G_k(a) \neq 0$ for $a \in W_k$
So f is regular on $\mathbf{P}^n \backslash V(G)$. $\square$

4.2.2 Example Let $X = \mathbb{P}^n$ and let $F_0, \ldots, F_m$ be homogeneous polynomials of the same degree. Let $Z \subset \mathbb{P}^n$ be $V(F_0, \ldots, F_m)$. Then the formula

$$[X_0 : \cdots : X_n] \mapsto [F_0(X_0, \ldots, X_n) : \cdots : F_m(X_0, \ldots, X_n)]$$

defines a regular map from $X \setminus Z$ to $\mathbb{P}^m$.

Prove this.

Proof. Without loss of generality, we can assume that $X_0 = 1$, because this argument also works for any $X_i = 1$, which must hold for some i, and for any polynomial F_i, F_i vanishes at $X \in \mathbf{A}^n \backslash 0$ if and only if it vanishes at the representation of X in $\mathbf{P}^n$ with one of the coordinates equal to 1.
Let $(1, \cdots, a_n)$ be a point in $\mathbf{A}^n$ that maps canonically to $[X_0 : \cdots : X_n] = [1 : \cdots : a_n]$.
Since $[X_0 : \cdots : X_n] \in X \backslash Z$, we can assume that $F_i[X_0 : \cdots : X_n] \neq 0$ because it will hold for some i. By previous results, it suffices to check if

$$(1, \cdots, a_n) \mapsto [X_0 : \cdots : X_n] \mapsto [F_1(X_0, \cdots, X_n), \cdots, F_m(X_0, \cdots, X_n)]$$
$$\mapsto \left(\frac{F_1(X_0, \cdots, X_n)}{F_i(X_0, \cdots, X_n)}, \cdots, \frac{F_m(X_0, \cdots, X_n)}{F_i(X_0, \cdots, X_n)} \right)$$

is regular, because we only need to check on *one* choice of charts for $[X_0 : \cdots : X_n]$ and $[F_1(X_0 : \cdots : X_n), \cdots, F_m(X_0, \cdots, X_n)]$. Now, note that because $F_1, \cdots, F_m$ are homogeneous, we have

$$\left(\frac{F_1(X_0,\cdots,X_n)}{F_i(X_0,\cdots,X_n)},\cdots,\frac{F_m(X_0,\cdots,X_n)}{F_i(X_0,\cdots,X_n)}\right)$$

$$=\left(\frac{F_1(1,\cdots,a_n)}{F_i(1,\cdots,a_n)},\cdots,\frac{F_m(1,\cdots,a_n)}{F_i(1,\cdots,a_n)}\right)$$

on the open set $\{a_1 \neq 0\} \cap \{F_i \neq 0\}$, and every component is a regular function from $\{a_1 \neq 0\} \cap \{F_i \neq 0\}$ to k. Open sets of the form $\{a_i \neq 0\} \cap \{F_j \neq 0\}$ cover $X \backslash Z$, so it follows that $[F_0 : \cdots : F_m]$ is regular on all of $X \backslash Z$. □

4.2.3 Example The natural map $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$ is regular.

4.2.4 Example (Automorphisms of $\mathbb{P}^n$) Consider the $n+1$-dimensional k-vector space V spanned by $X_0, \ldots, X_n$. Pick any basis $\ell_0, \ldots, \ell_n$ of this vector space. Then we have a regular map

$$L\colon \mathbb{P}^n \to \mathbb{P}^n$$
$$[X_0 : \cdots : X_n] \mapsto [\ell_0 : \cdots : \ell_n].$$

Explicitly, if we write

$$\ell_i = L_{i,0}X_0 + \cdots + L_{i,n}X_n$$

and write our homogenous vector as a column vector, then the map is

$$[X] \mapsto [LX].$$

In other words, it is induced by the invertible linear map $L\colon V \to V$. As a result, it has an inverse, induced by the inverse of the matrix M:

$$[X] \mapsto [MX].$$

In this way, we get an action of $GL_n(k)$ on $\mathbb{P}^n$. But notice that a matrix L and a scalar multiple λL induce the same map on $\mathbb{P}^n$. So the action descends to an action of the group $PGL_n(k) = GL_n(k)/\text{scalars}$.

4.2.5 Example (regular functions on $\mathbb{P}^1$) The previous example gave examples of regular functions on (strict) open subsets of the projective space. It turns out that there are *no* regular functions on $\mathbb{P}^n$ other than the constant functions!

1.2.5 Example (regular functions on $\mathbb{P}^1$) The previous example gave examples of regular functions on (strict) open subsets of the projective space. It turns out that there are *no* regular functions on $\mathbb{P}^n$ other than the constant functions!

> Prove this for $n = 1$. Then deduce it for all n using that through any two distinct points in $\mathbb{P}^n$ passes a projective line. — (4)

Proof:

We will first show this for $n = 1$.

We can split $\mathbb{P}^1 = \{[x\colon y]\}$ into two components, $\mathbb{P}^1 = U_0 \cup U_1$, where $U_0 = \{[1\colon y]\}$ is the set where the x coordinate is non-zero, and $U_1 = \{[x\colon 1]\}$ is the set where the y coordinate is non-zero.

Consider the map $\phi\colon U_0 = \{[1\colon y]\} \to \mathbb{A}^1$ by $\phi([1\colon y]) = y \in \mathbb{A}^1$. The map is regular since it is a polynomial function on its coordinates. Its inverse $\phi^{-1}\colon \mathbb{A}^1 \to U_0$ by $\phi^{-1}(y) = [1\colon y]$ is also regular for the same reason. Therefore, ϕ is an isomorphism, and $U_0, \mathbb{A}^1$ are isomorphic. Under this isomorphism, we have $k[U_0] = k[\mathbb{A}^1] = k[y]$. Similarly, $k[U_1] = k[x]$

Functions on $U_0 \cup U_1$ is a function on U_0, a function on U_1, and they must agree on $U_0 \cap U_1$. Consider $U_0 \cap U_1$, this is equivalent to taking away the origin from $U_0 \cong \mathbb{A}^1$, so $U_0 \cap U_1 \cong \mathbb{A}^1 \backslash \{\text{origin}\} \cong V(x)^C \subset \mathbb{A}^1$. By previous example in class, we have $U_0 \cap U_1 \cong V(x)^C \cong V(xy-1) \subset \mathbb{A}^2$, and $k[U_0 \cap U_1] \cong {k[x,y]}/{(xy-1)}$. In this quotient ring, we send y to x^{-1}, so ${k[x,y]}/{(xy-1)} \cong k[x, x^{-1}]$

Consider the image of $k[U_0]$ and $k[U_1]$ in ${k[x,y]}/{(xy-1)}$ by the obvious map (sends x to x, send y to $y = x^{-1}$). We have $k[U_0] = k[x^{-1}] \subset k[x, x^{-1}]$, $k[U_1] = k[x] \subset k[x, x^{-1}]$. Consider two regular functions $f \in k[U_0] = k[x^{-1}]$, $g \in k[U_1] = k[x]$. f is a polynomial with variable x^{-1}, g is a polynomial with variable x, and they must agree. In an algebraically closed field (which we assume), this happens only if f, g are the same constant polynomial. Therefore, $f, g \in k$, hence $k[\mathbb{P}^1] \subset k$

Also, every constant polynomial can be treated as a regular function on $\mathbb{P}^1$, so $k \subset k[\mathbb{P}^1]$. Therefore, $k[\mathbb{P}^1] = k$

Now, consider $\mathbb{P}^n, n > 1$. To prove that $k[\mathbb{P}^n] = k$, we will show that given any $f \in k[\mathbb{P}^n]$ and $p \neq q \in \mathbb{P}^n$, we have $f(p) = f(q)$.

$p, q \in \mathbb{P}^n$ are both non-zero 'vectors' in k^{n+1}, and they are not multiples of each other (which simply holds by definition of the projective space). Therefore, they span a two-dimensional vector space $V \subset k^{n+1}$, and we get a linear isomorphism between V and k^2. Now, consider all the lines passing through the origin in V. These forms a copy of $\mathbb{P}^1$, and p, q are in this $\mathbb{P}^1$. By previous part, $k[\mathbb{P}^1] = k$. For any regular function on $f \in k[\mathbb{P}^n]$, it must be regular on this copy of $\mathbb{P}^1$, so it must be a constant function on this $\mathbb{P}^1$. Therefore, $f(p) = f(q)$.

For every $f \in k[\mathbb{P}^n]$ and $p \neq q \in \mathbb{P}^n$, we have $f(p) = f(q)$. Therefore, $k[\mathbb{P}^n] = k$

4.3 Elementary properties of regular maps WEEK5

4.3.1 Proposition The identity map is regular. The composition of two regular maps is regular.

4.4 The Veronese embedding WEEK5

Let $n \geq 1$, and consider the k-vector space of degree n homogeneous polynomials in X, Y. This vector space has dimension $n+1$. Choose a basis, for example, let us take $X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n$. Then we have a regular map

$$\begin{aligned} v_n \colon \mathbb{P}^1 &\to \mathbb{P}^n \\ [X : Y] &\mapsto [X^n : \cdots : Y^n]. \end{aligned}$$

4.4.1 Proposition (Veronese curves) The image of v_n is a closed subset of C of $\mathbb{P}^n$. If we denote the homogeneous coordinates on $\mathbb{P}^n$ by $[U_0 : \cdots : U_n]$, then C is cut out by the equations

$$\{U_i U_j - U_k U_\ell \mid 0 \leq i, j, k, l \leq n \text{ and } i + j = k + \ell.$$

Prove this.

Proof. $\boxed{\subset}$ Let $U = [u_0 : \ldots : u_n] \in v_n(\mathbb{P}^1)$. Then by definition of v_n, we have for all $0 \leq i \leq n$ that $u_i = x^{n-i}y^i$ for some $x, y \in k$. Then for all $0 \leq i, j, k, l \leq n$ satisfying $i + j = k + l$ we have

$$\begin{aligned} u_i u_j - u_k u_l &= x^{n-i}y^i x^{n-j}y^j - x^{n-k}y^k x^{n-l}y^l \\ &= x^{2n-(i+j)}y^{i+j} - x^{2n-(k+l)}y^{k+l} \\ &= x^{2n-(i+j)}y^{i+j} - x^{2n-(i+j)}y^{i+j} && \text{by } i+j=k+l \\ &= 0. \end{aligned}$$

So $U \in v_n(\mathbb{P}^1)$ satisfies all the given equations and hence $U \in C$.

$\boxed{\supset}$ Given any element of C, we want to find an element of $\mathbb{P}^2$ which maps to U via v_n. I claim that elements of C can be categorised into three classes:

1. $U = [1 : 0 : \ldots : 0]$
2. $U = [0 : \ldots : 0 : 1]$
3. $U = [u_0 : \ldots : u_n]$ with all u_i nonzero.

Proof of classification. To see this, we first show that U cannot have both u_0 and u_n zero. Suppose this is the case with $u_0 = u_n = 0$. Then consider the following procedure which shows that every other u_i must be zero.

- Let $S = \{1, \ldots, n-1\}$ represent the induces for which u_i are nonzero.
- While S is nonempty:
  - Choose any $i \in S$.
  - Let $l, r \in \{0, \ldots, n\} \setminus S$ be the largest and smallest elements respectively such that $l \leq i \leq r$.
  - By definition of S, we have $u_l = u_r = 0$. So by the condition on C, we have $u_i u_{l+r-i} = u_l u_r = 0$, so either u_i or u_{l+r-i} is zero. Remove from S the corresponding index i or $l + r - i$.

Note that when this procedure terminates, S becomes nonempty and we get that $U = [0 : \ldots : 0]$ which is not a valid element of the projective space. It should be clear from construction that the procedure indeed terminates and is valid as in each step we can always find lower and upper bounds l, r not in S for any chosen i. Moreover, since no element u_i with $l \leq i \leq r$ has yet to be shown to be zero, each iteration of the while loop indeed removes an element of S as $l \leq l + r - i \leq r$ for $l \leq i \leq r$. Equation 12 illustrates an example of the procedure.

$$\begin{aligned}
&[0 : u_1 : u_2 : u_3 : u_4 : 0] \\
&[0 : u_1 : 0 : u_3 : u_4 : 0] \\
&[0 : u_1 : 0 : 0 : u_4 : 0] \\
&[0 : 0 : 0 : 0 : u_4 : 0] \\
&[0 : 0 : 0 : 0 : 0 : 0]
\end{aligned} \tag{12}$$

A similar argument can be used to classify elements of U as described above. For (1) and (2), suppose without loss of generality that $u_0 = 1$. Then the exact same procedure above still shows that $u_1 = \ldots = u_n = 0$ except we note that $0 \notin S$ no longer means that $u_0 = 0$ but is simply used to help argue that every other element is zero.

For (3), we now suppose u_0 and u_n are nonzero. Now suppose $u_j = 0$ for some $0 < j < n$ in order to derive a contradiction. But by the condition on C, we have $u_0 u_n = u_j u_{n-j} = 0$, implying that either u_0 or u_n is zero which contradicts our assumption. $\square$

Having classified the elements of C, we now show what elements map to them under v_n. In case (1), we have

$$v_n([1:0]) = [1:0:\ldots:0]$$

and similarly for case (2), we have

$$v_n([0:1]) = [0:\ldots:0:1].$$

For case (3), I claim that

$$v_n([u_0, u_1]) = [u_0 : \ldots : u_n].$$

We have that $v_n([u_0, u_1]) = [w_0 : \ldots : w_n]$ with $w_i = u_0^{n-i}u_1^i$. To show that $[u_0 : \ldots : u_n] = [w_0 : \ldots : w_n]$, we want to show that these elements viewed as vectors are linearly dependent, or equivalently

$$\begin{bmatrix} u_0 & \ldots & u_n \\ w_0 & \ldots & w_n \end{bmatrix}$$

has rank 1, or equivalently again in linear algebra that all 2×2 minors vanish. This is the same as showing that for all $0 \leq i, j \leq n$ that $u_i w_j = u_j w_i$. However since each u and hence w component is nonzero, it suffices to show that $u_i w_{i+1} = u_{i+1} w_i$ for all $0 \leq i < n$ as multiplying these equations together gives us

$$u_i u_{i+1} \ldots u_{j-1} w_{i+1} \ldots w_{j-1} w_j = w_i w_{i+1} \ldots w_{j-1} u_{i+1} \ldots u_{j-1} u_j$$

which when divided by the nonzero element $u_{i+1} \ldots u_{j-1} w_{i+1} \ldots w_{j-1}$ gives us $u_i w_j = u_j w_i$.

But for all $0 \leq i < n$ we have

$$\begin{aligned} u_i w_{i+1} &= u_i u_0^{n-i-1} u_1^{i+1} \\ &= u_0^{n-i-1} u_1^i \cdot u_i u_1 \\ &= u_0^{n-i-1} u_1^i \cdot u_{i+1} u_0 && \text{by condition from } C \\ &= u_{i+1} u_0^{n-i} u_1^i \\ &= u_{i+1} w_i \end{aligned}$$

so indeed we have $v_n([u_0, u_1]) = [w_0 : \ldots : w_n] = [u_0 : \ldots : u_n].$ □

4.4.2 Proposition (Veronese curves continued) The map $v_n \colon \mathbb{P}^1 \to C$ is in fact an isomorphism.

Define the inverse map.

Proof. The inverse map $w_n : C \to \mathbb{P}^1$ is defined as

$$\begin{aligned} w_n([U_0, \ldots, U_n]) =& [U_i : U_{i+1}] \\ & \text{if } U_i \neq 0 \text{ or } U_{i+1} \neq 0 \text{ for } i = 0, \ldots, n-1 \end{aligned}$$

To see that the map is well defined, observe that if $[U_0, \ldots, U_n] \in C$, then it must satisfy

$$U_i U_j - U_k U_l = 0 \text{ for } i + j = k + l$$

so in particular we have that for $i, j = 1, \ldots, n-1$,

$$[U_i : U_{i+1}] = [U_j : U_{j+1}]$$

since $U_i U_{j+1} - U_{i+1} U_j = 0$.

Now, I claim that w_n is the inverse map of v_n. To see this, notice that

$$\begin{aligned} w_n \circ v_n([X : Y]) &= w_n([X^n : X^{n-1}Y : \ldots : Y^n]) \\ &= [X^n : X^{n-1}Y] \\ &= [X : Y] \end{aligned}$$

where the second line follows from the fact that at least one of X or Y is nonzero. Thus, $w_n \circ v_n$ is the identity on $\mathbf{P}^1$.

For the other direction, we have

$$\begin{aligned} v_n \circ w_n([U_0 : \ldots : U_n]) &= v_n([U_i : U_{i+1}]) \\ &= [U_i^n : \ldots : U_{i+1}^n] \end{aligned}$$

Now I claim that in $\mathbf{P}^n$, $[U_0 : \ldots : U_n] = [U_i^n : \ldots : U_{i+1}^n]$. To check this, we need to show that all the cross terms are equal. Let $j, k \in \{0, \ldots, n\}$ and suppose without loss of generality that $k - j = m > 0$. Then we have that j^{th} and k^{th} cross terms are equal if and only if

$$\begin{aligned} U_j(U_i^{n-k} U_{i+1}^k) &= U_k(U_i^{n-j} U_{i+1}^j) \\ \iff U_j U_{i+1}^{k-j} &= U_k U_i^{k-j} \\ \iff U_j U_{i+1}^m &= U_{j+m} U_i^m \end{aligned}$$

But we know that $U_j U_{i+1} = U_{j+1} U_i$, by the construction of C. Hence, it follows by induction that $U_j U_{i+1}^m = U_{j+m} U_i^m$. Therefore,

$$U_j U_i^{n-k} U_{i+1}^k = U_k U_i^{n-j} U_{i+1}^j$$

so the j^{th} and k^{th} cross terms are equal. We can repeat the same argument for $k < j$. Thus, $[U_0 : \ldots : U_n] = [U_i^n : \ldots : U_{i+1}^n]$, so it follows that $v_n \circ w_n$ is the identity on C.

This concludes the proof that w_n is the inverse map of v_n. □

The proposition above generalises to all dimensions. Consider the k-vector space of degree n homogeneous polynomials in $X_0, \ldots, X_m$. It has dimension $N = \binom{n+m}{m}$. Choosing a basis gives a map $\mathbb{P}^m \to \mathbb{P}^N$. The image of this map is a closed subvariety Z and the map $\mathbb{P}^m \to Z$ is an isomorphism. The equations of Z and the description of the inverse map are analogous to the $m = 1$ case, but (understandably) somewhat more cumbersome.

4.5 Example: Conics in $\mathbb{P}^2$

WEEK5

The 2-nd Veronese embedding maps $\mathbb{P}^1$ isomorphically onto the zero-locus of a degree 2 equation in $\mathbb{P}^2$. More explicitly, the image of the map

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^2 \\ [X : Y] &\mapsto [X^2 : XY : Y^2] \end{aligned}$$

is the set $V(UW - V^2)$. Now recall a theorem from linear algebra. You may have proved this only over $\mathbb{C}$ or even over $\mathbb{R}$ (in which case, there are some signs you have to reckon with), but the same proof works for all algebraically closed fields of characteristic $\neq 2$.

4.5.1 Theorem (quadratic forms) Let k be an algebraically closed field of characteristic $\neq 2$ and let q be a quadratic form on a k-vector space V. Then there exists a basis $X_0, \ldots, X_n$ for V such that

$$q(X_0, \ldots, X_n) = X_0^2 + \cdots + X_\ell^2.$$

The form is called non-degenerate if $\ell = n$.

4.5.2 Corollary Let Q be a non-degenate conic in $\mathbb{P}^2$. Then Q is isomorphic to $\mathbb{P}^1$.

Proof. All non-degenerate conics are isomorphic to each other, and we know that at least one of them—the 2nd Veronese image of $\mathbb{P}^1$—is isomorphic to $\mathbb{P}^1$. □

4.5.3 Question What do the degenerate conics in $\mathbb{P}^2$ look like?

5 Products and the Segre embedding

5.1 Definition of the product variety WEEK6

If X and Y are algebraic varieties, then their product set $X \times Y$ is naturally an algebraic variety. This, in theory, should be completely straightforward (and it is), but you have to be slightly careful because the Zariski topology of $X \times Y$ is *not* the product topology.

First, suppose $X = \mathbb{A}^m$ and $Y = \mathbb{A}^n$, then $X \times Y = \mathbb{A}^{m+n}$ is an algebraic variety. Observe that the Zariski topology on $\mathbb{A}^{m+n}$ is *not* the product topology.

Second, if $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^m$ are both closed (or open), then $X \times Y \subset \mathbb{A}^{m+n}$ is closed (or open), so it is naturally an algebraic variety.

Prove that products of closed (or open) are closed (or open).

If X×Y is a subset of A^{m+n} such that X and Y are closed,
then there exists two finite sets of polynomials F and G with
V(F) = X and V(G) = Y.
Thus, we can define X×Y in A^{m+n} by the vanishing set of polynomials
F∪G.
This implies X×Y is closed in A^{m+n}.

For X and Y are open, we can prove X×Y is open using closed sets
X^C and Y^C.

Third, by combining the cases of closed/open and taking intersections, we get that if X and Y are locally closed, then $X \times Y \subset \mathbb{A}^{m+n}$ is also locally closed, and hence an algebraic variety. So the case of quasi-affine varieties is done.

In general, suppose X has the quasi-affine atlas $\{\phi_i \colon U_i \to V_i\}$ and Y has the quasi-affine atlas $\{\phi'_j \colon U'_j \to V'_j\}$. Then the product $X \times Y$ is covered by the sets $U_i \times U'_j$. We *declare* the product map $U_i \times U'_j \to V_i \times V'_j$ to be a homeomorphism; that is, we give $U_i \times U'_j$ the Zariski topology of $V_i \times V'_j$. Then, we declare a set $Z \subset X \times Y$ to be closed (or open) if and only if for all i, j, the intersection $Z \cap U_i \times U'_j$ is closed (or open) in $U_i \times U'_j$. It is easy to check that this gives $X \times Y$ a topology under which $U_i \times U'_j$ is an open cover, and the maps

$$\phi_i \times \phi'_j \colon U_i \times U'_j \to V_i \times V'_j$$

are a family of compatible charts.

5.1.1 Proposition The two projection maps $X \times Y \to X$ and $X \times Y \to Y$ are regular. A map $\phi \colon Z \to X \times Y$ is regular if and only if the two component maps $\phi_1 \colon Z \to X$ and $\phi_2 \colon Z \to Y$ are regular.

Proof. Skipped (for being easy). □

5.1.2 Remark If you have seen some category theory (in particular, Yoneda's lemma), you will see that the above proposition characterises the product "uniquely up to a unique isomorphism."

5.2 Example

WEEK6

Write down the charts of $\mathbb{P}^1 \times \mathbb{P}^1$, and the transition function between one pair of charts.

The charts of $P^1 \times P^1$ are:
$\phi_0 \times \phi_0 : ([1 : x], [1 : x']) \to (x, x')$
$\phi_0 \times \phi_1 : ([1 : x], [x' : 1]) \to (x, x')$
$\phi_1 \times \phi_0 : ([x : 1], [1 : x']) \to (x, x')$
$\phi_1 \to \phi_1 : ([x : 1], [x' : 1]) \to (x, x')$
One example of a transition map is

$$\phi_0 \times \phi_1) \circ (\phi_0 \times \phi_0)^{-1} : (x, x') \to (x, \frac{1}{x'}).$$

5.3 Closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$

WEEK6

Let $F \subset k[X_0, \dots, X_n, Y_0, \dots, Y_m]$ be a bi-homogeneous polynomial of bi-degree (a, b). This means that every term in F has X-degree a and Y-degree b. Or equivalently, for any $\lambda, \mu \in k$, we have

$$F(\lambda X_0, \dots, \lambda X_n, \mu Y_0, \dots, \mu Y_m) = \lambda^a \mu^b F(X_0, \dots, X_n, Y_0, \dots, Y_m).$$

Then $V(F) \subset \mathbb{P}^n \times \mathbb{P}^m$ is well-defined and is a closed subset. Same story for bi-homogeneous ideals.

5.4 The Segre embedding

WEEK6

The Segre embedding is a closed embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in a bigger projective space. It is a cool example, but it is also of theoretical importance. The most studied and the most well-behaved varieties are projective varieties (varieties isomorphic to closed subsets of projective space) or somewhat more generally quasi-projective varieties (varieties isomorphic to locally closed subsets of projective space). The Segre embedding shows that this class of varieties is closed under products.

Let $N = (m + 1)(n + 1) - 1$. Consider the Segre map $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ defined by

$$[X_0, \dots, X_n], [Y_0, \dots, Y_m] \mapsto [X_i \cdot Y_j].$$

It is easy to check that this map is regular.

A good way to think about this map is as follows. Think of elements of $\mathbb{P}^n$ as row vectors up to scaling, $\mathbb{P}^m$ as column vectors (up to scaling), and $\mathbb{P}^n$ as $(n+1) \times (m+1)$-matrices up to scaling. Then the product XY of $X \in \mathbb{P}^n$ and $Y \in \mathbb{P}^m$ is an $(n+1) \times (m+1)$ matrix, which taken up to scaling, defines an element of $\mathbb{P}^N$. Observe that matrix XY has rank 1, and hence the Segre map lands in the subspace $Z \subset \mathbb{P}^N$ corresponding to matrices of rank 1.

Now, a rank 1 matrix can be written as a product XY, and up to scaling, such an expression is unique. As a result, the Segre map is a bijection from $\mathbb{P}^n \times \mathbb{P}^m \to Z$. But more is true.

5.4.1 Theorem (Segre embedding) The rank 1 locus $Z \subset \mathbb{P}^N$ is closed, and the Segre map $\mathbb{P}^n \times \mathbb{P}^m \to Z$ is a bi-regular isomorphism.

Proof. Consider an $(n+1) \times (m+1)$ matrix M. Then M has rank 1 if and only if all 2×2 minors of M vanish. Hence, Z is the zero-locus of all 2×2-minors, which are homogeneous polynomials in the entries of the matrix.

To prove that the Segre map is an isomorphism onto Z, we must construct a regular inverse $Z \to \mathbb{P}^n \times \mathbb{P}^m$. $\#+\text{begin}_{\text{skipped}}$ Do it!

$\#+\text{begin}_{\text{proof}}$ We have that Z is the matrices of rank 1 taken up to scaling. Let $M \in Z$, and define a map $\phi : Z \to \mathbb{P}^n \times \mathbb{P}^m$ such that $\phi(M) = (ColM, RowM)$, where $ColM$ is any non-zero column in M and $RowM$ is any non-zero row.

To show that this map is well-defined, suppose there exist two distinct non-zero columns, $ColM$ and $Col'M$ in M, and also two distinct rows, $RowM$ and $Row'M$, in M. Since M has rank 1, all rows are linearly dependent, and all columns are independent. So $Col'M$ is a scalar multiple of $ColM$, and thus they define the same element of $\mathbb{P}^n$. Similarly, $Row'M$ is a scalar multiple of $RowM$ and so they define the same element of $\mathbb{P}^m$. So as elements of $\mathbb{P}^n \times \mathbb{P}^m$, $(Col'M, Row'M)$ is equal to $(ColM, RowM)$. So then our map is well-defined.

To check our map is an inverse, we define ψ to be the Segre map from $\mathbb{P}^n \times \mathbb{P}^m$ to Z.

Then $\psi \circ \phi(M) = \psi(ColM, RowM) = M$, since M has rank 1, so the product $ColM \cdot RowM$ defines M up to scaling.

Also, $\phi \circ \psi(X, Y) = \phi(XY) = (X, Y)$, since X and Y must be non-zero and the well-defined property of ϕ tells us we can take $X = Col(XY)$ and $Y = Row(XY)$.

So $\phi \circ \psi = id_{\mathbb{P}^n \times \mathbb{P}^m}$ and $\psi \circ \phi = id_Z$.

To show ϕ is regular, note that the component map $Z \to \mathbb{P}^n$ is regular since under any charts, ϕ defines a polynomial map. Similarly, the component map $Z \to \mathbb{P}^m$ is a polynomial map in affine coordinates and thus regular. So then ϕ is regular, and since both the component maps are polynomials in affine coordinates, ϕ is also a homomorphism.

So ϕ defines a regular inverse for the Segre map, and therefore the Segre map $\mathbb{P}^n \times \mathbb{P}^m \to Z$ is a bi-regular isomorphism. $\square$

#+end$_{\text{skipped}}$ #+end$_{\text{proof}}$

5.4.2 Definition (Projective and quasi-projective varieties) A *projective variety* is a variety isomorphic to a closed subset of projective space. A *quasi-projective variety* is a variety isomorphic to an open subset of a projective variety.

5.4.3 Proposition (All quasi-affines are quasi-projective) Every quasi-affine variety is quasi-projective.

Proof. The affine space $\mathbb{A}^n$ is (isomorphic to) an open subset of $\mathbb{P}^n$. So a locally closed subset of $\mathbb{A}^n$ is also a locally closed subset of $\mathbb{P}^n$. □

5.4.4 Corollary (of the Segre embedding) If X and Y are (quasi)-projective, then so is $X \times Y$.

Proof. Suppose X and Y are projective, say $X \subset \mathbb{P}^n$ is closed and $Y \subset \mathbb{P}^m$ is closed. Then $X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$ is closed. The Segre embedding shows that $\mathbb{P}^n \times \mathbb{P}^m$ is isomorphic to a closed subset of $\mathbb{P}^N$. Hence $X \times Y$ is isomorphic to a closed subset of $\mathbb{P}^n$. In other words, $X \times Y$ is projective.

In general, suppose X (resp. Y) is an open subset of a projective variety $\overline{X}$ (resp. $\overline{Y}$). Then $X \times Y$ is an open subset of $\overline{X} \times \overline{Y}$, which we proved is projective. So $X \times Y$ is quasi-projective. □

5.4.5 Exercise (Quadric surfaces) The Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ lives in $\mathbb{P}^3$.

Describe the equations that cut out the image. Conclude that every non-degenerate quadric in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Treat elements of \$ P^3 \$ as \$ 2×2 \$ matrices up to scaling, that is, of the form \$ $\left(X_0 \& X_1 \backslash\ \mathrm{X}_2\ \&\ \mathrm{X}_3\ \right)$ \$. The image of the Segre embedding is \$ $\mathrm{V(X_0X_3 - X_1X_2)}$ \$, that is, where the above matrix has zero determinant. \$ $\mathrm{P}^1 \times \mathrm{P}^1$ \$ is isomorphic to its image under the Segre embedding.

Now the polynomial \$ $\mathrm{X_0X_3 - X_1X_2}$ \$ is homogeneous of degree 2 (a quadratic form). In a field of characteristic not equal to 2, any quadratic form \$ $\sum_{\mathrm{i \leq j}} \mathrm{a_{ij}X_iX_j} = \sum_{\mathrm{i \neq j}} 1_{\overline{2a_{\mathrm{ij}}X_{\mathrm{i}}X_{\mathrm{j}}+\sum^{\mathrm{i}}}}\ \mathrm{a_{ii}X_i}^2$ \$. This can be written as \$ $\mathrm{x^TA\ x}$ \$, where \$ A \$ is a symmetric \$ $(\mathrm{n}+1) \times (\mathrm{n}+1)$ \$ matrix. Define a symmetric inner product \$ $\langle\ \backslash, , \rangle$ \$ by \$ $\langle\ \mathrm{x, y}\ \rangle = \mathrm{x^TAy}$ \$. This inner product can be diagonalised by Gram-Schmidt orthogonalisation.

In this case, we have

$$A = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}.$$

Hence, \$ rank(A) = 4 \$, which means that our quadratic form is non-degenerate. It can be written as \$ ~{X}$_0^2$ + ~{X}$_1^2$ + ~{X}$_2^2$ + ~{X}$_3^2$ \$, where \$ ~{X}$_0$ =

$$1\frac{}{2(X_0+X_3),\tilde{}\{X\}_1=\frac{1}{2}(X_1-X_2),\tilde{}\{X\}_2=-\frac{1}{2}i(X_1+X_2),\tilde{}\{X\}_3=-\frac{1}{2}i(X_0-X_3)\$.Theuseof\$i\$isjustifiedsinceourfieldisalgebraicallycl}$$

5.4.6 Exercise ($\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$)

Are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ isomorphic? Use whatever tools you have over your favourite field to answer this.

Proof. Suppose the base field is $\mathbb{C}$. Then every variety has a topology coming from the standard (Euclidean) topology on $\mathbb{C}$. Since polynomial functions are continuous in the Euclidean topology, regular maps between varieties over $\mathbb{C}$ are continuous functions in the Euclidean topology. A regular isomorphism between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ would give a homeomorphism between the two corresponding topological spaces $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$. But from topology, we know that these two topological spaces are not homeomorphic (one reason: they have non-isomorphic homology groups).

Surprisingly, the argument above **can** be made to work over an arbitrary field. There is a version of homology groups for varieties that can be defined purely algebraically, and hence over any field. These are called Chow groups. Once you develop this theory, it is quite easy to compute the Chow groups of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$, and see that they are non-isomorphic. Unfortunately, we won't get to the definition of Chow groups in this class.

A more elementary proof that we will get to is the following. We will prove that there do not exist any non-constant regular maps from $\mathbb{P}^n$ to $\mathbb{P}^m$ if $n > m$. Then it follows that $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are not isomorphic—the former has a non-constant map to $\mathbb{P}^1$ but the latter doesn't. $\square$

5.4.7 The diagonal embedding Consider the diagonal map $\Delta\colon \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$. The image of Δ is a closed subset. If we use homogeneous coordinates $[X_0 : \cdots : X_n]$ and $[Y_0 : \cdots : Y_n]$ on the two copies of $\mathbb{P}^n$, then the image is the vanishing set of the bi-homogeneous polynomials

$$X_iY_j - X_jY_i \text{ for } 0 \le i, j \le n.$$

Algebraic varieties X for which the image of the diagonal map $\Delta\colon X \to X \times X$ is closed are called *separated*. This condition is analogous to the Hausdorff condition in topology. Not all varieties are separated, but all quasi-projective varieties are.

5.4.8 Proposition All quasi-projective varieties are separated.

Proof. Let X be a quasi-projective variety. We may assume that $X \subset \mathbb{P}^n$. Let $\phi : X \to X \times X$ and $\psi : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ denote the diagonal maps, noting that $\phi(x) = \psi(x)$ for all $x \in X$.

Suppose $y \in \phi(X)$. Then $y \in X \times X$ and there is $x \in X$ such that $\phi(x) = y$. Hence $\psi(x) = y$, so $y \in \phi(\mathbb{P}^n)$. Therefore $\phi(X) \subset (X \times X) \cap \psi(\mathbb{P}^n)$.

Suppose now that $y \in (X \times X) \cap \psi(\mathbb{P}^n)$. Then there is $x \in \mathbb{P}^n$ such that $\psi(x) = y$. That is, $y = (x, x)$, so $x \in X$ because $y \in X \times X$. Thus $y = \phi(x)$, and hence $y \in \phi(X)$. Therefore $(X \times X) \cap \psi(\mathbb{P}^n) \subset \phi(X)$.

It follows that $\phi(X) = (X \times X) \cap \psi(\mathbb{P}^n)$, which is closed in $X \times X$ because $\psi(\mathbb{P}^n)$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$, by 1.4.7. That is, X is separated. $\square$