Regular maps

Recall the notion of regular maps for quasi-affine varieties –

$$X \subset \mathbb{A}^n, \quad Y \subset \mathbb{A}^m$$

open subsets of Zariski closed sets.

Then a map $f : X \to Y$ is regular if for every $a \in X$ there exist $f_1, \dots, f_m$, $g_1, \dots, g_m \in k[x_1, \dots, x_n]$, $g_i(x) \neq 0$ such that $f = \left(\frac{f_1}{g_1}, \dots, \frac{f_m}{g_m}\right)$

in a neighborhood of a.

Using charts, we extend the definition to arbitrary algebraic varieties

$f : X \to Y$ is regular if it is continuous and for every $a \in X$ there exist (eqv. for every) charts (U, V, φ) on X with $a \in U$ & (U', V', φ') on Y with $f(a) \in V'$ such that the map $\bar{f}$ below is regular.

$$\begin{tikzcd}
U \cap \varphi^{-1}(U') \arrow[r, "f"] \arrow[d] & U' \arrow[d] \\
\text{Open in } V \arrow[r, "\bar{f}"] & V'
\end{tikzcd}$$

When X and Y are quasi-projective, there is a more user-friendly criterion.

Say $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$.

<u>Prop</u> $f: X \to Y$ is regular if and only if for every $a \in X$ there exist homog. poly $F_0, \dots, F_m \in k[X_0, \dots, X_n]$ such that not all F_i are 0 at a and
$$f = [F_0 : \dots : F_m]$$
in a neighborhood of a.

<u>Pf</u>. ($\Rightarrow$) Suppose f is regular.

Let $a = [a_0 : \dots : a_n]$. wlog $a_n \neq 0$.
Then a lies in the affine chart
$$\{X_n \neq 0\} \xrightarrow{\sim} \mathbb{A}^n \text{ of } \mathbb{P}^n.$$

Let $b = f(a) = [b_0 : \dots : b_m]$ wlog $b_m \neq 0$.
Then b lies in the affine chart
$$\{Y_m \neq 0\} \xrightarrow{\sim} \mathbb{A}^m \text{ of } \mathbb{P}^m.$$

Restricting the charts to X & Y gives us charts

$$\begin{tikzcd} a \in X \cap \{X_n \neq 0\} \arrow[r, "f"] \arrow[d, "s"] & Y \cap \{Y_m \neq 0\} \arrow[d, "2"] \\ \mathbb{A}^n \supset U \arrow[r, "\bar{f}"] & V \subset \mathbb{A}^m \end{tikzcd}$$

$a \mapsto \bar{a}$

Since $\bar{f}$ is regular, there exist
$$f_0, \dots, f_{m-1}, \; g_0, \dots, g_{m-1} \in k[X_0, \dots, x_{n-1}]$$
such that $g_i(\bar{a}) \neq 0$ for any i and
$$\bar{f} = \left(\frac{f_0}{g_0}, \dots, \frac{f_{m-1}}{g_{m-1}}\right) \quad \text{around } \bar{a}.$$

Let us convert this back to homog. coordinates.
Set $f_m = g_0 \cdots g_{m-1}$ and rename
$$f_i \leftarrow \frac{f_i}{g_i} \cdot g.$$

Then f is given around a by
$$[x_0 : \dots : x_m : 1] \mapsto [f_0(x_0, \dots, x_{m-1}) : \dots : f_m(x_0, \dots, x_{m-1})]$$
We are almost done. We just have to homogenise. Set $d = \max \deg f_i$ and
$$F_i = X_m^d f_i\left(\frac{X_0}{X_m}, \dots, \frac{X_{m-1}}{X_m}\right)$$

Then f is given around a by
$$[X_0 : \dots : X_m] \mapsto [F_0(X_0, \dots, X_m), \dots, F_m(X_0, \dots, X_m)]$$

($\Leftarrow$) is even easier. Suppose we know that f has the stated form around a.
Let $a = [a_0 : \cdots : a_n]$ with $a_n \neq 0$ &
$f(a) = [b_0 : \cdots : b_m]$ with $b_m \neq 0$

Consider the restriction

$$X \cap \{X_n \neq 0\} \cap \{b_m \neq 0\} \xrightarrow{f} Y \cap \{Y_m \neq 0\}$$

The std chart identifies LHS as a quasi-affine in $\mathbb{A}^n$ & RHS as a quasi affine in $\mathbb{A}^m$?
In terms of the charts, the map f looks like

$$\bar{f} : (x_0, \ldots, x_{m-1}) \mapsto \left(\frac{F_0(x_0, \ldots, x_{m-1}, 1)}{F_m(x_0, \ldots, x_{m-1}, 1)}, \ldots, \frac{F_{m-1}(x_0, \ldots, x_{m-1}, 1)}{F_m(x_0, \ldots, x_{m-1}, 1)} \right)$$

which is regular.

$\square$.

Examples :

① $$f : \mathbb{P}^1 \to \mathbb{P}^2$$
$$f : [X:Y] \mapsto [X^2 : XY : Y^2]$$

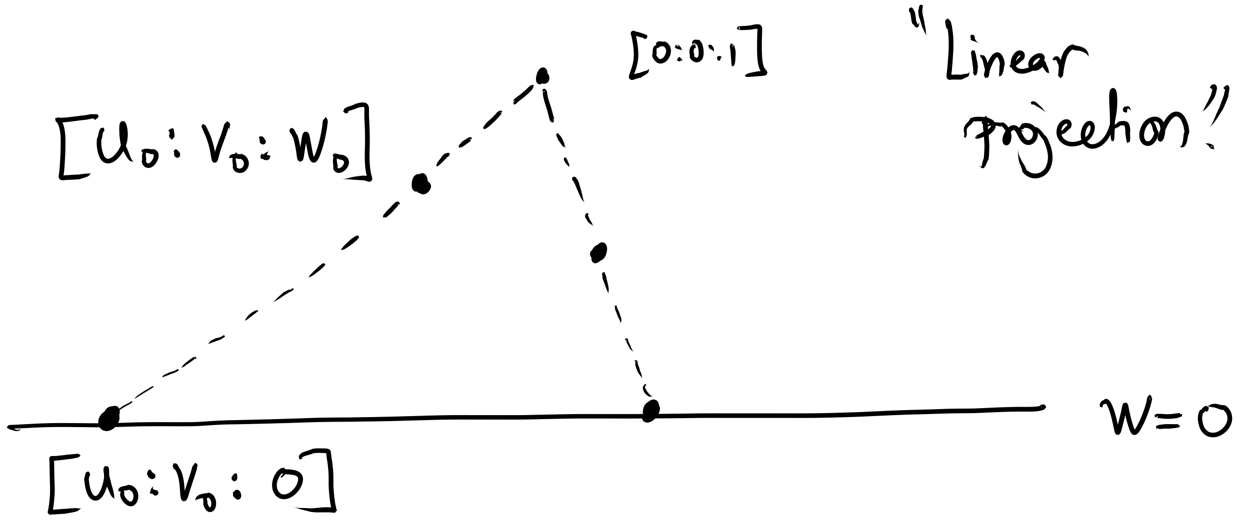
Image $\subset \{ [U:V:W] \mid UW - V^2 \}$.

Inverse

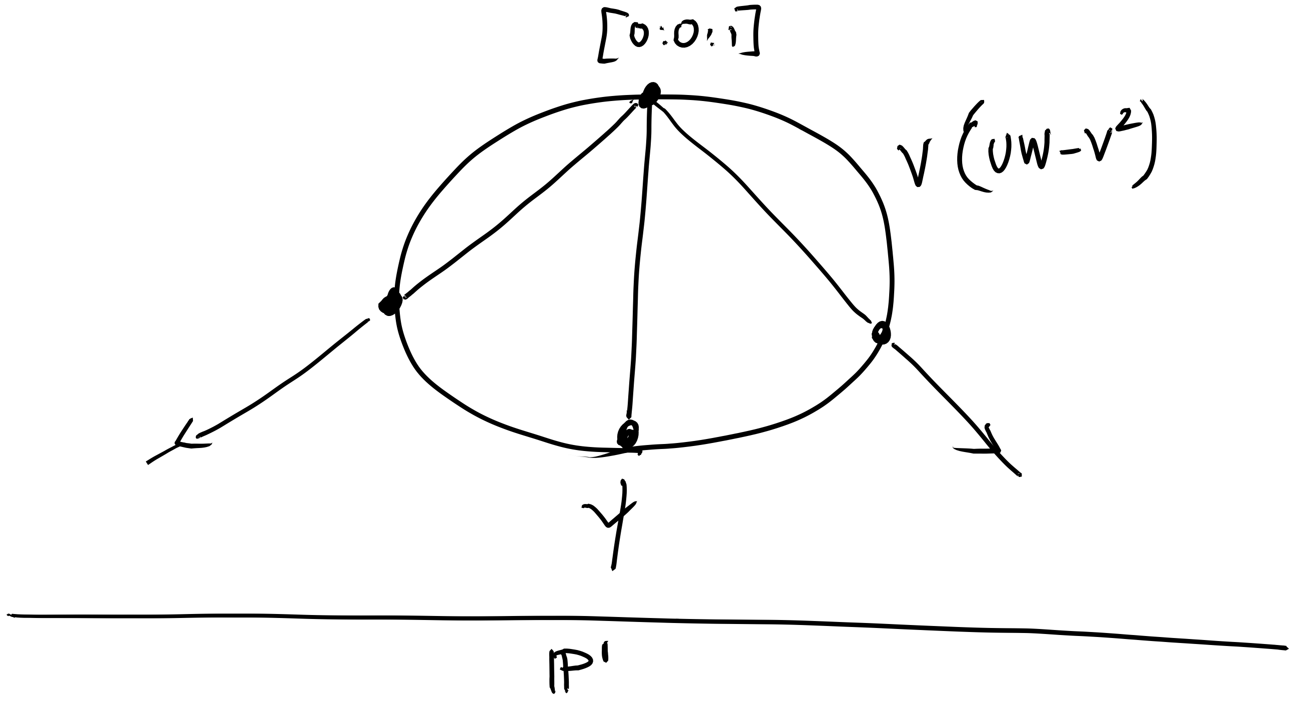
$$g : V(UW - V^2) \longrightarrow \mathbb{P}^1$$
$$\left.\begin{array}{l} g : [U:V:W] \longmapsto [U:V] \\ \quad [0:0:1] \longmapsto [1:0]. \end{array}\right\} \text{regular!}$$

$$\begin{aligned} g &= [U:V:W] \mapsto [U:V] \text{ on } \{W \neq 0\} \\ &= [U:V:W] \mapsto [V:W] \text{ on } \{U \neq 0\}. \end{aligned}$$

Geometry :- What is the map
$$[U:V:W] \longmapsto [U:V] \quad ?$$
$$\mathbb{P}^2 \setminus \{[0:0:1]\} \longrightarrow \mathbb{P}^1$$



"Linear projection"



② $f : \mathbb{P}^1 \to \mathbb{P}^3$

$f : [X:Y] \mapsto [X^3 : X^2Y : XY^2 : Y^3]$ regular!

Image C $\{[U_0:U_1:U_2:U_3] \mid$

$U_1^2 - U_0U_2,\ U_2^2 - U_1U_3,$
$U_1U_2 - U_0U_3$ $\} = X$

$g : X \to \mathbb{P}^1$

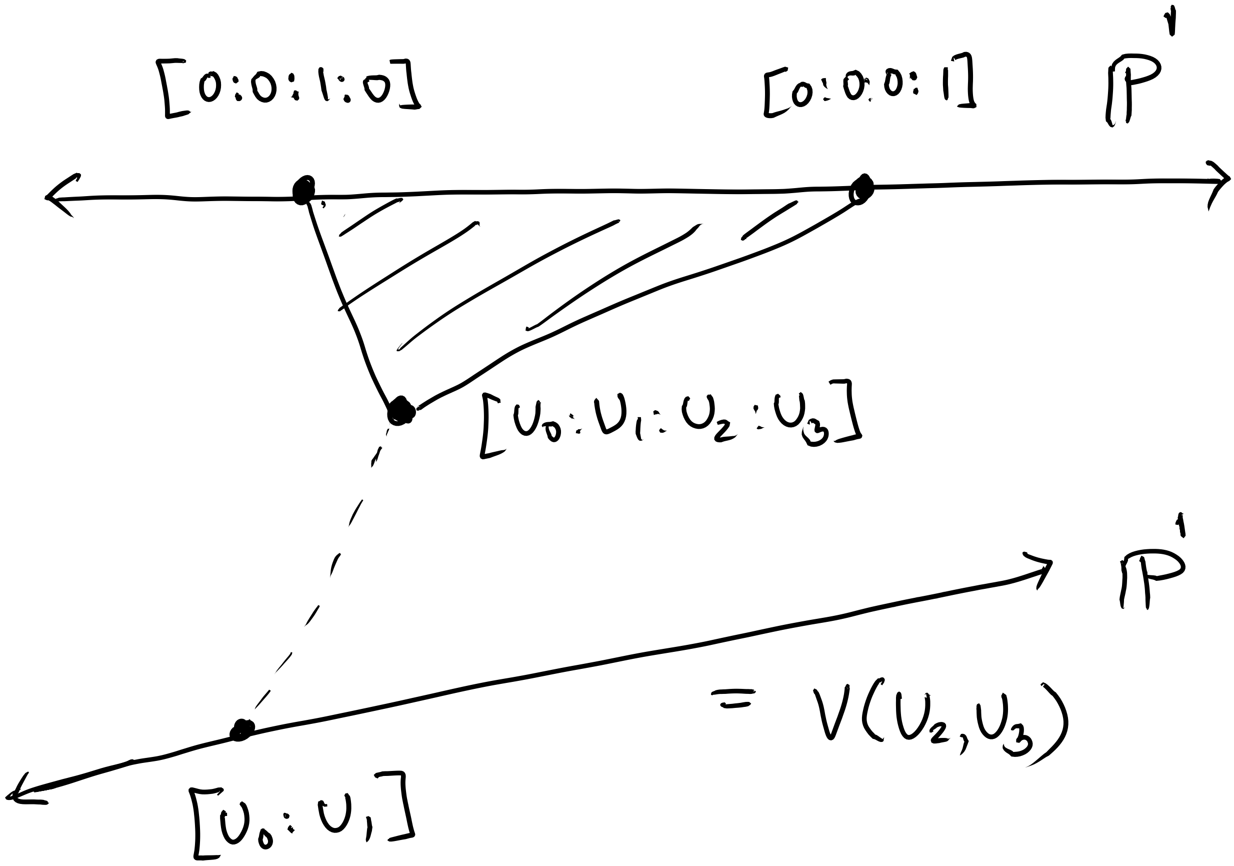
$g : [U_0:U_1:U_2:U_3] \mapsto [U_0:U_1]$ or
$[U_2:U_3]$

is an inverse!

Picture of g:

First $g : [U_0 : U_1 : U_2 : U_3] \mapsto [U_0 : U_1]$

$$\mathbb{P}^3 \setminus \underbrace{V(U_0, U_1)}_{\text{Copy of } \mathbb{P}^1.} \longrightarrow \mathbb{P}^1$$



so g = linear projection with "center of projection" $= V(U_0, U_1)$.

g is not defined along the center of proj. but $g|_X$ extends to $X \cap$ center of proj to a regular map!.

Generalisation

$$f : \mathbb{P}^1 \to \mathbb{P}^n$$
$$[X:Y] \longmapsto [X^n : X^{n-1}Y : \cdots : Y^n]$$

is regular and maps $\mathbb{P}^1$ isomorphically onto

$$\{ [U_0 : \cdots : U_n] \mid U_i U_j - U_l U_k = 0 \text{ if } i+j = l+k \}$$

Def: The image of f is called the <u>rational</u> <u>normal</u> <u>curve</u> in $\mathbb{P}^n$.

No reason to stop at curves

$$v : \mathbb{P}^2 \to \mathbb{P}^5$$
$$v : [X:Y:Z] \mapsto [X^2 : Y^2 : Z^2 : XY : YZ : XZ]$$

Then v is regular.

To find the image, it helps to label the homogeneous coordinates of $\mathbb{P}^5$ by $\{ (i,j,k) \,,\ i+j+k = 2 \ ;\ i,j,k \geq 0 \}$

so

$$\mathbb{P}^5 = \{ [u_{(2,0,0)} : u_{(0,2,0)} : u_{(0,0,2)} : u_{(1,1,0)} : u_{(0,1,1)} : u_{(1,0,1)}] \}$$

Then the image lies in

$$X = V (U_I U_J = U_K U_L \mid I+J = K+L).$$

<u>Thm</u>: $V : \mathbb{P}^2 \to X$ is an isomorphism

<u>Pf</u> (Sketch)

- V is a bijection
- X is covered by the charts $\{u_{(2,0,0)} \neq 0\}$, $\{U_{(0,2,0)} \neq 0\}$, $\{U_{(0,0,2)} \neq 0\}$.
- Inverse is given by $[U_I] \mapsto [u_{(2,0,0)} : u_{(1,1,0)} : u_{(1,0,1)}]$ on first chart & likewise on the other two charts.

$\square$

Def: $X \subset \mathbb{P}^5$ is called the Veronese surface.

$v: \mathbb{P}^2 \to \mathbb{P}^5$ is called the (2^{nd}) veronese embedding.

Why stop at 2^{nd} & why stop at a surface?

Define $v_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ by

$$[X_i] \mapsto [X^I \mid I = (i_0, \dots, i_n) \quad i_j \geq 0, \ \Sigma i_j = d]$$

$$N = \binom{n+d}{n} - 1$$

set $X = V\{[U_I] \mid U_I U_J = U_K U_L$ when $I+J = K+L\}$
$\subset \mathbb{P}^N$.

Thm: $V_d : \mathbb{P}^n \to X$ is an iso.

Pf: Similar to that of $\mathbb{P}^2$ (skipped).

$\square$.

V_d is called the d^{th} Veronese embedding of $\mathbb{P}^n$.

Linear maps, projections, linear subspaces.

Suppose $M : k^{n+1} \to k^{m+1}$ is an injective linear map. Then we get a regular induced map

$$M : \mathbb{P}^n \to \mathbb{P}^m.$$

The image of M is a linear subspace of $\mathbb{P}^m$, namely a set cut out by linear (homogeneous) equations.

In fact, the operation of taking the cone gives a bijection

Linear subspaces of $\mathbb{P}^n$ $\longleftrightarrow$ (Nonzero) vector subspaces of k^{n+1}

The <u>smallest</u> linear subspace containing a set $X \subset \mathbb{P}^n$ is called the <u>linear span</u> of $\mathbb{P}^n$. If $\tilde{x} \in k^{n+1}$ is any non-zero point on the line represented by $x \in X$, then the linear span of X corresponds (under the bijection above) to the vector space span of $\{\tilde{x} \mid x \in X\} \subset k^{n+1}$.

Now suppose $M : k^{n+1} \to k^{m+1}$ has a nonzero kernel $K \subset k^{n+1}$. Then the map

$$M : [x] \mapsto [Mx]$$

is regular on $\mathbb{P}^n \setminus \mathbb{P}K$.

If M is surjective, then M is called the <u>linear projection</u> of $\mathbb{P}^n$ onto $\mathbb{P}^m$ with center $\mathbb{P}K$.

Products and the Segre embedding

Let X and Y be algebraic varieties. The product set $X \times Y$ is naturally an algebraic variety in the following way.

Let $\{(U_i, V_i, \varphi_i)\}$ be an atlas for X and $\{(U_j', V_j', \varphi_j')\}$ be an atlas for Y.

The topology on $X \times Y$ is the following. First on $U_i \times U_j'$ we put the topology such that the bijection

$$\varphi_i \times \varphi_j' : U_i \times U_j' \longrightarrow V_i \times V_j'$$

is a homeomorphism. Here, $V_i \times V_j'$ is an affine variety, and has its Zariski topology (which is NOT the product topology). Now there is a UNIQUE way to define the topology on $X \times Y$ so that $U_i \times U_j'$ form an open cover — Declare $Z \subset X \times Y$ to be closed iff $Z \cap (U_i \times U_j')$ is closed for all i, j.

The charts for $X \times Y$ are

$$\varphi_i \times \varphi_j' : U_i \times U_j' \longrightarrow V_i \times V_j'.$$

With this definition, note that the two projections $P_1 : X\times Y \to X$ & $P_2 : X\times Y \to Y$ are regular. Moreover, the product satisfies the correct universal property :-

<u>Proposition</u> : A map $Z \xrightarrow{\varphi} X\times Y$ is regular if and only if the two maps $P_1\circ\varphi : Z\to X$ and $P_2\circ\varphi : Z\to Y$ are regular.

<u>Proof</u> : (only if) follows because composites of regular maps are regular.

(if) Suppose $\varphi_1 : Z\to X$ & $\varphi_2 : Z\to Y$ are regular.

Choose a chart $U_i\times U_j'$ of $X\times Y$.

Its preimage is $\varphi_1^{-1}(U_i)\cap\varphi_2^{-1}(U_j') = W$ which is an open in Z. Take an affine chart U in Z & consider the map

$$\varphi|_{U\cap W} : U\cap W \to U_i\times U_j'.$$

The lhs & rhs are quasi affine (via the chart maps). Suppose $U_i \subset \mathbb{A}^m$ & $U_j' \subset \mathbb{A}^n$. Then $\varphi|_{U\cap W}$ is regular iff its $(m+n)$ coordinate components are regular. But if φ_1 & φ_2 are regular then the first m

& the last n coord. are regular. But then all coordinates are regular.

$\square$

<u>Example</u> (Most important product)

$$\mathbb{P}^n \times \mathbb{P}^m$$
$$= \{([X_0 : \cdots : X_n], [Y_0 : \cdots : Y_m])\}.$$

What are the closed sets?

Def says – $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ is closed iff $Z \cap (\mathbb{A}^n_i \times \mathbb{A}^m_j)$ is closed for the various charts. But there is a more direct description –

<u>Prop</u>:

Closed sets of $\mathbb{P}^n \times \mathbb{P}^m$ are zero sets of <u>bihomogeneous</u> polynomials in $k[x_0, \ldots, x_n, y_0, \ldots, y_m]$.

A polynomial $p(X_0, \dots, X_m, Y_0, \dots, Y_n)$ is bihomogeneous of bidegree (d,e) if

$$p(\lambda x_0, \dots, \lambda x_n, \mu y_0, \dots, \mu y_n) = \lambda^d \mu^e \, p(x_0, \dots, x_n, y_0, \dots, y_n).$$

e.g. $x_0 y_1^2 + x_1 y_1 y_2$ is bihomog. of bidegree $(1,2)$ in x_0, x_1; y_0, y_1.

Pf of prop (sketch): Very similar to a similar statement about $\mathbb{P}^n$. Let us show that the topology on $\mathbb{P}^n \times \mathbb{P}^m$ defined by taking zero sets of bihom. systems as closed sets restricts to the Zariski topology on the charts.

Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ be the zero set of system of bihomog. equations. Then the set $Z \cap \{x_i \neq 0\} \times \{y_j \neq 0\} = Z \cap \mathbb{A}^n \times \mathbb{A}^m$ is cut out in $\mathbb{A}^n \times \mathbb{A}^m$ by the system obtained by dehomogenising (i.e. set $x_i = y_j = 1$), so it is closed.

Conversely if $Z \subset \{x_i \neq 0\} \times \{y_j \neq 0\}$ is a Zariski closed set, then it is the intersection of a set of $\mathbb{P}^n \times \mathbb{P}^m$ with $\{x_i \neq 0\} \times \{y_j \neq 0\}$. This bigger set is the zero set of the system obtained by homogenising wrt X_i & Y_j

$$P^{hom}(X_0, \dots, X_m ; Y_0, \dots, Y_n) =$$

$$X_i^{x\text{-}\deg P} \; Y_j^{Y\text{-}\deg P} \; p\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}, \frac{Y_0}{Y_j}, \dots, \frac{Y_m}{Y_j}\right)$$

$\square$.

<u>Rem</u>: Another description – the topology on $\mathbb{P}^n \times \mathbb{P}^m$ is the quotient topology from the Zariski topology on

$$(\mathbb{A}^{n+1} \setminus \{0\}) \times (\mathbb{A}^{m+1} \setminus \{0\})$$

The Segre embedding.

<u>Example</u>: $s : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$$([x:y],[u:v]) \longmapsto [xu : yu : xv : yv]$$

This map is regular (check on charts).

Image $\subset V(AC - BD)$.

<u>Claim</u>: $s : \mathbb{P}^1 \times \mathbb{P}^1 \to X$ is an isomorphism.

<u>Pf</u>: Inverse

$$[A:B:C:D] \mapsto ([A:B],[A:C])$$

or $([C:D],[A:C])$

or $([A:B],[B:D])$

or $([C:D],[A:C])$

— at least one formula makes sense!

$\square$.

The geometry of a quadric in $\mathbb{P}^3$

(char $k \neq 0$).

We saw in the tutorial that all non-deg. quadric hypersurfaces in $\mathbb{P}^3$ form one iso. class. We just saw that the quadric $V(AD - BC)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. So

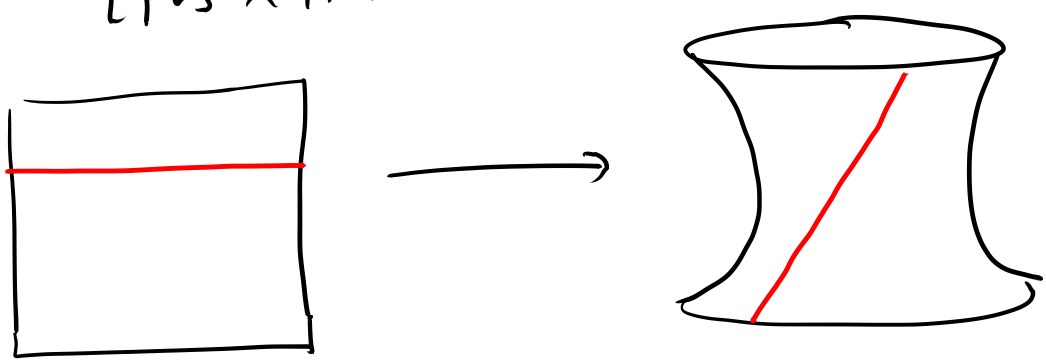
Thm: Any non-deg. quadric hypersurface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Lines on a quadric:

Let us restrict the map

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow X \subset \mathbb{P}^3$$

to $\{pt\} \times \mathbb{P}^1$.



$$[X_0 : Y_0] \times [U : V] \longmapsto [x_0 U : y_0 U : x_0 V : y_0 V]$$

Traces a line as $[U:V]$ varies.

Similarly, if we restrict to

$\mathbb{P}^1 \times \{pt\}$, we get

$$[x:y] \times [u_0:v_0] \longmapsto [xu_0 : yu_0 : xv_0 : yv_0]$$

also a line.

meets the previous line in a unique point (as expected), namely $[x_0u_0 : y_0u_0 : x_0v_0 : y_0v_0]$.

So $X \subset V(AD-BC)$ is <u>ruled</u> by two families of lines