ALGEBRAIC GEOMETRY: PRACTICE QUESTIONS

Here are some practice questions to test your understanding.

(1) Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes. Find $I(X)$.

(2) Let $I, J \subset k[x_1, \dots, x_n]$ be ideals. Denote byIJ the ideal generated by $\{fg \mid f \in I, g \in J\}$. Prove that $\sqrt{IJ} = \sqrt{I \cap J}$.

(3) Describe explicitly all the maximal ideals of the ring $k[x, y]/(x^2 - y^3)$.

(4) Give an example of a regular map $f\colon X \to Y$ that is a bijection, but the inverse map is not regular.

(5) Let $f\colon X \to Y$ be a regular map between two affine varieties. Prove that the graph of f, defined by

$$\Gamma = \{(x, f(x)) \mid x \in X\}$$

is a closed subset of $X \times Y$.

(6) char $k \neq 2$. Construct an isomorphism between the affine variety defined by $x^2 + y^2 = 1$ in $\mathbb{A}^2$ and the affine variety defined by $xy = 1$ in $\mathbb{A}^2$.

(7) Construct an isomorphism from $\mathbb{A}^1 \setminus \{0, 1\}$ to a Zariski closed subspace of $\mathbb{A}^n$ for some n.

(8) Let m, n be positive integers, and let $f_1, \dots, f_n \in k[x_1, \dots, x_m]$. Consider the map $f\colon \mathbb{A}^m \to \mathbb{A}^n$ defined by

$$f(p) = (f_1(p), \dots, f_n(p)).$$

Show that f is surjective if and only if for all $a_1, \dots, a_n \in k$, the ideal $\langle f_1 - a_1, \dots, f_n - a_n\rangle$ is not the unit ideal.

(9) Let $X = V(y^2 - x^3)$ and $U = X \setminus \{(0, 0)\}$. Show that the function y/x on U does not extend to a regular function on X.

(10) Prove that there does not exist an isomorphism between the hyperbola, defined by $xy = 1$ in $\mathbb{A}^2$, and the line $\mathbb{A}^1$.

(11) Prove that all isomorphisms $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ are of the form $f(x) = ax + b$.

(12) Prove that any two non-empty Zariski open subsets of $\mathbb{A}^n$ have a non-empty intersection.

(13) Prove that every affine variety is (isomorphic to) a quasi-projective variety.

(14) Let $v\colon \mathbb{P}^2 \to \mathbb{P}^5$ be the degree 2 Veronese map. Describe the image under v of the line $Z = 0$ in $\mathbb{P}^2$.

(15) Prove that every point in an algebraic variety has an open neighbourhood that is isomorphic to an affine variety.

(16) char $k \neq 2$. Construct an isomorphism from the quadric surface $X^2 + Y^2 + Z^2 + W^2 = 0$ in $\mathbb{P}^3$ to $\mathbb{P}^1 \times \mathbb{P}^1$.

(17) Let $C \subset \mathbb{P}^2$ be defined by

$$X^2 + Y^2 + Z^2 = 0.$$

Describe a quasi-affine atlas on C and compute the transition map between any two charts in your atlas.

(18) Let $C \subset \mathbb{P}^2$ be the Fermat cubic curve

$$C = V(X^3 + Y^3 + Z^3).$$

(a) Show that the map $[X : Y : Z] \mapsto [X + Y : Z]$ when restricted to C extends to a regular map $\phi\colon C \to \mathbb{P}^1$.

(b) Describe the fibers of ϕ.