Braids and the PL Sphere

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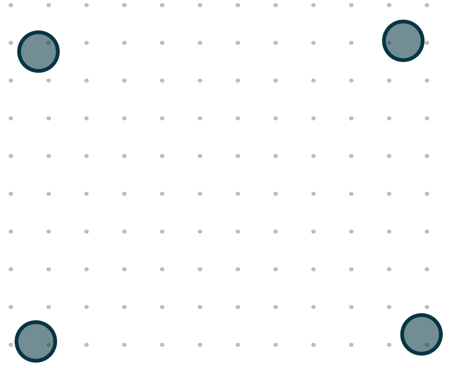
Braid group $\circlearrowright$ Sphere

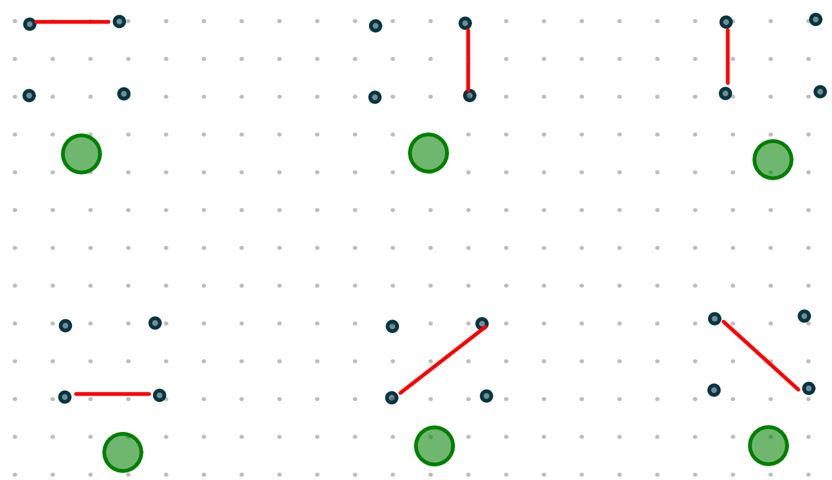
Fix n. For example $n=4$.



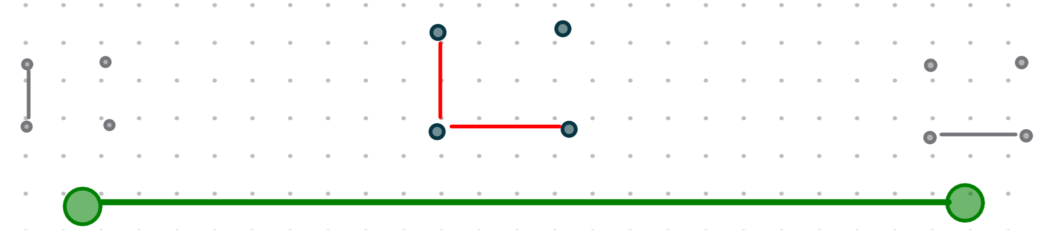
$\rightsquigarrow$ Simplicial complex Δ

Simplicial Complex Δ

0-cells =

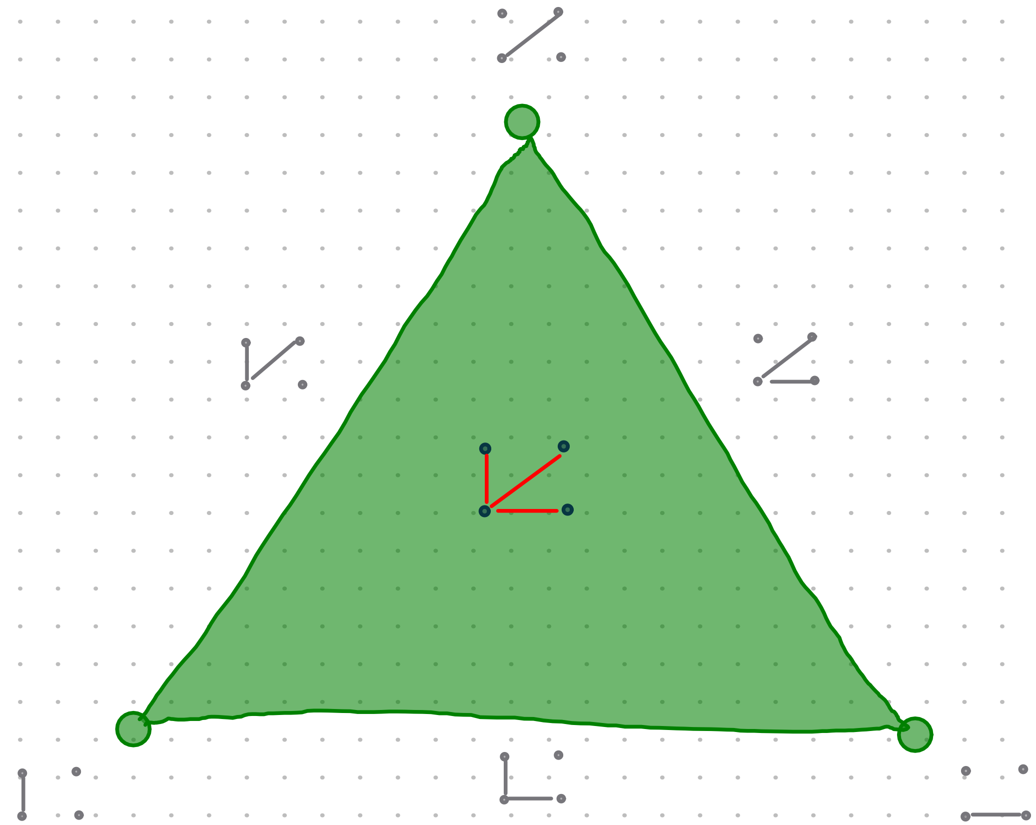


1-cells =



$\underline{\Delta}$

2-cells =



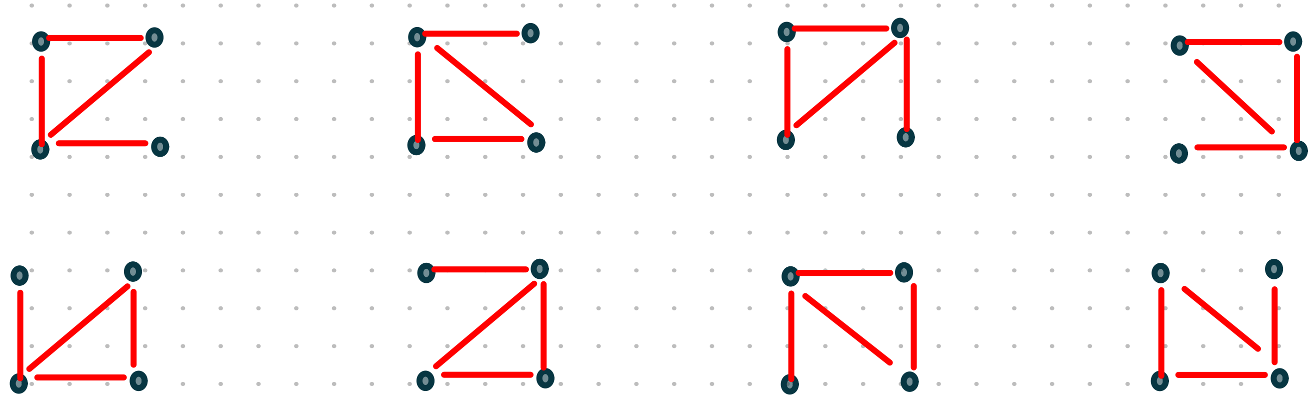
i-cells = { i+1 non crossing segments }

Theorem: For n points, the simplicial complex Δ is homeomorphic to B_{2n-4}.

(Tamari, Stascheff, Milnor 1950–60)

Let $\Sigma = \partial\Delta$

The top cells of Δ are

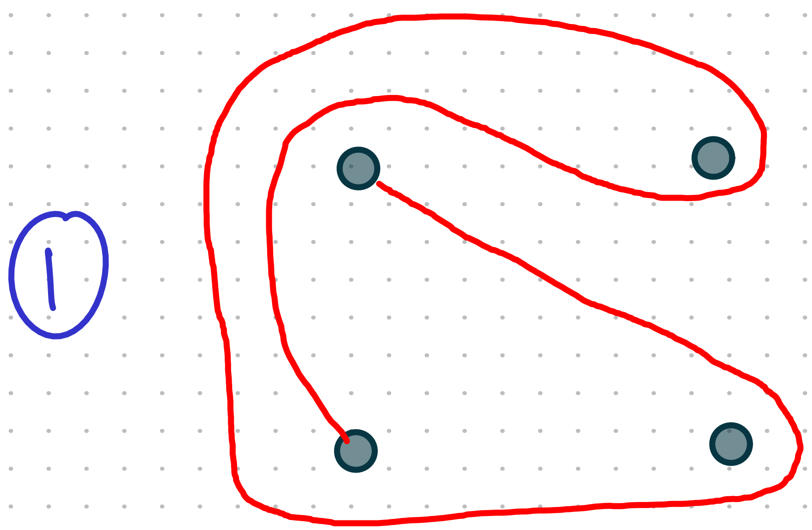


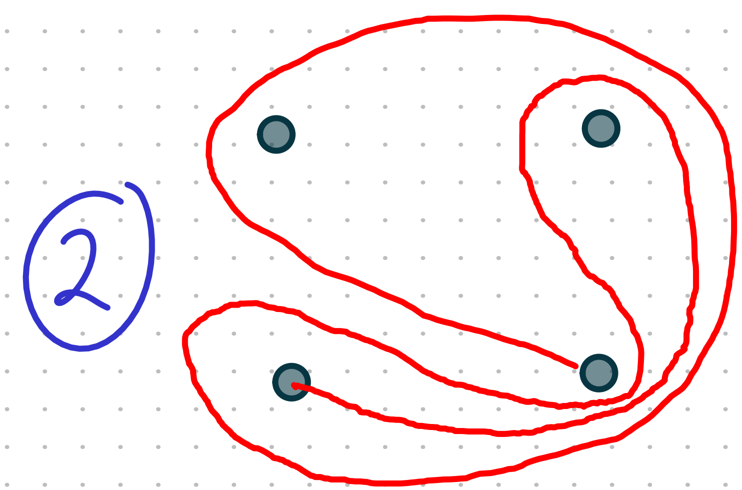
(Triangulations − external edge)

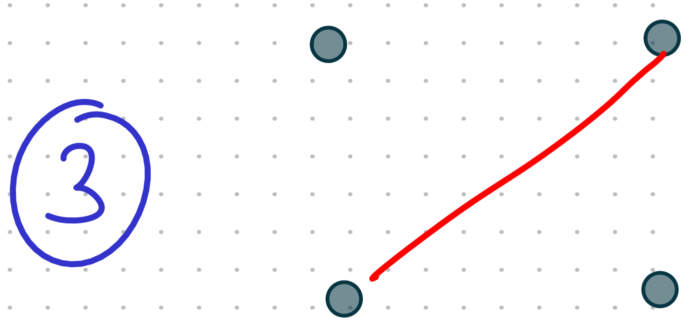
Braid group $\circlearrowright$ Sphere Σ

Arcs

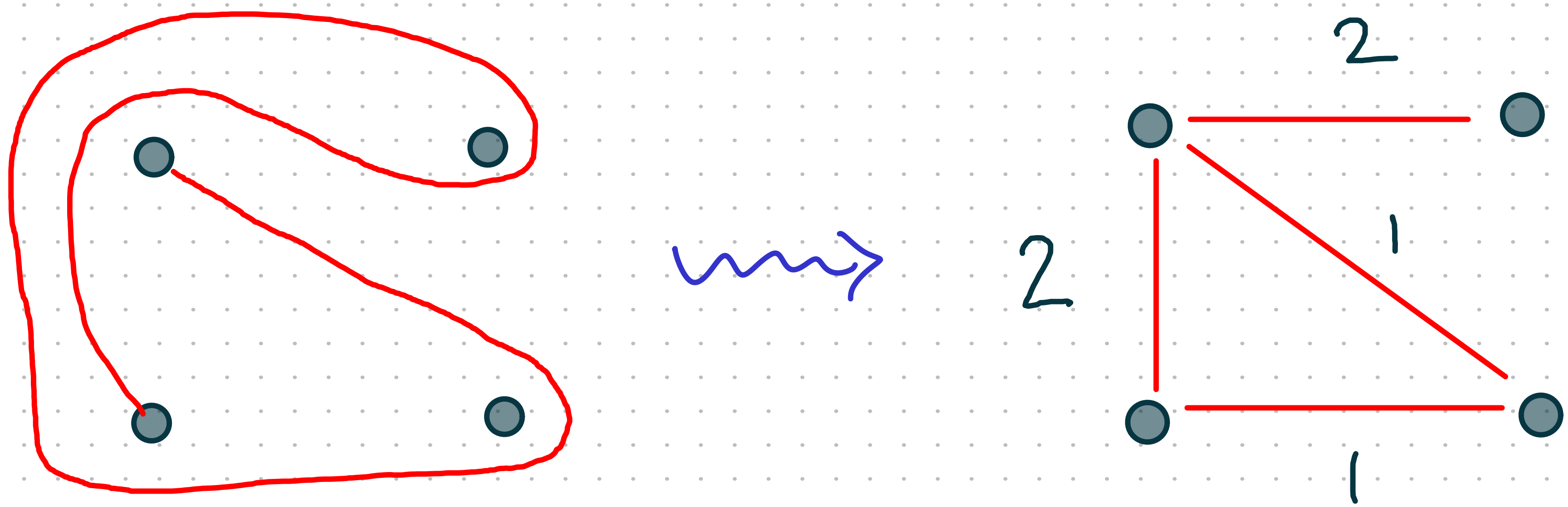
Simple curve connecting two of the n-points & otherwise staying away from them (up to isotopy)



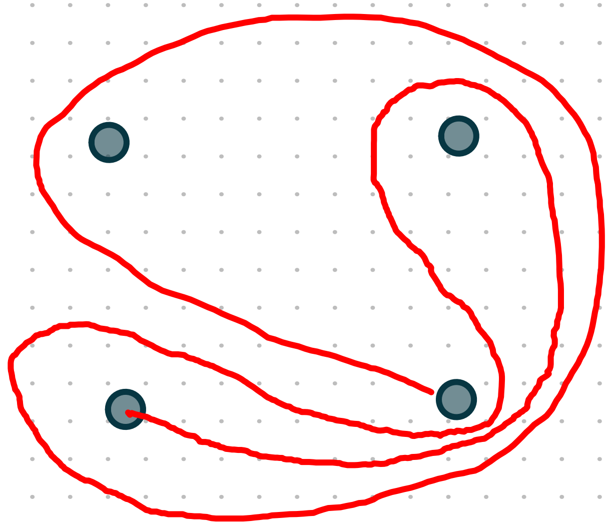


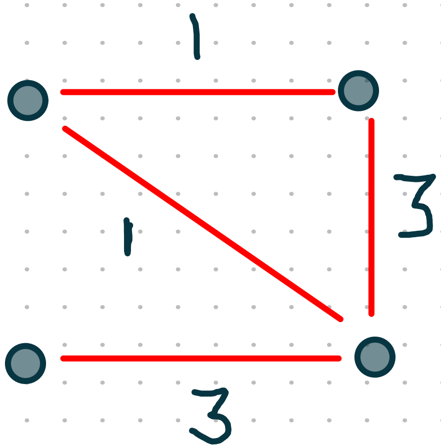


Arc $\rightsquigarrow$ Point on Σ



Arc $\rightsquigarrow$ Point on Σ



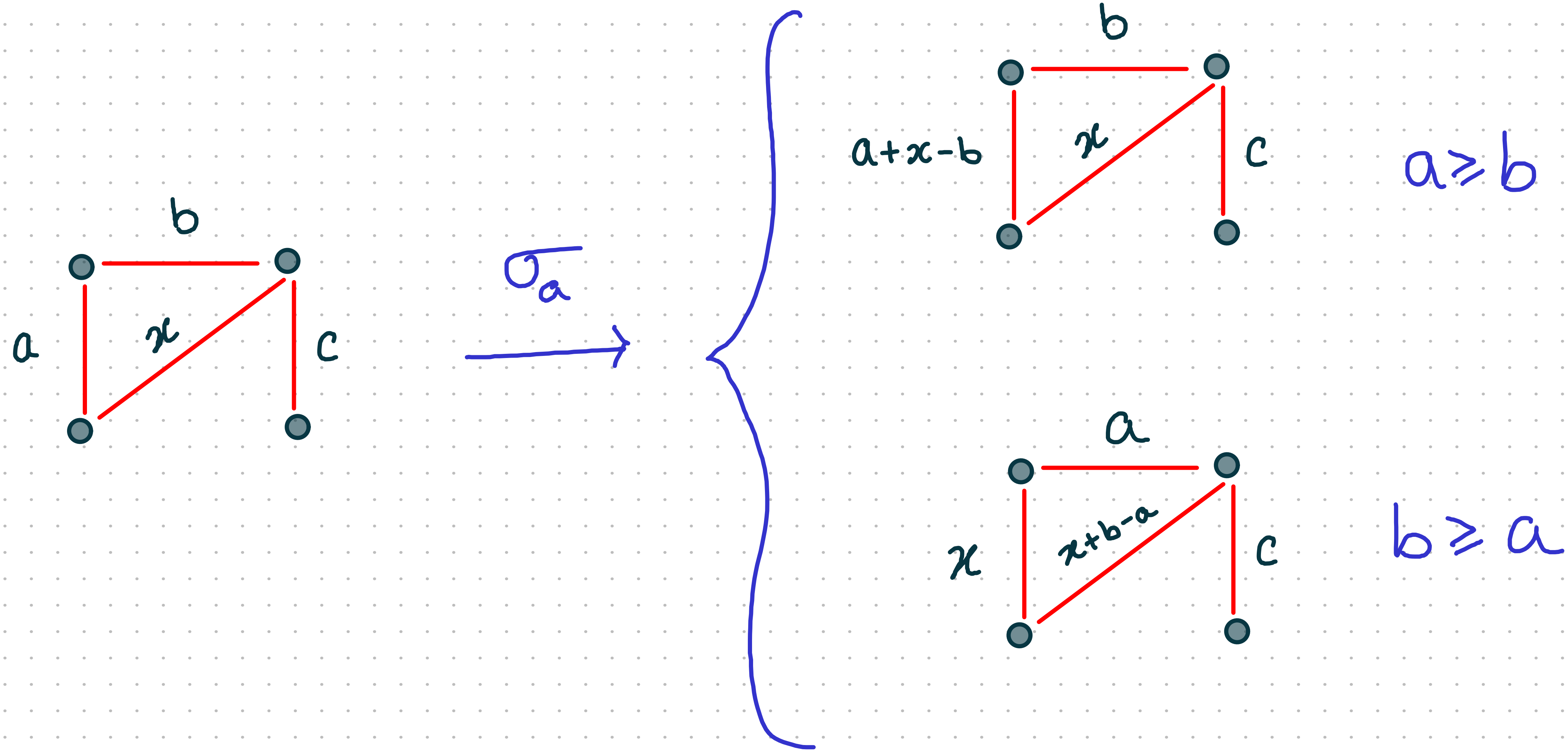


$$\begin{array}{ccc} \{\text{Arcs}\} & \longrightarrow & \Sigma \\ \circlearrowright & & \circlearrowright \\ \text{Braids} & & \text{Braids} \end{array}$$

Th (Bapat, –, Licata)

1. The map above is injective & its image is dense.

2. The B_{n-1} action on $\{\text{Arcs}\}$ by Dehn twists extends to an action on Σ by PL homeomorphisms



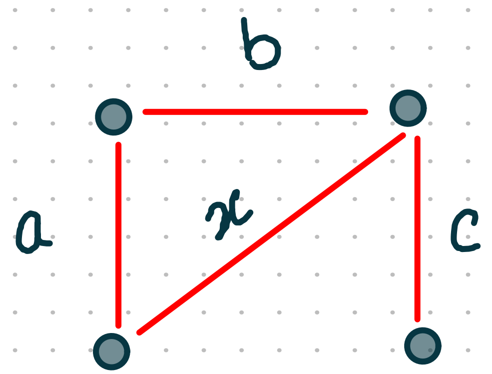
b
a
x
c
σ_a
b
a+x-b
x
c
a ≥ b
a
x
x+b-a
c
b ≥ a

Braid group $\circlearrowright$ Sphere Σ

Dynamics

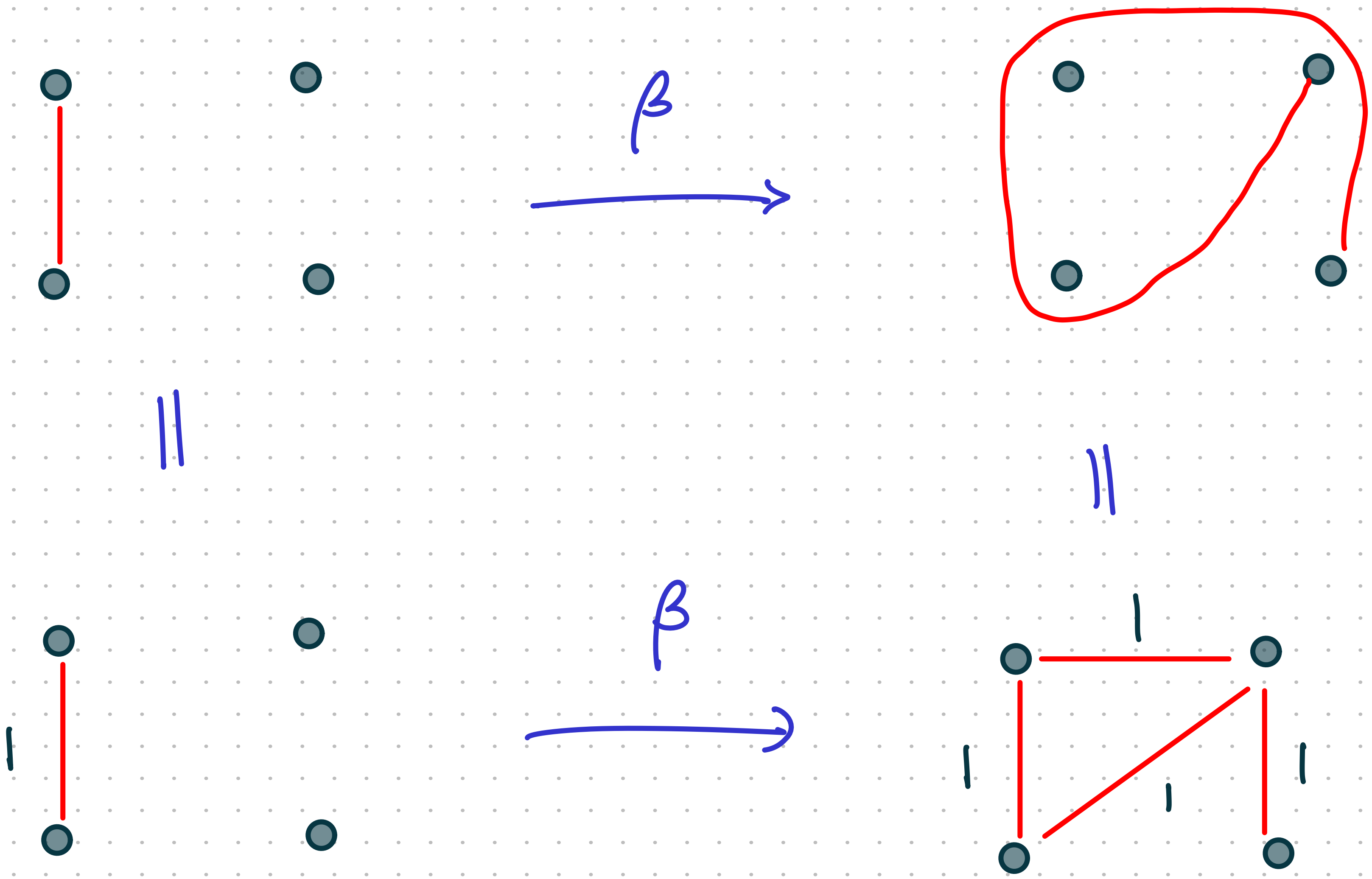
What happens when we iterate a braid?

Example :



$$\beta = \sigma_a \, \sigma_x^{-1} \, \sigma_c \, \sigma_b$$

$$\beta = \sigma_a \sigma_x^{-1} \sigma_c \sigma_b$$



Entropy of β

$$e(\beta, c) := \limsup_{n\to\infty} (\text{size } \beta^n c)^{1/n}$$

$$e(\beta) := \sup_c e(\beta, c)$$

Conjecture (algebraicity of entropy)

$e(\beta, c)$ and $e(\beta)$ are algebraic.

__Conjecture__ (algebraicity of entropy)

Question of

Dmitrov, Haiden, Katzarkov, Kontsevich

"Dynamical systems & categories"

(2014)

Dynamical systems & categories

$$F : \mathcal{C} \to \mathcal{C}$$

Q: Is the dynamics of F like the dynamics of a linear map?

Dynamics of $F : \mathcal{C} \to \mathcal{C}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{C} \arrow[d] \\ K(\mathcal{C}) \arrow[r, "F"] & K(\mathcal{C}) \end{tikzcd}$$

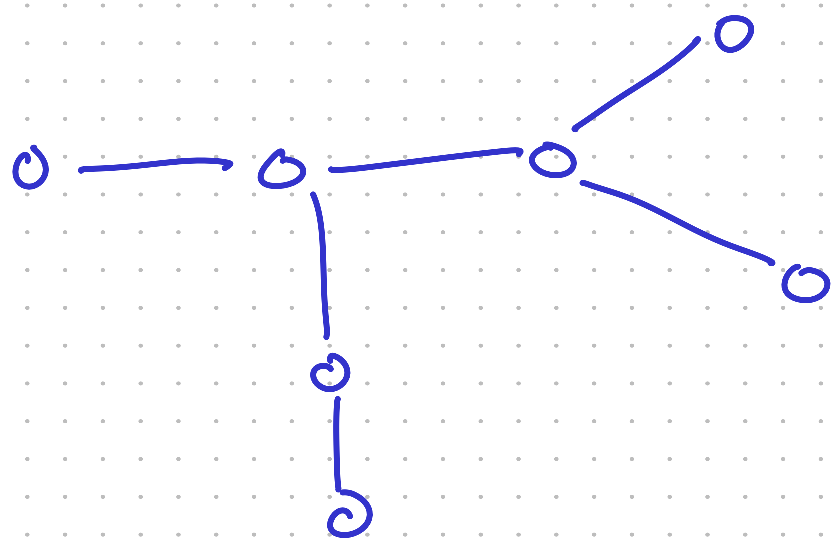
Dynamical systems & categories

$$F : \mathcal{C} \longrightarrow \mathcal{C}$$

Q: Is the dynamics of F like the dynamics of a linear map?

Define $$e(F) = \sup_{c \in \mathcal{C}} \limsup_{n \to \infty} |\beta_c^n|^{1/n}$$

$\hookrightarrow$ algebraic? "largest eigenvalue"



Γ

$\rightsquigarrow$ Category $\mathcal{C}_\Gamma$

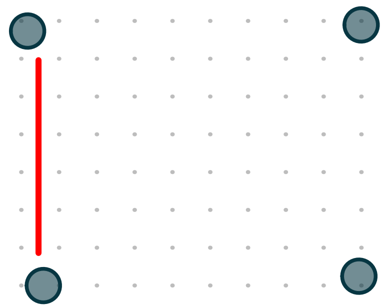
$$
\begin{tikzcd}
\mathcal{C}_\Gamma \arrow[d] & \circlearrowleft & B_\Gamma \arrow[d] \\
K(\mathcal{C}_\Gamma) & \circlearrowleft & W_\Gamma
\end{tikzcd}
$$

For $\Gamma = \circ - \circ - \circ \cdots - \circ$

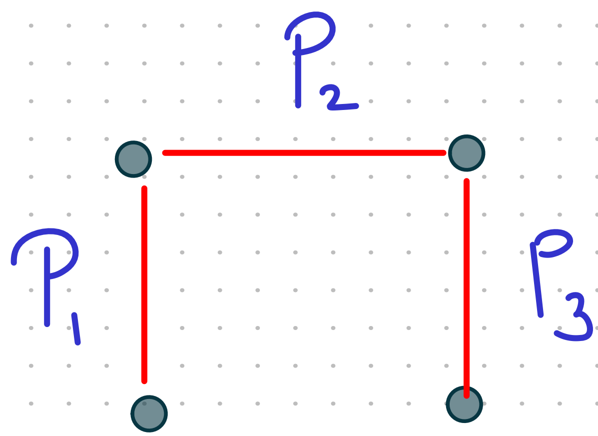
$$\text{Arcs in } n\text{-punctured plane} \rightsquigarrow \text{Objects of } \mathcal{C}_\Gamma$$

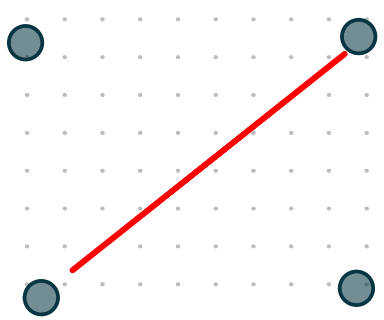
(Khovanov - Seidel)

Arcs $\longrightarrow$ Complexes



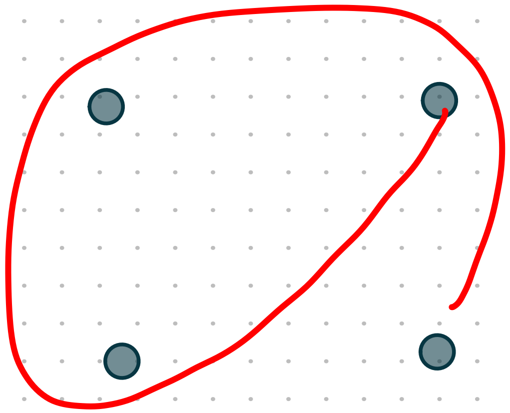
$\rightsquigarrow \quad P_1$





$\rightsquigarrow \quad P_1 \to P_2$

Arcs → Complexes

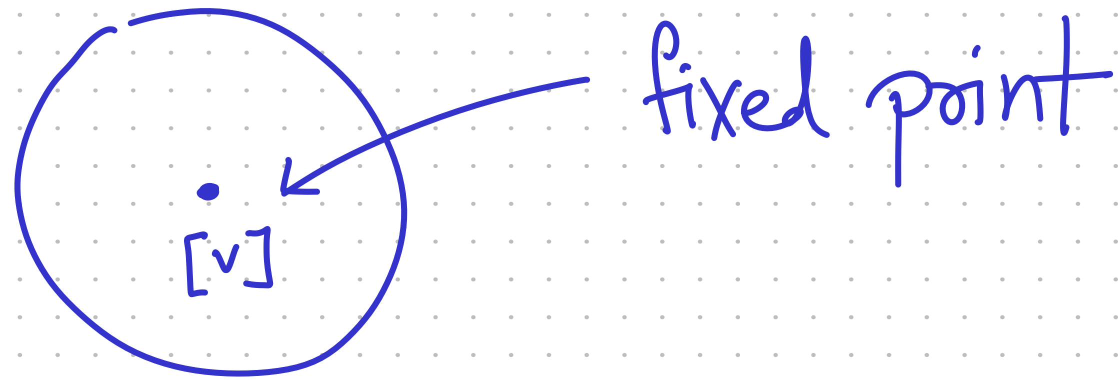
 $\rightsquigarrow$ $P_3 \to P_2 \to P_1 \to P_1 \to P_2$

Chopping up into segments $\longleftrightarrow$ Harder-Narasimhan filtration.

Topological size entropy $=$ Categorical size entropy

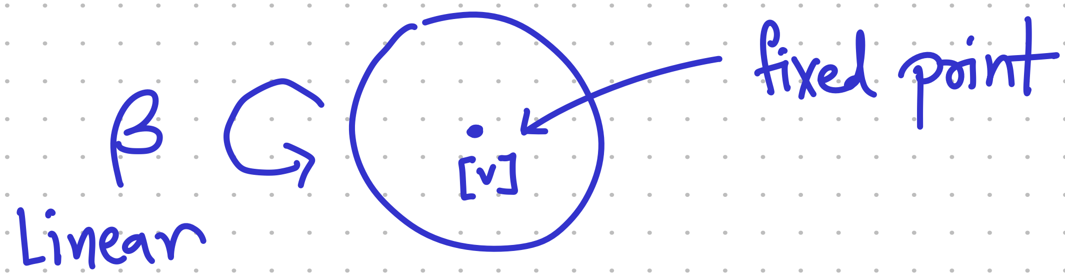
Algebraicity of entropy (expectation)

$\beta \circlearrowright \Sigma$ piecewise linear

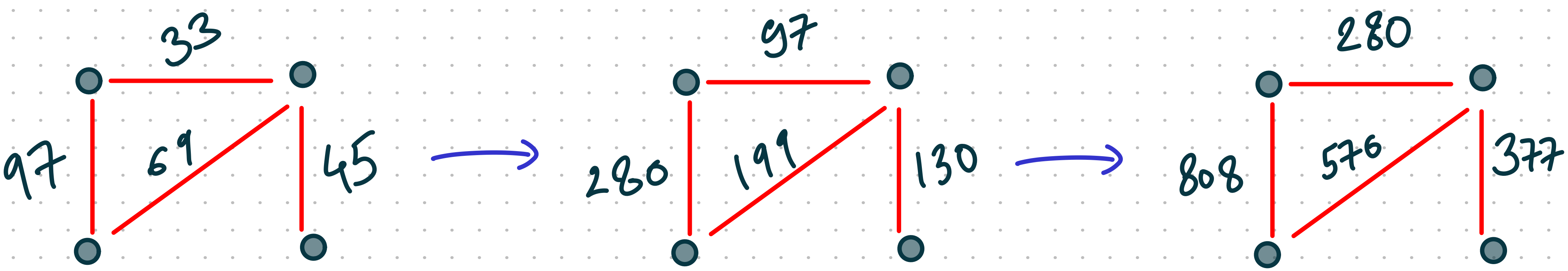
$\beta \circlearrowright$ 

Linear

Algebraicity of entropy (expectation)



Example – $\beta = \sigma_a \sigma_x^{-1} \sigma_c \sigma_b = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$



$$e = \frac{1}{2}\left(\sqrt{5} + \sqrt{2+2\sqrt{5}} + 1\right) \approx 2.9$$

Dynamics of Categories

$$F : \mathcal{C} \to \mathcal{C}$$

How does the HN filtration evolve ?

↳ Piecewise linear

Eventually linear

Categorical Dynamics / Group theory $\rightleftarrows$ Geometric Dynamics / Group theory

Thank you!