MATH 8320: ALGEBRAIC CURVES AND RIEMANN SURFACES — HOMEWORK 1

1. Hyperelliptic curves

(1) Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree $2n$ without repeated roots. Let $U \subset \mathbb{C}^2$ be the Riemann surface defined by $y^2 - f(x) = 0$. Construct explicitly a compact Riemann surface X containing U along with a map $X \to \mathbb{P}^1$.

(2) Let $D \subset \mathbb{C}$ be a small disk that does not contain any zeros of $f(x)$. Prove that the preimage $\pi^{-1}(D)$ is biholomorphic to a disjoint union of two disks. What if D contains a zero of $f(x)$?

(3) Compute the Euler characteristic and hence the genus of X.

(4) Prove that the field of meromorphic functions on X is isomorphic to

$$\mathbb{C}(x)[y]/(y^2 - f(x)).$$

2. Cyclic coverings

(5) Generalize as much of the above as you can to the curve defined by $y^n - f(x) = 0$, where $f(x) \in \mathbb{C}[x]$ is a polynomial of degree divisible by n without repeated roots.

(6) What happens in the analysis above if the degree of $f(x)$ is not divisible by n?

3. Complex Tori

(7) Let $X = \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice. Note that addition induces a (holomorphic) group law on X. Show that under this law, X is a divisible group. For a positive integer n, describe the group $X[n]$ of n-torsion points on X.

4. Plane curves

(8) Let $X \subset \mathbb{P}^2$ be a smooth plane curve of degree 1 or 2. Show that X is isomorphic to $\mathbb{P}^1$.

(9) Let $C \subset \mathbb{P}^2$ be the Fermat curve, defined by

$$X^d + Y^d + Z^d = 0.$$

By analyzing the ramification of the map $C \to \mathbb{P}^1$ given by $[X : Y : Z] \mapsto [X : Y]$, find the genus of C.

5. Line bundles

(10) We defined the line bundle $\mathcal{O}(m)$ on $\mathbb{P}^1$ in class. Identify the space of global holomorphic sections of $\mathcal{O}(m)$ with polynomials in $\mathbb{C}[x]$ of degree at most m, or equivalently, with homogeneous polynomials of degree m in $\mathbb{C}[x, y]$.