Regular functions and regular maps

k = Alg. closed field.

Recall from last time:

$X \subset \mathbb{A}^n_k$ affine algebraic set.

$f : X \to k$ regular if it is the restriction of a polynomial function.

$k[X]$ = k-algebra of regular functions on X

$\cong k[x_1, \dots, x_n] / I(X)$.

= Finitely generated nilpotent free k-algebra.

Observe - Any finitely generated nilpotent free k-algebra is of the form $k[X]$ for some X.

Why? Let A be such an algebra.

Let $a_1, \dots, a_n \in A$ be a set of generators.

Then we have a map

$$\varphi : k[x_1, \dots, x_n] \to A$$
$$x_i \mapsto a_i.$$

This map is surjective because $\{a_i\}$ generates A. By the first iso thm where $I = \operatorname{Ker} \varphi$.

Since A is nilpotent free, I is radical.

Then take $X = V(I)$.

By the Nullstellensatz,

$$\begin{aligned} k[X] &= k[x_1, \dots, x_n] / I(X) \\ &= k[x_1, \dots, x_n] / I \\ &\cong A \end{aligned}$$

$\square$

As a result we have the dictionary.

Algebra	Geometry
• Finitely generated reduced k-alg. A	• Alg of regular functions on affine alg set X.
• Max ideal of A	• Point of X
• Given $J \subset A$. $V(J) = \{ m \mid m \supset J \}$.	• Given $J \subset k[X]$ $V(J) = \{ x \mid f(x) = 0 \ \forall f \in J \}$

In particular $V(J) = \phi$ iff $J = (1)$.

Regular Maps

$X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ affine alg sets.
$f : X \to Y$ is a regular function if

$\exists\ f_1, \ldots, f_m \in k[X]$ such that

$$f(x) = (f_1(x), \ldots, f_m(x)) \quad \forall\, x \in X.$$

Equivalently, if there exist $F_1, \ldots, F_m$ in $k[X_1, \ldots, X_n]$ such that
$$f(x) = (F_1(x), \ldots, F_m(x)) \quad \forall\, x \in X.$$

Ex 1: $f : X \to \mathbb{A}^1$ regular map
$\iff$ f is a regular function.

Ex 2: $L : \mathbb{A}^n \to \mathbb{A}^m$ linear transf^n is regular.

Ex 3: Projections $\mathbb{A}^n \to \mathbb{A}^1$

Ex 4: Compositions of regular maps are regular

Ex 5: $X \subset \mathbb{A}^n$ Zariski closed.
The inclusion $X \to \mathbb{A}^n$ is regular.

Def: A regular $f: X \to Y$ is an isomorphism if there exists a regular inverse map $g: Y \to X$.

Ex 6: $X = \mathbb{A}^1$
$Y = \{y^2 - x^3 = 0\} \subset \mathbb{A}^2$

$f: X \to Y$
$t \mapsto (t^2, t^3)$ is a regular bijection but not an isomorphism!
How does one see that it's not an iso? Wait and see....

Let $\varphi: X \to Y$ be any map.
Then we get an induced map

$$\varphi^*: \text{Functions on } Y \to \text{Functions on } X$$
$$f \mapsto f \circ \varphi.$$

Proposition: φ is regular if and only if φ^* sends regular functions on Y to regular functions on X.

Pf: Suppose φ is regular.
If $f: Y \to \mathbb{A}^1$ is a regular function then $\varphi \circ f$ is regular because composition of regular maps is regular.
Conversely, suppose $\varphi^*(f)$ is regular for every regular f. Let $\varphi(x) = (\varphi_1(x), \dots, \varphi_m(x))$
We want to show each $\varphi_i(x)$ is regular
But $\varphi_i = \varphi^*(x_i)$ and $x_i \in k[Y]$ is regular. $\square$

Thus a regular map $\varphi: Y \to X$ induces a k-alg. hom $\varphi^*: k[Y] \to k[X]$.

Prop: Let $\alpha: k[Y] \to k[X]$ be a k-alg hom. Then there is a unique regular $\varphi: X \to Y$ such that $\alpha = \varphi^*$.

Pf: suppose $Y = V(J) \subset \mathbb{A}^m$

Then $k[Y] = k[y_1, \dots, y_m] / J$

$K[X] = K[x_1, \dots, x_n] / I$.

Let $\varphi_i = \alpha(y_i) \in k[X]$

Consider $\varphi := (\varphi_1, \dots, \varphi_m) : X \to \mathbb{A}^m$.

Let us check that φ maps X to Y.

To see this, we must show that

$$f(\varphi_1(x), \dots, \varphi_m(x)) = 0 \quad \forall\ x \in X, \ f \in J.$$

But $f(\varphi_1(x), \dots, \varphi_m(x))$

$= f(\alpha(y_1), \dots, \alpha(y_m))$

$= \alpha\ f(y_1, \dots, y_m)$

$= \alpha(0) = 0$.

So $\varphi : X \to Y$. Note $\varphi^*(y_i) = \alpha(y_i)$

so $\varphi^* = \alpha$ because $\{y_i\}$ generate $k[Y]$.

Finally, if $\varphi : X \to Y$ is such that $\varphi^* = \alpha$, and $\varphi = (\varphi_1, \dots, \varphi_m)$, then $\varphi^*(y_i) = \varphi_i = \alpha(y_i)$, so there is only one possible φ.

$\square$.

Conseq: $X \rightsquigarrow k[X]$ defines an equivalence of categories

$$\left\{\begin{array}{c}\text{Affine alg} \\ \text{sets with} \\ \text{regular maps}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Fin gen reduced} \\ k\text{-algebras} \\ \text{with } k\text{-alg.} \\ \text{homs}\end{array}\right\}$$

Ex: $X = \mathbb{A}^1$

$Y = V(y^2 - x^3) \subset \mathbb{A}^2$

$k[X] = k[t]$ $\quad k[Y] = k[x,y]/(y^2 - x^3)$

$\varphi : X \to Y$ $\quad \varphi(t) = (t^2, t^3)$

$\varphi^* : k[Y] \to k[X]$

$x \mapsto t^2$

$y \mapsto t^3.$

φ^* is not an isomorphism!

Any element in the image of φ^* has vanishing linear term.

<u>Next</u>: Algebraic varieties (more general spaces than affine algebraic sets).

To do that, we want to define the notion of regularity more locally.

Let $X \subset \mathbb{A}^n$ be an affine alg. set, $f: X \to k$ a function, and $x \in X$ a point. We say that f is <u>regular at x</u> if there exist $F, G \in k[x_1, \dots, x_n]$ with $G(x) \neq 0$ such that $f = F/G$ on the open set $X \cap \{G \neq 0\}$.

<u>Claim</u>: If f is regular at all $x \in X$, then f is regular (i.e. given by a polynomial).

<u>Pf</u>: $\forall\, x \in X$ $\exists$ F_x & G_x s.t. $G_x(x) \neq 0$ & $f = F_x/G_x$.
Then $\{G_x\}$ has no common zero on $X \Rightarrow \langle G_x \rangle = (1)$ in $k[X]$.
Write $1 = h_1 G_1 + \dots + h_\ell G_\ell$
Then X is the union of the opens

$$f = \frac{F_i}{G_i}.$$

Take $F = h_1F_1 + \cdots + h_\ell F_\ell \in k[X]$

Then $F = f$ on X.

$\square$

The above motivates the following.

Let $X \subset \mathbb{A}^n$ be an affine alg. set and $U \subset X$ an open set. A function $f : U \to k$ is regular on U if it is regular at every $x \in U$. That is for every $x \in U$ $\exists$ $F, G \in k[X_1, \ldots, X_n]$ $G(x) \neq 0$ such that

$$f = \frac{F}{G} \text{ on } U \cap \{G \neq 0\}.$$

Similarly $\varphi : U \to Y$ is regular if $\varphi = (\varphi_1, \ldots, \varphi_m)$ where each φ_i is a regular function on U.

Now we have

$$\left\{\begin{matrix}\text{affine algebraic}\\ \text{varieties}\end{matrix}\right\} = \left\{\begin{matrix}\text{affine alg. subsets}\\ \text{of } \mathbb{A}^n_k\end{matrix}\right\}$$

$$\cap \qquad\qquad\qquad\qquad \cap$$

$$\left\{\begin{matrix}\text{Quasi-affine}\\ \text{varieties}\end{matrix}\right\} = \left\{\begin{matrix}\text{open subsets of}\\ \text{affine alg subsets}\\ \text{of } \mathbb{A}^n_k\end{matrix}\right\}$$

Morphisms = Regular maps

Examples: 1

Let $X = \mathbb{A}^1 \setminus \{0\}$.

$Y = V(xy-1) \subset \mathbb{A}^2$.

Then we have an isomorphism

$X \xrightarrow{\sim} Y$.

In particular X is (isomorphic to) an affine algebraic variety!

The iso is given by $X \to Y$

and the inverse is

$$Y \to X$$
$$(x,y) \mapsto x.$$

(2) More generally, let $f \in k[X_1, \dots, X_n]$ and

$$X = \{ x \in \mathbb{A}^n \mid f(x) \neq 0 \}$$
$$= \mathbb{A}^n \setminus V(f).$$

Let $Y \subset \mathbb{A}^{n+1} = \{ (x_1, \dots, x_n, y) \}$

$$Y = V(y f(x_1, \dots, x_n) - 1).$$

Then we have an iso $X \xrightarrow{\sim} Y$ given by

$$(x_1, \dots, x_n) \mapsto \left(x_1, \dots, x_n, \frac{1}{f(x_1, \dots, x_n)}\right)$$

with inverse $(x_1, \dots, x_n, y) \mapsto (x_1, \dots, x_n)$.

In particular X is an affine alg. variety!

(3) Not all quasi-affine varieties are isomorphic to affine varieties.
To see an example, recall that affine alg. varieties satisfy the Nullstellensatz — there is a bijection between max ideals of $k[X]$ and points of X given by $m \mapsto V(m)$.

Take $X = \mathbb{A}^2 \setminus \{(0,0)\} \subset \mathbb{A}^2$

<u>Claim</u>: The k-algebra of regular functions on X is the same as $k[\mathbb{A}^2] = k[x,y]$.

<u>Pf</u>: Deferred.

But now $m = (x,y) \subset k[X]$ is a non-unit ideal such that $V(m) = \emptyset$ (in X). Therefore, X cannot be affine.