

What is an algebraic variety?

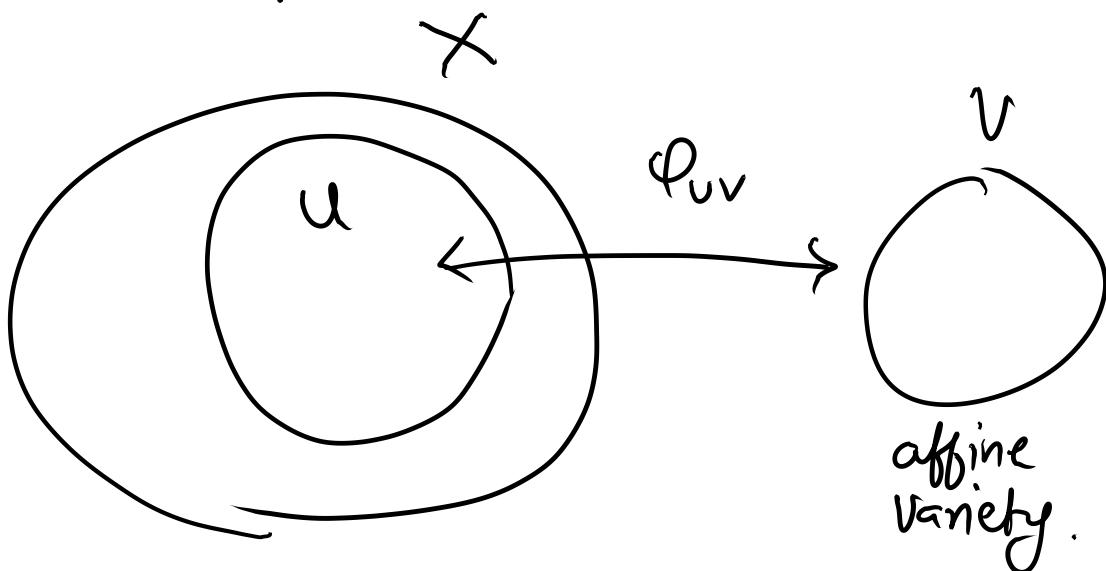
The definition of an algebraic variety is very similar to the definition of a manifold in differential geometry.

Def: An algebraic variety is a topological space with an affine atlas.

Affine atlas = Collection of compatible collection for X of affine charts that cover X .

Affine chart : (U, V, φ_{UV})

where $U \subset X$ is open, V is an affine variety and $\varphi_{UV} : U \rightarrow V$ is a homeomorphism.



Two charts (U_1, V_1, φ_1) , (U_2, V_2, φ_2)
are compatible if the map φ_{12}

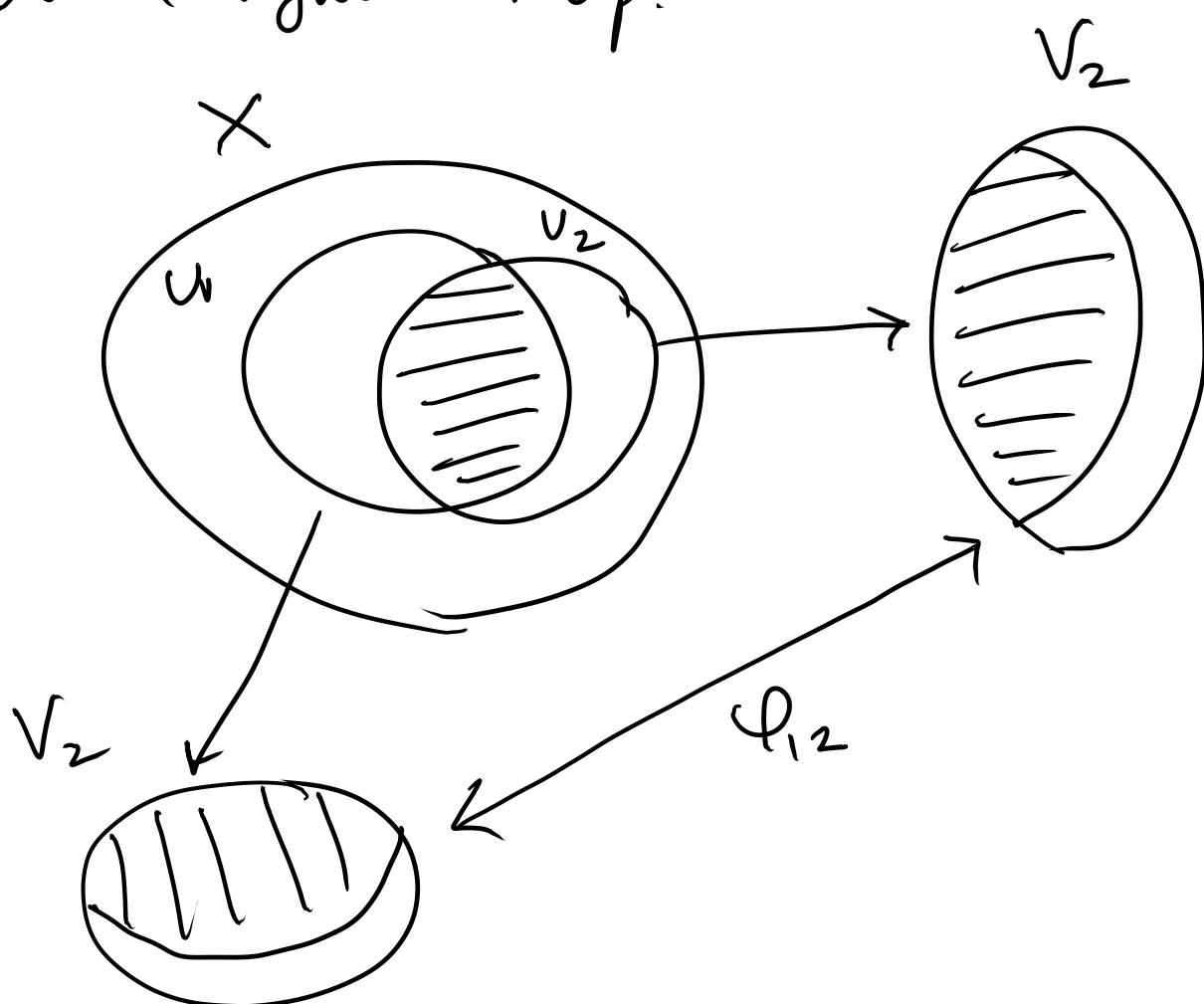
$$\varphi_1(U_1 \cap U_2) \xrightarrow{\varphi_{12}} \varphi_2(U_1 \cap U_2)$$



V_1

V_2

is a regular map.



An atlas is a collection of compatible
charts $\{(U_i, V_i, \varphi_i)\}$ such that
the open sets $\{U_i\}$ cover X .

Example: Quasi affine variety.

$$\begin{aligned} U &= X \setminus V(I), \quad X \text{ affine}, \quad I \subset k[x] \\ &= \bigcup_{f \in I} X_f, \quad \text{where} \end{aligned}$$

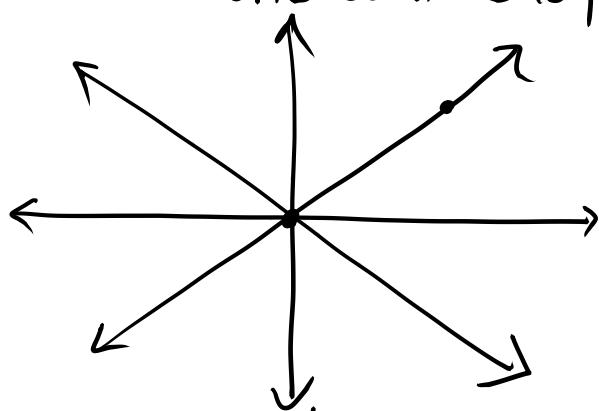
$$X_f = \{x \in X \mid f(x) \neq 0\} \leftarrow \text{Affine!}$$

so can take $\phi_f : X_f \xrightarrow{\text{id}} X_f$
as then all the transition maps
are just the identity.

Example (Important)
THE PROJECTIVE SPACE

\mathbb{P}^n = Set of lines in \mathbb{R}^{n+1}
one "dim. subspaces"

e.g. \mathbb{P}^1



$$\mathbb{P}^n = \{ (a_0, \dots, a_n) \mid a_i \in k, \text{not all } 0 \} / \sim$$

$$(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n) \quad \lambda \in k^\times.$$

Topology : Closed sets of \mathbb{P}^n
 = Subsets defined by collections
 of homogeneous polynomials.
 in $\mathbb{k}[X_0, \dots, X_n]$

$P(X_0, \dots, X_n)$ is homogeneous of degree

$$P(X_0, \dots, X_n) = \sum_I a_I X^I \quad \text{where } |I|=d \\ = \sum_{(i_0, \dots, i_n)} a_{i_0 \dots i_n} X_0^{i_0} \dots X_n^{i_n} \quad \sum i_j = d.$$

Equivalently, if $p(\lambda X_0, \dots, \lambda X_n)$
 $= \lambda^d p(X_0, \dots, X_n).$

Note : Even if P is homogeneous of $\deg P \geq 0$,
 P DOES NOT define a function on
 \mathbb{P}^n but it DOES define a "vanishing
 set"

$$V(P) = \{[x] \in \mathbb{P}^n \mid p(x) = 0\}$$

The equality $p(x) = 0$ does not depend
 on the chosen representative of $[x]$.

Since

$$p(\lambda x) = \lambda^d p(x),$$

both $p(\lambda x)$ & $p(x)$ are simultaneously zero
 or non-zero.

Affine charts: (See example of \mathbb{P}^2 two pages ahead.)

$$\begin{aligned}\mathbb{P}^n &= \{ [x_0 : \dots : x_n] \mid \text{Not all } x_i = 0 \} \\ &= U_0 \cup U_1 \cup \dots \cup U_n\end{aligned}$$

$$\begin{aligned}U_i &= \{ [x_0 : \dots : x_n] \mid x_i \neq 0 \} \\ &= \{ [x_0 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_n] \mid x_i \in \mathbb{k} \}\end{aligned}$$

$$\varphi_i: \mathbb{A}^n \xrightarrow{\quad \text{bijection} \quad}$$

$$\begin{aligned}\varphi_i: [x_0 : \dots : x_n] &\mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right)\end{aligned}$$

Claim: φ_i is a homeomorphism.

Pf: Let $Z \subset U_i$ be closed. Then

$Z = V(S) \cap U_i$ where S is a set of homogeneous polynomials in x_0, \dots, x_n

Then $\varphi_i(Z) =$

$$V(\{ p(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) \mid p \in S \}) \subset \mathbb{A}^n.$$



closed.

$$\{ (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \}$$

Conversely, consider

$$Y = V(T) \subset \mathbb{A}^n$$

where $T \subset k[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$.

We must construct a set T' of homogeneous polynomials in x_0, \dots, x_n such that

$$\varphi_i^*(Y) = V(T') \cap U_i.$$

T' is obtained from T by homogenizing w.r.t x_i . For $p \in T$, create

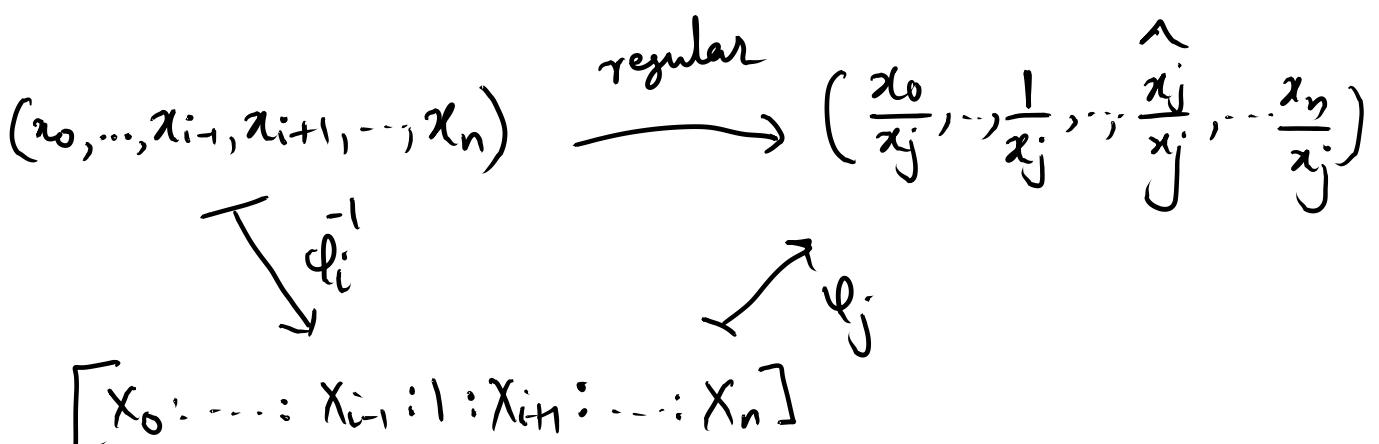
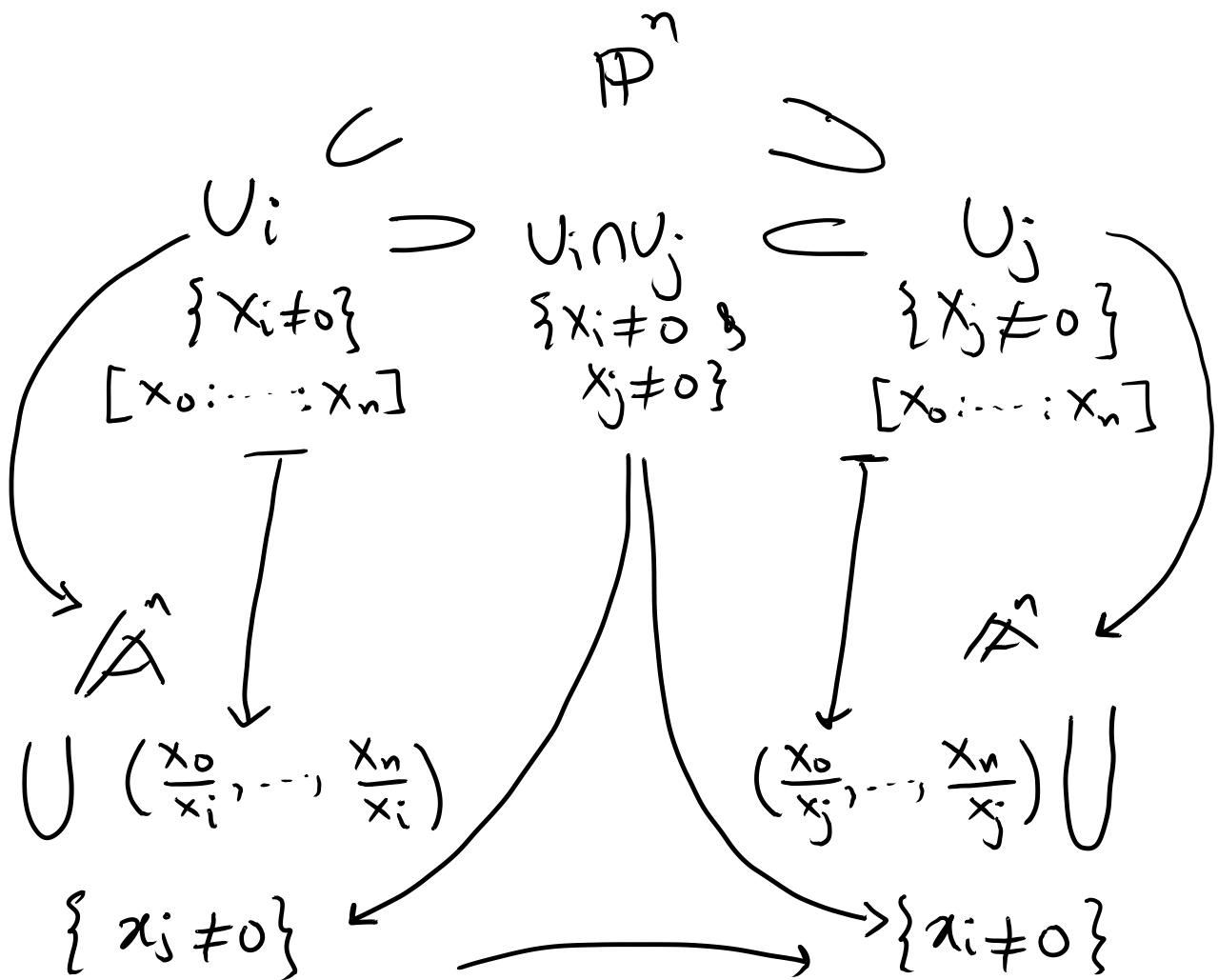
$$P^{\text{hom}} = X_i^{\deg p} P\left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

(e.g. $P = x_0^2 + x_1$,
 $P^{\text{hom}} = X_0^2 + X_1 X_i$)

Now set $T' = \{P^{\text{hom}} \mid p \in T\}$.

Then $\varphi^*(V(T)) = V(T') \cap U_i$, so $\varphi^*(V(T))$ is closed.

Transition functions.



([^] symbol = exclude this coordinate)

Example of \mathbb{P}^2

$$\mathbb{P}^2 = \{ [x_0 : x_1 : x_2] \mid \text{Not all } x_i = 0 \}$$

$$U_0 = \mathbb{P}^2 \setminus V(x_0)$$

$$= \{ [1 : x_1 : x_2] \mid x_1, x_2 \in k \}$$

$$\begin{aligned} \varphi_0 &([1 : x_1 : x_2]) = (x_1, x_2) \\ \varphi_0 &([x_0 : x_1 : x_2]) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0} \right). \end{aligned}$$

φ_0 takes closed sets to closed sets.

Example: Take $Z = V(x_0^2 - x_1 x_2, x_0^3 - x_2^3) \cap U_0$

Then

$$\varphi_0(Z) = V(1 - x_1 x_2, 1 - x_2^3) \subset \mathbb{A}^2.$$

φ_0^{-1} takes closed sets to closed sets.

Example: Take $Y = V(1 - x_1, x_2^2 - x_1^3) \subset \mathbb{A}^2$

Then

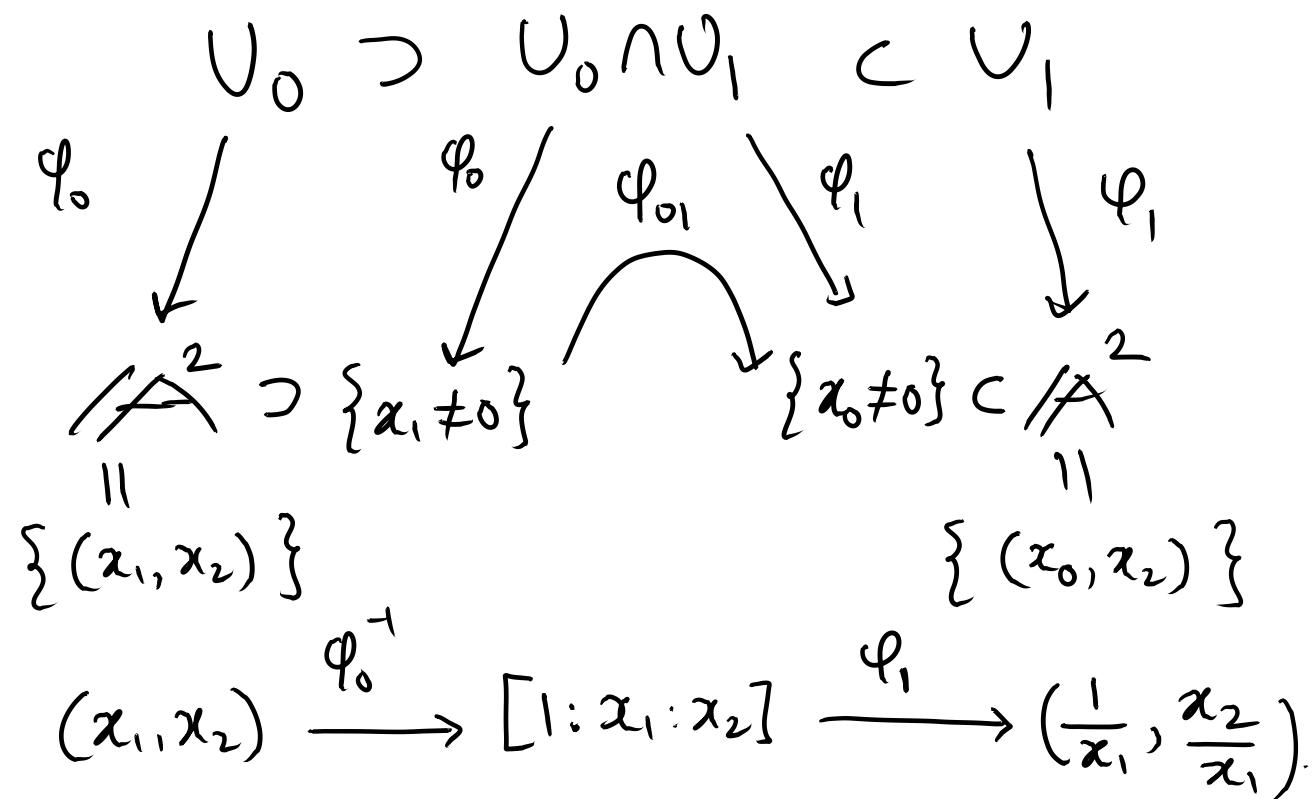
$$\varphi_0^{-1}(Y) = U_0 \cap V(x_0 - x_1, x_0 x_2^2 - x_1^3).$$

Example of transition functions

$$U_0 = \{ [x_0 : x_1 : x_2] \mid x_0 \neq 0 \}$$

$$U_1 = \{ [x_0 : x_1 : x_2] \mid x_1 \neq 0 \}$$

$$U_0 \cap U_1 = \{ [x_0 : x_1 : x_2] \mid x_0 \neq 0 \text{ and } x_1 \neq 0 \}.$$

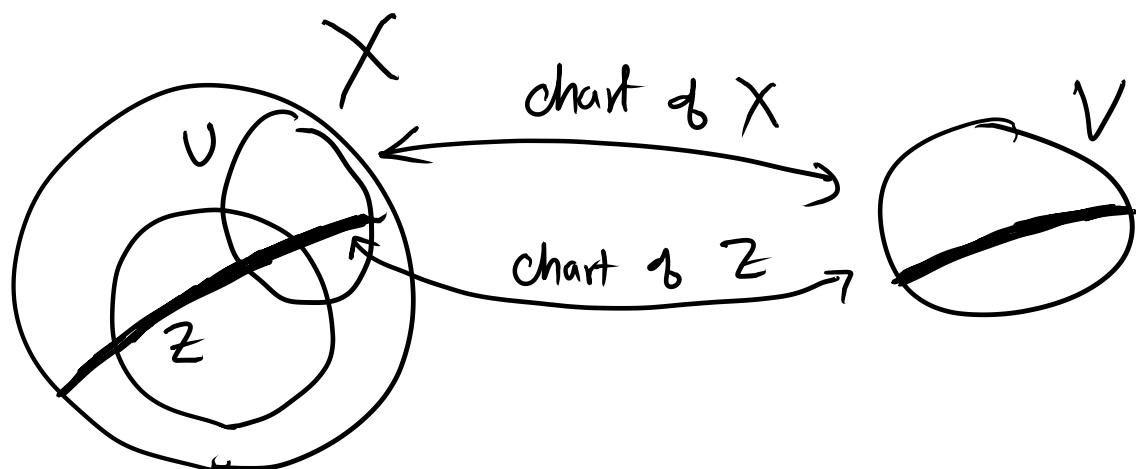


so $\varphi_{01}: \mathbb{A}^2 \setminus V(x_1) \rightarrow \mathbb{A}^2 \setminus V(x_0)$
 $(x_1, x_2) \mapsto \left(\frac{1}{x_1}, \frac{x_2}{x_1}\right)$

is regular.

Open & closed subvarieties

Let X be an algebraic variety, and $Z \subset X$ a closed subset. Then Z is naturally an algebraic variety. The atlas for Z is induced from the atlas for X . i.e. if $\varphi: U \rightarrow V$ is a chart for X , then $\varphi_Z: U \cap Z \rightarrow \varphi(U \cap Z)$ is a chart for Z . Note that the $\varphi(U \cap Z)$ is a closed subset of the affine variety V , so it is itself an affine variety.



Similarly, if $Y \subset X$ is an open subset, then Y is naturally an algebraic variety. The charts are again obtained by restricting the charts of X .

$$\varphi: U \rightarrow V \quad \text{chart of } X$$

$$\varphi_Y: U \cap Y \rightarrow \varphi(U \cap Y) \quad \text{chart of } Y.$$

(Note: $\varphi(U \cap Y)$ is a quasi-affine, so the latter is not necessarily an affine chart, only a "quasi-affine chart! But we can always write a quasi affine as a union of open affines, & these will provide affine charts).

Def: A Projective Variety is a closed subset of \mathbb{P}^n .

A quasi-Projective Variety is an open subset of a projective variety

Examples:

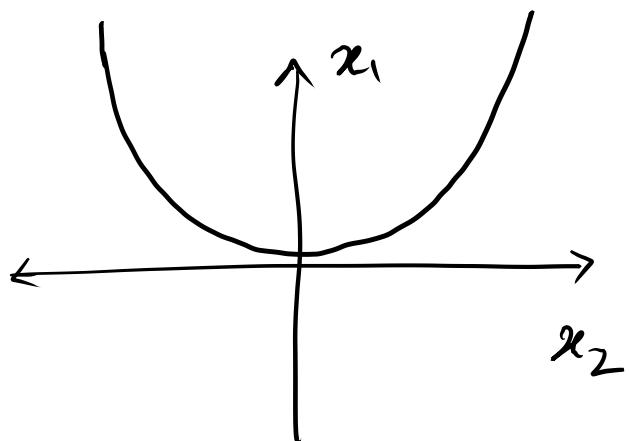
$$\textcircled{1} \quad X = V(x_0x_1 - x_2^2) \subset \mathbb{P}^2$$

Three affine charts, corresponding to
 $x_0 \neq 0$, $x_1 \neq 0$, and $x_2 \neq 0$.

$$\underline{\text{Chart}} \quad (x_0 \neq 0) = U_0 \cong \mathbb{A}^2$$

$$X \cap U_0 \subset U_0 \cong \mathbb{A}^2$$

$$\stackrel{||}{V}(x_1 - x_2^2).$$



$$\underline{\text{Chart}}: (x_1 \neq 0) = U_1 \cong \mathbb{A}^2$$

$$X \cap U_1 \subset U_1 \cong \mathbb{A}^2$$

$$\stackrel{||}{V}(x_0 - x_2^2)$$

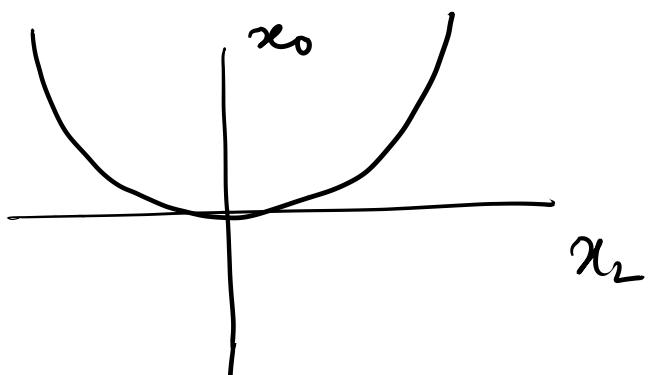
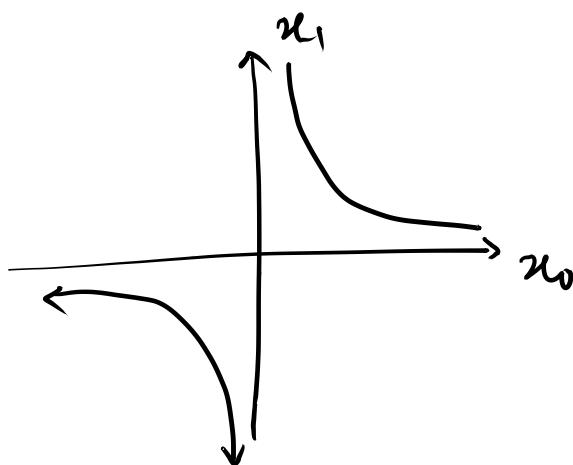


Chart $(X_2 \neq 0) = U_2 \cong A^2$

$$X \cap U_2 \subset U_2$$

$$\begin{matrix} \\ \parallel \\ V(x_0 x_1 - 1) \end{matrix}$$



Missing from Chart 1 :-

$$\begin{aligned} \mathbb{P}^2 \setminus U_0 &= \{ [0:x_1:x_2] \} \\ V(x_0) \cap X &\stackrel{\cong}{=} \mathbb{P}^1 \end{aligned}$$

So one point of X is missing from Chart 1.

This point is visible in Chart 2, but Chart 2 is missing $[1:0:0]$.

Chart 3 is missing both $[1:0:0]$ & $[0:1:0]$.

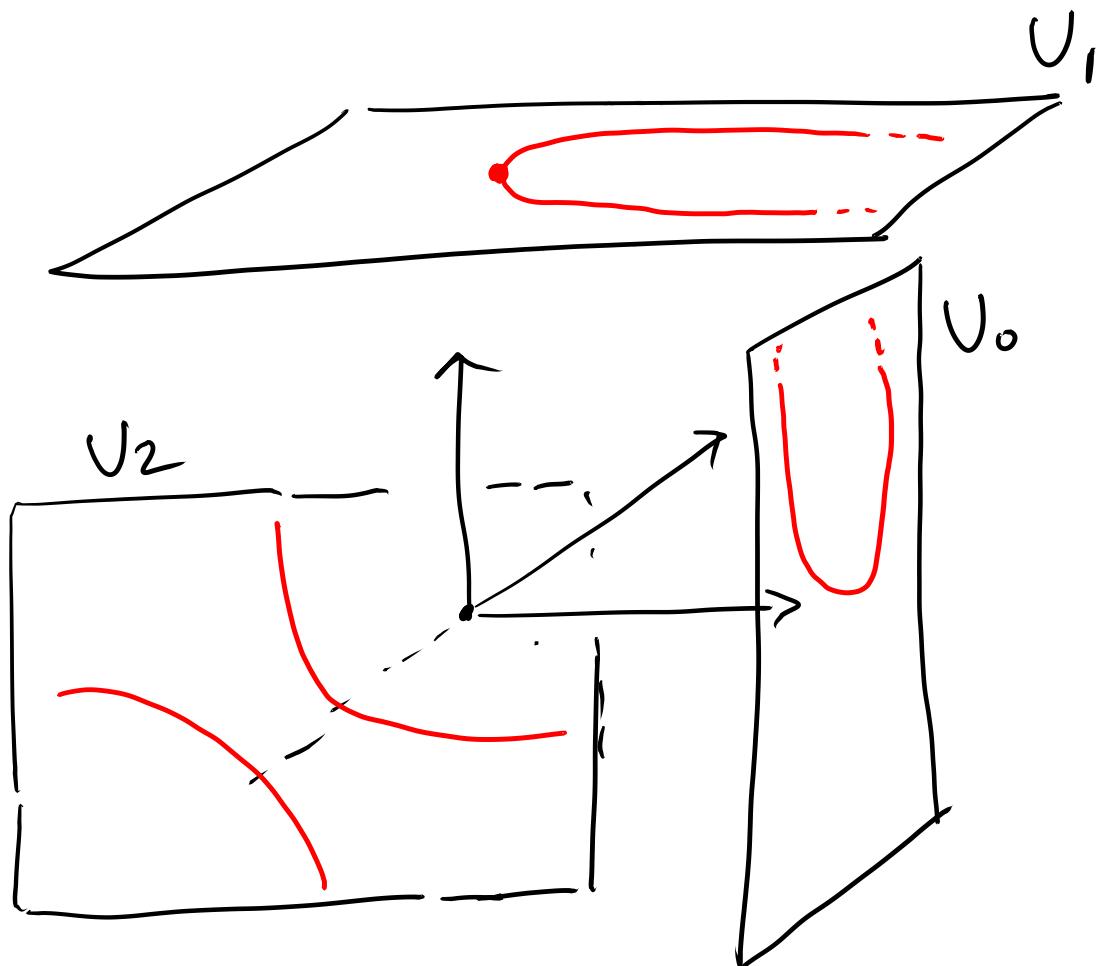
Visualization.

\mathbb{P}^2 = Lines in 3-space

$U_0 = \{\text{Lines meeting } X_0=1 \text{ plane}\}$

$U_1 = \{\text{Lines meeting } X_1=1 \text{ plane}\}$

$U_2 = \{\text{lines meeting } X_2=1 \text{ plane}\}$.



- Every projective variety is quasi proj.
- Every affine variety is quasi proj.

How?

We have $\mathbb{A}^n \subset \mathbb{P}^n$ open as U_n .

If $X \subset \mathbb{A}^n$ is closed, then

$$X = \overline{X} \cap \mathbb{A}^n$$

\hookrightarrow closure of X in \mathbb{P}^n

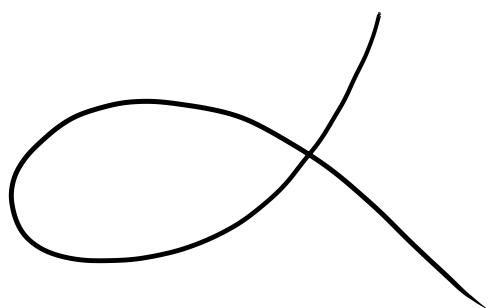
so $X \subset \overline{X}$ = Projective.

So every affine variety is an open in a projective variety.

$\overline{X} \subset \mathbb{P}^n$ is called the projective closure of $X \subset \mathbb{A}^n$.

Example:

$$X = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2$$



nodal cubic

View $\mathbb{A}^2 = \{(x,y)\}$ as
 $\{(x:y:1)\} = \{(x:y:z) \mid z \neq 0\} \subset \mathbb{P}^2$.

Then the projective closure of X is cut out by the homogenization of $f(x,y) = y^2 - x^3 - x^2$ with respect to z .

$$\text{i.e. by } F(x,y,z) = z^3 f\left(\frac{x}{z}, \frac{y}{z}\right) \\ = (yz^2 - x^3 - x^2 z).$$

"Points at infinity" = $\{\text{pts with } z=0\}$
 $= \{[0:1:0]\}$

Prop (Exercise) Let $I \subset k[x_0, \dots, x_n]$ be an ideal. Define

$$I^{\text{hom}} = \text{ideal generated by } \{p^{\text{hom}} \mid p \in I\} \subset k[x_0, \dots, x_n]$$

where $p^{\text{hom}}(x_0, \dots, x_n) = x_n^{\deg p} p\left(\frac{x_0}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right)$.

Then the proj. closure of $X = V(I) \subset \mathbb{A}^n$ in \mathbb{P}^n is $\bar{X} = V(I^{\text{hom}})$.

Example : $X \subset \mathbb{A}^3$

$$X = \{ (t, t^2, t^3) \mid t \in \mathbb{A}^1 \}$$

$$= V(y - x^2, z - x^3)$$

What is \overline{X} ?

$$\overline{X} = V(YW - x^2, ZW^2 - x^3) ??$$

Definitely \overline{X} is contained in this.

Points at infinity of $V(YW - x^2, ZW^2 - x^3)$

$$\text{are } \{ [0:Y:Z:0] \mid [Y:Z] \in \mathbb{P}^1 \},$$

Certainly, not all are in the closure of X !

Observe : $y^3 - z^2 \in I(X)$

so let's try

$$V(YW - x^2, ZW^2 - x^3, Y^3 - Z^2W).$$

Now, there is a unique point at infinity, namely $[0:0:1:0]$, and this point is in the closure of X . —④

How do you see \circledast ?

Move to a chart containing $[0:0:1:0]$

Then

$$\bar{X} \cap U_2 = V(yw-x^2, w^2-x^3, y^3-w)$$

$$\& X \cap U_2 = \bar{X} \setminus \{(0,0,0)\}.$$

To see that $(0,0,0)$ is in the closure of $X \cap U_2$, consider

$$\begin{aligned} \mathbb{A}^1 &\rightarrow \bar{X} \cap U_2 \\ s &\mapsto (s^2, s, s^3) \end{aligned}$$

then $\mathbb{A}^1 \setminus \{0\} \mapsto X \cap U_2$

& $0 \mapsto (0,0,0)$

Show that $(0,0,0) \in \text{closure of } X \cap U_2$.

□.

So $\bar{X} = V(Yw-x^2, zw^2-x^3, y^3-z^2w)$

↳ called the "twisted cubic".

