Algebraic Geometry Assignment 1

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Question 1

Firstly we shall show for a field k, that any Zariski closed subset of $\mathbb{A}^1_k$ is finite with the exception of A^1_k. This is since if a set $X \subset \mathbb{A}^1_k$ is closed then $X = V(S)$ for $S \subset k[x]$. But if $S \neq \{0\}$ then their exists $f \in S$ such that $f \neq 0$ and thus as f is a polynomial it has only finitely many roots and as $f(p) = 0$ for all $p \in X$ then X is finite unless $S = \{0\}$ which corresponds to $X = \mathbb{A}^1_k$.

We shall show that $U = \{z \in \mathbb{A}^1_{\mathbb{C}} : |z| = 1\} \subset \mathbb{A}^1_{\mathbb{C}}$ is not a algebraic subset of $\mathbb{A}^1_{\mathbb{C}}$. This is since U is a infinite set (the unit circle in $\mathbb{C}$), not equal to $\mathbb{A}^1_{\mathbb{C}}$ as $0 \notin U$. Thus as $\mathbb{A}^1_{\mathbb{C}}$ only has itself as a infinite closed subset in the Zariski topology then U is not closed and thus not a affine algebraic subset.

Question 2

Let R be a ring and let I be a ideal in R, we wish to show that $\sqrt{I}$ is a ideal. First we shall show that for all $a \in \sqrt{I}$, $b \in R$ that $ab \in \sqrt{I}$. This is since as $a \in \sqrt{I}$ their exists a positive integer n such that $a^n \in I$. Then as I is a ideal and $b^n \in R$ then $b^n a^n \in I$ and thus $(ab)^n \in I$ and thus $ab \in \sqrt{I}$ as required.

Now we wish to show that $a, b \in \sqrt{I}$ then $a + b \in \sqrt{I}$. Therefore we know their exists positive integers n, m such that $a^n, b^m \in I$. Then by the binomial expansion as the ring is commutative we know that

$$(a+b)^{m+n} = \sum_{i=0}^{m+n} \binom{m+n}{k} a^{m+n-i} b^i$$

Then for $i \leq m$ we know that $m + n - i \geq n$ and thus as I is closed under multiplication and as $a^n \in I$ then $a^{m+n-i}b^i \in I$. Whilst if $i > m$ then as $b^m \in I$ then $a^{m+n-i}b^i \in I$ and thus as these are all possible cases then each term of the sum $\sum_{k=0}^{m+n} \binom{m+n}{k} a^{m+n-i}b^i$ is in I where as I is closed under addition then $\sum_{k=0}^{m+n} \binom{m+n}{k} a^{m+n-i}b^i \in I$ and thus $(a+b)^{m+n} \in I$ and thus $a + b \in \sqrt{I}$ as required completing the proof that $\sqrt{I}$ is a ideal.

Now we wish to show that $\sqrt{\sqrt{I}} = \sqrt{I}$ we shall first show that $\sqrt{\sqrt{I}} \subset \sqrt{I}$. Therefore, let $a \in \sqrt{\sqrt{I}}$ then by definition their exists a positive integer n such that $a^n \in \sqrt{I}$. Thus by definition their exists a positive integer m such that $(a^n)^m = a^{mn} \in I$ and thus by definition $a \in \sqrt{I}$ thus proving $\sqrt{\sqrt{I}} \subset \sqrt{I}$. To prove the reverse inclusion $\sqrt{I} \subset \sqrt{\sqrt{I}}$ we let $a \in \sqrt{I}$ then taking $n = 1$ in the definition of radical ideal we have $a \in \sqrt{\sqrt{I}}$ thus completing the proof that $\sqrt{\sqrt{I}} = \sqrt{I}$.

Question 3

Let $f : \mathbb{A}^2_{\mathbb{C}} \longrightarrow \mathbb{A}^2_{\mathbb{C}}$ given by $f(x, y) = (x, yx)$. Then the image of f is the set of points $\{(a, b) \in \mathbb{A}^2_{\mathbb{C}} : a \neq 0\} \cup \{(0, 0)\}$. This is since if $(x, y) \in Im(f)$ then $(a, b) = (x, yx)$ for some complex numbers x, y. If $a \neq 0$ then $(a, b) \in \{(a, b) \in \mathbb{A}^2_{\mathbb{C}} : a \neq 0\} \cup \{(0, 0)\}$ automatically and thus we are left to check the case where

$a \neq 0$. Thus if $a = 0$ then $x = 0$ and thus $yx = b = 0$ and therefore $(a, b) \in \{(a, b) \in \mathbb{A}^2_{\mathbb{C}} : a \neq 0\} \cup \{(0, 0)\}$. Then for the reverse inclusion if $(a, b) \in \{(a, b) \in \mathbb{A}^2_{\mathbb{C}} : a \neq 0\} \cup \{(0, 0)\}$ if $a \neq 0$ then $f(a, a^{-1}b) = (a, b)$ whilst if $a = 0$ then $(a, b) = (0, 0) = f(0, 0)$ and thus in all cases $(a, b) \in Im(f)$ and thus we have shown $Im(f) = \{(a, b) \in \mathbb{A}^2_{\mathbb{C}} : a \neq 0\} \cup \{(0, 0)\}$. Then we know from lectures that any closed set in the Zariski topology is closed in the euclidean topology and thus by the contrapostive any set that is not closed in the Euclidean topology is not closed in the Zariski topology. Then we can show that $Im(f)$ is not closed in the euclidean topology. This is since we can consider the sequence $x_n = (\frac{1}{n}, 1) \in Im(f)$ but the limit in the Euclidean metric as $n \to \infty$ is $(0, 1)$ is not in the image of f and thus $Im(f)$ is not closed in the Euclidean topology and thus it is not closed in the Zariski topology.

Similarly a set A is not open in the Euclidean topology if and only if A^c is not closed in the Euclidean topology which implies A^c is not closed in the Zariski topology which is if and only if A is not open in the Zariski topology. Then $Im(f)$ is not open in the Euclidean topology as $(0, 0) \in Im(f)$ but $(0, r) \in B_r((0, 0))$ and $(0, r) \notin Im(f)$ for all $r > 0$ and thus $Im(f)$ is not open in the Euclidean topology and thus is not open in the Zariski topology.

We know that a set is closed in the Zariski topology implies it is closed in the Euclidean topology. We also know that the closure of a set $A \subset \mathbb{A}^2_{\mathbb{C}}$ in a given topology is the smallest closed set in that topology containing A. Then since $\overline{A}_{Zariski}$ is closed in the Euclidean topology as it is a closed Zariski set then $\overline{A}_{Euclidean} \subset \overline{A}_{Zariski}$ by definition of closure for the Eudclidean topology. Thus if U is dense in the Euclidean topology then it is dense in the Zariski topology as $\mathbb{A}^2_{\mathbb{C}} = \overline{U}_{Eulidean} \subset \overline{U}_{Zariski} \subset \mathbb{A}^2_{\mathbb{C}}$. Then we know that U is dense in the Euclidean topology as $\mathbb{A}^2_{\mathbb{C}}/U = \{(x, y) \in \mathbb{A}^2_{\mathbb{C}} : x = 0 \text{ and } y \neq 0\}$ and thus for any $(x, y) \in \mathbb{A}^2_{\mathbb{C}}/U$ we have $y_n = (1/n, y)$ is a sequence in U approaching (x, y) in the euclidean metric and thus $\mathbb{A}^2_{\mathbb{C}}/U = \partial U$ and thus U is dense in the Euclidean topology and thus is dense in the Zariski topology.

Question 4

We can identify the coefficients of a matrix in $M_n(\mathbb{C})$ with a point in $\mathbb{A}^{n^2}_{\mathbb{C}}$ and thus will consider whether the following sets are closed, open or neither in the Zariski topology.

Firstly we consider $GL_n(\mathbb{C})$ which are the $n \times n$ invertible matrices. We know A is not invertible if and only if $det(A) = 0$. We also know that the determinant is a polynomial equation in the coefficients of the matrix and thus $GL_n(\mathbb{C})^C = V(det)$ and thus $GL_n(\mathbb{C})$ is a Zariski open set.

Also, $GL_n(\mathbb{C})$ is not closed as $x_m = \frac{1}{m} I_n$ for I_n the identify matrix is a sequence in $GL_n(\mathbb{C})$. However, under the identification with $\mathbb{A}^{n^2}_{\mathbb{C}}$ with the Euclidean metric this is a sequence approaching the 0 matrix which is not a invertible matrix and thus $GL_n(\mathbb{C})$ is not closed in the Euclidean topology and thus it is not closed in the Zariski topology as required.

Now we wish to show that the set of nilpotent matrices X is closed in the Zariski topology. We know from linear algebra that a matrix A is nilpotent if and only if $A^k = 0$ for some $k \leq n$ and thus in particular $A^n = 0$. Therefore, as matrix multiplication is a polynomial equation in the coefficients then their are polynomial equations $f_{ij}(A_{11}, ..., A_{nn})$ for $1 \leq i, j \leq n$ such that $A^n_{ij} = f_{ij}(A_{11}, A_{12}, ..., A_{nn})$ and thus $X = V(\{f_{ij} : 1 \leq i, j \leq n\})$ and thus X is a closed set. Also X is not open as $x_m = \frac{1}{m} I_n$ is a sequence of non nilpotent matrices as $x^k_m = \frac{1}{m^k} I_n \neq 0$ but in the euclidean metric x_n limits to 0 which is a nilpotent matrix and thus the compliment of the set of nilpotent matrices is not Euclidean closed and thus the compliment of the set of nilpotent matrices is not Zariski closed and thus the set of nilpotent matrices is not Zariski open.

Now we shall show that the set of matrices of rank at most $r \leq n$ called X_r is a closed set. This is since if a matrix is of rank r then all sub matrices of size $r + i \times r + i$ for $i \geq 1$ are not invertable and thus have determinant equal to 0. Therefore a matrix is of rank at most r if and only if the determinant of every submatrix or size $r + i \times r + i$ equals 0. We shall labels these determinants f_j and thus $X_r = V(\{f_j : 1 \leq i \leq m\})$ for m the number of submatrices of size $r + i \times r + i$ for $i \geq 1$ and thus X_r is closed. Then X_r for

$r \neq n$ is not open as

$$x_m = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & \frac{1}{m} I_{n-r} \end{array}\right]$$

Then x_m is a matrix of rank n but its limit is

$$x = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

in the Euclidean metric and thus X_r^C is not closed in the Euclidean topology and thus X_r^c is not closed in the Zariski topology and thus X_r is not open in the Zariski topology. Then as $X_n = M_n(\mathbb{C}) = \mathbb{A}_{\mathbb{C}}^n$ as all matrices are of rank less than or equal to n then X_n is open.

We now wish to show the set of diagonalisable matrices Y is not open or closed. For the case where $n = 1$ we have $Y = \mathbb{A}_{\mathbb{C}}^1$ and thus Y is open and closed. Thus we now consider the case where $n \neq 1$. In this case Y is not closed as

$$x_m = \begin{bmatrix} 1 + \frac{1}{m} & 0 & \dots & 1 \\ 0 & 1 + \frac{2}{m} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 + \frac{n}{m} \end{bmatrix}$$

is not diagonalisable since it has distinct eigenvalues for every m given by the diagonal coefficients as the matrix is upper triangular. But the limit as m goes to ∞ is

$$x = \begin{bmatrix} 1 & 0 & \dots & 1 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is not diagonalisable as it has a single eigenvalue 1 of multiplicity n. Therefore the dimension of the eigenspace is the dimension of the nullspace of the matrix

$$x - I = \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is $n-1 \neq n$ and thus from linear algebra x is not diagonalisable. Thus, Y is not closed in the Euclidean topology and thus is not closed in the Zariski topology.

Now to show that Y is not open we shall show Y^c is not closed. This is since the matrices

$$y_m = \begin{bmatrix} 1 & 1/m & \dots & 1/m \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1/m \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which has a single eigenvalue 1 is multiplicity n. Then the dimension of the eigenspace is the dimension of the nullspace of the matrix

$$y_m - I = \begin{bmatrix} 0 & 1/m & \dots & 1/m \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1/m \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is 1 which is not equal to n and thus y_m is not diagonalisable. However, $y_m \to I$ in the Euclidean metric which is digonalisable. Thus Y^c is not closed in the Euclidean topology and thus not closed in the Zariski topology and thus Y is open in the Zariski topology.