1

(a). Let $X, Y \subset \mathbb{A}^n_k$ be affine algebraic sets. Let $f \in I(X \cup Y)$. Then $f(x) = 0$ for all $x \in X \cup Y$. So $f(x) = 0$ for all $x \in X$, therefore $f \in I(X)$. Symmetrically, $f \in I(Y)$ as well. So $f \in I(X) \cap I(Y) \implies I(X \cup Y) \subseteq I(X) \cap I(Y)$.

Conversely, if $f \in I(X) \cap I(Y)$, then $f(x) = 0$ for all $x \in X$ or $x \in Y$. Therefore $f \in I(X \cup Y) \implies I(X) \cap I(Y) \subseteq I(X \cup Y)$. Hence $I(X \cup Y) = I(X) \cap I(Y)$.

$\blacksquare$

For the next proof, we prove the following lemmas:

Lemma 1. *Let $I, J \subset k[x_1, \cdots, x_n]$ be ideals, then $V(I + J) = V(I) \cap V(J)$.*

Lemma 2. *A prime ideal is radical*

Proof of Lemma 1. Let $x \in V(I + J)$. Then for all $f \in I$, $g \in J$, $(f + g)(x) = 0$. If $g = -f$, then $I = J$ and we're done. Otherwise, we assume $f = 0$. Then $g(x) = 0$ for all x, hence $x \in V(J)$. Symetrically, $x \in V(I)$ as well. Therefore $x \in V(I) \cap V(J)$.

Conversely, let $x \in V(I) \cap V(J)$. Then for all $f \in I$, $g \in J$, $f(x) = g(x) = 0$, therefore $f(x) + g(x) = (f + g)(x) = 0$. Therefore $x \in V(I + J)$.

Hence $V(I + J) = V(I) \cap V(J)$. $\square$

Proof of Lemma 2. Let I be a prime ideal. Then let $x^n \in I$ where n is minimal. If $n = 1$, we are done. Otherwise, we see that $x \cdot x^{n-1} \in I$. If $x \in I$ again, we are done. Otherwise, if $x^{n-1} \in I$, we have a contradiction as n was assumed minimal. Hence I is radical. $\square$

Onto the main proof:

(b). Let $X, Y \subset \mathbb{A}^n_k$ be Zariski closed, with $X = V(P_X)$ and $Y = V(P_Y)$. Assume P_X and P_Y are radical ideals as $V(P) = V(\sqrt{P})$ for any Zariski closed set P. Then

$$\begin{aligned}
\sqrt{I(X) + I(Y)} &= \sqrt{I(V(P_X)) + I(V(P_X))} \\
&= \sqrt{\sqrt{P_X} + \sqrt{P_Y}} \\
&= \sqrt{P_X + P_Y} \\
&= I(V(P_X + P_Y)) \\
&= I(V(P_X) \cap V(P_Y)) \\
&= I(X \cap Y)
\end{aligned}$$

Where the second last line results from a use of the lemma above.

Let $P_X = \{y - x^2\}$ and $P_Y = \{y\}$ over $\mathbb{C}[x, y]$. Let $X = V(P_X)$, $Y = V(P_Y)$ and (with abuse of notation), let $(P_X), (P_Y)$ be the ideals generated by the polynomials in P_X, P_Y. $(P_Y) = (y)$ is radical. We see that $(P_X) = (y - x^2)$ is the kernel of the map $\phi : \mathbb{C}[x, y] \to C[t]$ where $(x, y) \mapsto (t, t^2)$. This map is surjective, hence $\mathbb{C}[x,y]/(y-x^2) \cong C[t]$. As $C[t]$ is an integral domain, $(y - x^2)$ is a prime ideal. By Lemma 2, it is radical.

Geometrically, X, Y represent the parabola $y = x^2$ and the line $y = 0$. We then see that $X \cap Y = \{0\}$, so $I(X \cap Y) = I(0) = (x, y) \subset \mathbb{C}[x, y]$. However $I(X) + I(Y) = (y - x^2) + (y) = (y - x^2, y) = (x^2, y) \neq (x, y)$. But $\sqrt{(x^2, y)} = (x, y)$. Hence the outer radical is necessary. ■

2

Proof. Let $X \subset A_k^n$, $f : X \to \mathbb{A}_k^1$ be a regular function. Let $x = (x_1, \cdots, x_n) \in X$ and $y \in \mathbb{A}_k^1$. Then let $X = V(F)$ where $F = \{F_i\}_{i \in I}$ for some indexing set I. Then define $G : X \times \mathbb{A}_k^1 \to \mathbb{A}_k^1$ with $G(x, y) = y - f(x)$. Viewing $\{F_i\}, f$ as regular functions in $\mathbb{A}_k^{n+1}$, with $(x, y) \in \mathbb{A}_k^{n+1}$, we claim that $\Gamma = \{(x, f(x)) \mid x \in X\} = V(F \cup \{G\})$. We see that:

$$
\begin{aligned}
(x, y) \in V(F \cup \{G\}) &\iff F_1(x) = F_2(x) = \cdots = 0, G(x, y) = y - f(x) = 0 \\
&\iff x \in X \text{ and } y = f(x) \\
&\iff (x, y) \in \Gamma
\end{aligned}
$$

Therefore Γ is Zariski closed. ■

3

Proof. Let $X \subset \mathbb{A}_k^n$, $Y \subset \mathbb{A}_k^m$ be Zariski closed.
Then let $X = V(P_X) \ni x = (x_1, \cdots, x_n)$ and $Y = V(P_Y) \ni y = (y_1, \cdots, y_m)$. Define $P_X = \{p_i\}_{i \in I} \subset k[x_1, \cdots, x_n], P_Y = \{q_j\}_{j \in J} \subset k[y_1, \cdots, y_m]$, with I, J indexing sets, such that $p_i(x) = 0$ and $q_j(y) = 0$ for all i, j.

We define $\hat{p}_i, \hat{q}_j \in k[x_1, \cdots, x_n, y_1, \cdots, y_m]$ such that

$$
\begin{aligned}
\hat{p}_i(x, y) &= p_i(x) = 0 \\
\hat{q}_j(x, y) &= q_i(y) = 0
\end{aligned}
$$

We let $\hat{P}_X = \{\hat{p}_i\}, \hat{P}_Y = \{\hat{q}_j\}$ as subsets of $k[x_1, \cdots, x_n, y_1, \cdots, y_m]$. Then $X \times Y = V(\hat{P}_X \cup \hat{P}_Y)$. Hence $X \times Y$ is Zariski closed. ■

4

We observe that points in $\mathbb{A}^n_{\mathbb{C}}$ correspond to maximal ideals (of the form $I = (x_1 - a_1, \cdots, x_n - a_n)$) of $\mathbb{C}[x_1, \cdots, x_n]$ by Hilbert's Nullstellensatz. We then observe that if $x \in V(P)$ for some set $P \subset \mathbb{C}[x_1, \cdots, x_n]$, then for all $f \in P$, $f(x) = 0$.

Looking at ${}^{\mathbb{C}[x_1,\cdots,x_n]}/_{I}$ for $(a_1, \cdots, a_n) \in V(P)$, we see that the corresponding evaluation map has kernel I. As $f(a_1, \cdots, a_n) = 0$, f is in the kernel as well, hence $(P) \subset I$ where (P) is the ideal generated by all $f \in P$. Therefore, the ideals containing (P) are the maximal ideals $(x_1 - a_1, \cdots, x_n - a_n)$ such that $f(a_1, \cdots, a_n) = 0$ for all $f \in P$.

For a ring R and an ideal $I \subset R$, we then see that the maximal ideals in ${}^{R}/_{I}$ are precisely the maximal ideals of R containing I. Therefore the maximal ideals of ${}^{\mathbb{C}[x_1,\cdots,x_n]}/_{I}$ are in correspondence with the points $x \in V(P)$.

(a) For ${}^{\mathbb{C}[x,y]}/_{(x^2+y^2-1,y+x)}$, we look for the solutions of the set of equations $x^2 + y^2 = 1, y = -x$. We observe that the solutions are:

$$V(\{x^2 + y^2 - 1, y + x\}) = \left\{\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right)\right\}$$

Defining the canonical homomorphism $\pi : \mathbb{C}[x, y] \to {}^{\mathbb{C}[x,y]}/_{(x^2+y^2-1,y+x)}$, we see that the vanishing set above correspond to the following maximal ideals in ${}^{\mathbb{C}[x,y]}/_{(x^2+y^2-1,y+x)}$:

$$\left\{\pi\left[\left(x - \frac{1}{\sqrt{2}}, y + \frac{1}{\sqrt{2}}\right)\right], \pi\left[\left(x + \frac{1}{\sqrt{2}}, y - \frac{1}{\sqrt{2}}\right)\right]\right\}$$

(b) For $\mathbb{C}[x, y]/(xy)$, we solve for $xy = 0$ and observe that:

$$V(xy) = \{(a, 0) \mid a \in \mathbb{C}\} \cup \{(0, b) \mid b \in \mathbb{C}^\times\}$$

With canonical homomorphism $\pi : \mathbb{C}[x, y] \to {}^{\mathbb{C}[x,y]}/_{(xy)}$, we see that the set of maximal ideals in ${}^{\mathbb{C}[x,y]}/_{(xy)}$ corresponding to $V(xy)$ is:

$$\{\pi\left[(x - a, y)\right], \mid a \in \mathbb{C}\} \cup \left\{\pi\left[(x, y - b)\right], \mid b \in \mathbb{C}^\times\right\}$$

(c) For $\mathbb{C}[x, y, z]/(xy, yz, zx)$, we solve for $xy = yz = xz = 0$ and observe that:

$$V(xy, yz, xz) = \{(a, 0, 0) \mid a \in \mathbb{C}\} \cup \{(0, b, 0) \mid b \in \mathbb{C}^\times\} \cup \{(0, 0, c) \mid c \in \mathbb{C}^\times\}$$

With canonical homomorphism $\pi : \mathbb{C}[x, y, z] \to {}^{\mathbb{C}[x,y,z]}/_{(xy,yz,xz)}$, we see that the set of maximal ideals in ${}^{\mathbb{C}[x,y,z]}/_{(xy,yz,xz)}$ corresponding to $V(xy, yz, xz)$ is:

$$\begin{aligned}\{\pi\left[(x-a, y, z)\right], \mid a \in \mathbb{C}\} \cup \left\{\pi\left[(x, y-b, z)\right], \mid b \in \mathbb{C}^{\times}\right\} \\ \cup \left\{\pi\left[(x, y, z-c)\right], \mid c \in \mathbb{C}^{\times}\right\}\end{aligned}$$

5

We prove the following theorem:

Theorem 1 (Chinese Remainder Theorem for Quotient Rings)**.** *Let R be a (commutative) ring and $I, J \subset R$ be ideals. Then if $I + J = R$, then:*

$$R/_{(I \cap J)} \cong R/_{I} \times R/_{J}$$

Proof of CRT. We define the map $\phi : R \to {}^{R}/_{I} \times {}^{R}/_{J}$ with $\phi(r) = (r+I, r+J)$. As each component of this map is a ring homomorphism (canonical homomorphisms w.r.t I and J), ϕ is a ring homomorphism. For surjectivity, assume $r + I \in {}^{R}/_{I}$ and $s + J \in {}^{R}/_{J}$. We see that there exists $i \in I, j \in J$ such that $i + j = 1$. So, letting $x = rj + si$, we see that:

$$\begin{aligned} x = rj + si \equiv rj \equiv rj + ri = r(i + j) \equiv r \pmod{I} \\ x = rj + si \equiv si \equiv sj + si = s(i + j) \equiv s \pmod{J} \end{aligned}$$

So for any $r + J \in {}^{R}/_{I}$ and $s + J \in {}^{R}/_{J}$, there exists an $x \in R$ such that $\phi(x) = (r+I, s+J)$. We observe that this solution is unique up to congruence in $I \cap J$ as if $x \equiv x' \pmod{I}$ and $x \equiv x' \pmod{J}$, then $x - x' \in I$ and $x - x' \in J$, therefore $x - x' \in I \cap J$. This implies that $\ker \phi = I \cap J$ as $\phi(x) = (I, J)$ iff $x \in I$ and $x \in J$.

So, by the First Isomorphism Theorem for Rings, we have that:

$$R/_{(I \cap J)} \cong R/_{I} \times R/_{J}$$

■

We observe that:

$$R/_{I} \times R/_{J} \cong R/_{I} \oplus R/_{J}$$

As a direct sum of a finite number of rings is isomorphic to the Cartesian product of the same rings.

We begin the main proof.

Main proof. We observe that $k[X] \cong {}^{k[x_1,\cdots,x_n]}/_{I(X)}$. So with the properties defined in question 1:

$$\begin{aligned} k[X \cup Y] &\cong {}^{k[x_1,\cdots,x_n]}\Big/_{I(X \cup Y)} \\ &\cong {}^{k[x_1,\cdots,x_n]}\Big/_{[I(X) \cap I(Y)]} \end{aligned}$$

We observe that if $X, Y \subset \mathbb{A}_k^n$ disjoint, then $\sqrt{I(X) + I(Y)} = I(X \cap Y) = I(\emptyset) = k[x_1, \cdots, x_n]$.

So $1 \in \sqrt{I(X) + I(Y)}$. But for any ideal J, $1 \in J \iff 1 \in \sqrt{J}$. So $I(X) + I(Y) = (1) = k[x_1, \cdots, x_n]$. We can apply the Chinese Remainder Theorem for Quotient Rings to see that

$$k[X \cup Y] \cong {}^{k[x_1,\cdots,x_n]}\Big/_{I(X)} \oplus {}^{k[x_1,\cdots,x_n]}\Big/_{I(Y)} \cong k[X] \oplus k[Y]$$

$\blacksquare$