

Claire Voisin on the question of rationality

February 27, 2019

Warm-up

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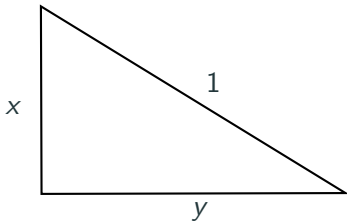
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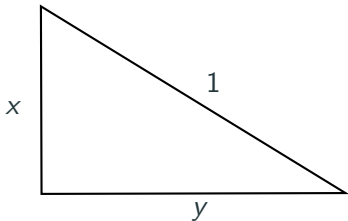
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All the solutions:

$$x = \frac{1 - t^2}{1 + t^2} \quad y = \frac{2t}{1 + t^2}.$$

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Two systems of equations . . .

Variables: x, y

Equations: $x^2 + y^2 = 1$.

Variables: t

Equations: None.

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\longleftarrow

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\longrightarrow

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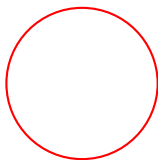
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$$\frac{y+1-x}{y+1+x}$$



$$x^2 + y^2 = 1$$



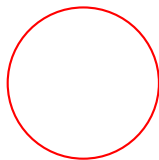
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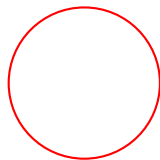


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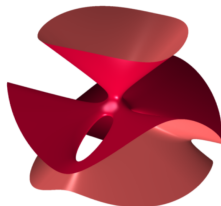
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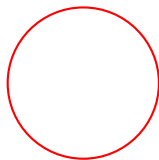


$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$$

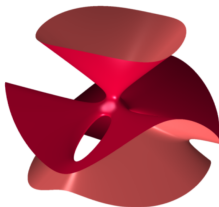
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A Kummer K3

Algebraic varieties

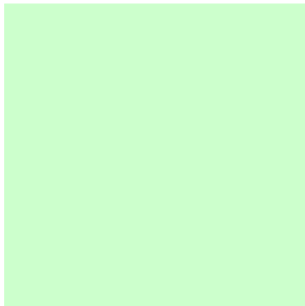
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Example (The best one)

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\mathbf{A}^n = The ambient space (no equations)!

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System of equations \longleftrightarrow Coördinate change \longleftrightarrow No equations!

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4. Varieties defined by **one cubic** ?
 - 4.1 Cubic curves: not rational (ancient)
 - 4.2 Cubic surfaces: rational (Castelnuovo, Enriques: Early 1900s)
 - 4.3 Cubic threefolds: not rational (Clemens–Griffiths: 1972)
 - 4.4 Cubic fourfolds and higher: ???

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as a candidate to detect non-rationality.

But $H^3(X, \mathbf{Z})_{\text{tors}} = 0$ for all interesting examples.



Photo credit: CNRS News Article "Claire Voisin, 2016 CNRS Gold Medal"

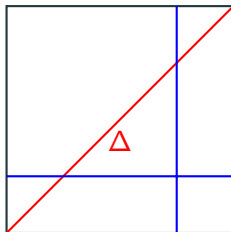
Decomposition of the diagonal

Definition (Voisin, 2015)

X admits a **decomposition of the diagonal** if in $\text{Chow}(X \times X)$,

$$[\Delta] \sim \{x\} \times X + \alpha$$

for some α supported on $X \times Z$ for $Z \subsetneq X$.



Theorem (Voisin, 2015)

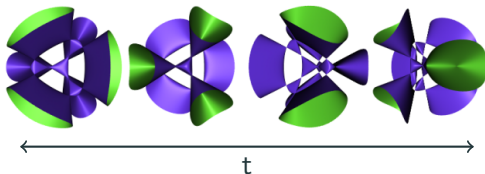
1. X rational $\implies X$ admits a decomp. of the diagonal.
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Theorem (Voisin, 2015)

1. X rational $\implies X$ admits a decomp. of the diagonal.
2. X admits decomp. of the diagonal $\implies H^3(X, \mathbf{Z})_{\text{tors}} = 0$.
3. If X_t is a family of varieties such that some X_{t_0} does not admit a decomp. of the diagonal, then neither does X_t for almost all t .

For example, $X_t = \{x^4 + y^4 + z^4 + w^4 - txyzw = 0\}$.



Decomposition of the diagonal

New technique for non-rationality theorems:

1. Consider a family X_t .
2. Find a t_0 such that X_{t_0} does not admit a decomposition of the diagonal (for example, show $H^3(X_{t_0}, \mathbf{Z})_{\text{tors}} \neq 0$).
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New technique for non-rationality theorems:

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 3. Theorem: Almost all X_t are not rational!
- Very general quartic double solids are not rational (Voisin, 2015).
 - Rationality is not deformation invariant (Hassett–Pirutka–Tschinkel, 2016).
 - Very general hypersurfaces in \mathbf{P}^{n+1} of degree $d \geq \log_2 n + 2$ are not rational (Schreieder, 2018).

There's a lot more...

1. Kodaira problem,
2. Green's conjecture for canonical curves,
3. Chow rings of K3 surfaces,
4. Many questions related to the Hodge conjecture.