VECTOR BUNDLES ON MODULI SPACES, THEIR POSITIVITY, AND APPLICATIONS TO GEOMETRY

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Introduction

This proposal outlines new approaches to studying the geometry of moduli spaces of polarized varieties and presents new applications of this study to classical questions in algebraic geometry. The theme that underlies the projects is the positivity of certain vector bundles on moduli spaces.

The first project addresses open questions about the cones of positive cycle classes on $\overline{M}_{g,n}$ using variations of Hodge structures on $\overline{M}_{g,n}$. The second project studies linear series on surfaces using the geometry of tautological bundles on its Hilbert scheme of points. The third project concerns the role of syzygies in the birational geometry of moduli spaces. These projects build on and substantially extend the techniques I have used in my past and current work, such as my work on the birational geometry of Hurwitz spaces [19, 20, 26], on covering constructions [18], on geometry of stacky curves [21], on GIT of syzygies [22, 23, 24], on slope bounds for fibrations [25], and on vector bundles arising from finite covers [27].

The rest of the introduction contains brief outlines of the projects. The numbered sections that follow develop these outlines. The sections are independent and can be read in any order.

Contents

Project 1: Cycle classes on $\overline{M}_{g,n}$ from variations of Hodge structures. An important open problem about $\overline{M}_{g,n}$ is to determine the cones of positive (effective, nef, pliant) cycle classes in its Chow groups. The answer is unknown even in the case of divisors, let alone in higher codimension, and is a subject of several outstanding conjectures. The problem is important for two reasons. First, it is crucial for understanding the birational geometry of $\overline{M}_{g,n}$, which is of intrinsic interest in algebraic geometry. Second, exploring this question for $\overline{M}_{g,n}$ will reveal fundamental structural properties of cones of positive cycle classes on projective varieties, especially in higher codimension. This is a fascinating new area of research with vigorous recent activity [14, 15, 36, 51].

The main challenge in this area is the construction of new positive cycle classes. The goal of this project is to address this challenge using Chern classes of bundles arising from variations of Hodge structures (VHS) of highly non-trivial families of projective varieties over $\overline{M}_{g,n}$. The broad long-term objectives of this project are the following.

- Construct families of projective varieties with normal crossings degenerations over $\overline{M}_{g,n}$. The proposed construction uses cyclic coverings of normal crossings pairs associated to pointed curves.
- Use the Chern classes of vector bundles in the VHS of the families above to construct infinitely many positive cycle classes on $\overline{M}_{g,n}$ in all dimensions.
- Understand the connection between these vector bundles and previously studied vector bundles on $\overline{M}_{g,n}$, such as the Verlinde bundles of conformal blocks. Use this connection as a geometric approach for understanding conformal blocks.
- Describe the cones generated by the cycle classes constructed above. Determine whether the cones are polyhedral, and identify extremal rays.

We highlight two features of the project. First, it gives an infinite class of nef cycle classes not just in codimension 1, but also in higher codimensions. Second, it yields nef bundles in all genera, whereas the bundles of conformal blocks are nef only in genus 0.

Project 2: Tautological bundles, linear series, and measures of irrationality. A fundamental question in algebraic geometry is to determine whether a given variety X admits a map to a given projective space $\mathbf{P}^r$ of a given degree d. As a culmination of work spanning centuries, we have a good understanding of this question for curves. The answers are collectively known as Brill–Noether theory. The situation in higher dimensions is much murkier. A new feature in higher dimensions is the distinction between regular and rational maps. The goal of this project is to systematically explore the basic question of Brill–Noether theory for surfaces, namely, for which r and d should there exist a *rational* map $X \dashrightarrow \mathbf{P}^r$ of degree d.

A particularly important case is when $r = \dim X$. The minimum d for which there exists a generically finite degree d rational map $X \dashrightarrow \mathbf{P}^r$ is called the *degree of irrationality* of X, denoted by $\operatorname{irr}(X)$. This generalization of the gonality of a curve measures how far X is from being rational, and has been studied by many authors [29, 53, 59, 63]. There has been exciting new progress in determining $\operatorname{irr}(X)$ for very general hypersurfaces of large degree by Ein, Lazarsfeld, and Ullery [29]. But the question remains open for a large class of varieties.

The concrete objectives of this project are the following.

- Determine when we expect a surface X to possess a linear series of rank r and degree d by reducing it a statement about degeneracy of maps on the Hilbert scheme of points on X.
- Prove the existence of linear series of a given rank and degree using intersection theory on the Hilbert scheme of points.
- Obtain upper bounds on the degree of irrationality of K3 surfaces. Determine how the irrationality of a very general K3 surface of genus g grows as a function of g.

The central question in our approach is of the vanishing or non-vanishing of a certain cohomology class on the Hilbert scheme of points. We highlight that either outcome—vanishing or non-vanishing—will lead to interesting mathematics. Vanishing will give non-trivial relations in the Chow ring; non-vanishing will give upper bounds on the degree of irrationality.

Project 3: Bundles of syzygies and the birational geometry of moduli spaces. The goal of this project is to understand vector bundles on moduli spaces constructed using syzygies. Our main motivation is completing the log minimal model program (MMP) for $\overline{M}_g$. However, the idea applies to the construction and study of moduli spaces of polarized varieties of all dimensions.

Consider a moduli space M with a universal family $\pi\colon X \to M$. Let L be a line bundle on X that is sufficiently ample on the fibers of π. For a pair of integers (p,q), consider the vector bundle $K_{p,q}$ on M whose fiber over $t \in M$ is the Koszul cohomology group $K_{p,q}(X_t, L_t)$, or equivalently, the vector space of pth syzygies of weight q of the homogeneous coordinate ring of X_t with respect to L_t. At the heart of this project is the connection between the positivity of $L_{p,q} = \det K_{p,q}$ and the geometry of the polarized varieties (X_t, L_t).

The remarkable observation for $M = \overline{M}_g$ is that the convex hull of the divisor classes of $L_{p,q}$ contains the canonical class $K_{\overline{M}_g}$. As a result, understanding $L_{p,q}$ gives an approach for understanding, and even constructing, the canonical model of $\overline{M}_g$. Our proposed approach is to explicitly construct the Proj of the section ring of $L_{p,q}$ using Geometric Invariant Theory (GIT), thereby obtaining a concrete description of the final steps in the log MMP for $\overline{M}_g$. The GIT construction uses the notion of *syzygy points* (defined in Section 3).

The concrete objectives of this project are the following.

- Prove GIT stability of syzygy points of generic curves.
- Give a new proof of Green's canonical syzygy conjecture proved by Voisin [61,62].
- Analyze the GIT unstable locus and use it to understand the base locus of $L_{p,q}$.
- Use the GIT analysis to give a modular interpretation of the canonical model of $\overline{M}_g$ and a description of the rational map from $\overline{M}_g$ to the canonical model.
- More broadly, understand the connection between the GIT stability of syzygies of projective varieties and the intrinsic geometry of the varieties.

The second objective will be a by-product of our approach towards the first objective.

1. Cycle classes on $\overline{M}_{g,n}$ from variations of Hodge structures

The goal of this project is to construct infinitely many positive cycle classes on $\overline{M}_{g,n}$ from variations of Hodge structures.

1.1. **Main construction in genus 0.**

We begin with the construction over $M_{0,n}$ for simplicity; see § 1.5 for the changes needed to extend to higher genus. Let C be a smooth curve of genus 0, and let $p_1, \dots, p_n$ be n distinct points on C. Let m be a positive integer. Set $Y = C^m$. For $1 \le i \le n$ and $1 \le j \le m$, define the divisor $D_{ij} \subset Y$ by

$$D_{ij} = \{(t_1, \dots, t_m) \in C^m \mid t_j = p_i\}.$$

For $1 \le k, l \le m$, define the diagonal divisor E_{kl} by

$$E_{kl} = \{(t_1, \dots, t_m) \in C^m \mid t_k = t_l\}.$$

Note that the union of all the divisors D_{ij} and E_{kl} is a normal crossings divisor of Y. Let p, a_{ij}, and b_{kl} be positive integers such that the divisor $\sum a_{ij} D_{ij} + \sum b_{kl} E_{kl}$ is divisible by p in $\mathrm{Pic}(Y)$. Let X be the normalization of the degree p cyclic covering of Y branched over $\sum a_{ij} D_{ij} + \sum b_{kl} E_{kl}$. More explicitly, if t_{ij} and s_{kl} are the local equations of D_{ij} and E_{kl}, then (locally) X is the normalization of the variety defined by the equation

$$u^p = \prod t_{ij}^{a_{ij}} \times \prod s_{kl}^{b_{kl}}.$$

Example 1.1. Take $m = 1$. Then there is are no diagonal divisors E_{kl} and X is simply a (normalized) cyclic cover of $C = \mathbf{P}^1$ branched over the marked points $p_1, \dots, p_n$ with assigned multiplicities.

Although X is not smooth, it only has cyclic quotient singularities. The standard Hodge theoretic statements about smooth projective varieties generalize to X (with $\mathbf{Q}$-coefficients); see [5]. In particular, the cohomology groups $H^i(Y)$ give a (pure, polarizable) Hodge structure of weight i in which the associated graded components are

$$\operatorname{gr}^k H^i(X) = H^{i-k}(X, \Omega_X^k),$$

where Ω_X^k is to be understood as the push-forward of Ω^k from the smooth locus. Our main focus will be on the middle cohomology, that is $i = m$.

As $(C, p_1, \dots, p_n)$ varies in $M_{0,n}$, we get a (pure, polarizable) VHS on $M_{0,n}$. Since X admits a natural action of the group μ_p of the pth roots of unity, so does the resulting VHS. As a result, it splits into μ_p eigen-summands.

1.2. **Extension to the boundary.** Since the complement of $M_{0,n}$ in $\overline{M}_{0,n}$ is a divisor with normal crossings, any VHS on $M_{0,n}$ with unipotent monodromy extends canonically to a (mixed) Hodge structure $\overline{M}_{0,n}$ by a theorem of Deligne. The VHS constructed in § 1.1 need not have unipotent monodromy. Nevertheless, after replacing $\overline{M}_{0,n}$ by an appropriate root stack along the boundary, we may assume that the monodromy along the boundary is unipotent, and we get a canonical extension après Deligne. Passing to a root stack in order to extend to the boundary is a well-known technique. For example, it is used in a similar context in [33] and also in my own work [18, 21, 25]. For simplicity of exposition, we suppress this and continue to denote the root stack by $\overline{M}_{0,n}$.

The utility of this construction comes from the inherent positivity properties of extensions of Hodge structures discovered by Griffiths, Fujita [35], and Kawamata [43]. For us, the key positivity property will be the fact that the lowest term in the Hodge filtration of is a nef vector bundle. Thus, the μ_p eigen-summands of the VHS described above give us nef vector bundles on $\overline{M}_{0,n}$.

To be able to effectively use these vector bundles to study $\overline{M}_{0,n}$, a mere existence theorem will not suffice. We must have an explicit description of the extension over the boundary.

Goal 1.2. *Find an explicit description for the extension to $\overline{M}_{0,n}$ of the VHS arising from the cyclic covering construction.*

For $m = 1$, the theory of admissible covers provides the answer. Using admissible covers, Fedorchuk in [33] extends the family of curves $\pi \colon \mathcal{X} \to M_{0,n}$ to a family of nodal curves $\overline{\pi} \colon \overline{\mathcal{X}} \to \overline{M}_{0,n}$. Then the extension of the VHS is simply $R^1\overline{\pi}_*\mathbf{Q}$ and the nef vector bundle is $R^1\overline{\pi}_*\omega_{\overline{\pi}}$. The analogous resolution for Goal 1.2 for $m > 1$ looks daunting at first, but the following approach gives a way forward.

1.2.1. *Approach for Goal 1.2.* Consider the space $\overline{M}_{0,m+n}$ and the forgetful map $\overline{M}_{0,m+n} \to \overline{M}_{0,n}$. Denote the $m+n$ marked points by $1, \dots, m$ and $1', \dots, n'$. Let Z be the fiber of the forgetful map over $(C, p_1, \dots, p_n)$. Then Z is a blow up of C^m, under which the proper transform of the divisor D_{ij} is the boundary divisor $\Delta_{ij'}$ and the proper transform of the divisor D_{kl} is the boundary divisor Δ_{kl}. Thus, instead of carrying out the cyclic covering construction with $Y = C^m$, we could have done it with $Y = Z$. It can be checked that this will not affect the middle Hodge structure. The key observation is that the fiber of $\overline{M}_{0,m+n} \to \overline{M}_{0,n}$ over a boundary point of $\overline{M}_{0,n}$ is a normal crossings degeneration of Z. Therefore, we can extend the family $\mathcal{X} \to M_{0,n}$ of cyclic covers constructed above to a family $\overline{\mathcal{X}} \to \overline{M}_{0,n}$ by taking a normalized cyclic cover of $\overline{M}_{0,n+m}$ branched along certain boundary divisors $\Delta_{ij'}$ and Δ_{kl} (this will require passing to a root stack). The resulting family $\overline{\pi}\colon \overline{\mathcal{X}} \to \overline{M}_{0,n}$ will have (orbifold) normal crossings singularities over the boundary. In particular, $R^m\overline{\pi_*}\mathbf{Q}$ will be the canonical extension of the VHS on $M_{0,n}$ and $\overline{\pi}_*(\omega_{\overline{\pi}})$ will be the nef vector bundle.

1.3. **Chern classes.** In § 1.1 and § 1.2, we described a multitude of nef vector bundles on $\overline{M}_{0,n}$ arising from cyclic coverings. Their Chern classes give nef cycle classes in $A^*(\overline{M}_{0,n})$.

Goal 1.3. *Compute the Chern classes of the nef vector bundles $\overline{\pi}_*(\omega_{\overline{\pi}})$, where $\overline{\pi}\colon \overline{\mathcal{X}} \to \overline{M}_{0,n}$ is an orbifold normal crossings completion of the family of cyclic coverings described in § 1.1.*

1.3.1. *Approach for Goal 1.3.* The geometric approach in § 1.2.1 also suggests a way to compute the Chern classes. The method consists of the following two steps.

(1) Note that the μ_p eigen-bundles of $\overline{\pi}_*(\omega_{\overline{\pi}})$ can be expressed as $f_*(\mathcal{L})$ for suitable line bundles $\mathcal{L}$ on $\overline{M}_{0,n+m}$. Here $f : \overline{M}_{0,n+m} \to \overline{M}_{0,n}$ is the forgetful map. This is a reflection of the fiber-wise identification of the eigen-spaces of $H^0(X, \omega_X)$ with $H^0(Y, \omega_Y(D))$ for a suitable divisor D (see [18] or [5]).
(2) Compute the Chern character $\operatorname{ch} f_*(\mathcal{L})$ using the Grothendieck–Riemann–Roch (GRR) formula for f. Note, however, that due to the root stacks, the fibers of f are not schemes, but Deligne–Mumford stacks. Therefore, we must use GRR for Deligne–Mumford stacks as developed in [28]. This involves a carefully understanding of the inertia stack. I have experience with such calculations, for example in [25] and [18].

This approach has already been successful in the case of $m = 1$ in my previous work [18], yielding a short proof of Fedorchuk's calculations in [33] (the note [18] will be an appendix to [33]).

Theorem 1.4 (D. [18], Fedorchuk [33])**.** *For Goal 1.3 for $m = 1$, the first Chern class of the summand of $\overline{\pi}_*(\omega_{\overline{\pi}})$ corresponding to the character $\zeta \mapsto \zeta^i$ is given by*

$$\frac{1}{2p^2}\left(\sum_i \langle ja_i\rangle_p \langle p - ja_i\rangle_p \psi_i - \sum_{I,J} \langle ja(I)\rangle_p \langle ja(J)\rangle_p \Delta_{I,J}\right).$$

Here $\langle a\rangle_p$ denotes the class of a modulo p in $\{0, \dots, p-1\}$ and $a(I) = \sum_{i\in I} a_i$.

Having outlined an approach for computing the Chern classes, we formulate the next goal.

Goal 1.5. *Describe the cone in $A^i(\overline{M}_{0,n})$ spanned by the ith Chern classes of the nef vector bundles arising from the cyclic covering constructions.*

For $m = 1$ and $i = 1$ the results of [33] show that the cone coincides with the cone of Chern classes of conformal block bundles of type A and level 1. However, even for $m = 1$, we will get new results in higher codimensions. Indeed, the higher Chern classes of the conformal blocks bundles are zero—type A and level 1 conformal blocks bundles have rank 1 on $\overline{M}_{0,n}$. In contrast, we expect the higher Chern classes of the cyclic covering bundles to be non-trivial.

There has been an explosion of activity regarding cones of higher codimension cycle classes on algebraic varieties, pioneered by Fulger and Lehmann [36]. The story of the structure and interpretation of these cones is in its early stages. Goal 1.5 will be invaluable in advancing our understanding in this regard.

1.4. **Relationship with conformal blocks.** Direct image bundles from cyclic covers are intimately connected with conformal blocks bundles. A conformal blocks bundle is a vector bundle $V_{g,n}(\mathfrak{g}, k, \overline{\lambda})$ on $\overline{M}_{g,n}$ associated to the data of a simple Lie algebra $\mathfrak{g}$, a non-negative integer k (called the *level*) and an n-tuple $\overline{\lambda} = (\lambda_1, \dots, \lambda_n)$ of dominant weights of $\mathfrak{g}$ of level k. We will not go into a precise definition of the bundle—see [60] or [8] (or [9] for an interpretation using global sections of tautological line bundles on moduli of space of principal bundles on curves). Understanding the conformal blocks bundles and using them to illuminate the geometry of $\overline{M}_{g,n}$ is a subject of extensive ongoing research; see [1,4,12,13,32,38,39,52].

The work of Schechtman-Varchenko, Ramadas [57], and Belkale [10] describes a connection between the vector space of conformal blocks over a point of $M_{0,n}$ and the space of differentials on a cyclic covering of $(\mathbf{P}^1)^n$ as in § 1.1. More precisely, Belkale in [10] shows that to the data of $\mathfrak{g}$ and $\overline{\lambda}$, we can associate multiplicities a_{ij} and b_{kl} such that we have an injective map

$$V_{0,n}(\mathfrak{g}, k, \overline{\lambda}) \to R\pi_*(\Omega^n_\pi) \tag{1}$$

where $\pi \colon \mathcal{X} \to M_{0,n}$ is the family of cyclic coverings of $(\mathbf{P}^1)^n$ described in § 1.1. Furthermore, the induced map to $R\pi_*\mathbf{C}$ is flat with respect to the KZ-Hitchin connection on the left and the Gauss-Manin connection on the right. Both sides of (1) admit natural extensions to the boundary.

Goal 1.6. *Describe what happens to the map* (1) *over the boundary. Use this connection to understand spaces of conformal blocks over singular curves.*

Achieving Goal 1.6 will give a geometric way of understanding the conformal blocks bundles over all of $\overline{M}_{0,n}$. For example, it will give an approach for understanding the algebra of conformal blocks, its finite generation, and a geometric interpretation by relating it to geometric section rings. Finite generation of the algebra of conformal blocks has proved to be a difficult open question; see [11,48,49,50].

1.5. **Higher genus.** Our discussion so far has been the case of pointed curves of genus 0. A natural objective is the following.

Goal 1.7. *Construct nef vector bundles on $\overline{M}_{g,n}$ by generalizing the cyclic covering construction. Describe the cones spanned by their Chern classes.*

This goal is interesting not only because it is an obvious generalization of the genus 0 case. It will give an unprecedented source of positive cycle classes on $\overline{M}_{g,n}$ (the conformal blocks bundles are *not nef* in higher genus). It will also be insightful to compare the cycle classes arising from Goal 1.7 to the cycle classes of geometric origin constructed by Chen–Coskun [15,16] and Chen–Patel [17].

1.5.1. *Approach towards Goal 1.7.* Recall that the approach outlined in § 1.2.1 for the genus zero case consists of the following two steps.

1. Construct a family of divisorially marked pairs (Y, D) with normal crossings degeneration over the boundary. This serves as the base of the cyclic covers.
2. Take a normalized cyclic covering of the family (Y, D), after passing to a root stack.

The first step outlined in § 1.2.1 for genus zero extends verbatim to higher genus. That is, the fibers of the forgetful map $\overline{M}_{g,n+m} \to \overline{M}_{g,n}$ along with boundary divisors of $\overline{M}_{g,n+m}$ flat over $M_{g,n}$ give family of a normal crossings pairs. The second step presents a slight difficulty: the cyclic p covering depends on a choice of a line bundle L such that $L^p = \mathcal{O}_Y(D)$. In genus 0, there is a unique such choice because $\operatorname{Pic}(Y)$ is torsion-free. This is not the case in higher genus. To overcome this, we propose to simply take the direct sum over all possible pth roots L of $\mathcal{O}_Y(D)$. In other words, the fiber of the VHS will be $\bigoplus_L H^*(X_L, \mathbf{Q})$, where the sum is taken over L such that $L^p = \mathcal{O}_Y(D)$ and where X_L is the normalization of

$$\operatorname{Spec}\left(\mathcal{O}_Y \oplus L^{-1} \oplus \cdots \oplus L^{-p+1}\right).$$

With this change, the approach proposed in § 1.2.1 will go through.

Having constructed the canonical extension to $\overline{M}_{g,n}$, we can generalize Goal 1.3.

Goal 1.8. *Compute the Chern classes of the canonical extensions of the cyclic covering bundles on $\overline{M}_{g,n}$. Describe the cones spanned by them, and identify extremal rays. Compare them with the Chern classes of Verlinde bundles computed in [52].*

Since the extension of the cyclic covering family over the boundary retains the recursive structure of the boundary, we expect the Chern classes arising from this construction to satisfy a recursive structure.

Goal 1.9. *Describe the recursive nature of the cyclic covering bundles. Do they satisfy analogues of the fusion rules for conformal blocks? Do their Chern classes form a cohomological field theory?*

We expect the answers to be "Yes".

2. Tautological bundles, linear series, and measures of irrationality

Let X be a smooth projective surface and L a line bundle on X. Let $V \subset H^0(X, L)$ be a k-dimensional subspace. We call the data of L and V a linear series of rank k, following classical terminology. Assume that V defines a generically finite map $\phi : X \dashrightarrow \mathbf{P}^{k-1}$. In this case, the *degree* of V is defined by

$$\deg V = \widetilde{\phi}^* H^2,$$

where $\widetilde{\phi} : \widetilde{X} \to \mathbf{P}^{k-1}$ is a resolution of ϕ and $H \subset \mathbf{P}^{k-1}$ the hyperplane class. Equivalently, $\deg V$ is the product $d_1 d_2$, where d_1 is the degree of the rational map $X \dashrightarrow \operatorname{im} X$, and d_2 is the degree of $\operatorname{im} X$ in $\mathbf{P}^{k-1}$.

This goal of this project is to address the following questions.

Question 2.1. *When does a surface X admit a linear series of rank k and degree d? In particular, what is the smallest d such that a surface X admits rank 3 linear series of degree d?*

Observe that the second question is equivalent to the following.

Question 2.2. *What is the degree of irrationality of X?*

There has been recent progress on Question 2.2 for hypersurfaces due to Ein, Lazarsfeld, and Ullery [29]. They show that for a very general hypersurface of a sufficiently large degree d, the degree of irrationality is $(d-1)$. The question of determining $\operatorname{irr} X$ for many interesting classes of varieties remains open and a subject of active current research; see [6, 59, 63].

Our main focus will be the case of K3 surfaces.

Question 2.3. *Let (X_g, L) be a very general polarized K3 surface of genus g. What is* $\operatorname{irr}(X_g)$*? In particular, is* $\lim_{g\to\infty} \operatorname{irr}(X_g)$ *bounded or unbounded? What is its order of growth?*

This project will be in collaboration with David Stapleton.

2.1. **Motivation: A sub-linear upper bound for K3 surfaces.** Let us describe a result that motivates the general idea.

Proposition 2.4 (Stapleton)**.** *The degree of irrationality of a general K3 surface of genus g is bounded above by a constant multiple of $\sqrt{g}$.*

Proof. Let (X, L) be a general polarized K3 surface of genus g, where L is the ample generator of $\operatorname{Pic} X$. Let r be a positive integer and $p \in X$ a point such that

$$\binom{r+1}{2} + 3 \leq g + 1. \tag{2}$$

Let $m_p \subset \mathcal{O}_X$ be the maximal ideal sheaf of p. The exact sequence

$$0 \to H^0(X, m_p^r \otimes L) \to H^0(X, L) \to H^0(X, L/m_p^r)$$

shows that we have 3 linearly independent global sections of L that vanish with multiplicity r at p. Note that the intersection multiplicity at p of any two such sections is at least r^2. Therefore, the rational map $X \dashrightarrow \mathbf{P}^2$ defined by the three sections has degree at most

$$d = 2g - 2 - r^2. \tag{3}$$

By combining (2) and (3), we see that we get a map $X \dashrightarrow \mathbf{P}^2$ of degree at most $r + 2$, which is about $\sqrt{2g}$. In other words, $\operatorname{irr}(X_g)$ grows as (at most) $O(g^{1/2})$. □

2.2. **A degeneracy locus on the Hilbert scheme.** They key idea in the proof of Proposition 2.4 is showing the existence of a linear series with a large base locus. Indeed, if a linear series V has a (finite) base locus of length n, then we have the (possibly strict) inequality

$$\deg(V) \leq c_1(L)^2 - n. \tag{4}$$

We are thus lead to a systematic search for linear series with a large base locus. Let $\operatorname{Hilb}^n X$ be the Hilbert scheme of length n subschemes of X. Denote by $L^{[n]}$ the tautological rank n bundle on $\operatorname{Hilb}^n X$ associated to L. This is a bundle whose fiber over $[\xi] \in \operatorname{Hilb}^n X$ is the vector space $H^0\left(L|_\xi\right)$. Evaluation of global sections gives a natural map

$$e \colon H^0(X, L) \otimes \mathcal{O}_{\operatorname{Hilb}^n X} \to L^{[n]}. \tag{5}$$

Question 2.1 is closely related to the following.

Goal 2.5. *Let (X, L) be a polarized surface. Determine pairs of positive integers (r, n) such that the locus*

$$D(r, n) = \{\xi \in \operatorname{Hilb}^n X \mid \operatorname{rk} e_\xi \leq r\}$$

is non-empty.

The precise relationship betwen the non-emptyness of $D(r, n)$ and the existence of linear series is the following.

Proposition 2.6. *Suppose all elements of the linear series $|L|$ are irreducible. If $D(r, n)$ is non-empty, then X admits a linear series of rank k and degree $d = c_1(L)^2 - n$ for any k in the range $2 \leq k \leq h^0(X, L) - r$.*

Proof. Let $\xi \in D(r,n)$. Let $V \subset H^0(X,L)$ be a linear series of rank k contained in the kernel of e_ξ. Using that all the elements of $|L|$ are irreducible, it is easy to check that V does not have a divisorial base locus, and the rational map defined by V is generically finite. By (4), we get $\deg(V) \le c_1(L)^2 - n$. □

The hypothesis that all elements of $|L|$ be irreducible is restrictive, but it is satisfied in the following important case: if (X,L) is such that $\mathrm{Pic} X = \mathbf{Z} \cdot L$. In particular, it holds when X is a general K3 surface of a given genus and L is the ample generator.

2.3. **A dimension count.** Set $\ell = h^0(X,L)$. Note that the map e in (5) is a map between vector bundles of rank ℓ and n on a $2n$-dimensional projective variety. The locus of points where it has rank at most r is expected to have dimension

$$2n-(\ell-r)(n-r). \tag{6}$$

If this number is non-negative, one may conjecture that the locus $D(r,n)$ is non-empty. The following example shows that this expectation is not necessarily correct.

Example 2.7. Let (X,L) be a general polarized K3 surface of genus g, where L is the ample generator of $\mathrm{Pic} X$. Then $\ell = g+1$. Take $r = g-2$. The inequality $2n-(\ell-r)(n-r) \ge 0$ is equivalent to

$$g-2 \le n \le 3g-6.$$

But Proposition 2.6 shows that $D(r,n)$ must be empty for $n \ge 2g-2$.

Remark 2.8. We highlight that the dimension count in the example above for $r = g-2$ leaves open the intriguing possibility that the irrationality of a general K3 surface of genus g is bounded by a constant independent of g!

The work of Fulton and Lazarsfeld [37] shows that under suitable positivity hypothesis, a degeneracy locus of maps of vector bundles necessarily has the expected dimension. That $D(r,n)$ is empty despite the positive expected dimension is a reflection of the fact that the tautological bundle $L^{[n]}$ is not ample. Thus, Goal 2.5 is intimately connected to the positivity (or lack thereof) of the tautological bundles.

2.4. **The degeneracy class.** We propose to approach Goal 2.5 using intersection theory. Denote by $d(r,n) \in A_m(\mathrm{Hilb}^n X)$ the Thom-Porteous class associated to the condition $\mathrm{rk}\, e \le r$. This is the determinant of the $(\ell-r)\times(\ell-r)$ matrix M_{ij} defined by

$$M_{ij} = c_{n-r+i-j}\left(L^{[n]}\right).$$

If $D(r,n)$ has the expected dimension, then $d(r,n)$ is its cycle class. More generally, $d(r,n)$ is the push-forward to $A_m(\mathrm{Hilb}^n X)$ of a class in $A_m(D(r,n))$. In particular, if $d(r,n)$ is non-zero, then $D(r,n)$ must be non-empty. Therefore, we can reduce Goal 2.5 to the following intersection-theoretic calculation.

Goal 2.9. *Let (X,L) be a polarized surface. Determine pairs of positive integers (r,n) such that the class $d(r,n) \in A(\mathrm{Hilb}^n X)$ is non-zero.*

2.4.1. *Possible approaches towards Goal 2.9.* Observe that Goal 2.9 is purely intersection-theoretic. It does not depend on the choice of a particular (X, L) from a family. In particular, for K3 surfaces, the answer depends only on the numerical invariant g. This allows us to make a particular choice of (X, L), such as a Kummer surface, or an elliptic fibration. If there exists some (X, L) for which $D(r, n)$ has the expected dimension, then $d(r, n)$ is non-zero, and hence $D(r, n)$ is non-empty for all (X, L).

The purely numerical nature of Goal 2.9 also suggests an approach by degeneration. Recall that the cobordism ring of polarized varieties is the ring generated by (X, L) modulo certain relations imposed by one-parameter normal crossings degenerations (see [44, 47]). By the work of Lee and Pandharipande [44], we know explicit generators for this ring. An effective tool for computing cohomological invariants is to assemble them into a generating function that factors as a homomorphism from the cobordism ring. Then, it suffices to compute the invariants for the generators of the cobordism ring. We are investigating whether this can be made to work for the classes $d(r, n)$.

Thanks to the work of many authors, we have explicit descriptions of the cohomology ring of the Hilbert scheme of surfaces and the Chern classe of tautological bundles [30, 45, 46]. Therefore, for a given X, r, and n, the question in Goal 2.9 can be solved (in principle) by an explicit calculation. As a first step towards a general answer, we will carry this out for K3 surfaces of low genus with the help of a computer.

2.5. **Consequences of vanishing and non-vanishing.** An attractive feature of Goal 2.9 is that its resolution either way will have interesting consequences. If $d(r, n)$ does not vanish, then we deduce the existence of linear series on X by Proposition 2.6. In particular, for $r = \ell - 3$, we deduce upper bounds on the degree of irrationality of X. If $d(r, n)$ vanishes, then we get a non-trivial relation in the Chow ring among the tautological classes. It would be interesting to deduce these relations using the explicit description of the Chow ring.

3. Bundles of syzygies and the birational geometry of moduli spaces

The goal of this project is to use syzygy bundles to illuminate birational geometry of moduli spaces. We motivate the main construction by its application to the log MMP for $\overline{M}_g$. But the construction applies more broadly, and leads to questions that are of interest independently of the log MMP for $\overline{M}_g$.

3.1. **The log MMP for $\overline{M}_g$.** The log MMP for $\overline{M}_g$, also known as the Hassett–Keel program, synthesizes two important streams in algebraic geometry—the study of moduli spaces and birational geometry. It seeks to describe the spaces $\overline{M}_g(\alpha)$ defined by

$$\overline{M}_g(\alpha) = \operatorname{Proj}\left(\bigoplus_n H^0(\overline{M}_g, n(K_{\overline{M}_g} + \alpha\Delta))\right)$$

for $\alpha \in [0, 1]$. The most sought after description is an interpretation of $\overline{M}_g(\alpha)$ as a good moduli space of a moduli stack of a class of curves depending on α. For $\alpha = 1$, we know that $\overline{M}_g(\alpha) = \overline{M}_g$ is the moduli space of Deligne–Mumford stable curves. For α close to 1 (specifically $\alpha > 7/10 - \epsilon$), the work of Schubert, Hassett, and Hyeon describes $\overline{M}_g(\alpha)$ as a moduli space of curves that include curves with cuspidal and tacnodal singularities [41, 42, 58]. As α decreases, our knowledge gets more speculative; see [34] for a survey.

3.2. **An encouraging calculation.** To achieve the next breakthrough, we need an approach to understanding sections of the line bundles $K_{\overline{M}_g} + \alpha\Delta$ on $\overline{M}_g$. For α close to 1, Hassett and Hyeon use GIT on Hilbert schemes of pluri-canonically embedded curves for this purpose. We need some analogue of this approach for α close to 0.

Let $p < \lfloor g/2 \rfloor$ be a positive integer. Associated to a curve C, we have the Koszul cohomology space $K_{p+1,1}(C, \omega_C)$ (see [2, 3] for Koszul cohomology). This is the vector space of $(p+1)$th syzygies among the generators of the ideal of the canonical embedding of C. Thanks to Voisin's proof of a generic version of Green's conjecture, we know that the spaces $K_{p+1,1}(C, \omega_C)$ define a vector bundle $K_{p+1,1}$ on the open subset U of M_g consisting of curves with the generic Clifford index. To extend this bundle to entire $\overline{M}_g$, observe that for a curve C in U, the space $K_{p+1,1}(C, \omega_C)$ is the only cohomology group in the Koszul complex

$$\begin{aligned} &\wedge^{p+2} H^0(C, \omega_C) \to \wedge^{p+1} H^0(C, \omega_C) \otimes H^0(C, \omega_C) \to \wedge^p H^0(C, \omega_C) \otimes H^0(C, \omega_C^2) \\ &\qquad \to \cdots \to \wedge^{p+2-i} H^0(C, \omega_C) \otimes H^0(C, \omega_C^i) \to \cdots \to H^0(C, \omega_C^{p+2}). \end{aligned} \tag{7}$$

As a result, we can extend $K_{p+1,1}$ on U to a virtual vector bundle on $\overline{M}_g$ defined by the formula

$$K_{p+1,1} = \sum_i (-1)^{i+1} \wedge^{p+2-i} \pi_*(\omega) \otimes \pi_*(\omega^i),$$

where $\pi \colon \overline{M}_{g,1} \to \overline{M}_g$ is the universal curve and ω the relative dualizing sheaf of π.

Proposition 3.1. *Up to scaling, the class of $L_p = \det K_{p+1,1}$ in $\operatorname{Pic}(\overline{M}_g)$ is $K_{\overline{M}_g} + \alpha_p \Delta$ where*

$$\alpha_p = \frac{g^2 - 3gp + 11g - 8p - 12}{7g^2 - 8gp - g - 4p - 6}.$$

Note that as p ranges from 1 (the minimum possible value) to approximately $g/2$ (the maximum possible value), α_p ranges from approximately $1/7$ to $-1/6$. In particular, the cone spanned by the classes of L_p for various p contains the canonical class $K_{\overline{M}_g}$. Therefore, understanding the section rings of the line bundles L_p is critical for the log MMP for $\overline{M}_g$.

3.3. **Constructing moduli spaces using syzygies and GIT.** We now propose to construct moduli spaces where the syzygy line bundle L_p is ample by design. The construction applies more breadly than canonical curves. It proceeds via the notion of a *syzygy point*, which is a point in a Grassmannian that encodes the vector space of syzygies. Before we define it precisely, we recall the property N_p.

Let $X \subset \mathbf{P}V$ be a projective variety. We say that X satisfies N_p if it is projectively normal, its homogeneous ideal is generated by quadrics, and its minimal free resolution is linear for at least p steps. This is equivalent to $K_{i,j}(X, \mathcal{O}(1)) = 0$ for $i \leq p$ and $j \geq 2$. For a fixed p, this can always be arranged by passing to powers of the line bundle $\mathcal{O}(1)$. For canonically embedded curves, the following celebrated conjecture of Green relates it to the Clifford index.

Conjecture 3.2 (Green [40])**.** *Let $C \subset \mathbf{P}^{g-1}$ be a smooth, canonically embedded curve. It satisfies property N_p if and only if $p \leq \operatorname{Cliff}(C)$.*

Suppose $X \subset \mathbf{P}V$ satisfies N_p. Then $K_{p+1,1}(X, \mathcal{O}(1))$ is the only cohomology group in the Koszul complex

$$\wedge^{p+2} V \to \wedge^{p+1} V \otimes V \to \wedge^p V \otimes H^0(X, \mathcal{O}(2)) \to \cdots \to H^0(X, \mathcal{O}(p+2)). \tag{8}$$

Set $\Gamma_p V = \wedge^{p+1} V \otimes V / \wedge^{p+2} V$. Truncating (8) yields the sequence

$$0 \to K_{p+1,1} \to \Gamma_p V \to Q_p \to 0. \tag{9}$$

Set $r = \dim Q_p$.

Definition 3.3. The pth syzygy point of $X \subset \mathbf{P}V$, denoted by $\mathrm{Syz}_p(X)$, is the point of $\mathbf{Gr}(\Gamma_p V, r)$ defined by the quotient $\Gamma_p V \to Q_p]$ in (9).

The following is an easy consequence of the definition.

Proposition 3.4. *If X satisfies property N_{p+1}, then the homogeneous ideal of X can be recovered from the point $\mathrm{Syz}_p(X) \in \mathbf{Gr}(\Gamma_p V, r)$.*

Let $\overline{\mathrm{Syz}}_p \subset \mathbf{Gr}(\Gamma_p V, r)$ be the closure of the set of points $\mathrm{Syz}_p(X)$ as X varies in a component of the Hilbert scheme of $\mathbf{P}V$. If the generic such X also satisfies N_{p+1}, then $\overline{\mathrm{Syz}_p}$ will be birational to the component of the Hilbert scheme by Proposition 3.4.

Goal 3.5. *Understand the geometry of $\overline{\mathrm{Syz}}_p$ and its relationship with the Hilbert scheme.*

Ein and Lazarsfeld have studied asymptotic numerical properties of the Koszul cohomology groups as the polarization becomes more and more positive. We propose to investigate the analogous question for geometric properties of $\overline{\mathrm{Syz}}_p$. Do these spaces show some uniform behavior? Do they stabilize? Or do they become arbitrarily bad?

In addition to asymptotics, we will explore Goal 3.5 for a class of varieties whose minimal free resolution is understood. This includes many varieties arising from representation theory:

(1) determinantal varieties, such as scrolls;
(2) symmetric spaces, such as Grassmannians, flag varieties, orthogonal Grassmannians, and their linear sections.

Specific instances of both types provide short-term, approachable, and interesting lines of investigation. An example of the first type is the following.

Goal 3.6. *Explicitly describe $\overline{\mathrm{Syz}_p}$ for rational normal curves in $\mathbf{P}^n$ along with the induced PGL_{n+1} action. Describe its relationship with other standard compactifications of the space of rational normal curves, such as the principal component of the Hilbert scheme, the Chow variety, and various GIT models such as $\mathrm{Hom}(\mathbf{C}^{n+1}, \mathrm{Sym}^n \mathbf{C}^2) /\!/ \mathrm{SL}_2$.*

Accomplishing this goal will extend the work of Ellingsrud, Piene, and Schlessinger [31, 56].

An example of the second type is the following.

Goal 3.7. *Describe $\overline{\mathrm{Syz}_p}$ for canonical curves, K3 surfaces, and Fano 3-folds of low genus.*

By the work of Mukai [54, 55], all these varieties can be realized as slices of symmetric spaces by linear subspaces.

Observe that the variety of syzygies $\overline{\mathrm{Syz}}_p \subset \mathbf{Gr}(\Gamma_p V, r)$ admits a natural $\mathrm{SL}\, V$ action. Our ultimate interest is in the quotient

$$\overline{\mathrm{Syz}}_p /\!/ \mathrm{SL}\, V.$$

The calculation in § 3.2 suggests the following.

Conjecture 3.8. *Denote by $\overline{\mathrm{Syz}}_p$ the closure of the locus of pth syzygy points of canonically embedded general curves of genus g. Then we have an isomorphism*

$$\overline{M}_g(\alpha_p) = \overline{\mathrm{Syz}_p} /\!/ \mathrm{SL}_g,$$

where α_p is as in Proposition 3.1. In particular, the GIT quotients of syzygies realize the final steps of the log MMP for $\overline{M}_g$.

A proof of Conjecture 3.8 will be a tremendous advance. The major steps towards the proof will be as follows.

(1) Prove that for a generic curve canonical curve $C \subset \mathbf{P}^{g-1}$, the point $\mathrm{Syz}_p(C)$ is stable.
(2) Find a geometric characterization of curves whose syzygy points are unstable.
(3) By a combination of stable reduction and GIT de-stabilization, identify the singular curves that replace the curves in step 2.
(4) Using steps 2 and 3, understand the rational map $\phi \colon \overline{M}_g \dashrightarrow \overline{\mathrm{Syz}_p} /\!/ \mathrm{SL}_g$, at least in codimension 1. In particular, show that it is a birational contraction. Furthermore, show that we have $\phi^*\mathcal{O}(1) = K_{\overline{M}_g} + \alpha_p \Delta - \sum a_i E_i$, where the E_i are the exceptional divisors for ϕ and $a_i \geq 0$.

We now discuss each of these steps, and formulate some broader questions.

My previous work with Fedorchuk and Swinarski achieves a part of the first step in this program.

Theorem 3.9 (D.–Fedorchuk–Swinarski [24])**.** *For a generic curve C of odd genus, the first syzygy point* $\mathrm{Syz}_1(C)$ *is semi-stable.*

Our proof gives a concrete approach for proving a generic semi-stability statement for all p. We show that the first syzygy point of a particular curve is semi-stable. The particular curve is not smooth; it is non-reduced! It is an example of a ribbon—a double structure over $\mathbf{P}^1$, studied by Eisenbud and Bayer in a different but related context [7]. Our technique for proving the semi-stability of Syz_p of a ribbon is explicit. For any given g, it can be checked with a computer. Our calculations with $g \leq 15$ show that for odd g a generic ribbon of genus g has a semi-stable pth syzygy point for all p. Proving this would accomplish the first step, at least for odd g.

Goal 3.10. *Show that for a generic ribbon C of genus g, the syzygy point* $\mathrm{Syz}_p(C)$ *is semi-stable.*

Implicit in Goal 3.10 is that $\mathrm{Syz}_p(C)$ is well-defined; that is, a generic ribbon satisfies property N_p. This was conjectured by Bayer and Eisenbud in [7]. I have recently proved their conjecture.

Theorem 3.11 (Green's conjecture for ribbons, D. [22])**.** *A ribbon C of genus g satisfies property N_p if and only if $p \leq$* $\mathrm{Cliff}(C)$*.*

My proof uses Green's conjecture for generic curves, proved by Voisin [61, 62]. I am investigating a more explicit approach that will give an independent proof and will also help in proving GIT semi-stability.

Goal 3.12. *Give an explicit description of the syzygies of a ribbons. As a consequence, obtain an independent proof of Green's conjecture for general curves (Voisin's theorem), and a also proof of GIT semi-stability.*

The next step will be to understand a connection between the instability of $\mathrm{Syz}_p(X)$ and intrinsic geometry of $X \subset \mathbf{P}V$. For canonical curves, this is a more refined question about syzygies, going beyond Green's conjecture. It will be fruitful to phrase this more generally.

Goal 3.13. *Identify geometric properties of $X \subset \mathbf{P}V$ that imply GIT instability of* $\mathrm{Syz}_p(X)$*.*

In addition to curves, I will investigate this question for polarized K3 surfaces and polarized abelian varieties. The first non-trivial case of Goal 3.13 is the case of canonical curves of genus 7. In this case, my following result gives a flavor of the kinds of answers we expect.

Theorem 3.14 (D. [23]). *Denote by* $\mathrm{Syz}_1(C)$ *the first syzygy point of a canonical curve of genus 7. We have the following implications*

1. *C generic $\implies$ $\mathrm{Syz}_1(C)$ is stable.*
2. *C generic with a g^1_4 $\implies$ $\mathrm{Syz}_1(C)$ is stable.*
3. *C generic with a g^2_6 $\implies$ $\mathrm{Syz}_1(C)$ is strictly semi-stable.*
4. *C bi-elliptic $\implies$ $\mathrm{Syz}_1(C)$ is unstable.*
5. *C trigonal or hyperelliptic $\implies$ $\mathrm{Syz}_1(C)$ is undefined.*

Note that stability of $\mathrm{Syz}_1(C)$ refines the Clifford index—it distinguishes between g^1_4 and g^2_6.

The next step will be to find semistable replacement for curves whose syzygy points are unstable. Recall that Syz_p is defined only for curves that satisfy property N_p. Thus, curves of Clifford index greater than p are excluded by design from the GIT quotient $\overline{\mathrm{Syz}_p} /\!\!/ \mathrm{SL}_g$.

Goal 3.15. *Describe the points that replace curves of Clifford index greater than p in $\overline{\mathrm{Syz}_p} /\!\!/ \mathrm{SL}_g$. As a first step, find the replacements of hyperelliptic and trigonal curves in genus 7.*

A more short-term goal towards carrying out the GIT of $\overline{\mathrm{Syz}}_p /\!\!/ \mathrm{SL}_g$ for all p and g will be to treat more manageable problems.

Goal 3.16. *Describe $\overline{\mathrm{Syz}_p} /\!\!/ \mathrm{SL}_{n+1}$ for the following moduli problems for small n: finite sets in $\mathbf{P}^n$, elliptic normal curves in $\mathbf{P}^n$, and low genus K3 surfaces in $\mathbf{P}^n$.*

The results will not only be interesting in themselves, but they will also give insight into the connection betwen GIT, syzygies, and geometry of moduli spaces.

BROADER IMPACTS

My research will have a positive impact on the community in the following ways.

3.4. **Organization.** I will be an enthusiastic organizer of research and educational activities. These activites will vary in their focus and target audience. The following are the activities I have planned in the near future.

(1) AIM Workshop on *Stability and Moduli Spaces* in San Jose, California. January 9–13, 2017.
With Fedorchuk, Morrison, and Wang, I am organizing a workshop on the role played by K-stability, KSBA stability, GIT stability, and Bridgeland stability in the construction of moduli spaces. Our main goal is to foster cross-disciplinary exchange of ideas. The workshop will include over 30 experts from all over the country and overseas, with many participants from underrepresented groups.
Assessment: We will assess the impact of this activity by looking at the new publications generated and the new collaborations formed.

(2) *MathCamp* at the University of Georgia. June, 2017.
I will be a faculty mentor for high school students working on exploratory projects.

(3) *High School Varsity Math Tournament* at the University of Georgia. October, 2016.
I will help conduct this annual tournament for high school students in Athens, GA.
Assessment: I will be able to assess the impact of the last two activities by following up with the students as they participate in future mathematical activities, and go on to further studies in college. The small size of the town of Athens makes this feasible.

The above is a continuation of my record of organizing activities such as the following.

(1) *Summer Workshop in Algebraic Geometry* at the University of Georgia. September 26–29, 2016. With Angela Gibney, Jesse Kass, and Nicola Tarasca, I organized a weekend workshop in algebraic geometry with a particular emphasis on accessibility to graduate students. Over 40 young mathematicians (mostly graduate students) attended the workshop.
(2) Poster session at *AGNES* at Boston College. October 25–27. With Anand Patel, I co-organized the poster session at AGNES, a bi-annual weekend conference in the North Eastern US, attended by over 150 participants.
(3) *Fairly Informal Reading Seminar and Tea* at the University of Georgia. I organize this weekly reading seminar, aimed at graduate students and young researchers, in which we discuss recent pre-prints.
(4) *Graduate Student Algebraic Geometry Seminar* at Columbia University. Spring 2015. With Johan de Jong, I organized the Graduate Student Seminar in algebraic geometry at Columbia in Spring 2015 on stable rationality and the decomposition of the diagonal.

3.5. **Education and mentoring.** In addition to organizing, I am actively involved in training and mentoring students at various stages.

(1) I am collaborating with and mentoring Changho Han, a graduate student at Harvard University on moduli spaces of log surfaces. I am also collaborating with David Stapleton, a graduate student at Stony Brook University on measures of irrationality of K3 surfaces.
(2) I designed and conducted the Undergraduate Seminar at Columbia on Young tableaux.
(3) I gave a reading course at Columbia from the book *Generatingfunctionology* by H. Wilf.
(4) I conducted *Putnam* preparation sessions for undergraduate students at Columbia.
(5) I gave expository lectures to graduate students in the *Workshop on birational geometry and stability of moduli stacks and spaces of curves* organized by *Vietnam Institute of Advanced Study in Mathematics* in Hanoi, Vietnam.
(6) I gave expository talks to high school students in Pune, India.

3.6. **Software development.** Many of the projects mentioned in this proposal have benefitted from computer experiments performed using computer algebra systems like `sage` or `macaulay2`. The following programs I have written for this purpose are openly available and are a valuable resource for the broader community.

(1) `Macaulay2` program to compute syzygies and to test their GIT stability.
(2) `Macaulay2` program to compute degeneracy loci of maps of some tautological bundles.
(3) `Sage` program to compute tropical moduli spaces of del Pezzo surfaces.
(4) `Sage` program to compute Chern classes of conformal blocks bundles in higher genus.

I will continue to develop mathematical software and will share my work broadly.
Assessment: I will assess the impact of my software by disseminating it through a public repository service like `github.com`. It will allow me to track the number of downloads, get feedback, and invite collaboration.

Results from prior NSF support

No prior NSF support.