Australian National University

Student Number:

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Mathematical Sciences Institute

EXAMINATION: Semester 2 — Mid-Semester, 2019

MATH3354/6216 — Algebra 3 (Classical Algebraic Geometry)

Exam Duration: 180 minutes.
Reading Time: 0 minutes.

Materials Permitted In The Exam Venue:

- Unmarked English-to-foreign-language dictionary (no approval from MSI required).
- No books, notes, reference materials are permitted.
- No electronic aids are permitted e.g. laptops, phones, calculators.

Materials To Be Supplied To Students:

- Scribble Paper.

Instructions To Students:

- The exam is worth a total of 80 points, with the value of each question as shown.
- A good strategy is to read through first and attack the problems in the order that allows you to make the most progress.
- Unless asked otherwise, you must justify your answers. Please be neat and concise.
- You may use any result from class, homework, or workshops as long as it does not trivialise the question. You must cite exactly what you use either by name ("Using the Hilbert basis theorem...") or by recalling the statement ("Since every ideal of a polynomial ring is finitely generated...").
- Througout, k denotes an algebraically closed field. All varieties are defined over k.

Q1	Q2	Q3	Q4	Q5	Q6
14	12	16	12	8	18

Total / 80

Question 1 *14 pts*

(a) State the definition of a *Zariski closed* subset of $\mathbb{A}^n$. *4 pts*

Write your solution here.

(b) Prove that every Zariski closed proper subset of $\mathbb{A}^1$ is finite. *4 pts*

Write your solution here.

(c) Find (with proof) the Zariski closure in $\mathbb{A}^2_{\mathbb{C}}$ of the set

$$S = \{(\cos t, \sin t) \mid t \in \mathbb{R}\}.$$

You may use (without proof) that $V(x^2 + y^2 - 1)$ is isomorphic to $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. *6 pts*

Write your solution here.

Question 2 ***12 pts***

Let $X \subset \mathbb{A}^n$ be a Zariski closed set, and let $f \in k[x_1, \ldots, x_n]$ be such that $f(x) \neq 0$ for every $x \in X$. Prove that there exists $g \in k[x_1, \ldots, x_n]$ such that

$$f(x)g(x) = 1 \text{ for every } x \in X.$$

Write your solution here.

Question 3 ***16 pts***

Let $U \subset \mathbb{A}^n$ be an open subset of a Zariski closed set.

(a) State the definition of a *regular function* on U. *4 pts*

Write your solution here.

(b) Let $U = \mathbb{A}^1 \setminus \{0, 1\}$. Describe (without proof) the ring of regular functions on U. *4 pts*

Write your solution here.

(c) Is $U = \mathbb{A}^1 \setminus \{0, 1\}$ isomorphic to an affine variety? If yes, describe an isomorphism between U and a Zariski closed subset of an affine space. If no, prove that such an isomorphism cannot exist. *8 pts*

Write your solution here.

Question 4 ***12 pts***

Let n be a positive integer. Let $X \subset \mathbb{A}^2$ be the zero set of the irreducible polynomial $y^2 - x^{2n+1}$.

(a) Construct a regular bijective map $\mathbb{A}^1 \to X$. No justification is necessary. *2 pts*

Write your solution here.

(b) Is your map an isomorphism? Why or why not? *10 pts*

Write your solution here.

Extra space for previous question

Question 5 ***8 pts***

Recall that a line in $\mathbb{P}^2$ is the zero locus of a degree 1 homogeneous polynomial.

(a) Prove that two distinct lines in $\mathbb{P}^2$ intersect in a unique point. *4 pts*

Write your solution here.

(b) Let L be the line $L = V(X + Y + Z)$. Construct an isomorphism $L \longrightarrow \mathbb{P}^1$. Your answer must specify a regular map $L \longrightarrow \mathbb{P}^1$ and a regular map $\mathbb{P}^1 \longrightarrow L$ inverse to the first map. No further justification is necessary. *4 pts*

Write your solution here.

Extra space for previous question

Question 6 ***18 pts***

Let $S \subset \mathbb{P}^4$ be the set of $[X_0 : \dots : X_4]$ satisfying the condition

$$\operatorname{rank}\begin{pmatrix} X_0 & X_1 & X_2 \\ X_2 & X_3 & X_4 \end{pmatrix} = 1.$$

(a) Show that S is a projective variety. *4 pts*

Write your solution here.

(b) Set $P = V(X_0, X_2) \cap S$. Show that the map $\pi\colon S \setminus P \longrightarrow \mathbb{P}^1$ defined by

$$\pi\colon [X_0 : \dots : X_4] \mapsto [X_0 : X_2]$$

extends to a regular map $\pi\colon S \longrightarrow \mathbb{P}^1$. *6 pts*

Write your solution here.

(c) Prove that all fibers of $\pi \colon S \to \mathbb{P}^1$ are isomorphic to $\mathbb{P}^1$. *8 pts*

Remark: S is known as the Hirzebruch surface $\mathbb{F}_1$, and is the simplest example of a non-trivial $\mathbb{P}^1$-fibration over $\mathbb{P}^1$.

Write your solution here.

Extra space for previous question