RESEARCH STATEMENT

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I am an algebraic geometer with broader interests in algebra, number theory, combinatorics, and topology. My work focuses on the construction and study of moduli spaces. I use diverse techniques such as geometric invariant theory, deformation theory, algebraic stacks, and birational geometry.

A moduli space M parametrizes a collection of related algebraic varieties (such as smooth curves of a fixed genus g) or, more generally, a collection of related mathematical objects. Moduli spaces of smooth objects are almost never compact. This severely limits their use in applications. A central theme in my work is finding ways to add singular degenerations to the original collection to obtain a *modular compactification*—a compact space $\overline{M}$ which is itself a moduli space for the enlarged collection. Modular compactifications have proven to be a powerful tool for solving problems about the objects of M by reducing them to easier problems about suitable degenerations. For example, we can prove that two general plane curves of degrees m and n intersect in mn points by degenerating them to the union of m and n lines. Finding a class of suitable degenerations is a delicate problem, which often benefits from a change of perspective. Recently, a new perspective on plane curves using stacky branched covers allowed me to answer an old question about their degenerations [10].

In the last couple of decades, we have realized that *many* modular compactifications $\overline{M}$ can arise from a single M. The idea of studying M through multiple $\overline{M}$ has broad applications. It gives spectacular examples of the Minimal Model Program (MMP) in birational geometry. It allows us to transport information from one space to another, leading to wall-crossing formulas in Gromov–Witten theory, mathematical physics, and representation theory. Studying how the many $\overline{M}$ relate to each other and what they tell us about M is a central theme in much of my work [8, 9, 12, 13].

My research explores the geometry of several moduli spaces. Here is an overview of my past and current work.

(1) On the moduli space of curves: birational models using syzygies and geometric invariant theory [12, 13, 14].
(2) On the Hurwitz spaces of branched covers: alternate compactifications and the MMP [8, 9], Picard group [16], effective divisors [15].
(3) On moduli spaces of maps to stacks and their applications [10].
(4) On moduli spaces in tropical geometry: tropical del Pezzo surfaces (in progress).

In the following sections, I discuss highlights from my work and my plans for future research.

1. Birational geometry of moduli spaces

Let X be a projective variety. Classifying all maps from X to other projective varieties is a fundamental question in algebraic geometry. It lies at the heart of the *Mori program*. Stated broadly, my goal is to achieve such classifications for various moduli spaces X.

Goal 1.1. *Let X be a projective moduli space. Describe the decomposition of the cone $\overline{\mathrm{Eff}}(X)$ of pseudo-effective divisors on X into Mori chambers. Describe the corresponding birational models as modular compactifications.*

The subject started with the discovery of several compactifications of the moduli space of curves M_g [27, 28, 39]. It is now known as the *Hassett–Keel program*.

1.1. Hassett–Keel program for the Hurwitz spaces.

The Hurwitz space $H_{d,g}$ is the moduli space of simply branched d-sheeted covers $\phi : C \to \mathbf{P}^1$, where C is a curve of genus g. Hurwitz spaces have been studied for over a century, not only by algebraic geometers, but also by number theorists, topologists, and representation theorists. They are the main tool in fundamental results about the moduli space of curves, such as the fact that it is connected, proved in the 1890s, or the fact that it is of general type, proved in the 1980s [18, 24, 29]. I study them from the standpoint of the Mori program. I construct a range of modular compactifications of $H_{d,g}$, described in the following theorem. My work begins a Hassett–Keel program for the Hurwitz spaces.

Theorem 1.2 (Deopurkar [8, Theorem B]). *Let ϵ be a rational number. There is a projective compactification $\overline{H}_{d,g}(\epsilon)$ of $H_{d,g}$ parametrizing ϵ-admissible covers. Roughly, these are branched covers $\phi : C \to P$ where P is a nodal rational curve on which up to $1/\epsilon$ of the branch points may coincide.*

For $\epsilon = 1$, I recover the spaces of admissible covers of Harris–Mumford [24]. For $d = 2$, I recover the spaces of hyperelliptic curves of Fedorchuk [20].

Using the ideas of Theorem 1.2, I explicitly describe the Mori chamber decomposition in the case of $d = 3$. Let T be the moduli space of pairs (C, D), where C is a curve of genus g and D a reduced divisor of degree 3 with $h^0(C, D) = 2$. The curves C appearing here are known as *trigonal curves*—they are degree 3 covers of $\mathbf{P}^1$.

Theorem 1.3 (Deopurkar [9, Theorem B+C]). *The space T of marked trigonal curves admits a sequence of modular compactifications*

$$\overline{T}^{g} \dashrightarrow \cdots \dashrightarrow \overline{T}^{\mu} \dashrightarrow \cdots \dashrightarrow \overline{T}^{0 \text{ or } 1},$$

indexed by an integer $\mu \equiv g \pmod 2$ between 0 and g. The sequence begins with the contraction of the hyperelliptic locus, continues through flips of the Maroni loci, and culminates in a Fano fibration. We describe the Mori chambers corresponding to $\overline{T}^{\mu}$.

Besides their role in the Mori program, the spaces of ϵ-admissible covers may play a role in Teichmüller dynamics. I discuss this direction in § 2.

1.2. **Varieties of stable limits.** Let X be a very singular algebraic variety. In a one-parameter smoothing of X, we can replace X by another variety with milder singularities by a process called stable reduction. My work addresses instances of the following goal, which lies at the crux of defining modular compactifications.

Goal 1.4. *Describe the possible stable replacements of a singular X.*

Hassett explored this question for curves and log surfaces [25,26]. Fedorchuk studied it for A and D curve singularities and Casalaina-Martin and Laza for ADE curve singularities [6,20]. Using Theorem 1.3, I get the answer for all triple point curve singularities.

My aim is to develop tools to approach Goal 1.4 for higher dimensions. This is an open-ended and long-term project. It will shed light on the geometry of moduli spaces of higher-dimensional varieties, an area which is currently mysterious.

1.3. **KSBA compactification of log surfaces.** Analogous to the Deligne–Mumford compactification of pointed curves, there is a compactification of divisorially marked surfaces of log-general type due to Kollár–Shepherd-Barron and Alexeev (KSBA) [2,31]. Unlike the case of curves, however, even the coarse geometric properties of these spaces remain unknown. Moduli of trigonal curves provide a bridge to this little understood area. A trigonal curve C embeds canonically in a Hirzebruch surface S such that the pair (S,C) is of log general type.

Goal 1.5. *Describe the KSBA compactification of pairs (S,C), where C is a trigonal curve canonically embedded in S.*

I will address this goal using the tools developed in my work. This project will give valuable examples of KSBA compactifications whose geometry we understand. In addition, it will deepen our understanding of stable reduction in higher dimensions.

2. Divisors, tautological classes, and cohomology

In the cohomology ring of a moduli space, we can construct many natural classes called *tautological classes*. While the tautological classes do not always generate the full cohomology ring, many geometrically important classes often lie in their span. My research addresses aspects of the following goal.

Goal 2.1. *Describe the cohomology ring of a moduli space X and the subring generated by the tautological classes. Describe which classes can be represented by effective cycles.*

The analysis of the cone of effective codimension one cycles has important implications for the birational geometry of the space.

Goal 2.2. *Determine the place of X in the birational classification of varieties. For example, determine if it is rational, rationally connected, Calabi–Yau, or of general type.*

2.1. **Cycle classes on Hurwitz spaces.** Despite enormous efforts, the questions in Goal 2.1 and Goal 2.2 remain open for Hurwitz spaces. Even the description of the Chow group $A^1(H_{d,g})$ is still conjectural.

Conjecture 2.3 (See [17])**.** *We have $A^1(H_{d,g}) \otimes \mathbf{Q} = 0$.*

Patel and I prove Conjecture 2.3 for d up to 5.

Theorem 2.4 (Deopurkar–Patel [16, Theorem A]). *We have $A^1(H_{d,g}) \otimes \mathbf{Q} = 0$ for $d \leq 5$.*

For $d > 2g-2$, this conjecture is equivalent to a similar statement about M_g, which was proved by Harer using transcendental techniques [23]. A purely algebraic proof of Conjecture 2.3 will give the first algebraic proof of a fundamental fact about M_g.

We also study the question of effectivity mentioned in Goal 2.1. Denote by λ and δ the pullbacks to $\overline{H}_{d,g}$ of the Hodge class and the total boundary class of $\overline{M}_g$. For $d = 3$, we describe which linear combinations of λ and δ are effective, extending prior work of Cornalba–Harris and Stankova-Frenkel [7, 40].

Theorem 2.5 (Deopurkar–Patel [15, Theorem 1.2]). *The smallest s such that a multiple of $s\lambda - \delta$ is effective on $\overline{H}_{3,g}$ is given by*

$$s = \begin{cases} 7 + 6/g & \text{for even } g \\ 7 + 20/(3g+1) & \text{for odd } g. \end{cases}$$

My future research in this direction will be guided by the following fundamental open questions about the birational geometry of $\overline{H}_{d,g}$.

Goal 2.6. *Determine the cones of effective and nef divisors on $\overline{H}_{d,g}$. In particular, analyze the positivity of the canonical divisor and find the birational type of $\overline{H}_{d,g}$.*

Patel and I are currently studying effective cycles on $\overline{H}_{d,g}$ that arise from the relative minimal free resolution of a branched cover. The first cycle in this series plays a crucial role in the proof of Theorem 2.5. I believe the later ones will be equally important.

The relative minimal free resolution of $f : X \to Y$ yields natural vector bundles on Y, which we call *syzygy bundles*. This construction suggests some open-ended questions.

Goal 2.7. *Understand the relationship between the geometry of $f : X \to Y$ and the properties of the syzygy bundles. In particular, relate via the syzygy bundles the moduli of branched coverings $X \to Y$ and the moduli of vector bundles on Y.*

2.2. **Hurwitz correspondences and Teichmüller dynamics.** Consider the more general Hurwitz space $\overline{H}_{d,g/h}$ that parametrizes genus g covers of genus h curves. It admits a map to $\overline{M}_{h,b}$ given by the configuration of the branch points. It also admits a map to $\overline{M}_{g,b}$ given by the configuration of the ramification points. We thus get a correspondence between $\overline{M}_{h,b}$ and $\overline{M}_{g,b}$, which we call a *Hurwitz correspondence*. It induces a map between the cohomology groups of $\overline{M}_{h,b}$ and $\overline{M}_{g,b}$. These are important maps to study.

Hurwitz correspondences for $g = h = 0$ arise in studying the dynamics of post-critically finite maps in Teichmüller theory [30, § 2]. Consider an orientation-preserving (topological) branched cover of the sphere, say $f : (S^2, P) \to (S^2, P)$, where P is the post-critical set of f. Assume that P is finite. A theorem of Thurston characterizes those f that arise from algebraic maps, using the induced map on the Teichmüller space of (S^2, P). The map on Teichmüller space rarely induces a map on the moduli space $M_{0,P}$, but it does induce a Hurwitz correspondence [30]. The eigenvalues of the induced map on cohomology are important dynamical invariants.

Rohini Ramadas has developed tools to compute the induced maps on cohomology in a range of degrees. We are working together on a complete solution using alternate compactifications of Hurwitz spaces.

3. Syzygies, free resolutions, and GIT

Let $X \subset \mathbf{P}^n$ be a projective variety. A time-honored theme in algebraic geometry is the connection between the algebraic information encoded in the homogeneous coordinate ring of X and the intrinsic geometry of X. I explore this connection from the point of view of moduli.

For suitable positive integers m and p, there are points in various Grassmannians associated to X, called the *mth Hilbert point* or the *pth syzygy point*. Let I be the homogeneous ideal of X. The mth Hilbert point corresponds to the vector space of degree m polynomials in I and the pth syzygy point to the vector space of pth syzygies among the generators of I. As X varies, the Hilbert and syzygy points for various m and p trace out loci in the respective Grassmannians. For sufficiently large m, the locus of mth Hilbert points is precisely the Hilbert scheme constructed by Grothendieck. For smaller m, however, we get variations that are more sensitive to the geometry of X. Already for twisted cubics in $\mathbf{P}^3$, these variations and their relationship with the Hilbert scheme have been a fascinating topic of study [19, 37].

My work addresses the geometry of the loci of syzygy points, an area which is completely unexplored. I expect that the geometry of these loci and their interaction with the natural action of SL_{n+1} will reflect subtle geometric properties of X. For example, I show the following for canonical curves in the first non-trivial case.

Theorem 3.1 (Deopurkar [12]). *Denote by $\mathrm{Syz}_1(C)$ the first syzygy point of a canonically embedded non-trigonal curve of genus 7. We have the following implications regarding the geometry of C and the GIT stability of $\mathrm{Syz}_1(C)$:*

(1) *C is generic $\implies$ $\mathrm{Syz}_1(C)$ is stable.*
(2) *C is generic with a g^1_4 $\implies$ $\mathrm{Syz}_1(C)$ is stable.*
(3) *C is generic with a g^2_6 $\implies$ $\mathrm{Syz}_1(C)$ is strictly semi-stable.*
(4) *C is bi-elliptic $\implies$ $\mathrm{Syz}_1(C)$ is unstable.*

For curves, the connection between the geometry and the GIT stability of syzygies is expected to play a key role in the log MMP for $\overline{M}_g$ (see § 3.1). I will explore such a connection systematically for curves as well as higher-dimensional varieties.

Goal 3.2. *Understand the locus of syzygy points of schemes $X \subset \mathbf{P}^n$ and its connection to the Hilbert scheme. Relate the SL_{n+1} stability/instability of the syzygy point of X to intrinsic geometric properties of X.*

My immediate focus will be on low genus curves, polarized K3 surfaces, and certain Fano 3-folds. Thanks to the work of Mukai, we can describe such varieties as linear sections of homogeneous spaces [33, 34].

Goal 3.3. *Describe the compact moduli spaces of low genus curves, polarized K3 surfaces, and Fano 3-folds using Mukai's descriptions. Relate these models to those constructed using Hilbert or syzygy points.*

With Han-Bom Moon, I am working out the Mukai model of the moduli space of genus 7 curves, which is the quotient of the Grassmannian $\mathbf{Gr}(7,16)$ by an action of Spin_{10}.

3.1. **Application to the log MMP for $\overline{M}_g$.** Denote by δ the total boundary class and by K the canonical class of $\overline{M}_g$. Pioneered by Hassett and Keel, the log MMP for $\overline{M}_g$ aims to give a geometric description of the spaces

$$\overline{M}_g(\alpha) = \mathrm{Proj} \bigoplus_{n \geq 0} H^0\left(\overline{M}_g, n(K + \alpha\delta)\right)$$

for $\alpha \in [0,1]$. As α goes from 1 to 0, the above sequence goes from the Deligne–Mumford model to the canonical model, realizing the minimal model program for $\overline{M}_g$. For α close to 1, the space $\overline{M}_g(\alpha)$ is known to be a GIT quotient of a locus of Hilbert points [27, 28, 39]. For α close to 0, it is expected to be a GIT quotient of a locus of syzygy points.

Conjecture 3.4. *Denote by Syz_p the closure of the locus of pth syzygy points of canonically embedded curves of genus g. The GIT quotients $\mathrm{Syz}_p /\!/ \mathrm{SL}_g$ are log canonical models of $\overline{M}_g$ for (explicitly computable) α-values that range from $(g+6)/(7g+6)$ to 0 as p varies.*

The construction of $\overline{M}_g(\alpha)$ as a GIT quotient of Syz_p was proposed by Keel and Farkas a couple of decades ago, but there was essentially no progress on this proposal. My coauthors and I prove the following generic semistability result, which provides an important step in this direction.

Theorem 3.5 (Deopurkar–Fedorchuk–Swinarski [13])**.** *A general curve of odd genus has a semistable first syzygy point.*

For small genera, we have verified that generic curves have semistable pth syzygy points for all p. Proving this will be a significant advance towards Conjecture 3.4.

Goal 3.6. *Show that a general curve of genus g has semistable pth syzygy points for all g and p. Relate instability with intrinsic geometric properties of the curve.*

We have an approach to Goal 3.6 using non-reduced curves with rich automorphism groups. We conjecture that a particular double structure on $\mathbf{P}^1$, namely the *balanced ribbon* will have a semistable syzygy point for all p. Going further, we would like to interpret $\overline{M}_g(\alpha)$ as a moduli space in its own right. In particular, the moduli interpretation of the canonical model of $\overline{M}_g$ will be extremely interesting.

3.2. **Relationship with Green's conjecture.** A necessary step for Goal 3.6 is related to a fascinating conjecture of Mark Green. Let R be the homogeneous coordinate ring of a canonically embedded curve $C \subset \mathbf{P}^{g-1}$. In 1984, Green made the following conjecture, which relates the shape of the minimal free resolution of R and the presence of linear series on C.

Conjecture 3.7 (Green [21]). *The Koszul cohomology group $K_{p,2}$ of R vanishes if and only if p is smaller than the Clifford index of C.*

In the early 2000s, Voisin proved the conjecture for a generic C using K3 surfaces [43, 44]. Subsequently, Farkas and Aprodu have settled it for a large class of C [3, 4].

In 1995, Bayer and Eisenbud made an analogous conjecture where C is replaced by a *ribbon*—a non-reduced curve obtained by appropriately gluing $\operatorname{Spec} \mathbf{C}[x, \epsilon]/\epsilon^2$ and $\operatorname{Spec} \mathbf{C}[y, \eta]/\eta^2$ [5]. In a recent paper, I prove their conjecture using Voisin's theorem.

Theorem 3.8 (Deopurkar [11]). *Green's conjecture (Conjecture 3.7) holds for all ribbons.*

As a corollary, I get a short proof that Green's conjecture holds for a non-empty open subset of curves of every genus and Clifford index. Theorem 3.8 also reinforces the viability of our approach for proving generic stability of syzygy points using ribbons.

4. Moduli of fibrations and maps to stacks

A powerful technique to study a complicated geometric object is to express it as a fibration where the fiber and the base are simpler. Consider a variety S fibered over a curve C by a map $\phi: S \to C$. We can interpret the data of $(\phi: S \to C)$ as a map from C to the moduli stack X of the fibers of ϕ. Seminal work Abramovich and Vistoli has made this a feasible approach [1].

Using insights from compactifications of Hurwitz spaces, I construct explicit and well-behaved compactifications of maps to X in the one-dimensional case.

Theorem 4.1 (Deopurkar [10, Theorem 2.5]). *Let X be a one-dimensional, smooth, and proper Deligne–Mumford stack of finite type. The space of finite maps from curves to X admits a smooth modular compactification with a normal crossings boundary divisor. The boundary points in this compactification parametrize admissible cover degenerations of finite covers of X.*

The theorem has immediate classical applications. First, taking $X = \overline{M}_{1,1}$ yields a nice compactification of elliptically fibered surfaces. Second, taking $X = [\overline{M}_{0,4}/S_4]$ yields a nice compactification of tetragonal curves on Hirzebruch surfaces. Using this compactification, I solve the first non-trivial case of a long-standing problem.

Problem 4.2. *Describe the stable limits of smooth plane curves of degree d. In other words, describe the closure in $\overline{M}_g$ of the locus of smooth plane curves.*

Theorem 4.3 (Deopurkar, [10, Theorem 1.1]). *Let $Q \subset \overline{M}_6$ be the locus of plane quintic curves and $\overline{Q}$ its closure. The boundary $\overline{Q} \setminus Q$ consists of 14 irreducible components, which we describe explicitly.*

In the proof of Theorem 4.3, I generalize the picture of tetragonal curves, trigonal curves, and theta characteristics described by Vakil [41].

Theorem 4.1 opens new directions for further research, many of which are approachable. Here is an attractive one.

Goal 4.4. *Understand the compactification of the space of elliptic fibrations, and more generally the space of genus 1 fibrations, using Theorem 4.1.*

The question is particularly interesting for the space of maps of degree 24, where the elliptically fibered surface is a K3 surface.

Another direction is to address higher-dimensional targets X. Abramovich and Vistoli have constructed a compactification $\overline{M}_g(X)$ of the space of maps from curves of genus g to X in great generality. Although the spaces $\overline{M}_g(X)$ have been studied from the point of view of Gromov–Witten theory, their geometric properties have been less explored.

Goal 4.5. *Study the geometric properties such as smoothness, connectedness, and irreducibility of the Abramovich–Vistoli compactifications $\overline{M}_g(X)$.*

Answers to these questions will give tools to study moduli spaces of higher-dimensional varieties. They are also likely to unify and illuminate many classical results as they did for $X = [\overline{M}_{0,4}/S_4]$ in my work [10]. Furthermore, it would be wonderful to construct compactifications with nice geometric properties for higher-dimensional X in the style of Theorem 4.1. This is a challenging question even when X is a scheme.

5. Moduli of surfaces and tropical geometry

Tropical and non-Archimedean geometry has had marvelous applications in many areas of algebraic geometry, such as Brill–Noether theory, Gromov–Witten theory, mirror symmetry, birational geometry, and real algebraic geometry. It has lead to remarkable connections between algebraic geometry and combinatorics, leading to progress in both fields.

Let X be an algebraic variety embedded in a toric variety T over a complete valued field K. We can associate to the data of $X \subset T$ a finite polyhedral complex X^{trop} [32]. This tropicalization procedure replaces questions about algebraic functions on X by questions about piecewise linear functions on X^{trop}. We can view X^{trop} as a polyhedral approximation of the Berkovich analytic space X^{an} associated to X [36].

In a joint project with Maria Angelica Cueto, I am studying moduli spaces of del Pezzo surfaces from the tropical perspective. Our current focus is on del Pezzo surfaces of degree 3, or cubic surfaces. We consider the smooth projective model Y of the moduli space of cubic surfaces constructed by Naruki using cross-ratios associated to root systems [35]. The space Y has a GIT interpretation and, conjecturally, a functorial interpretation as the moduli space of weighted stable log pairs [22].

Ren, Shaw, and Sturmfels use a $W(E_6)$-equivariant description of Y due to Yoshida to construct a tropicalization Y^{trop} along with a family of tropical surfaces $G^{\mathrm{trop}} \to Y^{\mathrm{trop}}$ [38, 45]. Our goal is to describe the tropical anti-canonical embeddings of the surfaces in this family.

Let X be a cubic surface. We consider the generators for $H^0(X, -K_X)$ given by the 45 tri-tangent planes and the resulting embedding $X \subset \mathbf{P}^{44}$. Our first goal is to answer a question of Sturmfels [38].

Goal 5.1. *Describe $X^{\mathrm{trop}} \subset T\mathbf{P}^{44}$ and $P^{\mathrm{trop}} \subset T\mathbf{P}^{44}$, where $P \cong \mathbf{P}^3 \subset \mathbf{P}^{44}$ is the span of X.*

We show that X^{trop} is isomorphic to the corresponding fiber of $G^{\text{trop}} \to Y^{\text{trop}}$. The problem of describing the tropical linear space TP is much more challenging. To solve it, we compute the Plücker coordinates of $P \subset \mathbf{P}^{44}$ as functions on Y. The expressions involve equations for the boundary divisors, the divisor of surfaces with an Eckardt point, and three other families of more mysterious divisors. Understanding these divisors and tropicalizing their equations is underway.

Our second goal is the following.

Goal 5.2. *Understand the tropical lines on $X^{\text{trop}} \subset T\mathbf{P}^{44}$ and relate them to the lines on tropical cubics in $T\mathbf{P}^3$.*

We show that the embedding $X^{\text{trop}} \subset T\mathbf{P}^{44}$ corrects the pathology of infinitely many lines described by Vigeland [42]. We can show in many cases that X^{trop} contains precisely 27 lines. We expect this to hold over an open dense subset in Y^{trop}.

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