

THE CANONICAL SYZYGY CONJECTURE FOR RIBBONS

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ABSTRACT. We prove the analogue for ribbons of Green’s canonical syzygy conjecture, formulated by Bayer and Eisenbud. Our proof uses the results of Voisin and Hirschowitz–Ramanan on Green’s conjecture for general smooth curves.

1. INTRODUCTION

Let X be a projective variety over a field \mathbb{K} and L a very ample line bundle on X . Consider the embedding $X \subset \mathbf{P}^n$ given by the complete linear system $|L|$. Associated to $X \subset \mathbf{P}^n$ is the homogeneous ideal $I \subset S = \mathbb{K}[x_0, \dots, x_n]$. The data of (X, L) and the data of (S, I) represent two different points of view of studying the same object—a geometer would rather work with (X, L) whereas an algebraist would rather work with (S, I) . But since (X, L) and (S, I) contain the same information, there ought to be a connection between the properties of (X, L) that a geometer would study with the properties of (S, I) that an algebraist would.

The question of finding relationships between the geometric properties of (X, L) and the algebraic properties of (S, I) has led to fascinating theorems and conjectures. One of the most deeply studied cases is where X is a smooth curve (but see [Gre84] and [EL12] for more general results, especially in an asymptotic setting). In this case, the work of Green [Gre84] and Green–Lazarsfeld [GL86] posits a simple relationship between the Clifford index of X and the shape of the minimal free resolution of S/I as an S module.

We now recall Green’s canonical syzygy conjecture, which deals with the case when the line bundle L is the canonical bundle. We state it in terms of the Koszul cohomology groups $K_{p,q}$. Recall that when $X \subset \mathbf{P}^n$ is projectively normal, $K_{p,q}(X, L)$ is naturally identified with the graded component of degree $(p + q)$ in the p th term of the minimal free resolution of S/I .

Conjecture 1.1 (Green’s canonical syzygy conjecture [Gre84]). *Let C be a smooth projective curve of genus $g \geq 2$ over a field of characteristic zero. The Koszul cohomology group $K_{p,2}(C, \omega_C)$ vanishes if and only if p is smaller than the Clifford index of C .*

The statement of the conjecture generalizes classical results of Noether and Petri, namely that a canonically embedded curve is projectively normal, and that its homogeneous ideal is generated by quadrics if it does not carry a g_3^1 or a g_5^2 .

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In 1995, Bayer and Eisenbud laid out an approach to prove a generic version of Green’s conjecture by a degeneration argument. They considered a special class of degenerate curves, called *ribbons*. These are double structures on \mathbf{P}^1 locally isomorphic to $\text{Spec } \mathbb{K}[s, \epsilon]/\epsilon^2$, and they arise as flat limits of families of canonically embedded curves as the curves approach a hyperelliptic curve. Despite being non-reduced, ribbons retain many of the nice features of smooth curves, such as a projectively normal canonical embedding and a sensible notion of Clifford index. In terms of this Clifford index, we have the following analogue of Conjecture 1.1 due to Bayer and Eisenbud.

Conjecture 1.2 (Canonical syzygy conjecture for ribbons [BE95]). *Let C be a ribbon of arithmetic genus g . Then $K_{p,2}(C, \omega_C)$ vanishes if and only if p is smaller than the (ribbon) Clifford index of C .*

By the semi-continuity of the Clifford index and the smoothability results of Fong [Fon93], Conjecture 1.2 implies Conjecture 1.1 for a general curve of every Clifford index (see the proof of Corollary 1.4).

Having described the conjectures, let us review what is known. Schreyer and Teixidor I Bigas settled Conjecture 1.1 for general p -gonal curves of genus g , where g is large compared to p . Schreyer settled the cases $g > (p-1)(p-2)$ [Sch89]; Teixidor I Bigas advanced it much further to handle $g \geq 3p+2$ [TIB02]. In two breakthrough papers, Voisin proved Conjecture 1.1 when C is general in moduli [Voi02, Voi05]. Combined with prior work of Hirschowitz and Ramanan [HR98], her work implies that Conjecture 1.1 holds for every smooth curve of odd genus and maximum Clifford index. Since then, several authors—Aprodu, Farkas, Lelli-Chiesa, and Pacienza—have made remarkable progress on Conjecture 1.1; it has been proved for general curves of every gonality, for curves lying on a K3 surface, and for certain curves on rational surfaces [Apr05, AP08, AF11, AF12, LC13].

In this paper, we prove Conjecture 1.2. Our proof uses the results of Voisin and Hirschowitz–Ramanan.

Theorem 1.3. *The canonical syzygy conjecture (Conjecture 1.2) holds for every ribbon.*

As a result, we get another proof of the following.

Corollary 1.4. *Green’s conjecture (Conjecture 1.1) holds for a non-empty open subset of curves of a given genus and Clifford index.*

Proof. That $K_{p,2}(C) \neq 0$ for $p \geq c$ is proved by Green and Lazarsfeld in the appendix to [Gre84] for every smooth curve C of Clifford index c .

We now show the converse for a non-empty open subset of curves with Clifford index c . Start with a ribbon C_0 of Clifford index c . By Theorem 1.3, we have $K_{p,2}(C_0, \omega_{C_0}) = 0$ for $p < c$. By [Fon93, Theorem 2], C_0 is the limit of a family of curves C_t whose generic member is a smooth curve of Clifford index c . Since the vanishing of $K_{p,2}(C_t, \omega_{C_t})$ is an open condition, and it holds for $t = 0$, it holds for the generic t , and hence for a non-empty open subset of curves with Clifford index c . \square

The proof of Theorem 2 of [Fon93] shows that a ribbon of Clifford index c is a limit of a family of curves whose generic member is a smooth curve of gonality $c + 2$. Therefore, the argument of Corollary 1.4 in fact yields the following stronger statement.

Corollary 1.5. *Green’s conjecture (Conjecture 1.1) holds for a dense open subset of curves of a given genus and gonality.*

It would be wonderful to have a proof of Conjecture 1.2 independent of Voisin’s proof of Conjecture 1.1 for general curves. It would give a new proof of Voisin’s difficult theorem and would address the original motivation of Bayer and Eisenbud behind studying ribbons. Nevertheless, any proof is better than no proof.

Knowing that the canonical syzygy conjecture holds for ribbons is important for another reason, motivated by recent progress in the log minimal model program for \overline{M}_g . For a canonical curve $C \subset \mathbf{P}^{g-1}$ with $K_{p,2}(C, \omega_C) = 0$, we can define a point in a Grassmannian called the *pth syzygy point* of C . This point encodes the vector space of p th syzygies among the generators of the homogeneous ideal of C (see [DFS16] for details). The GIT quotients of the loci of p th syzygy points are expected to lead to the canonical model of \overline{M}_g . We expect that the p th syzygy point of a general ribbon will be GIT semi-stable. The vanishing of $K_{p,2}$ shows that the p th syzygy point is at least well-defined!

The paper is organized as follows. Section 2 contains basic results about ribbons and their deformations. Section 3 proves Theorem 1.3 for ribbons of odd genus and maximum Clifford index. Section 4 extends the result to all ribbons. Without loss of generality, we take \mathbb{K} to be algebraically closed (of characteristic 0). All schemes and stacks are locally of finite type over \mathbb{K} .

2. RIBBONS

We quickly review the theory of ribbons from [BE95]. Let D be a reduced and connected scheme over \mathbb{K} . A *ribbon* over D is a scheme C with an isomorphism $D \rightarrow C_{\text{red}}$ such that the ideal I of $D \subset C$ satisfies $I^2 = 0$ and is locally free of rank 1 when considered as a sheaf on D . A ribbon over \mathbf{P}^1 is called a *rational ribbon*. All our ribbons will be rational, so we drop this adjective.

A ribbon C of arithmetic genus g gives an exact sequence

$$(1) \quad 0 \rightarrow \mathcal{O}_{\mathbf{P}^1}(-g-1) \rightarrow \Omega_C|_{\mathbf{P}^1} \rightarrow \Omega_{\mathbf{P}^1} \rightarrow 0.$$

We have $\Omega_C|_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(-a-2) \oplus \mathcal{O}_{\mathbf{P}^1}(-b-2)$ for some integers a and b with $0 \leq a \leq b \leq g-1$ and $a+b = g-1$. The *Clifford index* $\text{Cliff}(C)$ of C is defined to be the integer a . We say that C is *hyperelliptic* if the following equivalent conditions hold: (1) the inclusion $\mathbf{P}^1 \subset C$ admits a retraction $C \rightarrow \mathbf{P}^1$, (2) $\text{Cliff}(C) = 0$, (3) the sequence in (1) is split.

A ribbon of arithmetic genus g may be described explicitly as obtained from gluing $U_1 = \text{Spec } \mathbb{K}[s, \epsilon]/\epsilon^2$ and $U_2 = \text{Spec } \mathbb{K}[t, \eta]/\eta^2$ by isomorphisms

$$(2) \quad \begin{aligned} \epsilon &= t^{-g-1}\eta \\ s^{-1} &= t + F(t)\eta, \end{aligned}$$

on $U_1 \cap U_2$, where $F(t)$ lies in $\mathbb{K}[t, t^{-1}]/(\mathbb{K}[t] + t^{-g+1}\mathbb{K}[t^{-1}])$. The element $F(t)$ corresponds to the class of the extension (1) in $\text{Ext}^1(\Omega_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-g-1))$. In particular, $F(t) = 0$ if and only if C is hyperelliptic. Two ribbons C_1 and C_2 are isomorphic by a map that restricts to the identity on the underlying \mathbb{P}^1 if and only if $F_1(t) = cF_2(t)$ for some $c \in \mathbb{K}^*$.

Let $x \in C$ be a closed point, $\beta: C' \rightarrow C$ the blow up at x , and $E \subset C'$ the exceptional divisor. Then C' is a ribbon of arithmetic genus $g - 1$ and $\omega_{C'} = \beta^* \omega_C(-E)$. Blowing up is related to the Clifford index as follows.

Proposition 2.1 (See [BE95, Corollary 2.5]). *The Clifford index $\text{Cliff}(C)$ is the smallest number k such that there exists a sequence*

$$C_k \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 = C$$

where $C_{i+1} \rightarrow C_i$ is a blow-up at a closed point and C_k is hyperelliptic.

The usual Clifford index of smooth curves and the Clifford index for ribbons obey semi-continuity.

Theorem 2.2 (See [EG95, Theorem 2.1]). *Let $C \rightarrow \Delta$ be a proper flat family over a DVR Δ where the geometric general fiber $C_{\bar{\eta}}$ is smooth and the special fiber C_0 is a ribbon. Then $\text{Cliff}(C_{\bar{\eta}}) \leq \text{Cliff}(C_0)$.*

All ribbons of a given genus and Clifford index are related.

Theorem 2.3 (See [BE95, § 8]). *Let $c \geq 1$. There exists a surface $X_{g,c} \subset \mathbb{P}^g$ such that all canonically embedded ribbons of genus g and Clifford index c are hyperplane sections of $X_{g,c}$. In particular, the graded betti numbers $\dim K_{p,q}(C, \omega_C)$ depend only on the genus g and the Clifford index c of C .*

We need basic results about the deformation theory of ribbons.

Proposition 2.4. *A ribbon C has a smooth versal deformation space.*

Proof. For non-hyperelliptic ribbons, this follows from [BE95, Theorem 6.1]. Here is another proof that works for all ribbons, using deformation theory. Denote by T^i the deformation-obstruction functors of Lichtenbaum and Schlessinger for $i = 0, 1, 2$ (see [Har10, § 1.3]). Let T_C^i be the sheaf $T^i(C/\mathbb{K}, \mathcal{O}_C)$. Suppose $A \rightarrow A'$ is a surjection of local Artin \mathbb{K} -algebras with kernel \mathbb{K} . Let $C_{A'} \rightarrow \text{Spec} A'$ be a deformation of C . The obstructions to lifting $C_{A'} \rightarrow \text{Spec} A'$ to a deformation $C_A \rightarrow \text{Spec} A$ lie successively in $H^0(C, T_C^2)$, $H^1(C, T_C^1)$, and $H^2(C, T_C^0)$. Since C has dimension 1, we have $H^2(C, T_C^0) = 0$. Since C is a local complete intersection, we have $T_C^2 = 0$. To compute T_C^1 , we do a simple local computation. Consider $U = \text{Spec } \mathbb{K}[s, \epsilon]/\epsilon^2$. Set $S = \mathbb{K}[s, \epsilon]$, $J = \epsilon^2 S$, and $R = S/J$. The cotangent complex for U/\mathbb{K} is given by

$$L_\bullet: 0 \rightarrow J/J^2 \xrightarrow{d} \Omega_{S/\mathbb{K}} \otimes_S R.$$

By definition, $T_U^1 = H^1(\text{Hom}_R(L_\bullet, R))$. Let e be a generator of the free R module J/J^2 . Note that $\Omega_{S/\mathbb{K}} \otimes_S R = R\langle ds, d\epsilon \rangle$ and $d(e) = \epsilon d\epsilon$. Denote by ∂_s , ∂_ϵ , and e^* the dual generators of ds , $d\epsilon$, and e , respectively. Then

$$T_U^1 = \text{coker}(d^*: R\langle \partial_s, \partial_\epsilon \rangle \rightarrow R\langle e^* \rangle),$$

where $d^*(\partial_s) = 0$ and $d^*(\partial_\epsilon) = \epsilon e^*$. Therefore, we get $T_U^1 = R/\epsilon\langle e^* \rangle = \mathbb{K}[s]\langle e^* \rangle$. By a similar computation on $V = \text{Spec } \mathbb{K}[t, \eta]/\eta^2$ and gluing, we get $T_C^1 \cong \mathcal{O}_{\mathbb{P}^1}(2g+2)$. Therefore, $H^1(C, T_C^1) = 0$ and hence deformations of C are unobstructed. \square

Proposition 2.5. *Let $(U, 0)$ be a versal deformation space of C and $C_U \rightarrow U$ a versal family. Let $N \subset U$ be the closed subset consisting of $u \in U$ where C_u has worse than nodal singularities. Then the codimension of N in U at 0 is at least 2.*

Said differently, $N \subset U$ is the complement of the open set of $u \in U$ over which C_u is a semi-stable curve.

Proof. Note that all ribbons degenerate isotrivially to the hyperelliptic ribbon. Indeed, in the gluing description (2), we may replace $F(t)$ by $cF(t)$ and take c to 0. Therefore, it suffices to prove the proposition for the hyperelliptic ribbon. Furthermore, since $(U, 0)$ is irreducible, it suffices to exhibit an irreducible pointed scheme $(T, 0)$ and a map $\phi: (T, 0) \rightarrow (U, 0)$ such that the codimension of $\phi^{-1}(N)$ in T at 0 is at least 2. Said differently, it suffices to construct a family of curves $C_T \rightarrow T$ such that C_0 is a hyperelliptic ribbon and the locus of $t \in T$ such that C_t has worse than nodal singularities has codimension at least 2. Take $T = \mathbb{A}^{2g+3} = \mathbb{A}\langle a_0, \dots, a_{2g+2} \rangle$ with $0 = (0, \dots, 0)$ and let $C_T \rightarrow T$ be the family of hyperelliptic curves defined affine locally by

$$y^2 = a_{2g+2}x^{2g+2} + \dots + a_1x + a_0.$$

The locus of worse than nodal curves corresponds to set of (a_0, \dots, a_{2g+2}) for which the polynomial $a_{2g+2}x^{2g+2} + \dots + a_1x + a_0$ has a zero of multiplicity at least 3. It is easy to see that this locus has codimension 2. \square

3. PROOF FOR RIBBONS OF ODD GENUS AND MAXIMUM CLIFFORD INDEX

Let X be a projective scheme and L an invertible sheaf on X . Set $V = H^0(X, L)$. Recall that the Koszul cohomology group $K_{p,q}(X, L)$ is defined as the middle cohomology group in the sequence

$$\wedge^{p+1}V \otimes H^0(X, L^{q-1}) \rightarrow \wedge^pV \otimes H^0(X, L^q) \rightarrow \wedge^{p-1}V \otimes H^0(X, L^{q+1}).$$

Let $g = 2k + 1$ with $k \geq 1$. Let \mathcal{U} be the (non-separated) stack of all curves C that satisfy the following conditions:

- (1) C is Gorenstein of arithmetic genus g and $h^0(C, \mathcal{O}_C) = 1$.
- (2) ω_C is very ample and embeds C as an arithmetically Gorenstein subscheme in \mathbb{P}^{g-1} ,
- (3) a versal deformation space of C is smooth.

All three are open conditions and hence \mathcal{U} is an open substack of the stack of all curves constructed, for example, in [Hal10]. The first two conditions are more important than the third; the third is there just to avoid any problems about divisor theory. Denote by $\mathcal{U}^{\text{nodal}} \subset \mathcal{U}$ the open substack parametrizing curves with at worst nodes as singularities. Then $\mathcal{U}^{\text{nodal}} \subset \overline{M}_g$. In fact, $\mathcal{U}^{\text{nodal}}$ is precisely $\overline{M}_g^{\text{va}}$, the stack of stable curves with a very ample dualizing sheaf; it is studied in [Apr05]. Note that $M_g \cap \overline{M}_g^{\text{va}}$ is the complement in M_g of the hyperelliptic locus; we denote it by M_g^{nh} . Also note that

all non-hyperelliptic ribbons satisfy the three conditions—the first is clear; the second is [BE95, Theorem 5.3]; and the third is Proposition 2.4.

Let $\pi: \mathcal{C} \rightarrow \mathcal{U}$ be the universal curve. Set $\mathcal{V} = \pi_*(\omega_\pi)$. Then \mathcal{V} is locally free of rank g on \mathcal{U} . Consider the Koszul complex

$$K_\bullet: \wedge^{k+1} \mathcal{V} \xrightarrow{\delta_1} \wedge^k \mathcal{V} \otimes \mathcal{V} \xrightarrow{\delta_2} \wedge^{k-1} \mathcal{V} \otimes \pi_*(\omega_\pi^2) \xrightarrow{\delta_3} \wedge^{k-2} \mathcal{V} \otimes \pi_*(\omega_\pi^3) \xrightarrow{\delta_4} \cdots \rightarrow \wedge^0 \mathcal{V} \otimes \pi_*(\omega_\pi^{k+1}).$$

Define \mathcal{E} and \mathcal{F} by

$$\begin{aligned} \mathcal{E} &= \text{coker } \delta_1 \\ \mathcal{F} &= \ker \delta_3. \end{aligned}$$

Proposition 3.1. *\mathcal{E} and \mathcal{F} are locally free on \mathcal{U} of the same rank.*

Proof. \mathcal{E} is clearly locally free. Consider a point $[C] \in \mathcal{U}$. Since the homogeneous coordinate ring of the canonical embedding of C is Gorenstein, we have the duality

$$K_{p,q}(C, \omega_C) \cong K_{g-2-p, 3-q}(C, \omega_C)^*.$$

In particular, we have $K_{p,q}(C, \omega_C) = 0$ for $p \geq 4$ and for $p = 3$ and $q < g - 2$. Therefore, the complex K_\bullet is exact from the third place onward: $\ker(\delta_{q+1}) = \text{im}(\delta_q)$ for all $q \geq 3$. Since K_\bullet is a finite complex of locally free sheaves, it follows that $\mathcal{F} = \ker \delta_3$ is locally free.

It is easy to see that the rank of both \mathcal{E} and \mathcal{F} is $\frac{2 \cdot g!}{(k-1)!(k+1)!}$. □

The map δ_2 in K_\bullet induces a map $\delta: \mathcal{E} \rightarrow \mathcal{F}$. Observe that

$$(3) \quad \text{coker } \delta|_{[C]} = K_{k-1,2}(C, \omega_C).$$

Let $D \subset \mathcal{U}$ be defined by the vanishing of $\det(\delta)$.

Let $D_{k+1} \subset \mathcal{U}$ be the closure of the locus of $(k+1)$ -gonal curves in M_g^{nh} .

Proposition 3.2. *Let C be a non-hyperelliptic ribbon of odd genus $2k+1$. We have the equality $D_{k+1} = D$ in an open subset containing $[C] \in \mathcal{U}$.*

Proof. The results of Voisin [Voi05] and Hirschowitz–Ramanan [HR98] give $D_{k+1} = D$ on M_g^{nh} . Consider the open inclusion $M_g^{\text{nh}} \subset \mathcal{U}^{\text{nodal}}$. Note that $\mathcal{U}^{\text{nodal}} \setminus M_g^{\text{nh}}$ has one divisorial component Δ_0 whose general fiber corresponds to an irreducible nodal curve. In [Voi05], Voisin shows that $K_{k-1,2}(C, \omega_C) = 0$ for every C in a linear series on a K3 surface, which includes irreducible nodal curves. Therefore D does not contain Δ_0 as a component. Since D_{k+1} is the closure of a divisor on M_g^{nh} , it does not contain Δ_0 as a component either. Therefore, the equality of divisors $D = D_{k+1}$ holds in codimension 1 on $\mathcal{U}^{\text{nodal}}$ and hence on all of $\mathcal{U}^{\text{nodal}}$. The same reasoning (Proposition 2.5) implies that it holds around every point in \mathcal{U} corresponding to a ribbon. □

Corollary 3.3. *For a ribbon C of genus $2k+1$ with the maximum Clifford index k , we have $K_{k-1,2}(C, \omega_C) = 0$.*

Proof. Note that the smooth curves parametrized by D_{k+1} have Clifford index at most $k - 1$. By upper semi-continuity (Theorem 2.2), we see that $[C] \notin D_{k+1}$, and hence $[C] \notin D$. By (3), this is equivalent to $K_{k-1,2}(C, \omega_C) = 0$. \square

4. PROOF FOR ALL RIBBONS

We now deduce the canonical syzygy conjecture for all ribbons from Corollary 3.

Lemma 4.1. *Let C be a ribbon and $\beta: C' \rightarrow C$ the blow up at a closed point. Then we have an inclusion $K_{k,1}(C', \omega_{C'}) \subset K_{k,1}(C, \omega_C)$.*

Proof. This follows by the same argument as in [Voi02, Corollary 1]. We reproduce the details for completeness.

Since $\omega_{C'} = \beta^* \omega_C(-E)$ where E is the exceptional divisor of β , we have inclusions $H^0(C', \omega_{C'}^l) \rightarrow H^0(C, \omega_C^l)$ for $l \geq 0$.

Set $V' = H^0(C', \omega_{C'})$ and $V = H^0(C, \omega_C)$. Consider the Koszul complexes

$$\begin{array}{ccccccc} \wedge^{k+1} V' & \xrightarrow{\delta'} & \wedge^k V' \otimes V' & \longrightarrow & \wedge^{k-1} V' \otimes H^0(C', \omega_{C'}^2) & \longrightarrow & \dots \\ \downarrow j_1 & & \downarrow j_2 & & \downarrow & & \\ \wedge^{k+1} V & \xrightarrow{\delta} & \wedge^k V \otimes V & \longrightarrow & \wedge^{k-1} V \otimes H^0(C, \omega_C^2) & \longrightarrow & \dots \end{array}$$

For any vector space U , the Koszul differential $\delta: \wedge^{k+1} U \rightarrow \wedge^k U \otimes U$ has a retract $\wedge: \wedge^k U \otimes U \rightarrow \wedge^{k+1} U$ given by the wedge product. Precisely, the two are related by

$$\wedge \circ \delta = (k+1) \text{id}.$$

Suppose $\alpha' \in \wedge^k V' \otimes V'$ is such that $j_2(\alpha') = \delta(\beta)$ for some $\beta \in \wedge^{k+1} V$. Then

$$\wedge \circ j_2(\alpha') = \wedge \circ \delta(\beta) = (k+1)\beta.$$

It is easy to see that $\wedge \circ j_2(\alpha') = j_1 \circ \wedge(\alpha')$. Set $\beta' = \wedge(\alpha')/(k+1)$. Then

$$j_2(\delta'(\beta')) = \delta(j_1(\beta')) = \delta(\beta) = j_2(\alpha').$$

Since j_2 is injective, we get $\delta'(\beta') = \alpha'$. In other words, any element of $\wedge^k V' \otimes V'$ that becomes a coboundary in $\wedge^k V \otimes V$ is already a coboundary. Therefore the map on cohomology $K_{k,1}(C', \omega_{C'}) \rightarrow K_{k,1}(C, \omega_C)$ is injective. \square

Theorem 4.2. *Let C be a ribbon of genus g and Clifford index c . Then $K_{p,2}(C, \omega_C) = 0$ if and only if $p < c$.*

Proof. The “only if” direction is the ‘easy’ direction; it is the content of [BE95, Corollary 7.3].

We now prove the other direction. For $c = 0$, there is nothing to prove. Henceforth, we assume $c \geq 1$. In particular, C is non-hyperelliptic. Recall the following:

- (1) $K_{p,q}(C, \omega_C) = 0$ implies $K_{p',q}(C, \omega_C) = 0$ for all $p' \leq p$,
- (2) we have the duality $K_{p,q}(C, \omega_C) = K_{g-2-p,3-q}(C, \omega_C)^*$.

Since all canonically embedded ribbons of genus g and Clifford index c have the same graded betti numbers (Theorem 2.3), it suffices to prove the theorem for one such ribbon. That is, we must show that $K_{c-1,1}(C, \omega_C) = 0$ for one ribbon of genus g and Clifford index c .

By definition of the Clifford index, we have $2c \leq g - 1$. Write $g = 2k + 1 - i$ and $c = k - i$ for integers k and i with $k \geq 1$ and $k \geq i \geq 0$. Let \overline{C} be a ribbon of genus $2k + 1$ and maximum Clifford index k . By Proposition 2.1, there is a sequence of blow-ups

$$C_k \rightarrow \cdots \rightarrow C_i \rightarrow \cdots \rightarrow C_0 = \overline{C},$$

where C_k is hyperelliptic. In this sequence, $C = C_i$ is a ribbon of genus $g = 2k + 1 - i$ and Clifford index $c = k - i$. By Corollary 3, we know that $K_{k-1,2}(\overline{C}, \omega_{\overline{C}}) = 0$, which gives $K_{k,1}(\overline{C}, \omega_{\overline{C}}) = 0$ by duality. By repeated applications of Lemma 4.1, we deduce that $K_{k,1}(C, \omega_C) = 0$. Note that $g - 2 - k = k - i - 1 = c - 1$. Therefore, we get $K_{c-1,2}(C, \omega_C) = 0$ by duality. The proof of the theorem is thus complete. \square

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