Theorem on dimensions of fibers

$\varphi : X \to Y$ map between irreducible varieties.

f surjective

$\dim Y = m$ $\qquad X_y := \varphi^{-1}(y)$

$\dim X = n$

Thm: ① $m \leq n$

② For every $y \in Y$,

$$\dim X_y \geq n - m.$$

③ $\exists$ open $U \subset Y$ s.t. $\forall\, y \in U$

$$\dim X_y = n - m.$$

For the proof we need some preparation.

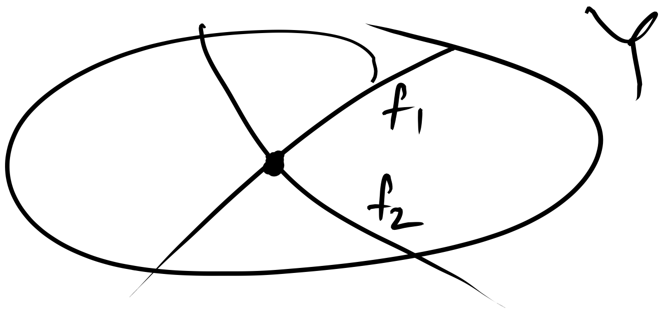
1. Slicing by hyperplanes.

Let Y be affine of dim m & $y \in Y$.

Then there exist $f_1, \ldots, f_m \in k[Y]$

s.t. $V(f_1, \ldots, f_m) \subset Y$ is finite &

$y \in V(f_1, \ldots, f_m)$.



2. Chevalley's thm

$f: X \to Y$ dominant map of irred. var.

$\Rightarrow$ $f(X)$ contains an open subset of Y

(Pf skipped – Black box.)

(BUT we won't even use this!

Pf of theorem: ① We have seen already.

② Suffices to take Y affine. Take $y \in Y$

Now $\exists f_1, \dots, f_m \in k[Y]$ s.t.

$y \in V(f_1, \dots, f_m)$ & $V(f_1, \dots, f_m)$ is finite.

By shrinking Y, assume

$$V(f_1, \dots, f_m) = \{y\}.$$

Then
$$X_y = X \cap \left\{ \begin{array}{c} \phi^*(f_1) = 0 \\ \vdots \\ \phi^*(f_m) = 0 \end{array} \right\}$$

By the principal ideal thm,

$$\dim X_y \geq n-m.$$

③ Take $V \subset X$ open affine

We'll show that $\exists$ non ϕ open $U_V \subset X$

s.t. $\forall$ $y \in U_V$, the fiber

V_y is (either ϕ or) $n-m$ dim.

By taking $U = \bigcap U_V$ as V ranges over a finite open cover of X, we get the theorem.

Now, consider $V \to Y$.

The idea of the proof is easy.

Write $k[V] = k[Y][t_1, \ldots, t_N] / I$

Since $\operatorname{trdeg}_k k(V) = m$

$\operatorname{trdeg}_k k(Y) = n$

$\operatorname{trdeg}_{k(V)} k(Y) = n-m.$

So, wlog, assume $t_1, \ldots, t_r \in k[V]$ are alg. indep over $k(Y)$ and $\exists$ $P_i \in k[Y][t_1, \ldots, t_{n-m}, t_i]$ not identically 0 such that

$P_i(t_1, \ldots, t_{n-m}, t_i) \equiv 0$ on V.

(So $P_i(t_1, \ldots, t_{n-m}, t_i) \in I$).

Think of P_i as a polynomial in the t-variables with coeff in $k[Y]$.

Given $y \in Y$, we can evaluate

these coeff. at y & get a polynomial

$$ev_y(P_i) \in k[t_1, \dots, t_{n-m}, t_i].$$

Let $U \subset Y$ be the open set where $ev_y(P_i)$ is not the zero polynomial $\forall i$.

For $y \in Y$, consider $k[Z]$ where Z is a component of V_y.

Then $k[Z]$ is a quotient of $k[V_y]$

And $k[V_y]$ is

$$k[V_y] = k[Y][t_1, \dots, t_N] \Big/ \sqrt{(I + I(y))}$$

$$\cong k[t_1, \dots, t_N] \Big/ \sqrt{ev_y(f) \text{ for } f \in I}$$

$= $ A quotient of

$$k[t_1, \dots, t_N] \Big/ \left(ev_y(P_i) \text{ for } i = n-m+1, \dots, N\right)$$

So (if $V_y \neq \emptyset$) if $Z \subset V_y$ is an irr. comp., then $k(Z)$ is gen. over k by the images of $t_1, \dots, t_N$ and $t_{n-m+1}, \dots, t_N$ are algebraic over the subfield gen by the images of $t_1, \dots, t_{n-m}$. Indeed, t_i for $i > 0$ satisfies the <u>non-zero</u> polynomial

$$ev_y(P_i) \in k[t_1, \dots, t_{n-m}, t_i]$$

on Z.

so $\dim Z \leq n-m$.

$\square$