FUNDAMENTALS OF ALGEBRAIC GEOMETRY: QUESTIONS FOR MIDTERM

(1) (a) Describe the points of Spec $\mathbb{C}[x,y]/\langle xy\rangle$.
  (b) What are the stalks of the structure sheaf at the points $\langle x,y\rangle$, $\langle x-1,0\rangle$, and $\langle y\rangle$?
  (c) Which of these stalks are integral domains?
  (d) For which of these stalks is the natural map from $\mathbb{C}[x,y]/\langle xy\rangle$ injective?
  (e) What is the closure of the point $\langle y\rangle$?

(2) Let $f : A \to B$ be a map of rings.
  (a) Prove that the preimage of a prime ideal is a prime ideal.
  (b) Is the preimage of a maximal ideal a maximal ideal?
  (c) Is the answer "yes" under some hypotheses on A and B?
  (d) Use the above to prove that, under the same hypotheses on A, the nilradical of A is the intersection of all maximal ideals of A.
  (e) Can you state the previous result in terms of functions on Spec A?
  (f) Take $A = \mathbb{C}[x]$ and $B = \mathbb{C}[x,y]/(y^2-x)$ with the obvious map $f : A \to B$. Describe the sets Spec A, Spec B and the map between them induced by f.

(3) Let A be a ring.
  (a) What is a distinguished open subset of Spec A?
  (b) Prove that the distinguished open subsets form a base of the Zariski topology.
  (c) Is every open subset a distinguished open subset?
  (d) Is the intersection of two distinguished open subsets a distinguished open subset?
  (e) Use the distinguished open subsets to show that Spec A is quasi-compact.

(4) (a) Define a scheme.
  (b) Define an affine scheme.
  (c) Give an example of a scheme which is not an affine scheme.
  (d) Let X be the one point set. Let O_X be the sheaf of rings on X defined by $O_X(X) = \mathbb{Z}$. Is this a scheme?

(5) Let $A = \mathbb{C}[x,y]/(xy)$ and $B = \mathbb{C}[x,y]/(xy^2)$.
  (a) Are Spec A and Spec B isomorphic as schemes?
  (b) What is the relationship between the underlying topological spaces?
  (c) Draw pictures of Spec A and Spec B.
  (d) There are (non-zero) $f \in A$ and $g \in B$ such that A_f and B_g are isomorphic. Using your pictures (or otherwise), find them.

(6) Let $A = \mathbb{C}[t]$ and $P = \text{Proj}\, A[X,Y]$, where X, Y are in degree 1.
  (a) What are the points of P?
  (b) Describe the distinguished open D_{XY} and the ring $O_P(D_{XY})$.
  (c) Let $Z \subset P$ be the closed subscheme cut out by $Y^2 - tX^2$. what are the points of Z and where are they mapped by the map $Z \to \text{Spec}\, A$?
  (d) Write Z as the Proj of a graded algebra.
  (e) Write down an affine open cover of Z.
  (f) Is Z affine?