# 1 Irreducibility and rational maps **WEEK8**

## 1.1 Irreducible topological spaces

A topological space $X$ is *reducible* if it can be written as a union of two proper closed subsets. It is *irreducible* if it is not reducible.

We have encourtered this property many times before, even though we have not named it yet.

**1.1.1 Example** The space $X = V(xy) \subset \mathbb{A}^2$ is reducible (in the Zariski topology). We can write $X$ as the union $V(x) \cup V(y)$. On the other hand, we will soon see that $X = V(xy - 1)$ is irreducible (the real picture is misleading!).

— In the usual Euclidean topology, we rarely encounter irreducible spaces. In fact, it is not hard to show that $X \subset \mathbb{R}^n$ is irreducible (in the Euclidean topology) if and only if $X$ is a single point. But irreducibility turns out to be an important notion in algebraic geometry.

**1.1.2 Proposition (Equivalent conditions for irreducibility)** The following are equivalent

1. $X$ is irreducible.
2. Every non-empty open subset of $X$ is dense.
3. Any two non-empty open subsets of $X$ have a non-empty intersection.

*Proof.* Let us prove $1 \implies 2 \implies 3 \implies 1$.

For $1 \implies 2$, suppose $X$ is irreducible, and $U \subset X$ is a non-empty open. Let $Y = X - Z$. Then $Y \subset X$ is a proper closed subset. Let $\overline{U}$ be the closure of $U$. Then $X = Y \cup \overline{U}$. Since $X$ is irreducible and $Y \subset X$ is a proper closed subset, we must have $\overline{U} = X$.

For $2 \implies 3$, assume that every non-empty open is dense and let $U, V \subset X$ be non-empty open subsets. Pick a $v \in V$. Then $v$ lies in the closure of $U$, so any open subset containing $v$ must intersect $U$. In particular, $V$ intersects $U$.

For $3 \implies 1$, assume that any two non-empty opens have a non-empty intersection. Suppose $X = Y \cup Z$, where $Y, Z \subset X$ are open and $Y \neq X$. We show that $Z = X$. By taking complements, we have $Y^c \cap Z^c = \varnothing$, and hence either $Y^c$ or $Z^c$ is empty. But by assumption $Y^c$ is non-empty, so $Z^c$ must be empty. In other words, we have $Z = X$. $\square$

**1.1.3 Proposition (Closure and image of irreducible is irreducible)**

1. Suppose $U \subset X$ is dense. Then $U$ is irreducible if and only if $X$ is irreducible.
2. If $f\colon X \to Y$ is a surjective continuous map and $X$ is irreducible, then $Y$ is irreducible.

Prove this.

For affine varieties, irreducibility is (unsurprisingly) related to a well-known property of the ring of regular functions.

**1.1.4 Proposition (Irreducibility of affines)** Let $X \subset \mathbb{A}^n$ be a Zariski closed subset. Then the following are equivalent.

1. $X$ is irreducible.
2. $I(X)$ is a prime ideal.
3. $k[X]$ is an integral domain.

Prove this. — (2)

**1.1.5 Corollary (Grassmannians are irreducible)** The Grassmannians (and in particular, the projective spaces) are irreducible.

*Proof.* There is a surjective regular map from an open subset $\mathbb{A}^{mn}$ to $\mathbf{Gr}(m, n)$. □

## 1.2 Irreducible components

If $X$ is reducible, it has a unique decomposition into irreducible components. The idea is simple: we start by writing $X = Y \cup Z$, where $Y$ and $Z$ are proper closed subsets. If either $Y$ or $Z$ or both are reducible, we further write them as unions of proper closed subsets, and continue. We need something to ensure that the process stops (it does not stop, for example, in the usual topology).

**1.2.1 Definition (Noetherian topological space)** A topological space $X$ is Noetherian if every nested sequence of closed subsets

$$X \supset X_1 \supset X_2 \supset X_3 \supset \cdots$$

stabilises.

A consequence of the Hilbert basis theorem is that every affine variety is Noetherian. It is easy to check that if $X$ has a finite open cover by Noetherian topological spaces, then $X$ is Noetherian. As a result, every algebraic variety of finite type is Noetherian. (A variety is of *finite type* if it has an atlas consisting of finitely many charts.)

**1.2.2 Proposition (Irreducible decomposition)** Let $X$ be a Noetherian topological space. We can write

$$X = X_1 \cup \cdots \cup X_n,$$

where $X_i \subset X$ are irreducible closed subsets with $X_i \not\subset X_j$ for $i \neq j$. Furthermore, this decomposition is unique (up to permutation of the factors).

The factors $X_i$ are called *irreducible components* of $X$.

*Proof.* The idea is to keep decomposing until we reach irreducible pieces. The Noetherian hypothesis ensures that the process terminates. Uniqueness is also quite straightforward when we observe the following characterisation of an irreducible component: it is an irreducible closed subset of $X$ which is not contained in a (strictly) bigger irreducible closed subset. I will skip the details. □

**1.2.3 Example (Hypersurfaces)** Let $X = V(f) \subset \mathbb{A}^n$. Then the unique decomposition of $X$ into irreducible components corresponds precisely to the unique factorisation of $f$ into prime factors.

### 1.3 Rational maps and rational functions

Recall our notation $f\colon X \dashrightarrow Y$ for a map $f$ defined only on an open subset. This notion becomes really useful when $X$ is irreducible. Let $X$ be irreducible and $Y$ separated. A *rational map* from $X$ to $Y$, denoted by $f\colon X \dashrightarrow Y$ is a map from an open subset of $X$ to $Y$. More precisely, it is a pair $(U, f)$ where $U \subset X$ is a (non-empty) open and $f\colon U \to Y$ is a regular map. Two pairs $(U, f)$ and $(V, g)$ are considered equivalent if $f$ and $g$ are equal on $U \cap V$.

> Show that this is an equivalence relation. — (4)
> You will have to use that $Y$ is separated.

We say that a rational map $X \dashrightarrow Y$ is *defined* (or *regular*) at $x$ if there exists a representative $(U, f)$ such that $U$ contains $x$. The subset of $X$ where a rational map is defined is an open subset, called the *domain of definition* of the rational map.

Suppose we have rational maps $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow X$, we have to be a bit careful while composing them. After all, it could happen that $g$ is not defined at any point in the image of $f$! But if the domain of $g$ contains a point in the image of $f$, then the composition makes sense and defines a rational map $g \circ f\colon X \dashrightarrow Z$.

> Define the composition precisely. Produce an example where the composition is not defined. — (5)

We say that a rational map $f\colon X \dashrightarrow Y$ is a *birational isomorphism* (or *birational*) if there exists $g\colon Y \dashrightarrow X$ such that $g \circ f$ and $f \circ g$ are defined and equivalent to the identity on $X$ and $Y$ respectively. We say that two varieties are *birational* if there exists a birational isomorphism between them. Classifying varieties up to birational isomorphism is a major open problem in algebraic geometry.

**1.3.1 Examples (birational isomorphisms)** In the following, all varieties are assumed to be irreducible and separated.

1. Any variety is birational to any of its open subsets.
2. The affine space $\mathbb{A}^n$, the projective space $\mathbb{P}^n$, any product $\mathbf{P}^a \times \mathbf{P}^b$ with $a + b = n$ (and any triple product etc.) are in the same birational isomorphism class.
3. The group of biregular automorphisms of $\mathbf{P}^n$ turns out to be quite easy to understand—it is just $\mathrm{PGL}_n$—but the group of birational automorphisms is huge and very poorly understood (except when $n = 1$, where it agrees with the biregular automorphisms group by one of the homework questions). Here is an example of a birational automorphism of $\mathbf{P}^2$, called a 'Cremona transformation':
$$\phi\colon [X : Y : Z] \mapsto [1/X : 1/Y : 1/Z].$$

**1.3.2 Definition (field of fractions)** Let $X$ be an irreducible variety. The set of rational maps $X \dashrightarrow \mathbb{A}^1 = k$ is naturally a ring. But in fact, it is actually a field, called the *fraction field* of $X$, and is denoted by $k(X)$.

If $X$ is affine, then we really do have
$$k(X) = \operatorname{frac} k[X].$$

> Prove this. — (6)

It is easy to check that a birational isomorphism $f\colon X \dashrightarrow Y$ induces an isomorphism of fields over $k$:
$$f^*\colon k(Y) \to k(X).$$
(and conversely).