Better title

GIT Theory and the action of SL_n^3 on $n \times n \times n$ cubal matrices

Benjamin John Leedom

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For someone or something or whatever

Declaration

The work in this thesis is my own except where otherwise stated.

Benjamin John Leedom

Acknowledgements

Abstract

This thesis provides an exposition for the basic building blocks of Geometric Invariant Theory. We develop the theory to explain and prove the Hilbert-Mumford Numerical Criterion for the stability of points. We then explain how this Criterion can be manipulated into a convex geometry problem, and discuss an implementation of this problem into code in the Sage Programming Language. Utilising this code, we then find unstable points in some elementary problems before turning to the problem of the action of SL_n^3 on $n \times n \times n$ cubic matrices, for which we provide a potential generic solution for all n.

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Notation and terminology

This section details some basic notation for the thesis:

Notation

k	Refers to an algebraically closed field where the characteristic is zero unless otherwise stated
$\mathcal{O}(X)$	refers to the ring of regular functions for an algebraic variety, although $R(X)$ may also be used explicitly when the variety is projective.
$O(n)$	Used in chapter three and refers to the computational complexity of a piece of code
SL_n	refers to the special linear group of size n, the field k being implicit.
$\mathbb{A}^n$	refers to Affine n-space over a field k which is again implicit
$\mathbb{P}^n$	refers to Projective n-space over k.
X	refers to an algebraic variety, unless otherwise stated.

Chapter 1

Introduction

1

2

If we consider the action of the field k on $\mathbb{A}^2$, we see that the orbit space (acquired by quotienting out by the equivalence relation $x \sim y$ if $y = \lambda x$ for some $\lambda \in k$) behaves oddly around 0. In fact, the point 0 is contained in the closure of the element representing the orbit of 1, for example. That is, the orbit space of a group action on an algebraic variety is often not an algebraic variety. However, if we consider the same action on $\mathbb{A}^2 \setminus 0$, we see that the orbit space is a variety, it is in fact $\mathbb{P}^1$. Here 0 is an "unstable point", whilst the remaining points of $\mathbb{A}^2$ are "stable". This concept is the primary differentiator between a categorical notion of the quotient of a variety X by G and the orbit space.

3

This dissonance between the orbit space and the category of algebraic varieties is the primary motivation for this thesis. We can first resolve this tension of not receiving an algebraic variety by defining the categorical version of a quotient, the GIT quotient. From this, we then ask how similar the GIT quotient is to the orbit space. Chapter Two is dedicated to exploring these ideas, strongly based in Hoskins work in [5], and utilising the theory of Lie algebras and affine algebraic groups from Humphreys in [6] to describe the affine and projective GIT quotients, the affine for affine varieties, and the projective working more generally. With the help of some basic concepts in category theory, we then justify why they are a better definition than the orbit space, following which we build up and discuss the concept of the stability of points. This idea of stability allows us to understand the level of similarity between the GIT quotients and the orbit space. The chapter finishes by using the theory we have extablished, along with some results from linear algebra due to Nagata and Cohen and algebraic completions of curves 4

① Start more gently and broadly.

Introduce objects one at a time.

For example:

" Let X be a geometric object, for example a top. space, a smooth manifold, or an algebraic variety. Let G be a group with an action on X. Let X/G be the set of orbits of the action. We would like to endow X/G with a similar geometric structure as X, for example, make it a topological space, a manifold, or an alg. variety." - - - - -

② Accuracy: k does not act

$k^\times$ does!

③ Readers cannot understand what you mean in this sentence.

④ A categorical quotient is not the same as a GIT quotient. (GIT $\Rightarrow$ categorical but not iff) to prove the Hilbert Mumford Numerical Criterion, a strong theorem that allows for algorithmic identification of point stability.

In chapter three, we will use this criterion along with some convex geometry to create a program which we can use to identify which points are unstable in our varieties. We then justify the use of this particular algorithm over a more naive approach within the context of our main example in chapter four. The chapter is rounded out with the use of this program on some basic examples of unstable point identification relating to homogeneous polynomials.

In Chapter four, we move onto our main example - the action of SL_n on cubal $n \times n \times n$ matrices. We use the program, and work done by Bhargava and Ho [1] [2] to explore some basis-free descriptions of the unstable points before providing a conjecture for integer generality, of which one direction is proven (points that share this property are unstable).

This thesis should be accessible to anyone with basic understanding of Lie algebras, affine algebraic groups, algebraic varieties, representation theory. The discussion of the complexity of the program in Chapter three may require some basic computer science knowledge.

5 This is on the whole good.
Perhaps a little more polish, but
we can do that later.

Chapter 2

GIT Quotients and Stable Points

2.1 Algebraic Groups, Lie Algebras and Reductivity

2.1.1 Algebraic Groups

Before we discuss some actual GIT theory such as the GIT quotient or stable points, we first need to build up a strong background in some algebraic group theory, as well as a brief discussion of Lie Algebras - since they will be useful in proving that in characteristic zero the main group used in this thesis, SL_n, is linearly reductive.

6 **Definition 2.1.** A group G is an algebraic group if it is also an algebraic variety

7 **Examples 2.2.** A very simple example is the complex numbers. More importantly for our exploration into GIT, various matrix groups are also algebraic - $n \times n$ matrices over a field k are isomorphic to $\mathbb{A}^{n^2}$ and are therefore a variety. Further, GL_n is the $n \times n$ matrices with a zariski open condition, $\det(M) \neq 0$, and so is also a variety and therefore an Algebraic Group. Lastly, and the main object of study for us, SL_n is clearly a variety, as it is GL_n modulo the zariski closed condition that the determinant of the matrix is 1.

8 It is useful to think of GL_n as a sort of fundamental example here for two reasons. The first is that for any affine algebraic group (group is an affine variety), G is isomorphic to a closed subgroup of some GL_n as seen in [6]. As previously discussed SL_n is an obvious case of this, and it turns out that some key facts we will need to prove the linear reductivity of SL_n using Lie Algebras are derived directly from observations made about GL_n.

(6) Not precise!

(7) All excellent examples.

But separate them, at least in different paragraphs, if not also as separate environments.

Write them clearly.

- What is the set
- How is this a variety?
- What is the group operation?
- How is this a map of varieties?

Also give "trivial" examples (optional) e.g. finite groups.

Also give non-reductive examples.

(8) Excellent point!

But again - take your time to develop the ideas.

Ex. "For us, it will be useful to think of GLn as a fundamental example. In this thesis, we exclusively study affine algebraic groups. These are algebraic groups that are affine alg. varieties. By a standard theorem ----"

2.1.2 Lie Algebras

9

The purpose of this diversion into Lie Algebras is to prove that SL_n has no nontrivial normal subgroups which, as we will see later, is the main fact we will use to prove SL_n is linearly reductive. To expose this structure, we will look at its Lie Algebra, which we will soon prove are $n \times n$ matrices of trace zero. First, we need to understand what a Lie Algebra is, and in particular, how to think about the Lie Algebra of a specific group. Humphreys fantastic book [6] is a fantastic reference and will be adhered to closely in this discussion.

Definition 2.3. A *Lie Algebra* over a field k is a k-algebra subspace which is closed under the *lie bracket*:

10

$$[x, y] = xy - yx$$

This is an important tool to tell whether or not something is a Lie Algebra, but it doesn't tell us what the Lie Algebra of a specific group is - and to do that we need to first understand what a derivation is.

12

Definition 2.4. For E a field and L an extension, a *derivation* δ is a map from E to L which satifies:

13

- $\delta(x + y) = \delta(x) + \delta(y)$
- $\delta(xy) = x\delta(y) + \delta(x)y$

Furthermore, if we have a subfield F of E andd $\delta(x) = 0$ for all $x \in F$, δ is an *F-derivation*

Now, to define the Lie Algebra of an algebraic group G, we need to note two more important facts. The first is that we can see that we have a left action λ_g of G on the ring of regular functions $\mathcal{O}(G)$: $(g \cdot f)(y) = f(g^{-1}y)$. Respectively, we have a right action ρ_g: $(f \cdot g)(y) = f(yg)$. It is also not to hard to note that the bracket (lie bracket) of two derivations of $\mathcal{O}(G)$ (these are k-derivations) is again a derivation, and so the derivations of $\mathcal{O}(G)$ form a Lie Algebra. Thus, we can define the Lie Algebra of G:

14

Definition 2.5. The *Lie Algebra* of an algebraic group G is the subspace of Der A that are left invariant. That is, the algebra is the set:

$$\{\delta \in \text{Der } A \mid \delta\lambda_g = \lambda_g\delta \text{ for all } g \in G\}$$

(9) Lie algebras are useful more broadly than this. So it is good to advertise the better than for this specific purpose.

(10) This definition does not make sense.

Give examples after the definition.

(12) You are making an assertion here –
" To every alg. group, one can associate a Lie algebra. "

But this is left implicit – don't do this. If you do it, then only the readers who already know the material will be able to follow.

(13) You define a derivation for a field and use it on $\mathcal{O}(G)$ $\leftarrow$ Not a field.

(14) What does this mean?

This however, is not the most useful way to think about the Lie Algebra. 15
Most often it is much more convenient to identify it with a tangent space:

Definition 2.6. For an affine variety X defined by polynomials $f_i(t_1, \cdots, t_n)$, the tangent space $\mathrm{Tan}_x(X)$ at a point $x = (x_1, \cdots, x_n) \in X$ is the affine variety is defined as the zero locus of the polynomials

$$\sum_{i=1}^{n} \frac{df_i}{dt_i}(x)(t_i - x_i)$$

Note that this definition views the tangent space as a vector space with the zero vector being identified with e. If we want to view this tangent space more geometrically, with zero as the zero vector, we simply take the shift map $t_i \to (t_i - x_i)$.

In general, the tangent space is defined via a local ring construction. While we 16
will use the above definition to compute the Lie Algebra of SL_n, to prove that
19 the traditional definition of the Lie Algebra is equivalent to the definition via the tangent space, we will need to explore this local ring construction somewhat.
Assume G is an affine algebraic group and let $R = \mathcal{O}(G)$, M be the maximal 17
ideal in R that vanishes at x. We can identify R/M with K, and so we can view
M/M^2 as a vector space over K, since it is an R/M-module. Since we can think k?
18 of $d_x f$ (for any functions in $\mathcal{O}(\mathbb{A}^n)$) as a linear function on $\mathbb{A}^n$, and $\mathrm{Tan}(X)_x$ is a vector subspace of $\mathbb{A}^n$, we can also think of $d_x f$ as a linear function on the tangent space. Now since, by definition, if f is in M, it vanishes on the tangent space, and so $d_x f$ is determined by where f gets sent to in the quotient of $R = \mathcal{O}(\mathbb{A}^n)/I(G)$. Thus, d_x is a k-linear map from R to the dual space of the tangent space at x. Next, since as a vector space we can write $R = k + M$, and d_x of a constant function is zero, we can think of d_x as a map from M to the dual space of the tangent space. It is then not hard to show that the kernel of d_x is M^2. However, this identification of the Tangent space relies too heavily on the embedding of G into some $\mathbb{A}^n$, and we want this kind of construction to work for algebraic groups in general, so we pass to the local ring, and define the tangent space as:

20 **Definition 2.7.** For an arbitrary algebraic group G, we can define the *tangent space of G at some x* to be the dual vector space $(m_x/m_x^2)^*$ over $k = \mathcal{O}_x/m_x$ where $m_x = M\mathcal{O}_x$.

From here, we can move toward the more useful (at least for our purposes) way to view the tangent space, as the set of point derivations $\delta : \mathcal{O}_x \to k$

why do you need an alg. gp?

(15) Then why did you introduce it?

If there are multiple ways of thinking about an object, say so at the beginning, and write some commentary on why you are doing things the way you are doing.

(16) Again, the same comment as before.

(17) Grammar

(18) What is $d_x f$?

(19) What's the goal of this paragraph?

<u>General comment</u> - Provide plenty of "guide posts" to the reader.

" Our next goal is to define ----
There are two definitions, which turn out to be equivalent ----
Having seen the two definitions, let us show they are equivalent ----

Now that we have defined the vector space underlying the Lie algebra, let us define the Lie bracket"

At every point the reader must know where they are in the logical flow.

21 **Definition 2.8.** A *point derivation* $\delta : \mathcal{O}_x \to k$ is a k-linear map satisfying the following equation:

$$\delta(fg) = \delta(f) \cdot g(x) + f(x) \cdot \delta(g)$$

Proposition 2.9. *The set of point derivations at x are naturally isomorphic to the tangent space at x.*

Proof. It is clear that the set of point derivations form a vector space over k. Now, if f is constant or in m_x^2 it is zero by the above equation (let g be the identity map in the dual space), and so δ is determined by what it does to maps in m_x/m_x^2, and so we have an injection from the point derivations into the tangent space. In the other direction, we can take a map $m_x/m_x^2 \to k$ to a map from $m_x \to k$ by composing with the projection map $m_x \to m_x/m_x^2$, but we can then extend this map from $m_x \to K$ to a map from $\mathcal{O}_x$ to K by sending constants to zero. It then remains to show that this map satisfies the equation found in the definition of the point derivation.

Not reading the proof

For $f \in \mathcal{O}_x$, we can write $f = c + g$ where c is a constant function and $g(x) = 0$. Note that we extended $\delta : m_x \to K$ to $\mathcal{O}_x$ by setting $\delta(\text{const}) = 0$, ,and since δ is k-linear, ew hhave that

$$\begin{aligned} \delta(f) &= \delta(c + g) \\ &= \delta(c) + \delta(g) \\ &= \delta(g) \end{aligned}$$

Now:

$$\begin{aligned} \delta(f_1 f_2) &= \delta((c_1 + g_1) \cdot (c_2 + g_2)) \\ &= \delta(c_1 c_2) + \delta(c_1 g_2) + \delta(c_2 g_1) + \delta(g_1 g_2) \\ &= c_1 \delta(g_2) + c_2 \delta(g_1) + \delta(g_1 g_2) \end{aligned}$$

However, $g_1, g_2 \in m_x$ so $g_1 g_2 \in m_x^2$, and so

$$\begin{aligned} \delta(f_1 f_2) &= c_1 \delta(g_2) + c_2 \delta(g_1) \\ &= \delta(f_2) f_1(x) + \delta(f_1) f_2(x) \end{aligned}$$

□

21 In the same theme as 19, motivate this definition, or at least explain where it fits in the logical flow.

With this information in hand, we will prove the following theorem:

22 **Theorem 2.10.** *The Lie Algebra $\mathfrak{g}$ of G is isomorphic to the tangent space $Tan_e(G)$*

Proof. To prove this theorem, we construct the map sending a tangent vector x to a derivation $*$x which we define in the following way:

$$(f * \mathrm{x})(x) = \mathrm{x}(\lambda_{x^{-1}} f)$$

We first check that it is a derivation

$$\begin{aligned}(fg * \mathrm{x})(x) &= \mathrm{x}(\lambda_{x^{-1}}(fg)) \\ &= \mathrm{x}((\lambda_{x^{-1}} f)(\lambda_{x^{-1}} g)) \text{ but x is a point derivation} \\ &= \mathrm{x}(\lambda_{x^{-1}} f)g(x) + f(x)\mathrm{x}(\lambda_{x^{-1}} g) \\ &= ((f * \mathrm{x})g + f(g * \mathrm{x}))(x)\end{aligned}$$

We then need to make sure it is left invariant

$$\begin{aligned}\lambda_y(f * \mathrm{x})(x) &= (f * \mathrm{x})(y^{-1}x) \\ &= \mathrm{x}(\lambda_{x^{-1}y} f) \\ &= \mathrm{x}(\lambda_{x^{-1}}(\lambda_y f)) \\ &= (\lambda_y f) * \mathrm{x}(x)\end{aligned}$$

Thus, we have a well defined map in one direction. We now need an inverse map. We again turn to our definition of the tangent space in terms of point derivations. Since as previously discussed, the derivations are determined by their effect on $\mathcal{O}(G)$, we can define our inverse map θ as

$$(\theta\delta)(f) = (\delta f)(e)$$

for δ a derivation and f a function in $\mathcal{O}(G)$. We can then check that the composites are the identity

$$\begin{aligned}f * \theta(\delta)(x) &= \theta(\delta)(\lambda_{x^{-1}} f) \\ &= (\delta\lambda_{x^{-1}} f)(e) \text{ but we know the action commutes with derivations so} \\ &= \lambda_{x^{-1}}(\delta f)(e) \\ &= (\delta f)(x)\end{aligned}$$

(22) As a vector space?

Why is this a Theorem rather than a proposition?

Then we just need to show the other composition also gives the identity:

$$\begin{aligned}\theta(*\mathrm{x})(f) &= f * \mathrm{x}(e)\\ &= \mathrm{x}(\lambda_{e^{-1}} f)\\ &= \mathrm{x}(f)\end{aligned}$$

□

We must also note that differentiation also operates on functions in the following way:

22

Theorem 2.11. *If $\phi : G \to G'$ is a morphism of algebraic groups, $d\phi_e : \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of Lie Algebras*

Proof. Let $\mathrm{x}' = d_\phi(\mathrm{x})$, $\mathrm{y}' = d_\phi(\mathrm{y})$, $f = \phi^* f'$. Now

$$\begin{aligned}[\mathrm{x}', \mathrm{y}'](f') &= (\mathrm{x}'\mathrm{y}' - \mathrm{y}'\mathrm{x}')(f')\\ &= (f' * \mathrm{y}' * \mathrm{x}')(e) - (f' * \mathrm{x}' * \mathrm{y}')(e)\\ &= \mathrm{x}'(f' * \mathrm{y}') - \mathrm{y}'(f' * \mathrm{x}')\\ &= \mathrm{x}(\phi^*(f' * \mathrm{y}')) - \mathrm{y}(\phi^*(f' * \mathrm{x}'))\end{aligned}$$

In a similar way we have that

$$d_\phi[x, y] = \mathrm{x}(f * \mathrm{y}) - \mathrm{y}(f * \mathrm{x})$$

and so all it remains to show is that $\phi^*(f' * \mathrm{x}') = (f * \mathrm{x})$. That is, that $(\phi^* f') * \mathrm{x} = \phi^*(f' * d_\phi \mathrm{x})$. Let's evaluate each side at $x \in G$.

$$\begin{aligned}(\phi^* f') * \mathrm{x}(x) &= x(\lambda_{x^{-1}} \phi^* f')\\ \phi^*(f' * d_\phi \mathrm{x})(x) &= (f' * d_\phi \mathrm{x})\phi(x)\\ &= d_\phi \mathrm{x}(\lambda_{\phi(x)^{-1}} f')\\ &= \mathrm{x}(\phi^*(\lambda_{\phi(x)^{-1}} f'))\end{aligned}$$

So we need to show that

$$\lambda_{x^{-1}} \phi^* f' = \phi^*(\lambda_{\phi(x)^{-1}} f') \qquad (\star)$$

(22) What is a "homomorphism of Lie algebras" ?

We evaluate at some $y \in G$.

$$\begin{aligned}(\lambda_{x^{-1}}\phi^* f')(y) &= \phi^* f'(xy) \\ &= f'\phi(xy)\end{aligned}$$

We now evaluate the right hand side of $\star$

$$\begin{aligned}\phi^*(\lambda_{\phi(x)^{-1}} f')(y) &= (\lambda_{\phi(x)^{-1}} f')\phi(y) \\ &= f'(\phi(x)\phi(y)) \\ &= f'(\phi(xy))\end{aligned}$$

as required. □

Now that we have a more illuminating fundamental structure defined by the concept of differentiation, we find an important representation that relates G and $\mathfrak{g}$, the Adjoint Representation Ad.

23 **Definition 2.12.** The *Adjoint Representation* Ad g is defined to be the differential of the *Inner Automorphism* Int $g(h) = ghg^{-1}$. THat is, if we think of the Lie ALgebra as the space of left derivations, Ad $\delta = \rho_x \delta \rho_{x^{-1}}$

Before we prove some basic properties of the Adjoint representation, we need to know what the Lie Algebra of the general linear group is:

24 **Theorem 2.13.** *The Lie Algebra $\mathfrak{gl}_n$ of GL_n is the set of $n \times n$ matrices*

Proof. Since the ring of regular functions of an affine open variety (embedded in $\mathbb{A}^{n^2}$, say) is just the ring of regular functions on $\mathbb{A}^{n^2}$, we see that the tangent space at e of GL_n has canonical basis $\delta/\delta T_{i,j}$, evaluated at e, and so we can write any tangent vector x as a set of numbers defined by what it does to each of the $T_{i,j}$. That is, x is defined by the $x_{i,j} = \mathrm{x}(T_{i,j})$ where we arrange the $x_{i,j}$ in a square matrix. One can check that the multiplication of these tangent vector matches the matrix multiplication and so we can identify the tangent vectors as a subset of $M_n(k)$. This map is injective, since x gets sent to zero if it kills all the $T_{i,j}$. That is, it is the zero matrix. It is surjective since the dimensions on either side are the same, and so we can indeed identify the Lie Algebra of GL_n with $M_n(k)$. □

(23) <u>Clarity</u>: Declare the "type."

what is Ad? It is a map $\mathrm{Ad}: G \to Gl(\mathfrak{g})$.

Too many things happening in the definition.
What is Int?

(24) And what is the Lie bracket?

Now we can prove some basic properties about the Adjoint Representation in relation to GL_n. These will be useful since, as previously discussed, any affine algebraic group is a closed subgroup of GL_n and will therefore inherit any properties we can prove about GL_n (we will discuss this more rigorously below). In particular, they will inherit the structure on the differential of Ad, which will be paramount in comparing structure between a group and its Lie Algebra. Denote the coordinate functions on GL_n by $T_{i,j}$ and let T be the matrix whose (i, j) entry is $T_{i,j}$.

Lemma 2.14. *For $x \in GL_n$ $x \in \mathfrak{gl}_n$, $\rho_x T_{ij}$ is the (i, j) entry of Tx. Further, $(T_{i,j} * x)$ is the i, j entry of Tx.*

Proof. We want to show that $\rho_x T_{i,j}(y) = (Tx)_{i,j}(y)$. From evaluating the group action we have that $\rho_x T_{i,j}(y) = T_{i,j}(xy)$, which we can think of as just the i, j entry of the matrix xy. This is something we know from matrix multiplication laws - it is $\sum_k y_{i,k} x_{k,j}$ but the i, k entry of y is the coordinate matrix $T_{i,k}$ evaluated on y, so we actually have $\sum_k T_{i,k}(y) x_{k,j}$, which if we roll back our matrix multiplication we have $(Tx)_{i,j}(y)$. In a similar vein we can prove the result for the convolution when applied to some arbitrary y:

$$\begin{aligned}
(T_{i,j} * \mathrm{x})(y) &= \mathrm{x}(\lambda_{y^{-1}} T_{i,j}) \\
&= \mathrm{x}(\sum_k y_{i,k} T_{k,j}) \\
&= \mathrm{x}(\sum_k T_{i,k}(y) T_{k,j}) \\
&= (T\mathrm{x})_{i,j}(y)
\end{aligned}$$

□

Lemma 2.15. *Let $x \in GL_n$, $x \in \mathfrak{gl}_n$. Then $Ad\, x(x) = xxx^{-1}$.*

Is the $x \in \mathfrak{gl}_n$ a y?

Proof. Since the coordinate functions act as a basis for $\mathfrak{gl}_n$, it suffices to show the result when applied to one of the $T_{i,j}$. Within this proof we will apply both sides of the previous lemma, as well as the same splitting through matrix multiplication step used in the previous proof.

$$\begin{aligned}
\text{Ad } x(\mathrm{x})(T_{i,j}) &= \rho_x(*\mathrm{x})\rho_{x^{-1}}(T_{i,j}) \\
&= \rho_x(*\mathrm{x})(Tx^{-1})_{i,j}) \\
&= \rho_x(*\mathrm{x})\sum_k T_{i,k}x^{-1}_{k,j} \\
&= \rho_x\sum_k (T\mathrm{x})_{i,k}x^{-1}_{k,j} \\
&= \rho_x\sum_k\sum_l T_{i,l}\mathrm{x}_{l,k}x^{-1}_{k,j} \\
&= \sum_k\sum_l\sum_m T_{i,m}x_{m,l}\mathrm{x}_{l,k}x^{-1}_{k,j} \\
&= (Tx\mathrm{x}x^{-1})_{i,j} \\
&= x\mathrm{x}x^{-1}(T_{i,j})
\end{aligned}$$

□

We can then conclude that Ad : $GL_n \to GL_{n^2}$ is a morphism of algebraic groups and since restriction commutes with differentiation. Then, based on this, and our isomorphism between G and a closed subgroup of GL_n, we can show

Proposition 2.16. *$Ad : G \to GL(\mathfrak{g})$ is a morphism of algebraic groups. When G is a closed subgroup of GL_n, $Ad\ x$ is conjugation by x for $x \in G$.*

Proof. Firstly, by the proposition that all affine algebraic groups are contained in GL_n we have that $G \subseteq GL_n$, and so we have that Int_G is the restriction of Int_{GL_n} and that $\mathfrak{g} \subseteq \mathfrak{gl}_n$. Then, since restriction commutes with differentiation, we have that Ad_G is the restriction of Ad_{GL_n}, and so by the above conclusion about Ad : $GL_n \to Gl_{n^2}$, taking the restriction gives us the morphism of algebraic groups we require. □

The last piece of Lie Algebra theory we need before a technical proof about the Lie Algebra of SL_n is some way to relate structure between the group and its Lie Algebra. We will do this by looking at the differential of Ad. To do so we must first compute the differentials of some more simple maps:
We first look at the multiplication map $\mu : G\times G \to G$. It is not hard to show that the tangent space of $G\times G$ at (e,e): $\mathcal{T}(G\times G)_{(e,e)}$ is isomorphic to $\mathcal{T}(G)_e\oplus\mathcal{T}(G)_e$ (take the direct sum of the projection maps), and so we can therefore think about the differential of this multiplication map as $d\mu_{(e,e)}(\mathrm{x},\mathrm{y})$

Proposition 2.17. $d\mu_{(e,e)}(x, y) = x + y$

Proof. For $f \in \mathcal{O}_G$, $\mu^*(f) = \sum f_i \otimes g_i$, and so we have $f(xy) = \sum f_i(x)g_i(y)$, and in particular $f = \sum f_i(e)g_i = \sum f_i g_i(e)$ Now

$$\begin{aligned} d\mu_{(e,e)}(\mathrm{x}, \mathrm{y})(f) &= (\mathrm{x}, \mathrm{y})(\mu^* f) \\ &= (\mathrm{x}, \mathrm{y})(\sum f_i \otimes g_i) \\ &= \sum \mathrm{x}(f_i)g_i(e) + \sum f_i(e)\mathrm{y}(g_i) \end{aligned}$$

where we get the last line since $(\mathrm{x}, \mathrm{y}) = (\mathrm{x}, e)+(e, \mathrm{y})$. Now, considering $(\mathrm{x}+\mathrm{y})(f)$:

$$\begin{aligned} (\mathrm{x} + \mathrm{y})(f) &= \mathrm{x}(f) + \mathrm{y}(f) \\ &= \mathrm{x}(\sum f_i g_i(e)) + \mathrm{y}(\sum g_i f_i(e)) \\ &= \sum \mathrm{x}(f_i)g_i(e) + \sum f_i(e)\mathrm{y}(g_i) \end{aligned}$$

□

We now consider the differential of the inverse map: ι.

Proposition 2.18. $d\iota(x) = -x$

Proof. We know that $d(\mathrm{id}) = \mathrm{id}$ and that $d(\mathrm{id}, \iota) = (d(\mathrm{id}), d(\iota))$. Further, we also know that $0 = d(\mu \circ (\mathrm{id}, \iota))(x)$ and so:

$$\begin{aligned} d(\mu \circ (\mathrm{id}, \iota))(\mathrm{x}) &= d(\mu d(\mathrm{id}, \iota)(\mathrm{x})) \\ &= d\mu(\mathrm{x}, d(\iota)(\mathrm{x})) \\ &= \mathrm{x} + d\iota(\mathrm{x}) \end{aligned}$$

so $d\iota\mathrm{x} = -\mathrm{x}$. □

This is helpful information, but to find the differential of the adjoint map, we consider yet another formulation of the tangent space - this time in relation to the dual numbers:

Definition 2.19. The *Dual Numbers* are the algebra $k[t]/t^2$ but will be referred to notationally as $k[\epsilon]$. That is, elements are $a+b\epsilon$, such that $a, b \in k$, and $\epsilon^2 = 0$.

The key idea here is that we can identify the tangent space at x with the set of $k[\epsilon]$-valued points that satisfy some reasonable conditions. To understand what this means, let X be affine, with $R = \mathcal{O}(X)$. Now, a point of X, say $x = (c_1, \cdots, c_n)$ is an element such that for the f_i that generate the ideal I whereby $\mathcal{O}(X) = k[t_1, \cdots, t_n]/I$, $f_i(x) = 0$. Thus, from x we can derive a map $\alpha_x : R \to k$, where $\alpha_x(t_i) = c_i$. We call x a k-valued point. With this information, we see another way to identify the tangent space as a subset of the $k[\epsilon]$-valued points. In particular, the $k[\epsilon]$-valued points that lift the map $\alpha_x : R \to k$. That is, k-algebra homomorphisms t such that:

$$\begin{tikzcd} R \arrow[r, "t"] \arrow[dr, "x \mapsto c"'] & k[\epsilon] \arrow[d, "\epsilon \mapsto 0"] \\ & k \end{tikzcd}$$

commutes.

Proposition 2.20. *The tangent space at a point x are in bijection with the $k[\epsilon]$-valued points that lift α_x*

Proof. We know that the tangent space at a point X is the vector space $\mathrm{Hom}(m/m^2, k)$. Take a homomorphism in this set, say, h. Now, we need a k-algebra homomorphism from $R \to k[\epsilon]$ that is a lift of α_x. As a k-vector space, we can write our ring of regular functions R as $m \oplus k$. In particular, for any $r \in R$, we have $(r, -r(x), r(x))$. Note $r - r(x) \in m$, since $(r - r(x))(x) = 0$. Thus, given h, we take the homomorphism that sends r to $r(x) + h(r - r(x))\epsilon$, where we understand h here to really be h composed with the projection from $m \to m/m^2$. This is clearly a lift of α_x, since if we send ϵ to 0, we get $r(x) = \alpha_x(r)$. It remains to show this is a k-algebra homomorphism however. Let $r, s \in R$. Then:

$$\begin{aligned} r + s \mapsto & (r+s)(x) + h(r + s - (r+s)(x))\epsilon \\ &= r(x) + s(x) + h(r + s - r(x) - s(x))\epsilon \\ &= r(x) + h(r - r(x))\epsilon + s(x) + h(s - s(x))\epsilon \end{aligned}$$

$$\begin{aligned}
r \cdot s \mapsto & (r(x) + h(r - r(x))\epsilon) \cdot (s(x) + h(s - s(x))\epsilon) \\
&= r(x)s(x) + r(x)h(s - s(x))\epsilon + s(x)h(r - r(x))\epsilon \\
&= rs(x) + h(r(x)s - r(x)s(x))\epsilon + h(s(x)r - r(x)s(x))\epsilon \text{ since h is k-linear} \\
&= rs(x) + h(r(x)s + s(x)r - 2rs(x))\epsilon
\end{aligned}$$

We want $h(r(x)s + s(x)r - 2rs(x)) = h(rs - rs(x))$, so subtracting them should give us zero.

$$\begin{aligned}
h(r(x)s + s(x)r - 2rs(x) - rs + rs(x)) &= h(r(x)s + s(x)r - rs - rs(x)) \\
&= h((r - r(x)) \cdot (s - s(x))) \\
&= h(0)
\end{aligned}$$

since $(r - r(x)) \cdot (s - s(x)) \in m^2$. Thus, we have a k-algebra homomorphism for h that lifts α_x.
For the reverse direction, let $f : R \to k[\epsilon]$ be a k-algebra homomorphism. For some $c \in m/m^2$, choose a lift of c, say $\tilde{c} \in m$. Now, $m \subset R$, so we have $f(\tilde{(c)}) = \alpha_x(c) + h(\tilde{c})\epsilon$. This h will be our homomorphism from m/m^2 to k. To ensure this however, there are two things we need to check:

1. h is zero on elements in m^2

2. This process is independent of our choice of lift.

For the first, let $c' = cd \in m^2$, $c, d \in m$. Now, $f(c') = f(c)f(d) = h(c)\epsilon h(d)\epsilon = h(c')\epsilon^2 = 0$. However, $h(c')\epsilon = f(c')$, so $h(c') = 0$.
For the second, we note that if we have two separate lifts of c, say $\tilde{c}$ and $\tilde{c}'$, that since both map to c in the quotient m/m^2, their difference $\tilde{c} - \tilde{c'}$ must be contained in m^2. Then:

$$\begin{aligned}
f(\tilde{c} - \tilde{c}') &= h(\tilde{c} - \tilde{c}')\epsilon \\
&= 0
\end{aligned}$$

Checking that each compose with each other to give the identity is clear. $\square$

However, the above proof only demonstrates there is a bijection as sets. Ideally, for us to be able to think of this dual numbers construction as a tangent space, it should also have the structure of a vector space. Since k-algebra homomorphisms are k-linear, it simply remains to check that we have an additive structure:

Proposition 2.21. *The numbers identification of the tangent space is a vector space*

Proof. Let $f_1, f_2 : R \to k[\epsilon]/\epsilon^2$ be lifts of α_x derived from h_1, h_2. Within this context, define $f_1 + f_2$ to be

$$(f_1 + f_2)(r) = r(x) + h_1(r - r(x))\epsilon + h_2(r - r(x))\epsilon$$

We now check that this is a k-algebra homomorphism

$$\begin{aligned}(f_1 + f_2)(r + s) &= (r + s)(x) + h_1(r + s - (r + s)(x))\epsilon + h_2(r + s - (r + s)(x))\epsilon \\ &= r(x) + h_1(r - r(x))\epsilon + h_2(r - r(x))\epsilon + s(x) \\ &+ h_1(s - s(x))\epsilon + h_2(s - s(x))\epsilon \\ &= (f_1 + f_2)(r) + (f_1 + f_2))(s)\end{aligned}$$

$$\begin{aligned}(f_1 + f_2)(r) \cdot (f_1 + f_2)(s) &= (r(x) + h_1(r - r(x))\epsilon + h_2(r - r(x))\epsilon) \\ &\cdot (s(x) + h_1(s - s(x))\epsilon + h_2(s - s(x))\epsilon) \\ &= rs(x) + r(x)h_1(s - s(x))\epsilon + r(x)h_2(s - s(x))\epsilon \\ &+ s(x)h_1(r - r(x))\epsilon + s(x)h_2(r - r(x))\epsilon \\ &= rs(x) + h_1(r(x)s + s(x)r - 2rs(x))\epsilon \\ &+ h_2(r(x)s + s(x)r - 2rs(x))\epsilon\end{aligned}$$

But we proved in proving that each of these f_i was a k-algebra in the previous proposition that $h_i(r(x)s + s(x)r - 2rs(x)) = h_i(rs - rs(x))$, so we have

$$rs(x) + h_1(rs - rs(x))\epsilon + h_2(rs - rs(x))\epsilon = (f_1 + f_2)(rs)$$

□

In the case of the tangent space at the identity to GL_n, we actually have an even more concrete way to think about the lifts. We know that the tangent space at the identity for GL_n are homorphisms $\phi : R \to k[\epsilon]$ that lift the map sending

x to the identity. That is, the set of $k[\epsilon]$ points whereby they are sent to the identity matrix when ϵ is sent to 0. Thus, $T_e(GL_n)$ is the set of matrices in GL_n (b_{ij}) where $b_{ii} = 1 + a_{ii}\epsilon$ and $b_{ij} = a_{ij}\epsilon$. THat is, we can think of the matrix (b_{ij}) as $I + (a_{ij})\epsilon$.
Finally, we can prove that:

Theorem 2.22. *The differential of Ad is ad where ad $x(y) = [x, y]$*

Proof. As in the case of the computation of Ad, 2.16, we will restrict to $\mathfrak{gl}_n$ and GL_n. Ad maps GL_n to GL_{n^2}, so the differential of Ad, ad : $T_e(GL_n) \to T_e(GL_{n^2})$. That is, we send $x = \mathrm{id} + M\epsilon$ in the tangent space to the map sending $y \to xyx^{-1}$. We know from the addition law on the tangent space, along with the formula for the differential of the inverse map that this is the map that sends

$$\begin{aligned} y &\to (\mathrm{id} + M\epsilon)y(\mathrm{id} - M\epsilon) \\ &= y + My\epsilon - ym\epsilon \\ &= y + (My - yM)\epsilon \end{aligned}$$

But this is the map $\mathrm{id} + [M, -]\epsilon$ applied to y. Therefore, $d\mathrm{Ad} = \mathrm{ad}$ sends the tangent vector $\mathrm{id}+M\epsilon$ to the map $\mathrm{id}+[M, -]\epsilon$. That is, ad$x$ is the lie bracket. □

We will then use this to show some greater structure on $\mathfrak{h}$ corresponding to H in the case where H is normal. First we show what H corresponds to as a closed subgroup:

Lemma 2.23. *For H a closed subgroup of G, I the ideal of $\mathcal{O}(G)$ vanishing on H, $\mathfrak{h} = \{x \in \mathfrak{g} \mid I * x \subset I\}$*

Proof. Suppose $\mathrm{x} \in \mathfrak{h}$, $f \in I$, $x \in H$. Then $(f * \mathrm{x})(x) = \mathrm{x}(\lambda_{x^{-1}} f)$, but $\lambda_{x^{-1}} f$ still vanishes on H since $(xy) \in H$ if $y \in H$ so $\mathrm{x}(\lambda_{-1} f) = 0$, and so $I * \mathrm{x} \subset I$.
Now suppose $\mathrm{x} \in \mathfrak{g}$ such that $I * \mathrm{x} \subset I$. THat is, we have that for all $f \in I$,

$$\begin{aligned} (f * \mathrm{x})(e) &= \mathrm{x}(\lambda_{e^{-1}} f) \\ &= \mathrm{x}(f) \end{aligned}$$

That is, $\mathrm{x}f \in I$, so $\mathrm{x}(f)$ vanishes on H for all $f \in I$, so $\mathrm{x} \in \mathfrak{h}$ □

Then, when H is normal, $\mathfrak{h}$ becomes an ideal:

Lemma 2.24. *For H a closed normal subgroup of G, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. That is, for $x \in \mathfrak{g}$, $y \in \mathfrak{h}$, $[x, y] \in \mathfrak{h}$.*

Proof. If H is normal, it is invariant under Int, and so its corresponding lie subalgebra $\mathfrak{h}$ is invariant under Ad. In particular, if we have a basis for $\mathfrak{g}$ extended from a basis for $\mathfrak{h}$, we see that Ad will have the matrix form $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$, where the first block is the first m columns that make up the basis of $\mathfrak{h}$. Differentiating will not change this form, and so ad also fixes $\mathfrak{h}$, which precisely gives us that $\mathfrak{h}$ is an ideal, since ad $x(\mathfrak{h}) = [x, \mathfrak{h}] = \mathfrak{h}$. □

Thus, we have a correspondence between normal subgroups H of G and ideals $\mathfrak{h}$ in $\mathfrak{g}$. In particular, if we have no nontrivial ideals $\mathfrak{h}$ in $\mathfrak{g}$; that is, $\mathfrak{g}$ is *simple*, G has no nontrivial normal subgroups. It is this fact that we will use to prove that SL_n is linearly reductive. That is, we will prove the simplicity of the Lie Algebra of SL_n, the trace zero matrices:

Lemma 2.25. *The Lie Algebra $\mathfrak{sl}_n$ of SL_n are the $n \times n$ matrices of trace zero.*

Proof. We prove this using our first definition of the tangent space, and note that we identify the lie algebra with the vector space with the zero matrix in $\mathbb{A}^{n^2}$ as the zero vector. That is, following the computation of the tangent space, we apply the shift map $t_{i,j} \to t_{i,j} - e_{i,j} = t'_{i,j}$, and so we see that, since SL_n is defined by the function $f(x) = \det(x) - 1$, has is defined on the $t'_{i,j}$ with vanishing polynomial

$$\sum \frac{\delta f}{\delta t_{i,j}}(e) t'_{i,j}$$

Thus, all that remains is to compute the partial derivatives of the determinant at e. We can write the determinant using the cofactor expansion:

$$\det(T) = t_{1,1}\det(T_{1,1}) \pm t_{1,2}\det(T_{1,2}) \pm \cdots \pm t_{1,n}\det(T_{1,n})$$

where $T_{i,j}$ are the minors with the ith row and jth column of M omitted. We see that differentiating at $t_{1,1}$ gives determinant of $T_{1,1}$ which is the identitity matrix of size $n-1 \times n-1$, and so is 1. For the remainder of the $t_{1,i}$, since in the minors the first row of T is omitted but the first column is not, the first column is all zeroes, and as such the determinant of the minors are zero, and thus the partial derivative is also. For the remainder of the $t_{i,j}$ their partial differentials will have elements in each component of the sum except for the component $t_{1,j}\det(T_{1,j})$, and therefore will be written as some poly in the other $t_{i,j}$:

$$t_{1,1}g_{1,1} \pm \cdots \pm t_{1,j-1}g_{1,j-1}t_{1,j+1}g_{1,j+1} \pm \cdots \pm t_{1,n}g_{1,n}$$

However, at e all the $t_{1,j}$ where $j \neq 1$ are zero, so the partial derivative is fully contained in the summand $t_{1,1}\det(T_{1,1})$ and $t_{1,1} = 1$, so we may as well consider the partial derivative in the minor. However, the fact we're considering the minor which is the identity matrix one dimension lower allows us to apply the above reasoning inductively, and we therefore find that $\frac{\delta f}{\delta t_{i,j}} = 1$ if $i = j$ and 0 otherwise. Thus, we find that the tangent space is defined by the polynomial

$$\sum_{i=1}^{n} t'_{i,i} = 0$$

That is, the trace zero matrices. □

To prove that this algebra is simple, we will utilise a proof idea explored by Yung [8], and will first need to prove a short technical lemma about eigenspaces:

Lemma 2.26. *Suppose V is a finite dimensional vector space and $T : V \to V$ is a diagonalisable linear map. If Λ is the set of eigenvalues of T and V_λ is the eigenspace of T associated with an eigenvalue λ, and W is a T-invariant subspace, then*

$$W = \bigoplus_{\lambda \in \Lambda} (W \cap V_\lambda)$$

Proof. Since T is diagonalisable, we have an eigenbasis for V, and therefore we can write $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$. Thus, for any vector, and in particular vectors $w \in W$, we can write

$$w = \sum_{w_\lambda \in V_\lambda} w_\lambda$$

In particular, if we look at the λ_i such that $w_\lambda \neq 0$, we write

$$w = w_{\lambda_1} + \cdots + w_{\lambda_m}$$

. That is, we get that

$$\begin{aligned} w &= w_{\lambda_1} + \cdots + w_{\lambda_m} \\ Tw &= \lambda_1 w_{\lambda_1} + \cdots + \lambda_m w_{\lambda_m} \\ &\vdots \\ T^{m-1}w &= \lambda_1^{m-1} w_{\lambda_1} + \cdots + \lambda_m^{m-1} w_{\lambda_m} \end{aligned}$$

Taking $x = (w_{\lambda_1}, \cdots, w_{\lambda_m})$, $b = (w, Tw, \cdots, T^{m-1}w)$, we can write a coefficient matrix

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_m \\ \vdots & \ddots & \vdots \\ \lambda_1^{m-1} & \cdots & \lambda_m^{m-1} \end{bmatrix}$$

and so if A is invertible, we can then write the w_{λ_i} as a linear combination of the $T^j w$, but the $T^j w$ are in W since W is T-invariant, and so we would be done. Thus, all it remains to show is that the determinant of A is nonzero. However, A is a square vandermonde matrix, so we have that

$$\det(A) = \prod_{i<j} \lambda_j - \lambda_i$$

But, since T is diagonalisable, the eigevalues are distinct and so the determinant is nonzero. □

Finally, we can prove the simplicity of $\mathfrak{sl}_n$:

Theorem 2.27. *$\mathfrak{sl}_n$ is simple.*

Proof. Consider an ideal $J \subseteq \mathfrak{sl}_n$. Since $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus (k \cdot I)$, where I is the identity, since $k \cdot I$ is contained in teh center of $\mathfrak{gl}_n$, any ideal of $\mathfrak{sl}_n$ is also an ideal of $\mathfrak{gl}_n$. Now, consider the matrix $s = \sum_{k=1}^{n} 2^k E_{kk}$ where the E_{ij} are the standard basis for $n \times n$ matrices. We see that for any of the E_{ij}:

$$\text{ad } s(E_{ij}) = [s, E_{ij}] = (2^i - 2^j)E_{ij}$$

and hence ad s is a diagonal matrix in $GL(\mathfrak{gl}_n)$, and therefore (obviously) is diagonalizable. Now, since J is an ideal of $\mathfrak{gl}_n$, it must be ad$s$ invariant, and so we can apply the previous techincal lemma. That is, $J = \bigoplus_{\lambda \in \Lambda} V_\lambda \cap J$. In this instance, since the eigenvalues are 0 and $2^i - 2^j$ $i \neq j$, this means that

$$J = J \cap V_0 \bigoplus_{i \neq j} V_{2^i - 2^j}$$

Since V_0 is the diagonal matrices, we see that if $J \cap V_0 \neq \{0\}$, then J contains a diagonal matrix of trace zero. That is, a matrix $a = \sum_{i=1}^{n} a_i E_{ii}$ where $a_i \neq a_j$ for some i and j. But this means that $[a, E_{ij}] \in I$, but $b = [a, E_{ij}] = (a^i - a^j)E_{ij}$, and so for $a' = \frac{2}{a^i - a^j} E_{ii} + \frac{1}{a^i - a^j} E_{jj}$ $[a', b] = E_{ij}$, and so $E_{ij} \in I$.

Similarly, if $J \cap V_0 = \{0\}$, then there exist some $i \neq j$ such that $J \cap V_{2^i - 2^j}$ is nonempty - but the eigenspace $V_{2^i-2^j}$ is the span of some E_{ij}, and so $E_{ij} \in J$.

Now, since

$$[E_{jk}, E_{ij}] = -E_{ik}$$

for $k \neq i$ and

$$[E_{ki}, E_{ij}] = E_{kj}$$

for $k \neq j$, if $l \neq i$, then we have $E_{il} \in J$, and so by the second bracket for $k \neq l$, we have $E_{kl} \in J$. Similarly, if $k \neq j$, then we have that $E_{kj} \in J$ and so by the second bracket for $k \neq l$ we have $E_{kl} \in J$. This covers all basis matrices where $i \neq j$ except for E_{ji}. We can get E_{ji} by getting some E_{ik} $k \neq i, j$ in the following way

$$\begin{aligned} [E_{jk}, E_{ij}] &= -E_{ik} \\ [E_{ji}, E_{ik}] &= E_{jk} \\ [E_{ki}, E_{jk}] &= -E_{ji} \end{aligned}$$

where the last step is allowed since $i \neq j$. Thus, we have all E_{kl} for $k \neq l$. Thus, we also have

$$[E_{kl}, E_{lk}] = E_{kk} - E_{ll}$$

for all $k \neq l$, but this means we have all $\mathfrak{sl}_n$, and so $I = \mathfrak{sl}_n$, as required. $\square$

2.1.3 Reductivity

To understand reductivity, we first need to understand unipotency and semisimplicity. We begin by defining unipotency and semisimplicity for elements of GL_n.

Definition 2.28. An element $g \in GL_n$ is *unipotent* if all of its eigenvalues are 1. An element $g \in GL_n$ is *semisimple* if it is diagonalisable.

With respect to this definition, we can define unipotency and semisimplicity for a general element $g \in G$.

Definition 2.29. An element $g \in G$ is *unipotent* if there is a faithful linear representation $\rho : G \to GL_n$ such that $\rho(g)$ is unipotent. Similarly, an element $g \in G$ is *semisimple* if $\rho(g)$ is.

This leads us to one of the most famous theorems of linear algebra: Jordan Decomposition.

Theorem 2.30 (Jordan Decomposition). *For an affine algebraic group G over k, every element $g \in G$ has a unique decomposition into unipotent and semisimple elements. That is, there exist unique g_u unipotent and g_{ss} semisimple such that $g = g_u g_{ss} = g_{ss} g_u$. Furthermore, if ϕ is a morphism of affine algebraic groups, $\phi(g) = \phi(g_u)\phi(g_{ss})$.*

From here, we can define the unipotent radical, and reductivity:

Definition 2.31. The *unipotent radical* $R_u(G)$ is the unique closed maximal connected normal subgroup consisting of unipotent elements. A group G is *reductive* if this radical is trivial.

From here, we will define linear reductivity, and quote an important theorem due to many mathematicians that will finally allow us to demonstrate that our main group, SL_n, is linearly reductive

Definition 2.32. An algebraic group G is said to be *linearly reductive* if, every finite dimensional linear representation $\rho : G \to GL(V)$ decomposes as a sum of irreducibles.

Example 2.33. A basic example is $\mathbb{G}_m$. We can prove this by showing that $k[\mathbb{G}_m] = k[t, t^{-1}] \cong k[x, y]/(xy = 1)$ is semisimple. However, $k[t, t^{-1}]$ is a division ring, and so we can think about $k[t, t^{-1}]$ as $1x1$ dimensional matrices over itself. Then, the Artin-Wedderburn theorem gives that it is semisimple (in terms of representations) and therefore linearly reductive.

Example 2.34. A non-example of a linearly reductive group is the complex numbers under addition. To see this, consider the representation $a \to \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.

Whilst we're discussing reductivity, it will be helpful later to understand a related concept: geometric reductivity.

Definition 2.35. A group G is *geometrically reductive* for every finite dimensional linear representation $\rho : G \to GL(V)$ and every non-zero G-invariant point $v \in V$, there is a G-invariant non-constant homogeneous polynomial $F \in \mathcal{O}(V)$ such that $f(v) \neq 0$.

Examples 2.36. As we are about to see, every linearly reductive group is geometrically reductive. A further example is SL_n for a field of any characteristic. Thus we see that the converse is not true.

We now state one of the most important theorems surrounding reductivity. It is due to many people - Nagata, Mumford, Weyl and Haboush

Theorem 2.37. *1. Every linearly reductive group is reductive*

2. Every geometrically reductive group is reductive and vice versa

3. If the characteristic of the base field is 0*, reductive groups are linearly reductive.*

We will prove below that every linearly reductive group is geometrically reductive, and so will provide some indication as to why some of these statements might be true - but otherwise would require far more work and background than would be appropriate. It is however, useful for our purposes, since it allows us to prove that SL_n is linearly reductive in characteristic zero.

Theorem 2.38. *SL_n is linearly reductive in characteristic zero fields*

Proof. We have seen already (2.27) that in characteristic zero, the lie algebra of SL_n, $\mathfrak{sl}_n$ is simple. Further, we know that since normal subgroups correspond to ideals in the lie algebra, that since we have no ideals in the lie algebra, we have no normal subgroups. In particular therefore, we have no normal solvable subgroups, and so SL_n is reductive and therefore linearly reductive. $\square$

For the remainder of this section as well as the next, we will follow the work of Hoskins in [5] as well as a few other contributers I will mention where relevant. The proofs are based heavily upon her work although the understanding is my own.

Proposition 2.39. *Every linearly reductive group is geometrically reductive.*

Proof. We will prove the implication with a chain of implications pertaining to the following facts. That is, for the following:

1. G is linearly reductive

2. For any finite dimensional linear representation $\rho : G \to GL(V)$, any G-invariant subspace V' admits a G-stable complement.

3. For any surjection of finite dimensional G-representations $\phi : V \to W$, the induced map on G-invariants is surjective

4. For any finite dimensional linear representation $\rho : G \to GL(V)$ and every nonzero G-invariant point v, there is a G-invariant linear form $f : V \to k$ such that $f(v) = 0$

We will prove that $1 \implies 2$, $2 \implies 3$ and $3 \implies 4$. Furthermore, note that a group that has property 4 is geometrically reductive.
For $1 \implies 2$, since G is linearly reductive, and V' is G-invariant, both V' and V decompose as a direct sum of irreducibles. However, since $V' \hookrightarrow V$, $\bigoplus_{\text{irred}} V' \hookrightarrow \bigoplus_{\text{irred}} V$ and so letting $V'' = V/V'$ provides a G-stable complement.
For $2 \implies 3$, let $V' = \ker(f)$. Then $V'' \cong W$ and since both are G-invariant $V^G = V'^G \oplus V''^G$. That is, we have a surjective map from V^G to $V''^G \cong W^G$.
For $3 \implies 4$, pick a nonzero G-invariant vector v. This determines a G-invariant linear form $\phi : V^\vee \to k$. If we let G act trivially on k, ϕ is a surjection G-representations and by 3 then, we have a surjection $(V^\vee)^G \to k^G = k$. Thus, taking the preimage of 1, there will be an f in the preimage such that $f(v) = 1$. □

2.2 A Category Theoretic Approach

Given an algebraic variety X, a group G and an action of the group G on X, $\phi : G \times X \to X$, by $\phi(g, x) = g \cdot x$, one might reasonably want to consider some kind of quotient of X by G. What might initially spring to mind is the orbit space. That is, given the equivalence relation $x \sim y$ if $y = g \cdot x$ for some $g \in G$, identify points that are equal. However, for a full variety X, this orbit space may not actually be a variety such as the case of: the action of k on $\mathbb{A}^2$, since 0 is contained in the closure of any other point. Therefore, we need to look for some other machinery to properly define a quotient in the category of algebraic varieities. We will call such a quotient a categorical quotient. We can then ascribe further definitions to understand how well behaved such a quotient is. A "good quotient" has several nice properties that make it desirable whilst a "geometric quotient" tells us that, in fact, the categorical quotient has lined up with our orbit space.

We begin by constructing the categorical quotient. Thankfully, with category theory underlying the definition, we can make use of its machinery. In particular, we have the Yoneda Lemma for locally small categories, which we define as follows:

Definition 2.40. A *locally small category* is a category which for all pairs of objects A and B, $\text{Hom}(A, B)$ is a set.

For locally small categories, we have the *hom-functor* for each object A.

Definition 2.41. For an object A in a locally small category $\mathcal{C}$, the *hom-functor* from $\mathcal{C}$ to **Set** which maps each object X in $\mathcal{C}$ to $\text{Hom}(A, X)$ and each morphism $f : X \to Y$ to the function which maps $\text{Hom}(A, X)$ to $\text{Hom}(A, Y)$ by $g \mapsto f \circ g$.

With this, we can state the Yoneda Lemma:

Theorem 2.42. *For F a functor from $\mathcal{C}$ to **Set**, natural transformations from the hom-functor of A to F are in a one-to-one correspondence with the elements of $F(A)$.*

The Yoneda Lemma is important in our case for our definition of our categorical formulation of the GIT quotient, which we define as the functor $\text{Mor}(X/G, -)$. That is, for an object y, and for $\pi : G \times X \to X$ projection onto the second factor,

$$\text{Fun}_{X/G} = \{\psi : X \to Y | \psi \circ \phi = \psi \circ \pi\}.$$

From this, we can see why the Yoneda Lemma matters. We are defining X/G indirectly here as the algebraic variety Z such that $\text{Fun}_{X/G} = \text{Mor}(Z, -)$, and the Yoneda Lemma guarantees that if such a Z exists, it is unique up to isomorphism. To see this, suppose $\text{Mor}(A, -)$ and $\text{Mor}(B, -)$ are isomorphic. Then the Yoneda Lemma and the natural transformation from $\text{Mor}(A, -)$ to $\text{Mor}(B, -)$ gives us an element of $\text{Mor}(B, A)$ whilst the inverse transformation gives us an element of $\text{Mor}(A, B)$. Then, the fact that the natural transformations are indeed inverses provides that the composition of the two resultant maps are also the identity. That is, $A \cong B$.

2.3 The GIT Quotient

Now that we have this definition, it would be reasonable to ask when such a Z exists. Indeed, there is a set of conditions encapsulated in the following theorem:

Theorem 2.43. *For G linearly reductive and X affine. If $\mathcal{O}(X) = A$, (that is $X = spec(A)$) then A^G (the functions of A fixed by G) is finitely generated and reduced. Furthermore, for $Z = spec(A^G)$, Z is the categorical quotient of X by G. In otherwords, $spec(A^G)$ is the Z for which $Fun_{X/G} = Mor(Z, -)$*

To fully understand this theorem, we must understand what the spec functor is. Put simply, it is a functor from the category of rings to the category of algebraic varieties that, for a given ring A, it returns an X such that the ring of regular functions on X is A.

Now that we understand what the theorem is trying to say, we first need to build up a few definitions in order to prove it; both are definitions of particular types of quotient maps from X to its appropriate quotient space. However, we note that these morphism based formulations have equivalent data to our formulation in terms of functors above.

Definition 2.44. A *categorical quotient* for the action of a group G on an algebraic variety X is a G-invariant regular map $\phi : X \to Y$ such that for every other G-invariant regular map $f : X \to Z$ there is a unique map $h : Y \to Z$ such that $f = h \circ \phi$.

To prove that our spec(A^G) variety is such a quotient, we will instead prove that it is something slightly stronger: a good quotient.

Definition 2.45. With G and X above, a regular map ϕ is a *good quotient* if the following conditions hold:

1. ϕ is G-invariant
2. ϕ is surjective
3. The image of every G-invariant closed subset $W \subset X$ is closed in Y.
4. For any two disjoint G-invariant closed subsets, their images are disjoint
5. For every affine open subset of Y, its preimage under ϕ is affine
6. For an open subset U of Y, the morphism between the ring of regular functions of U over Y and the ring of regular functions of the preimage of U over X is an isomorphism onto the G-invariant functions.

We can also define exactly what we mean when we say geometric quotient

Definition 2.46. A geometric quotient is a quotient whereby the preimage of ecah point is a single orbit. Equivalently, it is a good quotient whereby the action of G on X is closed in the sense that the orbits are closed. If the orbits are closed but the preimage gives two separate orbtis, then this contradits property 4 of the good quotient.

Remark 2.47. With this definition of a geometric quotient in hand, it is not hard to see that in the case of the GIT quotient as defined below that if a GIT quotient is geometric, it is the same as the orbit space.

Lemma 2.48. *If* 2 *holds,* 3 *and* 4 *are equivalent to the statement: "If* W_1 *and* W_2 *are disjoint, closed,* G*-invariant subsets, then the closures of* $\phi(W_1)$ *and* $\phi(W_2)$ *are."*

Proof. That 2, 3 and 4 imply the statement is clear. Further, if the closures of $\phi(W_1)$ and $\phi(W_2)$ are disjoint, then so are $\phi(W_1)$ and $\phi(W_2)$. Thus, for the statement implies 3 and 4 if 2 holds direction, it remains to show 3. Without loss of generality, we prove that $\phi(W_1)$ is closed. Take a point p in the closure of $\phi(W_1)$. Since ϕ is surjective, this has a preimage W. Now, since the closures of $\phi(W)$ and $\phi(W_1)$ are not disjoint, W and W_1 are not disjoint. Thus, $p \in \phi(W_1)$ and thus $\phi(W_1)$ is closed. □

Lemma 2.49. *Every good quotient is a categorial quotient. Furthermore, a map satisfying* 1, 3, 4, *and* 6 *of the properties above is a categorical quotient.*

Proof. Suppose a map $\phi : X \to Y$ satsifies at least properties 1, 3, 4 and 6. Then, since property 1 gives G-invariance, it remains to show the "universal" or unique factoring property. Consider a map $f : X \to Z$, we will construct the map h as follows:
Take a finite affine open cover V_i of Z and use it to get an open cover U_i of Y, with regular maps $h_i : U_i \to V_i$ in the following way:
Let $W_i = X - f^{-1}(V_i)$. Note this is closed since $f^{-1}(V_i)$ is open and G-invariant, since otherwise f would not be. Thus, by property 3, $\phi(W_i)$ is closed in Y. Now, let $U_i = Y - \phi(W_i)$. To show that this is an open cover, we must prove that $\cap_i \phi(W_i) = \emptyset$. Thus, in order to gain a contradiction, suppose othwerise. Then, there exists a point $p \in \cap_i \phi(W_i)$. Now, let P denote $\phi^{-1}(p)$, and take a closed orbit W in P. Then, since $\phi(W) \cap \phi(W_i) \neq \emptyset$ by the contrapostive of property 4, we get that $W \cap W_i \neq \emptyset$. But each W_i is G-invariant, which means that $W \subseteq W_i$. Therefore, $W \subseteq W_i$, which is a contradiction since the V_i are a cover of Z and so the intersection of the W_i must be empty. Thus, we have our open cover. Now,

by property 6 and the fact that f is G-invariant we get the following diagram:

$$\begin{tikzcd} \mathcal{O}_Z(V_i) \arrow[r, "h_i^*"] \arrow[d] & \mathcal{O}_Y(U_i) \arrow[d, "\sim"] \\ \mathcal{O}_X(f^{-1}(V_i))^G \arrow[r] & \mathcal{O}_X(\phi^{-1}(U_i))^G \end{tikzcd}$$

and h_i^* is the unique map that makes the diagram commute. Then, since each of the U_i etc are affine, the standard categorical equivalence provides us with a map $h_i : U_i \to V_i$, which, by construction, factors f restricted to $\phi^{-1}(U_i)$, and by inputting $U_i \cap U_j$ and similar data into the diagram, we see that h_i and h_j agree on the intersection and thus glue correctly. Therefore, we have the map $h : Y \to Z$ such that $f = h \circ \phi$ as required. $\square$

Now, to prove our theorem, we prove that the map $\phi : X \to X//G$ induced from the inclusion of $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$ is a good quotient. However, to do this we need one other important theorem theorem and for that we require a few more definitions. The first is the definition of a Reynolds Operator:

Definition 2.50. For a group G acting on a k-algebra A, a linear map $R_A : A \to A^G$ is called a *Reynolds Operator* if it is a projection onto A^G and, for $a \in A^G$ and $b \in A$, we have $R_A(ab) = aR_A(b)$.

Example 2.51. A simple example is to take $\mathbb{Q}$ (under multiplication) acting on $\mathbb{C}$ in the following way: $g \cdot (a + bi) = a + gbi$. Then the invariant ring is all real numbers A, and we have a reynolds operator taking $a + bi$ to a. To see this note that $R(c(a + bi)) = R(ca + cbi) = ca = cR(a + bi)$.

To tie this into the present discussion, and to prove the lemma, we define what a *rational action* is:

Definition 2.52. A *rational action* of G on X is an action such that every element of X is contained in a finite dimensional G-invariant linear subspace of X.

Now we state and prove a lemma and corollary that will prove useful:

Lemma 2.53. *Let G be a linearly reductive group acting rationally on a finitely generated k-algebra A. Then, there exists a Reynolds operator $R_A : A \to A^G$.*

Proof. A is finitely generated. As such, it has a countable basis b_i, $i \in I$. Consider any element that is a j-fold product $(1 \leq j \leq n)$ of the first n basis elements. Since the action of G is rational, it is contained in a finite dimensional G-invariant subspace. Further, since the action of a particular $g \in G$ must be a k-algebra homomorphism, we have that the direct sum of any such subspaces are G-invariant, as $g(a_1 + a_2) = ga_1 + ga_2$. Therefore, let B_n be the direct sum of all such invariant subspaces for the up to j-fold products of the first n basis elements. We note that $B_n \subset B_m$ if $m > n$, and that $A = \bigcup_{n \geq 0} B_n$. We know from our proof that linearly reductive implies geometrically reductive that for linearly reductive groups, any G-invariant subspace V' admits a G-stable complement. Thus, we can write $B_n = B'_n \oplus B_n^G$ where B'_n is a direct sum of irreducible representations of G. Thus, we have a surjection $R_n : B_n \to B_n^G$. To make this into a potential candidate for a Reynolds operator from $A \to A^G$, we need to show the maps for each A_n are compatible. Note that for $m > n$ we have

$$\begin{array}{ccc} B_n & \xrightarrow{R_n} & B_n^G \\ \big\downarrow & & \big\downarrow \\ B_m & \xrightarrow{R_m} & B_m^G \end{array}$$

as, since $B_n \subset B_m$, $B'_n \subset B'_m$ and $B_n^G \subset B_m^G$. Thus our projections are compatible. It remains to show that $R_A(ab) = aR_a(b)$ for $a \in A^G$, $b \in A$. Take an n such that a and b are contained in B_n, and let $m \geq n$ be such that $a(B_n) \subset B_m$. We can write B_n as $B_n^G + B'_n$ where B'_n is a direct sum of non-trivial irreducible representations - say $W_1, \cdots, W_k$. Consider l_a, left multiplication by a. By Schur's Lemma, $l_a(W_i)$ must either be trivial or an isomorphism, and therefore $l_a(B'_n) \subset B'_m$. Thus, writing $b = b^G + b'$, we have that $l_a(b) = l_a(b^G) + l_a(b')$. Furthermore, $a \in A^G$, so $ab^G \in A^G$. Thus, we have that $ab = l_a(b)$ can be written as a sum of a G-fixed component and otherwise in B_m. Thus, $R_A(l_a(b)) = R_A(ab^G + ab') = ab^G = aR_a(b)$ as required. □

Corollary 2.54. *Let A and B be k-algebras with a rational action of a linearly reductive group G which have Reynolds operators R_A and R_B. THen any G-equivariant homomorphism $h : A \to B$ of these k-algebras commutes with the Reynolds operators. That is, $R_B \circ h = h \circ R_A$.*

Finally, we require a lemma about noetherianness:

Lemma 2.55. *Let A be a k-algebra with a rational G-action and suppose that A has a Reynolds operator $R_A : A \to A^G$. Then, for any ideal $I \subset A^G$, we have $IA \cap A^G = I$. More generally, if $\{I_j\}_{j \in J}$ are a set of ideals in A^G, then we have*

$$(\sum_{j \in J} I_j A) \cap A^G = \sum_{j \in J} I_j$$

Furthermore, if A is Noetherian, then so is A^G.

Proof. Since $I \subset A^G$, we have that $I \subset IA \cap A^G$. Now, consider $x \in IA \cap A^G$, which we can write as the sum of some $i_j a_j$. Then, since the Reynolds operator fixes elements of A^G and is linear:

$$x = R_A(x) = R_A(\sum i_j a_j) = \sum R_A(i_j a_j) = \sum i_j R_A(a_j) \in I$$

Now suppose A is Noetherian. Then for any ascending chain of ideals $I_1 \subset I_2 \subset \cdots$, in A^G $I_1 A \subset I_2 A \subset \cdots$ is also an ascending chain of ideals in A and must terminate with $I_n A$. However, $I_n A \cap A^G = I_n$ and so the chain in A^G must also terminate. $\square$

Now we can prove a theorem fundamental to the proof of theorem 2.43:

Theorem 2.56. *[Hilbert, Mumford] For A a finitely generated k-algebra, G a linearly reductive group acting on A, A^G is a finitely generated algebra.*

Proof. Let $a_1, \cdots, a_n$ be the generators of A. Then, since the action of G on A is rational, we have that each a_i is contained in a finite-dimensional G-invariant subspace V_i, the $b_{i,j}$ a basis (the j ranges and i stays fixed). Thus, the subspace $V = \bigoplus_{i=1}^n V_i$ is a finite-dimensional subspace containing the a_i with the $b_{i,j}$ as the basis (now i and j both range). Since the action of G must be a k-algebra homomorphism, we have the action is linear. In particular, $g(v_1 + v_2) = g(v_1) + g(v_2)$, and $g(\lambda v) = \lambda g(v)$, and so V is also G-invariant, since $b_{i,j} = \sum_{i=1}^n \lambda_i a_i$, and so $g(b_{i,j}) = \sum_{i=1}^n \lambda_i g(a_i)$, and the $g(a_i)$ are in V_i. Furthermore, we know that $g(b_{i,j}) = \sum_{k,l} \lambda_{k,l} g(b_{k,l})$, and so the induced action of G on $\mathrm{Sym}^*(V)$ preserves polynomial degree, and in particular, the surjection from $\mathrm{Sym}^*(V) \to A$ is G-equivariant. Thus, by Lemma 2.53 and its corollary, we have a surjection from $\mathrm{Sym}^*(V)^G \to A^G$, and so we can consider the case of $\mathrm{Sym}^*(V)$ instead of A, as finite generation remains true under a quotient.

From here, we assume A is a polynomial algebra with the action G preserving homogeneous degree. Since k is a field, we have that A is Noetherian, and then by the most recent lemma, A^G is Noetherian. Further, since A is graded by homogeneous polynomial degree, and the action of G preserves this degree, we have a natural grading on A^G also by homogeneous polynomial degree. Thus, we have an ideal containing all the polynomials with no degree 0 term: A^G_+. But A^G is Noetherian, so A^G_+ is finitely generated. Therefore, A^G must be. □

We require two last lemmas:

To prove the main lemma, we require some prior work, encapsulated by this short lemma:

Lemma 2.57. *Let G be an affine algebraic group acting on an affine variety X. Then any $f \in \mathcal{O}(X)$ is contained in a finite dimensional G-invariant subspace of $\mathcal{O}(X)$.*

Proof. First, (from Hoskins) we note that the action of G on X gives us a homomorphism of k-algebras $\sigma^* : \mathcal{O}(X) \to \mathcal{O}(G \times X) \cong \mathcal{O}(G) \otimes_k \mathcal{O}(X)$ given by $f \mapsto \sum h_i \otimes f_i$. This, in turn provides a homomorphism from G to the automorphisms on $\mathcal{O}(X)$ where the k-algebra automorphism of $O(X)$ corresponding to each g is given by $f \mapsto \sum h_i(g) f_i$. We can use this automorphism in particular to note that if we take the vector space spanned by the f_i given above, not only does it contain f, but it is also G-invariant. □

Lemma 2.58. *Let G be a geometrically reductive group acting on an affine variety X. if W_1 and W_2 are disjoint G-invariant closed subsets of X, then there is an invariant function $f \in \mathcal{O}(X)^G$ which sepeartes these sets. That is, $f(W_1) = 0$, $f(W_2) = 1$.*

Proof. First note that $(1) = I(\emptyset) = I(W_1) \cap I(W_2) = I(W_1) + I(W_2)$. Thus, we can write $1 = f_1 + f_2$ such that $f_1(W_1) = 0$ and $f_1(W_2) = 1$. Then, since f_1 is contained in a finite dimensional G-invariant subspace $V = \mathrm{Span}(G \cdot f_1)$ of $\mathcal{O}(X)$, we have an injective, G-equivariant map $V \hookrightarrow A$. Then, since the action on V induces an action on $\mathrm{Sym}^*(V) = \mathcal{O}(V^\vee)$, we get a G-equivariant map $\mathrm{Sym}^*(V) \to A$. However, both these spaces are rings of regular functions, and so by the equivalence of categories, this map then induces a G-equivariant map $h : X \to V^\vee$ whereby $W_1 \mapsto 0$ and $W_2 \mapsto p$, $p \neq 0$. Now, since G is geometrically reductive there exist a homogeneous polynomial (non-constant), say F, such that

$F(p) \neq 0$ and $F(0) = 0$. Thus, the function $f = cF \circ h$ is a sufficient function, whereby $c = 1/F(p)$. □

Proof of Theorem 2.43. We will prove that the GIT quotient is a categorical quotient by proving that it is a good quotient. To see that the GIT quotient satisfies properties 1 and 3, we note that since G is linearly reductive, then by Theorem 2.56 $\mathcal{O}(X)^G$ is a finitely generated k-algebra. Then since the quotient is constructed from the map on the function rings, it is affine, and G-invariant as it is an inclusion from a finitely generated G-invariant k-algebra.
For surjectivity, we need to show that any given point in $X//G$ is mapped to by a point in X. To do this, consider the maximal ideal associated to any point y in $X//G$ and take its generators $f_1, \cdots, f_n$. Then we note that by Lemma 2.55 that

$$(\sum f_i \mathcal{O}(X)) \cap \mathcal{O}(X)^G = \sum f_i \mathcal{O}(X)^G$$

and thus $\sum (f_i \mathcal{O}(X))$ contains an element that corresponds to such an x.
As proved in Lemma 2.48, since we already have surjectivity, 4 and 5 are equivalent to proving that for two disjoint invariant closed subsets have the closures of the images disjoint. By Lemma 2.58, there is a function f in $\mathcal{O}(X)^G$ such that $f(W_1) = 0$ and $f(W_2) = 1$. Note however, that this is a function in $\mathcal{O}(X)^G$ and is thus a regular function on $X//G$. Therefore, we have that $f(\phi(W_1)) = 0$ and $f(\phi(W_2)) = 1$ and therefore the closures of their images must also be disjoint.
It remains to prove 6. Firstly, for all $f \in \mathcal{O}(X)^G$, the subsets $X//G_f$ act as a basis for the open subsets of $X//G$. Thus, it suffices to prove the result simply for theses sets. Since f is G-invariant we have:

$$\mathcal{O}_X(\phi^{-1}(X//G_f))^G = \mathcal{O}_X(X_f)^G = (\mathcal{O}(X)^G)_f = \mathcal{O}_Y(Y_f)$$

and thus the GIT quotient is a good quotient. □

For future discussion, we denote the GIT quotient as $X//G$.

Example 2.59. If we take the trivial action of an algebraic group on itself, we see that $G//G = G$. One can also end up with the trivial variety. For example consider the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}$ via multiplication by 1 on the identity and -1 not on the identity. The ring of regular functions is $\mathbb{C}[x]$ but the invariant ring is $\mathbb{C}$ and thus our variety $\mathbb{C}//\mathbb{Z}/2\mathbb{Z} = 0$. Further examples appear in the next chapter.

2.4 Stable Points

We have a geometric object that one could reasonably describe as a quotient space, the GIT quotient. Recall however, that we have another quotient space, the set theoretic (or geometric) quotient. Clearly then, the next reasonable question to address is the "sameness" of these two formulations of a quotient space. The answer, as one might expect, is fairly different (and indeed from a geometric context extremely different), and this is best illustrated with an example.

Example 2.60. Consider the action of GL_2 on the set of 2×2 matrices by conjugation. Our ring of regular functions is just polynomials in 4 variables and we have the following fact about the invariant ring:

Remark 2.61. The invariant ring is generated by the trace and the determinant

We see then that the matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ are identified in the GIT quotient, even though they are not in the same orbits

Thus, these two objects are quite different, and this difference is encapsulated in the property of stability; it is the subvariety of a type of point called a stable point X^s for which we get an equivalence between the GIT quotient and geometric quotient. We define a stable point on an affine GIT quotient as follows:

Definition 2.62. A point $x \in X$ is *stable* if its orbit is closed in X and $\dim(G_x) = 0$. The set of all stable points of X is denoted by X^s.

Example 2.63. Returning to the example above, we see that since the Jordan normal form of a matrix $M = PAP^{-1}$, and is obtained by conjugation, we can find which points are stable by just considering Jordan normal forms. For any diagonal matrix, we see that the stabilizer is all of the other diagonal matrices - this stabilizer is 2-dimensional and so no diagonal matrix is stable. Scalar matrices have stabilizer GL_2 and so their stabilizer is also not zero dimensional, and thus scalars are unstable. What remains then are non-diagonalizable matrices. That is, matrices with linearly dependent eigenvectors. These are matrices of the form $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$. The stabilizer of such matrices is the scalar matrices - which is a 1 dimensional subgroup. Thus no points are stable

Example 2.64. An example for which the stable points are not the emptyset is the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$. Note that every point except the origin has a zero dimensional stabiliser, so the stable points are the points in $\mathbb{A}^2 \setminus \{(0,0)\}$ that have a closed orbit. Note that no points on the axis have a closed orbit, since limiting t to 0 or infinity gives that the origin is in the closure, while the orbit of the origin is itself. However, every other point is stable. To see this, note that for a point (a,b), t cuts out the curve, $xy = ab$. This is zariski closed, as it is a polynomial equation. Thus $X^s = \{(a,b) \mid a, b \neq 0\}$

Proposition 2.65. *For a linearly reductive group G acting on an affine variety X and with $\phi : X \to Y := X//G$ the affine GIT quotient, X^s is open and G-invariant. $Y^s = \phi(X^s)$ is open in Y and $X^s = \phi^{-1}(Y^s)$. Furthermore, $\phi : X^s \to Y^s$ is a geometric quotient*

To prove this proposition, we need three facts which we will not prove. They can be found in Hoskins but require some mathematics beyond the scope of this thesis:

Proposition 2.66. *Let G be an affine algebraic group acting group acting on a variety X. Then, for every n, the sets*

$$\{x \in X \mid dim\ G_x \geq n\} \qquad \{x \in X \mid dim\ (G \cdot x) \leq n\}$$

are closed in X

Moreover,

Lemma 2.67. $dim(G) = dim(G_x) + dim(G \cdot x)$.

Lemma 2.68. *The boundary of an orbit is a union of strictly lower dimensional orbits.*

Proof. Suppose x is stable. Then the dimension of its stabiliser is zero. Since we know that the set $\{x \mid \dim(G_x) > 0\}$ is closed, if we can show it is G-invariant then, since G is linearly (and therefore geometrically) reductive, there exists a G-invariant function f such that

$$f(G \cdot x) = 1, \qquad f(\{x \mid \dim(G_x) > 0\}) = 0$$

However, the set is G-invariant, since if $y = g \cdot X$, then $G \cdot y = G \cdot x$, and in particular their dimensions are the same, so the dimensions of their stabilisers are the same.

Thus, we have a function f such that $x \in X_f$ where X_f is open. Now, suppose $y \in X_f$. The dimension of the stabiliser of y must be zero, since otherwise

$f(y) = 0$. Furthermore, since the boundary of an orbit is a union of strictly lower dimensional orbits, and elements of the boundary of the orbit of y must be in X_f as X_f is an open set, the stabiliser of elements on the boundary of y also have dimension zero, and so their orbits have the same dimension as the orbit of y, and so the orbit of y must be closed. Hence $y \in X^s$ and X_f is open in X^s and so x has a neighbourhood of points around it also in X^s. This is true for any $x \in X^s$ however, and so X^s must be open.
We note from this that these X_f will cover X^s, and so since $\phi(X_f) = Y_f$, and Y_f is open, then Y^s must be open. In a similar vein therefore $\phi^{-1}(Y^s) = X^s$, since $\phi^{-1}(Y_f) = X_f$.
It remains to show that $\phi^s : X^s \to Y^s$ is a geometric quotient. However, it is not hard to check that the restriction of a good quotient to open sets remains a good quotient, and from there since we know that the orbits of elements in X^s are closed, we can conclude that ϕ^s is a geometric quotient. □

To discuss a better criterion for stability, we need to move onto projective varieties and to do this we need to know what a linear G-action on a projective variety means:

Definition 2.69. Let X be a projective variety with an action of an affine algebraic group G. A *linear G-equivariant projective embedding* of X is a group homomorphism $G \to GL_{n+1}$ and a G-equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$. We will reduce this to saying that the G-action on X is *linear* to mean that we have a linear G-equivariant projective embedding of X.

So now suppose we have such a linear action of a linearly reductive group G on some X. Then since $\mathbb{P}^n$ lifts to the afffine cone, the action of G must as well. Furthermore, since the projective embedding is G-equivariant we get an induced action of G on the affine cone of X, $\tilde{X}$. That is, if X associates to the homogeneous ideal $I(X)$. Then $\tilde{X} = \mathrm{Spec}R(X)$ where $R(X) = k[x_0, \cdots, x_n]/I(X)$. Note that since $R(X)$ is graded by homogeneous degree and the G-action is linear, we get a grading on $R(X)^G$ by the homogeneous degree. Note further since G is linearly redutive, $R(X)^G$ is finitely generated and thus the inclusion $R(X)^G \hookrightarrow R(X)$ induces a rational map of projective varieties

$$X \dashrightarrow \mathrm{Proj}R(X)^G.$$

This motivates the following definition:

Definition 2.70. For a linear action of a reductive group G on a projective variety X, we define the *nullcone* to be the closed subvariety of X defined by $R(X)^G_+$. Furthermore, we define the *semistable set* as the open set $X^{ss} = X - N$. That is, $x \in X$ is *semistable* if there exists a G-invariant homogeneous function $f \in R(X)^G_r$ for $r > 0$ such that $f(x) \neq 0$. Also, we see that via this construction, the semistable set is the set for which the rational map mentioned earlier is defined.
Thus, the morphism $X^{ss} \to X//G = \mathrm{Proj}R(X)^G$ is the *projective GIT quotient*

This is all well and good, but we want to be sure like its affine counterpart that it is a good quotient

Theorem 2.71. *For a linear action of a reductive group G on a projective variety X, the GIT quotient $\phi : X^{ss} \to X//G$ is a good quotient.*

Proof. To prove that the projective GIT quotient is a good quotient, note that being a good quotient is a local property. Then, for $f \in R(X)^G_+$, note that (if our GIT quotient maps $X^{ss} \to Y$) the sets Y_f form a basis of open sets on Y. Furthermore, we have that $\phi^{-1}(Y_f) = X_f$. Then, if we consider the affine cones of X_f and Y_f, say $\tilde{X}_f$ and $\tilde{Y}_f$. Then we have that

$$\mathcal{O}(Y_f) \cong \mathcal{O}(\tilde{Y}(f))_0 \cong ((R(X)^G)_f)_0 \cong ((R(X)_f)_0)^G \cong (\mathcal{O}(\tilde{X}_f)_0)^G \cong \mathcal{O}(X_f)^G$$

Thus, the localisation of the quotient to the affine schemes $\phi_f : X_f \to Y_f$ is an affine GIT quotient, and is therefore a good quotient. Then, as mentioned earlier, since being a good quotient is a local property and we can get to our original quotient ϕ by gluing all the ϕ_f morpisms, ϕ is also a good quotient. □

However, we were trying to understand how different the GIT quotient was from the geometric quotient but with semistability in hand we can define stability for projective varieties:

Definition 2.72. For a linear action of a reductive group G on a closed projective variety X, a point $x \in X$ is

1. *stable* if the dimenstion of its stabiliser $\dim G_x = 0$ and there is a G-invariant homogeneous polynomial $f \in R(X)^G_+$ such that $x \in X_f$ and the action of G on X_f is closed

2. *unstable* if it is not semistable

Thus, all that remains to prove is that on the stable points, the GIT quotient restricts to the geometric one, and to do that we need the following lemma:

Lemma 2.73. *The stable and semistable sets are open in X*

Proof. Firstly, we note that since we defined X^{ss} as the complement of the Null-cone, it is clearly open. Then, consider the set $X' = \bigcup X_f$ for f in $R(X)_+^G$ where the union preserves the closed nature of the action of G on X_f Then, since X' is a union of open sets, it is an open set in X, and thus if X^s is open in X', we're done. However, the above fact gives us this openness property since $\{x \in X' \mid \dim G_x \geq 1\}$ is a closed set whose complement is precisely the set with zero dimensional stabilisers. $\square$

Theorem 2.74. *For a linear action of a reductive group G on a closed projective variety X, let $\phi : X^{ss} \to X//G$ denote the GIT quotient. Then there is an open subvariety of $X//G$ - $X//G^s$ such that $\phi^{-1}(X//G^s) = X^s$ and that the GIT quotient restricts to a geometric quotient*

Proof. First, note that any geometric quotient is also a good quotient. Furthermore, note that for any good (or geometric) quotient $\phi : X \to Y$ and any open subset U of Y, then the restriction of the quotient to the preimage of U is also a good (or geometric) quotient. For a final piece of setup, also note that if we have a cover of Y be open sets U_i such that the restriction of ϕ to the preimages of these U_i are good (or set theoretic) quotients then ϕ is a good (or set theoretic) quotient. With this information in hand, consider the set $Y' = \bigcup Y_f$ where $f \in R(X)_+^G$ and that the action of G on X_f is closed. Furthermore, let X' be the preimage under ϕ of Y'. Then the map $\phi : X' \to Y'$ can be constructed via the gluing of the ϕ_f. Note that each ϕ_f is a good quotient since they are the restriction of the image of an open set under a good quotient. Additionally, since the G-action on X_f is closed, we have that this is a geometric quotient. Therefore, we have that ϕ is a set theoretic quotient. Thus, it remains to show that $\phi(X^s)$ is open. Note that since X^s is a G-invariant subset, $\phi(X' - X^s) = Y' - Y^s$. Remember that by the fact mentioned in Lemma 2.73, $X' - X^s$ is closed, and therefore since ϕ is a good quotient, so is $Y' - Y^s$ via property 5. Thus, Y^s is open in Y' and since Y' is open in Y, so is Y^s. $\square$

2.5 The Numerical Criterion

So we now know how close the GIT quotient is to the geometric one for both affine and projective GIT quotients, but it would be better if we had a nice way to detect unstable points. Such a detection method exists, and is known as the Hilbert-Mumford Numerical Criterion. To prove that it works, and makes sense we need to provide some less nice criteria to build from:

Lemma 2.75. *Let G be a reductive group acting linearly on X. A point x is stable if and only if x is semistable, its orbit is closed in X^{ss} and its stabiliser G_x is zero dimensional*

Proof. First, note that if x is stable, then there exists an $f \in R(X)_+^G$ such that $x \in X_f$ so x is semistable. We also already know that the dimension of the stabiliser G_x is zero. Thus, it remains to show the orbit of x is closed in X^{ss}. Suppose $x' \in \overline{G \cdot x}$. Then, since the quotient ϕ is G-invariant, $\phi(x') = \phi(x)$, and so $x' \in X^s$, and therefore in X^{ss}. Thus, it remains to show x' is in the orbit. However, we know that orbits on the boundary are unions of orbits of strictly lower dimension, but x' is stable, and so the dimension of its stabiliser is zero, and so its orbit has dimension the same as the dimension of x, so x' must be in the orbit of x.

Now, suppose $x \in X^{ss}$, $G \cdot x$ is closed in X^{ss}, and the dimension of the stabiliser G_x is zero. Then, since $x \in X^{ss}$, there exists some $f \in R(X)_+^G$ such that $x \in X_f$. Now, since $G \cdot x$ is closed in X^{ss}, it is closed in X_f. Further, it is disjoint from the set of points

$$Z = \{y \in X_f | \dim(G_y) > 0\}$$

Thus, there exists some $h \in O(X_f)^G$ such that $h(G \cdot x) = 1$ and $h(Z) = 0$. Now, $O(X_f) = O(\tilde{X}_f)_0$ and, in particular, is the quotient of some algebra $A = (k[x_0, \cdots, x_n]_f)_0$, where the kernel I is some ideal. Then, by the lemma below, we have that there exists a positive integer r such that $h^r \in A^G/(I \cap A^G)$, but A is localised at f so in particular we have that $h^r = h'/f^s$, where $h' \in O(X)^G$. Now, $fg = h^r f^{s+1}$, and in particular $x \in X_{fg}$ and for $y \in X_f$, $y \in X_{fg}$ only if the stabiliser of y is dimension zero. Therefore, $h' = fg$ is a polynomial in $O(X)^G$ whereby $x \in X_{h'}$ and all orbits of elements in $X_{h'}$ are closed, and so x is stable. □

Lemma 2.76. *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra A. For a G-invariant ideal I of A and $a \in (A/I)^G$, there is a positive integer r such that $a^r \in A^G/(I \cap A^G)$.*

Proof. Let $b \in A$ be an element in the preimage of a and further assume $a \neq 0$. We have that G acts rationally and so b is contained in a finite dimensional

G-invariant vector space V spanned by $g \cdot b$. Since a is nonzero $b \notin V \cap I$ but $g{\cdot}b - b \in V \cap I$ as $\pi(g{\cdot}b - b) = \pi(g{\cdot}b) - \pi(b) = a - a$. Thus, dim $(V) =$ dim $(V \cap I) + 1$ and we can write any $v \in V$ as $\lambda b + b'$ for a scalar λ and an element in $V \cap I$. The linear projection from V to k onto the line spanned by V is G-equivariant, and so in the dual representation, the projection corresponds to a nonzero fixed point, say l^*. Furthermore, since G is geometrically reductive (and has an induced action on the dual representation), there exists a G-invariant homogeneous function F of positive degree r which doesn't vanish at l^*. To complete the proof, note that first, we have the algebra homomorphism as in the proof of theorem 2.56

$$\mathcal{O}(V^\vee) = \mathrm{Sym}^*(V) \to A$$

and second we can take a basis of V such that the first basis vector corresponds to b. Then, $F = \lambda x_1^r + \cdots$, $\lambda \neq 0$. Then $F - \lambda x_1^r$ is a polynomial in which every monomial contains a power of x_i, $i \neq 1$ Thus, for b_0 is the image of F in A^G, we have that $b_0 - \lambda b^r \in I$ as the images of the remaining x_i are elements in $V \cap I$ and so $a^r \in A^G/(I \cap A^G)$ □

So with that criterion set up, we can move to a topological criterion.

Proposition 2.77. *Let x be a point of X and choose a nonzero lift into the affine cone $\tilde{x}$. Then*

1. *x is semistable iff $0 \notin \overline{G \cdot \tilde{x}}$*
2. *x is stable if and only if $dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$.*

Proof. First, suppose that x is semistable. Then, there exists f such that $f(x) \neq 0$ and so $f(\tilde{x}) \neq 0$. But G-invariant functions are constant on orbits and their closures, so $f(\overline{G \cdot \tilde{x}}) \neq 0$ and so $0 \notin \overline{G \cdot \tilde{x}}$
Suppose $0 \notin \overline{G \cdot \tilde{x}}$. Then there exists f such that $f(\overline{G \cdot \tilde{x}}) \neq 0$ and $f(0) = 0$. We can write $f = \sum f_i$ where the f_i are homogeneous, and the action of G is linear, so the f_i are G-invariant, and at least one must be nonvanishing at $\tilde{x}$ and so is nonvanishing at x. Hence, x is semistable.

Suppose x is stable. Since $G_{\tilde{x}} \subset G_x$, the dimension of $G_{\tilde{x}}$ is zero. There also exists f homogeneous such that $f(x) \neq 0$, and so $f(\tilde{x}) \neq 0$. Let

$$Z = \{z \in \tilde{X} \mid f(z) = f(\tilde{x})\}$$

and let $\pi : Z \to X_f$ be the map obtained from restricting the surjection $\tilde{X} \setminus 0 \to X$. Suppose $y \in \overline{G \cdot \tilde{x}}$. Then $\pi(y) \in G \cdot X$, and so the dimension of $G_{\pi(y)} = 0$, but that means that G_y has dimension 0 and so the orbit of y must have dimension $\dim G$ and so if y isn't in the orbit of $\tilde{x}$, then $G \cdot \tilde{x}$ must have dimension bigger than that of G, which is a contradiction.
For the reverse direction, we know that $0 \notin G \cdot \tilde{x}$, since otherwise $\tilde{x} \in G \cdot 0$. Thus, $0 \notin \overline{G \cdot \tilde{x}}$ since $G \cdot \tilde{x}$ is closed, and so by part 1 of this proposition, x is semistable, and so there exists f such that $f(x) \neq 0$. Let Z be as before , and since f is homogeneous of degree d, say, $Z \cap \pi^{-1}(x)$ is the set of points λx such that $\lambda^d f(x) = 1$. That is, π has finite fibers. Now, for $g \in G_x$, since x has finite preimages say $\tilde{x}, \tilde{x_1}, \cdots, \tilde{x_n}$, $g\tilde{x} = \tilde{x}$ or $\tilde{x_i}$. If we denote $G_i = \{g \mid g\tilde{x} = \tilde{x_i}\}$ and $g_i \in G_i$, then G_i is isomorphic (as a variety) to $G_{\tilde{x}}$ by left multiplication by g_i, and since $G_{\tilde{x}}$ has dimension 0, all the G_i do. But $G_x = G_{\tilde{x}} \sqcup G_1 \sqcup \cdots \sqcup G_n$, so G_x has dimension 0. Finally, pick $y \in \overline{G \cdot x}$ strictly on the boundary. Then, since $\pi(G \cdot \tilde{x}) = G \cdot x$, then $\tilde{y} \notin G \cdot \tilde{x}$ That is, there exists h such that $h(\tilde{y}) \neq 0$, but $h(G \cdot \tilde{x}) = 0$. THus, there exists a homogeneous component h_i for which $h_i(\tilde{y}) \neq h_i(G \cdot \tilde{x})$. That is, $h_i(y) \neq h_i(G \cdot x)$, but y is in the closure, and so this is a contradiction. Hence $G \cdot x$ is closed in X_f, and so $G \cdot x$ is closed in $X^{ss} = \bigcup X_f$ and therefore x is stable by Lemma 2.75. □

Now we can begin to discuss the Hilbert-Mumford criterion but first we need to understand 1-parameter subgroups:

Definition 2.78. A *1-parameter subgroup* of G is a non-trivial group homomorphism $\lambda : \mathbb{G}_m \to G$.

Now, we can state the Hilbert-Mumford Criterion:

Theorem 2.79. *Let G be a (linearly) reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$. Then, for $x \in X$, we have*

1. $x \in X^{ss} \iff \lim_{t\to 0} \lambda(t) \cdot \tilde{x}$ *is either nonzero or doesn't exist for all* 1*-PSs* λ *of* G
2. $x \in X^s \iff \lim_{t\to 0} \lambda(t) \cdot \tilde{x}$ *doesn't exist for all* 1*-PSs* λ *of* G.

In order to prove the criterion, we must first recall a few definitions. The first is of a local ring at a point:

Definition 2.80. A ring is a *local ring* if it has a unique maximal ideal. The local ring of a variety X at a point P is the ring

$$\{f/g \mid f, g \in \mathcal{O}(X), g(P) \neq 0\}$$

The next is the normalization of a variety

Definition 2.81. A variety is normal if at every point $x \in X$ the local ring $\mathcal{O}(X)_x$ is an integrally closed domain. The normalization of a variety is the unique morphism $\phi : Y \to X$ such that Y is normal and for any other normal variety Z and a map $\psi : Z \to X$ dominant, we have a unique morphism $\theta : Z \to \tilde{Y}$ with $\phi = \psi \circ \theta$. Intuitively, this can be thought of (at least for curves) as the removal of singularities.

Lastly, we have the completion of a ring

Definition 2.82. Let A be a ring with a descending filtration

$$A = F^0 A \supset F^1 A \supset \cdots$$

of subrings. The completion A' of the ring A is the inverse limit

$$A' = \lim_{\leftarrow}(A/F^n A)$$

Next, we need to prove two lemmas. For each let G be a linearly reductive group acting linearly on $\mathbb{A}^n$ and let $z \in \mathbb{A}^n$ be a k-point. Further, suppose 0 lies in the orbit closure of z:

Lemma 2.83. *There exists an irreducible curve $C_1 \subset G \cdot z$ that contains the origin in its closure*

Proof. Begin by fixing an embedding of affine n-space into Projective n-space, and let p denote the image of the origin. Let $Y = \overline{G \cdot z} \subset \mathbb{P}^n$ and let $d = \dim(Y)$. Note that we can assume that $d > 1$ as if $d = 1$, we have a curve that we can obtain C_1 from via removing Z-points. For $n > 1$, we can use an intersection of hyperplanes to produce the required curve. Given the set of hyperplanes containing p is a codimension 1 subspace $\mathcal{H}_p$, taking $(d-1)$ copies of this subspace that contain hypersurfaces $H_1, \cdots H_{d-1}$ with the following two conditions:

1. $\cap_i H_i \cap Y$ is a curve (that is has dimension 1)

2. $\cap_i H_i \cap Y$ is not entirely contained in Z

Gives us a curve $C_1' = \cap_i H_i \cap Y$ that isn't entirely contained in Z, since the two conditions are nonempty open conditions, and the dimension of $\mathcal{H}_p^{d-1}$ is greater than zero. Then, we retrieve the required curve C_1 by removing the Z-points. $\square$

Lemma 2.84. *There exists a curve $C_2 \subset G$ which dominates C_1 under the group action morphism σ_z.*

Proof. To prove such a curve exists, use the same technique as in the previous lemma except on the preimage σ_z^{-1} to construct a curve $C_2' \subset \sigma_z^{-1}(C_1)$. Since G acts linearly, dimensions are preserved. Furthermore, open sets are also preserved under preimage since the action morphism is a morphism in the category of varieties. □

Now, we can begin the proof:

Proof. To start with, note that because of Lemma 2.77 the Hilbert Mumford Criterion reduces to the following theorem.

Theorem 2.85. *A point $x \in X$ with lift $\tilde{x}$ is unstable if and only if there exists a one parameter subgroup λ such that*

$$\lim_{t \to 0} \lambda(t) \cdot \tilde{x} = 0$$

The reverse direction is clear. If such a subgroup exists, then zero is clearly in the closure of the orbit, as it is in the closure of the one parameter subgroup. Thus, by Lemma 2.77, the point is unstable. The forward direction requires considerably more work:

To start with, we use Lemma 2.83 to provide us with a curve $C_1 \subset G \cdot z$ that contains zero in its closure.

Then, we use Lemma 2.84, to get the curve C_2. Then, we get a curve C by taking the projective completion of the normalization of C_2, $\tilde{C_2} \to C_2$. Then, we have a rational map $\rho : C \dashrightarrow G$ defined by the maps $\tilde{C_2} \to C_2 \to G$, and thus we have a preimage of the origin in C (since the origin is contained in the closure of C_1), say c_0. Then $\lim_{c \to c_0} \rho(c) \cdot z = \lim_{c \to c_0} \sigma_z(\rho(c)) = 0$.

Since C is smooth (we took the normalization), the completion of its local ring at c_0, $\mathcal{O}_{C,c_0}$ is isomorphic to the power series ring $k[[t]]$ by the Cohen Structure Theorem [3]. The field of fractions of the power series ring is the Laurent series ring $k((t))$, since ρ was defined in a punctured neighbourhood of c_0 we get an induced morphism on Spec Frac of the completion of the local ring down:

$$q : \mathrm{Spec}k((t)) \cong \mathrm{SpecFrac}\hat{\mathcal{O}}_{C,c_0} \to \mathrm{SpecFrac}\mathcal{O}_{C,c_0} \xrightarrow{\rho} G$$

Thus $\lim_{t \to 0} q(t) \cdot z = 0$.

Next, let $R = \mathrm{Spec}k[[t]]$, $K = \mathrm{Spec}k((t))$. Note that there is a morphism $K \to R$

induced by the inclusion of the ring of regular functions and so the R-valued points of G form a subgroup of the K-valued ones. Further the limit as $t \to 0$ of the R-valued G-points exist, since the powers of t in R are all positive. Now, for any one parameter system λ, we can define its Laurent series expansion (in $G(K)$) in the following way: there is a natural morphism from $K \to \mathbb{G}_m$ induced from the morphism on their group rings $k[s, s^{-1}] \to k((t))$ by $s \mapsto t$. Define the expansion then to be the composition of λ with this map from $K \to \mathbb{G}_m$
Now, we relate our map q to such a Laurent series expansion using the Cartan-Iwahori decomposition. That is, every double coset in $G(K)$ for $G(R)$ is represented by a Laurent series expansion of a one parameter system $\langle\lambda\rangle$. Following this proof, we will prove this decomposition for $G = SL_n$, the main group of study in this thesis. A more general proof can be found in [7], but for now we will use this decomposition without proof. From it, we see that there exists a Laurent expansion $< \lambda >$ and two $G(R)$ points l_1, l_2^{-1} such that $l_1 q l_2^{-1} =< \lambda >$. That is, $l_1 q =< \lambda > l_2$. Note that since $q \notin G(R)$, $< \lambda >$ cannot be trivial.
Let $g_i = l_i(0) \in G$. Then, we have that

$$0 = g_1 \cdot 0 = \lim_{t\to 0} l_1[q(t) \cdot z] = \lim_{t\to 0}(l_2 < \lambda >)(t) \cdot z$$

Now, the action of λ (not $< \lambda >$) decomposes into weight spaces V_r, $r \in \mathbb{Z}$. Furthermore, since $l_2 \in G(R)$ and $g_2 = \lim_{t\to 0} l_2(t)$, we know that l_2 decomposes into g_2 plus some positive powers of t, say $\epsilon(t)$ which go to zero when we take the limit. Under our weight space decomposition:

$$l_2(t) \cdot z = g_2 \cdot z + \epsilon(t) = \sum_{r\in\mathbb{Z}} (g_2 \cdot z)_r + \epsilon(t)_r$$

However, we have that $\lim_{t\to 0}(l_2 < \lambda >)(t) \cdot z = 0$, and so $(g_2 \cdot z)_r = 0$ for $r \leq 0$. That is, there are no negative powers of t in $g_2 \cdot z$, and so the limit $\lim_{t\to 0} \lambda(t) \cdot g_2 \cdot z = 0$ and so we have a one parameter system $\lambda' = g_2^{-1}\lambda g_2$ such that $\lim_{t\to 0} \lambda'(t) \cdot z = 0$, completing the proof. □

The final element that remains is the proof of the Cartan-Iwahori decomposition in the case where $G = SL_n$. That is,

Theorem 2.86. *Every double coset in n by n matrices whose entries are Laurent series for n by n matrices whose entries are power series is represented by a laurent series expansion of a one parameter system. That is, there exist matrices of power series M and N, such that $MKN = K'$ where K' is a diagonal matrix with laurent series entries.*

Proof. First, note that if this is possible in GL_n, say with $\langle\lambda\rangle = l_1 q l_2$, then it is possible in SL_n with $\frac{l_1}{\det(l_1)} q \frac{l_2}{\det(l_2)}$, and furthermore that row and column operations are matrices in GL_n. Next, note that K is the fraction field of R, so entries in elements of $G(K)$ are in fact $\frac{p(x)}{q(x)}$, for p and q power series. Now, power series can all be written in the form $x^\alpha p(x)$ whereby $p(0) \neq 0$. That is, $p(0)$ is invertible. Thus

$$\begin{aligned} f(x) = \frac{p(x)}{q(x)} &= \frac{x^{\alpha_p} p'(x)}{x^{\alpha_q} q'(x)} \\ &= x^{\alpha_p - \alpha_q} \frac{p'x}{q'x} \end{aligned}$$

But $p'(x)$ and $q'(x)$ are nonzero at 0, so $\frac{p'(x)}{q'(x)}$ is a power series $f'(x)$ which is also nonzero at 0. Thus, every entry in an element of $G(K)$ is of the form $x^\alpha p(x)$ where $\alpha \in \mathbb{Z}$ and $p(x)$ is a power series where $p(0) \neq 0$.

Now, using row and column operations, move the entry with the smallest negative power into the top left corner (for notation we label the ij-th entry of the matrix as $x^{\alpha_{ij}} p_{ij}(x)$)) Now for column j, multiply the column by $x^{\alpha_{11} - \alpha_{1j}} p_{1j}(x)^{-1} p_{11}(x)$. The entry in space $1, j$ then becomes the same as the entry in $1, 1$. Then subtract column 1 from column j. Doing this for all columns j zeroes every element in the first row except the diagonal entry. We can then do the same process to zero the first column, instead working with rows i. This creates a matrix whose first diagonal entry is nonzero, the first column and row are otherwise zero with an $n-1 \times n-1$ minor. We can then repeat this process with the minor provided that the row and column operations preserve the form of the entries. That is, provided the new laurent series in entry i, j, say, still has $p_{ij}(0) \neq 0$.

Without loss of generality, we can check this for a single operation. Consider an entry i, j and assume that the multiplication step has been done, to take $x^{\alpha_{ij}} p_{ij}(x)$ to some $x^\beta q(x)$. Now subtract (again without loss of generality) column 1. We then have the i, j entry is $x^\beta q(x) - x^{\alpha_{i,1}} p_{i,1}(x)$. One of the powers β or $\alpha_{i,1}$ must have larger (or equal) absolute value. Since the only difference is a factor of -1, assume β is the larger absolute value. Then our i, j entry is $f(x) = x^\beta (q(x) - x^{\alpha_{i,1} - \beta} p_{i,1}(x))$ where the factor inside the bracket is now a power series. Label this power series $q'(x)$. If $|\beta|$ is directly larger, $q'(0) = q(0) \neq 0$, so the only case is where $\beta = \alpha_{i1}$. Then $q'(0) = 0$ oly when $q(0) = p_{i,1}(0)$. Thus,

$$\begin{aligned} q'(x) &= (a + \sum_{i=1}^{\infty} a_i x^i - (a + \sum_{i=1}^{\infty} b_i(x^i))) \\ &= \sum_{i=1}^{\infty} (a_i - b_i) x^i \end{aligned}$$

Notice that if the two power series are completely equal, then subtracting them gives zero - which is also fine. Thus, there must exist some power of x whereby $a_i \neq b_i$, say m. Then we can write $f(x) = x^{\beta+m}(a_m - b_m + \sum_{i=1}^{\infty}(a_i - b_i)^{x^i}) = x^{\beta+m} q'(x)$. This now has $q'(0) \neq 0$ and so the form of the entries is maintained.

$\square$

Chapter 3

Computation and Classification

3.1 Some Basic Convex Geometry

In the last chapter we established a powerful numeric criterion for stability. However, it was a check on individual points, of which there may be infinitely many. Therefore finding the set of unstable points simply using the criterion naively is next to impossible. The section following on from this one will explain a method of translating a numeric problem with limits into a convex geometry problem with halfspaces. For that reason, we will take a brief interlude into some convex geometry, covering some basic definitions and one or two useful facts - in particular, about polyhedra.
For what follows, all sets are contained in $\mathbb{R}^n$. We first define a convex set:

Definition 3.1. A set A is *convex* if, for every two points $x, y \in A$, $\lambda x + \mu y \in A$ for any $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$

Given a set we can also define its convex hull

Definition 3.2. The convex hull conv A of a set A is the smallest convex set containing A.

These definitions are fairly intuitive - take two points, and take the set which includes the line between them. A basic example is the sphere S^n and its convex hull D^n.
From here, we can define a polyhderon

Definition 3.3. A *polyhedron* is the intersection of a finite faimly of closed halfspaces. That is, it is the set of all points $(x_1, \cdots, x_n)$ which satisfy a finite system of linear inequalities. The set of such halfspaces for a polyhedron P is called the *halfspace representation* of P.

This idea of a halfspace representation is what we will rely on heavily for our computation, but before we describe the program, we must illuminate in greater detail how one moves from the Numerical Criterion to a question about convex sets.

3.2 The Connection with Convex Geometry

As we established at the end of the previous chapter - we have a numerical criterion for the instability of points of X under the action of G, the Hilbert-Mumford Numerical Criterion:

Theorem 3.4. *For a point $x \in X$, and a lift $\tilde{x}$ into the affine cone, x is unstable if there exists a one parameter subgroup λ_t such that*

$$\lim_{t \to 0} \lambda_t \tilde{x} = 0$$

To translate this into a problem involving convex geometry, we need the following four facts:

1. G has (at least one) maximal torus T
2. Any two maximal tori are conjugate
3. Each one parameter subgroup is contained in some maximal torus T
4. V decomposes into a finite direct sum of one-dimensional representations

Since every maximal torus is conjugate to any other, we can solve the problem of unstable points up to conjugation by fixing a torus T. If we now fix a torus that is say, n-dimensional, any one dimensional representation of this torus will be parametrised by n distinct integers defined by what power of themselves each of the n generators of T get sent to. Thus, as V decomposes into a finite direct sum of one-dimensional representations, we are provided with m (say) n-tuples in $\mathbb{Z}^n$.

Now consider a one parameter subgroup λ factoring through T, and note that because of our one dimensional representation, we can write any vector $v \in V$ as a linear combination of generators for each of our one dimensional representations.

That is, we can write $v = (v_1, \cdots, v_m)$, with $a_1, \cdots, a_m$ being the m n-tuples defining our one dimensional representations. Writing our torus as a matrix, our one parameter subgroup maps into T as follows:

$$t \mapsto \begin{pmatrix} t^{\lambda_1} & 0 & \cdots & 0 \\ 0 & t^{\lambda_2} & \ddots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & t^{\lambda_n} \end{pmatrix}$$

Again, we can write this map as an n-tuple $\lambda = (\lambda_1, \cdots, \lambda_n)$. We therefore have the action of λ on v as

$$\lambda(t).v \mapsto (t^{\lambda \cdot a_1} v_1, \cdots, t^{\lambda \cdot a_m} v_m)$$

Therefore, for the point to be unstable, for each nonzero v_i we need $(\lambda_1, \lambda_2) \cdot a^i$ to be positive. Thus, the criterion for instability becomes the following:

Theorem 3.5. *A point x is unstable if and only if for each nonzero v_i in its lift $\tilde{x}$, the convex hull of the associated a_i does not.*

For our proof, recall the Hahn Banach Separation Theorem for a point and a set:

Theorem 3.6 (Hahn-Banach)**.** *Let X be a real topological vector space, A a non-empty convex open subset of X, and $x_0 \notin A$. Then there exists a continuous $\lambda \in X^*$ such that $\lambda a - x_0 > 0$ for every $a \in A$*

Proof of 3.5. If a point is unstable then $(\lambda_1, \lambda_2) \cdot a^i$ is positive for all i. If zero is contained in the convex hull, we can write it as a linear combination of the a^i $0 = \sum c_i a^i$ such that the c_i are positive and sum to 1. Then, multiplying by (λ_1, λ_2) provides a contradiction.

If zero is not contained in the convex hull of the a^i then by Hahn Banach there exists a linear functional such that for all v' in the convex hull of the a^i $\lambda(v' - 0) > 0$. In particular, this is true for the a^i. □

Thus, we have formed a clear connection between a numerical criterion, and a finite problem in convex geometry - if we can find all the maximal convex sets of points that do not include the origin, we have found all possible unstable points up to conjugation. The beauty of this method is that it is a finite search problem - and so we can utilise a computer in order to find the convex sets in complicated, multidimensional cases.

Remark 3.7. One may be concerned here that through this search - a real-valued line may cut out a larger convex set of points than an integer valued one. However, this is not the case - the rationals are dense within the reals and a rational line can be turned into an integer one by clearing denominators

3.3 The Program

In the previous section, we established that there is a clear goal for computing unstable points - simply find the maximal convex sets which do not contain the origin. However, this comes with some computational challenges. Since the cardinality of the power set is 2^n for n the cardinality of X (as a set), computing all possible combinations of points manually is very computationally expensive. For this reason, computing maximal convex sets (not containing zero) via working with polyhedra is the method used in the program that does the computational work within this thesis. This also has the added benefit of being able to utilise a powerful python add-on known as Sage, which provides functions to do some of the more brute-force computation, such as identifying whether the origin was present in a given convex set.
The initial attempt at creating such a program was used with the specific example of GL_3 acting on degree n homogeneous polynomials (where n could be varied). This particular example was simpler, as through a simple affine transformation, one is able to take the points in $\mathbb{R}^3$ and convert them into points in $\mathbb{R}^2$, whereby the algorithm boils down to rotating a hyperplane around $\mathbb{R}^2$. Then, since the points also formed a triangle, one could then compute a minimal increment of rotation for the hyperplane, as well as prevent excess computation due to the rotational symmetry of the points. The code is attached in appendix 3.
The second program was designed to be for any group acting (under the assumed conditions) on any algebraic variety X. Here, a more complicated algorithm was necessary:

1. Begin with a hyperplane not containing any of the points that are being considered

2. Add this polyhedron to a list of polyhedra required to be checked

3. Take a polyhedron H from the list, and for each point $x \in X$, add each point individually to the polyhedron, creating a new polyhedron H', take

its halfspace representation and for each halfspace check whether zero is contained inside it

4. If not, add the halfspace to the list of new polyhedra (note this list is specific to each H)

5. If the list of new polyhedra is empty, this means that no extra points could be added without adding the origin, and so we have a maximal convex set. Add this to a list of maximal sets.

6. If the list is not empty, check it against the list of polyhedra to check for duplicates and for non-duplicates, add them to the list of polyhedra to check.

7. Repeat this for every polyhedron in the list of polyhedra to check

Note this algorithm will eventually terminate, since the number of points is finite, and so the number of polyhedra must also be and it will provide a list of polyhedra that cover the maximal sets. Furthermore, by building polyhedra in this way, the ordering of the points does not matter, and so we avoid checking a large number of potential sets, creating increased efficiency.
Again, sage is instrumental here. It allows for the creation of Polyhedron objects (via both their hyperplane and vertex representations) (which polyhedra are considered to be) as well as then sage contains functions that allow to check whether zero is contained in the polyhedron. The code for this iteration of the program is attached in appendix 2. It is also worth noting that points are read in from a csv file, allowing for simple user input.
Since the primary example we work with is cubal $n \times n \times n$ matrices acted on by SL_n^3, we have symmetries of $S_n \times S_3$ and so some regular expression analysis is used to cut out these additional hyperplanes, dramatically reducing the output data and thereby making the remaining human computation simpler. It is worthwhile, however, to check that this program indeed has lower complexity than the naive algorithm (found in Appendix 2). We assume that we have n points in m dimensions. The complexity of the naive algorithm depends upon the size of m and n. In short, it depends on whether the process of checking the containment of zero is faster than selecting the maximal subsets. The containment check is:

```
for v in vertex_sets:
    check_poly = Polyhedron(vertices = v)
```

```
    if check_poly.contains(zero) == False:
        sets_without_zero +=[v]
```

This has complexity $O(2^n)(O(n)(O(m))) = O(2^n nm)$ since there are 2^n possible maximal sets, each containing a linear number of points relative to n, and checking containment of 0 requires m dot products with each possible hyperplane in the polyhedra, the number of hyperplanes also a linear function relative to n (see Euler's Formula). Alternatively it might be that the removal of non-maximal sets causes the greatest amount of complexity:

```
for v in sets_without_zero:
    contained = False
    for w in sets_without_zero:
        if v != w:
            if set(v).issubset(w):
                contained = True
                break
    if contained == False:
        maximal_sets += [v]
```

Since all the sets could possibly not contain zero, this has complexity $O(2^n 2^n) = O(2^{2n})$. Thus, the naive algorithm either has complexity $O(2^n nm)$ or $O(2^{2n})$.

For the algorithm utilising hyperplanes, the complexity depends on m and the number of points contained in a single large, convex set. For the number of points, all points could be possibly in one convex set for which zero is not included. Since we are working with combinations of points, this means that the number of choices made is

$$\sum_{k=0}^{\frac{n}{2}-1} \binom{(n-k)}{(2+k)}$$

The largest term will be where k is zero, $\frac{n^2-n}{2}$. Note that the point with the greatest possible computational complexity outside of the regex analysis will be the main loop.

```
while i < len(check_list):
    new_planes = []
    poly_points = []
```

```
    current_half = check_list[i]
    poly_to_check = Polyhedron(ieqs = [current_half])
    for y in data_points:
        if poly_to_check.contains(y):
            poly_points +=[y]
    for y in data_points:
        if not poly_to_check.contains(y):
            new_poly = Polyhedron(vertices = (poly_points + [y]))
            new_halves = new_poly.Hrepresentation()
            j = 0
            while j < len(new_halves):
                h = new_halves[j]
                if h.is_inequality():
                    check_half = Polyhedron(ieqs = [h])
                    if not check_half.contains(zero):
                        new_planes = new_planes + [h]
                else:
                    new_ineq = Polyhedron(ieqs =
                    ↪  [tuple(h.vector())]).Hrepresentation()
                    new_halves = new_halves + new_ineq
                    new_ineq = Polyhedron(ieqs = [tuple(-1 *
                    ↪  (h.vector()))]).Hrepresentation()
                    new_halves = new_halves + new_ineq
                j += 1
    if new_planes == []:
        maximal_halves = maximal_halves + [current_half]
    for h in new_planes:
        if h not in check_list:
            check_list = check_list + [h]
    i += 1
```

Since the program will search for the largest hyperplane by adding in points as the choice formula above, the maximum possible length of the hyperplane list will be a quadratic form, and so whatever executes inside the main loop will have complexity $O(n^2)$ times the complexity of the code within the loop. The next largest complexity term will come from the second for loop, since it contains another nested loop, and the first has two $O(1)$ complexity steps. Whatever is inside this loop is then multiplied by $O(n)$. To check containment, the program must execute a dot product of vectors, which has $O(m)$ complexity. Then, since the number of faces of a polyhedron is a linear expression in relation to the number of points, we see that the complexity of the loop is the complexity of the embedded loop times $O(n) + O(m)$. The only nonlinear execution time step inside this last loop is the check for zero being contained, which is another $O(m)$ step. Thus, our program's complexity is:

$$O(n^2)(O(n)(O(m) + O(nm))) = O(n^3m) + O(n^4m)$$

Thus, our program has $O(n^4m)$ complexity. It remains to check that the component of the program doing the regular expression analysis doesn't dwarf this complexity. Recall however, that this regular expression analysis only works for the action of SL_n^3 on $n \times n \times n$ matrices, and would otherwise be removed, as it relies on the symmetries of that particular action

```
for h in maximal_halves:
    ieq_regex = tuple_regex.search(str(h))
    ieq_tuple = ieq_regex.groups()
    ieq_list = list(ieq_tuple)

    k = len(ieq_list)
    m = k/3
    ieq_list_1 = []
    ieq_list_2 = []
    ieq_list_3 = []

    i = 0

    while i < m:
        ieq_list_1 += [ieq_list[i]]
        i+=1
    while i < (2*m):
        ieq_list_2 += [ieq_list[i]]
        i+=1
    while i < (3*m):
        ieq_list_3 += [ieq_list[i]]
        i+=1
```

```
    i = 0
    while i < len(ieq_list_1):
        ieq_list_1[i] = int(ieq_list_1[i])
        i +=1
    sorted_list_1 = sorted(ieq_list_1)
    print(sorted_list_1)
    i = 0
    while i < len(ieq_list_2):
        ieq_list_2[i] = int(ieq_list_2[i])
        i +=1

    sorted_list_2 = sorted(ieq_list_2)

    i = 0
    while i < len(ieq_list_3):
        ieq_list_3[i] = int(ieq_list_3[i])
        i +=1
    sorted_list_3 = sorted(ieq_list_3)

    ieq_list = [sorted_list_1, sorted_list_2, sorted_list_3]
    same_elts = False
    sorted_list = sorted(ieq_list)

    for l in number_list:
        if l == sorted_list:
            same_elts = True
            bad_ineqs +=[h]
            break
    if same_elts == False:
        number_list += [sorted_list]

    j+=1
```

However, at worst the maximal sets are all the initial n^2 hyperplanes, and since the function sorted has m log m complexity, and we apply it to 3 tuples of size $m/3$, we get that the sorting component of the inside of the loop has m log $m/3$

complexity. That is, it has $m \log m$ complexity (since we round to the nearest order of magnitude). The inside for loop will have complexity on its final iteration of the number of unique hyperplanes - which in the worst case (given we still have to check the last plane) is $n^2 - 1$, which we call n^2, again, since we round to the nearest order of magnitude. The operations inside are linear, so the complexity of the sorting algorithm is $O(n^2)(O(n^2) + O(m \log m))$, and $n^4 > n^2 m \log m$ for $m > 1$ $n > \sqrt{m \log m}$ whilst $m > \sqrt{m \log m}$, so as long as $n > m$ the sorting component has n^4 complexity, which is less than $n^4 m$ and otherwisse has $n^2 m \log m$ complexity, and so to check what the complexity of the overall program is in this case, we write $n^4 m > n^2 m \log m$. That is $n^2 > \log m$. For the main example discussed in the thesis, $n > m$, and since $m > \sqrt{m \log m}$ for all positive integers m, we get that $n^2 > \log m$ for all n, m. Thus, for our main example, we have complexity $O(n^4 m)$, whilst in other cases, one may have complexity $O(n^2 m \log m)$. Computing these various combinations of inequalities (even what are possible) is beyond the subject of this thesis. In our main examples, we know that for k the dimension of SL_k, $n = k^3$, and $m = 3k-3 \approx k$. Thus, we substitute $n = m^3$ to get complexities:
Naive:

- $2^{m^3} m^4$
- 4^{m^3}

Hyperplanar:

- m^{13}
- $m^7 \log m$

$2^{m^3} m^4$ is not greater than 4^{m^3} for any natural number. Meanwhile, m^{13} is greater than $m^7 \log m$ for all natural numbers, so we now check when $4^{m^3} > m^{13}$ and this is the case for all natural m. .

3.4 A Quick Explainer on Algebraic Curves

For this section, we will mostly follow Fulton's work in [4]. We provide a short, definition heavy exposition of these objects, since computational examples will appear in the GIT examples in the next section. We will be working over algebraically closed fields k

Put simply, algebraic (or plane) curves are the zero loci of equivalence classes of non-constant polynomials. This equivalence is defined in terms of scaling. That is, $F \sim G$ if $F = \lambda G$ for some $\lambda \in k$. These curves can be both affine or projective, but for the properties of such curves that we are concerned with, the projective case reduces to the affine, and so we will start there.
Consider an affine plane curve f in $\mathbb{A}^n$ (the zero locus of a polynomial in $x_1, \cdots, x_n$). We have a few definitions associated to such a curve:

Definition 3.8. A point P on the curve f is called simple if at least one of the derivatives at P $\frac{df}{dx_i}(P)$ is nonzero. If a point is not simple it is *multiple.* A curve with only simple points is *nonsingular* or *smooth.* A curve with one or more multiple points is called *singular.*

What we are concerned with is the multiplicity (and type) of these multiple points.

Definition 3.9. Write a curve $f = f_m + f_{m+1} + \cdots + f_n$ where f_i is a polynomial of degree i, and f_m is nonzero. We define the *multiplicity* of f at P, the origin, to be m. Notably, if $m = 2$ P is called a double point, and if $m = 3$, P is called a triple point.

To determine the multiplicity of a curve f at a point P not at the origin, simply take the affine transformation that moves the point to the origin and apply it to the curve in the obvious way. Then, use the definition as described above.

Example 3.10. A simple example is the curve $x^2 + y^2 = 0$ (with our field as $\mathbb{C}$). It is 0 at $(0,0)$ as are both its derivatives. Since the curve is made up of only one quadratic form, we see that f has multiplicity 2 at $(0,0)$. That is, $(0,0)$ is a double point.

We can also describe the type of multiple point. We do this using tangent lines.

Definition 3.11. For a curve f with multiplicity m at P the origin, write $F_m = \prod L_i^{r_i}$ we say that P is an *ordinary multiple point* if the r_i are all 1. Otherwise, it is a *non-ordinary multiple point*

This describes the situation with affine curves, but the examples in the next section are all projective. In the instance of a projective plane curve (or indeed for a multihomogeneous curve in m copies of $\mathbb{P}^n$), we simply dehomogenise the curve, setting one (or multiple for the multihomogeneous curve) variable to 1. We can then describe (assume we dehomogenise with respect to the last variable or variables), the multiplicity of the point $[0 : 0 : \cdots : 0 : 1]$, or $([0 : \cdots : 0 : 1], \cdots , [0 : \cdots : 0 : 1])$.

3.5 Three Elementary Examples

Now we have a program that is capable of providing the unstable points of an action of a group on a variety, we look at two rudimentary examples to illustrate the power of the algorithm, but also to demonstrate that often there will be a clean geometric classifier for the set of unstable points. The first example is a simpler form of the second, and the second will be instrumental in considering the work of Bhargava and Ho.

Consider $G = SL_2$ acting on degree 3 homogeneous polynomials in two variables in the standard way.

Theorem 3.12. *The semistable points are the curves that split completely into linear factors.*

Proof. First, note that every point is a linear combination of the following monomials:

- x^3
- y^3
- x^2y
- xy^2

Considering the action of the maximal torus

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

We see that the breakdown into one dimensional representations provides us with the following integers corresponding to each basis element:

- x^3 - 3
- y^3 - -3

- x^2y - 1
- xy^2 - -1

Thus searching for maximal convex sets gives us that polynomials conjugate to $ax^3 + bx^2y = x^2(ax + by)$ are unstable. Note that since the action of G on the curve is $g \cdot P(X,Y) = P(g^{-1} \cdot (X,Y))$ where the action on points of $\mathbb{P}^1$ is induced from the action on polynomials and thus double points will be preserved.

For the reverse direction, consider the polynomial $P(X,Y) = (ax+by)^2(cx+dy)$. If $a = c$, $b = d$ the map $ax + by- > x$ is sufficient to map to an unstable point, and if not take the map $ax + by- > x$, $cx + dy- > y$. □

Thus, we see that for two variable homogeneous cubics, the unstable points are simply the points that have a double root.
Adding an additional variable creates a slightly more subtle classification for unstable points:

Theorem 3.13. *Consider $G = SL_3$ acting on degree 3 homogeneous polynomials in 3 variables in the standard way. Then the unstable points of this action are polynomials with either a non-ordinary double point or a triple point.*

Proof. Consider the action of the maximal torus:

$$\begin{pmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & s^{-1}t^{-1} \end{pmatrix}$$

Again, the breakdown into one dimension representation provides us with data corresponding to each basis elements, this time 2-tuples of integers corresponding to the powers of s and t:

- x^3 - $(3,0)$
- y^3 - $(0,3)$
- z^3 - $(-3,-3)$
- x^2y - $(2,1)$
- x^2z - $(1,-1)$

- xy^2 - $(1,2)$
- y^2z - $(-1,1)$
- xz^2 - $(-1,-2)$
- yz^2 - $(-2,-1)$
- xyz - $(0,0)$

By plotting these points (or using our program to compute the sets), we see that up to conjugation there are two maximal convex sets for which any linear combination of monomials within the sets (that is all monomials are taken from set 1, or all monomials are taken from set 2) will result in an unstable polynomial

- $\{x^3, x^2y, xy^2, y^3, y^2z\}$
- $\{x^3, x^2y, z^3, xz^2, x^2z\}$

Now since sending y to z gets us to every monomial in the second set from the first except for x^2y it suffices to check all possible combinations of monomials in the top set, and then check only the combination of all the monomials in the bottom set. A simple algebraic check of these combinations will demonstrate that each either has a triple point or a non-ordinary double point. For this direction it remains to show that this is conserved under the group action. This is clear however, as if the plane curve has a repeated linear factor as its degree two form at a given point, any affine transformation τ will preserve the linearity of the factor, and so there will be a non-ordinary double point and $\tau(p)$. A triple point is clearly unchanged under affine transformation.

Now consider a plane curve $P(x, y, z)$ with a triple point. Without loss of generality, we can assume that this point is at $[0 : 0 : 1]$, so we can simply make an affine transformation. Dehomogenizing with respect to z, the plane curve must look like $ax^3 + by^3 + cx^2y + dxy^2$, but this is already an unstable point.
Considering a plane curve with a non-ordinary double point, we again can assume the point is at $[0 : 0 : 1]$. Dehomogenizing with repsect to z, the plane curve must look like the triple root curve above plus a repeated root in two variables $(\alpha x + \beta y)^2$. Taking the affine transformation $\alpha x + \beta y \to y$ gives a curve of the form $a'x^3 + b'y^3 + c'x^2y + d'xy^2 + y^2$ - which after homogenization is an unstable point. $\square$

For our last example, consider $G = SL_2^2$ acting on X: bidegree $(2,2)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 3.14. *The unstable points of the action of G on X are curves that possess a non-ordinary double point.*

Proof. Consider the action of the maximal torus:

$$(\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})$$

Yet again, the breakdown into one dimensional representations gives 2-tuples corresponding to the powers of s and t. Our basis (and their corresponding 2-tuples) (with variables (x,y) and (α,β)) are:

- $x^2\alpha^2$ - $(2,2)$
- $x^2\alpha\beta$ - $(2,0)$
- $x^2\beta^2$ - $(2,-2)$
- $y^2\alpha^2$ - $(-2,2)$
- $y^2\alpha\beta$ - $(-2,0)$
- $y^2\beta^2$ - $(-2,-2)$
- $xy\alpha^2$ - $(0,2)$
- $xy\alpha\beta$ - $(0,0)$
- $y^2\alpha\beta$ - $(0,-2)$

Again, we use our program or simply plot in two dimensions to see that up to conjugation we have two maximal convex sets:

- $\{x^2\alpha^2.x^2\alpha\beta, xy\alpha^2.y^2\alpha^2\}$
- $\{x^2\alpha^2, xy\alpha^2, y^2\alpha\beta, y^2\alpha^2\}$

First, let $f = ax^2\alpha^2 + bx^2\alpha\beta + cxy\alpha^2 + dy^2\alpha^2$. Dehomogenising with respect to y and β we see that

$$\begin{aligned} f' &= ax^2\alpha^2 + bx^2\alpha + cx\alpha^2 + d\alpha^2 \\ \frac{df'}{dx} &= 2ax\alpha^2 + 2bx\alpha + c\alpha^2 \\ \frac{df'}{d\alpha} &= 2ax^2\alpha + bx^2 + 2cx\alpha + 2d\alpha \end{aligned}$$

This has a singular point at $(0,0)$ - $([0:1],[0:1])$ after homogenising. We see that the degree 2 form of f' is $d\alpha^2$ and so we have a non-ordinary double point. Now let $f = ax^2\alpha^2 + bxy\alpha^2 + cy^2\alpha\beta + dy^2\alpha^2$, and deohomogenise with respect to x and β.

$$\begin{aligned} f' &= a\alpha^2 + by\alpha^2 + cy^2\alpha + dy^2\alpha^2 \\ \frac{df'}{dy} &= b\alpha^2 + 2cy\alpha + 2dy\alpha^2 \\ \frac{df}{d\alpha} &= 2a\alpha + 2by\alpha + cy^2 + 2dy^2\alpha \end{aligned}$$

So we have a singular point at $(0,0)$ - $([1:0],[0:1])$ and the degree two form $a\alpha^2$ has a repeated root, so again we have a non-ordinary double point. To see that this is closed under the action of SL_2^2, note that as in the last two examples, affine transformations do not affect the multiplicity of a form, and so we always have a non-ordinary double point.

□

For our last example, we will see a slightly more specific geometric criterion. Consider $G = SL_2^2$ acting on X: bidegree $(2,2)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 3.15. *The unstable points of the action of G on X are reducible curves containing a fiber and a curve which intersects the fiber only once.*

Proof. Consider the action of the maximal torus:

$$\left(\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)$$

Yet again, the breakdown into one dimensional representations gives 2-tuples corresponding to the powers of s and t. Our basis (and their corresponding 2-tuples) (with variables (x,y) and (α,β)) are:

- $x^2\alpha^2$ - $(2,2)$
- $x^2\alpha\beta$ - $(2,0)$
- $x^2\beta^2$ - $(2,-2)$
- $y^2\alpha^2$ - $(-2,2)$
- $y^2\alpha\beta$ - $(-2,0)$
- $y^2\beta^2$ - $(-2,-2)$
- $xy\alpha^2$ - $(0,2)$
- $xy\alpha\beta$ - $(0,0)$
- $y^2\alpha\beta$ - $(0,-2)$

Again, we use our program or simply plot in two dimensions to see that up to conjugation we have two maximal convex sets:

- $\{x^2\alpha^2.x^2\alpha\beta, xy\alpha^2.y^2\alpha^2\}$
- $\{x^2\alpha^2, xy\alpha^2, y^2\alpha\beta, y^2\alpha^2\}$

First, let $f = ax^2\alpha^2 + bx^2\alpha\beta + cxy\alpha^2 + dy^2\alpha^2$. Then f is divisible by α. That is, its zero locus contains the fiber $\alpha = 0$. Furthermore, $f = \alpha g$, where $g = ax^2\alpha + bx^2\beta + cxy\alpha + dy^2\alpha$. At $\alpha = 0$, $g = bx^2\beta$, and so $g = 0$ at precisely one point when $\alpha = 0$, $x = 0$. or $[0:1], [0:1]$.
Second, let $f = ax^2\alpha^2 + bxy\alpha^2 + cy^2\alpha\beta + dy^2\alpha^2$. Then $f = \alpha g$ where $g = ax^2\alpha + bxy\alpha + cy^2\beta + dy^2\alpha$. Again, setting $\alpha = 0$, $g = cy^2\beta$ and so its zero locus only has a single point of intersection, this time at $[1:0], [0:1]$.
Note that the property of having a zero fiber and one point of intersection is invariant under the action of G, since the action of the second factor only moves where the fiber is, and the action of the first factor only moves where the point of intersection is.
To begin, note that if we have the same curve, except swapping x with α and y with β, it is still unstable, since one can just swap the order of the two factors of SL_2^2. Thus, without loss of generality, we can assume that a curve f with a fiber has a fiber at $(a\alpha + b\beta)$, and then without further loss of generality, we can assume the fiber is at $\alpha = 0$, since we can take the curve under the affine transformation $a\alpha + b\beta - > \alpha$, and so $f = \alpha g$. Further, suppose the curve has

only one point of intersection at $\alpha = 0$. This must be at $\beta = 1$, and so we must have a double point in g at $\beta = 1$. Thus, the quadratic form of g homogenised at β must look like $(ax + by)^2$, but then we simply take the affine transformation $ax + by \mapsto x$ to get a curve in the form of the first unstable point. $\square$

Chapter 4

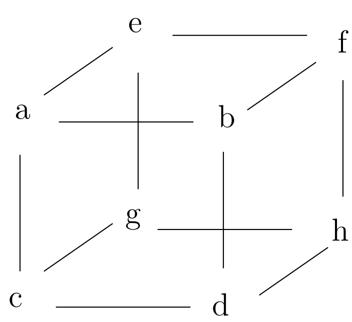
Cubal Matrices

4.1 $2 \times 2 \times 2$ and $3 \times 3 \times 3$ Cubal Matrices

The remainder of this thesis will be dedicated to studying the GIT quotient that is the action of SL_n^3 on $n \times n \times n$ cubal matrices for an algebraically closed field k where $\text{char}(k) = 0$. We will first describe the action in the simple case of $n = 2$, after which we will consider the simplest interesting case for unstable point computation: $n = 3$. Finally, using computed examples for $n = 4$ and $n = 5$, we come up with a generalised solution for all n.

4.1.1 $2 \times 2 \times 2$ Matrices

We first provide some background from Bhargava on the $2 \times 2 \times 2$ cubal matrices. We take C_2 to be $k^2 \otimes k^2 \otimes k^2$. :



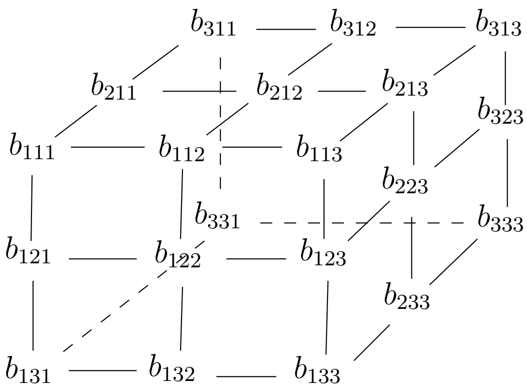
The cube can be partitioned 3 different ways into $2x2$ matrices:

$$M_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad N_1 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$
$$M_2 = \begin{bmatrix} a & c \\ e & g \end{bmatrix}, \qquad N_2 = \begin{bmatrix} b & d \\ f & h \end{bmatrix}$$
$$M_3 = \begin{bmatrix} a & e \\ b & f \end{bmatrix}, \qquad N_3 = \begin{bmatrix} c & g \\ d & h \end{bmatrix}$$

We define our action of SL_2^3 in the following way: for $\Gamma_i = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ the ith component of the tuple, we replace (M_i, N_i) by $(rM_i + sN_i, tM_i + uN_i)$. One can check that the actions of the components Γ_i commute, and so the action is well-defined and natural.

4.1.2 $3 \times 3 \times 3$ Matrices

In the same way we can consider the action of SL_3^3 on $3 \times 3 \times 3$ cubes ($V_i \otimes V_j \otimes V_k$, all 3 dimensional vector spaces over k) which we label as follows:



Again, like the $2 \times 2 \times 2$ case, the action is defined with respect to partitions on the cube. The partitions are:

1. $M_1 = (b_{1jk})$, $N_1 = (b_{2jk})$, $P_1 = (b_{3jk})$
2. $M_2 = (b_{i1k})$, $N_1 = (b_{i2k})$, $P_2 = (b_{i3k})$
3. $M_3 = (b_{ij1})$, $N_3 = (b_{ij2})$, $P_3 = (b_{ij3})$.

Here, the 1 slices represent the basis elements of V_i tensor the rest, the 2 slices representing the basis elements of V_j tensor the rest and the 3 slices representing the basis elements of V_k. Thus, the action of a group element

$$A = (\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix})$$

is

$$(M_1, N_1, P_1) \mapsto (a_{11}M_1+a_{12}N_1+a_{13}P_1, a_{21}M_1+a_{22}N_1, a_{23}P_1, a_{31}M_1+a_{32}N_1+a_{33}P_1)$$

for (M_1, N_1, P_1), then (the new) (M_2, N_2, P_2) gets mapped in the same way with the b_{ij} and (M_3, P_3, N_3) gets acted on in the same way again with the c_{ij}.
A maximal torus for SL_3^3 is:

$$(\begin{bmatrix} s_1 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & s_1^{-1}t_1^{-1} \end{bmatrix}, \begin{bmatrix} s_2 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & s_2^{-1}t_2^{-1} \end{bmatrix}, \begin{bmatrix} s_3 & 0 & 0 \\ 0 & t_3 & 0 \\ 0 & 0 & s_3^{-1}t_3^{-1} \end{bmatrix})$$

Thus, we have a one dimensional representation decomposition made up of 27 elements whereby the "matrices" B_{ijk} are the matrices where $b_{ijk} = 1$ and all else 0. Let $f : \{1, 2, 3\} \to \{1, 0, -1\}$ be a function defined as $f(1) = 1$, $f(2) = 0$, $f(3) = -1$ and let $g : \{1, 2, 3\} \to \{1, 0, -1\}$ be a function defined as $g(1) = 0$, $g(2) = 1$, $g(3) = -1$ Then our representations are:

$$B_{ijk} \to s_1^{f(i)}t_1^{g(i)}s_2^{f(j)}t_2^{g(j)}s_3^{f(k)}t_3^{g(k)}B_{ijk}$$

This provides us with the following 27 tuples:

$$\begin{aligned}
\{(1,0,1,0,1,0),(1,0,1,0,0,1),(1,0,1,0,-1,-1),(1,0,0,1,1,0),\\
(1,0,0,1,0,1),(1,0,0,1,-1,-1),(1,0,-1,-1,1,0),(1,0,-1,-1,0,1),\\
(1,0,-1,-1,-1,-1),(0,1,1,0,1,0),(0,1,1,0,0,1),(0,1,1,0,-1,-1),\\
(0,1,0,1,1,0),(0,1,0,1,0,1),(0,1,0,1,-1,-1),(0,1,-1,-1,1,0),\\
(0,1,-1,-1,0,1),(0,1,-1,-1,-1,-1),(-1,-1,1,0,1,0),(-1,-1,1,0,0,1),\\
(-1,-1,1,0,-1,-1),(-1,-1,0,1,1,0),(-1,-1,0,1,0,1),(-1,-1,0,1,-1,-1)\\
(-1,-1,-1,-1,1,0),(-1,-1,-1,-1,0,1),(-1,-1,-1,-1,-1,-1)\}
\end{aligned}$$

The program then provides us with the three types of unstable points. For further simplicity, we will write cubal matrices as separating the forward to back slices of the cube by lines, rather than drawing it in three dimensions. We will also represent (potentially) nonzero elements as asterisks. The three types are as follows:

$$A : \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B : \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}$$

$$C : \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Type A (up to coordinate change) is cut out by the hyperplane parametrised by $(1, 1, -2)$, $(0, 0, 0)$, $(0, 0, 0)$. Type B by $(2, -1, -1)$, $(0, 0, 0)$, $(2, -1, -1)$ and Type C by $(3, 0, -3)$, $(2, -1, -1)$, $(2, -1, -1)$. Thus, we can conclude the following linear algebraic condition for the instability of points:

Proposition 4.1. *A $3 \times 3 \times 3$ matrix is unstable under the action of SL_3^3 as described above if and only if it either:*

1. *There exists a codimension 1 subspace $S \in V_i^*$ such that $\phi(S \otimes V_i^*) = 0$*
2. *There exists a codimension m subspace S in V_j^*, a codimension $k - m$ subspace T in V_k^*, $k \leq 1$ such that $\phi(S \otimes T)$ is at most $(3-k-1)$-dimensional.*
3. *There exists a codimension m subspace S in V_j^*, a codimension $k - m$ subspace T in V_k^*, $k \leq 2$ such that for $\overline{S} = S \otimes V_k^*$, $\overline{T} = V_j^* \otimes T$, $\phi(\overline{S} \cap \overline{T})$ is at most $(3-k)$-dimensional and $\phi(\overline{S} \cup \overline{T})$ is at most $(n-1)$-dimensional.*

4.2 Higher Dimensions and a Conjecture for Integer Generality

For general n, we write $n \times n \times n$ matrices B as $V_i \otimes V_j \otimes V_k$, n-dimensional k vector spaces. We can partition these matrices up into 3 sets of n slices $(A_{11}, A_{12}, \cdots, A_{1n})$, $(A_{21}, \cdots, A_{2n})$, $(A_{31}, \cdots, A_{3n})$ in the same way by taking subspaces - fixing basis elements of the Vs, and we again get the same natural action. For (A, B, C) $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $A_{1j} = \sum_{i=1}^{n} a_{ij} A_{1i}$, $A_{2j} =$

$\sum_{i=1}^{n} b_{ij}A_{2i}$, $A_{3j} = \sum_{i=1}^{n} c_{ij}A_{3i}$. A maximal torus for SL_n^3 is the set of 3 diagonal matrices whose diagonals are $(s_1, s_2, \cdots, s_{n-1}, s_1^{-1} \cdots s_{n-1}^{-1})$, $(t_1, t_2, \cdots, t_{n-1}, t_1^{-1} \cdots t_{n-1}^{-1})$, and $(v_1, v_2, \cdots, v_{n-1}, v_1^{-1} \cdots v_{n-1}^{-1})$ respectively. Thus, we have a one dimensional representation decomposition made up of n^3 elements whereby the "matrices" B_{ijk} are the matrices where $b_{ijk} = 1$ and all else 0. Much like the $3 \times 3 \times 3$ case*, our representations split into orderered triples of $n-1$-tuples corresponding to an element's position in each of the slices - the first tuple corresponding to its position front to back in the cube, the second top to bottom, and the third left to right. No matter which choice of partition, being in the ith slice makes the tuple for that partition either a 1 at the ith component of the tuple and zero elswhere for $i \leq n-1$, or, if $i = n$ -1 everywhere.

Doing this for $n = 4$, and running the 64 points through the program gives us 5 types of unstable points:

$$A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$C = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

*You could again use functions for this but that would be fairly unsightly to write out

Here, the types are cut out by the following tuples:

- A – $(1,1,1,-1)$, $(0,0,0,0)$, $(0,0,0,0)$
- B – $(2,-2,-2,2)$, $(3,-1,-1,-1)$, $(0,0,0,0)$
- C – $(3,-1,-1,-1)$, $(3,-1,-1,-1)$, $(-1,3,-1,-1)$
- D – $(3,3,-1,-5)$, $(3,-1,-1,-1)$, $(3,-1,-1,-1)$
- E – $(4,0,0,-4)$, $(3,-1,-1,-1)$, $(2,2,-2,-2)$

We can see a pattern starting to emerge when we compare the 3 and 4-dimensional cases - there is always a type that has all but one matrix full, then types where some number of rows and columns are full, and are repeated along all not entirely full slices, and finally types where all but one of the non full slices are the same, with the one different one being the "intersection" of the rows and columns of the non-full slices. This will be confirmed in the $n = 5$ case.
For $n = 5$, we run the 125 points through the program and it produces 8 types:

$$A = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

$$C = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

$$D = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

$$E = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & 0 & 0 & * \\ * & * & 0 & 0 & * \\ * & * & 0 & 0 & * \\ * & * & 0 & 0 & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & 0 & 0 & * \\ * & * & 0 & 0 & * \\ * & * & 0 & 0 & * \\ * & * & 0 & 0 & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & 0 & 0 & * \\ * & * & 0 & 0 & * \\ * & * & 0 & 0 & * \\ * & * & 0 & 0 & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Bigg| \begin{bmatrix} * & * & * & * & * \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The types are cut out by the following tuples:

- A – $(1, 1, 1, 1, 1)$, $(0, 0, 0, 0, 0)$, $(0, 0, 0, 0, 0)$
- B – $(3, -2, -2, -2, 3)$, $(0, 0, 0, 0, 0)$, $(3, -2, -2, -2, 3)$
- C – $(2, 2, -3, -3, 2)$, $(0, 0, 0, 0, 0)$, $(4, -1, -1, -1, -1)$
- D – $(3, -2, -2, -2, 3)$, $(4, -1, -1, -1, -1)$, $(4, -1, -1, -1, -1)$
- E – $(5, 0, 0, 0, -5)$, $(4, -1, -1, -1, -1)$, $(2, 2, -3, -3, 2)$
- F – $(4, 4, -1, -1, -6)$, $(4, -1, -1, -1, -1)$, $(3, -2, -2, -2, 3)$
- G – $(5, 0, 0, 0, -5)$, $(3, 3, -2, -2, -2)$, $(3, -2, -2, -2, 3)$
- H – $(3, 3, 3, -2, -7)$, $(4, -1, -1, -1, -1)$, $(4, -1, -1, -1, -1)$

Thus, we again see the same types emerging, and a pattern that is being kept to as dimension increases. Indeed, we conjecture that:

Conjecture A. *For n dimensional vector spaces V_i, V_j, V_k, consider the map $\phi : V_i^* \otimes V_j^* \to V_k$*[†]*. A matrix M in $V_i \otimes V_j \otimes V_k$ $n \times n \times n$ dimensional is unstable with respect to the action of SL_n^3 if and only if either:*

1. *There exists a codimension 1 subspace $S \in V_i^*$ such that $\phi(S \otimes V_i^*) = 0$*

2. *There exists a codimension m subspace S in V_j^*, a codimension $k - m$ subspace T in V_k^*, $k \leq n - 2$ such that $\phi(S \otimes T)$ is at most $(n - k - 1)$-dimensional.*

3. *There exists a codimension m subspace S in V_j^*, a codimension $k - m$ subspace T in V_k^*, $k \leq n - 1$ such that for $\overline{S} = S \otimes V_k^*$, $\overline{T} = V_j^* \otimes T$, $\phi(\overline{S} \cap \overline{T})$ is at most $(n - k)$-dimensional and $\phi(\overline{S} \cup \overline{T})$ is at most $(n - 1)$-dimensional.*

Since we have such a regular pattern, the if (or forward) direction is very simple:

Proof of forward direction. For unstable points, assume that all full matrices are grouped in the forward slices, and and that subspaces in V_k^* correspond to the leftmost $k - m$ columns, and subspaces in V_j^* correspond to the topmost m rows.
Type 1 can be destabilised by $(1, 1, \cdots, 1, 1 - n), (0, \cdots, 0), (0, \cdots, 0)$
Type 2 can be destabilised by [‡]

1. $(k + 1, \cdots k + 1, k + 1 - n, \cdots, k + 1 - n)$

2. $(n - m, \cdots, n - m, -m, \cdots, -m)$

3. $(n + m - k, \cdots, n + m - k, m - k, \cdots, m - k)$

Type 3 can be destabilised by[§]

1. $(k + 1, \cdots, k + 1, k + 1 - n, \cdots, k + 1 - n, k + 1 - 2n)$

2. $(n - m, \cdots, n - m, -m, \cdots, -m)$

3. $(n + m - k, \cdots, n + m - k, m - k, \cdots, m - k)$

[†]note i j k may vary to account for swapping of matrix (in SL_n^3) order

[‡]in the first tuple the first grouping lasts for $n - k - 1$ many entries, in the second tuple the first grouping lasts for m many entries adn in the third the first grouping lasts for $k - m$ many entries

[§]in the first tuple the first group lasts for $n - k$ many entries, the second group lasting for $k - 1$ many entries. The other groupings last as long as in the previous footnote.

□

However, the proof of the reverse direction is unclear. For $n = 3$, 4 and 5, the program has proved the result for us. However, even for $n = 5$, the execution time became quite long (around 14 hours), and testing for higher dimensions would be impractical, not to mention it would not prove the result in general. The obvious method to try would be to replicate what the program is doing for arbitrary n, but the cases at first glance seem to be too numerous to deal with in an efficient way.

Appendix A

Complete Program

```
import csv
import time
import re

start_time = time.time()

data_points = []
with open('data_points.txt') as csv_file:
    csv_reader = csv.reader(csv_file, delimiter=',')
    for row in csv_reader:
        new_point = ()
        for x in row:
            num_x = int(x)
            new_point = new_point + (num_x,)
        data_points = data_points + [new_point]

default_points =
↪  [(3,0),(2,1),(1,2),(0,3),(-1,1),(-2,-1),(-3,-3),(-1,-2),(0,0),(1,-1)]

if data_points == []:
    data_points = default_points

min_x = 10000000
for y in data_points:
    if y[0] < min_x:
```

```
        min_x = y[0]

entry = min_x - 1
print(entry)

point_poly = Polyhedron(vertices = data_points)
center = point_poly.center()
dimension = len(data_points[0])
number_tuple = (entry,-1)
zero_tuple = (0,)*(dimension - 1)
zero = (0,)*dimension
default_half = number_tuple + zero_tuple

regex_string = '\(' + '([-]?[0-9]+), '*(dimension - 1) +
↪  '([-]?[0-9]+)\)'
tuple_regex = re.compile(regex_string)

testing = Polyhedron(ieqs = [default_half])
points_in_testing = []
for x in data_points:
    if testing.contains(x):
        points_in_testing +=[x]

print(points_in_testing)

check_list = [default_half]

i = 0
maximal_halves = []
while i < len(check_list):
    new_planes = []
    poly_points = []
    current_half = check_list[i]
    poly_to_check = Polyhedron(ieqs = [current_half])
    for y in data_points:
        if poly_to_check.contains(y):
            poly_points +=[y]
```

```
    for y in data_points:
        if not poly_to_check.contains(y):
            new_poly = Polyhedron(vertices = (poly_points + [y]))
            new_halves = new_poly.Hrepresentation()
            j = 0
            while j < len(new_halves):
                h = new_halves[j]
                if h.is_inequality():
                    check_half = Polyhedron(ieqs = [h])
                    if not check_half.contains(zero):
                        new_planes = new_planes + [h]
                else:
                    new_ineq = Polyhedron(ieqs =
                     ↪  [tuple(h.vector())]).Hrepresentation()
                    new_halves = new_halves + new_ineq
                    new_ineq = Polyhedron(ieqs = [tuple(-1 *
                     ↪  (h.vector()))]).Hrepresentation()
                    new_halves = new_halves + new_ineq
                j += 1
    if new_planes == []:
        maximal_halves = maximal_halves + [current_half]
    for h in new_planes:
        if h not in check_list:
            check_list = check_list + [h]
    i += 1

number_list = []
bad_ineqs = []
j = 0
for h in maximal_halves:
    ieq_regex = tuple_regex.search(str(h))
    ieq_tuple = ieq_regex.groups()
    ieq_list = list(ieq_tuple)

    k = len(ieq_list)
    m = k/3
    ieq_list_1 = []
```

```
ieq_list_2 = []
ieq_list_3 = []

i = 0

while i < m:
    ieq_list_1 += [ieq_list[i]]
    i+=1
while i < (2*m):
    ieq_list_2 += [ieq_list[i]]
    i+=1
while i < (3*m):
    ieq_list_3 += [ieq_list[i]]
    i+=1

i = 0
while i < len(ieq_list_1):
    ieq_list_1[i] = int(ieq_list_1[i])
    i +=1
sorted_list_1 = sorted(ieq_list_1)
print(sorted_list_1)
i = 0
while i < len(ieq_list_2):
    ieq_list_2[i] = int(ieq_list_2[i])
    i +=1

sorted_list_2 = sorted(ieq_list_2)

i = 0
while i < len(ieq_list_3):
    ieq_list_3[i] = int(ieq_list_3[i])
    i +=1
sorted_list_3 = sorted(ieq_list_3)

ieq_list = [sorted_list_1, sorted_list_2, sorted_list_3]
same_elts = False
sorted_list = sorted(ieq_list)
```

```
    for l in number_list:
        if l == sorted_list:
            same_elts = True
            bad_ineqs +=[h]
            break
    if same_elts == False:
        number_list += [sorted_list]

    j+=1

maximal_halves = [ele for ele in maximal_halves if ele not in
↪   bad_ineqs]

message = " contains points "
print(len(maximal_halves))
for h in maximal_halves:
    j = Polyhedron(ieqs =[h])
    points_in = []
    for x in data_points:
        if j.contains(x):
            points_in +=[x]
    str_h = str(h)
    str_pts = str(points_in)
    print(str_h + message + str_pts)

print("--- %s seconds ---" % (time.time() - start_time))
```

Appendix B

Naive Algorithm

```
import csv
import time
import re

start_time = time.time()

data_points = []
with open('data_points.txt') as csv_file:
    csv_reader = csv.reader(csv_file, delimiter=',')
    for row in csv_reader:
        new_point = ()
        for x in row:
            num_x = int(x)
            new_point = new_point + (num_x,)
        data_points = data_points + [new_point]

default_points = [(3,0),(2,1),(1,2),(0,3),(-1,1),(-2,-1),(-3,-3),(-1,-2),(0,0)

if data_points == []:
    data_points = default_points

vertex_sets = powerset(data_points)
sets_without_zero = []

zero = (0,)*dimension
```

```
for v in vertex_sets:
    check_poly = Polyhedron(vertices = v)
    if check_poly.contains(zero) == False:
        sets_without_zero +=[v]

maximal_sets = []

for v in sets_without_zero:
    contained = False
    for w in sets_without_zero:
        if v != w:
            if set(v).issubset(w):
                contained = True
                break
    if contained == False:
        maximal_sets += [v]

print("--- %s seconds ---" % (time.time() - start_time))
```

Appendix C

Specific Algorithm

```
import itertools
import math
import matplotlib.pyplot as plt
import time
import copy

degree_string = input("What degree: ")
#graph_string = input("Do you want to see the graphs? Answer 'y'
↪  or 'n': ")
graph_string = 'n'

#Sets up our list of homogeneous 3 tuples
degree = int(degree_string)

range_list = list(range(degree + 1))

before_list = list(itertools.product(range_list, range_list,
↪  range_list))

tuple_list = [x for x in before_list if x[0] + x[1] + x[2] ==
↪  degree]

working_list = [list(x) for x in tuple_list]
```

```
#Take our list of 3-tuples in R^3 and take the plane containing
↪ them and transform that into a copy of R^2 centered at the
↪ middle of the triangle
a = (2 * math.sqrt(3) * degree)/3

def plane_convert(list_in):
    list_in[0] = -a/(2 * degree) * list_in[0] + a/(2 * degree) *
     ↪ list_in[2]
    list_in[1] = list_in[1] - degree/3.0
    return list_in

def plane_inverse(list_in):
    list_in[0] = list_in[2] - ((2 * degree)/a) * list_in[0]
    list_in[1] = list_in[1] + degree/3.0
    return list_in

convert_list = [plane_convert(x) for x in working_list]

#Set up this triangle for graphing if wanted by getting a list of
↪ x and y coordinates.
x_list = []
y_list = []
for x in convert_list:
    x_list = x_list + [x[0]]
    y_list = y_list + [x[1]]

#Since we're only checking the first 60 degrees, the bottom right
↪ quadrant will never show up so computationally we can delete
↪ it and not have to worry. This is also why we get our list of
↪ graphed points before this step.
for x in convert_list:
    if x[0] > 0 and x[1] < 0:
        convert_list.remove(x)

x_mul = 0
```

```
#We can calculate the minimum angular distance between any two
↪  points in our triangle by looking at the bottom right and
↪  next closest outer point.
corner_point3 = plane_convert([degree, 0, 0])
near_point3 = plane_convert([degree-1 , 1, 0])
corner_point = [corner_point3[0], corner_point3[1] ]
near_point = [near_point3[0], near_point3[1]]

#long_side is the long side of the triangle, near_side is the
↪  side to the near point and between_side is the side between
↪  the two points
long_side = math.sqrt(corner_point[0]**2 + corner_point[1]**2)
near_side = math.sqrt(near_point[0]**2 + near_point[1]**2)

x_diff = corner_point[0] - near_point[0]
y_diff = corner_point[1] - near_point[1]

between_side = math.sqrt(x_diff**2 + y_diff**2)

big_value = 1/(2* long_side * near_side) * (long_side**2 +
↪  near_side**2 - between_side**2)
minimum_angle = math.acos(big_value)

extreme_x_right = (plane_convert([degree, 0, 0]))[0]
extreme_x_left = (plane_convert([0, 0, degree]))[0]

point_list = []
max_set_list = []
angle = 0

#Now, while the angle is less than 60, we check the points in the
↪  "left" halfspace generated by the line through the origin at
↪  that angle. If these points are contained in a set of points
↪  for a previous halfspace, we discard them. If not, they are
↪  added to a general list of points, the length of which tells
↪  us the number of different unstable polynomials we have.
while angle <= math.pi/3:
```

```
for x in convert_list:
    if x[1] - x_mul*x[0] > 0:
        point_list = point_list + [x]

if max_set_list == []:
    max_set_list = max_set_list + [point_list]

for x in max_set_list:
    contains = all(elem in x for elem in point_list)
    if contains == True:
        break
    rev_contain = all(elem in point_list for elem in x)
    if rev_contain == True:
        max_set_list.remove(x)

if contains == False:
    max_set_list = max_set_list + [copy.deepcopy(point_list)]


x_listp = []
y_listp = []
for x in point_list:
    x_listp = x_listp + [x[0]]
    y_listp = y_listp + [x[1]]

#Optional graphing - it's a little bit awkward at the moment
↪  since you have to close each window as it arrives.
if graph_string == "y":
    plt.plot([extreme_x_left, extreme_x_right],
     ↪  [x_mul*extreme_x_left, x_mul*extreme_x_right], '-g')
    plt.plot(x_list,y_list, 'ro')
    plt.plot(x_listp, y_listp, 'bo')
    plt.axis('equal')
    plt.grid()
    plt.show()

angle = angle + minimum_angle
```

```
    x_mul = math.tan(angle)
    point_list = []

for x in max_set_list:
    for y in x:
        y = plane_inverse(y)

print(max_set_list)
```

Bibliography

[1] M. Bhargava. Higher composition laws i: A new view on gauss composition, and quadratic generalizations. *Annals of Mathematics*, 159(1):217–250, 2004.

[2] M. Bhargava and W. Ho. Coregular spaces and genus one curves. *Cambridge Journal of Mathematics*, 4(1):1–119, 2016.

[3] I. S. Cohen. On the structure and ideal theory of complete local rings. *Transactions of the American Mathematical Society*, 59(1):54–106, Jan 1946.

[4] W. Fulton. *Algebraic curves: an introduction to algebraic geometry.* Addison-Wesley, 1989.

[5] V. Hoskins. Moduli problems and geometric invariant theory.

[6] J. E. Humphreys. *Linear algebraic groups.* Springer, 2004.

[7] D. Mumford, J. Fogarty, and F. C. Kirwan. *Geometric invariant theory.* Springer, 1994.

[8] P.-L. Yung. A proof of simplicity of sln.