On the Critical loci of finite maps

§1. Critical points of rational functions.

§2. The general setup.

§3. Criteria for finite degree

↳ Background - Varieties of minimal degree

§4. Degrees

- ↳ Case of rat. norm. curves.
- ↳ Case of quadric hypersurface
- ↳ Case of Veronese $\mathbb{P}^2$
- ↳ Case of surface scrolls.
- ↳ Speculation — higher dim scrolls.

§5. Broader points.

→ Why can we do curves but not higher dimensional varieties?

§1. Rational functions

$$f(z) = \frac{p(z)}{q(z)} \qquad p, q \in \mathbb{C}[z] \text{ of degree } n.$$

Critical Locus (f) = Ramification Divisor (f) = $\mathrm{Ram}(f)$

$$:= \{ z \mid f'(z) = 0 \}.$$

<u>Count</u>: $f'(z) = \dfrac{p'q - q'p}{q^2}$

$$\deg(p'q - q'p) = 2n-2.$$

Expect $\# \mathrm{Ram}(f) = 2n-2.$

i.e. $\mathrm{Ram}(f) \in \mathrm{Sym}^{2n-2}\mathbb{C}$

$\#$ coefficients of $f = 2n+1$,

i.e. $f \in \mathbb{C}^{2n+1}.$

Obs. f & $\varphi \circ f$ have same ram. div.
where $\varphi \in PGL_2(\mathbb{C}) = \left\{ \dfrac{az+b}{cz+d} \right\}.$

So

$$\begin{array}{ccc}
\{f\}/PGL_2 & \overset{\mathrm{Ram}}{\underset{\mathrm{Crit}}{\dashrightarrow}} & \{2n-2 \text{ points on } \mathbb{C}\} \\
\wr\wr & & \wr\wr \\
\mathbb{C}^{2n+1}/PGL_2 & \dashrightarrow & \mathbb{C}^{2n-2} \\
\downarrow & & \downarrow \\
\dim = 2n-2 & & \dim = 2n-2.
\end{array}$$

$\Rightarrow$ Given a general $D \in \mathbb{C}^{2n-2}$, there are finitely many f s.t. $\mathrm{Ram}(f) = D$.

Thm (Castelnuovo 1889, perhaps earlier) :-
Given a general $D \in \mathrm{Sym}^{2n-2}(\mathbb{C})$, there are $\frac{1}{n}\binom{2n-2}{n-1}$ rational functions f with $D = \mathrm{Ram}(f)$.

i.e. The degree of Ram_n is $\mathrm{Catalan}_{n-1}$.

Proof - Compactify everything.

Homogenize f to $F : \mathbb{P}^1 \to \mathbb{P}^1$ degree n
Then
$$\mathrm{Ram}(F) \in \mathrm{Sym}^{2n-2}(\mathbb{P}^1) \cong \mathbb{P}^{2n-2}.$$

$$\{F : \mathbb{P}^1 \to \mathbb{P}^1\} / PGL_2(\text{Target}) \overset{\mathrm{Ram}}{\dashrightarrow} \mathbb{P}^{2n-2}$$

$$F = [F_0 : F_1], \quad F_i \in \mathrm{Sym}^n(\mathbb{C}^2)$$

$$F / PGL_2 \longleftrightarrow \mathrm{Span}\langle F_0, F_1 \rangle \subset \mathrm{Sym}^n(\mathbb{C}^2)$$
i.e. $\in Gr(2, \mathrm{Sym}^n \mathbb{C}^2)$

So $Gr(2, \mathrm{Sym}^n \mathbb{C}^2) \overset{\mathrm{Ram}}{\dashrightarrow} \mathbb{P}^{2n-2}$

Key - Ram extends to a regular map!

$$\mathrm{Ram} : H^*(\mathbb{P}^{2n-2}) \to H^*(G(2, \mathrm{Sym}^n \mathbb{C}^2))$$

$$\mathrm{degree}(\mathrm{Ram}) = \mathrm{degree}\ \mathrm{Ram}^* c_1(\mathcal{O}(1))^{2n-2}$$
$\hookrightarrow$ Schubert calculus.

$\square$

E. Generalization

$k = \mathbb{C}$. Variety = integral scheme / k.

X a smooth proj. variety of dim n
L a very ample line bundle on X.

(imagine $X \subset \mathbb{P}^N$ & $L = \mathcal{O}(1)|_X$).

Set $V = H^0(X, L)$.
$(n+1)$ generic elements $s_0, \dots, s_n \in V$ give

$$\varphi : X \longrightarrow \mathbb{P}^n$$
$$x \longmapsto [s_0(x) : \dots : s_n(x)]$$

Define $\mathrm{Ram}(\varphi) \subset X$ divisor as follows.

$$d\varphi : T_X \to \varphi^* T_{\mathbb{P}^n}$$

$\mathrm{Ram}(\varphi)$ = Zero locus of $\det(d\varphi)$.
$\det(\varphi)$ = section of $K_X \otimes \varphi^* K_{\mathbb{P}^n}^\vee$
$$\qquad\qquad\qquad\qquad \| $$
$$K_X \otimes L^{n+1}$$

So we have a rational map

$$pr : \underbrace{Gr(n+1, H^0(X, L))}_{G} \dashrightarrow \underbrace{\mathbb{P}H^0(X, K_X \otimes L^{n+1})}_{P}$$

$$G \dashrightarrow P$$

<u>Main result</u> - when is this finite?
<u>Question</u> - what is the degree?

<u>Ex.</u> $X = \mathbb{P}^1$, $L = \mathcal{O}(n)$, then the map is finite of degree $\frac{1}{n}\binom{2n-2}{n-1}$

Thm 1: X a sm. proj variety & L a very ample line bundle. We have

$$\dim G \leq \dim P$$

with equality if and only if (X, L) is a variety of minimal degree.

E. Varieties of minimal degree

$X \subset \mathbb{P}^N$ a non-degenerate sm proj variety. Then

$$\deg(X) \geq \operatorname{codim}(X) + 1$$

Varieties of minimal degree are those where equality holds. They are —

(0) $X = \mathbb{P}^N$.
(1) $X \subset \mathbb{P}^N$ a quadric hypersurface
(2) $X \cong \mathbb{P}^2 \subset \mathbb{P}^5$ by $\mathcal{O}(2)$.
(Veronese surface)
(3) X is a rational normal scroll.

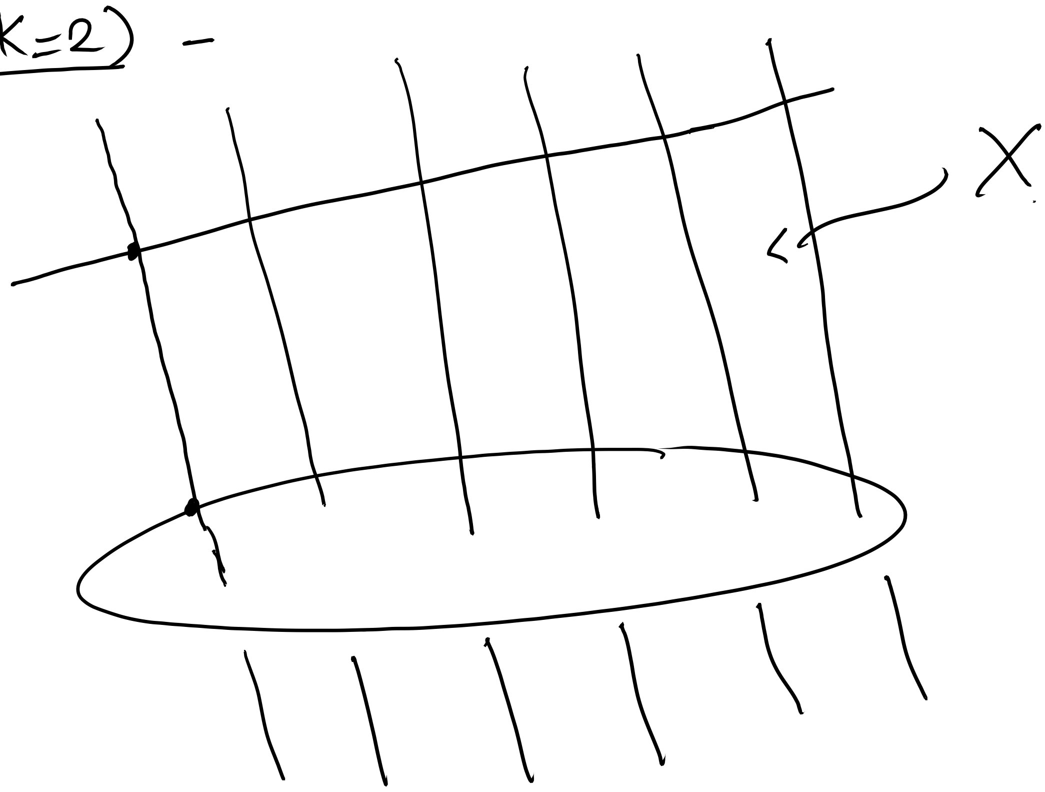
Simplest (3): $X \subset \mathbb{P}^n$ a rational normal curve i.e. $X \cong \mathbb{P}^1$ embedded by $\mathcal{O}(n)$.

General (3) - $\mathbb{P}^n = \mathbb{P}V$, $V = V_1 \oplus \cdots \oplus V_K$

$X_i \hookrightarrow \mathbb{P}V_i$ a rational normal curve.
Fix iso $\varphi_i : \mathbb{P}^1 \xrightarrow{\sim} X_i$

$$X = \bigcup_{t \in \mathbb{P}^1} \operatorname{Span}(\varphi_1(t), \ldots, \varphi_K(t))$$

Picture ($K=2$) -



Alternatively - $X \cong \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is an ample vector bundle on $\mathbb{P}^1$, embedded in $\mathbb{P}^N$ by a line bundle which is $\mathcal{O}(1)$ on the fibers of $X \to \mathbb{P}^1$.

$$\mathcal{E} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_k) \qquad \text{ample} \Leftrightarrow a_i > 0$$

$$\updownarrow$$

$X_i \subset \mathbb{P}V_i$ by $\mathcal{O}(a_i)$.

Pr for varieties of minimal degree

(0) $X = \mathbb{P}^N$, $G = \bullet$, $P = \bullet$

(1) $X \subset \mathbb{P}^N = \mathbb{P}V$ a quadric hypersurface

$G = Gr(N, V) \cong \mathbb{P}V^\vee$

$P = \mathbb{P}V$.

$Pr : \mathbb{P}V \dashrightarrow \mathbb{P}V^\vee$

is the duality induced by X.

(2) $X \cong \mathbb{P}^2 \subset \mathbb{P}^5$ by $\mathcal{O}(2)$.

$G = Gr(3, H^0(\mathcal{O}(2)))$ ← "Net of conics"

$\downarrow$

$P = \mathbb{P}H^0(\mathcal{O}(3))$. ← cubic

Thm: (Cayley, Steiner)

$Pr : G \dashrightarrow P$ is generically finite of degree 3

For $[C] \in P$, we have a bijection

$Pr^{-1}([C]) \overset{\sim}{\longleftrightarrow}$ Non-trivial étale double covers of C

$\updownarrow 2$

Non-zero homs : $\pi_1(C) \to \mathbb{Z}/2\mathbb{Z}$

$\pi_1(C) = \mathbb{Z} \oplus \mathbb{Z}$.

§. Pr for scrolls

For every $(a_1, \dots, a_k)$ with $a_i \in \mathbb{Z}_{>0}$, we have a number
$$\varphi(a_1, \dots, a_k) := \deg_X (\mathrm{Pr}: G \dashrightarrow P) \text{ for } X = \mathbb{P}(\mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_k)).$$

(1) $\varphi(n) = \frac{1}{n}\binom{2n-2}{n-1}$

(2) $\varphi(1,1,1,\dots,1) = 1.$

(3) If $\sum a_i = \sum b_i = d$
$(a_1, \dots, a_k) \prec (b_1, \dots, b_k)$
in the dominance order of length k partitions of d, then
$$\varphi(a_1, \dots, a_k) \leq \varphi(b_1, \dots, b_k)$$

(4) For each k and d
$\varphi(a_1, \dots, a_k) \neq 0$ for the maximal (i.e. most balanced) partition of d.

§. Numerics for $k=2$ (Magma)

d	$\mathcal{E}$	φ
2	(1,1)	1
3	(1,2)	1
4	(2,2)	2
5	(2,3)	6
6	(3,3)	22
7	(3,4)	92
8	(4,4)	422

$\Big\}$ OEIS A001181

# of Baxter permutations of length n.

ξ Broader point.

$$\{X \to \mathbb{P}^n\} \longrightarrow \{(X,R)\}$$

$n=1$: Compactification $\longrightarrow$ Compactification.

$n=2$ & higher: ?? ??

Suitable compactifications of moduli of curves, maps of curves etc.

$\rightsquigarrow$ Answers to enumerative problems, GW. theory, etc.

Missing for surfaces, threefolds, etc.