0.1 The Zariski topology

WEEK2

The notion of affine algebraic sets allows us to define a topology on $\mathbb{A}^n_k$. Recall that we can specify a topology on a set by specifying what the open subsets are, or equivalently, what the closed subsets are. In our case, it is more convenient to do the latter. The collection of closed subsets must satisfy the following properties.

1. The empty set and the entire set are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

   We define the *Zariski topology* on $\mathbb{A}^n_k$ by setting the closed subsets to be the affine algebraic sets, namely, the sets of the form $V(A)$ for some $A \subset k[x_1, \dots, x_n]$.

0.1.1 Proposition The collection of affine algebraic subsets satisfies the three conditions above.

Proof. — (1)

0.1.2 Proposition The Zariski topology on $\mathbb{A}^1_k$ is the *finite complement topology*. The only closed sets are the finite sets (or the whole space). In other words, the only open sets are the complements of finite sets (or the empty set).

Proof. We saw that the subsets $V(A) \subset \mathbb{A}^1_k$ are either the whole $\mathbb{A}^1_k$ or finite sets. □

0.1.3 Comparison between Zariski and Euclidean topology over $\mathbb{C}$. Every Zariski closed (open) subset of $\mathbb{A}^n_{\mathbb{C}}$ is also closed (open) in the usual Euclidean topology. The converse is not true.

Proof. It suffices to prove that $V(A)$ is closed in the usual topology. We have $V(A) = \cap_{f \in A} V(f)$, so it suffices to show that $V(f)$ is closed. But $V(f) = f^{-1}(0)$ is closed, because it is the pre-image of a closed set under a continuous function. □

0.1.4 Proposition (Polynomials are continuous) Let f be a polynomial function on $\mathbb{A}^n_k$, viewed as a map $f \colon \mathbb{A}^n_k \to \mathbb{A}^1_k$. Then f is continuous in the Zariski topology.

Proof. We check that pre-images of closed sets are closed. The only closed sets of $\mathbb{A}^1_k$ is the whole space and finite sets. The pre-image of $\mathbb{A}^1_k$ is $\mathbb{A}^n_k$, which is closed. Since finite unions of closed sets are closed, it suffices to check that the pre-image of a point $a \in \mathbb{A}^1_k$ is closed. But the pre-image of a under f is just $V(f - a)$, which is closed by definition. □

— The Zariski topology has very few open sets, and as a result has terrible separation properties. It is not even Hausdorff (except in very small examples). Nevertheless, we will see that it is extremely useful. For one, it makes sense over every field!

0.2 The Nullstellensatz

WEEK2

We associated a set $V(A)$ to a subset A of the polynomial ring $k[x_1, \dots, x_n]$. If we think of A as a system of equations $\{f = 0 \mid f \in A\}$, then $V(A)$ is the set of solutions. We can also define a reverse operation. The Nullstellensatz says that if k is algebraically closed, then these two operations are mutually inverse. That is, the data of a system of equations is equivalent to the data of its set of solutions. This pleasant fact allows us go back and forth between algebra (equations) and geometry (the solution set).

We start with a straightforward definition.

0.2.1 Definition (Ideal vanishing on a set) Let $S \subset \mathbb{A}^n_k$ be a set. The *ideal vanishing on* S, denoted by $I(S)$, is the set

$$I(S) = \{f \in k[x_1, \dots, x_n] \mid f(a) = 0 \text{ for all } a \in S\}$$

— Recall that an ideal $I \subset k[x_1, \dots, x_n]$ is *radical* if it has the property that whenever $f^n \in I$ for some $n > 1$, then $f \in I$.

0.2.2 Proposition The set $I(S)$ is a radical ideal of $k[x_1, \dots, x_n]$.

Proof. We leave it to you to check that $I(S)$ is an ideal. To see that it is radical, see that if f^n vanishes on S, then so does f. □

0.2.3 Proposition (Easy properties of radical ideals)

1. $I \subset R$ is radical if and only if R/I has no (non-zero) nilpotents.
2. All prime ideals are radical. In particular, all maximal ideals are radical.

Proof. Consider $f \in R$ and its image $\overline{f} \in R/I$. Then $\overline{f}$ is a nilpotent of R/I if and only if $f^n \in I$ and $\overline{f} = 0$ in R/I if and only if $f \in I$. From this, the result follows. If I is prime, then R/I is an integral domain, so it has no nilpotents (it does not even have zero divisors). □

0.2.4 Proposition (Radical of an ideal) Let I be an ideal, and set $\sqrt{I} = \{f \mid f \in I \text{ for some } n > 0\}$. Then $\sqrt{I}$ is a radical ideal.

Proof. — (2)

0.2.5 Definition (Radical of an ideal) The ideal $\sqrt{I}$ is called the radical of I.

0.2.6 Proposition (V is unchanged by radicals) We have $V(I) = V(\sqrt{I})$.

Proof. — (3)

— We now state a string of important theorems, all called the "Nullstellensatz", starting with the most comprehensive one.

0.2.7 Theorem Let k be an algebraically closed field. Then we have a bijection

$$\text{Radical ideals of } k[x_1, \ldots, x_n] \leftrightarrow \text{Zariski closed subsets of } \mathbb{A}^n_k$$

where the map from the left to the right is $I \mapsto V(I)$ and the map from the right to the left is $S \mapsto I(S)$. The correspondence is inclusion reversing.

0.2.8 Theorem Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. If $V(I) = \emptyset$, then $I = (1)$.

0.2.9 Theorem Let k be an algebraically closed field. Then all the maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $\langle x_1 - a_1, \ldots, x_2 - a_n \rangle$ for some $(a_1, \ldots, a_n) \in \mathbb{A}^n_k$.

— Theorem 1.2.8 says that we have a dichotomy: either a system of equations $f_i = 0$ has a solution, or there exist polynomials g_i such that

$$\sum f_i g_i = 1.$$

0.2.10 Theorem Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. If f is identically zero on $V(I)$, then $f^n \in I$ for some n.

0.3 Proof of the Nullstellensatz

WEEK2

The proof of Theorem 1.2.7 actually goes via the proofs of the subsequent theorems. We use the following result from algebra, whose proof we skip.

0.3.1 Theorem Let K be any field and let L be a finitely generated K-algebra. If L is a field, then it must be a finite extension of K.

- Proof See `https://web.ma.utexas.edu/users/allcock/expos/nullstellensatz3.pdf`

0.3.2 Proof of Theorem 1.2.9

Let $m \subset k[x_1, \ldots, x_n]$ be a maximal ideal. Taking $K = k$ and $L = k[x_1, \ldots, x_n]/m$ in Theorem 1.3.1, and using that k is algebraically closed, we get that the natural map $k \to k[x_1, \ldots, x_n]/m$ is an isomorphism. Let $a_i \in k$ be the pre-image of x_i under this isomorphism. Then we have $m = (x_1 - a_1, \ldots, x_n - a_n)$.

> Explain this and prove the last statement. — (4)

0.3.3 Proof of Theorem 1.2.8

Suppose I is not the unit ideal. We show that $V(I)$ is non-empty. To do so, we use that every proper ideal is contained in a maximal ideal.

> Finish the proof. — (5)

0.3.4 Proof of Theorem 1.2.10

We consider the system $g = 0$ for $g \in I$ and $f \neq 0$. Notice that the last one is not an equation, but there is a trick that allows us to convert it into an equation. Let y be a new variable, and consider the polynomial ring $k[x_1, \ldots, x_n, y]$. In the bigger ring, consider the system of equations $g = 0$ for $g \in I$ and $yf - 1 = 0$. By our assumption, this system of equations has no solutions.

> Explain this and finish the proof using Theorem 1.2.8. — (6)

0.3.5 Proof of Theorem 1.2.7.

We show that the maps $I \to V(I)$ and $S \to I(S)$ are mutual inverses. That is, we show that $I(V(I)) = I$ if I is a radical ideal, and $V(I(S)) = S$ if S is a Zariski closed subset of $\mathbb{A}^n_k$.

Let us first show that for any ideal I, we have $I(V(I)) = \sqrt{I}$. Suppose $f \in \sqrt{I}$, then $f^n \in I$ for some $n > 0$. But then f^n is identically zero on $V(I)$, and hence so is f; that is, $f \in I(V(I))$. It remains to show that $I(V(I)) \subset \sqrt{I}$. Let $f \in I(V(I))$. Then f is identically zero on $V(I)$. By 1.2.10, there is some n such that $f^n \in I$, and hence $f \in \sqrt{I}$.

Let us now show that $V(I(S)) = S$. Since S is Zariski closed, we know that $S = V(J)$ for some ideal J. So $I(S) = I(V(J)) = \sqrt{J}$. But we know that $V(J) = V(\sqrt{J})$, and hence $V(I(S)) = S$. The proof of Theorem 1.2.7 is then complete.

0.4 Affine and quasi-affine varieties

WEEK2

An *affine variety* is a subset of the affine space that is closed in the Zariski topology. A *quasi-affine variety* is a subset of the affine space that is locally closed in the Zariski topology. (A locally closed subset of a topological space is a set that can be expressed as an intersection of an open set and a closed set).