# MATH 8320: ALGEBRAIC CURVES AND RIEMANN SURFACES — HOMEWORK 3

(1) Compute the cohomology groups $H^i(X, \mathbb{Z})$ using Cech cohomology for $X = \mathbb{P}^1$.

(2) Let $L$ be a vector bundle on $X$. Establish a bijection between (a) the set of sections of $L$, and (b) the set of maps of vector bundles from the trivial bundle $\mathbb{C} \times X$ to $L$. Conclude that a line bundle is trivial if and only if it has a nowhere vanishing section.

(3) Let $D$ be a divisor of degree $0$ on a compact Riemann surface $X$. Show that

$$\dim H^0(X, \mathcal{O}(D)) = \begin{cases} 1 & \text{if } D \sim 0 \\ 0 & \text{otherwise.} \end{cases}$$

(4) Let $D$ be a non-zero effective divisor on a compact Riemann surface $X$ such that

$$\dim H^0(X, \mathcal{O}(D)) = \deg D + 1.$$

Show that $X \cong \mathbb{P}^1$
*Hint:* Show that there exists $p \in X$ such that $\dim H^0(X, \mathcal{O}(p)) = 2$.

(5) Let $X$ be a compact Riemann surface and $U = X \setminus S$, where $S$ is a finite set. Show that a every holomorphic map $U \to \mathbb{P}^n$ given by homogeneous coordinates which are meromorphic functions on $X$ extends to a holomorphic map $X \to \mathbb{P}^n$.

(6) Let $D$ be a very ample divisor on $X$. Show that the meromorphic functions in $H^0(X, \mathcal{O}(D))$ separate the points and tangent vectors on $X$. Conclude that they generate $\mathcal{M}_X$.

(7) Let $X$ be a compact Riemann surface and $S \subset X$ a finite set. Use the Riemann-Roch theorem to show that there exists a meromorphic function on $X$ that is holomorphic on $X \setminus S$.

(8) (Counts as 2 problems) Let $U$ be the Riemann surface in $\mathbb{C}^2$ defined by $y^3 = x^5 - 1$. We know that there exists a compact Riemann surface $X$ that contains $U$, and the complement $X \setminus U$ consists of a finite set of points. Let $\pi \colon X \to \mathbb{P}^1$ be the extension of the projection $(x, y) \mapsto x$.
   (a) Show that there is a unique point in $X \setminus U$, call it $\infty$.
   (b) Find the ramification divisor of $\pi$ and hence the genus of $X$.
   (c) Write down the canonical series $H^0(X, K)$.
   (d) Show that the canonical map yields an embedding $X \to \mathbb{P}^3$.
   (e) Show that the canonical image of $X$lies on a singular quadric hypersurface.

(9) Let $D$ be a divisor on a smooth algebraic curve $X$ of genus $g$ such that $\deg D = 2g - 2$ and $\dim H^0(X, \mathcal{O}(D)) = g$. Show that $D$ is a canonical divisor (that is, $\mathcal{O}_X(D) \cong K_X$).