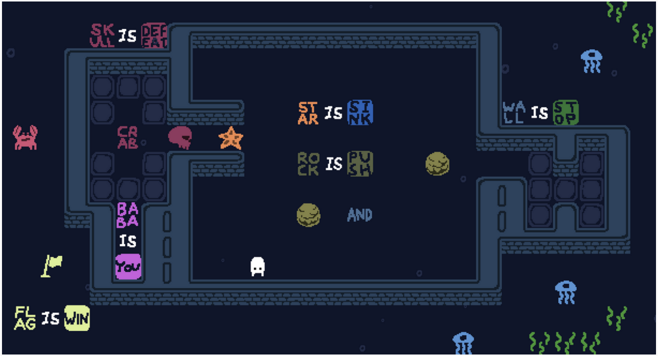
Games, graphs, and machines



October 22, 2024

Grundy value: warm-up

Find the Grundy value of PosetChomp for the following poset.

The Sprague-Grundy theorem

Theorem Two games have the same Grundy value if and only if they are equivalent.

Why? First a lemma.

Lemma If $G \sim H$, then $G + A \sim H + A$.

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So $G + H$ is a P-game.
Then $G + G + H$ is equivalent to both G and H.
So $G \sim H$.