CLASS XII (2019-20)
MATHEMATICS (041)
SAMPLE PAPER-1

Time : 3 Hours **Maximum Marks : 80**

General Instructions :

(i) All questions are compulsory.
(ii) The questions paper consists of 36 questions divided into 4 sections A, B, C and D.
(iii) Section A comprises of 20 questions of 1 mark each. Section B comprises of 6 questions of 2 marks each. Section C comprises of 6 questions of 4 marks each. Section D comprises of 4 questions of 6 marks each.
(iv) There is no overall choice. However, an internal choice has been provided in three questions of 1 mark each, two questions of 6 marks each. You have to attempt only one of the alternatives in all such questions.
(v) Use of calculators is not permitted.

Section-A

DIRECTION : (Q 1-Q 10) are multiple choice type questions. Select the correct option.

1. The operation $*$ is defined as $a * b = 2a + b$, then $(2 * 3) * 4$ is. [1]

(a) 18 (b) 17
(c) 19 (d) 21

Ans : (a) 18

We have,

$$\begin{aligned} a * b &= 2a + b \\ (2 * 3) * 4 &= (2 \times 2 + 3) * 4 = 7 * 4 \\ &= 2 \times 7 + 4 = 18 \end{aligned}$$

2. $\cos^{-1}\dfrac{1-x^2}{1-x^2} =$ [1]

(a) $2\cos^{-1}x$ (b) $2\sin^{-1}x$
(c) $2\tan^{-1}x$ (d) $\cos^{-1}2x$

Ans : (c) $2\tan^{-1}x$

Let, $$f(x) = \cos^{-1}\frac{1-x^2}{1+x^2} \quad ....(i)$$

Let, $x = \tan\theta$ and substituting in equation (i)

$$f(x) = \cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

But $$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

Therefore, $$f(x) = \cos^{-1}\cos 2\theta$$

$$\left[\cos^{-1}(\cos t) = t\right]$$

$$= 2\theta = 2\tan^{-1}x$$

3. $A = \begin{bmatrix} 3 & 6 \\ 5 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}$, $2A + 3B = ?$ [1]

(a) $\begin{bmatrix} 27 & 24 \\ 22 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 27 & 36 \\ 25 & 10 \end{bmatrix}$

(c) $\begin{bmatrix} 27 & 36 \\ 25 & 15 \end{bmatrix}$ (d) $\begin{bmatrix} 27 & 36 \\ 35 & 10 \end{bmatrix}$

Ans : (b) $\begin{bmatrix} 27 & 36 \\ 25 & 10 \end{bmatrix}$

We have,

$$\begin{aligned} 2A + 3B &= 2\begin{bmatrix} 3 & 6 \\ 5 & -4 \end{bmatrix} + 3\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 12 \\ 10 & -8 \end{bmatrix} + \begin{bmatrix} 21 & 24 \\ 15 & 18 \end{bmatrix} \\ &= \begin{bmatrix} 27 & 36 \\ 25 & 10 \end{bmatrix} \end{aligned}$$

4. If 7 and 2 are two roots of the equation $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ then the third root is [1]

(a) -9 (b) 14
(c) $\dfrac{1}{2}$ (d) None of these

Ans : (a) -9

We have $$\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

$$\begin{aligned} x(x^2 - 12) - 3(2x - 14) + 7(12 - 7x) &= 0 \\ x^3 - 12x - 6x + 42 + 84 - 49x &= 0 \\ x^3 - 67x + 126 &= 0 \end{aligned}$$

$$(x-7)(x^2+7x-18) = 0$$
$$(x-7)(x^2+9x-2x-18) = 0$$
$$(x-7)(x-2)(x+9) = 0$$
$$x = 7, 2, -9$$

5. $\dfrac{d(2^x)}{d(3^x)} =$ [1]

(a) $\left(\dfrac{2}{3}\right)^x$ (b) $\dfrac{2^{x-1}}{3^{x-1}}$

(c) $\left(\dfrac{2}{3}\right)^x \log_3 2$ (d) $\left(\dfrac{2}{3}\right)^x \log_2 3$

Ans : (c) $\left(\dfrac{2}{3}\right)^x \log_3 2$

We have $f(x) = \dfrac{d(2^x)}{d(3^x)}$ ...(i)

$$\frac{d}{dx}(2^x) = 2^x \log 2$$

$$\frac{d}{dx}(3^x) = 3^x \log 3 \left[\frac{d}{dx}(a^x) = a^x \log a\right]$$

Putting values in eq.(i),

$$f(x) = \frac{2^x \log 2}{3^x \log 3}$$
$$= \left(\frac{2}{3}\right)^x \log_3 2 \qquad \left[\frac{\log a}{\log b} = \log ba\right]$$

6. If $y = x^2+3x-4$, then the slope (gradient) of the normal to the curve at (1, 1) is [1]

(a) 5 (b) $-\dfrac{1}{5}$

(c) 8 (d) $-\dfrac{1}{8}$

Ans : (b) $-\dfrac{1}{5}$

We have, $y = x^2+3x-4$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2x+3$$

$$\left.\frac{dy}{dx}\right|_{1,1} = 2+3 = 5$$

Hence, Slope of normal

$$= -\frac{1}{\left(\frac{dy}{dx}\right)_{(1,1)}} = -\frac{1}{5}$$

7. Integrating factor of the differential equation $\dfrac{dy}{dx} + y\sec x = \tan x$ is- [1]

(a) $\sec x + \tan x$ (b) $\sec x - \tan x$

(c) $\sec x$ (d) $\tan x \sec x$

Ans : (a) $\sec x + \tan x$

We have,

$$\frac{dy}{dx} + y\sec x = \tan x$$
$$\text{I.f.} = e^{\int \sec x\, dx} = e^{\log(\sec x + \tan x)}$$
$$= \sec x + \tan x$$

8. $x\hat{i} - 3\hat{j} + 5\hat{k}, -x\hat{i} + x\hat{j} + 2\hat{k}$ are perpendicular to each other then $x =$ [1]

(a) -2, 5 (b) 2, 5

(c) -2, -5 (d) 2, -5

Ans : (d) 2, -5

Let, $\vec{a} = x\hat{i} - 3\hat{j} + 5\hat{k}$

and $\vec{b} = -x\hat{i} + x\hat{j} + 2\hat{k}$

For $\vec{a} \perp \vec{b}$, $\vec{a}\cdot\vec{b} = 0$

$$\vec{a}\cdot\vec{b} = -x^2 - 3x + 10$$
$$= -(x^2+3x-10)$$
$$\vec{a}\cdot\vec{b} = 0$$
$$x^2+3x-10 = 0$$
$$x^2+5x-2x-10 = 0$$
$$x(x+5) - 2(x+5) = 0$$
$$(x-2)(x+5) = 0$$
$$x = 2, -5$$

9. The coordinates of the midpoint of the line segment joining the points (2, 3, 4) and $(8, -3, 8)$ are [1]

(a) (10, 0, 12) (b) (5, 6, 0)

(c) (6, 5, 0) (d) (5, 0, 6)

Ans : (d) (5, 0, 6)

We have $A = (2,3,4)$ B $(8,-3,8)$

Mid point of line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$P = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Here, $x_1 = 2$ $y_1 = 3$ $z_1 = 4$

$x_2 = 8$ $y_2 = -3$ $z_2 = 8$

$$P = \left(\frac{2+8}{2}, \frac{3-3}{2}, \frac{4+8}{2}\right)$$
$$P = (5, 0, 6)$$

10. If A and B are two events such that $P(A) \neq 0$ and $P\left(\dfrac{B}{A}\right) = 1$ [1]

(a) $B \subset A$ (b) $A \subset B$

(c) $B = \phi$ (d) $A \cap B = \phi$

Ans : (b) $A \subset B$

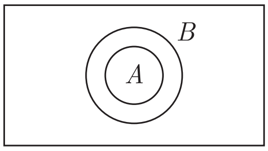
$$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}$$

[Conditional probability]

$$\frac{P(B \cap A)}{P(A)} = 1 \qquad \left[P\left(\frac{B}{A}\right) = 1\right]$$

$$\Rightarrow \quad P(B \cap A) = P(A)$$

$$\Rightarrow \quad A \subset B$$



Q. 11-15 (Fill in the blanks)

11. If the binary operation $*$ defined on Q, is defined as $a * b = 2a + b - ab$, $\forall\, a,\ b \in Q$, then the value of $3 * 4$ is .... [1]

Ans :

We have, sd $a * b = 2a + b - ab$

$$\therefore \quad 3 * 4 = 2(3) + 4 - (3)(4)$$

$$= 6 + 4 - 12 = -2 \qquad \textbf{(1)}$$

12. If the vectors: $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{c} = 3\hat{i} + \lambda\hat{j} + 5\hat{k}$ are coplanar, then the value of λ is .... [1]

Ans :

Given that, $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

and $\vec{c} = 3\hat{i} + \lambda\hat{j} + 5\hat{k}$ are coplanar.

$$\therefore \quad \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & \lambda & 5 \end{vmatrix} = 0 \qquad [\because [\vec{a}\,\vec{b}\,\vec{c}] = 0]$$

(1/2)

$$2(10 - 3\lambda) + 1(5 - 9) + 1(\lambda - 6) = 0$$

[expanding along R_1]

$$20 - 6\lambda - 4 + \lambda - 6 = 0$$

$$10 = 5\lambda$$

$$\therefore \quad \lambda = 2 \qquad \textbf{(1/2)}$$

13. $\sin\left(2\sin^{-1}\frac{3}{5}\right)$ is equal to .... [1]

Ans :

We have, $\sin\left(2\sin^{-1}\frac{3}{5}\right)$

Let, $\sin^{-1}\frac{3}{5} = x$

$$\sin x = \frac{3}{5}$$

and $\cos x = \sqrt{1 - \sin^2 x}$

$$= \sqrt{1 - \frac{9}{25}} = \frac{4}{25} \qquad \textbf{(1/2)}$$

Now, $\sin\left(2\sin^{-1}\frac{3}{5}\right) = \sin 2x$

$$\left[\because \sin^{-1}\frac{3}{5} = x\right]$$

$$= 2\sin x \cos x$$

$$= 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$$

Hence, $\sin\left(2\sin^{-1}\frac{3}{5}\right) = \frac{24}{25}$ **(1/2)**

14. $\cos^{-1}(2x - 1) =$ .... [1]

(a) $2\cos^{-1}x$ (b) $\cos^{-1}\sqrt{x}$

(c) $2\cos^{-1}\sqrt{x}$ (d) None of these

Ans : (c) $2\cos^{-1}\sqrt{x}$

Let, $x = \cos^2 A$

or $A = \cos^{-1}\sqrt{x}$ ...(i)

Then given equation

$$\cos^{-1}(2x - 1) = \cos^{-1}(2\cos^2 A - 1)$$

$$= \cos^{-1}(\cos 2A) = 2A \quad \text{...(ii)}$$

From Eq.(i),

$$\cos^{-1}(2x - 1) = 2\cos^{-1}\sqrt{x}$$

or

The principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is ....

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

Ans : (d) $\frac{\pi}{3}$

Let, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \theta$

$$\frac{\sqrt{3}}{2} = \sin\theta$$

$$\sin\theta = \sin 60$$

$$\theta = 60$$

or $60 \times \frac{\pi}{180} = \frac{\pi}{3}$

15. If $y = x^2 + 3x - 4$, then the slope (gradient) of the normal to the curve at (1, 1) is .... [1]

(a) 5 (b) $-\frac{1}{5}$

(c) 8 (d) $-\frac{1}{8}$

Ans : (b) $-\frac{1}{5}$

We have, $y = x^2 + 3x - 4$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2x + 3$$

$$\left.\frac{dy}{dx}\right|_{1,1} = 2 + 3 = 5$$

Hence, Slope of normal

$$= -\frac{1}{\left(\frac{dy}{dx}\right)_{(1,1)}} = -\frac{1}{5}$$

or

The angle which the tangent to curve $y = x^2$ at $(0, 0)$ makes with the positive direction of x-axis is .... [1]

(a) 90° (b) 0°

(c) 45° (d) 30°

Ans : (b) 0°

Given: $y = x^2$

Differentiating both sides w.r.t., we get

$$\frac{dy}{dx} = 2x$$

$$\left.\frac{dy}{dx}\right|_{(0,0)} = 0$$

Thus, tangent is parallel to x axis.

So, angle between tangent at (0,0) with x -axis is 0°.

16. Find matrix X, if $X + \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$ [1]

Ans :

Let $A = \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$

Then, the given matrix equation is $X + A = B$

$\therefore$ $X + A = B$

$\Rightarrow$ $X = B + (-A)$

$$= \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix} + \begin{bmatrix} -4 & -6 \\ 3 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + (-4) & -6 + (-6) \\ 5 + 3 & -8 + (-7) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}$$

Hence, $X = \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}$

or

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then find the value of $\theta\left(\text{where}, \theta\,\varepsilon\left(0, \frac{\pi}{2}\right)\right)$ satisfying the equation $A^T + A = I_2$.

Ans :

We have, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^T + A = \begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $2\cos\theta = 1$

$$\cos\theta = \frac{1}{2}$$

$\therefore$ $\theta = \frac{\pi}{3}$

17. Evaluate $\int_0^{\pi/2} x\sin x dx$ [1]

Ans :

Let $I = \int_0^{\pi/2} \underset{\text{I}}{x}\underset{\text{II}}{\sin} x dx$

$$= [-x\cos x]_0^{\pi/2} - \int_0^{\pi/2} 1 \times (-\cos x)\,dx$$

[Integrating by parts]

$$= [-x\cos x]_0^{\pi/2} + [\sin x]_0^{\pi/2}$$

$$= \left(-\frac{\pi}{2}\cos\frac{\pi}{2} + 0\cos 0\right) + \left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= 1$$

18. Examine the continuity of the function [1]

$$f(x) = \begin{cases} \frac{|\sin x|}{x}, & x \neq 0 \\ 1, & x = 0 \text{ at } x = 0 \end{cases}$$

Ans :

We have, $f(0) = 1$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} \frac{|\sin(0 + h)|}{(0 + h)}$$

$$= \lim_{h \to 0} \frac{|\sin h|}{h} = \lim_{h \to 0} \frac{\sin h}{h}$$

$$= 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$$

$$= \lim_{h \to 0} \frac{|\sin(-h)|}{-h}$$

$$= \lim_{h \to 0} \frac{|-\sin h|}{-h}$$

$$= \lim_{h \to 0} \frac{\sin h}{-h}$$

$$= -1$$

$\therefore$ $\lim_{h \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

So, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

19. Find the volume of a parallelopiped whose sides are given by [1]
$-3\hat{i}+7\hat{j}+5\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$.

Ans :

Let
$$\vec{a} = -3\hat{i}+7\hat{j}+5\hat{k}$$
$$\vec{b} = -5\hat{i}+7\hat{j}-3\hat{k}$$
and
$$\vec{c} = 7\hat{i}-5\hat{j}-3\hat{k}.$$

We know that, the volume of a parallelopiped whose three adjacent edges are, $\vec{a}, \vec{b}, \vec{c}$, is equal to $|[\vec{a}\ \vec{b}\ \vec{c}]|$.

$$\text{Here,} [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$$\Rightarrow \quad [\vec{a}\vec{b}\vec{c}] = -3(-21-15)-7(15+21) +5(25-49)$$

$$[\vec{a}\vec{b}\vec{c}] = 108-252-120$$
$$= -264$$

Hence, volume of the parallelopiped

$$= |[\vec{a}\vec{b}\vec{c}]| = |-264|$$
$$= 264 \text{ cu units}$$

20. Two coins are tossed. What is the probability of coming up two heads if it is known that at least one head comes up. [1]

Ans :

Consider the event

A = Getting at least one head

B = Getting two heads

$$A = \{HT, TH, HH\}$$
$$B = \{HH\}$$
$$A \cap B \ \{HH\}$$
$$P(A) = 3/4$$
$$P(B) = \frac{1}{4},\ P(A \cap B) = \frac{1}{4}$$

$\therefore$ Required probability

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Section B

21. Evaluate $\int_0^1 \frac{2x}{5x^2+1}dx$. [2]

Ans :

Given, $$I = \int_0^1 \frac{2x}{5x^2+1}dx$$

Let $$5x^2+1 = t$$

Then, $$d(5x^2+1) = dt$$
$$10x dx = dt$$

When, $x = 0$, $t = 5x^2+1 \Rightarrow t = 1$

When, $x = 1$, $t = 5x^2+1 \Rightarrow t = 6$ **(1)**

$$\therefore \quad \int_0^1 \frac{2x}{5x^2+1}dx = \int_1^6 \frac{2x}{t} \times \frac{dt}{10x}$$
$$= \frac{1}{5}\int_1^6 \frac{dt}{t} = \frac{1}{5}[\log t]_1^6$$
$$= \frac{1}{5}[\log 6 - \log 1] = \frac{1}{5}\log 6$$ **(1)**

or

Evaluate $\int_0^2 [x^2]dx$, where $[\cdot]$ is the greatest integer function. [2]

Ans :

$$\text{Here,} \quad [x^2] = \begin{cases} 0, \text{ when } 0 < x < 1 \\ 1, \text{ when } 1 < x < \sqrt{2} \\ 2, \text{ when } \sqrt{2} < x < \sqrt{3} \\ 3, \text{ when } \sqrt{3} < x < 2 \end{cases}$$ **(1/2)**

$$\int_0^2 [x^2]dx = \int_0^1 0dx + \int_1^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{\sqrt{3}} 2dx + \int_{\sqrt{3}}^2 3dx$$ **(1/2)**

$$= 0 + [x]_1^{\sqrt{2}} + [2x]_{\sqrt{2}}^{\sqrt{3}} + [3x]_{\sqrt{3}}^2$$
$$= (\sqrt{2}-1)+2(\sqrt{3}-\sqrt{2})+3(2-\sqrt{3})$$ **(1/2)**
$$= \sqrt{2}-1+2\sqrt{3}-2\sqrt{2}+6-3\sqrt{3}$$
$$= 5-\sqrt{2}-\sqrt{3}$$ **(1/2)**

22. Find the value of k, if the area of a triangle is 4 sq units and vertices are $(k,0)$, $(4,0)$ and $(0,2)$. [2]

Ans :

Give vertices are $(k,0)$, $(4, 0)$ and $(0, 2)$.

Also, Area of a triangle = 4 sq units.

$$\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \pm 4$$ **(1)**

$$[k(0-2)+1(8-0)] = \pm 8$$

[expanding along R_1]

$$-2k+8 = \pm 8$$
$$-2k+8 = 8 \text{ or } -2k+8 = -8$$

$$-2k = 0 \text{ or } = -2k = -16$$
$$k = 0 \text{ or } k = 8$$

Hence, the value of k is 0 or 8. **(1)**

23. Using properties of determinants, prove that

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$ [2]

Ans :

To prove,

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^2$$

Let, $$\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

On taking $(a+b+c)$ common from R_1, we get

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$ **(1)**

On applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$\Delta = (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b+c+a & -(a+b+c) & 2b \\ 0 & c+a+b & c-a-b \end{vmatrix}$$

On taking $(a+b+c)$ common from C_1 and C_2 we get

$$\Delta = (a+b+c)^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

On expanding along C_1, we get

$$\Delta = (a+b+c)^3[0-1(0-1)+0]$$
$$= (a+b+c)^3$$ **(1)**

24. Show that, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$. [2]

Ans :

We have, $\text{LHS} = \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$

Let, $\sin^{-1}\frac{3}{4} = x \Rightarrow \sin x = \frac{3}{4}$

and $$\cos x = \sqrt{1-\sin^2 x}$$
$$= \sqrt{1-\frac{9}{16}} = \frac{\sqrt{7}}{4}$$ **(1)**

We know that,

$$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$$

$$\therefore \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{1-\frac{\sqrt{7}}{4}}{\frac{3}{4}}$$

$$= \frac{4-\sqrt{7}}{3} = \text{R.H.S.}$$

Hence proved. **(1)**

or

Prove that, $\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right) = 7$.

Ans :

We have, $\text{LHS} = \cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right)$

$$\because \quad 2\cot^{-1}3 = \cot^{-1}\left(\frac{9-1}{6}\right)$$

$$\left[\because 2\cot^{-1}x = \cot^{-1}\left(\frac{x^2-1}{2x}\right)\right]$$

$$2\cot^{-1}3 = \cot^{-1}\frac{4}{3}$$ **(1)**

$$\therefore \quad \cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right) = \cot\left(\frac{\pi}{4} - \cot^{-1}\frac{4}{3}\right)$$

$$= \frac{\cot\frac{\pi}{4}\cot\left(\cot^{-1}\frac{4}{3}\right)+1}{\cot\left(\cot^{-1}\frac{4}{3}\right) - \cot\frac{\pi}{4}}$$

$$= \frac{\frac{4}{3}+1}{\frac{4}{3}-1} = 7 = \text{RHS}$$

Hence proved. **(1)**

25. Rajeev appears for an interview for two posts A and B for which selection is independent. The probability of his selection for post A is 1/5 and for post B is 1/6. He prepare well for two posts by getting all the possible informations. What is the probability that he is selected for atleast one of the post? [2]

Ans :

Given that,

$$P \text{ (selection for post } A) = P(A) = \frac{1}{5}$$

and $$P \text{ (selection for post } B) = P(B) = \frac{1}{6}$$

$$\therefore P \text{ (not selected for post } A) = P(\overline{A})$$
$$= 1 - P(A)$$
$$= 1 - \frac{1}{5} = \frac{4}{5}$$

and P (not selected for post B) $= P(\overline{B})$

$$= 1 - P(B)$$
$$= 1 - \frac{1}{6}$$
$$= \frac{5}{6} \qquad \textbf{(1)}$$

$\therefore$ P (Rajeev is selected for atleast one post)
$= 1 - P$ (Rajeev is selected for none of them)

$$= 1 - P(\overline{A} \cap \overline{B})$$
$$= 1 - P(\overline{A}) \cdot P(\overline{B})$$
$$= 1 - \frac{4}{5} \times \frac{5}{6} = 1 - \frac{2}{3} = \frac{1}{3} \qquad \textbf{(1)}$$

26. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, then find a vector of magnitude 6 units, which is parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$. [2]

Ans :

Given that,

$$\vec{a} = \hat{i} + \hat{j} + \hat{k},$$
$$\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$$

and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$

Let the vector, $\vec{r} = \lambda(2\vec{a} - \vec{b} + 3\vec{c})$

$\Rightarrow$ $\vec{r} = \lambda(2\hat{i} + 2\hat{j} + 2\hat{k} - 4\hat{i} + 2\hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k})$

$\Rightarrow$ $\vec{r} = \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ **(1)**

$\therefore$ $|\vec{r}| = |\lambda||\hat{i} - 2\hat{j} + 2\hat{k}|$

As, $|\vec{r}| = 6$ [given]

$\therefore$ $|\lambda|\sqrt{1+4+4} = 6 \Rightarrow |\lambda| \cdot 3 = 6$

$\Rightarrow$ $\lambda \pm 2$

Therefore, $\vec{r} = \pm 2(\hat{i} - 2\hat{j} + 2\hat{k})$

$$= \pm(2\hat{i} - 4\hat{j} + 4\hat{k})$$

which is the required vector. **(1)**

Section C

27. Differentiate the equation $y = e^{\sin x} + (\tan x)^x$ w.r.t. x. [4]

Ans :

Given that, $y = e^{\sin x} + (\tan x)^x$

Let, $u = e^{\sin x}$ and $v = (\tan x)^x$

Then, $y = u + v$ ...(i)

On differentiating both side of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{...(ii)}$$

Now, consider $u = e^{\sin x}$ **(1)**

On differentiating both sides of w.r.t. x, we get

$$\frac{du}{dx} = e^{\sin x} \cdot \frac{d}{dx}(\sin x)$$

$\Rightarrow$ $\frac{du}{dx} = e^{\sin x} \cdot \cos x$ ...(iii)

and $v = (\tan x)^x$

Taking log on both sides, we get

$$\log v = x \log \tan x \qquad \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = x \cdot \frac{1}{\tan x} \cdot \sec^2 x + \log \tan x \cdot 1$$

[by product rule of derivative]

$\Rightarrow$ $\frac{1}{v} \cdot \frac{dv}{dx} = \frac{x \sec^2 x}{\tan x} + \log \tan x$

$\Rightarrow$ $\frac{1}{v} \cdot \frac{dv}{dx} = \frac{x}{\sin x \cdot \cos x} + \log \tan x$ **(1)**

$\Rightarrow$ $\frac{1}{v} \cdot \frac{dv}{dx} = 2x \operatorname{cosec} 2x + \log \tan x$

$$\left[\because \frac{1}{\sin x \cos x} = \frac{2}{\sin 2x} = 2 \operatorname{cosec} 2x\right]$$

$\Rightarrow$ $\frac{dv}{dx} = (\tan x)^x (\log \tan x + 2x \operatorname{cosec} 2x)$ ...(iv)

$$[\text{put } v = (\tan x)^x]$$

On putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from Eqs. (iii) and (iv) in Eq. (ii), we get

$$\frac{dy}{dx} = e^{\sin x} \cdot \cos x + (\tan x)^x [\log \tan x + 2x \operatorname{cosec} 2x]$$

(1)

28. Evaluate $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$. [4]

Ans :

Let, $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ ...(i)

$\Rightarrow$ $I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a - x) dx\right]$$

$\Rightarrow$ $I = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x} dx$ ...(ii) **(1)**

On adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx$$

$\Rightarrow$ $2I = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

$$[\because \sin^2 x + \cos^2 x = 1]$$

$$\Rightarrow \qquad 2I = \int_0^{\pi/2} \frac{dx}{\dfrac{2\tan x/2}{1+\tan^2 x/2} + \dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}}$$

$$\left[\because \sin x = \frac{2\tan x/2}{1+\tan^2 x/2} \text{ and } \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right]$$

$$2I = \int_0^{\pi/2} \frac{\sec^2 x/2}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}\, dx$$

$$\left[\because 1+\tan^2\frac{x}{2} = \sec^2\frac{x}{2}\right] \quad \textbf{(1)}$$

Put, $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$\therefore \qquad 2I = 2\int_0^1 \frac{dt}{1-t^2+2t}$$

$$\left[\begin{aligned} &\text{when } x = \frac{\pi}{2}, \text{then } t = \frac{\pi}{4} = 1 \\ &\text{and when } x = 0, \text{then } t = \tan 0 = 0 \end{aligned}\right]$$

$$\Rightarrow \qquad 2I = 2\int_0^1 \frac{dt}{-[t^2-2t+1]+2}$$

[adding and subtracting 1 from denominator] **(1)**

$$\Rightarrow \qquad 2I = -2\int_0^1 \frac{dt}{(t-1)^2 - (\sqrt{2})^2}$$

$$\Rightarrow \qquad 2I = -\frac{2}{2\sqrt{2}}\left[\log\frac{t-1-\sqrt{2}}{t-1+\sqrt{2}}\right]_0^1$$

$$\left[\because \int\frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$\Rightarrow I = -\frac{1}{2\sqrt{2}}\left[\log\left|\frac{1-1-\sqrt{2}}{1-1+\sqrt{2}}\right| - \log\left|\frac{-1-\sqrt{2}}{-1+\sqrt{2}}\right|\right]$$

$$\therefore \qquad I = -\frac{1}{2\sqrt{2}}\left[\log 1 - \log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right]$$

$$= -\frac{1}{2\sqrt{2}}\left[0 - \log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right]$$

$$[\because \log 1 = 0]$$

$$= \frac{1}{2\sqrt{2}}\log\left[\frac{\sqrt{2}+1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\right]$$

[multiplying numerator and denominator by $(\sqrt{2}+1)$]

$$= \frac{1}{2\sqrt{2}}\log\frac{(\sqrt{2}+1)^2}{2-1}$$

$$= \frac{1}{\sqrt{2}}\log(\sqrt{2}+1) \quad \textbf{(1)}$$

or

Evaluate $\int_0^3 f(x)\,dx$, where

$f(x) = |x| + |x-1| + |x-2|$.

Ans :

Let, $\qquad I = \int_0^3 f(x)\,dx$

where, $f(x) = |x| + |x-1| + |x-2|$

Now,

$$I = \int_0^3 |x|\,dx + \int_0^3 |x-1|\,dx + \int_0^3 |x-2|\,dx$$

$$= \int_0^3 x\,dx + \int_0^1 -(x-1)\,dx + \int_1^3 (x-1)\,dx$$

$$+ \int_0^2 -(x-2)\,dx + \int_2^3 (x-2)\,dx \quad \textbf{(1)}$$

$$\left[\because |x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}\right]$$

$$= \left[\frac{x^2}{2}\right]_0^3 + \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^3 + \left[2x - \frac{x^2}{2}\right]_0^2$$

$$+ \left[\frac{x^2}{2} - 2x\right]_2^3 \quad \textbf{(1)}$$

$$= \frac{9}{2} + \left[\left(1-\frac{1}{2}\right) - (0-0)\right] + \left[\left(\frac{9}{2}-3\right) - \left(\frac{1}{2}-1\right)\right]$$

$$+ \left[\left(4-\frac{4}{2}\right) - (0-0)\right] + \left[\left(\frac{9}{2}-6\right) - \left(\frac{4}{2}-4\right)\right] \quad \textbf{(1)}$$

$$= \frac{9}{2} + \frac{1}{2} + \frac{3}{2} + \frac{1}{2} + \frac{4}{2} - \frac{3}{2} + \frac{4}{2} = \frac{19}{2} \quad \textbf{(1)}$$

29. For the matrices A and B, prove that $(AB)' = B'A'$, if $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$ and $B = [-1\ 2\ 1]$. [4]

Ans :

Given that, $\qquad A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$

and $\qquad B = [-1\ \ 2\ \ 1]$

To prove, $\quad (AB)' = B'A'$

Now, $\qquad AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}_{3\times 1} [-1\ \ 2\ \ 1]_{1\times 3}$ **(1)**

$$\Rightarrow \qquad AB = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$$

$$\therefore \qquad (AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \quad ...(i)$$

[Interchange rows and columns] **(1)**

Now, $\qquad B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $A' = [1\ -4\ \ 3]$

$$\therefore \quad B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} [1 \ -4 \ \ 3] \qquad \textbf{(1)}$$

$$\Rightarrow \quad B'A' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \qquad ...(ii)$$

From Eq. (i) and (ii), we get

$$(AB)' = B'A' \qquad \textbf{(1)}$$

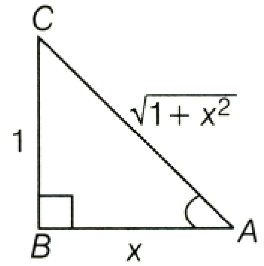
30. Prove that, [4]

$$\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right] = \sqrt{\frac{1+x^2}{2+x^2}}.$$

Ans :

To prove,

$$\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right] = \sqrt{\frac{1+x^2}{2+x^2}}$$



Now, $\quad \text{LHS} = \cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right]$ ...(i)

Put, $\quad \cot^{-1}x = A \Rightarrow \cot A = x$

$$\therefore \quad \sin A = \frac{1}{\sqrt{1+x^2}}$$

$$A = \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) \qquad \textbf{(1)}$$

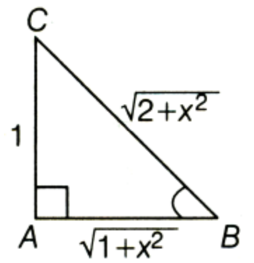
From Eq. (i), we get

$$\text{LHS} = \cos\left[\tan^{-1}\left\{\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right\}\right]$$

$$\text{LHS} = \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$$

$$\left[\because \sin\left(\sin^{-1}\theta\right) = \theta\right]$$

...(ii) **(1)**



Again putting,

$$\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = B$$

$$\tan B = \frac{1}{\sqrt{1+x^2}}$$

$$\therefore \quad \cos B = \sqrt{\frac{1+x^2}{2+x^2}} \qquad \textbf{(1)}$$

$$\Rightarrow \quad B = \cos^{-1}\left(\sqrt{\frac{1+x^2}{2+x^2}}\right)$$

From eq. (ii), we get

$$\text{LHS} = \cos\left(\cos^{-1}\sqrt{\frac{1+x^2}{2+x^2}}\right)$$

$$= \sqrt{\frac{1+x^2}{2+x^2}} = \text{RHS}$$

$$\left[\because \cos\left(\cos^{-1}\theta\right) = \theta\right] \qquad \textbf{(1)}$$

31. Evaluate $\int \frac{dx}{5+4\cos x}$ [4]

Ans :

Let, $\quad I = \int \frac{dx}{5+4\cos x}$

$$= \int \frac{dx}{5+4\frac{\left(1-\tan^2\frac{x}{2}\right)}{\left(1+\tan^2\frac{x}{2}\right)}}$$

$$\left[\because \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)}$$

$$\left[\because 1+\tan^2\frac{x}{2} = \sec^2\frac{x}{2}\right]$$

$$\Rightarrow \quad I = \int \frac{\sec^2\frac{x}{2}\,dx}{9+\tan^2\frac{x}{2}} \qquad \textbf{(1)}$$

Put, $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$\therefore \quad I = 2\int \frac{dt}{9+t^2} = 2\int \frac{dt}{(3)^2+t^2} \qquad \textbf{(1)}$$

$$= \frac{2}{3}\tan^{-1}\frac{t}{3} + C$$

$$\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right] \textbf{(1)}$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right) + C$$

$$\left[\text{Put } t = \tan\frac{x}{2}\right] \textbf{(1)}$$

or

Evaluate $\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)}dx.$

Ans :

Let, $I = \int \frac{\sin 2x}{(1+\sin x)(2+\sin x)} dx$

$= 2\int \frac{\sin x \cdot \cos x \, dx}{(1+\sin x)(2+\sin x)}$

$[\because \sin 2x = 2\sin x \cos x]$ **(1)**

Put, $\sin x = t \Rightarrow \cos x \, dx = dt$

$\therefore$ $I = 2\int \frac{t}{(1+t)(2+t)} dt$

Let, $\frac{t}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$ **(1)**

$t = A(2+t) + B(1+t)$

$t = (2A+B) + (A+B)t$

On comparing the coefficients of t and constant term, we get

$2A + B = 0$ and $A + B = 1$ **(1)**

On solving the above equations, we get

$A = -1$ and $B = 2$

$\therefore$ $I = 2\left[\int \frac{-dt}{1+t} + \int \frac{2dt}{2+t}\right]$

$= -2\log|1+t| + 4\log|2+t| + C$

$= -2\log(1+\sin x) + 4\log(2+\sin x) + C$

$[\text{put } t = \sin x]$ **(1)**

32. Show that the differential equation:
$\sin x \frac{dy}{dx} + (\cos x) \cdot y = \cos x \cdot \sin^2 x$ is linear and also solve the differential equation. [4]

Ans :

Given differential equation is,

$\sin x \frac{dy}{dx} + (\cos x) y = \cos x \cdot \sin^2 x$

$\frac{dy}{dx} + (\cot x) y = \cos x \cdot \sin x$

[dividing both sides by $\sin x$]

which is a linear differential equation of the form,

$\frac{dy}{dx} + Py = Q,$

where P and Q are the functions of x **(1)**

Here, $P = \cot x$ and $Q = \cos x \cdot \sin x$

Now, $\text{IF} = e^{\int P dx} = e^{\int \cot x dx} = e^{\log_e \sin x}$

$= \sin x$ **(1)**

Therefore, the required solution is,

$y \cdot \text{IF} = \int (\text{IF} \cdot Q) dx + C$

$y \cdot \sin x = \int \sin x \cdot \cos x \sin x \, dx + C$

$y \cdot \sin x = \int \sin^2 x \cos x \, dx + C$ **(1)**

Put, $\sin x = t \Rightarrow \cos x \, dx = dt$

$y \cdot \sin x = \int t^2 dt + C$

$y \cdot \sin x = \frac{t^3}{3} + C = y \cdot \sin x$

$= \frac{\sin^3 x}{3} + C$ **(1)**

$[\text{Put } t = \sin x]$

Section D

33. Find the area of the region bounded by the paraboles $y^2 = 4ax$ and $x^2 = 4ay$. [6]

Ans :

Given parabolas are,

$y^2 = 4ax$ and $x^2 = 4ay$.

For points of intersection of these parabolas, consider

$\left(\frac{x^2}{4a}\right)^2 = 4ax$ $[\because x^2 = 4ay]$

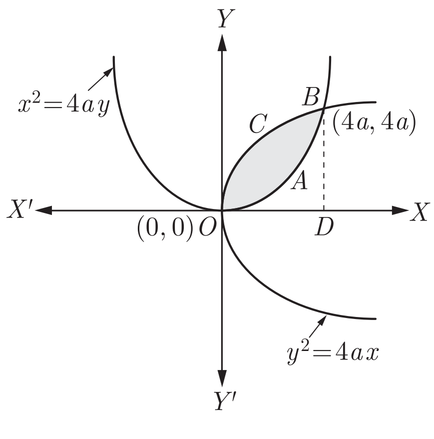
$\Rightarrow$ $x^4 = 64a^3 x$

$\Rightarrow$ $x(x^3 - 64a^3) = 0 \Rightarrow x = 0, 4a$

when $x = 0$, then $y = 0$

when $x = 4a$, then $y = 4a$ **(1)**

So, the points of intersection are (0,0) and $(4a, 4a)$.



(2)

$\therefore$ Area of the shaded region

$= \text{Area of region } ODBCO$

$- \text{Area of region } ODBAO$

$= \int_0^{4a} y(\text{parabola } y^2 = 4ax) dx$

$- \int_0^{4a} y(\text{parabola } x^2 = 4ay) dx$ **(1)**

$= \int_0^{4a} \sqrt{4ax} \, dx - \int_0^{4a} \frac{x^2}{4a} dx$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= 2\sqrt{a}\,\frac{(4a)^{3/2}}{\frac{3}{2}} - \frac{1}{4a}\left[\frac{64a^3}{3}\right]$$

$$= \frac{4}{3}\times 8a^2 - \frac{16}{3}a^2$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

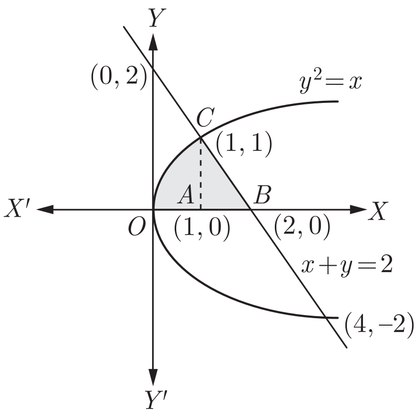
$$= \frac{16}{3}a^2 \text{ sq units.} \qquad \textbf{(2)}$$

or

Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$ and the X-axis.

Ans :

Given parabola is $y^2 = x$ and line $x + y = 2$.



(2)

$$y^2 = 2 - y \qquad [\because x + y = 2]$$

$$\Rightarrow \quad y^2 + y - 2 = 0$$

$$\Rightarrow \quad y^2 + 2y - y - 2 = 0$$

$$\Rightarrow \quad y(y+2) - 1(y+2) = 0$$

$$\Rightarrow \quad (y-1)(y+2) = 0$$

$$\therefore \quad y = 1, -2$$

When $y = 1$, then $x = 1$ and when $y = -2$, then $x = 4$.

So, the points of intersection are (1,1) and $(4, -2)$. **(1)**

$\therefore$ Area of the shaded region

= Area of region $OACO +$ Area of region $ABCA$

$$= \int_0^1 y(\text{parabola})\,dx + \int_1^2 y(\text{line})\,dx$$

$$= \int_0^1 \sqrt{x}\,dx + \int_1^2 (2-x)\,dx \qquad \textbf{(1)}$$

$$= \frac{2}{3}[x^{3/2}]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{2}{3}(1-0) + \left[\left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{2}{3} + \frac{4}{2} - \frac{3}{2}$$

$$= \frac{2}{3} + \frac{1}{2} = \frac{7}{6} \text{ sq units.} \qquad \textbf{(2)}$$

34. A merchant plant to sell two types of personal computers, a desktop model and a portable model that will cost ₹25000 and ₹40000, respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Find the number of units of each type of computers which the merchant should stock to get maximum profit, if he does not want to invest more than ₹70 lakh and his profit on the desktop model is ₹4500 and on the portable model is ₹5000. Make an LPP and solve it graphically. [6]

Ans :

Let merchant stock x units of desktop model and y unit of portable model.

Total profit to maximise,

$$Z = 4500x + 5000y$$

Subject to constraints are,

$$x + y \le 250 \qquad \text{(i)}$$

$$25000x + 40000y \le 7000000$$

$$\Rightarrow \quad 5x + 8y \le 1400 \qquad \text{...(ii)}$$

and $\quad x \ge 0,\ y \ge 0$ **(1)**

Firstly, draw the graph of line $x + y = 250$

x	0	250
y	250	0

On putting (0,0) in the inequality.

$x + y \le 250$, we get

$0 + 0 \le 250 \Rightarrow 0 \le 250$, which is true.

So, the half plane is towards the origin. **(1)**

Secondly, draw the graph of line $5x + 8y = 1400$

x	0	280
y	175	0

On putting (0,0) in the inequality $5x + 8y \le 1400$, we get

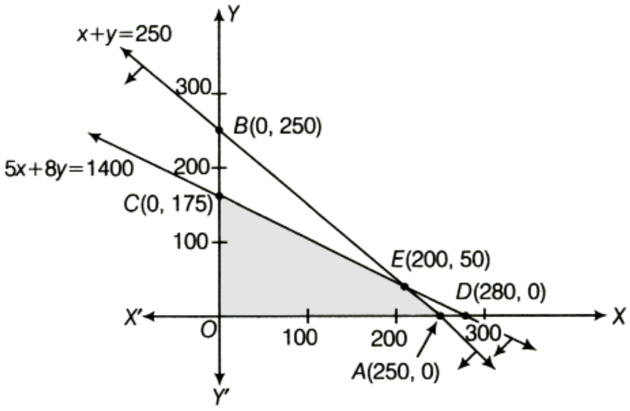
$$5(0) + 8(0) \le 1400$$

$\Rightarrow \quad 0 \le 1400$. which is true. **(1)**

So, the half plane is towards the origin.

Also, $x \ge 0$, $y \ge 0$, it means feasible region

lies in 1st quadrant.
The point of intersection of lines (1) and (2) is $E(200, 50)$.



(2)

Now, we draw all the lines on a graph paper, the feasible region is $OAECO$. The corner points of the feasible region are $O(0,0)$, $A(250,0)$, $E(200,50)$ and $C(0,175)$.

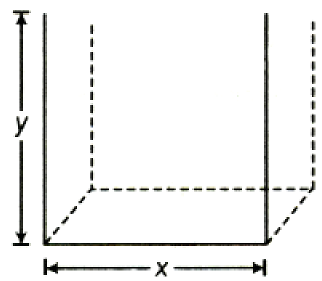
Corner Points	$\boldsymbol{Z = 4500x + 5000y}$
$A(250,0)$	$Z = 4500 \times 250 + 5000 \times 0$ $= 1125000$
$E(200,50)$	$Z = 4500 \times 200 + 5000 \times 50$ $= 1150000$
$C(0,175)$	$Z = 4500 \times 0 + 5000 \times 175$ $= 875000$
$O(0,0)$	$Z = 4500 \times 0 + 5000 \times 0$ $= 0$

Clearly, the maximum profit is ₹1150000 at $E(200,50)$, i.e., when 200 desktops and 50 portable models are in stock. **(1)**

35. An open box with a square base is to be made out of a given quantity of metal sheet of area c^2. Show that maximum volume of the box is $c^3/6\sqrt{3}$. [6]

Ans :

Let, x be the side of the square base and y be the height of the open box.



Then, c^2 = Surface area of the open box $= x^2 + 4xy$ **(1)**

$$\Rightarrow \quad y = \frac{c^2 - x^2}{4x}$$

Now, V = Volume of the box

$$\Rightarrow \quad V = x^2 y = x^2 \cdot \left(\frac{c^2 - x^2}{4x}\right)$$

$$\Rightarrow \quad V = \frac{1}{4}(c^2 x - x^3) \quad \text{...(i)} \quad \textbf{(1)}$$

On differentiating both sides twice, w.r.t. x, we get

$$\frac{dV}{dx} = \frac{1}{4}(c^2 - 3x^2)$$

and $\dfrac{d^2V}{dx^2} = \dfrac{1}{4}(-6x) = \dfrac{-3}{2}x$ ...(ii) **(1)**

For maximum or minimum value, put $\dfrac{dV}{dx} = 0$.

$$\Rightarrow c^2 - 3x^2 = 0 \Rightarrow x = \pm\frac{c}{\sqrt{3}} \quad \textbf{(1)}$$

As x (length of side) can never be negative, so $x = \dfrac{c}{\sqrt{3}}$.

On putting $x = \dfrac{c}{\sqrt{3}}$ in Eq.(ii), we get

$$\frac{d^2V}{dx^2} = \frac{-3}{2} \cdot \frac{c}{\sqrt{3}}$$

$$= \frac{-\sqrt{3}}{2}c < 0 \quad [\because c > 0] \quad \textbf{(1)}$$

So, V is maximum at $x = \dfrac{c}{\sqrt{3}}$

$$\therefore \text{ Maximum value of } V = \frac{1}{4}\left[c^2 \cdot \frac{c}{\sqrt{3}} - \left(\frac{c}{\sqrt{3}}\right)^3\right]$$

[from eq. (i)]

$$= \frac{c^3}{6\sqrt{6}} \quad \textbf{(1)}$$

or

Find all the points of local maxima and local minima of $f(x) = -x + 2\sin x$ on $[0, 2\pi]$. Also, find local maximum and minimum values.

Ans :

We have, $f(x) = -x + 2\sin x$

On differentiating both sides w.r.t. x, we get

$$f'(x) = -1 + 2\cos x \quad \text{...(i)} \quad \textbf{(1)}$$

For local maxima and local minima, put $f'(x) = 0$.

$$\Rightarrow \quad -1 + 2\cos x = 0 \Rightarrow \cos x = \frac{1}{2}$$

$$\Rightarrow \quad x = 2n\pi \pm \frac{\pi}{3} \Rightarrow x = \frac{\pi}{3},$$

$$\frac{5\pi}{3} \in [0, 2\pi] \quad \textbf{(1)}$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$f''(x) = -2\sin x$$

At $\quad x = \frac{\pi}{3},\ f''\left(\frac{\pi}{3}\right) = -2\sin\frac{\pi}{3}$

$$= -\sqrt{3} < 0 \quad \textbf{(1)}$$

$\therefore\ x = \frac{\pi}{3}$ is a point of local maxima

At, $\quad x = \frac{5\pi}{3},\ f''\left(\frac{5\pi}{3}\right)$

$$= -2\sin\frac{5\pi}{3} = \sqrt{3} > 0$$

$\therefore\ x = \frac{5\pi}{3}$ is a point of local minima.

Hence, the points of local maxima is $\frac{\pi}{3}$ and local minima is $\frac{5\pi}{3}$. **(1)**

On putting $x = \frac{\pi}{3}$ in $f(x)$, we get

$$f\left(\frac{\pi}{3}\right) = -\frac{\pi}{3} + 2\sin\frac{\pi}{3}$$

$$= -\frac{\pi}{3} + 2 \times \frac{\sqrt{3}}{2}$$

$$= \frac{-\pi}{3} + \sqrt{3} \quad \textbf{(1)}$$

which is the required local maximum value.

On putting $x = \frac{5\pi}{3}$ in $f(x)$, we get

$$f\left(\frac{5\pi}{3}\right) = -\frac{5\pi}{3} + 2\sin\frac{5\pi}{3}$$

$$= \frac{-5\pi}{3} + 2 \times \left(\frac{-\sqrt{3}}{2}\right)$$

$$= \frac{-5\pi}{3} - \sqrt{3}$$

which is the required local minimum value. **(1)**

36. A manufacturer produces three models of toys in the form of bikes say X, Y and Z. Model X takes as 10 man-hour to make per unit, Model Y takes 5 man-hour per unit and model Z takes 4 man-hour per unit. There are a total 212 man-hour available per week. Handling and marketing costs are ₹ 20, ₹30 and ₹40 per unit for models X, Y and Z respectively. The total funds available for these purposes are ₹920 per week. Profits per unit for models X, Y and Z are ₹40, ₹10 and ₹70 respectively, but at the end of the week, company get a profit of ₹810. Solve the system of equations by matrix method. [6]

Ans :

let x, y and z denote the number of bikes of models, X, Y and Z, respectively.

(i) Given conditions in linear equations are,

$$10x + 5y + 4z = 212,$$

$$20x + 30y + 40z = 920$$

$$\text{and} \quad 40x + 10y + 70z = 810$$

Above system of equation can be written in matrix form as.

$$\begin{bmatrix} 10 & 5 & 4 \\ 20 & 30 & 40 \\ 40 & 10 & 70 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 212 \\ 920 \\ 810 \end{bmatrix}$$

i.e. $\quad AX = B \Rightarrow X = A^{-1}B \quad$ ...(i)

where, $\quad A = \begin{bmatrix} 10 & 5 & 4 \\ 20 & 30 & 40 \\ 40 & 10 & 70 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $\quad B = \begin{bmatrix} 212 \\ 920 \\ 810 \end{bmatrix} \quad$ **(1)**

Now, $\quad |A| = \begin{vmatrix} 10 & 5 & 4 \\ 20 & 30 & 40 \\ 40 & 10 & 70 \end{vmatrix}$

$$= 10(2100 - 400) - 5(1400 - 1600)$$
$$+ 4(200 - 1200) \text{ [expanding along } R_1]$$
$$= 10(1700) - 5(-200) + 4(-1000)$$
$$= 17000 + 1000 - 4000$$
$$= 14000 \quad \textbf{(1)}$$

Now, the cofactors of determinant A are

$$A_{11} = \begin{vmatrix} 30 & 40 \\ 10 & 70 \end{vmatrix} = 2100 - 400$$
$$= 1700$$

$$A_{12} = -\begin{vmatrix} 20 & 40 \\ 40 & 70 \end{vmatrix}$$
$$= -(1400 - 1600)$$
$$= 200$$

$$A_{13} = \begin{vmatrix} 20 & 30 \\ 40 & 10 \end{vmatrix} = 200 - 1200$$
$$= -1000$$

$$A_{21} = -\begin{vmatrix} 5 & 4 \\ 10 & 70 \end{vmatrix} = -(350 - 40)$$
$$= -310$$

$$A_{22} = \begin{vmatrix} 10 & 4 \\ 40 & 70 \end{vmatrix} = 700 - 160$$

$$= 540$$

$$A_{23} = -\begin{vmatrix} 10 & 5 \\ 40 & 10 \end{vmatrix} = -(100 - 200)$$

$$= 100$$

$$A_{31} = \begin{vmatrix} 5 & 4 \\ 30 & 40 \end{vmatrix} = 200 - 120$$

$$= 80$$

$$A_{32} = -\begin{vmatrix} 10 & 4 \\ 20 & 40 \end{vmatrix} = -(400 - 80)$$

$$= -320$$

$$A_{33} = \begin{vmatrix} 10 & 5 \\ 20 & 30 \end{vmatrix} = 300 - 100$$

$$= 200 \qquad \textbf{(2)}$$

$$\therefore \quad \text{adj.}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1700 & 200 & -1000 \\ -310 & 540 & 100 \\ 80 & -320 & 200 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1700 & -310 & 80 \\ 200 & 540 & -320 \\ -1000 & 100 & 200 \end{bmatrix}$$

Now, $A^{-1} = \dfrac{\text{adj}(A)}{|A|}$

$$= \frac{1}{14000}\begin{bmatrix} 1700 & -310 & 80 \\ 200 & 540 & -320 \\ -1000 & 100 & 200 \end{bmatrix} \qquad \textbf{(1)}$$

From Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14000}\begin{bmatrix} 1700 & -310 & 80 \\ 200 & 540 & -320 \\ -1000 & 100 & 200 \end{bmatrix}\begin{bmatrix} 212 \\ 920 \\ 810 \end{bmatrix}$$

$$= \frac{1}{14000}\begin{bmatrix} 360400 & -285200 & +64800 \\ 42400 & +496800 & -259200 \\ -212000 & +92000 & +162000 \end{bmatrix}$$

$$= \frac{1}{14000}\begin{bmatrix} 140000 \\ 280000 \\ 42000 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 3 \end{bmatrix}$$

$$\therefore \qquad x = 10,\ y = 20 \text{ and } z = 3 \qquad \textbf{(1)}$$