CLASS XII (2019-20)
MATHEMATICS (041)
MOCK TEST-1

Time : 3 Hours **Maximum Marks : 80**

General Instructions :

(i) All questions are compulsory.
(ii) The questions paper consists of 36 questions divided into 4 sections A, B, C and D.
(iii) Section A comprises of 20 questions of 1 mark each. Section B comprises of 6 questions of 2 marks each. Section C comprises of 6 questions of 4 marks each. Section D comprises of 4 questions of 6 marks each.
(iv) There is no overall choice. However, an internal choice has been provided in three questions of 1 mark each, two questions of 6 marks each. You have to attempt only one of the alternatives in all such questions.
(v) Use of calculators is not permitted.

Section-A

DIRECTION : (Q 1-Q 10) are multiple choice type questions. Select the correct option.

1. Let R be the relation in the set $\{1,2,3,4\}$ given by $R = \{(1,2),(2,2),(1,1),(4,4),(1,3),(3,3),(3,2)\}$. Then, [1]

(a) R is reflexive and transitive but not symmetric
(b) R is reflexive and symmetric but not transitive
(c) R is symmetric and transitive but not reflexive
(d) R is an equivalence relation

Ans : (a) R is reflexive and transitive but not symmetric

Here, $R = \{(1,2),(2,2),(1,1),(4,4),(1,3),(3,3),(3,2)\}$

Since, $(a,a) \in R$ for every $a \in \{1,2,3,4\}$.
$\therefore$ R is reflexive.
Now, since $(1,2) \in R$ but $(2,1) \notin R$
$\therefore$ R is not symmetric.
Also, it is observed that

$$(a,b)(b,c) \in R$$
$$(a,c) \in R$$

For all $a,b,c \in \{1,2,3,4\}$
$\therefore$ R is transitive.
Hence, R is reflexive and transitive but not symmetric.

2. The normal at the point (0, 1) on the curve $y = e^{2x} + x^2$ is [1]

(a) $x + y = 0$ (b) $x + 2y = 2$
(c) $x + 2y + 1 = 0$ (d) $x - y + 1 = 0$

Ans : (b) $x + 2y = 2$

The given curve is $y = e^{2x} + x^2$
On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2e^{2x} + 2x$$

Slope of normal at (0, 1) is

$$\frac{-1}{\left(\frac{dy}{dx}\right)_{(0,1)}} = \frac{-1}{2}$$

Equation of the normal at (0, 1) is

$$y - 0 = \left(\frac{-1}{2}\right)(x - 1)$$

$\therefore$ $x + 2y = 2$

3. The probability of obtaining an even prime number on each die when a pair of dice is rolled, is [1]

(a) zero (b) $\frac{1}{3}$
(c) $\frac{1}{12}$ (d) $\frac{1}{36}$

Ans : (d) $\frac{1}{36}$

When a pair of dice is rolled
Total number of outcomes $= 6^2 = 36$
The only even prime is 2.
Let A be the event getting an even prime number of each die i.e., $A = \{2,2\}$.

$\therefore$ Required probability $= \frac{1}{36}$

4. If $\vec{a}$ is a non-zero vector of magnitude $|\vec{a}|$ and λ is a non-zero scalar, then λa is unit vector, if [1]

(a) $\lambda = 1$ (b) $\lambda = -1$

(c) $|\vec{a}| = |\lambda|$ (d) $|\vec{a}| = \frac{1}{|\lambda|}$

Ans : (d) $|\vec{a}| = \frac{1}{|\lambda|}$

Vector λa is a unit vector.

$$\therefore \quad |\lambda \vec{a}| = 1$$

$$|\lambda||\vec{a}| = 1$$

$$|\vec{a}| = \frac{1}{|\lambda|}$$

5. $\int_{-5}^{-5} |x+2| dx$ is equal to [1]

(a) 22 (b) 29

(c) 35 (d) 15

Ans : (b) 29

Let $I = \int_{-5}^{5} |x+2| dx$

It can be seen that $(x+2) \leq 0$ on $[-5, -2]$ and $(x+2) \geq 0$ on $[-2, 5]$.

$$\therefore I = \int_{-5}^{-2} -(x+2)\,dx + \int_{-2}^{5} (x+2)\,dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right]$$

$$I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right]$$

$$+ \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$$

$$= -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$$

$$= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4$$

$$= 29$$

6. The number of arbitrary constants in the particular solution of differential equation of third order is [1]

(a) 3 (b) 2

(c) 1 (d) 0

Ans : (d) 0

We know that in particular solution of differential equation is free from arbitrary constant.

7. The total revenue in rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue when $x = 15$ is [1]

(a) 116 (b) 96

(c) 90 (d) 126

Ans : (d) 126

Given, $R(x) = 3x^2 + 36x + 5$

Marginal revenue, $MR = \frac{dR}{dx}$

$$\therefore \quad \frac{dR}{dx} = 6x + 36$$

when $x = 15$,

$$MR = 6 \times 15 + 36$$

$$= 90 + 36 = 126$$

Hence, the required marginal revenue is 126 at $x = 15$.

8. $\int_0^2 x\sqrt{2-x}\,dx$ is equal to [1]

(a) $\frac{16\sqrt{2}}{15}$ (b) $\frac{3\sqrt{2}}{5}$

(c) $\frac{4\sqrt{3}}{5}$ (d) $\frac{6\sqrt{5}}{7}$

Ans : (a) $\frac{16\sqrt{2}}{15}$

Let $I = \int_0^2 x\sqrt{2-x}\,dx$ ...(i)

Also, $I = \int_0^2 (2-x)\sqrt{2-(2-x)}\,dx$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^2 (2-x)\sqrt{x}\,dx$$

$$= \int_0^2 (2x^{1/2} - x^{3/2})\,dx$$

$$= \left[\frac{2x^{(1/2)+1}}{\left(\frac{1}{2}\right)+1} - \frac{x^{(3/2)+1}}{\left(\frac{3}{2}\right)+1}\right]_0^2$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^2$$

$$= \frac{4}{3} \cdot 2^{3/2} - \frac{2}{5} \cdot 2^{5/2} - 0$$

$$= \frac{4}{3}2\sqrt{2} - \frac{2}{5}4\sqrt{2}$$

$$= \left(\frac{8}{3} - \frac{8}{5}\right)\sqrt{2} = \left(\frac{40-24}{15}\right)\sqrt{2}$$

$$= \frac{16\sqrt{2}}{15}$$

9. For the function $f(x) = xe^x$, the point [1]

(a) $x = 0$ is a maximum

(b) $x = 0$ is a minimum

(c) $x = -1$ is a maximum

(d) $x = -1$ is a minimum

Ans : (d) $x = -1$ is a minimum

Given, $f(x) = xe^x$

$$f'(x) = e^x + xe^x$$
$$f''(x) = e^x + xe^x + e^x$$
$$= 2e^x + xe^x$$

For maxima or minima, put $f'(x) = 0$

$$e^x(1+x) = 0$$
$$x = -1$$

At $x = -1$, $f''(x) > 0$

So, at $x = -1$, $f(x)$ is minimum.

10. $\int_0^2 \{x\} dx$ is equal to (where $\{x\}$ is fraction part of x) [1]

(a) 2 (b) 1

(c) 5 (d) 4

Ans : (b) 1

Let $I = \int_0^2 \{x\} dx$

$$I = \int_0^2 (x - [x]) dx \quad [\because \{x\} = x - [x]]$$
$$= \int_0^2 x\,dx - \int_0^2 [x] dx$$
$$= \int_0^2 x\,dx - \int_0^1 0\,dx - \int_1^2 dx$$
$$= \left[\frac{x^2}{2}\right]_0^2 - [x]_1^2$$
$$= \left(\frac{4}{2} - 0\right) - (2 - 1)$$
$$= 2 - 0 - 2 + 1 = 1$$

DIRECTION : **(Q 11-Q 15) fill in the blanks**

11. A feasible solution which leads to an optimal value of the objective function is called .... [1]

Ans : Optimal solution

12. The range of $\cos^{-1}x$ is .... [1]

Ans : $[0, \pi]$

13. Every differentiable function is continuous. But a continuous function may or may not be .... [1]

Ans : Differentiable

or

Let $f : [a, b] \to R$ be a continuous function on $[a, b]$ and differential function in $[a, b]$. By mean value theorem, there exists atleast one c in $[a, b]$ such that $f'(c) =$ .... [1]

Ans :

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

14. If A and B are square matrices such that $AB = BA$, then $(A + B)^2 =$ .... [1]

Ans : $A^2 + 2AB + B^2$

or

Transpose of a column matrix is a ....

Ans : Row matrix

15. $\int_0^{2a} f(x) dx = \int_0^a f(x) dx +$ .... [1]

Ans : $\int_0^a f(2a - x) dx$

DIRECTION : **(Q 16-Q 20) Answer the following questions.**

16. If $y = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - ...\infty$, then prove that $\frac{d^2y}{dx^2} - y = 0$. [1]

Ans :

We have,

$$y = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - ...\infty$$
$$y = e^{-x}$$
$$\therefore \quad \frac{dy}{dx} = -e^{-x}$$
$$\frac{d^2y}{dx^2} = e^{-x}$$
$$\frac{d^2y}{dx^2} = y$$
$$\frac{d^2y}{dx^2} - y = 0$$

Hence proved.

17. If A and B are matrices of order 3 and $|A| = 5$, $|B| = 3$, then find $|3AB|$. [1]

Ans :

Given, $|A| = 5, \ |B| = 3$

We know that,

$$|kA| = k^n|A|$$

where A is matrix of order n.

$$\therefore \quad |3AB| = 3^3|A| \cdot |B|$$
$$= 27 \times 5 \times 3 = 405$$

18. Find the direction cosines of the line passing through the two points (–2, 4, –5) and (1, 2, 3). [1]

Ans :

Let $(x_1, y_1, z_1) \equiv (-2, 4, -5)$ and

$$(x_2, y_2, z_2) \equiv (1, 2, 3)$$

DR's of the line are

$$1-(-2),\ 2-4,\ 3-(-5) = 3, -2, 8$$

[$\because$ DR's of the line are $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$]

$\therefore$ DC's are

$$= \frac{3}{\sqrt{(3)^2+(-2)^2+(8)^2}}, \frac{-2}{\sqrt{(3)^2+(-2)^2+(8)^2}}, \frac{8}{\sqrt{(3)^2+(-2)^2+(8)^2}}$$

$$= \frac{3}{\sqrt{9+4+64}}, \frac{-2}{\sqrt{9+4+64}}, \frac{8}{\sqrt{9+4+64}}$$

$$= \frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$$

or

Find the distance of the point whose positive vector is $(2\hat{i}+\hat{j}-\hat{k})$ from the plane $\vec{r}(\hat{i}-2\hat{j}+4\hat{k}) = 9$.

Ans :

Here, $\vec{a} = 2\hat{i}+\hat{j}-\hat{k}$

$\vec{n} = \hat{i}-2\hat{j}+4\hat{k}$ and $d = 9$

So, the required distance is

$$\frac{\left|(2\hat{i}+\hat{j}-\hat{k})(\hat{i}-2\hat{j}+4\hat{k})-9\right|}{\sqrt{1+4+16}} = \frac{|2-2-4-9|}{\sqrt{21}}$$

$$= \frac{13}{\sqrt{21}}$$

19. Evaluate $\int_0^1 3^{x-[x]}dx$. [1]

Ans :

Let $I = \int_0^1 3^{x-[x]}dx$

$$= \int_0^1 3^{x-0}dx \quad [\because [x] = 0, x \in (0,1)]$$

$$= \left[\frac{3^x}{\log 3}\right]_0^1 \quad \left[\because \int a^x dx = \frac{a^x}{\log a}\right]$$

$$= \left[\frac{3^1}{\log 3} - \frac{3^0}{\log 3}\right] = \frac{3}{\log 3} - \frac{1}{\log 3}$$

$$= \frac{2}{\log 3}$$

20. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'number is even' and B be the event, 'number is red'. Are A and B are independent ? [1]

Ans :

When a die is thrown, the sample space

$$S = \{1,2,3,4,5,6\}$$

Also, A = Number is even,

B = Number is red

$$A = \{2,4,6\},$$

$$B = \{1,2,3\},$$

$$A \cap B = \{2\}$$

$$P(A) = \frac{3}{6},\quad P(B) = \frac{3}{6},$$

$$P(A \cap B) = \frac{1}{6}$$

$$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(A) \times P(B) \neq P(A \cap B)$$

Thus, A and B are not independent event.

Section B

21. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B, respectively, find the position vector of a point C on BA produced such that $BC = 1.5BA$. [2]

Ans :

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$

We have, $\overrightarrow{BC} = 1.5\,\overrightarrow{BA}$

$\because$ $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

and $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

$$\therefore \quad \overrightarrow{OC} - \overrightarrow{OB} = 1.5(\overrightarrow{OA} - \overrightarrow{OB})$$

$$\overrightarrow{OC} - \vec{b} = 1.5(\vec{a} - \vec{b}) \quad [1]$$

$$\overrightarrow{OC} = 1.5\vec{a} - 1.5\vec{b} + \vec{b}$$

$$= 1.5\vec{a} - 0.5\vec{b}$$

$$= \frac{3}{2}\vec{a} - \frac{1}{2}\vec{b}$$

$$\therefore \quad \vec{c} = \frac{3\vec{a} - \vec{b}}{2} \quad [1]$$

22. Show that the function $f(x)$ given by $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at $x = 0$. [2]

Ans :

At $x = 0$,

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} -h\sin\left(\frac{1}{-h}\right)$$

$$= 0 \times$$

(An oscillating number between –1 and 1)

$$= 0 \quad \textbf{[1/2]}$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} h \sin \frac{1}{h}$$

$$= 0 \times$$

(An oscillating number between –1 and 1)

$$= 0 \quad \textbf{[1/2]}$$

and $\quad f(0) = 0$

Thus, $f(0) = \text{LHL} = \text{RHL}$

$\therefore \quad f(x)$ is continuous at $x = 0$. **[1/2]**

or

Differentiate $(\log \sin x)$ with respect to $\sqrt{\cos x}$.

Ans :

Let $u = \log \sin x$ and $v = \sqrt{\cos x}$

Then, $\qquad \dfrac{du}{dx} = \cot x$

and $\qquad \dfrac{dv}{dx} = -\dfrac{\sin x}{2\sqrt{\cos x}}$ **[1]**

$$\therefore \qquad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = -\frac{\cot x}{\left(\frac{\sin x}{2\sqrt{\cos x}}\right)}$$

$$= -2\sqrt{\cos x} \cot x \operatorname{cosec} x$$

[1]

23. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in R. [2]

Ans :

We have, $\qquad f(x) = x^3 - 3x^2 + 3x - 100$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 3x^2 - 6x + 3$$

$$= 3(x^2 - 2x + 1)$$

$$= 3(x-1)^2 \geq 0$$

for real values of x **[1]**

$\therefore \quad f(x)$ is increasing in R. **[1]**

Hence proved.

24. A fair die is rolled. Consider the following events $A = \{2,4,6\}$, $B = \{4,5\}$ and $C = \{3,4,5,6\}$. Find

(i) $P\left(\dfrac{A \cup B}{C}\right)$,

(ii) $P\left(\dfrac{A \cap B}{C}\right)$. [2]

Ans :

Given, events are

$$A = \{2,4,6\}$$

$$B = \{4,5\}$$

and $\qquad C = \{3,4,5,6\}$

Sample space, $\quad S = \{1,2,3,4,5,6\}$

Now, $\qquad A \cup B = \{2,4,5,6\}$

$$A \cap B = \{4\}$$

$$A \cup B \cap C = \{2,4,5,6\} \cap \{3,4,5,6\}$$

$$= \{4,5,6\}$$

and $\qquad A \cap B \cap C = \{4\} \cap \{3,4,5,6\} = \{4\}$

$\therefore \qquad n(S) = 6$

$$n(A \cup B \cap C) = 3$$

$$n(A \cap B \cap C) = 1$$

and $\qquad n(C) = 4$

(i) $\qquad P\left(\dfrac{A \cup B}{C}\right) = \dfrac{P(A \cup B \cap C)}{P(C)}$

$$= \frac{\frac{n(A \cup B \cap C)}{n(S)}}{\frac{n(C)}{n(S)}}$$

$$= \frac{\frac{3}{6}}{\frac{4}{6}} = \frac{3}{4} \quad \textbf{[1]}$$

(ii) $\qquad P\left(\dfrac{A \cap B}{C}\right) = \dfrac{P(A \cap B \cap C)}{P(C)}$

$$= \frac{\frac{n(A \cap B \cap C)}{n(S)}}{\frac{n(C)}{n(S)}}$$

$$= \frac{\frac{1}{6}}{\frac{4}{6}} = \frac{1}{4} \quad \textbf{[1]}$$

25. Show that the determinant value of a skew-symmetric matrix of odd order is always zero. [2]

Ans :

Let A be a skew-symmetric matrix of order n.

Then, $\qquad A' = -A$

$$|A'| = |-A|$$

$$|A'| = (-1)^n |A| \quad \textbf{[1]}$$

$$|A'| = -|A| \qquad [\because \ n \text{ is odd}]$$

$$|A| = -|A| \qquad [\because \ |A'| = |A|]$$

$$2|A| = 0$$

$$|A| = 0 \quad \textbf{[1]}$$

Hence, determinant value of a skew-symmetric matrix of odd order is always zero.

or

Without expanding, show that

$$\Delta = \begin{vmatrix} \text{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \text{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0$$

Ans :

We have,

$$\Delta = \begin{vmatrix} \text{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \text{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$$

On applying $C_1 \to C_1 - C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} \text{cosec}^2\theta - \cot^2\theta & \cot^2\theta & 1 \\ \cot^2\theta - \text{cosec}^2\theta + 1 & \text{cosec}^2\theta & -1 \\ 42-40-2 & 40 & 2 \end{vmatrix} \quad \textbf{[1]}$$

$$\Delta = \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \text{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix}$$

$\Delta = 0$ Hence proved. **[1]**

26. Find the minimum value of n for which $\tan^{-1}\dfrac{n}{\pi} > \dfrac{\pi}{4},\ n \in N$. [2]

Ans :

We have,

$$\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}$$

$$\tan\left(\tan^{-1}\frac{n}{\pi}\right) > \tan\frac{\pi}{4}$$

$$\frac{n}{\pi} > 1 \quad \textbf{[1]}$$

$$n > \pi$$

$$n > 3.14$$

$$n = 4, 5, 6, \ldots \quad [\because n \in N]$$

Hence, the minimum value of n is 4. **[1]**

Section C

27. Find the equation of a curve passing through the point (0, 1), if the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x-coordinate (abscissa) and the product of the x-coordinate and y-coordinate (ordinate) of that point. [4]

Ans :

We know that, slope of tangent to the curve $y = f(x)$ at any point (x, y) is $\dfrac{dy}{dx}$.

According to the given condition, we have

$$\frac{dy}{dx} = x + xy$$

or $$\frac{dy}{dx} - xy = x$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where, $P = -x$ and $Q = x$

Now, $IF = e^{\int P dx} = e^{\int(-x)dx} = e^{-x^2/2}$ **[1]**

Now, the solution is given by

$$y \cdot (IF) = \int (Q \cdot IF)\,dx + c$$

$$y \cdot e^{-x^2/2} = \int x \cdot e^{-x^2/2}\,dx + c \quad \textbf{[1/2]}$$

Now, put $\dfrac{x^2}{2} = t \Rightarrow xdx = dt$

$$\therefore \quad y \cdot e^{-x^2/2} = \int e^{-t}\,dt + c \quad \textbf{[1]}$$

$$= -e^{-t} + c = -e^{-x^2/2} + c$$

$$y = -1 + ce^{x^2/2} \quad \ldots\text{(i)}$$

Since, the curve passes through the point (0, 1).

We have, $1 = -1 + ce^0$

$c = 2$ **[1/2]**

Hence, the required equation of curve is

$$y = -1 + 2e^{x^2/2} \quad \textbf{[1]}$$

28. Evaluate $\displaystyle\int \frac{1+x^2}{1+x^4}\,dx$. [4]

Ans :

Let $$I = \int \frac{1+x^2}{1+x^4}\,dx = \int \frac{\frac{1}{x^2} + \frac{x^2}{x^2}}{\frac{1}{x^2} + \frac{x^4}{x^2}}\,dx$$

[1/2]

[divide numerator and denominator by x^2]

$$= \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\,dx \quad \textbf{[1/2]}$$

$$= \int \frac{1 + \frac{1}{x^2}}{\left(x^2 + \frac{1}{x^2} - 2 + 2\right)}\,dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2}\,dx \quad \textbf{[1/2]}$$

Now, put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

$\therefore \quad I = \int \frac{dt}{t^2+2}$

$= \int \frac{dt}{t^2+(\sqrt{2})^2}$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)$

$\left[\because \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$ **[1/2]**

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)$

$\left[\because t = x-\frac{1}{x}\right]$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right)+c$

[1/2]

or

Evaluate $\int x\cdot(\log x)^2\,dx$.

Ans :

Let $\quad I = \int \underset{\text{II}}{x}\cdot\underset{\text{I}}{(\log x)^2}\,dx$

On applying integration by parts, we get

$I = (\log x)^2\int x dx - \int\left[\frac{d}{dx}(\log x)^2\cdot\int x dx\right]dx$

[1]

$= (\log x)^2\cdot\frac{x^2}{2} - \int\left[\frac{2\log x}{x}\times\frac{x^2}{2}\right]dx$

$= \frac{x^2}{2}(\log x)^2 - \int \underset{\text{II}}{x}\,\underset{\text{I}}{\log x}\,dx$ **[1]**

Again applying integration by parts, we get

$I = \frac{x^2}{2}(\log x)^2 - \left[\log x\int x dx - \int\left(\frac{d}{dx}\log x\int x dx\right)dx\right]$

$= \frac{x^2}{2}(\log x)^2 - \left[\log x\cdot\frac{x^2}{2} - \int\frac{1}{x}\cdot\frac{x^2}{2}dx\right]$ **[1]**

$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\cdot\log x + \int\frac{x}{2}dx$

$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + c$ **[1]**

29. A can hit target 4 times out of 5 times, B can hit target 3 times out of 4 times and C can hit target 2 times out of 3 times.
They fire simultaneously. Find the probability that
(i) any two out of A, B and C will hit the target.
(ii) none of them will hit the target.

Ans :

Here, $P(A) = P$ (A hit the target) $= \frac{4}{5}$

$P(B) = P$ (B hit the target) $= \frac{3}{4}$

and $P(C) = P$ (C hit the target) $= \frac{2}{3}$ **[1/2]**

Then, $\quad P(\overline{A}) = 1-P(A) = 1-\frac{4}{5} = \frac{1}{5}$

$P(\overline{B}) = 1-P(B) = 1-\frac{3}{4} = \frac{1}{4}$

and $\quad P(\overline{C}) = 1-P(C) = 1-\frac{2}{3} = \frac{1}{3}$

[1/2]

(i) P (any two of them hit the target)

$= P(A\cap B\cap\overline{C}) + P(A\cap\overline{B}\cap C) + P(\overline{A}\cap B\cap C)$

$= P(A)P(B)P(\overline{C}) + P(A)P(\overline{B})P(C) + P(\overline{A})P(B)P(C)$

$= \left(\frac{4}{5}\times\frac{3}{4}\times\frac{1}{3}\right)+\left(\frac{4}{5}\times\frac{1}{4}\times\frac{2}{3}\right) +\left(\frac{1}{5}\times\frac{3}{4}\times\frac{2}{3}\right)$ **[1½]**

$= \frac{12}{60}+\frac{8}{60}+\frac{6}{60} = \frac{26}{60} = \frac{13}{30}$ **[1/2]**

(ii) P (none of them hit the target)

$= P(\overline{A})P(\overline{B})P(\overline{C})$

$= \frac{1}{5}\times\frac{1}{4}\times\frac{1}{3} = \frac{1}{60}$ **[1]**

or

In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that, a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that a student knows the answer given that he answered it correctly ?

Ans :

Let E_1 : the event that the student knows the answer and E_2 : the event that the student guesses the answer.
Therefore, E_1 and E_2 are mutually exclusive and exhaustive.

$\therefore \quad P(E_1) = \frac{3}{4}$

and $\quad P(E_2) = \frac{1}{4}$

Let E : the answer is correct. **[1]**

The probability that the student answered correctly, given that he knows the answer, is 1 i.e. $P\left(\frac{E}{E_1}\right) = 1$

Probability that the students answered correctly, given that the guessed, is $\frac{1}{4}$ i.e., $P\left(\frac{E}{E_2}\right) = \frac{1}{4}$. **[1]**

By using Baye's theorem,

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)} \quad \textbf{[1]}$$

$$= \frac{1 \times \frac{3}{4}}{1 \times \frac{3}{4} + \frac{1}{4} \times \frac{1}{4}}$$

$$= \frac{\frac{3}{4}}{\frac{3}{4} + \frac{1}{16}} = \frac{\frac{3}{4}}{\frac{12+1}{16}}$$

$$= \frac{3}{4} \times \frac{16}{13}$$

$$= \frac{12}{13} \quad \textbf{[1]}$$

30. Let $\vec{a} = 2\hat{i} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$ be three vectors. Find a vector $\vec{r}$ which satisfies $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a} = 0$. [4]

Ans :

Given,
$$\vec{a} = 2\hat{i} + \hat{k},$$
$$\vec{b} = \hat{i} + \hat{j} + \hat{k},$$
$$\vec{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$$

and for a vector $\vec{r}$,
$$\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$$

and
$$\vec{r} \cdot \vec{a} = 0$$

Now, consider $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$
$$\vec{r} \times \vec{b} - \vec{c} \times \vec{b} = \vec{0}$$
$$(\vec{r} - \vec{c}) \times \vec{b} = \vec{0} \quad \textbf{[1]}$$

$\Rightarrow$ $\vec{r} - \vec{c}$ is parallel to $\vec{b}$.

Let $\vec{r} - \vec{c} = \lambda\vec{b}$ for some scalar λ.
$$\vec{r} = \vec{c} + \lambda\vec{b} \quad \text{...(i)} \quad \textbf{[1]}$$

Also, it is given that,
$$\vec{r} \cdot \vec{a} = 0$$
$$\therefore \quad (\vec{c} + \lambda\vec{b}) \cdot \vec{a} = 0 \quad \text{[using eq.(i)]}$$
$$\vec{c} \cdot \vec{a} + \lambda(\vec{b} \cdot \vec{a}) = 0$$

$$\lambda = \frac{\vec{c} \cdot \vec{a}}{\vec{b} \cdot \vec{a}}$$
$$= \frac{-\left[(4\hat{i} - 6\hat{j} + 7\hat{k}) \cdot (2\hat{i} + \hat{k})\right]}{\left[(\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} + \hat{k})\right]}$$
$$= \frac{-(8+7)}{2+1} = \frac{-15}{3} = -5 \quad \textbf{[1]}$$

Now, putting $\lambda = -5$ in eq. (1), we get
$$\vec{r} = \vec{c} - 5\vec{b}$$
$$= (4\hat{i} - 3\hat{j} + 7\hat{k}) - 5(\hat{i} + \hat{j} + \hat{k})$$
$$= -\hat{i} - 8\hat{j} + 2\hat{k} \quad \textbf{[1]}$$

31. A toy company manufactures two types of dolls, A and B. Market tests and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is almost half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by almost 600 units. If the company makes profit of ₹12 and ₹16 per doll, respectively on dolls A and B, then how many of each should be produced weekly in order to maximise the profit ? [4]

Ans :

Let the company manufactures x dolls of type A and y dolls of type B. Then, objective function is maximum profit, $Z = 12x + 16y$.

Subject to the constraints

$$x + y \leq 1200, \quad y \leq \frac{x}{2}$$
$$x - 2y \geq 0 \text{ and } x \leq 3y + 600$$
$$x - 3y \leq 600 \quad \textbf{[1]}$$

Consider the given constraints as equations, we get
$$x + y = 1200 \quad \text{...(i)}$$
$$x - 2y = 0 \quad \text{...(i)}$$
and
$$x - 3y = 600 \quad \text{...(iii)}$$

Table for $x + y = 1200$ is

x	400	800
y	200	400

So, the line $x + y = 1200$ passes through the points (0, 1200) and (1200, 0).

On putting (0, 0) in the inequality $x + y \leq 1200$, we get
$$0 + 0 \leq 1200$$
$$0 \leq 1200 \quad \text{[true]}$$

So, the half plane is towards the X-axis.

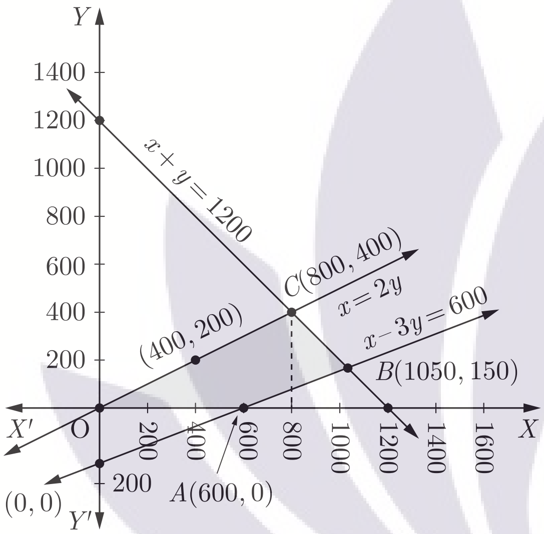
Table for $x-3y=600$ is

x	0	600
y	–200	0

So, the line $x-3y=600$ passes through the points (0, –200) and (600, 0).
On putting (0, 0) in the inequality $x-3y\le 600$, we get

$$0-3(0)\le 600$$
$$0\le 600 \qquad \text{[true]}$$

So, the half plane is towards the origin.
Now, intersection point of eqs.(i) and (ii) is $C(800,400)$ and intersection point of eqs.(iii) and (i) is $B(1050,150)$.



[1]

Now, plotting the graph of equations, the shaded portion $OABC$ represents the feasible region which is bounded and coordinates of the corner points are $O(0,0)$, $A(600,0)$, $B(1050,150)$ and $C(800,400)$.

Now, the value of Z at each corner point is given below

Corner points	$Z=12x+16y$
$O(0,0)$	$Z=12(0)+16(0)$ $=0+0=0$
$A(600,0)$	$Z=12(600)+16(0)$ $=7200$
$B(1050,150)$	$Z=12(1050)+16(150)$ $=12600+2400=15000$
$C(800,400)$	$Z=12(800)+16(400)$ $=9600+6400=16000$ (maximum)

$\therefore$ Maximum value of Z is 16000 at the point $C(800,400)$.

Hence, maximum profit is ₹ 16000 when 800 dolls of type A and 400 dolls of type B are produced. **[1]**

or

If $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}, \vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ and $\vec{a}\neq\vec{0}$, then prove that $\vec{b}=\vec{c}$.

Ans :

Given,

$$\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c} \text{ and } \vec{a}\neq\vec{0}$$
$$\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}=\vec{0}\ \vec{a}\neq\vec{0}$$
$$\vec{a}(\vec{b}-\vec{c})=\vec{0} \text{ and } \vec{a}\neq\vec{0}$$
$$\vec{b}-\vec{c}=\vec{0} \text{ or } \vec{a}\perp(\vec{b}-\vec{c})$$
$$\vec{b}=\vec{c} \text{ or } \vec{a}\perp(\vec{b}-\vec{c})\text{...(i)}1\tfrac{1}{2}$$

Also, given $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ and $\vec{a}\neq\vec{0}$

$$\vec{a}\times\vec{b}-\vec{a}-\vec{c}=\vec{0} \text{ and } \vec{a}\neq\vec{0}$$
$$\vec{a}\times(\vec{b}-\vec{c})=\vec{0} \text{ and } \vec{a}\neq\vec{0}$$
$$\vec{b}-\vec{c}=\vec{0} \text{ or } \vec{a}\,||\,(\vec{b}-\vec{c})$$
$$\vec{b}=\vec{c} \text{ or } \vec{a}\,||\,(\vec{b}-\vec{c})\text{ ...(ii)}1\tfrac{1}{2}$$

From Eqs. (i) and (ii), we get

$$\vec{b}=\vec{c}\ [\vec{a}\perp(\vec{b}-\vec{c}) \text{ and } \vec{a}\,||\,(\vec{b}-\vec{c}) \text{ cannot hold simultaneously}]$$

Hence proved

32. Show that $f:R-(-1)\to R-\{1\}$ given by $f(x)=\dfrac{x}{x+1}$ is invertible. Also, find f^{-1}. [4]

Ans :

In order to prove the invertibility of $f(x)$, it is sufficient to show that it is a bijection.
f is one-one
For any $x,\ y\in R-\{-1\}$

$$f(x)=f(y)$$
$$\frac{x}{x+1}=\frac{y}{y+1}$$

So, f is one-one. **[1]**
f is onto
Let $y\in R-\{-1\}$

Then, $$f(x)=y$$
$$\frac{x}{x+1}=y$$
$$x=\frac{y}{1-y} \qquad \textbf{[1]}$$

Clearly, $x\in R$ for all $y\in R-\{1\}$.
Also, $x\neq 1$. Because,

$$x=-1$$

$$\frac{y}{1-y} = -1$$

$$y = -1 + y$$

which is not possible.

Thus, for each $y \in R - \{1\}$ there exists $x = \frac{y}{1-y} \in R - \{-1\}$ such that

$$f(x) = \frac{x}{x+1}$$

$$= \frac{\frac{y}{1-y}}{\frac{y}{1-y}+1} = y$$

So, f is onto. **[1]**

Thus, f is both one-one and onto. Consequently it is invertible.

Now, $fof^{-1}(x) = x$ for all $x \in R - \{1\}$

$$f(f^{-1}(x)) = x$$

$$\frac{f^{-1}(x)}{f^{-1}(x)+1} = x$$

$$f^{-1}(x) = \frac{x}{1-x} \text{ for all } x \in R - \{1\}$$

[1]

Section D

33. Show that the normal at any point θ to the curve $x = a\cos\theta + a\theta\sin\theta$ and $y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin. [6]

Ans :

Given, curves are

$$x = a\cos\theta + a\theta\sin\theta \quad ...(i)$$

and $$y = a\sin\theta - a\theta\cos\theta \quad ...(ii)$$

On differentiating both sides of eq.(i) w.r.t. θ, we get

$$\frac{dx}{d\theta} = -a\sin\theta + a(\theta\cos\theta + \sin\theta)$$

$$= -a\sin\theta + a\theta\cos\theta + a\sin\theta$$

$$= a\theta\cos\theta \quad \textbf{[1/2]}$$

On differentiating both sides of eq.(ii) w.r.t. θ, we get

$$\frac{dy}{d\theta} = a\cos\theta - a[\theta(-\sin\theta) + \cos\theta]$$

$$= a\cos\theta + a\theta\sin\theta - a\cos\theta$$

$$= a\theta\sin\theta \quad \textbf{[1/2]}$$

$\therefore$ Slope of the tangent at θ,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta}$$

$$= \tan\theta \quad \textbf{[1]}$$

Then, slope of the normal at $\theta = \frac{-1}{\frac{dy}{dx}}$

$$= \frac{-1}{\tan\theta}$$

$$= -\cot\theta \quad \textbf{[1]}$$

Thus, the equation of the normal at a given point (x, y) is given by

$$y - (a\sin\theta - a\theta\cos\theta) = -\cot\theta[x - (a\cos\theta + a\theta\sin\theta)]$$

$$y - (a\sin\theta - a\theta\cos\theta) = \frac{-\cos\theta}{\sin\theta}[x - (a\cos\theta + a\theta\sin\theta)]$$

$$y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta \quad \textbf{[1]}$$

$$x\cos\theta + y\sin\theta = a(\sin^2\theta + \cos^2\theta)$$

$$x\cos\theta + y\sin\theta = a \quad [\because \sin^2 x + \cos^2 x = 1]$$

$$x\cos\theta + y\sin\theta - a = 0 \quad \textbf{[1]}$$

Now, the perpendicular distance of the normal from the origin $= \frac{|0 + 0 - a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = \frac{|-a|}{\sqrt{1}} = a$, which is a constant.

Hence, the normal at point θ to the given curve is at a constant distance from the origin. **[1]**

or

If the length of three sides of a trapezium other than base are equal to 10 cm, find the area of the trapezium when it is maximum.

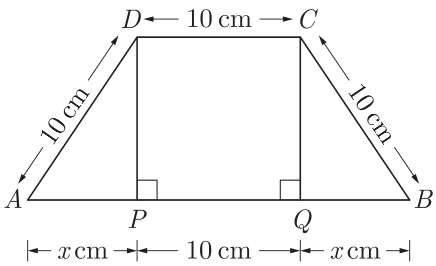
Ans :

Let $ABCD$ be a trapezium such that DC is parallel to AB and $AD = 10\text{ cm} = DC = BC$

Now, draw perpendiculars DP and CQ from D and C, on AB, respectively.

$\therefore \quad \Delta APD \sim \Delta BQC$

Therefore, $\quad PA = QB = x$ cm



In right angled ΔAPD, we have

$$AD^2 = AP^2 + PD^2$$

[by Pythagoras theorem]

$$PD^2 = AD^2 - AP^2$$
$$PD\ \sqrt{AD^2 - AP^2}$$
$$= \sqrt{100 - x^2}\text{ cm} \quad [1]$$

Similarly, in ΔBQC,

$$QC = \sqrt{100 - x^2}\text{ cm}$$

If A denotes the area of the trapezium $ABCD$, then

$$A = f(x) = \frac{1}{2}\text{ (Sum of parallel sides)} \times \text{Height}$$

$$f(x) = \frac{1}{2}(AB + DC) \times PD$$
$$= \frac{1}{2}[(2x+10)+10] \times \sqrt{100 - x^2}$$
$$= (x+10)\sqrt{100 - x^2} \quad \text{...(i)} \quad [1]$$

On differentiating both sides of eq. (i) w.r.t. x, we get

$$f'(x) = 1\sqrt{100 - x^2} + (x+10)\left(\frac{-2x}{2\sqrt{100 - x^2}}\right)$$
$$f'(x) = \frac{(100 - x^2) - x^2 - 10x}{\sqrt{100 - x^2}}$$
$$= \frac{100 - 2x^2 - 10x}{\sqrt{100 - x^2}}$$

Again, differentiating both sides w.r.t. x, we get

$$f''(x) = \frac{\left[\begin{array}{l}\sqrt{100 - x^2}(-4x - 10) \\ -(100 - 2x^2 - 10x)\left(\frac{-2x}{2\sqrt{100 - x^2}}\right)\end{array}\right]}{(100 - x^2)}$$
$$= \frac{\left[\begin{array}{l}(100 - x^2)(-4x - 10) \\ +(100x - 2x^3 - 10x^2)\end{array}\right]}{(100 - x^2)\sqrt{100 - x^2}} \quad [2]$$
$$= \frac{\left[\begin{array}{l}(-400x - 1000 + 4x^3 + 10x^2) \\ +(100x - 2x^3 - 10x^2)\end{array}\right]}{(100 - x^2)\sqrt{100 - x^2}} \quad \text{...(ii)}$$

For maxima or minima, put

$$f'(x) = 0$$
$$\frac{100 - 2x^2 - 10x}{\sqrt{100 - x^2}} = 0$$
$$100 - 2x^2 - 10x = 0$$
$$2x^2 + 10x - 100 = 0$$
$$2(x^2 + 5x - 50) = 0$$
$$x^2 + 5x - 50 = 0$$
$$x^2 + 10x - 5x - 50 = 0$$
$$x(x+10) - 5(x+10) = 0$$
$$(x+10)(x-5) = 0$$
$$\therefore \quad x = -10,\ x = 5$$
$$x = 5$$

[$\because$ x represents distance, so it cannot be negative] [1]

On putting $x = 5$ in eq.(ii), we get

$$[f''(x)]_{\text{at } x=5}$$
$$= \frac{\left(\begin{array}{l}[-400(5) - 1000 + 4(5)^3 + 10(5)^2] \\ +[100(5) - 2(5)^3 - 10(5)^2]\end{array}\right)}{(100 - 25)\sqrt{100 - 25}}$$
$$= \frac{(-3000 + 500 + 250) + (500 - 250 - 250)}{75\sqrt{75}}$$
$$= \frac{-2250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$$

Hence, the area of trapezium is maximum, when $x = 5$ and the area is given by

$$f(5) = (5+10)\sqrt{100 - 25} = 15\sqrt{75}$$
$$= 75\sqrt{3}\text{ cm}^2 \quad [1]$$

34. Find the image of the point (1, 6, 3) on the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$. Also, write the equation of the line joining the given point and its image and find the length of segment joining the given point and its image. [6]

Ans :

Given, point $P = (1,6,3)$ and equation of line AB is $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

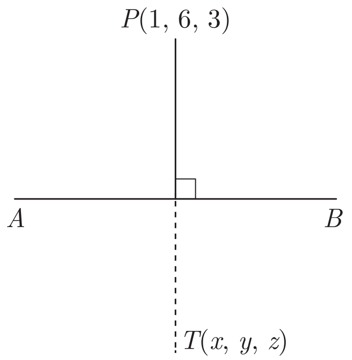
Let $T(x,y,z)$ be the image of the point $P(1,6,3)$ and Q be the foot of perpendicular PQ on line AB.

Then, $$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda \text{ (say)} \quad \text{...(i)}$$

$$x = \lambda,\quad y = 2\lambda + 1,\quad z = 3\lambda + 2$$

Then, coordinates of

$$Q = (\lambda, 2\lambda + 1, 3\lambda + 2) \quad \text{...(ii)}$$



[1]

Now, DR's of line

$$PQ = (\lambda - 1, 2\lambda + 1 - 6, 3\lambda + 2 - 3)$$
$$= (\lambda - 1, 2\lambda - 5, 3\lambda - 1) \quad ...(iii)$$

Since, $PQ \perp AB$.

$$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$$

where, $a_1 = \lambda - 1,\ b_1 = 2\lambda - 5,\ c_1 = 3\lambda - 1$
and $a_2 = 1,\ b_2 = 2,\ c_2 = 3$

$$\therefore 1(\lambda - 1) + 2(2\lambda - 5) + 3(3\lambda - 1) = 0 \quad [1]$$
$$\lambda - 1 + 4\lambda - 10 + 9\lambda - 3 = 0$$
$$14\lambda - 14 = 0$$
$$\lambda = 1 \quad [1]$$

On putting $\lambda = 1$ in eq.(ii), we get

Coordinates of $Q = (1, 2+1, 3+2)$
$$= (1,3,5)$$

Also, Q is the mid-point of PT.

$$\therefore \quad Q = \left(\frac{x+1}{2}, \frac{y+6}{2}, \frac{z+3}{2}\right) \quad [1]$$

[$\because$ mid-point of (x_1, y_1, z_1) and (x_2, y_2, z_2)
$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$]

But $Q = (1,3,5)$

$$\therefore \quad \left(\frac{x+1}{2}, \frac{y+6}{2}, \frac{z+3}{2}\right) = (1,3,5)$$
$$\frac{x+1}{2} = 1,\ \frac{y+6}{2} = 3,\ \frac{z+3}{2} = 5$$
$$x = 2 - 1,\ y = 6 - 6,\ z = 10 - 3$$
$$x = 1,\ y = 0 \text{ and } z = 7$$

So, coordinates of $T = (x, y, z) = (1, 0, 7)$
Hence, coordinates of image of the point $P(1,6,3)$ is $T(1,0,7)$. **[1]**

Length of the segment joining P and T,

$$PT \sqrt{(1-1)^2 + (0-6)^2 + (7-3)^2}$$
$$= \sqrt{0 + 36 + 16}$$
$$= \sqrt{52} = 2\sqrt{13} \text{ units}$$

DR's of line $PT = (0, -6, 4)$

DR's of line $PQ = [1 - 1, 2(1) - 5, 3(1) - 1]$
[from eq. (iii)]
$$= (0, -3, 2)$$

Hence, the equation of line PT is

$$\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2} \quad [1]$$

or

Find the foot of the perpendicular from the point $(0,2,3)$ on the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$

Also, find the length of the perpendicular.

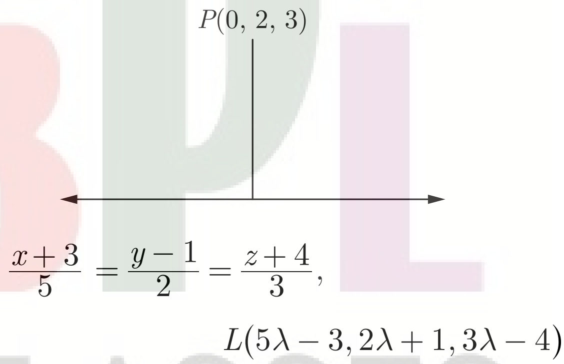
Ans :

Let L be the foot of the perpendicular drawn from the point $P(0,2,3)$ to the given line. The coordinates of a general point of the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$ are given by

$$\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3} = \lambda$$

or
$$x = 5\lambda - 3,$$
$$y = 2\lambda + 1,$$
$$z = 3\lambda - 4$$

Let the coordinates of L be $(5\lambda - 3,\ 2\lambda + 1,\ 3\lambda - 4)$. Therefore direction ratios of PL are proportional to $5\lambda - 3 - 0,\ 2\lambda + 1 - 2,\ 3\lambda - 4 - 3$ i.e., $5\lambda - 3,\ 2\lambda - 1,\ 3\lambda - 7$. **(2)**



Direction ratios of the given line are proportional to 5, 2, 3. But, PL is perpendicular to the given line.

$$\therefore \quad 5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0$$
$$\lambda = 1$$

(2)

Putting $\lambda = 1$ in $(5\lambda - 3,\ 2\lambda + 1,\ 3\lambda - 4)$, the coordinates of L are (2, 3, –1).

$$\therefore \quad PL = \sqrt{(2-0)^2 + (3-2)^2 + (-1-3)^2}$$
$$= \sqrt{21} \text{ units}$$

Hence, length of the perpendicular from P on the given line is $PL = \sqrt{21}$ units. **(2)**

35. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$. [6]

Ans :

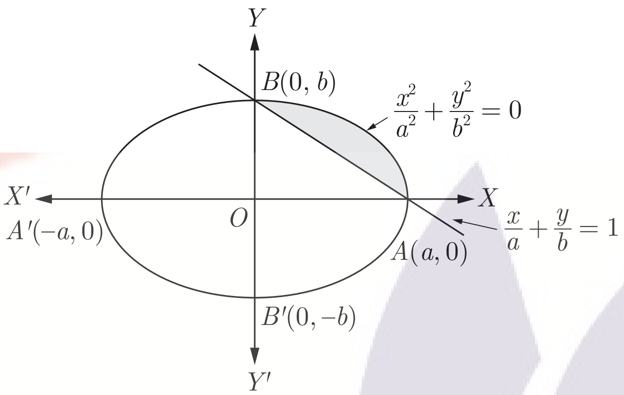
Given equations of ellipse and the straight line are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad ...(i)$$

and $$\frac{x}{a}+\frac{y}{b}=1 \quad ...(ii)$$

Ellipse with eq.(i) has vertices $(\pm a,0)$ and $(0,\pm b)$ and centre $(0,0)$. While the line with eq.(ii) has x-intercept a and y-intercept b. So, line passes through the points $(a,0)$ and $(0,b)$. **[1]**

$\therefore$ Graph of the above region is given below



Clearly, points of intersection are $A(a,0)$ and $B(0,b)$.

$\therefore$ Required area $=\int_0^a \left[y_{(\text{ellipse})}-y_{(\text{line})}\right]dx$ ...(iii) **[1]**

Now, equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$$

$$y^2=\frac{b^2}{a^2}(a^2-x^2)$$

$$y=\frac{b}{a}\sqrt{a^2-x^2} \quad ...(iv)$$

and equation of line is $\frac{x}{a}+\frac{y}{b}=1$

$$\frac{y}{b}=1-\frac{x}{a}=\frac{a-x}{a}$$

$$y=\frac{b}{a}(a-x) \quad ...(v) \quad \textbf{[1]}$$

Hence, from eqs.(iii), (iv) and (v),

Required area

$$=\int_0^a\left[\frac{b}{a}\sqrt{a^2-x^2}-\frac{b}{a}(a-x)\right]dx$$

$$=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx-\frac{b}{a}\int_0^a(a-x)\,dx \quad \textbf{[1]}$$

We know that,

$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$$

$\therefore$ $$\text{Area}=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a-\frac{b}{a}\left[ax-\frac{x^2}{2}\right]_0^a$$

$$=\frac{b}{a}\left[\frac{a^2}{2}\sin^{-1}1\right]-\frac{b}{a}\left[a^2-\frac{a^2}{2}\right] \quad \textbf{[1]}$$

$$=\frac{ba^2}{2a}\sin^{-1}\left(\sin\frac{\pi}{2}\right)-\frac{b}{a}\left(\frac{a^2}{2}\right)$$

$$\left[\because 1=\sin\frac{\pi}{2}\Rightarrow \sin^{-1}1=\sin^{-1}\left(\sin\frac{\pi}{2}\right)\right]$$

$$=\left(\frac{ba}{2}\times\frac{\pi}{2}\right)-\frac{ab}{2}=\frac{\pi ab}{4}-\frac{ab}{2}$$

$$=\left(\frac{\pi}{4}-\frac{1}{2}\right)ab \text{ sq units} \quad \textbf{[1]}$$

36. Solve the following system of equations by matrix method, where $x\neq 0$, $y\neq 0$ and $z\neq 0$.

$$\frac{2}{x}-\frac{3}{y}+\frac{3}{z}=10,$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=10$$

and $$\frac{3}{x}-\frac{1}{y}+\frac{2}{z}=13$$

[6]

Ans :

Given system of equations can be written in matrix form as

$$AX=B \quad ...(i)$$

where $A=\begin{bmatrix}2&-3&3\\1&1&1\\3&-1&2\end{bmatrix}$,

$$X=\begin{bmatrix}1/x\\1/y\\1/z\end{bmatrix}$$

and $B=\begin{bmatrix}10\\10\\13\end{bmatrix}$

Now, $|A|=\begin{vmatrix}2&-3&3\\1&1&1\\3&-1&2\end{vmatrix}$

$$=2(2+1)+3(2-3)+3(-1-3)$$

[expanding along R_1]

$$=6-3-12=-9\neq 0$$

$\therefore$ A^{-1} exists. **[1/2]**

Now, cofactors of elements of $|A|$ are

$$C_{11}=(-1)^{1+1}\begin{vmatrix}1&1\\-1&2\end{vmatrix}=2+1=3$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix}1 & 1\\ 3 & 2\end{vmatrix} = -(2-3) = 1$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix}1 & 1\\ 3 & -1\end{vmatrix} = -1-3 = -4$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix}-3 & 3\\ -1 & 2\end{vmatrix} = -(-6+3) = 3$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix} = 4-9 = -5$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix}2 & -3\\ 3 & -1\end{vmatrix}$$

$$= -(-2+9) = -7$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix}-3 & 3\\ 1 & 1\end{vmatrix} = -3-3 = -6$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix}2 & 3\\ 1 & 1\end{vmatrix} = -(2-3) = 1$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix}2 & -3\\ 1 & 1\end{vmatrix} = 2+3 = 5$$

[1½]

$$\therefore \quad \text{adj}(A) = \begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}^T$$

$$= \begin{bmatrix}3 & 1 & -4\\ 3 & -5 & -7\\ -6 & 1 & 5\end{bmatrix}^T$$

$$= \begin{bmatrix}3 & 3 & -6\\ 1 & -5 & 1\\ -4 & -7 & 5\end{bmatrix}$$ **[1]**

$$\because \quad A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

$$A^{-1} = \frac{1}{-9}\begin{bmatrix}3 & 3 & -6\\ 1 & -5 & 1\\ -4 & -7 & 5\end{bmatrix}$$

Now, eq.(1) can be written as $X = A^{-1}B$.

i.e., $$\begin{bmatrix}1/x\\ 1/y\\ 1/z\end{bmatrix} = \frac{1}{-9}\begin{bmatrix}3 & 3 & -6\\ 1 & -5 & 1\\ -4 & -7 & 5\end{bmatrix}\begin{bmatrix}10\\ 10\\ 13\end{bmatrix}$$ **[1]**

$$= \frac{-1}{9}\begin{bmatrix}30+30-78\\ 10-50+13\\ -40-70+65\end{bmatrix}$$

$$= \frac{-1}{9}\begin{bmatrix}-18\\ -27\\ -45\end{bmatrix}$$

$$= \begin{bmatrix}2\\ 3\\ 5\end{bmatrix}$$ [dividing each element by -9]

$$\therefore \quad \begin{bmatrix}1/x\\ 1/y\\ 1/z\end{bmatrix} = \begin{bmatrix}2\\ 3\\ 5\end{bmatrix}$$ **[1]**

On comparing the corresponding elements, we get

$$\frac{1}{x} = 2 \Rightarrow x = \frac{1}{2}$$

$$\frac{1}{y} = 3 \Rightarrow y = \frac{1}{3}$$

and $$\frac{1}{z} = 5 \Rightarrow z = \frac{1}{5}$$

Hence, $$x = \frac{1}{2},\ y = \frac{1}{3},\ z = \frac{1}{5}$$ **[1]**