CLASS XII (2019-20)
MATHEMATICS (041)
SAMPLE PAPER-3

Time : 3 Hours **Maximum Marks : 80**

General Instructions :

(i) All questions are compulsory.

(ii) The questions paper consists of 36 questions divided into 4 sections A, B, C and D.

(iii) Section A comprises of 20 questions of 1 mark each. Section B comprises of 6 questions of 2 marks each. Section C comprises of 6 questions of 4 marks each. Section D comprises of 4 questions of 6 marks each.

(iv) There is no overall choice. However, an internal choice has been provided in three questions of 1 mark each, two questions of 6 marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is not permitted.

Section-A

DIRECTION : (Q 1-Q 10) are multiple choice type questions. Select the correct option.

1. If $f : R \to R$ such that $f(x) = 3x - 4$ then which of the following is $f^{-1}(x)$? [1]

(a) $\frac{x+4}{3}$ (b) $\frac{1}{3}x - 4$

(c) $3x - 4$ (d) $3x + 5$

Ans : (a) $\frac{x+4}{3}$

Let, $f(x) = y = 3x - 4$

$$\frac{y+4}{3} = x$$

Replacing $x \to y$ and $y \to x$

we get, $f^{-1}(x) = \frac{x+4}{3}$

2. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, then- [1]

(a) $(x = -2, y = 8)$ (b) $(x = 2, y = -8)$

(c) $(x = 3, y = -6)$ (d) $(x = -3, y = 6)$

Ans : (b) $(x = 2, y = -8)$

We have $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

$$\begin{bmatrix} 6+1 & 8+y \\ 10+0 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8+y \\ 10 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$8 + y = 0 \text{ and } 2x + 1 = 5$$

$$y = -8 \text{ and } x = 2$$

3. The matrix $\begin{bmatrix} 3 & 5 \\ 2 & k \end{bmatrix}$ has no inverse if the value of k is [1]

(a) 0 (b) 5

(c) $\frac{10}{3}$ (d) $\frac{4}{9}$

Ans : (c) $\frac{10}{3}$

$$A = \begin{bmatrix} 3 & 5 \\ 2 & k \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

A^{-1} will not exist if $|A| = 0$.

Therefore, $3k - 10 = 0$

$$k = \frac{10}{3}$$

Hence, inverse will not exist if $k = \frac{10}{3}$.

4. $\frac{d}{dx}[\log(\sec x + \tan x)] =$ [1]

(a) $\frac{1}{\sec x + \tan x}$ (b) $\sec x$

(c) $\tan x$ (d) $\sec x + \tan x$

Ans : (b) $\sec x$

Let, $y = \log(\sec x + \tan x)$

$t = \sec x + \tan x$

Now, $y = \log t$

Differentiating both sides, with respect to t, we get

$$\frac{dy}{dt} = \frac{1}{t}$$

Now, $t = \sec x + \tan x$

Differentiating both sides with respect to x

, we get

$$\frac{dt}{dx} = \sec x \cdot \tan x + \sec^2 x$$
$$= \sec x(\tan x + \sec x)$$

Now, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$= \frac{1}{t} \times \sec x(\tan x + \sec x)$$

$$= \frac{1}{(\sec x + \tan x)} \times \sec x(\tan x + \sec x)$$

$$= \sec x$$

5. The slope of the tangent to the curve, $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is- [1]

(a) $\frac{12}{7}$ (b) $\frac{-6}{7}$

(c) $\frac{6}{7}$ (d) $\frac{-12}{7}$

Ans : (c) $\frac{6}{7}$

We have,

$$t^2 + 3t - 8 = 2 \text{ and } 2t^2 - 2t - 5 = -1$$
$$t^2 + 3t - 10 = 0 \text{ and } 2t^2 - 2t - 4 = 0$$
$$(t+5)(t-2) = 0 \text{ and } t^2 - t - 2 = 0$$
$$t = -5, 2$$

and $(t-2)(t+1) = 0$

$$t = -5, 2 \text{ and } t = 2, -1$$
$$t = 2$$

Now, $\frac{dy}{dt} = 4t^2 - 2$ and $\frac{dx}{dt} = 2t + 3$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right) = \frac{4t^2 - 2}{2t + 3}$$

Slope of the tangent

$$\left(\frac{dy}{dx}\right)_{(2,-1)} = \frac{4t - 2}{2t + 3}$$

$$= \frac{4(2) - 2}{2 \times 2 + 3} \qquad [t = 2]$$

$$= \frac{8 - 2}{4 + 3} = \frac{6}{7}$$

6. $\int_0^1 \frac{(\tan^{-1} x)^2}{1 + x^2} dx =$ [1]

(a) 1 (b) $\frac{\pi^3}{64}$

(c) $\frac{\pi^2}{192}$ (d) None of these

Ans : (d) None of these

Let, $I = \int_0^1 \frac{(\tan^{-1} x)^2}{1 + x^2} dx$

Put, $\tan^{-1} x = t$

and $\frac{1}{1 + x^2} dx = dt$

When, $x = 0, \ t = 0$

and $x = 1, \ t = \frac{\pi}{4}$

Thus, $I = \int_0^{\frac{\pi}{4}} t^2\, dt = \frac{t^3}{3}\Big|_0^{\pi/4}$

$$= \frac{1}{3}\left(\frac{\pi}{4}\right)^3 - \frac{0}{3} = \frac{\pi^3}{192}$$

7. Solution of the differential equation $ydx - xdy = xydx$ is [1]

(a) $\frac{y^2}{2} - \frac{x^2}{2} = xy + c$ (b) $x = kye^x$

(c) $x = kye^y$ (d) None of these

Ans : (b) $x = kye^x$

$$ydx - xdy = xydx$$

Dividing both sides by y^2

$$\frac{ydx - xdy}{y^2} = \frac{xy}{y^2} dx \qquad ...(1)$$

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

Putting value in eq. (1),

$$d\left(\frac{x}{y}\right) = \frac{x}{y} dx \qquad ...(2)$$

Putting $\frac{x}{y} = t$ in eq.(2),

$$dt = tdx$$

$$\frac{1}{t} dt = dx$$

Integrate in both sides

$$\int \frac{1}{t} dt = \int dx$$

$$\ln t = x + c$$

$$t = e^{x+c} = e^x \cdot e^c = ke^x$$

$$[e^c = k = \text{constant}]$$

Substituting $t = \frac{x}{y}$,

$$\frac{x}{y} = ke^x$$

$$x = kye^x$$

8. If $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$, then the value of $(\vec{a} + 3\vec{b}).(2\vec{a} - \vec{b})$ is- [1]

(a) 15 (b) 18

(c) -18 (d) -15

Ans : (d) -15

We have,

$$(\vec{a}+3\vec{b}).(2\vec{a}-\vec{b}) = 2\vec{a}.\vec{a}+6\vec{a}.\vec{b}-\vec{a}.\vec{b}-3\vec{b}.\vec{b}$$
$$= 2|\vec{a}|^2+5\vec{a}.\vec{b}-3|\vec{b}|^2$$

Here $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$

$\vec{b} = 3\hat{i}+2\hat{j}-\hat{k}$

$$= 2(\sqrt{1^2+1^2+2^2})^2+5(1\times 3+1\times 2$$
$$+2\times(-1))-3(\sqrt{3^2+2^2+(-1)^2})^2$$
$$= 2\times 6+5\times 3-3\times 14$$
$$= 12+15-42 = 27-42 = -15$$

9. The direction ratios of a straight line are 1,3,5. Its direction cosines are [1]

(a) $\frac{1}{\sqrt{35}},\frac{3}{\sqrt{35}},\frac{5}{\sqrt{35}}$ (b) $\frac{1}{9},\frac{1}{3},\frac{5}{9}$

(c) $\frac{3}{\sqrt{35}},\frac{5}{\sqrt{35}},\frac{1}{\sqrt{35}}$ (d) $\frac{5}{\sqrt{35}},\frac{3}{\sqrt{35}},\frac{1}{\sqrt{35}}$

Ans : (a) $\frac{1}{\sqrt{35}},\frac{3}{\sqrt{35}},\frac{5}{\sqrt{35}}$

If l, m, n are direction cosine of line and a, b, c are direction ratios then

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}$$
$$m = \frac{b}{\sqrt{a^2+b^2+c^2}}$$
$$n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Given direction ratios are

$$a = 1,\ b = 3,\ c = 5$$

Then, $a^2+b^2+c^2 = 1+9+25 = 35$

Therefore, $l = \frac{1}{\sqrt{35}}$, $m = \frac{3}{\sqrt{35}}$ and

$$n = \frac{5}{\sqrt{35}}$$

10. If $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{3}$ and $P(A\cap B) = \frac{1}{4}$ then $P(A'\cap B') =$ [1]

(a) $\frac{13}{8}$ (b) $\frac{13}{4}$

(c) $\frac{13}{24}$ (d) $\frac{13}{9}$

Ans : (c) $\frac{13}{24}$

We have $P(A) = \frac{3}{8}$

$$P(B) = \frac{1}{3}$$
$$P(A\cap B) = \frac{1}{4}$$

$$P(A'\cap B') = P(\overline{A\cup B})$$

We know that,

$$= P(A)+P(B)-P(A\cap B)$$
$$= \frac{3}{8}+\frac{1}{3}-\frac{1}{4} = \frac{11}{24}$$

$$P(\overline{A\cup B})+P(A\cap B) = 1$$
$$P(\overline{A\cup B}) = 1-P(A\cup B)$$
$$= 1-\frac{11}{24} = \frac{24-11}{24}$$
$$= \frac{13}{24}$$

Q. 11-15 (Fill in the blanks)

11. Let $\vec{a}$ and $\vec{b}$ be two given vectors such that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a}\cdot\vec{b} = 1$. The angle between $\vec{a}$ and $\vec{b}$ .... [1]

Ans :

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a}\cdot\vec{b} = 1 \Rightarrow |\vec{a}||\vec{b}|\cos\theta = 1$$

$$\Rightarrow \quad (2\times 1)\cos\theta = 1\left[\because |\vec{a}| = 2 \text{ and } |\vec{b}| = 1\right]$$

(1/2)

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

Hence the angel between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$.**(1/2)**

12. If I_3 is the identity matrix of order 3 then the value of $(3I_3)$ will be .... [1]

Ans :

Since the matrix I_3 is of order 3×3, therefore,

$$|3I_3| = 3^3|I_3| \quad \textbf{(1/2)}$$
$$\Rightarrow \quad |3I_3| = 27|I_3|$$
$$= 27(1) = 27 \quad \textbf{(1/2)}$$

13. The principal value of $\text{cosec}^{-1}(2)$ will be .... [1]

Ans :

Let $\text{cosec}^{-1}(2) = \theta$

$\Rightarrow \quad \text{cosec}\,\theta = 2$

i.e. $\text{cosec}\,\theta = \text{cosec}\left(\frac{\pi}{6}\right)$ **(1/2)**

So, principle value of $\text{cosec}^{-1}(2)$ is $\frac{\pi}{6}$ as principal value of y

$$= \text{cosec}^{-1}x \text{ is } \left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}.\textbf{(1/2)}$$

14. If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \forall x_1, x_2 \in A$, then the function $f: A\to B$ is .... [1]

(a) one-one (b) constant

(c) onto (d) many one

Ans : (a) one-one

If $f(x_1) = f(x_2)$

$$x_1 = x_2 \ \forall \ x_1 \ x_2 \in A$$

Since there exist a unique value of $f(x)$ in set B Therefore $f: A \to B$ is one-one.

or

If function $f: N \to N$ be defined by $f(x) = 4x+3$ then $f^{-1}(x) =$ ....

(a) $4x-3$ (b) $\dfrac{4x-3}{2}$

(c) $\dfrac{x+3}{2}$ (d) $\dfrac{x-3}{4}$

Ans : (d) $\dfrac{x-3}{4}$

$$f: N \to N$$

$$f(x) = 4x+3$$

Let, $y = 4x+3$

Then, $y-3 = 4x$

$$x = \frac{y-3}{4}$$

$$y \to x \text{ and } x \to y$$

We get $f^{-1}(x) = \dfrac{x-3}{4}$

15. The order of the differential equation $\left(\dfrac{dy}{dx}\right)^2 + y = x$ is .... [1]

(a) 0 (b) 1

(c) 2 (d) 3

Ans : (b) 1

Order of differential equation is the order of highest differential coefficient occurring in it.

Therefore, order of $\left(\dfrac{dy}{dx}\right)^2 + y = x$ is 1

or

The differential equation of family of lines passing through the origin is

(a) $x\dfrac{dy}{dx} = y$ (b) $y\dfrac{dy}{dx} = x$

(c) $\dfrac{dy}{dx} = y$ (d) $\dfrac{dy}{dx} = x$

Ans : (a) $x\dfrac{dy}{dx} = y$

Equation of any line passing through origin is given by,

$$y = mx \qquad \text{...(i)}$$

on differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = m \qquad \text{...(ii)}$$

From Eq. (i) and (ii), we have

$$y = x\frac{dy}{dx}$$

which is required differential equation.

16. If A is a matrix of order 2×3 and B is a matrix of order 3×5, then what is the order of matrix $(AB)'$ or $(AB)^T$? [1]

Ans :

Given, A is a matrix of order 2×3 and B is a matrix of order 3×5, therefore the product AB is a matrix of order 2×5. Then, the order of matrix $(AB)'$ or $(AB)^T$ is 5×2.

17. Find the value of λ, so that the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + \lambda\hat{j} + 3\hat{k}$ are perpendicular to each other. [1]

Ans :

Given, $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$

and $\vec{b} = \hat{i} + \lambda\hat{j} + 3\hat{k}$

Since, vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

$$\therefore \quad \vec{a} \cdot \vec{b} = 0$$

$$(3\hat{i} + 2\hat{j} + 9\hat{k}) \cdot (\hat{i} + \lambda\hat{j} + 3\hat{k}) = 0$$

$$3 + 2\lambda + 27 = 0$$

$$2\lambda = -30$$

$$\Rightarrow \quad \lambda = -\frac{30}{2}$$

$$= -15$$

Hence, the required value of λ is -15.

18. Let $f: R \to R$, $f(x) = (x^2 - 3x + 2)$. Find $fof(x)$. [1]

Ans :

$$\begin{aligned} fof(x) &= f\{f(x)\} = f\{x^2 - 3x + 2\} \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 \\ &= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 \\ &\qquad - 3x^2 + 9x - 6 + 2 \\ &= x^4 - 6x^3 + 10x^2 - 3x \end{aligned}$$

19. Prove that the function f given by $f(x) = \log\cos x$ is strictly decreasing. [1]

Ans :

Given, $f(x) = \log\cos x$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x)$$

$$= \frac{1}{\cos x} \cdot (-\sin x)$$
$$= -\tan x$$

We know that, for $x \varepsilon \left(0, \frac{\pi}{2}\right), \tan x > 0$

$\therefore \quad f'(x) = -\tan x < 0$

Hence, $f(x)$ is strictly decreasing.

Hence proved

20. Maximise $Z = 3x + 4y$, subject to the constraints $x + y \leq 1$, $x \geq 0$, $y \geq 0$. [1]

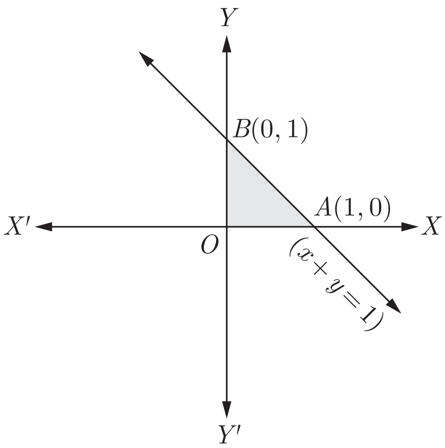
Ans :

Maximize $Z = 3x + 4y$

Subject to the constraints

$$x + y \leq 1,\ x \geq 0,\ y \geq 0$$

The shaded region shown in the figure as OAB is bounded and the coordinates of corner points O, A and B are (0,0), (1,0) and (0,1), respectively.



Corner points	Corresponding value of Z
(0, 0)	0
(1, 0)	3
(0, 1)	4 ← Maximum

Hence, the maximum value of Z is 4 at (0,1).

Section B

21. Solve for x $\cos(2\sin^{-1}x) = \frac{1}{9}$, $x > 0$ [2]

Ans :

We have, $\cos(2\sin^{-1}x) = \frac{1}{9}$

We know that,

$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$\cos(2\sin^{-1}x) = 1 - 2\sin^2(\sin^{-1}x)$$
$$= \frac{1}{9} \quad \textbf{(1)}$$
$$1 - 2x^2 = \frac{1}{9}$$
$$2x^2 = 1 - \frac{1}{9}$$
$$2x^2 = \frac{8}{9}$$
$$x^2 = \frac{4}{9} \Rightarrow x = \frac{2}{3} \quad \textbf{(1)}$$

or

Evaluate $\cos\left[\sin^{-1}\frac{1}{4} + \sec^{-1}\frac{4}{3}\right]$

Ans :

We have, $\cos\left[\sin^{-1}\frac{1}{4} + \sec^{-1}\frac{4}{3}\right]$

$$= \cos\left[\sin^{-1}\frac{1}{4} + \cos^{-1}\frac{3}{4}\right] \quad \textbf{(1/2)}$$
$$= \cos\left(\sin^{-1}\frac{1}{4}\right)\cos\left(\cos^{-1}\frac{3}{4}\right) - \sin\left(\sin^{-1}\frac{1}{4}\right)\sin\left(\cos^{-1}\frac{3}{4}\right)$$
$$= \frac{3}{4}\sqrt{1 - \left(\frac{1}{4}\right)^2} - \frac{1}{4}\sqrt{1 - \left(\frac{3}{4}\right)^2} \quad \textbf{1/2)}$$
$$= \frac{3}{4} \times \frac{\sqrt{15}}{4} - \frac{1}{4} \times \frac{\sqrt{7}}{4}$$
$$= \frac{3\sqrt{15} - \sqrt{7}}{16} \quad \textbf{(1)}$$

22. Find the derivative of $\log \sin x$ w.r.t. x. [2]

Ans :

Let $y = \log \sin x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) \text{ (by chain rule)} \quad \textbf{(1/2)}$$
$$= \frac{1}{\sin x} \cdot \cos x \quad \textbf{(1/2)}$$
$$\Rightarrow \quad \frac{dy}{dx} = \cot x \quad \textbf{(1)}$$

23. Evaluate $\int (3\,\mathrm{cosec}^2 x - 5x + \sin x)\,dx$. [2]

Ans :

Let

$$I = \int (3\,\mathrm{cosec}^2 x - 5x + \sin x)\,dx$$
$$= 3\int \mathrm{cosec}^2 x\,dx - 5\int x\,dx + \int \sin x\,dx \quad \textbf{(1/2)}$$
$$= 3(-\cot x) - 5\frac{x^2}{2} - \cos x + C \quad \textbf{(1)}$$
$$= -3\cot x - \frac{5x^2}{2} - \cos x + C \quad \textbf{(1/2)}$$

24. If the function $f(x) = \frac{1}{x+2}$, find the points of discontinuity of the composite function $y = f(f(x))$. [2]

Ans :

Given, $f(x) = \frac{1}{x+2}$

Clearly, $f(x)$ is not continuous at $x = -2$ **(1/2)**

[$\because$ rational functions are continuous for all real numbers except at those points, where the denominator is zero]

Now, for $x \neq -2$

$$f(f(x)) = \frac{1}{f(x)+2} = \frac{1}{\frac{1}{(x+2)}+2}$$

$$= \frac{(x+2)}{1+2(x+2)}$$

$$= \frac{x+2}{2x+5}$$

which is discontinuous at $x = -\frac{5}{2}$ **(1)**

Hence, the points of discontinuity are

$$x = -2 \text{ and } x = -\frac{5}{2}$$ **(1/2)**

or

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ and $x \neq y$, prove that $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$.

Ans :

Given $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$x\sqrt{1+y} = -y\sqrt{1+x}$$

$$x^2(1+y) = y^2(1+x)$$

(squaring both sides)

$$x^2 - y^2 = y^2x - x^2y$$ **(1/2)**

$$(x+y)(x-y) = -xy(x-y)$$

$\Rightarrow$ $$x+y = -xy$$

$$x = -y - xy$$

$\Rightarrow$ $$y(1+x) = -x$$

$$y = -\frac{x}{1+x}$$ **(1/2)**

$\therefore$ $$\frac{dy}{dx} = -\left[\frac{(1+x)\cdot 1 - x(0+1)}{(1+x)^2}\right]$$

$$= -\frac{1}{(1+x)^2}$$ Hence proved **(1)**

25. Without expanding, show that [2]

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0$$

Ans :

Applying $C_1 \to C_1 - C_2 - C_3$, we have

$$\Delta \begin{vmatrix} \operatorname{cosec}^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \operatorname{cosec}^2\theta + 1 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix}$$ **(1)**

$$= \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0$$ **(1)**

26. Show that $\Delta = \begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x-p)(x^2+px-2q^2)$ [2]

Ans :

Applying $C_1 \to C_1 - C_2$, we have

$$\Delta = \begin{vmatrix} x-p & p & q \\ p-x & x & q \\ 0 & q & x \end{vmatrix} = (x-p)\begin{vmatrix} 1 & p & q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$$ **(1)**

Applying $R_1 \to R_1 + R_2$

$$= (x-p)\begin{vmatrix} 0 & p+x & 2q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$$ **(1/2)**

Expanding along C_1, we have

$$\Delta = (x-p)(px+x^2-2q^2)$$

$$= (x-p)(x^2+px-2q^2)$$ **(1/2)**

Section C

27. Let $f : R \to R$ defined by $f(x) = \frac{2x-1}{3}$, $x \in R$, where x is the number of students in a class and $f(x)$ is money collected by the class for girl child welfare, Show that f is invertible. [4]

Ans :

(1) Let $x_1, x_2 \in R$ such that

$$f(x_1) = f(x_2)$$

$\Rightarrow$ $$\frac{2x_1-1}{3} = \frac{2x_2-1}{3}$$

$\Rightarrow$ $$2x_1 - 1 = 2x_2 - 1$$ **(1/2)**

$\Rightarrow$ $$x_1 = x_2$$

Thus, f is one-one **(1/2)**

Now, consider $y \in R$, such that

$\Rightarrow$ $$y = f(x)$$

$\Rightarrow$ $$y = \frac{2x-1}{3}$$ **(1/2)**

$\Rightarrow$ $$3y = 2x - 1$$

$\Rightarrow$ $$x = \frac{3y+1}{2} \in R$$ **(1/2)**

Also, $$f(x) = f\left(\frac{3y+1}{2}\right)$$

$$= \frac{2\left(\frac{3y+1}{2}\right)-1}{3}$$

$$= \frac{3y+1-1}{3} = y \qquad \textbf{(1/2)}$$

$\therefore$ For every $y \in R$, there exists $x \in R$ such that $f(x) = y$ **(1/2)**

$\Rightarrow$ Every element in co-domain has its pre-image in domain.

Thus, f is onto.

Hence, f is both one-one and onto

$\Rightarrow$ f is invertible. **(1)**

28. Solve the differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2$. [4]

Ans :

The equation is of the type $\frac{dy}{dx} + Py = Q$, which is a linear differential equation. **(1)**

Now, I.F. $= e^{\int \frac{1}{x}dx} = e^{\log x} = x$. **(1)**

Therefore, solution of the given differential equation is

$$y \cdot x = \int x x^2\, dx, \qquad \textbf{(2)}$$

$$yx = \frac{x^4}{4} + c$$

Hence, $y = \frac{x^3}{4} + \frac{c}{x}$

or

Solve $x^2\frac{dy}{dx} - xy = 1 + \cos\left(\frac{y}{x}\right)$, $x \neq 0$ and $x = 1$, $y = \frac{\pi}{2}$.

Ans :

Given equation can be written as

$$x^2\frac{dy}{dx} - xy = 2\cos^2\left(\frac{y}{2x}\right),\ x \neq 0 \qquad \textbf{(1/2)}$$

$$\Rightarrow \quad \frac{x^2\frac{dy}{dx} - xy}{2\cos^2\left(\frac{y}{2x}\right)} = 1$$

$$\Rightarrow \quad \frac{\sec^2\left(\frac{y}{2x}\right)}{2}\left[x^2\frac{dy}{dx} - xy\right] = 1 \qquad \textbf{(1)}$$

Dividing both sides by x^3, we get

$$\frac{\sec^2\left(\frac{y}{2x}\right)}{2}\left[\frac{x\frac{dy}{dx} - y}{x^2}\right] = \frac{1}{x^3}$$

$$\Rightarrow \quad \frac{d}{dx}\left[\tan\left(\frac{y}{2x}\right)\right] = \frac{1}{x^3} \qquad \textbf{(1)}$$

Integrating both sides, we get

$$\tan\left(\frac{y}{2x}\right) = \frac{-1}{2x^2} + k \qquad \textbf{(1)}$$

Substituting $x = 1$, $y = \frac{\pi}{2}$, we get

$k = \frac{3}{2}$, therefore, $\tan\left(\frac{y}{2x}\right) = -\frac{1}{2x^2} + \frac{3}{2}$ is the required solution. (1/2)

29. Find the values of x which satisfy the equation: [4]

$$\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x.$$

Ans :

From the given equation, we have

$$\sin\left(\sin^{-1}x + \sin^{-1}(1-x)\right) = \sin\left(\cos^{-1}x\right)$$

$$\Rightarrow \sin\left(\sin^{-1}x\right)\cos\left(\sin^{-1}(1-x)\right) + \cos\left(\sin^{-1}x\right)$$

$$\sin\left(\sin^{-1}(1-x)\right) = \sin\left(\cos^{-1}x\right) \qquad \mathbf{1\tfrac{1}{2}}$$

$$\Rightarrow x\sqrt{1-(1-x)^2} + (1-x)\sqrt{1-x^2} = \sqrt{1-x^2}$$

$$\Rightarrow x\sqrt{2x-x^2} + \sqrt{1-x^2}(1-x-1) = 0 \qquad \mathbf{(1\tfrac{1}{2})}$$

$$\Rightarrow \quad x\left(\sqrt{2x-x^2} - \sqrt{1-x^2}\right) = 0$$

$$\Rightarrow x = 0 \text{ or } 2x - x^2 = 1 - x^2 \qquad \textbf{(1/2)}$$

$$\Rightarrow x = 0 \text{ or } \quad x = \frac{1}{2} \qquad \textbf{(1/2)}$$

30. Find the equation of the plane passing through the points $(2,1,-1)$ and $(-1,3,4)$ and perpendicular to the plane $x - 2y + 4z = 10$.[4]

Ans :

The equation of any plane through $(2,1,-1)$ is **(1)**

$$a(x-2) + b(y-1) + c(z+1) = 0 \qquad \text{...(i)}$$

If it passes through $(-1,3,4)$, then

$$a(-1-2) + b(3-1) + c(4+1) = 0$$

$$\Rightarrow \quad -3a + 2b + 5c = 0 \qquad \text{...(ii)}$$

If plane (i) is perpendicular to the plane

$$x - 2y + 4x = 10, \text{then}$$

$$a - 2b + 4c = 0 \ (\because a_1a_2 + b_1b_2 + c_1c_2 = 0)$$

...(iii) **(1)**

On solving eqs. (ii) and (iii) by the method of cross-multiplication, we get

$$\frac{a}{8+10} = \frac{b}{5+12} = \frac{c}{6-2}$$

$$\Rightarrow \quad \frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda \text{ (say)}$$

$$\Rightarrow \quad a = 18\lambda,\ b = 17\lambda,\ c = 4\lambda \ \textbf{(1)}$$

On putting $a = 18\lambda$, $b = 17\lambda$ and $c = 4\lambda$ in eq. (i), we get

$$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1)=0$$

$$\Rightarrow \qquad 18x+17y+4z=49$$

(1)

This is the required equation of the plane.

31. Find the unit vector in the direction of the sum of the vectors $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}+3\hat{k}$. [4]

Ans :

Let $\vec{c}$ denote the sum of $\vec{a}$ and $\vec{b}$. We have

$$\vec{c}=(2\hat{i}-\hat{j}+2\hat{k})+(-\hat{i}+\hat{j}+3\hat{k})$$

$$=\hat{i}+5\hat{k} \qquad \textbf{(1)}$$

Now, $|\vec{c}|=\sqrt{1^2+5^2}=\sqrt{26}$ **(1)**

Thus, the required unit vector is

$$\hat{c}=\frac{\vec{c}}{|\vec{c}|}=\frac{1}{\sqrt{26}}(\hat{i}+5\hat{k}) \qquad \textbf{(1)}$$

$$=\frac{1}{\sqrt{26}}\hat{i}+\frac{5}{\sqrt{26}}\hat{k} \qquad \textbf{(1)}$$

or

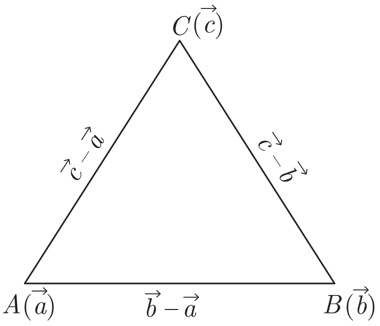
If $\vec{a}$, $\vec{b}$ and $\vec{c}$ determine the vertices of a triangle, show that $\frac{1}{2}[\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}]$ gives the vector area of me triangle. Hence, deduce the condition that the three points $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear. Also, find the unit vector normal to the plane of the triangle.

Ans :

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the vertices of a ΔABC.

$$\overrightarrow{AB}=\vec{b}-\vec{a}$$

and $$\overrightarrow{AC}=\vec{c}-\vec{a}$$



Now, area of $\Delta ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$$=\frac{1}{2}|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})| \qquad \textbf{(1)}$$

$$=\frac{1}{2}|\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}|$$

$$=\frac{1}{2}|\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}+\vec{0}|$$

$[\because \vec{b}\times\vec{a}=-\vec{a}\times\vec{b}, \vec{a}\times\vec{c}=-\vec{c}\times\vec{a}$ and $\vec{a}\times\vec{a}=0]$

$$=\frac{1}{2}|\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}| \qquad \textbf{(1)}$$

Hence, $\frac{1}{2}[\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}]$ gives the vector area of the triangle.

If these three points are collinear, the area of ΔABC should be equal zero.

$$\frac{1}{2}[\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}]=0$$

$$\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}=0 \qquad \textbf{(1)}$$

This is the required condition for collinearity of three points $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Let $\hat{n}$ be the unit vector normal to the plane of the ΔABC.

Then, $$\hat{n}=\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|}$$

$$=\frac{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})}{|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})|}$$

$$=\frac{\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}}{|\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}|}$$

$$=\frac{\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}}{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|} \qquad \textbf{(1)}$$

32. Find the vector equation of a line passing through a point with position vector $2\hat{i}-\hat{j}+\hat{k}$, and parallel to the line joining the points $-\hat{i}+4\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+2\hat{k}$. Also, find the Cartesian equivalent of this equation. [4]

Ans :

Let A, B, C be the points with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $-\hat{i}+4\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+2\hat{k}$ respectively.

We have to find the equation of a line passing through the point A and parallel to $\overrightarrow{BC}$ **(1)**

We have,

$$\overrightarrow{BC}=\text{Position vector of } C$$

$$-\text{ Position vector of } B$$

$$\Rightarrow \qquad \overrightarrow{BC}=(\hat{i}+2\hat{j}+2\hat{k})-(-\hat{i}+4\hat{j}+\hat{k})$$

$$=2\hat{i}-2\hat{j}+\hat{k}$$

We know that the equation of a line passing through a point $\vec{a}$ and parallel to $\vec{b}$ is

$$\vec{r}=\vec{a}+\lambda\vec{b} \qquad \textbf{(1/2)}$$

Here, $\vec{a}=2\vec{i}-\vec{j}+\vec{k}$ and $\vec{b}=2\vec{i}-\vec{j}+\vec{k}$.

So, the equation of the required line is

$$\vec{r}=(2\vec{i}-\vec{j}+\vec{k})+\lambda(2\vec{i}-2\vec{j}+\vec{k})$$

...(i) **(1/2)**

Reduction to cartesian form: Putting $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ in (1), we obtain

$$x\vec{i} + y\vec{j} + z\vec{k} = (2+2\lambda)\vec{i} + (-1-2\lambda)\vec{j} + (1+\lambda)\vec{k}$$

$$\Rightarrow \quad x = 2+2\lambda,\ y = -1-2\lambda,\ z = 1+\lambda$$

(1)

$$\Rightarrow \quad \frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1},$$

which is the Cartesian equivalent of equation (i) **(1)**

Section D

33. Show that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix. [6]

Ans :

Let A be a square matrix. Then,

$$A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$$

$$= P + Q \text{ (say)}, \qquad \mathbf{(1\tfrac{1}{2})}$$

where $P = \frac{1}{2}(A+A^T)$ and $Q = \frac{1}{2}(A-A^T)$

Now, $P^T = \left(\frac{1}{2}(A+A^T)\right)^T = \frac{1}{2}(A+A^T)^T$

$[\because (kA)^T = kA^T]$ **(1/2)**

$$\Rightarrow \quad P^T = \frac{1}{2}\left(A^T + (A^T)^T\right)$$

$[\because (A+B)^T = A^T + B^T]$ **(1)**

$$\Rightarrow \quad P^T = \frac{1}{2}(A^T + A) \qquad [\because (A^T)^T = A]$$

$$\Rightarrow \quad P^T = \frac{1}{2}(A + A^T) = P$$

[By commutative prop. of matrix over odd.]

Therefore, P is a symmetric matrix.

Also, $Q^T = \left(\frac{1}{2}(A-A^T)\right)^T = \frac{1}{2}(A-A^T)^T$

$$= \frac{1}{2}\left(A^T - (A^T)^T\right) = \frac{1}{2}(A^T - A)$$

$$= -\frac{1}{2}(A - A^T) = -Q \qquad \mathbf{(1)}$$

Therefore, Q is a skew-symmetric matrix.

Thus, $A = P + Q$, where P is a symmetric matrix and Q is a skew-symmetric matrix.

Hence A is expressible as the sum of a symmetric and a skew-symmetric matrix. **(1)**

Uniqueness: If possible, let $A = R + S$, where R is symmetric and S is skew-symmetric.

Then, $A^T = (R+S)^T = R^T + S^T$

$$\Rightarrow \quad A^T = R - S \ [\because R^T = R \text{ and } S^T = -S]$$

Now, $A = R + S$ and $A^T = R - S$

$$\Rightarrow \quad R = \frac{1}{2}(A + A^T) = P$$

$$S = \frac{1}{2}(A - A^T) = Q. \qquad \mathbf{(1)}$$

Hence, A is uniquely expressible as the sum of a symmetric and a skew-symmetric matrix. Thus, we can say that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

or

If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying $AA^T = 9I_3$, then find the values of a and b.

Ans :

We have, $$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$$

$$\Rightarrow \quad A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} \qquad \mathbf{(1)}$$

Now, $AA^T = 9I_3$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{(2)}$$

$$\Rightarrow \quad \begin{bmatrix} 9 & 0 & a+2b+4 \\ 0 & 9 & 2a+2-2b \\ a+2b+4 & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \qquad \mathbf{(1)}$$

$$\Rightarrow \quad a + 2b + 4 = 0$$

$$2a + 2 - 2b = 0$$

and $$a^2 + 4 + b^2 = 9 \qquad \mathbf{(1)}$$

$$\Rightarrow \quad a + 2b + 4 = 0$$

$$a - b + 1 = 0$$

and $$a^2 + b^2 = 5$$

Solving $$a + 2b + 4 = 0$$

and $$a - b + 1 = 0$$

We get $a = -2$ and $b = -1$ **(1)**

34. A manufacturer produces two types of steel trunks. He has two machines A and B. The first type of trunk requires 3h on machine A and 3h on machine B. The second type of trunk requires 3h on machines A and 2h on machine B. Both machines are run daily for 18h and 15h, respectively. There is a profit of ₹30 on first type of trunk and ₹25 on the second type of trunk. How many trunks of each type should be produced and sold to make maximum profit? [6]

Ans :

Let number of trunks of first type $= x$ and number of trunks of second type $= y$

Now, according to the question, required linear programming problem is maximize **(1/2)**

$$Z = 30x + 25y$$

Subject to the constraints are

$$3x + 3y \le 18 \Rightarrow x + y \le 6 \quad ...(i)$$

and $$3x + 2y \le 15 \quad ...(ii)$$

$$x, y \ge 0 \quad ...(iii)$$ **(1/2)**

Now, we draw the graph of above lines.

Table for $x + y = 6$,

x	0	6
y	6	0

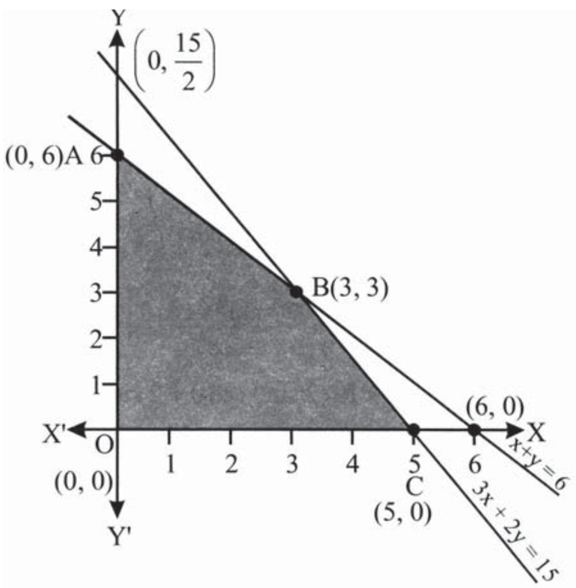
i.e., eq. (i) passes through the points (0,6) and (6, 0).

Table for $3x + 2y = 15$

x	5	0
y	0	15/2

i.e. eq., (ii) passes through the points (5, 0) and $\left(0, \frac{15}{2}\right)$. **(1)**

$\therefore$ Graph is as follows. **(1)**



The region $OABC$ is the feasible region. To find the corner points of the region, we solve eqs. (i) and (ii), we get

$$3 \times (x + y = 6)$$
$$1 \times (3x + 2y = 15)$$

$$\Rightarrow \quad \begin{array}{r} 3x + 3y = 18 \\ 3x + 2y = 15 \\ - \quad - \quad - \\ \hline y = 3 \end{array}$$ **(1)**

Putting $y = 3$ in eq. (i), we get

$$x + y = 6$$
$$x + 3 = 6$$
or $$x = 3$$

$\therefore$ Point B is $B(3,3)$ **(1)**

Now, we find value of Z at various corner points $O(0,0)$, $A(0,6)$, $B(3,3)$ and $C(5,0)$

Corner Points	$\boldsymbol{Z = 30x + 25y}$
0 (0, 0)	$30(0) + 25(0) = 0$
$A(0,6)$	$30(0) + 25(6) = 150$
$B(3,3)$	$30(3) + 25(3) = 90 + 75$ $= 165$ (maximum)
$C(5,0)$	$30(5) + 25(0) = 150$

(1/2)

Hence, the maximum profit ₹165 which is achieved, when 3 units of each type of trunk is produced. **(1/2)**

35. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the straight line $\frac{x}{a} + \frac{y}{b} = 1$. [6]

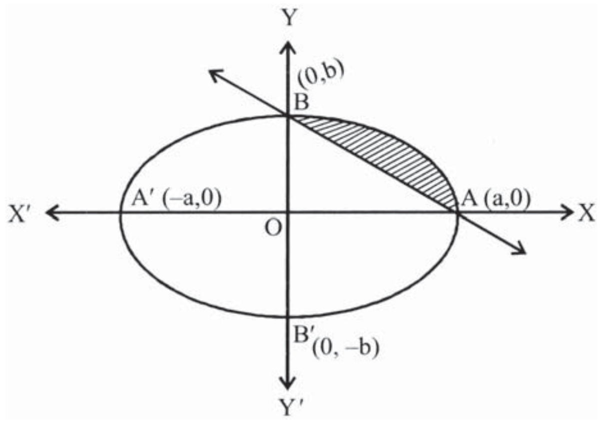
Ans :

The equation of the given curves are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(i)$$

and $$\frac{x}{a} + \frac{y}{b} = 1 \quad ...(ii)$$

Ellipse and equation of a straight line cutting x and y-axes at $(a,0)$ and $(0,b)$ respectively.



(2)

Required area $=$ Area of the shaded region

$$= \int_0^a \left\{ \frac{b}{a}\sqrt{a^2 - x^2} - \frac{b}{a}(a - x) \right\} dx \qquad \textbf{(1)}$$

$$= \frac{b}{a}\left[\int_0^a \sqrt{a^2 - x^2}\, dx - \int_0^a (a - x)\, dx \right]$$

$$= \frac{b}{a}\left[\left[\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) \right]_0^a - \left[ax - \frac{x^2}{2} \right]_0^a \right]$$

($1\frac{1}{2}$)

$$= \left(\frac{\pi ab}{4} - \frac{1}{2}ab \right) \text{ sq. units.} \qquad \textbf{(}1\tfrac{1}{2}\textbf{)}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 1 \right) ab \text{ sq. units.}$$

or

Evaluate $\int_a^b x dx$ using integration as limit of sum. [6]

Ans :

Let $I = \int_a^b x dx$

Comparing I with $\int_a^b f(x) dx$, we get: $a = a$, $b = b$, $f(x) = x$

Let $h = \frac{b - a}{n} \Rightarrow nh = b - a,\ n \in N$

$$\therefore I = \lim_{h \to 0} h[f(a) + f(a + h) + f(a + 2h) + \ldots.. + f\{a + (n - 1)h\}]$$

$$= \lim_{h \to 0} h[a + (a + h) + (a + 2h) + \ldots.. + \{a + (n - 1)h\}] \quad [\because f(x) = x]$$

$$= \lim_{h \to 0} h[(a + a + a + \ldots.. + a) + h(1 + 2 + 3 + \ldots.. + (n - 1)\}]$$

$$= \lim_{h \to 0} h\left[na + h \cdot \frac{n(n - 1)}{2} \right]$$

$$\left[\because [1 + 2 + 3 + \ldots.. + (n - 1)] = \frac{n(n - 1)}{2} \right]$$

$$= \lim_{h \to 0}\left[nha + \frac{nh(nh - h)}{2} \right]$$

$$= \lim_{h \to 0}\left[(b - a)a + \frac{(b - a)(b - a - h)}{2} \right]$$

$$[\because nh = (b - a)]$$

$$= a(b - a) + \frac{(b - a)(b - a)}{2}$$

$$= (b - a)\left[a + \frac{b - a}{2} \right] = \frac{(b - a)(b + a)}{2}$$

$$= \frac{b^2 - a^2}{2}$$

36. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere. [6]

Ans :

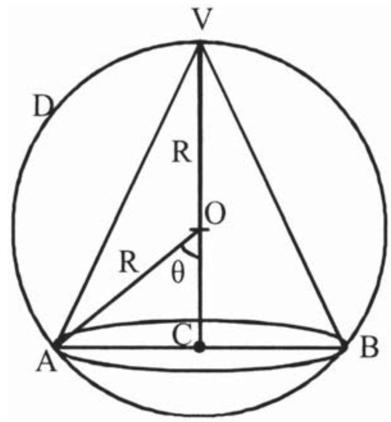
Let a cone. VAB of greatest volume be inscribed in the sphere.

Let $AOC = \theta$

$\therefore AC$, radius of the base of the cone $= R\sin\theta$

and $VC = VO + OC = R(1 + \cos\theta)$
$= R + R\cos\theta$
$=$ height of the cone

Volume of the cone, $V = \frac{1}{3}\pi(AC)^2(VC)$



$$\Rightarrow \quad V = \frac{1}{3}\pi R^3 \sin^2\theta(1 + \cos\theta)$$

$$= \frac{1}{3}\pi R^3[\sin^2\theta + \sin^2\theta\cos\theta] \qquad \textbf{(1)}$$

$$\therefore \frac{dV}{d\theta} = \frac{1}{3}\pi R^3(2\sin\theta\cos\theta + 2\sin\theta\cos^2\theta - \sin^3\theta)$$

(1)

For maximum and minimum, we have

$$\frac{dV}{d\theta} = 0 \Rightarrow \cos\theta = \frac{1}{3} \text{ or } \cos\theta = -1 \qquad \textbf{(1)}$$

But $\cos\theta \neq -1$ as $\cos\theta = -1$

$\Rightarrow \theta = \pi$, which is not possible. Also $\sin\theta \neq 0$

$\Rightarrow \theta = 0$ (not possible) $\therefore \cos\theta = \frac{1}{3}$

When $\quad \cos\theta = \frac{1}{3}, \sin\theta = \sqrt{1-\cos^2\theta}$

$$= \frac{2\sqrt{2}}{3} \qquad \textbf{(1/2)}$$

$$\left(\frac{d^2V}{d\theta^2}\right) \text{ at } \left[\theta = \cos^{-1}\left(\frac{1}{3}\right)\right] < 0$$

Hence V is maximum at $\theta = \cos^{-1}\left(\frac{1}{3}\right)$ **(1)**

Now, $\quad \cos\theta = \frac{1}{3}, \sin\theta = \frac{2\sqrt{2}}{3}$

$\therefore$Maximum volume of cone

$$= \frac{1}{3}\pi R^3\left(\frac{2\sqrt{2}}{3}\right)^2\left(1+\frac{1}{3}\right)$$

$$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right) \qquad \textbf{(1)}$$

Max. volume of cone $= \frac{8}{27} \times$ volume of the sphere **(1/2)**