CLASS XII (2019-20)
MATHEMATICS (041)
SAMPLE PAPER-4

Time : 3 Hours **Maximum Marks : 80**

General Instructions :

(i) All questions are compulsory.

(ii) The questions paper consists of 36 questions divided into 4 sections A, B, C and D.

(iii) Section A comprises of 20 questions of 1 mark each. Section B comprises of 6 questions of 2 marks each. Section C comprises of 6 questions of 4 marks each. Section D comprises of 4 questions of 6 marks each.

(iv) There is no overall choice. However, an internal choice has been provided in three questions of 1 mark each, two questions of 6 marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is not permitted.

Section-A

DIRECTION : (Q 1-Q 10) are multiple choice type questions. Select the correct option.

1. The domain of the function $\cos^{-1}(2x-1)$ is [1]

(a) [0, 1] (b) [–1, 1]

(c) (–1, 1) (d) $[0, \pi]$

Ans : (a) [0, 1]

We have,

$$f(x) = \cos^{-1}(2x-1)$$

$$\therefore \quad -1 \le 2x-1 \le 1$$

$$0 \le 2x \le 2$$

$$0 \le x \le 1$$

$$x \in [0,1]$$

2. If $2x+3y = \sin x$, then $\frac{dy}{dx}$ is equal to [1]

(a) $\frac{\cos x+2}{3}$ (b) $\frac{\cos x-2}{3}$

(c) $\cos x+2$ (d) None of these

Ans : (b) $\frac{\cos x-2}{3}$

Given, $2x+3y = \sin x$

On differentiating w.r.t. x, we get

$$2+3\frac{dy}{dx} = \cos x$$

$$3\frac{dy}{dx} = \cos x - 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x-2}{3}$$

3. The value of $\int \frac{1}{x-\sqrt{x}}dx$ is [1]

(a) $2\log\sqrt{x}+c$ (b) $\frac{x}{x-\sqrt{x}}+c$

(c) $2\log(\sqrt{x}-1)+c$ (d) $\log(\sqrt{x}-1)+c$

Ans : (c) $2\log(\sqrt{x}-1)+c$

We have, $I = \int \frac{dx}{x-\sqrt{x}}$

$$I = \int \frac{dx}{\sqrt{x}(\sqrt{x}-1)}$$

Put $\sqrt{x}-1 = t$

$$\frac{1}{2\sqrt{x}}dx = dt$$

$$I = 2\int \frac{dt}{t}$$

$$I = 2\log t + c$$

$$= 2\log(\sqrt{x}-1)+c$$

4. The function $f(x) = [x]$, where $[x]$ denotes the greatest integer function is continuous at [1]

(a) 4 (b) –2

(c) 1 (d) 1.5

Ans : (d) 1.5

The greatest integer function $[x]$ is discontinuous at all integral value of x.

5. $\int e^x(\cos x - \sin x)dx$ is equal to [1]

(a) $e^x\cos x + c$ (b) $e^x\sin x + c$

(c) $-e^x\cos x + c$ (d) $-e^x\sin x + c$

Ans : (a) $e^x\cos x + c$

$$\int e^x(f(x)+f'(x))\,dx = e^x f(x)+c$$

$$\therefore \quad f(x) = \cos x$$

$$f'(x) = -\sin x$$

Hence,

$$\int e^x(\cos x-\sin x)\,dx = e^x\cos x+c$$

6. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2$, $|\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=4$, then $|\vec{a}-\vec{b}|$ is equal to [1]

(a) $\sqrt{3}$ (b) $\sqrt{5}$
(c) $\sqrt{7}$ (d) $2\sqrt{2}$

Ans : (b) $\sqrt{5}$

$$\begin{aligned}|\vec{a}-\vec{b}|^2 &= (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})\\ &= |\vec{a}|^2+|\vec{b}|^2-2(\vec{a}\cdot\vec{b})\\ &= 4+9-2(4)\\ |\vec{a}-\vec{b}| &= \sqrt{13-8}=\sqrt{5}\end{aligned}$$

7. $\int_{-\pi/2}^{\pi/2}\frac{1}{1+e^{\sin x}}\,dx$ is equal to [1]

(a) π (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

Ans : (b) $\frac{\pi}{2}$

$$I = \int_{-\pi/2}^{\pi/2}\frac{1}{1+e^{\sin x}}\,dx \quad \text{...(i)}$$

$$= \int_{-\pi/2}^{\pi/2}\frac{1}{1+e^{\sin\left(\frac{\pi}{2}-\frac{\pi}{2}-x\right)}}\,dx$$

$$= \int_{-\pi/2}^{\pi/2}\frac{1}{1+e^{-\sin x}}\,dx$$

$$= \int_{-\pi/2}^{\pi/2}\frac{1}{1+\frac{1}{e^{\sin x}}}\,dx$$

$$I = \int_{-\pi/2}^{\pi/2}\frac{e^{\sin x}}{1+e^{\sin x}}\,dx \quad \text{...(ii)}$$

On adding eqs.(i) and (ii), we get

$$2l = \int_{-\pi/2}^{\pi/2}\left(\frac{1}{1+e^{\sin x}}+\frac{e^{\sin x}}{1+e^{\sin x}}\right)dx$$

$$= \int_{-\pi/2}^{\pi/2}\frac{1+e^{\sin x}}{1+e^{\sin x}}\,dx$$

$$2l = \int_{-\pi/2}^{\pi/2}dx$$

$$= [x]_{-\pi/2}^{\pi/2} = \frac{\pi}{2}+\frac{\pi}{2}=\pi$$

$$I = \frac{\pi}{2}$$

8. If $P(A)=\frac{7}{13}$, $P(B)=\frac{9}{13}$ and $P(A\cap B)=\frac{4}{13}$, then $P\left(\frac{A}{B}\right)$ is equal to [1]

(a) $\frac{2}{9}$ (b) $\frac{4}{9}$
(c) $\frac{5}{9}$ (d) $\frac{1}{9}$

Ans : (b) $\frac{4}{9}$

We know that,

$$P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)}$$

$$= \frac{\frac{4}{13}}{\frac{9}{13}} = \frac{4}{9}$$

9. $\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ is equal to [1]

(a) $\frac{\pi}{4}$ (b) $\frac{2\pi}{3}$
(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

Ans : (a) $\frac{\pi}{4}$

$$I = \int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \quad \text{...(1)}$$

$$= \int_0^{\pi/2}\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

$$I = \int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \quad \text{...(2)}$$

On adding eqs. (1) and (2), we get

$$2I = \int_0^{\pi/2}\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$= \int_0^{\pi/2}dx$$

$$2I = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{4}$$

10. $\int_0^{\pi}\frac{x}{1+\sin x}\,dx$ is equal to [1]

(a) 3π (b) π
(c) 2π (d) 0

Ans : (b) π

$$I = \int_0^{\pi}\frac{x}{1+\sin x}\,dx \quad \text{...(i)}$$

$$= \int_0^{\pi}\frac{(\pi-x)}{1+\sin(\pi-x)}\,dx$$

$$= \int_0^{\pi} \frac{(\pi - x)}{1+\sin x}\,dx \qquad (2)$$

On adding Eqs. (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x+\pi-x}{1+\sin x}\,dx$$

$$= \pi \int_0^{\pi} \frac{dx}{1+\sin x}\,dx$$

$$= \pi \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}\,dx$$

$$= \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \pi \int_0^{\pi} (\sec^2 x - \sec x \tan x)\,dx$$

$$= \pi[\tan x - \sec x]_0^{\pi}$$

$$= \pi[(\tan \pi - \sec \pi) - (\tan 0 - \sec 0)]$$

$$= \pi[(0+1)-(0-1)]$$

$$2I = 2\pi$$

$$\therefore \quad I = \pi$$

DIRECTION : (Q 11-Q 15) fill in the blanks

11. The linear inequalities of equation or restrictions on the variables of a linear programming problem are called .... [1]

Ans : Constraints.

or

Variables of the objective function of the linear programming problem are ....

Ans : Zero or positive.

12. If A and B are square matrices of the same order, then $(AB)' =$ .... [1]

Ans : $(B'A')$

We know that, $(AB)' = B'A'$

or

A square matrix whose diagonal element is unity and other elements are zero is called ....

Ans : Identity matrix.

13. Another name for the mean of a probability distribution is .... [1]

Ans : Expected value,

$$E(x) = \Sigma x P(x) = \mu = \text{Mean}$$

14. The number of arbitrary constants in the general solution of a differential equation of order three is .... [1]

Ans : (3)

In differential equation of order three has three arbitrary constants.

15. If $f'(x)$ changes sign from negative to positive as x increases through c, then c is a point of .... [1]

Ans : Local minima.

DIRECTION : (Q 16-Q 20) Answer the following questions.

16. If $X + \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$, then matrix X is ? [1]

Ans :

Let $A = \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$

Then, the given matrix equation is $X + A = B$

$$X + A = B$$

$$\Rightarrow \quad X = B + (-A)$$

$$= \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix} + \begin{bmatrix} -4 & -6 \\ 3 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4 & -6-6 \\ 5+3 & -8-7 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}$$

17. Find a vector in the direction of a vector $\vec{a} = \hat{i} - \hat{j} + \hat{k}$, which has magnitude 8 units. [1]

Ans :

Given, $\vec{a} = \hat{i} - \hat{j} + \hat{k}$

Now, unit vector in the direction of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{(1)^2 + (-1)^2 + (1)^2}}$$

$$= \frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}$$

$\therefore$ Vector of magnitude 8 units in direction of

$$\hat{a} = \frac{8(\hat{i} - \hat{j} + \hat{k})}{\sqrt{3}}$$

$$= \frac{8}{\sqrt{3}}\hat{i} - \frac{8}{\sqrt{3}}\hat{j} + \frac{8}{\sqrt{3}}\hat{k}$$

or

If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + \vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

Ans :

Given, $|\vec{a}| = |\vec{b}| = 1 = |\vec{a}+\vec{b}|$

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|a||b|\cos\theta$$
$$1 = 1+1+2\cos\theta$$
$$2\cos\theta = -1$$
$$\cos\theta = \frac{-1}{2}$$
$$\therefore \quad \theta = \frac{2\pi}{3}$$

18. Let $f: R \to R$, $f(x) = (x^2 - 3x + 2)$. Find $fof(x)$. [1]

Ans :

$$fof(x) = f\{f(x)\} = f\{x^2 - 3x + 2\}$$
$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$
$$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$$
$$= x^4 - 6x^3 + 10x^2 - 3x$$

19. If x changes from 3 to 3.3, find the approximate change in $\log_e(1+x)$. [1]

Ans :

Let $y = \log_e(1+x)$, $x = 3$ and $\Delta x = 0.3$

Now, $\dfrac{dy}{dx} = \dfrac{1}{1+x}$

$$\therefore \quad dy = \left(\frac{dy}{dx}\right)_{x=3} \times \Delta x = \frac{1}{1+3} \times 0.3$$
$$= \frac{0.3}{4} = 0.075$$

20. Show that the function: $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima. [1]

Ans :

We have,

$$f(x) = 4x^3 - 18x^2 + 27x - 7$$
$$f'(x) = 12x^2 - 36x + 27$$
$$f'(x) = 3(4x^2 - 12x + 9) = 3(2x-3)^2$$
$$f'(x) = 0$$
$$x = \frac{3}{2} \quad \text{(critical point)}$$

Since, $f'(x) > 0$ for all $x < \dfrac{3}{2}$ and for all $x > \dfrac{3}{2}$.

Hence, $x = \dfrac{3}{2}$ is a point of inflexion.

i.e., neither a point of maxima nor a point of minima.

Section B

21. Show that the points $(a+5, a-4)$, $(a-2, a+3)$ and (a, a) do not lie on a straight line for any value of a. [2]

Ans :

Given, points are $(a+5, a-4)$, $(a-2, a+3)$ and (a, a).

Now, consider

$$\Delta = \frac{1}{2}\begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix} \quad [1]$$

$$= \frac{1}{2}\begin{vmatrix} a+5 & a-4 & 1 \\ -7 & 7 & 0 \\ -5 & 4 & 0 \end{vmatrix}$$

[applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$= \frac{1}{2}(-28 + 35)$$

[expanding along C_3]

$$= \frac{7}{2} \neq 0$$

Which is also independent of a.

Hence, the given points form a triangle i.e., given points do not lie on a straight line for any value of a. **[1]**

22. If $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{5}$ and $P(A \cap B) = \frac{1}{7}$, find $P\left(\frac{\overline{A}}{\overline{B}}\right)$. [2]

Ans :

Given, $P(A) = \dfrac{1}{4}$, $P(B) = \dfrac{1}{5}$

and $P(A \cap B) = \dfrac{1}{7}$

$$\therefore \quad P\left(\frac{\overline{A}}{\overline{B}}\right) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{1 - P(A \cup B)}{1 - P(B)}$$
$$= \frac{1 - [P(A) + P(B) - P(A \cap B)]}{1 - \frac{1}{5}}$$
$$= \frac{1 - \left(\frac{1}{4} + \frac{1}{5} - \frac{1}{7}\right)}{\frac{4}{5}}$$
$$= \frac{1 - \left(\frac{35 + 28 - 20}{140}\right)}{\frac{4}{5}} \quad [1]$$
$$= \frac{1 - \frac{43}{140}}{\frac{4}{5}} = \frac{(140 - 43)}{140} \times \frac{5}{4}$$

$$= \frac{97}{28 \times 4} = \frac{97}{112}$$ **[1]**

23. Determine $f(0)$, so that the function $f(x)$ defined by $f(x) = \frac{(4^x - 1)^3}{\sin\frac{x}{4}\log\left(1+\frac{x^2}{3}\right)}$ becomes continuous at $x = 0$. [2]

Ans :

For $f(x)$ to be continuous at $x = 0$, we must have $\lim_{x \to 0} f(x) = f(0)$

$$f(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{(4^x - 1)^3}{\sin\frac{x}{4}\log\left(1+\frac{x^2}{3}\right)}$$ **[1]**

$$= \lim_{x \to 0} \frac{\left(\frac{4^x - 1}{x}\right)^3}{\left(\frac{\sin\frac{x}{4}}{4 \times \frac{x}{4}}\right)\left(\frac{\log\left(1+\frac{x^2}{3}\right)}{\frac{x^2}{3} \times 3}\right)}$$

$$= \frac{(\log_e 4)^3}{\frac{1}{4} \times \frac{1}{3}} = 12(\log_e 4)^3$$ **[1]**

or

If $y = b\tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$, find $\frac{dy}{dx}$.

Ans :

Given, $$y = b\tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$$

$$\frac{y}{b} = \tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$$

$$\tan\frac{y}{b} = \frac{x}{a} + \tan^{-1}\frac{y}{x}$$ **[1]**

On differentiating both sides w.r.t. x, we get

$$\frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx} = \frac{1}{a} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{x\frac{dy}{dx} - y}{x^2}$$

$$\frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx} = \frac{1}{a} + \frac{x\frac{dy}{dx} - y}{x^2 + y^2}$$

$$\frac{dy}{dx}\left\{\frac{1}{b}\sec^2\left(\frac{y}{b}\right) - \frac{x}{x^2+y^2}\right\} = \frac{1}{a} - \frac{y}{x^2+y^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{1}{a} - \frac{y}{x^2+y^2}}{\frac{1}{b}\sec^2\left(\frac{y}{b}\right) - \frac{x}{x^2+y^2}}$$ **[1]**

24. Solve for x

$$\cos(2\sin^{-1}x) = \frac{1}{9}, \quad x > 0$$ [2]

Ans :

We have,

$$\cos(2\sin^{-1}x) = \frac{1}{9}$$

We know that,

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\therefore \quad \cos(2\sin^{-1}x) = 1 - 2\sin^2(\sin^{-1}x) = \frac{1}{9}$$ **[1]**

$$1 - 2x^2 = \frac{1}{9}$$

$$2x^2 = 1 - \frac{1}{9}$$

$$2x^2 = \frac{8}{9}$$

$$x^2 = \frac{4}{9}$$

$$x = \frac{2}{3}$$ **[1]**

or

Evaluate $\cos\left[\sin^{-1}\frac{1}{4} + \sec^{-1}\frac{4}{3}\right]$.

Ans :

We have, $\cos\left[\sin^{-1}\frac{1}{4} + \sec^{-1}\frac{4}{3}\right]$

$$= \cos\left[\sin^{-1}\frac{1}{4} + \cos^{-1}\frac{3}{4}\right]$$ [1/2]

$$= \cos\left(\sin^{-1}\frac{1}{4}\right)\cos\left(\cos^{-1}\frac{3}{4}\right) - \sin\left(\sin^{-1}\frac{1}{4}\right)\sin\left(\cos^{-1}\frac{3}{4}\right)$$

$$= \frac{3}{4}\sqrt{1-\left(\frac{1}{4}\right)^2} - \frac{1}{4}\sqrt{1-\left(\frac{3}{4}\right)^2}$$ **[1/2]**

$$= \frac{3}{4} \times \frac{\sqrt{15}}{4} - \frac{1}{4} \times \frac{\sqrt{7}}{4}$$

$$= \frac{3\sqrt{15} - \sqrt{7}}{16}$$ **[1]**

25. Evaluate $\int \tan(x-\theta)\tan(x+\theta)\tan 2x\, dx$. [2]

Ans :

We know that,

$$2x = (x-\theta) + (x+\theta)$$

$$\tan 2x = \tan\{(x-\theta) + (x+\theta)\}$$

$$\tan 2x = \frac{\tan(x-\theta) + \tan(x+\theta)}{1 - \tan(x-\theta)\tan(x+\theta)}$$ **[1]**

$$\tan 2x - \tan(x-\theta)\tan(x+\theta)\tan 2x = \tan(x-\theta) + \tan(x+\theta)$$

$$\tan(x-\theta)\tan(x+\theta)\tan 2x = \tan 2x - \tan(x-\theta) - \tan(x+\theta)$$

$$\therefore I = \int \tan(x-\theta)\tan(x+\theta)\tan 2x\, dx$$

$$= \int \{\tan 2x - \tan(x-\theta) - \tan(x+\theta)\}\, dx$$

$$I = -\frac{1}{2}\log|\cos 2x| + \log|\cos(x-\theta)| + \log|\cos(x+\theta)| + c$$ **[1]**

26. Find the position vector of a point R which divides the line joining the points $P(\hat{i}+2\hat{j}-\hat{k})$ and $Q(-\hat{i}+\hat{j}+\hat{k})$ in the ratio $2:1$

(i) internally

(ii) externally [2]

Ans :

Given, $\overrightarrow{OP} = \hat{i}+2\hat{j}-\hat{k}$

and $\overrightarrow{OQ} = -\hat{i}+\hat{j}+\hat{k}$

(i) Let R divides PQ internally in the ratio $2:1$. Then, position vector of R

$$= \frac{2(-\hat{i}+\hat{j}+\hat{k}) + 1(\hat{i}+2\hat{j}-\hat{k})}{2+1}$$

$$= \frac{-\hat{i}+4\hat{j}+\hat{k}}{3}$$ **[1]**

(ii) Let R divides PQ externally in ratio $2:1$. Then, position vector of R

$$= \frac{2(-\hat{i}+\hat{j}+\hat{k}) - 1(\hat{i}+2\hat{j}-\hat{k})}{2-1}$$

$$= -3\hat{i}+3\hat{k}$$ **[1]**

Section C

27. Find the shortest distance between lines

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

and $$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$ [4]

Ans :

Given, equations of lines are

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad ...(i)$$

and $$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \quad ...(ii)$$

On comparing above equations with one point form of equation of line which is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

we get, $a_1 = 1$, $b_1 = -2$, $c_1 = 1$, $y_1 = 5$, $z_1 = 7$

and $a_2 = 7$, $b_2 = -6$, $c_2 = 1$, $x_2 = -1$, $y_2 = -1$, $z_2 = -1$

[1]

We know that the shortest distance between two lines is given by

$$d = \left| \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2 + (c_1a_2-c_2a_1)^2 + (a_1b_2-a_2b_1)^2}} \right|$$

$$= \left| \frac{\begin{vmatrix} -4 & -6 & -8 \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix}}{\sqrt{(-2+6)^2 + (7-1)^2 + (-6+14)^2}} \right|$$

[1]

$[\because x_2 - x_1 = -1-3 = -4,\ y_2 - y_1 = -1-5 = -6,\ z_2 - z_1 = -1-7 = -8]$

$$= \left| \frac{-4(-2+6) + 6(1-7) - 8(-6+14)}{\sqrt{(4)^2 + (6)^2 + (8)^2}} \right|$$

$$= \left| \frac{-4(4) + 6(-6) - 8(8)}{\sqrt{16+36+64}} \right|$$

$$= \left| \frac{-16-36-64}{\sqrt{116}} \right|$$ **[1]**

$$= \left| \frac{-116}{\sqrt{116}} \right| = \frac{116}{\sqrt{116}} = \frac{(\sqrt{116})^2}{\sqrt{116}} = \sqrt{116}$$

Hence, the required shortest distance is $\sqrt{116}$ units. **[1]**

28. Find the particular solution of the differential equation $(1+e^{2x})\,dy + (1+y^2)e^x\,dx = 0$, given that $y = 1$ when $x = 0$. [4]

Ans :

Given, differential equation is

$$(1+e^{2x})\,dy + (1+y^2)e^x\,dx = 0$$

$$(1+e^{2x})\,dy = -(1+y^2)e^x\,dx$$

$$\frac{dy}{1+y^2} = -\frac{e^x}{1+e^{2x}}\,dx$$

On integrating both sides, we get

$$\int \frac{dy}{1+y^2} = -\int \frac{e^x}{1+e^{2x}}\,dx$$ **[1]**

Put $e^x = t$, $e^x\,dx = dt$

We have, $$\int \frac{dy}{1+y^2} = -\int \frac{dt}{1+t^2}$$

$$\tan^{-1}y = -\tan^{-1}t + c$$

$$\tan^{-1}y + \tan^{-1}e^x = c \qquad [\because\ t = e^x]$$

[1½]

Now, it is given that, $y = 1$ when $x = 0$.

$$\therefore \quad \tan^{-1}(1) + \tan^{-1}(1) = c$$

$$\frac{\pi}{4} + \frac{\pi}{4} = c$$

$$c = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$$

Hence, the required particular solution is

$$\tan^{-1}y + \tan^{-1}e^x = \frac{\pi}{2}$$

$$\tan^{-1}y = \frac{\pi}{2} - \tan^{-1}e^x$$

$$\tan^{-1}y = \cot^{-1}e^x$$

$$y = \tan(\cot^{-1}e^x)$$

$$= \tan\left(\tan^{-1}\frac{1}{e^x}\right)$$

$$\therefore \qquad y = \frac{1}{e^x}$$ **[1½]**

or

Show that the differential equation that represents the family of all parabolas having their axis of symmetry coincident with the axis of x is $yy_2 + y_1^2 = 0$.

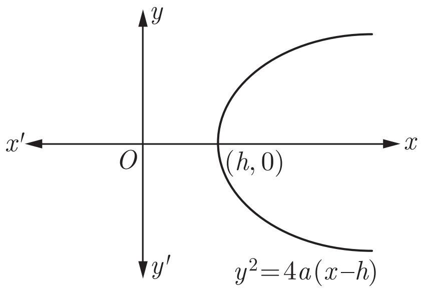
Ans :

The equation that represents a family of parabolas having their axis of symmetry coincident with the axis of x is

$$y^2 = 4a(x-h) \qquad \text{...(i)}$$ **[1]**

where, a and h are parameters.

This equation contains two parameters a and h, so we will differentiate it twice to obtain a second order differential equation.



[1]

On differentiating eq.(i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a$$

$$y\frac{dy}{dx} = 2a \qquad \text{...(ii)}$$ **[1]**

On differentiating eq.(ii) w.r.t. x, we get

$$y\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\therefore \qquad yy_2 + y_1^2 = 0$$

which is the required differential equation.

[1]

29. Evaluate $\int \sqrt{3-4x-4x^2}\,dx$. [4]

Ans :

Let $$I = \int \sqrt{3-4x-4x^2}\,dx$$

$$= \int \sqrt{-4\left(x^2+x-\frac{3}{4}\right)}\,dx$$

$$= \sqrt{4}\int \sqrt{-\left(x^2+x-\frac{3}{4}\right)}\,dx$$ **[1]**

$$= 2\int \sqrt{-\left(x^2+x-\frac{3}{4}+\frac{1}{4}-\frac{1}{4}\right)}\,dx$$

$$= 2\int \sqrt{\left[-\left(x+\frac{1}{2}\right)^2+\left(\frac{3}{4}+\frac{1}{4}\right)\right]}\,dx$$

$$\left[\because\ x^2+x+\frac{1}{4} = \left(x+\frac{1}{2}\right)^2\right]$$

$$= 2\int \sqrt{-\left[\left(x+\frac{1}{2}\right)^2-1\right]}\,dx$$

$$= 2\int \sqrt{(1)^2-\left(x+\frac{1}{2}\right)^2}\,dx$$ **[1]**

$$= 2\left[\frac{x+\frac{1}{2}}{2}\sqrt{(1)^2-\left(x+\frac{1}{2}\right)} + \right.$$

$$\left. + \frac{1^2}{2}\sin^{-1}\left(\frac{x+\frac{1}{2}}{1}\right)\right] + c$$

$$\left[\because\ \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

[1]

$$= 2\left[\frac{2x+1}{4}\sqrt{\frac{3-4x-4x^2}{4}} + \right.$$

$$\left. + \frac{1}{2}\sin^{-1}\left(\frac{2x+1}{2}\right)\right] + c$$

$$\left[\because\ \sqrt{1-\left(x+\frac{1}{2}\right)^2} = \sqrt{1-x^2-x-\frac{1}{4}}\right.$$

$$\left. = \sqrt{\frac{3-4x^2-4x}{4}}\right]$$

$$= \frac{2x+1}{4}\sqrt{3-4x-4x^2} + \sin^{-1}\left(\frac{2x+1}{2}\right) + c$$

[1]

or

Evaluate $\int \frac{\sin(x-\alpha)}{\sin(x+\alpha)}\,dx$.

Ans :

Let $\quad I = \int \frac{\sin(x-\alpha)}{\sin(x+\alpha)} dx$

Put $\quad x+\alpha = t$

$$x = t-\alpha$$

$$dx = dt \qquad [\mathbf{1}]$$

Then, given integral reduces to

$$I = \int \frac{\sin(t-2\alpha)}{\sin t} dt$$

$$= \int \frac{\sin t\cos 2\alpha - \cos t\sin 2\alpha}{\sin t} dt$$

$$[\because \sin(x-y) = \sin x\cos y - \cos x\sin y] \quad [\mathbf{1}]$$

$$= \int \left(\frac{\sin t\cos 2\alpha}{\sin t} - \frac{\cos t\sin 2\alpha}{\sin t}\right) dt$$

$$= \int \cos 2\alpha\, dt - \int \cot t\sin 2\alpha\, dt$$

$$\left[\because \frac{\cos x}{\sin x} = \cot x\right]$$

$$= \cos 2\alpha \int dt - \sin 2\alpha \int \cot t\, dt \quad [\mathbf{1}]$$

$$= t\cos 2\alpha - \sin 2\alpha \cdot \log|\sin t| + c$$

$$\left[\because \int \cot x\, dx = \log|\sin x|\right]$$

$\therefore \quad I = (x+\alpha)\cos 2\alpha$

$$- \sin 2\alpha \cdot \log|\sin(x+\alpha)| + c$$

$$[\text{put } t = x+\alpha] \quad [\mathbf{1}]$$

30. An aeroplane can carry a maximum of 200 passengers. A profit of ₹1000 is made on each executive class ticket and a profit of ₹600 is made on each economy class ticket. The airline donate its 5% of total profit in welfare fund for poor girls. The airline reserves atleast 20 seats for executive class. However, atleast 4 times as many passengers prefer to travel by economy class, then by executive class. Determine how many tickets of each type must be sold in order to maximise profit for the airline ? What is the maximum profit? [4]

Ans :

Let number of executive class tickets $= x$
and number of economy class tickets $= y$
Now, required linear programming problem is given by

Maximise $Z = 1000x + 600y$
Subject to the constraints

$$x+y \le 200,\ x \ge 20,\ y \ge 4x$$

$$4x - y \le 0 \ \text{ and } x, y \ge 0 \qquad [\mathbf{1}]$$

On considering the constraints as equation, we get

$$x + y = 200 \qquad ...(1)$$

$$x = 20 \qquad ...(2)$$

and $\quad 4x - y = 0 \qquad ...(3)$

Table for $x+y = 200$ is

x	0	200
y	200	0

So, line $x+y=200$ passes through the points (0, 200) and (200, 0).
Put (0, 0) in the inequality $x+y \le 200$, we get

$$0 + 0 \le 200 \qquad \text{[true]}$$

$\therefore$ Shaded region is towards the origin.
Table for $4x - y = 0$ is

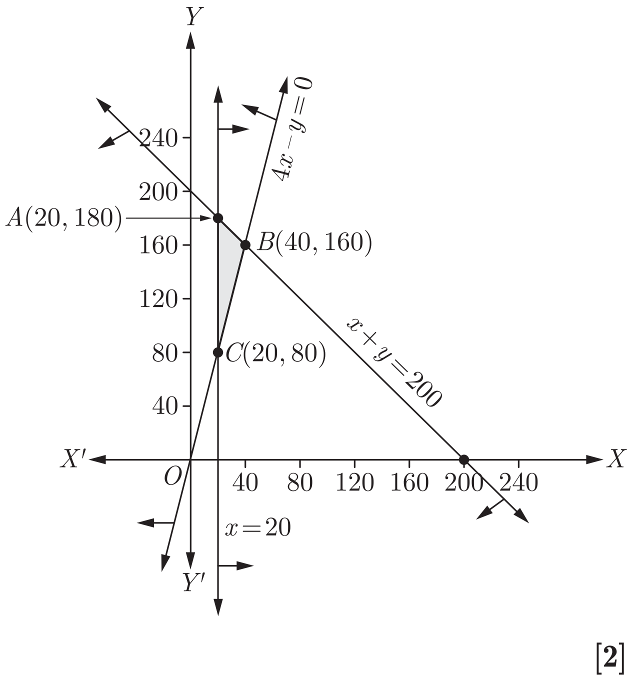
x	40	20
y	160	80

So, line $4x - y = 0$ passes through the points (40, 160) and (20, 80).
Put (1, 0) in the inequality $4x - y \le 0$, we get

$$4(1) - 0 \le 0$$

$$4 \le 0 \qquad \text{[false]}$$

$\therefore$ Required shaded region is the region not containing (1, 0).
Also, the line $x = 20$ is parallel to Y-axis, so it passes through the point (20, 0).
Put (0, 0) in the inequality $x \ge 20$, we get $0 \ge 20$ [false]
$\therefore$ Shaded region is away from the origin.
On plotting the above points, we get the required feasible region.



[2]

The intersection point of lines (1) and (2) is (20, 180), intersection point of lines (1) and

(3) is (40, 160) and intersection point of lines (2) and (3) is (20, 80). Thus, corner points of the region are $A(20, 180)$, $B(40, 160)$ and C (20, 80).

Now, consider value of Z at corner points which are given below

Corner points	$Z = 1000x + 600y$
$A(20, 180)$	$1000(20) + 600(180)$ $= 20000 + 108000 = ₹\,128000$
$B(40, 160)$	$1000(40) + 600(160)$ $= 40000 + 96000 = ₹\,136000$ (maximum)
$C(20, 80)$	$1000(20) + 600(80)$ $= 20000 + 48000 = ₹\,68000$

Hence, maximum profit is ₹ 136000 and it is achieved when 40 tickets of executive class and 160 tickets of economy class are sold. **[1]**

31. Find the mean and variance of number of tails when a coin is tossed thrice. [4]

Ans :

Given, a coin is tossed thrice, so its sample space is

$$S = \{HHH, TTT, HHT, HTH, THH, TTH, THT, HTT\}$$

Let X be a random variable that denotes the number of tails. Then, X can take values 0, 1, 2 and 3.

Now, $P(X = 0) = P$ (no tail occurs) $= \frac{1}{8}$

$P(X = 1) = P$ (only one tail occurs) $= \frac{3}{8}$

$P(X = 2) = P$ (exactly two tails occur) $= \frac{3}{8}$

and $P(X = 3) = P$ (three tails occur) $= \frac{1}{8}$

$\therefore$ The probability distribution of number of tails is as follows

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

[1]

Now, we find mean and variance using the following table

X	$P(X)$	$X.P(X)$	$X^2.P(X)$
0	$\frac{1}{8}$	0	0
1	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
2	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{12}{8}$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{9}{8}$
		$\Sigma X \cdot P(X)$ $= \frac{12}{8} = \frac{3}{2}$	$\Sigma X^2 \cdot P(X)$ $= \frac{24}{8} = 3$

[1]

Clearly, mean $= \Sigma X \cdot P(X) = \frac{3}{2}$ **[1]**

and

$$\text{variance} = \Sigma X^2 \cdot P(X) - [\Sigma X \cdot P(X)]^2$$

$$= 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4}$$

$$= \frac{12 - 9}{4} = \frac{3}{4}$$ **[1]**

or

An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter, a car and a truck are 0.01, 0.03 and 0.15, respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

Ans :

Let us define the following events

E_1 = Insured person is a scooter driver

E_2 = Insured person is a car driver

E_3 = Insured person is a truck driver

A = Insured person meets with an accident

Here, total number of insured vehicles

$$= 2000 + 4000 + 6000 = 12000$$

$$P(E_1) = \frac{2000}{12000} = \frac{1}{6}$$

$$P(E_2) = \frac{4000}{12000} = \frac{1}{3}$$

and $P(E_3) = \frac{6000}{12000} = \frac{1}{2}$

Also given,

$P\left(\frac{A}{E_1}\right)$ = Probability that scooter driver meets with an accident $= 0.01$ **(1)**

$P\left(\frac{A}{E_2}\right)$ = Probability that car driver meets with an accident $= 0.03$

$P\left(\frac{A}{E_3}\right)$ = Probability that truck driver meets with an accident $= 0.15$ **(1)**

Now, P (person meets with an accident is a scooter driver),

$$P\left(\frac{E_1}{A}\right)$$

$$= \frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right)+P(E_2)\cdot P\left(\frac{A}{E_2}\right)+P(E_3)\cdot P\left(\frac{A}{E_3}\right)}$$

[by Baye's theorem] **(1)**

On putting all the above values, we get

$$P\left(\frac{E_1}{A}\right)$$

$$= \frac{\frac{1}{6}\times 0.01}{\left(\frac{1}{6}\times 0.01\right)+\left(\frac{1}{3}\times 0.03\right)+\left(\frac{1}{2}\times 0.15\right)}$$

$$= \frac{\frac{1}{6}\times\frac{1}{100}}{\left(\frac{1}{6}\times\frac{1}{100}\right)+\left(\frac{1}{3}\times\frac{3}{100}\right)+\left(\frac{1}{2}\times\frac{15}{100}\right)}$$

$$= \frac{\frac{1}{6}}{\frac{1}{6}+1+\frac{15}{2}} = \frac{\frac{1}{6}}{\frac{1+6+45}{6}} = \frac{1}{52}$$ **(1)**

32. Let T be the set of all triangles in a plane. Let us define a relation $R = \{(T_1, T_2): T_1$ is similar to T_2, $T_1, T_2 \in T\}$. Show that R is an equivalence relation. [4]

Ans :

Given, relation is $R = \{(T_1, T_2): T_1$ is similar to T_2; $T_1, T_2 \in T\}$, where T is the set of all triangles in a plane. We know that two triangles are said to be similar, if they have same shape.

Reflexive : Let $T_1 \in T$ be any arbitrary element. As we know that, every triangle is similar to itself. **[1]**

So, $(T_1, T_1) \in R$

$\therefore$ R is reflective.

Symmetric : Let $T_1, T_2 \in T$ such that $(T_1, T_2) \in R$

$\Rightarrow$ T_1 is similar to T_2.

$\Rightarrow$ T_2 is similar to T_1. **[1]**

[$\because$ two triangles are similar to each other]

$\Rightarrow$ $(T_2, T_1) \in R$

$\therefore$ R is symmetric.

Transitive : Let $T_1, T_2, T_3 \in T$ such that

$(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

$\Rightarrow$ T_1 is similar to T_2 and T_2 is similar to T_3. **[1]**

$\Rightarrow$ T_1 is similar to T_3.

$\Rightarrow$ $(T_1, T_3) \in R$

$\therefore$ R is transitive.

Thus, relation R is reflexive, symmetric and transitive, so R is an equivalence relation. **[1]**

Section D

33. Find the area of the region bounded by the parabola $x^2 = 4y$ and the line $x = 4y - 2$. [6]

Ans :

Given, equations of curves are

$$x^2 = 4y \quad \text{...(i)}$$

and $$x = 4y - 2 \quad \text{...(ii)}$$

Eq.(i) represents a parabola which is open upward having vertex (0, 0) and eq.(ii) represents a straight line.

$$x = x^2 - 2$$
$$x^2 - x - 2 = 0$$
$$x^2 - 2x + x - 2 = 0$$
$$x(x-2) + 1(x-2) = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1, 2$$ **[1/2]**

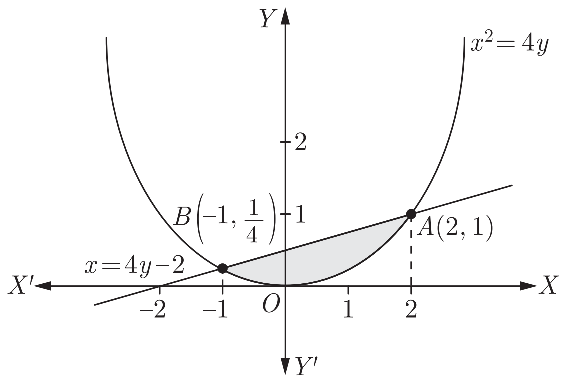
When $x = -1$, then from eq.(i), we get

$$y = \frac{1}{4}$$

and when $x = 2$, then from eq.(i), $y = 1$

$\therefore$ Points of intersection of given curves are $\left(-1, \frac{1}{4}\right)$ and (2, 1). **[1/2]**

Now, the graph of given curves is as follows



[2]

$\therefore$Required area = Area of shaded region $BOAB$

$$= \int_{-1}^{2}\left[y_{(\text{line})} - y_{(\text{parabola})}\right]dx$$

$$= \int_{-1}^{2}\left[\left(\frac{x+2}{4}\right) - \frac{x^2}{4}\right]dx$$

$$= \frac{1}{4}\int_{-1}^{2}(x + 2 - x^2)\,dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} \quad [1]$$

$$= \frac{1}{4}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left(6 - \frac{8}{3} + 2 - \frac{5}{6}\right)$$

$$= \frac{1}{4}\left(8 - \frac{8}{3} - \frac{5}{6}\right) \quad [1]$$

$$= \frac{1}{4}\left(\frac{48 - 16 - 5}{6}\right) = \frac{1}{4}\cdot\frac{27}{6}$$

$$= \frac{27}{24} = \frac{9}{8}$$

Hence, the required area is $\frac{9}{8}$sq units. [1]

34. Show that of all the rectangles with a given perimeter, the square has the largest area. [6]

Ans :

Let x and y be the lengths of two sides of a rectangle. Also, let P denotes the perimeter and A denotes the area of rectangle.

Given, $P = 2(x + y)$

[$\because$ perimeter of rectangle $= 2(l + b)$]

$$P = 2x + 2y$$

$$y = \frac{P - 2x}{2} \quad \text{...(i)} \quad [1]$$

Also, we know that area of rectangle is given by

$$A = xy$$

$$A = x\left(\frac{P - 2x}{2}\right) \quad \text{[from eq.(i)]}$$

$$A = \frac{Px - 2x^2}{2} \quad [1]$$

On differentiating both sides w.r.t. x, we get

$$\frac{dA}{dx} = \frac{P - 4x}{2} \quad [1]$$

Now, for maxima and minima, put $\frac{dA}{dx} = 0$

$$\therefore \quad \frac{P - 4x}{2} = 0$$

$$P = 4x$$

$$2x + 2y = 4x \quad [\because P = 2x + 2y]$$

$$x = y$$

$\because$ $x = y$, so the rectangle is a square. [1]

Also, $$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{P - 4x}{2}\right)$$

$$= -\frac{4}{2} = -2 < 0 \quad [1]$$

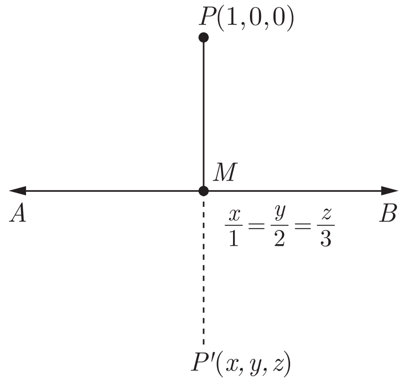
$\therefore$ A is maximum.

Hence, area is maximum, when rectangle is a square. [1]

35. Find the image of point (1, 0, 0) on the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$. [6]

Ans :

Let $P'(x, y, z)$ be the image of point $P(1,0,0)$ and M be the foot of perpendicular PM on the line AB.



[1]

Given, equation of line AB is

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = \lambda \text{ (say)}$$

$$x = \lambda,\ y = 2\lambda \text{ and } z = 3\lambda$$

$\therefore$ Coordinates of M are $(\lambda, 2\lambda, 3\lambda)$ for some $\lambda \in R$ ...(i) [1]

Now, DR's of PM are $\lambda - 1$, 2λ and 3λ and DR's of AB are 1, 2 and 3.

$\because$ $PM \perp AB$

$\therefore$ $a_1a_2 + b_1b_2 + c_1c_2 = 0$ [1]

$$1(\lambda - 1) + 2(2\lambda) + 3(3\lambda) = 0$$

[$\because a_1 = 1,\ b_1 = 2,\ c_1 = 3,\ a_2 = (\lambda - 1),\ b_2 = 2\lambda,\ c_2 = 3\lambda$]

$$\lambda - 1 + 4\lambda + 9\lambda = 0$$

$$14\lambda - 1 = 0$$

$$\lambda = \frac{1}{14}$$

Thus, the coordinates of point M are $\left(\frac{1}{14}, \frac{1}{7}, \frac{3}{14}\right)$. [1]

Now, as M is the mid-point of line PP'.

$\therefore$ Coordinates of M = Mid-point of $P(1,0,0)$

and $$P'(x, y, z) = \left(\frac{1 + x}{2}, \frac{0 + y}{2}, \frac{0 + z}{2}\right)$$

$$= \left(\frac{1 + x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

But coordinate of M are $\left(\frac{1}{14}, \frac{1}{7}, \frac{3}{14}\right)$

$$\therefore \quad \left(\frac{x + 1}{2}, \frac{y}{2}, \frac{z}{2}\right) = \left(\frac{1}{14}, \frac{1}{7}, \frac{3}{14}\right) \quad [1]$$

$$\frac{x+1}{2} = \frac{1}{14}$$

$$14x + 14 = 2$$

$$14x = -12$$

$$x = -\frac{6}{7}$$

$$\frac{y}{2} = \frac{1}{7}$$

$$y = \frac{2}{7}$$

and $$\frac{z}{2} = \frac{3}{14}$$

$$z = \frac{3}{7}$$

Hence, the image of point $P(1,0,0)$ is $P'\left(-\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$. **[1]**

or

Find the equation of the plane that contains the point (1, –1, 2) and is perpendicular to both the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$. Hence, find the distance of point $P(-2,5,5)$ from the plane obtained above.

Ans :

Equation of plane containing the point (1, –1, 2) is

$$a(x-1) + b(y+1) + c(z-2) = 0 \quad ...(i)$$

$\because$ Plane (1) is perpendicular to plane

$$2x + 3y - 2z = 5$$

$$2a + 3b - 2c = 0 \quad ...(ii)$$

Also, plane (1) is perpendicular to plane

$$x + 2y - 3z = 8$$

$$\therefore \quad a + 2b - 3c = 0 \quad ...(iii) \quad \textbf{[1]}$$

From eqs. (ii) and (iii), we get

$$\frac{a}{-9+4} = \frac{b}{-2+6} = \frac{c}{4-3} \quad \textbf{[1]}$$

$$\frac{a}{-5} = \frac{b}{4} = \frac{c}{1} = \lambda \text{ (say)}$$

$$a = -5\lambda,\ b = 4\lambda \text{ and } c = \lambda$$

[1]

On putting these values in eq.(i), we get

$$-5\lambda(x-1) + 4\lambda(y+1) + \lambda(z-2) = 0$$

$$-5(x-1) + 4(y+1) + (z-2) = 0$$

$$-5x + 5 + 4y + 4 + z - 2 = 0$$

$$-5x + 4y + z + 7 = 0$$

$$5x - 4y - z - 7 = 0$$

...(iv) **[1]**

It is the required equation of plane.

Now, if d is the distance of point (–2, 5, 5) from plane (4).

Then,

$$d = \left| \frac{5 \times (-2) + (-4) \times 5 + (-1) \times 5 - 7}{\sqrt{5^2 + (-4)^2 + (-1)^2}} \right|$$

$$= \left| \frac{-10 - 20 - 5 - 7}{\sqrt{25 + 16 + 1}} \right|$$

$$= \frac{42}{\sqrt{42}} = \sqrt{42} \text{ units} \quad \textbf{[2]}$$

36. Show that ΔABC is an isosceles triangle, if the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos B \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$$

[6]

Ans :

We have,

$$\Delta \begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos B \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$$

On applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$\begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1+\cos C \\ \cos^2 A + \cos A - \cos^2 C - \cos C & \cos^2 B + \cos B - \cos^2 C - \cos C & \cos^2 C + \cos C \end{vmatrix} = 0$$

$$\left[\begin{aligned} &\because \cos^2 A - \cos A - \cos^2 C - \cos C \\ &\quad = \cos^2 A - \cos^2 C + \cos A - \cos C \\ &\quad = (\cos A - \cos C)(\cos A + \cos C) + \cos A - \cos C \\ &\quad = (\cos A - \cos C)(\cos A + \cos C + 1) \\ &\text{similarly, } \cos^2 B + \cos B - \cos^2 C - \cos C \\ &\qquad = (\cos B - \cos C)(\cos B + \cos C + 1) \end{aligned}\right]$$

[1]

On taking $(\cos A - \cos C)$ common from C_1 and $(\cos B - \cos C)$ common from C_2, we get $(\cos A - \cos C) \cdot (\cos B - \cos C)$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1+\cos C \\ \cos A + \cos C + 1 & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix}$$

$$= 0 \quad [1]$$

$$\Rightarrow (\cos A - \cos C) \cdot (\cos B - \cos C)$$

$$[(\cos B + \cos C + 1) - (\cos A + \cos C + 1)] = 0$$

[expanding along R_1] **[1]**

$$\Rightarrow (\cos A - \cos C) \cdot (\cos B - \cos C)$$

$$(\cos B + \cos C + 1 - \cos A - \cos C - 1) = 0$$

$$\Rightarrow (\cos A - \cos C) \cdot (\cos B - \cos C)$$

$$(\cos B - \cos A) = 0$$

$\Rightarrow \cos A = \cos C$ or $\cos B = \cos C$ or $\cos B = \cos A$

$\Rightarrow A = C$ or $B = C$ or $B = A$

Hence, ΔABC is an isosceles triangle. **[2]**

Hence proved.

or

If $A + B + C = \pi$, show that

$$\begin{vmatrix} \sin^2 A & \sin A \cos A & \cos^2 A \\ \sin^2 B & \sin B \cos B & \cos^2 B \\ \sin^2 C & \sin C \cos C & \cos^2 C \end{vmatrix} = \sin(A-B)\sin(B-C)\sin(C-A)$$

Ans :

The given determinant

$$= \begin{vmatrix} 1 & \left(\frac{1}{2}\right)\sin 2A & \left(\frac{1}{2}\right)(1+\cos 2A) \\ 1 & \left(\frac{1}{2}\right)\sin 2B & \left(\frac{1}{2}\right)(1+\cos 2B) \\ 1 & \left(\frac{1}{2}\right)\sin 2C & \left(\frac{1}{2}\right)(1+\cos 2C) \end{vmatrix}$$

$[C_1 \to C_1 + C_3]$

$$= \frac{1}{4}\begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 1 & \sin 2B & 1+\cos 2B \\ 1 & \sin 2C & 1+\cos 2C \end{vmatrix} \quad [1]$$

$$= \frac{1}{4}\begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 0 & \sin 2B - \sin 2A & \cos 2B - \cos 2A \\ 0 & \sin 2C - \sin 2A & \cos 2C - \cos 2A \end{vmatrix}$$

$$\begin{Bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{Bmatrix} \quad [1]$$

$$= \frac{1}{4}\begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 0 & 2\cos(A+B)\sin(B-A) & 2\sin(A+B)\sin(A-B) \\ 0 & 2\cos(A+C)\sin(C-A) & 2\sin(A+C)\sin(A-C) \end{vmatrix}$$

$$= \sin(A-B)\sin(A-C)\begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 0 & -\cos(A+B) & \sin(A+B) \\ 0 & -\cos(A+C) & \sin(A+C) \end{vmatrix}$$

[taking $2\sin(A-B)$ and $2\sin(A-C)$ common from R_2 and R_3, respectively] **[1]**

$$= \sin(A-B)\sin(A-C)\begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 0 & -\cos(\pi - C) & \sin(\pi - C) \\ 0 & -\cos(\pi - B) & \sin(\pi - B) \end{vmatrix}$$

$[\because A + B + C = \pi]$ **[1]**

$$= \sin(A-B)\sin(A-C)\begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 0 & \cos C & \sin C \\ 0 & \cos B & \sin B \end{vmatrix}$$

[1]

$$= \sin(A-B)\sin(A-C)[\sin B\cos C - \cos B \sin C]$$

$$= \sin(A-B)\sin(A-C)\sin(B-C)$$

$$= -\sin(A-B)\sin(B-C)\sin(C-A)$$

Hence proved. **[1]**