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Accelerated Concurrent Learning Algorithms via Data-Driven Hybrid Dynamics and Nonsmooth ODEs

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Abstract

We introduce a novel class of data-driven accelerated concurrent learning algorithms. These algorithms are suitable for the solution of high-performance system identification and parameter estimation problems with *convergence certificates*, in settings where the standard persistence of excitation (PE) condition is difficult to verify *a priori*. In order to achieve (uniform) fast convergence, the proposed algorithms exploit the existence of information-rich data sets, as well as certain non-smooth regularizations that generate a family of non-Lipschitz dynamics modeled as data-driven ordinary differential equations (DD-ODEs) and/or data-driven hybrid dynamical systems (DD-HDS). In each case, we provide stability and convergence certificates via Lyapunov theory. Moreover, to illustrate the advantages of the proposed algorithms, we consider an online estimation problem in Lithium-Ion batteries where the satisfaction of the PE condition is difficult to verify.

Keywords: Concurrent learning, adaptive control, hybrid systems, Lyapunov theory.

1. Introduction

In this paper, we study efficient algorithms for online parameter estimation problems which can be cast as linear parametric models of the form

$$y(t) = \phi(t)^\top \theta^*, \tag{1}$$

where $y : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a measurable signal, $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a bounded vector-valued regressor function, and $\theta^* \in \mathbb{R}^n$ is an unknown parameter that we want to estimate. This problem plays an important role in different areas, such as adaptive control Ioannou and Sun (2012), model-free optimization Teel (2000), and reinforcement learning Kaelbling et al. (1996), to name just a few. To achieve online parameter estimation with convergence and robustness certificates, different feedback-based algorithms have been proposed during the last three decades; see Boyd and Sastry (1986); Narendra and Annaswamy (1987, 2012). Today, it is well-known that most of the adaptive estimation dynamics that achieve uniform convergence to the true parameter θ^* require a persistence of excitation (PE) condition in the regressor ϕ, of the form

$$\int_t^{t+T} \phi(t)^\top \phi(t) \succ \epsilon I, \quad \text{where} \quad T, \varepsilon > 0. \tag{2}$$

Indeed, the PE condition has been shown to be *sufficient* and *necessary* in order to achieve (uniform) exponential convergence in several adaptive estimation dynamics, including the so-called gradient method Praly (2017):

$$\dot{\hat{\theta}} = -\sigma\phi(t)\left(\phi(t)^\top\hat{\theta} - y(t)\right), \qquad \sigma \in \mathbb{R}_{>0}, \tag{3}$$

which has been widely used in academic and industrial applications. To relax the PE condition, the works Chowdhary and Johnson (2010), Chowdhary et al. (2012b), and Chowdhary et al. (2012a) introduced a class of concurrent learning (CL) adaptive dynamics that incorporate a sequence of recorded data $\{\phi(t_k)\}_{k=1}^N$ that is "sufficiently rich", resulting in a data-driven ordinary differential equation (DD-ODE) of the form

$$\dot{\hat{\theta}} = -\sigma\phi(t)\left(\phi(t)^\top\hat{\theta} - y(t)\right) - \rho\sum_{k=1}^{N}\phi(t_k)\left(\phi(t_k)^\top\hat{\theta} - y(t_k)\right), \tag{4}$$

where $\sigma \in \mathbb{R}_{\geq 0}$ and $\rho \in \mathbb{R}_{>0}$ are tunable gains. These types of algorithms have been extended in several directions to develop PE-free adaptive dynamics in the context of model-reference adaptive control Chowdhary and Johnson (2010), reinforcement learning Kamalapurkar et al. (2014), and extremum seeking control Poveda et al. (2020), to name just a few examples. However, by removing (or relaxing) the PE condition, these types of data-driven algorithms can also suffer from poor transient performance in terms of slow rates of convergence, specially when the matrix of recorded data is ill-conditioned. This behavior stems from the fact that systems of the form (3) or (4) can be cast as *time-varying* gradient flows for which the Hessian matrix might be degenerate whenever the PE condition is relaxed. Indeed, the slow learning rates that may emerge in CL have limited its application in practical engineering problems that require fast adaptation and/or estimation.

Motivated by this background, in this paper we introduce a novel class of concurrent learning algorithms able to achieve acceleration and/or fixed-time convergence properties. The dynamics make use of different types of regularization mechanisms that have been explored during the last years to design fast optimization algorithms and feedback controllers with fixed-time stability properties, but which have never been studied in the context of CL. Given that the proposed dynamics are non-smooth, they are modeled either as non-Lipschitz ODEs Khalil (2002) or as hybrid dynamical systems Goebel et al. (2012). For these systems, we exploit hybrid Lyapunov-based methods to establish suitable stability and convergence properties. Moreover, to illustrate the performance of the proposed algorithms, we study a battery estimation problem where the satisfaction of the PE condition is usually difficult to verify *a priori*, and where standard concurrent learning algorithm of Chowdhary and Johnson (2010) lead to prohibitively slow convergence rates. As evidenced by the numerical experiments, the proposed algorithms significantly outperform the standard CL dynamics in terms of transient performance and steady state error.

The rest of this paper is organized as follows: Section 2 introduces the notation used in the paper, as well as some preliminaries on hybrid dynamical systems. Section 3 presents the main results, Section 4 presents the numerical experiments, and Section 5 ends with some conclusions. The proofs of the results are presented in the Appendix of the extended technical report Ochoa et al. (2020).

2. Notation and Preliminaries on Hybrid Dynamical Systems

We define $\mathbf{c}_n \in \mathbb{R}^n$ as the vector with all entries equal to $c \in \mathbb{R}$, and use $|\cdot|$ as the Euclidean norm. We use $|x|_{\mathcal{A}} := \min_{y\in\mathcal{A}} |x-y|$ to denote the distance of a vector $x \in \mathbb{R}^n$ with respect to a compact set $\mathcal{A}$. To simplify notation, and given vectors $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, and $z \in \mathbb{R}^{n_3}$, we use (x, y, z) to denote the column vector $[x^\top, y^\top, z^\top]^\top$. We use $I_n \in \mathbb{R}^{n\times n}$ to denote the identity matrix, and S^n to denote the nth-Cartesian product of the set S.

2.1. Hybrid Dynamical Systems

In this paper, we will model our algorithms as Hybrid Dynamical Systems with time-varying flows, of the form (see Goebel et al. (2012)):

$$(x, s) \in C \times \mathbb{R}_{\geq 0}, \qquad \dot{x} = F(x, s), \qquad \dot{s} = 1, \tag{5a}$$

$$(x, s) \in D \times \mathbb{R}_{\geq 0}, \qquad x^+ = G(x), \qquad s^+ = s, \tag{5b}$$

where $x \in \mathbb{R}^m$ is the main state of the system, s is an auxiliary state used to model the evolution of the continuous time, $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}^m$ is called the flow map, and $G : \mathbb{R}^m \to \mathbb{R}^m$ is called the jump map. The sets C and D, called the flow set and the jump set, respectively, condition the points in $\mathbb{R}^m$ where the system can *flow* or *jump* via equations (5a) or (5b), respectively. In this way, the HDS can be represented by the notation $\mathcal{H} := \{C, F, D, G\}$. Systems of the form (5) can be seen as generalizations of purely continuous-time systems ($D = \emptyset$) and purely discrete-time systems ($C = \emptyset$). Solutions to HDS of the form (5) are parameterized by both a continuous-time index $t \in \mathbb{R}_{\geq 0}$, which increases continuously during flows, and a discrete-time index $j \in \mathbb{Z}_{\geq 0}$, which increments by one during jumps. Thus, the notation $\dot{x}$ in (5a) represents the derivative of x with respect to time t, i.e., $\frac{dx(t,j)}{dt}$; and x^+ in (5b) represents the value of x after an instantaneous jump, i.e., $x(t, j + 1)$. Naturally, solutions $x : \text{dom}(x) \to \mathbb{R}^m$ to (5) are defined on hybrid time domains. For a precise definition of hybrid time domains and solutions to HDS (5) we refer the reader to (Goebel et al., 2012, Ch.2). In some cases, system (5) will not depend on s.

2.2. Stability and Convergence Notions

To model the different convergence properties of our algorithms, we make use of class $\mathcal{KL}$ functions β, which are continuous functions that satisfy $\lim_{r\to 0^+} \beta(r, \nu) = 0$ for each fixed $\nu \in \mathbb{R}_{\geq 0}$, $\lim_{s\to\infty} \beta(r, \nu) = 0$ for each fixed $r \in \mathbb{R}_{\geq 0}$, and which are non-decreasing in its first argument, and non-increasing in the second argument. Class $\mathcal{KL}$ functions are standard in the analysis of feedback control; see Khalil (2002). Moreover, these functions can model different types of convergence properties depending on the structure of β.

Definition 1 *Let $\mathcal{A} \subset \mathbb{R}^m$ be a closed set, and let $x : dom(x) \to \mathbb{R}^m$ be a solution to (5) that satisfies $|x(t, j)|_{\mathcal{A}} \leq \beta(|x(0, 0)|_{\mathcal{A}}, t + j), \quad \forall \quad (t, j) \in dom(x)$. The set $\mathcal{A}$ is said to be:*

- *(a) Uniformly globally asymptotically stable (UGAS) if β is of class $\mathcal{KL}$.*
- *(b) Uniformly globally exponentially stable (UGES) if $\exists c_1, c_2 > 0$ s.t. $\beta(r, s) = c_1 r e^{-c_2 s}$.*
- *(c) Uniformly globally finite-time stable (UGFS) if it is UGAS and there exists a continuous function $T : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\lim_{s\to T(r)} \beta(r, s) = 0$.*
- *(d) Uniformly globally fixed-time stable (UGFXS) if it is UGFS and there exists $T^* > 0$ such that $T(r) < T^*$ for all $r \in \mathbb{R}_{\geq 0}$.*

Note that all the properties listed in Definition 1 are stronger than standard convergence notions used in offline optimization or estimation algorithms. In particular, UGAS implies not only convergence in the standard limiting sense, but also *uniform* global stability (in the sense of Lyapunov) *and uniform* global attractivity.

3. Accelerated Adaptive Concurrent Learning Dynamics

To characterize the accelerated concurrent learning (ACL) dynamics considered in this paper, let the mappings $\Psi : \mathbb{R} \to \mathbb{R}^n$ and $B : \mathbb{R}^n \to \mathbb{R}^n$ be defined as

$$\Psi(t) = \frac{\phi(t)}{\left(1 + \phi(t)^\top \phi(t)\right)^2}, \quad \text{and} \quad B(\hat{\theta}) \coloneqq \sum_{k=1}^{N} \Psi(t_k)(\phi(t_k)^\top \hat{\theta} - y(t_k)), \tag{6}$$

respectively. Using these mappings, as well as (1), the DD-ODE (4) can be written as a time-invariant dynamical system of the form

$$(\hat{\theta}, s) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}, \qquad \dot{\hat{\theta}} = -\sigma A(s, \hat{\theta}) - \rho B(\hat{\theta}), \quad \dot{s} = 1, \tag{7}$$

where $A(s, \hat{\theta}) \coloneqq \Psi(s)(\phi(s)^\top \hat{\theta} - y(s))$. Taking system (7) as a benchmark, we will construct four different data-driven CL dynamics that will achieve (uniform) global asymptotic convergence, exponential convergence, finite-time convergence, and fixed-time convergence, respectively, to the true parameter $\mathcal{A}_0 \coloneqq \{\theta^*\}$. The convergence properties of these dynamics will rely on the following assumption on the available data, which uses the notion of "sufficient richness"; see Astrom and Wittenmark (1989).

Assumption 3.1 *Let $\{\phi(t_k)\}_{k=1}^N$ be a sequence of recorded data. Then, the matrix $D \coloneqq [\phi(t_1),\ \phi(t_2),\ \cdots,\ \phi(t_N)] \in \mathbb{R}^{n \times N}$ satisfies rank$(D) = p$.*

Sequences of data satisfying Assumption 3.1 are said to be *sufficiently-rich* (SR). The following lemma provides an equivalent (and instrumental) characterization of SR data.

Lemma 2 *Let $\{\phi(t_k)\}_{k=1}^N$ be a sequence of recorded data, and let $P \coloneqq \sum_{k=1}^N \frac{\phi(t_k)\phi(t_k)^\top}{\left(1+\phi(t_k)^\top\phi(t_k)\right)^2}$. Then $\{\phi(t_k)\}_{k=1}^N$ is SR if and only if there exists $\gamma \in \mathbb{R}_{>0}$ such that $P \succeq \gamma I_n$.*

We call the constant γ the *level of richness* of the data $\{\phi(t_k)\}_{k=1}^N$.

3.1. Data-Driven Accelerated Hybrid Dynamics with Periodic Restarting

The first dynamical system that we consider is inspired by Nesterov's ODEs studied in the context of accelerated optimization; see Su et al. (2016) and Wibisono et al. (2016). Such algorithms can induce suitable acceleration properties by incorporating dynamic momentum, emulating in continuous time the acceleration properties of Nesterov's accelerated optimization algorithm; see Nesterov (2004). However, unlike the results of Su et al. (2016) and Wibisono et al. (2016), in the setting of CL we are also interested in establishing suitable robustness properties that are relevant in applications where noisy measurements are unavoidable. Such robustness properties can be obtained by endowing the dynamics with discrete-time restarting mechanisms that persistently reset the momentum coefficient/state of the dynamics. The combination of continuous-time and discrete-time dynamics leads to a hybrid regularization of the Nesterov's algorithm which has been modeled as a HDS of the form (5) in Poveda and Li (2019). Based on this work, the *hybrid accelerated concurrent learning* (HACL) dynamics that we consider are modeled by a HDS with state $x := (\hat{\theta}, p, \tau)$, where $p \in \mathbb{R}^n$ is the momentum state, and $\tau \in \mathbb{R}_{>0}$ is a resetting state, and dynamics given by

$$(x,s) \in C_1 := \left\{x \in \mathbb{R}^{2n+2} : \tau \in [\delta, \Delta]\right\}, \quad \begin{pmatrix} \dot{x} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} \dot{\hat{\theta}} \\ \dot{p} \\ \dot{\tau} \\ \dot{s} \end{pmatrix} = F_1(x,s) := \begin{pmatrix} \frac{2}{\tau}\left(p - \hat{\theta}\right) \\ -2k\tau\left(\sigma A(s,\hat{\theta}) + \rho B(\hat{\theta})\right) \\ \frac{1}{2} \\ 1 \end{pmatrix}, \tag{8a}$$

$$(x,s) \in D_1 := \left\{x \in \mathbb{R}^{2n+2} : \tau = \Delta\right\}, \quad \begin{pmatrix} x^+ \\ s^+ \end{pmatrix} = \begin{pmatrix} \hat{\theta}^+ \\ p^+ \\ \tau^+ \\ s^+ \end{pmatrix} = G_1(x) := \begin{pmatrix} \hat{\theta} \\ (1-q)p + q\hat{\theta} \\ \delta \\ s \end{pmatrix}, \tag{8b}$$

where $s \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{R}_{>0}$ is a tunable gain, $\infty > \Delta > \delta > 0$ are tunable parameters that describe how frequently the algorithm restarts, and $q \in \{0, 1\}$ is a Boolean variable that characterizes the restarting policy of the algorithm. In particular, when $q = 0$ the HACL only restarts the coefficient τ, whereas when $q = 1$ the algorithm also restarts the momentum state p. By construction, the discrete-time updates of the system are periodic and separated by intervals of flow of duration $2(\Delta - \delta)$. To guarantee suitable convergence properties, we will impose the following "data-driven" condition on the parameters (δ, Δ) and the gains (k, ρ).

Assumption 3.2 *Let γ be generated as in Lemma 2. Then, the tunable parameters (δ, Δ, k) satisfy $2k\rho(\Delta^2 - \delta^2) > \frac{1}{\gamma}$.*

The following theorem, which is the first main result of this paper, characterizes the convergence properties of the HACL dynamics. In the rest of the paper, we use $\tilde{\theta} := \hat{\theta} - \theta^*$ to denote the estimation error.

Theorem 3.1 *Suppose that Assumptions 3.1 and 3.2 hold. Then, every solution of system (8) has an unbounded time domain, and the closed set $\mathcal{A} := \mathcal{A}_0 \times \mathcal{A}_0 \times [\delta, \Delta] \times \mathbb{R}_{\geq 0}$ is UGAS. Moreover, for each compact set of initial conditions $K \subset C \cup D$, the following convergence properties hold for all $(t, j) \in dom(x, s)$:*

(a) If $q = 0$, then for each $j \in \mathbb{Z}_{\geq 0}$ there exists $\beta_j \in \mathbb{R}_{>0}$ such that each trajectory of the system satisfies the bound

$$|\tilde{\theta}(t,j)|^2 \leq \frac{\beta_j}{k\rho\tau(t,j)^2}. \tag{9}$$

(b) If $q = 1$, the set $\mathcal{A}$ is UGES, and each trajectory of the system satisfies the bound

$$|\tilde{\theta}(t,j)| \leq k_0\tilde{\gamma}^j|\tilde{\theta}(0)|, \quad \text{where} \quad \tilde{\gamma} = \sqrt{\frac{1}{k\rho\Delta^2}\left(\frac{1}{2\gamma} + k\rho\delta^2\right)} \in (0,1), \quad k_0 = \frac{\Delta}{\delta}. \tag{10}$$

The result of Theorem 3.1 establishes two main convergence properties: item (a) establishes that the estimation error decreases at a rate of approximately $O(1/t^2)$ during flows. On the other hand, item (b) establishes exponential convergence with a convergence rate adjustable via the values of $(\delta, \Delta, k, \rho)$. In particular, note that information-rich data sets ($\gamma \gg 1$) lead to faster rates of convergence. Optimal restarting periods, similar to those studied in O'Donoghue and Candes (2013), can also be derived for system (8).

3.2. Data-Driven Accelerated Hybrid Dynamics with Adaptive Restarting

The HACL dynamics (8) implement a periodic restarting mechanism that is coordinated by the state τ. In this subsection, we now consider an alternative approach based on *adaptive restarting*, where the momentum state is reset whenever a certain state-dependent condition is satisfied. Such type of mechanisms have been studied in the optimization literature; see Su et al. (2016), O'Donoghue and Candes (2013), Teel et al. (2019). However, in the context of CL, these types of mechanisms remain mostly unexplored. Using the function $H(\hat{\theta}, p) = \frac{|\tilde{\theta}|_P^2}{2} + \frac{1}{2}|p|^2$, where $\tilde{\theta} := \hat{\theta} - \theta^*$, and the data-induced norm $|\cdot|_P : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is defined as $|u|_P^2 := u^\top P u$ for all $u \in \mathbb{R}^n$, with P as defined in Lemma 2, we consider the *Hybrid Hamiltonian Concurrent Learning* (HHCL) dynamics, in which have state $x = (\hat{\theta}, p, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, and dynamics given by

$$x \in C := C_0 \times [0,\ \Delta], \qquad \begin{pmatrix} \dot{\hat{\theta}} \\ \dot{p} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} 0 & k\rho I_p & 0 \\ -k\rho I_p & 0 & 0 \\ 0 & 0 & k\rho \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \hat{\theta}} \\ 1 \end{pmatrix}, \tag{11a}$$

$$x \in D := (C_0 \times \{\Delta\}) \cup (D_0 \times [0, \Delta]), \qquad \begin{pmatrix} \hat{\theta}^+ \\ p^+ \\ \tau^+ \end{pmatrix} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\theta} \\ p \\ \tau \end{pmatrix}, \tag{11b}$$

where $\Delta := \frac{n\pi}{2k\rho\sqrt{\gamma}}$, γ is the level of richness of the data, and where

$$C_0 := \left\{(\hat{\theta}, p)\ :\ \langle B(\hat{\theta}), p\rangle \leq 0\right\}, \quad D_0 := \left\{(\hat{\theta}, p)\ :\ \langle B(\hat{\theta}), p\rangle = 0\ \&\ |p|^2 \geq \left|B(\hat{\theta})\right|^2/\overline{\lambda}\right\},$$

with $\overline{\lambda} \geq \lambda_{\max}(P)$. Given that $\frac{\partial H}{\partial \hat{\theta}} = B(\hat{\theta})$ and $\frac{\partial H}{\partial p} = p$, the construction of the sets C_0 and D_0 indicate that system (11) is allowed to flow whenever there is no increase in the *potential* energy of the data-induced Hamiltonian function H.

Remark 3.1 *The role of the timer τ in system (11) is to guarantee the existence of an initial reset after an interval of flow of duration $\Delta > 0$. Once this reset has occurred, the update $p^+ = 0$ will guarantee that next reset of the system will happen before $\tau = \Delta$, i.e., due to the condition $|p|^2 \geq \left|B(\hat{\theta})\right|^2/\overline{\lambda}$ for all $x \in D$. Such types of bounds on the reset times have been used in Teel et al. (2019) to solve standard static optimization problems. However, in the context of CL, their application is new. In particular, note that the parameter Δ in the jump set D is data-dependent.*

The following theorem is the second main result of this paper.

Theorem 3.2 *Suppose that Assumption 3.1 holds. Then, system (11) renders the set $\mathcal{A}_H = \mathcal{A}_0 \times \{0\} \times [0, \Delta T]$ UGES, and every solution is complete and satisfies the bound*

$$|\hat{\theta}(t,j)| \leq \sqrt{\frac{2c_0}{\gamma}} \min\left\{1, e^{-\frac{\alpha}{2}(t-\Delta)}\right\} |\hat{\theta}(0,0) - \theta^*| \tag{12}$$

for all $(t,j) \in dom(x)$, where $\alpha = \frac{1}{\Delta} \ln\left(1 + \frac{\gamma}{\lambda}\right)$.

3.3. Finite-Time and Fixed-Time Concurrent Learning Dynamics

While the hybrid CL dynamics (8) and (11) can induce sublinear and linear convergence rates, the convergence properties of the algorithms are still of *asymptotic* nature, i.e., $\theta(t) \to \theta^*$ only as $t \to \infty$. In this subsection, consider a different class of learning dynamics able to achieve exact convergence to the true parameter θ^* in a finite amount of time. Moreover, in some cases this finite time can be upper bounded by a constant independent of the initial conditions of the estimate $\hat{\theta}$, which leads to fixed-time convergence guarantees.

In particular, to achieve finite time convergence we consider the *Finite-Time Concurrent Learning* (FTCL) dynamics modeled by the non-smooth DD-ODE:

$$\hat{\theta} \in C := \mathbb{R}^n, \quad \dot{\hat{\theta}} = -k\frac{\sigma A(s,\hat{\theta}) + \rho B(\hat{\theta})}{|\rho B(\hat{\theta})|^{\frac{1}{2}}}, \quad \dot{s} = 1, \tag{13}$$

where $(k, \sigma, \rho) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ are tunable gains, and where the pair (A, B) is defined in (6) and (7). Note that this system is not Lipschitz continuous. However, the vector field is continuous in $\hat{\theta}$ due to the linearity of A and B with respect to $\hat{\theta}$. Thus, the existence of solutions is guaranteed from every initial condition. For this system we establish the following result, which is the third main contribution of the paper.

Theorem 3.3 *Suppose that Assumption 3.1 holds. Then, system (13) renders the set $\mathcal{A}_0$ UGFTS, and the settling time function satisfies $T(\hat{\theta}(0)) \leq \frac{2}{k\rho\gamma}|P|\sqrt{|\hat{\theta}(0) - \theta^*|}$.*

The result of Theorem 3.3 guarantees that $\hat{\theta}(t) = \theta^*$ for all $t \geq T(\hat{\theta}(0))$, where the constant $T(\hat{\theta}(0))$ depends on the initial conditions of the estimate $\hat{\theta}(0)$, as well as the level of richness γ of the data. To remove the dependence on $\hat{\theta}(0)$ we can further consider a class of *Fixed-Time Concurrent Learning* (FXCL) dynamics, modeled by the following nonsmooth DD-ODE:

$$\hat{\theta} \in C := \mathbb{R}^n, \quad \dot{\hat{\theta}} = -k\frac{\sigma A(s,\hat{\theta}) + \rho B(\hat{\theta})}{|B(\hat{\theta})|^{a}} - k\frac{\sigma A(s,\hat{\theta}) + \rho B(\hat{\theta})}{|B(\hat{\theta})|^{-a}}, \quad \dot{s} = 1, \tag{14}$$

where $(k, \sigma, \rho) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ are tunable gains, and $a \in (0, 1)$ is a tunable exponent. System (14) is not Lipschitz continuous. However, it is everywhere continuous in $\hat{\theta}$, and the existence of solutions is guaranteed from every initial condition. The following theorem is the fourth main result of this paper.

Theorem 3.4 *Suppose that Assumption 3.1 holds. Then, system (14) renders UGFXS the set $\mathcal{A}_0$, and every solution satisfies $\hat{\theta}(t) = \theta^*$ for all $t \geq T^*$, where $T^* = \frac{\pi}{2a\gamma\rho k}$.*

Remark 3.2 *Note that the fixed-time T^* is independent of the initial estimate $\hat{\theta}(0)$, but dependent on the level of richness of the data of the regressor, i.e., dependend on $\gamma > 0$ (c.f. Remark 2).*

3.4. Discussion

The results of Theorems 3.1-3.4 establish new convergence bounds for CL algorithms that explicitly show the dependence on the richness of the data, i.e., γ. In particular, while the standard CL dynamics achieve exponential convergence with rate of convergence proportional to the level or richness of the matrix P (cf. Lemma 2), under suitable tuning of the restarting parameters the data-driven hybrid dynamics introduced in this paper can achieve rates of convergence proportional to the squared root of the level of richness of the matrix P, see Poveda and Li (2019) and Teel et al. (2019). This acceleration property is induced by the addition of momentum to the dynamics, and the design of the flow set and the jump set. Similarly, for the non-smooth DD-ODES (13) and (14), our results establish finite and fixed-time convergence bounds that are new in the context of CL, with an explicit characterization in terms of the level of richness γ. Finally, note that the bound (9) is independent of γ. Thus, for applications where $\gamma \approx 0$, the bound (9) establishes a semi-acceleration property similar to those established in Su et al. (2016) for optimization problems that are convex, but not necessarily strongly convex.

4. Numerical Experiments: Li-ion battery parameter estimation

To illustrate the methods we presented in Section 3 in a practical context, we utilize the developed methods to estimate the impedance parameters of an equivalent circuit model of a Lithium-Ion (Li-Ion) battery. Li-Ion batteries specifically have strict requirements to achieve safe and reliably life-time operation, e.g. (Reddy and Linden, 2011, Part 4) or (Plett, 2016, Volume II, Chapter 1). In order to guarantee that the battery achieves its performance targets safely, it is customary to design real-time algorithms that control the battery and estimate potentially internal states based on a dynamical model of the battery. In applications with limited computational capability, circuit based models are typically used in battery management systems. A first order equivalent circuit model (see Liaw et al. (2004), Feng et al. (2015) for more details) for a Li-ion battery is described below

$$\dot{soc} = \frac{I}{C_0}, \quad \dot{i}_1 = \frac{I - i_1}{R_1 C_1}, \quad V(t) = \Phi(soc(t)) + I(t)R_0 + i_1(t)R_1, \tag{15}$$

where $C_0[As]$ denotes to the battery capacity, $R_0[\Omega]$,$R_1[\Omega]$,$C_1[F]$ denotes to the battery impedance parameters, soc denotes to the state of charge of the battery, i.e. the relative capacity, $I[A]$ denotes the input current, $i_1[A]$ is the current through the parallel resistor R_1 and Φ represents state of charge to open circuit voltage mapping. The impedance parameters R_0, R_1, C_1 are typically functions of SOC as noted in Hannan et al. (2017) and Liaw et al. (2004).

For ease of presentation and to retain a linear parameter formulation of the estimation problem, we make some simplifying assumptions regarding the estimation problem. We note that the methods described here can be extended to the case without such assumptions whereas standard methods like RLS (Recursive least squares) are not suitable due to structure of the parameter dependency on battery state of charge. The battery capacity C_0 evolves on a slower time scale and is typically estimated using rest voltage measurements as in Subbaraman et al. (2019), which decouples it from battery impedance parameter estimation which can change over a single charge/discharge cycle due to state of

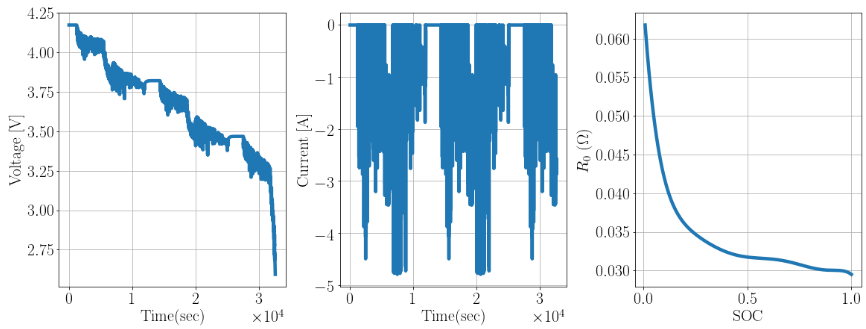

Figure 1: Battery model simulation with nominal value of R_0

charge, temperature variations. If battery capacity, initial state of charge (obtained from rest voltage), OCV-SOC relation is known at least over a cycle, the battery SOC can be calculated accurately using coulomb counting. So, we assume the battery capacity, initial SOC is known and estimation is restricted to battery impedance parameters of which the functional dependency of SOC is restricted to R_0 whereas R_1 and C_1 are constant. We also assume an estimate of the time constant R_1C_1 is available from fitting relaxation data. While the assumptions above are restrictive, it is to ensure the parameter estimation problem can be formulated in a linear parametric form for ease of presentation. Future work will analyze the performance of these algorithms without such assumptions.
Let the series resistance R_0 be expressed as $R_0(soc) = \sum_{i=0}^{N} \alpha_i soc^i$. The parameter vector $w = [\alpha_0, ..., \alpha_N, R_1]$. Then, the system dynamics can be rewritten as follows

$$(V - \Phi(soc)) = [I, .., soc^N I, i_1]w. \tag{16}$$

The simulation model parameters are as follows: The initial SOC=1.0, $C_0 = 3.4$Ah, $R_1C_1 = 100$. A sample drivecycle behavior and nominal value of R_0 as a function of SOC is illustrated in Figure 1.
Simulation Results: Having established the problem of parameter estimation in the battery system setup, we choose the regressor vector as

$$\phi(t) = \left[I(t), I(t)soc(t), \cdots, I(t)soc(t)^N, i_1(t)\right]^\top,$$

with $N = 7$, and $I(t), i_1(t)$ are the input-current and the current through resistor R_1.
On the other hand, in order to achieve good numerical performance in the practical setup, in addition to being compliant with Assumption (3.1) so that we can guarantee the stability of the set $\mathcal{A}_0$, we pay special attention to the convergence rates of the different dynamics and how they are closely related to the magnitude of the minimum eigenvalue of the matrix P. With this in mind, we aim to select data that is both SR, and at the same time maximizes said eigenvalue. In order to do so, we follow the procedure presented in Chowdhary and Johnson (2011), and find a subset of 50 data points from the battery sample drivecycle behavior. We highlight that the resulting $\lambda_{\min}(P) \approx 10^{-6}$, and even though this value can increase slightly when larger data-sets are considered, the size is kept small in order to facilitate faster computation, specially for the FTCL and FXCL variants which involve calculating the norm of B. With the selected data-set we implement the different methods

presented in Section 3 and compare them with the standard concurrent learning dynamics (4). The results are presented in Figure 2. The novel proposed dynamics outperform the standard CL approach for the number of iterations considered. Moreover, and given that in this setting we have $\gamma \approx 0$ since $\gamma \leq \lambda_{\min}(P)$, we confirm the behaviour predicted in Section 3.4.

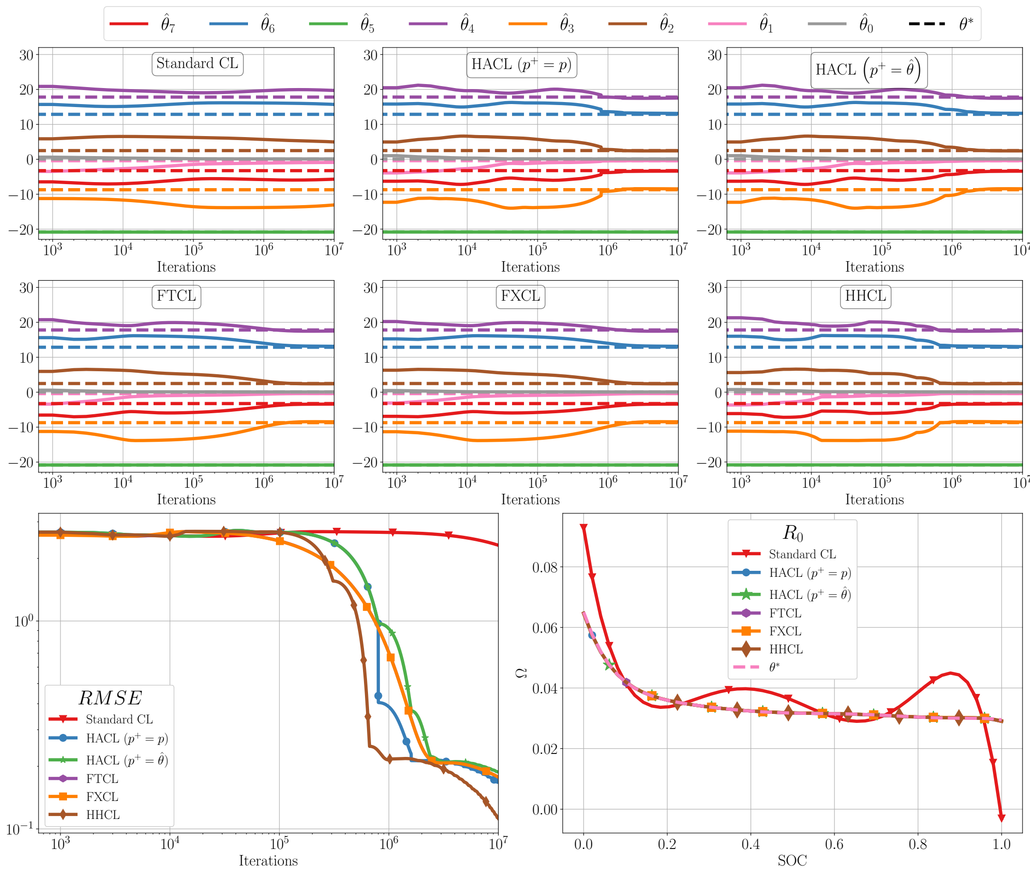

Figure 2: Battery model parameter estimation of R_0 using the different proposed dynamics

5. Conclusions

In this paper, we introduced a new class of concurrent learning algorithms with acceleration and finite/fixed-time convergence properties. The algorithms are suitable for identification and parameter estimation problems that arise in the context of adaptive control, model-free optimization, and reinforcement learning. The proposed algorithms are modeled as nonsmooth ODEs or hybrid dynamical systems, for which suitable stability, convergence, and robustness properties can be established via Lyapunov-based tools and invariance principles. We illustrated the advantages of the method via numerical examples in a Lithium-Ion battery estimation problem where standard concurrent learning dynamics lead to prohibitively slow rates of convergence. Future directions will consider data-driven accelerated adaptive controllers for dynamical systems. Such results will harness the efficient real-time identification results of Theorems 1-4 established in this paper.

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Appendix A. Proofs

In this section, we present the main stability and convergence proofs of the algorithms.

A.1. Proof of Theorem 3.1

We will establish the UGAS result by using hybrid Lyapunov functions. In particular, we consider the function

$$V(x)=\frac{|p-\hat{\theta}|^2}{4}+\frac{|p-\theta^*|^2}{4}+\frac{k\rho\tau^2}{2}|\tilde{\theta}|_P^2. \tag{17}$$

Using the definition of P and $|\cdot|_P$, we have

$$\frac{|\tilde{\theta}|_P^2}{2}=\frac{|\tilde{\theta}^\top P\tilde{\theta}|}{2}\leq\frac{|\tilde{\theta}||P\tilde{\theta}|}{2}\leq\frac{\lambda_{\max}(P)|\tilde{\theta}|^2}{2}. \tag{18}$$

Moreover, $\frac{\gamma|\tilde{\theta}|^2}{2}\leq\frac{|\tilde{\theta}|_P^2}{2}$. Therefore, for all $x\in C_1\cup D_1$, we obtain

$$\begin{aligned}V(x)&\geq\frac{\left|p-\hat{\theta}\right|^2}{4}+\frac{|p-\theta^*|^2}{4}+\frac{k\rho\delta^2}{2}|\tilde{\theta}|_P^2\geq\frac{\left|p-\hat{\theta}\right|^2}{4}+\frac{|p-\theta^*|^2}{4}+\gamma k\rho\delta^2\frac{\left|\hat{\theta}-\theta^*\right|^2}{2}\\&\geq\frac{\left|p-\hat{\theta}\right|^2}{4}+\min\left\{\frac{1}{4},\ \frac{\gamma k\rho\Delta^2}{2}\right\}\left(|p-\theta^*|^2+\left|\hat{\theta}-\theta^*\right|^2\right)\geq\underline{c}|x|_{\mathcal{A}}^2,\end{aligned} \tag{19}$$

where $\underline{c}:=\min\left\{\frac{1}{4},\ \frac{\gamma k\rho\delta^2}{2}\right\}$. On the other hand, using (18) and the fact that $\tau\leq\Delta$, we obtain

$$V(x)\leq\frac{\left|p-\hat{\theta}\right|^2}{4}+\frac{|p-\theta^*|^2}{4}+\frac{k\rho\Delta^2}{2}\left|\tilde{\theta}\right|_P^2\leq\frac{\left|p-\hat{\theta}\right|^2}{4}+\frac{|p-\theta^*|^2}{4}+k\rho\lambda_{\max}(P)\Delta^2\frac{\left|\hat{\theta}-\theta^*\right|^2}{2}. \tag{20}$$

Furthermore, using the inequality $\left|p-\hat{\theta}\right|^2\leq 2\left(\left|\hat{\theta}-\theta^*\right|^2+|p-\theta^*|^2\right)$, we obtain

$$\begin{aligned}V(x)&\leq\frac{1}{2}\left(\left|\hat{\theta}-\theta^*\right|^2+|p-\theta^*|^2\right)+\frac{|p-\theta^*|^2}{4}+k\rho\lambda_{\max}(P)\Delta^2\frac{\left|\hat{\theta}-\theta^*\right|^2}{2}\\&=\frac{3}{4}|p-\theta^*|+\frac{1}{2}\left(1+k\rho\lambda_{\max}(P)\Delta^2\right)\left|\hat{\theta}-\theta^*\right|^2\leq\overline{c}|x|_{\mathcal{A}}^2,\end{aligned}$$

where $\overline{c}:=\max\left\{\frac{3}{4},\frac{1}{2}\left(1+k\rho\lambda_{\max}(P)\Delta^2\right)\right\}$. Similarly, using $|p-\theta^*|^2\leq 2\left(\left|p-\hat{\theta}\right|^2+\left|\hat{\theta}-\theta^*\right|^2\right)$, we obtain:

$$\begin{aligned}V(x)&\leq\frac{\left|p-\hat{\theta}\right|^2}{4}+\frac{1}{2}\left(\left|p-\hat{\theta}\right|^2+\left|\hat{\theta}-\theta^*\right|^2\right)+k\rho\lambda_{\max}(P)\Delta^2\frac{\left|\hat{\theta}-\theta^*\right|^2}{2},\\&=\frac{3}{4}\left|p_{\hat{\theta}}-\hat{\theta}\right|^2+\frac{1}{2}\left(1+k\rho\lambda_{\max}(P)\Delta^2\right)\left|\hat{\theta}-\theta^*\right|^2\leq\overline{c}\left(\left|p-\hat{\theta}\right|^2+\left|\hat{\theta}-\theta^*\right|^2\right),\end{aligned}$$

$$\implies \left|p-\hat{\theta}\right|^2+\left|\hat{\theta}-\theta^*\right|^2 \geq \frac{V(z)}{\bar{c}}. \tag{21}$$

Now, let $\eta := 1-\frac{\delta}{\Delta^2}-\frac{1}{2k\gamma\rho\Delta^2}$, $\tilde{\rho} := \min\left\{\frac{1}{\Delta},\ \frac{k\gamma\rho\delta}{2}\right\}$ and $\lambda := \min\left\{\frac{\tilde{\rho}}{\bar{c}}, -\log(1-\eta)\right\}$. Notice that $\lambda \in \mathbb{R}_{>0}$, since $\frac{\rho}{\bar{c}} > 0$ and $\eta \in (0,1)$ by Assumption 3.2. Additionally, note that we can write B as

$$B(\tilde{\theta}) = \sum_{i=1}^{N} \frac{\phi(t_i)}{\left(1+\phi(t_i)^\top\phi(t_i)\right)^2}\left(\phi(t_i)^\top\hat{\theta}-\phi(t_i)\theta^*\right) = \sum_{i=1}^{N} \frac{\phi(t_i)\phi(t_i)^\top}{\left(1+\phi(t_i)^\top\phi(t_i)\right)^2}\tilde{\theta} = P\tilde{\theta}. \tag{22}$$

and recall that $A(s,\tilde{\theta}) := \Psi(s)\phi(s)^\top\tilde{\theta}$. By defining $\rho f(s,\tilde{\theta}) := \sigma A(s,\tilde{\theta}) + \rho P\tilde{\theta}$, we can compute the time-derivative of V along the trajectories generated by the flows of $\mathcal{H}_1$:

$$\begin{aligned}
\dot{V}(x) &= \nabla V(x)^\top \dot{x},\\
&= \left[-\frac{(p-\hat{\theta})}{2}+\rho\tau^2 f(s,\tilde{\theta}),\ \frac{(p-\hat{\theta})}{2}+\frac{(p-\theta^*)}{2},\ \rho\tau\left|\tilde{\theta}\right|_P^2\right]^\top \dot{x},\\
&= -\frac{\left|p-\hat{\theta}\right|^2}{\tau} + 2k\rho\tau(p-\hat{\theta})^\top f(s,\tilde{\theta}) - k\rho\tau(p-\hat{\theta})^\top f(s,\tilde{\theta}) - k\rho\tau(p-\theta^*)^\top f(s,\tilde{\theta}) + \frac{k\rho\tau}{2}\left|\tilde{\theta}\right|_P^2,\\
&= -\frac{\left|p-\hat{\theta}\right|^2}{\tau} + \rho k\tau\left((p-\hat{\theta})^\top f(s,\tilde{\theta}) - (p-\theta^*)^\top f(s,\tilde{\theta}) + \frac{\left|\tilde{\theta}\right|_P^2}{2}\right),\\
&= -\frac{\left|p-\hat{\theta}\right|^2}{\tau} + \rho k\tau\left(-\tilde{\theta}^\top f(s,\tilde{\theta}) + \frac{\left|\tilde{\theta}\right|_P^2}{2}\right)\\
&= -\frac{\left|p-\hat{\theta}\right|^2}{\tau} + \rho k\tau\left(-\tilde{\theta}^\top\left(\frac{\sigma}{\rho}A(s,\tilde{\theta})+P\tilde{\theta}\right) + \frac{\left|\tilde{\theta}\right|_P^2}{2}\right).
\end{aligned}$$

Since $\tilde{\theta}^\top A(s,\tilde{\theta}) \geq 0$, and using the fact that the data is SR with level of richness $\gamma > 0$, we obtain:

$$\begin{aligned}
\dot{V}(x) &\leq -\frac{\left|p-\hat{\theta}\right|^2}{\tau} - \rho k\tau\frac{\left\|\tilde{\theta}\right\|_P^2}{2} \leq -\frac{\left|p-\hat{\theta}\right|^2}{\tau} - \frac{k\rho\gamma\tau}{2}\left|\hat{\theta}-\theta^*\right|^2,\\
&\leq -\frac{\left|p-\hat{\theta}\right|^2}{\Delta} - \frac{k\rho\gamma\delta}{2}\left|\hat{\theta}-\theta^*\right|^2 \leq -\min\left\{\frac{1}{\Delta},\ \frac{k\rho\gamma\delta}{2}\right\}\left(\left|p-\hat{\theta}\right|^2+\left|\hat{\theta}-\theta^*\right|^2\right) \leq -\frac{\tilde{\rho}}{\bar{c}}V(x).
\end{aligned} \tag{23}$$

Since by definition $\lambda \leq \frac{\tilde{\rho}}{\bar{c}}$, we obtain

$$\dot{V}(x) \leq -\lambda V(x) \leq -\lambda\underline{c}|x|_{\mathcal{A}}^2 < 0 \qquad \forall x \in C \setminus \mathcal{A}. \tag{24}$$

Note that this bound implies that V does not increase during flows, i.e., $V(x(t,j)) \leq V(x(s,j))$ for all $t > s$ and all j such that $(t,j) \in \mathrm{dom}(x)$. This generates the bound (9).

On the other hand, during jumps the restarting policy $q = 0$ generates changes in the Lyapunov function given by

$$\begin{aligned} V(x^+) - V(x) &= \frac{\left|p-\hat{\theta}\right|^2}{4} + \frac{|p-\theta^*|^2}{4} + \frac{k\rho\delta^2}{2}|\tilde{\theta}|_P^2 - \frac{\left|p-\hat{\theta}\right|^2}{4} - \frac{|p-\theta^*|^2}{4} - \frac{k\rho\Delta^2}{2}|\tilde{\theta}|_P^2 \\ &= -k\rho\frac{\Delta^2-\delta^2}{2}|\tilde{w}|_P^2 \leq 0. \end{aligned} \tag{25}$$

When $q = 1$, the change of the Lyapunov function during jumps is given by

$$\begin{aligned} V(x^+) - V(x) &= \frac{\left|\hat{\theta}-\theta^*\right|^2}{4} + \frac{k\rho\delta^2}{2}|\tilde{\theta}|_P^2 - \frac{\left|p-\hat{\theta}\right|^2}{4} - \frac{|p-\theta^*|^2}{4} - \frac{k\rho\tau^2}{2}|\tilde{\theta}|_P^2 \\ &\leq \frac{|\theta|_P^2}{4\gamma} + \frac{k\rho\delta^2}{2}|\tilde{\theta}|_P^2 - \frac{\left|p-\hat{\theta}\right|^2}{4} - \frac{|p-\theta^*|^2}{4} - \frac{k\rho\tau^2}{2}|\tilde{\theta}|_P^2 \\ &= -\frac{\left|p-\hat{\theta}\right|^2}{4} - \frac{|p-\theta^*|^2}{4} - \left(1-\frac{\delta^2}{\tau^2}-\frac{1}{2k\rho\gamma\tau^2}\right)\frac{k\tau^2|\theta|_P^2}{2} \\ &\leq -\eta V(x) \\ &\implies V(x^+) \leq e^{-\lambda}V(x) \qquad \forall x \in D,\ x^+ \in G(x), \end{aligned} \tag{26}$$

where we used $(1-\eta) \leq e^{-\lambda}$ which follows from the definition of η. Inequality (26) induces the bound (10). Inequalities (23) and (26) imply UGES of the set $\mathcal{A}$. Similarly, inequalities (23) and (25) imply UGAS via the hybrid invariance principle (Goebel et al., 2012, Thm. 8.8). UGES of $\mathcal{A}_H$ follows by the fact that $c_1|(\tilde{\theta},p)|^2 \leq H(\hat{\theta},q) \leq c_2|(\tilde{\theta},p)|^2$.

A.2. Proof of Theorem 3.2

First, notice that the Hamiltonian function H is invariant in time, since

$$\dot{H}(\hat{\theta},p) = \nabla H^\top \begin{pmatrix}\dot{\hat{\theta}} \\ \dot{p}\end{pmatrix} = k\rho\left(\frac{\partial H}{\partial\hat{\theta}}\right)^\top\left(\frac{\partial H}{\partial p}\right) - k\rho\left(\frac{\partial H}{\partial p}\right)^\top\left(\frac{\partial H}{\partial\hat{\theta}}\right) = 0,$$

where we have used that $\frac{\partial H}{\partial\hat{\theta}} = B(\hat{\theta})$. To analyze the behaviour of H during jumps, we first note that for any pair of vectors $u, v \in \mathbb{R}^n$ we have $\frac{|u|_P^2}{2} = \frac{|v|_P^2}{2} + v^\top P(u-v) + \frac{1}{2}(u-v)^\top P(u-v)$. Therefore, since the data is SR with level of richness $\gamma > 0$, we obtain:

$$\begin{aligned} \frac{|u|_P^2}{2} &\geq \frac{|v|_P^2}{2} + v^\top P(u-v) + \frac{\gamma}{2}|u-v|^2, \\ &\geq \min_{u\in\mathbb{R}^n}\left(\frac{|v|_P^2}{2} + v^\top P(u-v) + \frac{\gamma}{2}|u-v|^2\right) = \frac{|v|_P^2}{2} - \frac{1}{2\gamma}|Pv|^2, \end{aligned}$$

which implies that

$$\min_{u\in\mathbb{R}^n}\frac{|u|_P^2}{2} \geq \frac{|v|_P^2}{2} - \frac{1}{2\gamma}|Pv|^2 \implies |Pv|^2 \geq \gamma|v|_P^2.$$

With this in mind, let $\epsilon := \frac{1}{\frac{\bar{\lambda}}{\gamma}+1}$, and note that after a jump the value of the Hamiltonian is given by

$$H(\hat{\theta}^{+},p^{+}) = H(\hat{\theta},0) = \frac{|\tilde{\theta}|_P^2}{2} \leq (1-\epsilon)\frac{|\tilde{\theta}|_P^2}{2} + \epsilon\frac{|\tilde{\theta}|_P^2}{2},$$

$$\leq (1-\epsilon)\frac{|\tilde{\theta}|_P^2}{2} + \epsilon\frac{\left|P\tilde{\theta}\right|^2}{2\gamma} = (1-\epsilon)\frac{|\tilde{\theta}|_P^2}{2} + \epsilon\frac{|B(\hat{\theta})|^2}{2\gamma}.$$

Using the definitions of the jump set and the constant ϵ, we get

$$H(\hat{\theta}^{+},p^{+}) \leq (1-\epsilon)\frac{|\tilde{\theta}|_P^2}{2} + \epsilon\frac{\bar{\lambda}}{2\gamma}|p|^2 = \epsilon\frac{\bar{\lambda}}{2\gamma}\left(\frac{|\tilde{\theta}|_P^2}{2} + \frac{|p|^2}{2}\right),$$

$$\implies H(\hat{\theta}^{+},p^{+}) \leq \epsilon\frac{\bar{\lambda}}{\gamma}H(\hat{\theta},p) \qquad \forall x \in D.$$

Therefore, from initial conditions $(\hat{\theta}_0, p_0, \tau_0)$, after j jumps we have

$$H(\hat{\theta}(t,j),p(t,j)) \leq \left(\epsilon\frac{\bar{\lambda}}{\gamma}\right)^j H(\hat{\theta}_0,p_0) = (1-\epsilon)^j H(\hat{\theta}_0,p_0) = e^{-j\Delta T\beta}H(\hat{\theta}_0,p_0), \tag{27}$$

where $\alpha := \frac{1}{\Delta}\ln\left(\frac{1}{1-\epsilon}\right) = \frac{1}{\Delta}\ln\left(1+\frac{\gamma}{\bar{\lambda}}\right)$. As described in Remark 3.1 we have that $\tau(t,j) < \Delta$ for all $t+j>0$, and thus $t < (j+1)\Delta \implies e^{-j\Delta\alpha} < e^{-\alpha(t-\Delta)}$. Using this inequality in (27), we obtain

$$H(\hat{\theta}(t,j),p(t,j)) \leq \min\left\{1, e^{-\alpha(t-\Delta T)}\right\} H(\hat{\theta}_0,p_0).$$

A.3. Proof of Theorem 3.3

We consider the Lyapunov function $V(\hat{\theta}) = \frac{1}{2}\left|B(\hat{\theta})\right|^2$, which satisfies

$$\dot{V}(\hat{\theta}) = \nabla V(\hat{\theta})^\top\dot{\hat{\theta}} = -k\tilde{\theta}^\top PP\frac{\sigma A(s,\hat{\theta})+\rho P\tilde{\theta}}{|P\tilde{\theta}|^{\frac{1}{2}}} = -k\sigma\tilde{\theta}^\top P^\top P\frac{A(s,\hat{\theta})}{|P\tilde{\theta}|^{\frac{1}{2}}} - k\rho\tilde{\theta}^\top P^\top P\frac{P\hat{\theta}}{|P\tilde{\theta}|^{\frac{1}{2}}}.$$

Using the definition of A, we obtain $\tilde{\theta}^\top P^\top PA(s,\hat{\theta}) \geq 0$, therefore

$$\dot{V}(\hat{\theta}) \leq -k\rho\frac{\tilde{\theta}^\top P^\top PP\tilde{\theta}}{|P\tilde{\theta}|^{\frac{1}{2}}} = -k\rho\frac{|P\tilde{\theta}|_P^2}{|P\tilde{\theta}|^{\frac{1}{2}}}, \tag{28}$$

and since the data is SR with level of richness $\gamma > 0$, we get

$$\dot{V}(\hat{\theta}) \leq -k\rho\gamma\frac{|P\tilde{\theta}|^2}{|P\tilde{\theta}|^{\frac{1}{2}}} = -k\rho\gamma|P\tilde{\theta}|^{\frac{3}{2}} = -k\rho\gamma\left(2V(\hat{\theta})\right)^{\frac{3}{4}}, \tag{29}$$

and it follows by (Bhat and Bernstein, 2000, Thm. 4.2) that the set $\mathcal{A}_0$ is UGFTS with settling time function $T : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ bounded as

$$T(\hat{\theta}(0)) \leq \frac{2}{k\rho\gamma}|P|\sqrt{|\hat{\theta}(0)-\theta^*|}. \tag{30}$$

A.4. Proof of Theorem 3.4

We consider the Lyapunov function $V(\hat{\theta}) = \frac{1}{2}\left|B(\hat{\theta})\right|^2$. Using (22) and (14), we obtain

$$\dot{V}(\hat{\theta}) = \nabla V(\hat{\theta})^\top \dot{\hat{\theta}} = (P \cdot P\tilde{\theta})^\top \dot{\hat{\theta}} = -k(P \cdot P\tilde{\theta})^\top \left(\frac{\sigma A(s,\hat{\theta}) + \rho P\tilde{\theta}}{\left|B(\hat{\theta})\right|^a} + \frac{\sigma A(s,\hat{\theta}) + \rho P\tilde{\theta}}{\left|B(\hat{\theta})\right|^{-a}} \right).$$

Using the definition of A and the fact that $\tilde{\theta}^\top P^\top P A(s,\hat{\theta}) \geq 0$, we obtain:

$$\begin{aligned}
\dot{V}(\hat{\theta}) &\leq -k\rho(P\tilde{\theta})^\top P(P\tilde{\theta}) \left(\frac{1}{\left|B(\hat{\theta})\right|^a} + \frac{1}{\left|B(\hat{\theta})\right|^{-a}} \right) \leq -k\gamma\rho \left|P\tilde{\theta}\right|^2 \left(\frac{1}{\left|P\tilde{\theta}\right|^a} + \frac{1}{\left|P\tilde{\theta}\right|^{-a}} \right) \\
&= -k\gamma\rho \left(|P\tilde{\theta}|^{2-a} + |P\tilde{\theta}|^{2+a} \right) = -k\gamma\rho \left((2V(\hat{\theta}))^{1-\frac{a}{2}} + ((2V(\hat{\theta}))^{1+\frac{a}{2}} \right), \\
&= -k(\gamma\rho 2^{1-\frac{a}{2}})V(\hat{\theta})^{1-\frac{a}{2}} - k(\gamma\rho 2^{1+\frac{a}{2}})V(\hat{\theta})^{1+\frac{a}{2}}.
\end{aligned}$$

The last inequality implies UGFXS via (Polyakov, 2012, Lemma 1). Moreover, by (Parsegov et al., 2012, Lemma 2), a sharp bound T^* on the settling time function T can be computed as

$$T^* = \frac{\pi}{2a\gamma\rho k}. \tag{31}$$