Simultaneous Safe Screening of Features and Samples in Doubly Sparse Modeling

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Introduction

We consider regularized empirical risk minimization induces feature/sample sparsity.

- **Motivation:** To reduce computational cost for training
- **Approch:** Identifying non-active features/samples at the optimal solution

Previous works :

- **Safe feature screening**[1]**:** Identifying non-active features for feature sparse models
- **Safe sample screening**[2]**:** Identifying non-active samples for sample sparse models

Safe screening has been individually studied either for feature or sample screening

- **Main contribution (simultaneous safe screening of features and samples) :** Safely screening features and samples simultaneously by alternatively iterating feature and sample screening steps for feature and sample (doubly) sparse models

Preliminaries

Safe feature screening (for Elastic net penalty)

KKT condtion:

Safe fature screening rule :

$$\frac{1}{\lambda n}X_{:j}^\top\alpha^* \in \begin{cases}[-1,1] & (w_j^*=0) \Rightarrow UB(|X_{:j}^\top\alpha^*|) \le \lambda n \Rightarrow w_j^* = 0\\ \frac{w_j^*}{|w_j^*|}+w_j^* & (w_j^*\neq 0),\end{cases}$$

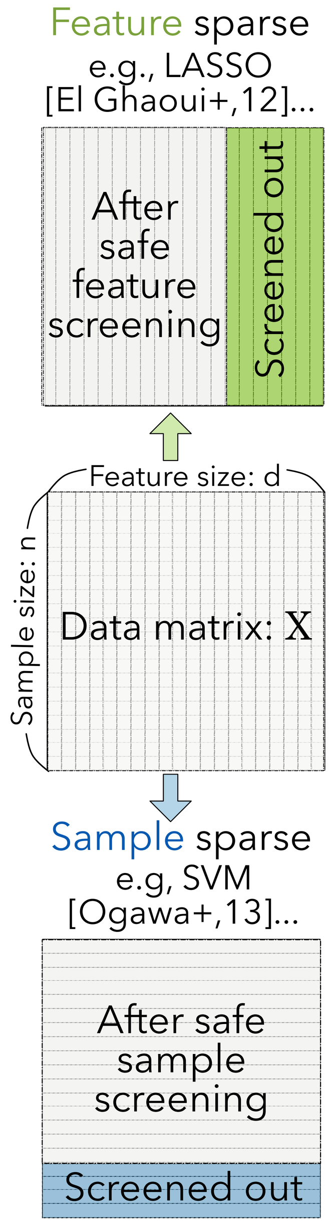
$$|X_{:j}^\top\alpha^*| \le UB(|X_{:j}^\top\alpha^*|) := \max_{\alpha}|X_{:j}^\top\alpha| \ \text{ s.t. } \ \alpha\in\Theta_{\alpha^*}$$
$$= |X_{:j}^\top\hat\alpha| + \|X_{:j}\|_2\sqrt{2n(P_\lambda(\hat w)-D_\lambda(\hat\alpha))/\gamma}$$

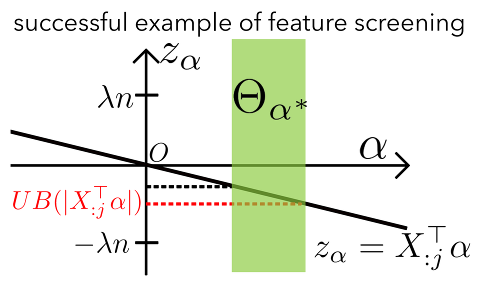
Θ_{α^*}: region of dual optimal solution

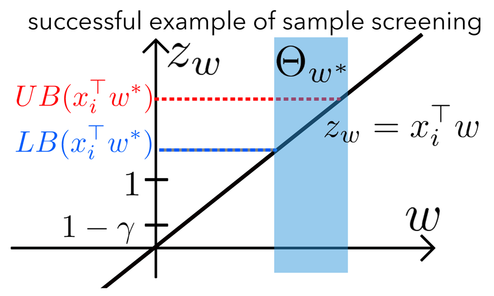
[Ndiaye+, 15] If the D_λ is γ/n-strongly concave then

$$\alpha^*\in\Theta_{\alpha^*} := \{\,\alpha \mid \|\hat\alpha-\alpha\|_2 \le \sqrt{2n(P_\lambda(\hat w)-D_\lambda(\hat\alpha))/\gamma}\,\},$$

for any $\hat w\in\mathrm{dom}P_\lambda, \hat\alpha\in\mathrm{dom}D_\lambda$







Safe sample screening (for smoothed hinge loss)

KKT condtion:

Safe sample screening rules:

$$x_i^\top w^* \in \begin{cases}[1,\infty) & (\alpha_i^*=0) \ \Rightarrow\ LB(x_i^\top w^*)\ge 1 \Rightarrow \alpha_i^*=0,\\ (-\infty,1-\gamma] & (\alpha_i^*=1) \ \Rightarrow\ UB(x_i^\top w^*)\le 1-\gamma \Rightarrow \alpha_i^*=1\\ -\gamma\alpha_i^*+1 & (\alpha_i^*\in(0,1))\end{cases}$$

$$x_i^\top w^* \ge LB(x_i^\top w^*) := \min_{w\in\Theta_{w^*}} x_i^\top w = x_i^\top\hat w - \|x_i\|_2\sqrt{2(P_\lambda(\hat w)-D_\lambda(\hat\alpha))/\lambda}$$
$$x_i^\top w^* \le UB(x_i^\top w^*) := \max_{w\in\Theta_{w^*}} x_i^\top w = x_i^\top\hat w + \|x_i\|_2\sqrt{2(P_\lambda(\hat w)-D_\lambda(\hat\alpha))/\lambda}$$

Θ_{w^*}: region of primal optimal solution

P_λ is λ-strongly convex $\Rightarrow w^*\in\Theta_{w^*} := \{w \mid \|\hat w - w\|_2 \le \sqrt{2(P_\lambda(\hat w)-D_\lambda(\hat\alpha))/\lambda}\}$

Dynamic screening [Bonnefoy+, 14]

We need good accurate solution $\hat w$ and $\hat\alpha$ for good safe screening performances !

While convergence do;

1. Safe screening using $(\hat w_t, \hat\alpha_t)$
2. $(\hat w_{t+1}, \hat\alpha_{t+1}) \leftarrow$ Optimization update$(\hat w_t, \hat\alpha_t)$

Abstract

The problem of learning a sparse model is conceptually interpreted as the process of identifying *active* features/samples and then optimizing the model over them. Recently introduced *safe screening* allows us to identify a part of non-active features/samples. So far, safe screening has been individually studied either for feature/sample screening. In this paper, we introduce a new approach for safely screening features and samples *simultaneously* by alternatively iterating feature and sample screening steps. A advantage of considering them simultaneously rather than individually is that they have a *synergy* effect in the sense that the results of the previous safe feature screening can be exploited for improving the next safe sample screening performances, and vice-versa.

Simultaneous Safe Screening

- Results of safe feature/sample screening can improve a safe sample/feature screening

safe feature screening using the result of sample screening

- We know $\alpha_i^* = \{0,\pm1\}$ for $i\in\mathcal{S}$ by safe sample screening ($\bar{\mathcal{S}} := [n]\setminus\mathcal{S}$)
- We can get the tighter upper bound of $|X_{:j}^\top\alpha^*|$:

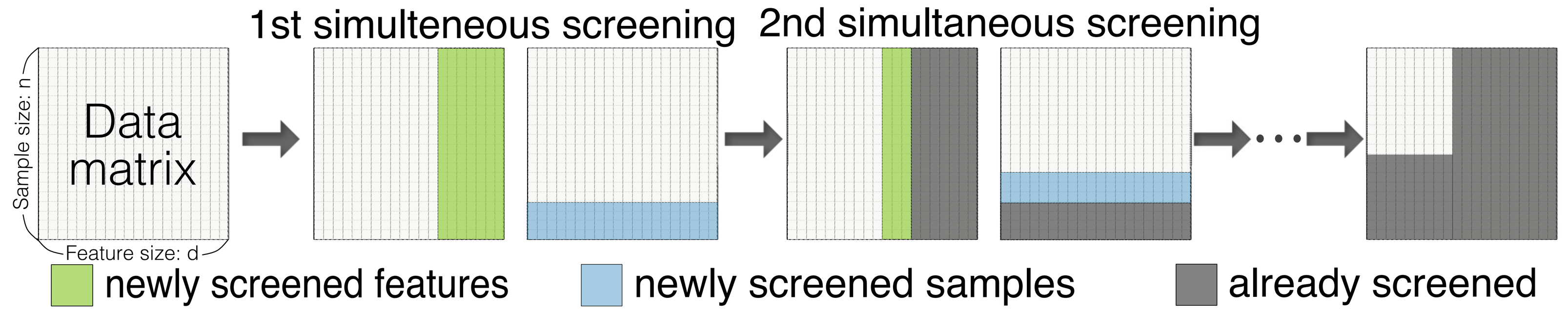
$$\tilde{UB}(|X_{:j}^\top\alpha^*|) := \max_{\alpha}|X_{:j}^\top\alpha| \ \text{ s.t. } \ \alpha\in\Theta_{\alpha^*}, \alpha_i = \alpha_i^*\ \forall i\in\mathcal{S}$$
$$= |X_{\bar{\mathcal{S}},j}^\top\alpha_{\bar{\mathcal{S}}}| - \|X_{\bar{\mathcal{S}},j}\|_2\sqrt{2n(P_\lambda(\hat w)-D_\lambda(\hat\alpha))/\gamma - \|\hat\alpha_{\mathcal{S}}-\alpha_{\mathcal{S}}^*\|_2^2}$$

safe sample screening using the result of feature screening

- We know $w_j^*=0$ for $j\in\mathcal{F}$ by safe feature screening ($\bar{\mathcal{F}} := [d]\setminus\mathcal{F}$)
- We can get the tighter bounds of $x_i^\top w^*$:

$$\tilde{LB}(x_i^\top w^*) = \min_{w} x_i^\top w \ \text{ s.t. } \ w\in\Theta_{w^*}, w_j = w_j^*\ \forall j\in\mathcal{F}$$
$$= x_{i\bar{\mathcal{F}}}^\top\hat w_{\bar{\mathcal{F}}} - \|x_{i\bar{\mathcal{F}}}\|_2\sqrt{2(P_\lambda(\hat w)-D_\lambda(\hat\alpha))/\lambda - \|\hat w_{\mathcal{F}}\|_2^2}$$

- $\tilde{UB}(x_i^\top w^*)$ as well as $\tilde{LB}(x_i^\top w^*)$
- More and more features and samples could be screened out by alternately iterating feature and sample screening



Safe keeping

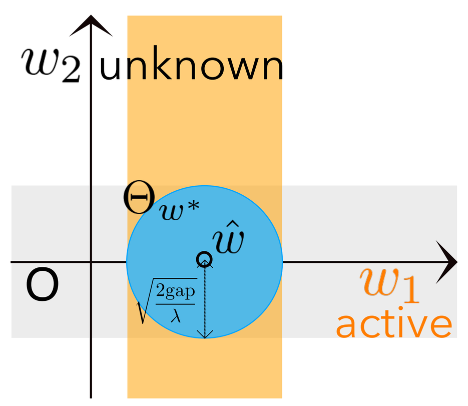
- allows us to identify a part of active features/samples

Safe feature keeping: If P_λ is λ-strongly convex then

$$|\hat w_j| - \sqrt{2(P_\lambda(\hat w)-D_\lambda(\hat\alpha))/\lambda} > 0 \ \Rightarrow\ w_j^*\neq 0$$

Safe sample keeping: If D_λ is γ/n-strongly convex then

$$|\hat\alpha_i| - \sqrt{2n(P_\lambda(\hat w)-D_\lambda(\hat\alpha))/\gamma} > 0 \text{ and}$$
$$|\hat\alpha_i| + \sqrt{2n(P_\lambda(\hat w)-D_\lambda(\hat\alpha))/\gamma} < 1 \Rightarrow \alpha_i^*\notin\{0,\pm1\}$$



Advantages:

- We do not have to waste the screening rule evaluation costs for active features/samples
- By combining safe screening and safe keeping:

#(features/samples aren't determined to be active or non-active)

can be also used as a stopping criteria of dynamic screening and simultaneous screening

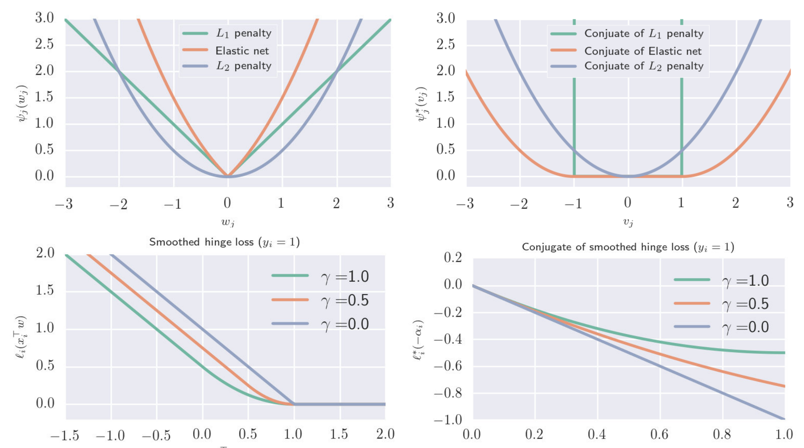
Summary

Primal space: safe sample screening and safe feature keeping

Dual space: safe feature screening and safe sample keeping

Problem Setup

- Data: $\{(x_i,y_i)\}_{i\in[n]}$, Data matrix $(n\times d)$: X
- (Primal) $w^* = \arg\min_{w\in\mathbb{R}^d} P_\lambda(w) := \lambda\psi(w) + \frac{1}{n}\sum_{i\in[n]}\ell_i(x_i^\top w)$
- (Dual) $\alpha^* = \arg\max_{\alpha\in\mathrm{dom}D_\lambda} D_\lambda(\alpha) := -\lambda\psi^*(\frac{1}{\lambda n}X^\top\alpha) - \frac{1}{n}\sum_{i\in[n]}\ell_i^*(-\alpha_i),$



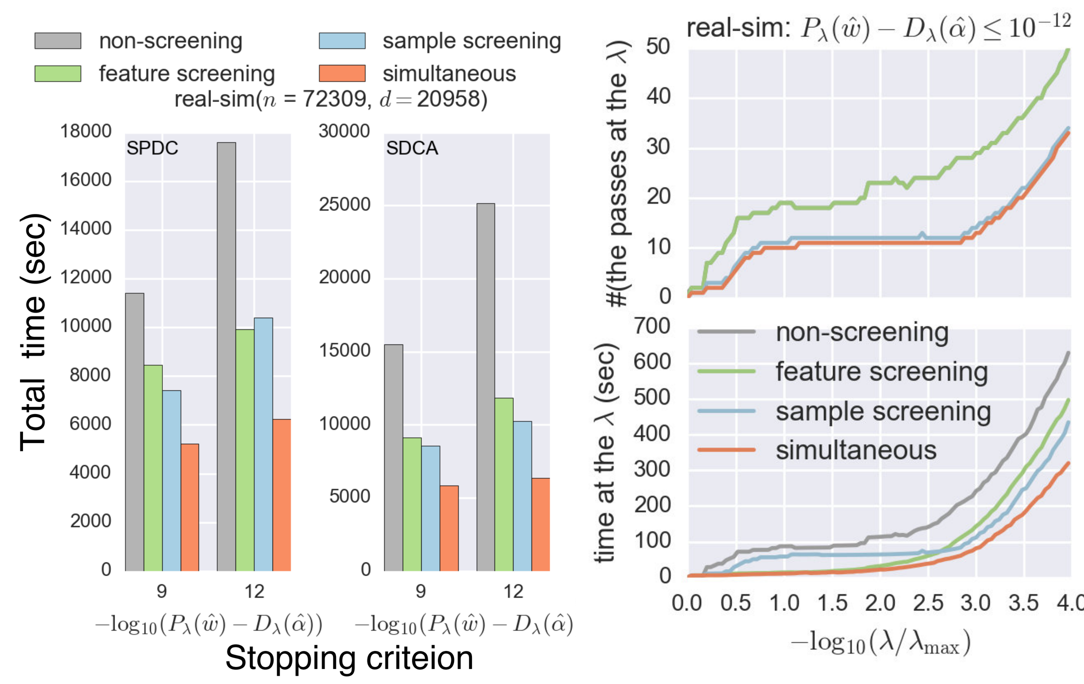
- Elastic net: $\psi(w) := \|w\|_1 + \frac{1}{2}\|w\|_2^2$
- Smoothed hinge loss:

$$\ell_i(a) := \begin{cases}0 & (y_i a > 1),\\ 1 - y_i a - \frac{\gamma}{2} & (y_i a < 1-\gamma),\\ \frac{1}{2\gamma}(1-y_i a)^2 & (\text{otherwise}),\end{cases}$$

Experiments

Elastic net + smoothed hinge (P_λ: λ-strongly convex, D_λ: γ/n-strongly concave)

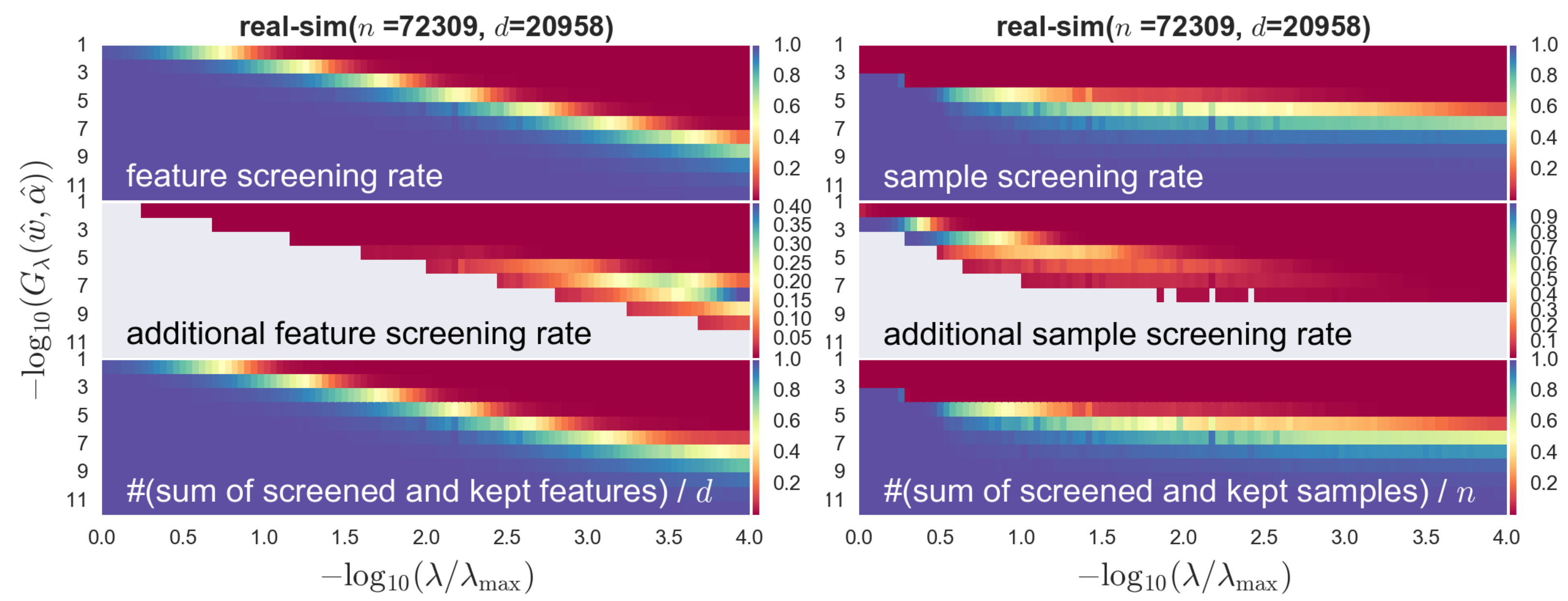
- Computation time savings



- $\gamma = 0.5$
- $\lambda_{\max} := \|\mathrm{diag}(y)X^\top\mathbf{1}\|_\infty$
- train at 100 different λ evenly allocated in $[10^{-4}\lambda_{\max},\ \lambda_{\max}]$ in the logarithmic scale
- we used warm-start
- Solvers:
  - Stochastic Primal-Dual Coordinate
  - Stochastic Dual Coordinate Ascent

- Screening and keeping rates

- screening rate := #(screened features or samples) / #($w_j^*=0$ or $\alpha_i^*=\{0,\pm1\}$)
- additional screening rate: the addition by simulatenous screening



[1] L. El Ghaoui, V. Viallon, and T. Rabbani. Safe feature elimination for the lasso and sparse supervised learning problems. Pacific Journal of Optimization, 2012. **[2]** K. Ogawa, Y. Suzuki, and I. Takeuchi. Safe screening of non-support vectors in pathwise svm computation. ICML, 2013. **[3]** E. Ndiaye, O. Fercoq, A. Gramfort, and J. Salmon. Gap safe screening rules for sparse multi-task and multi-class models. NIPS, 2015.