Proof by Computer

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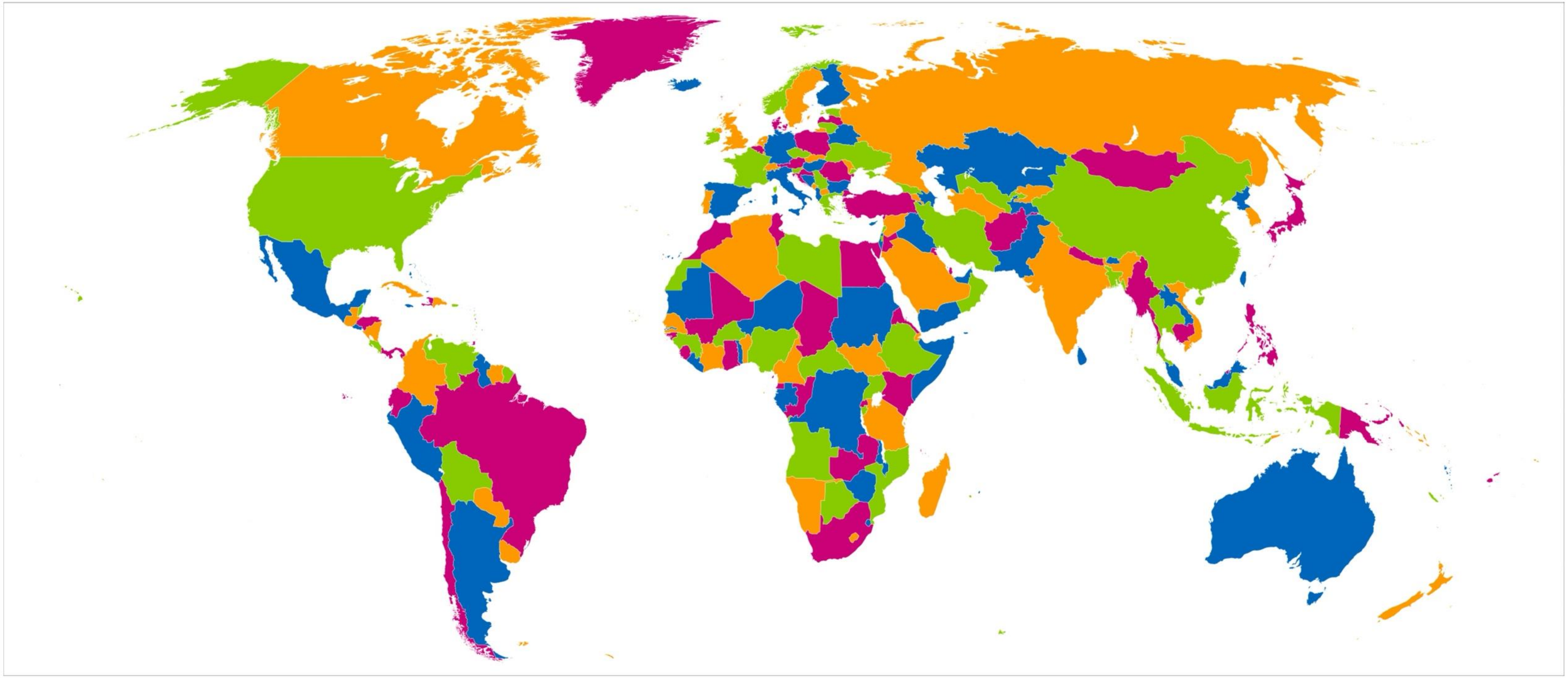
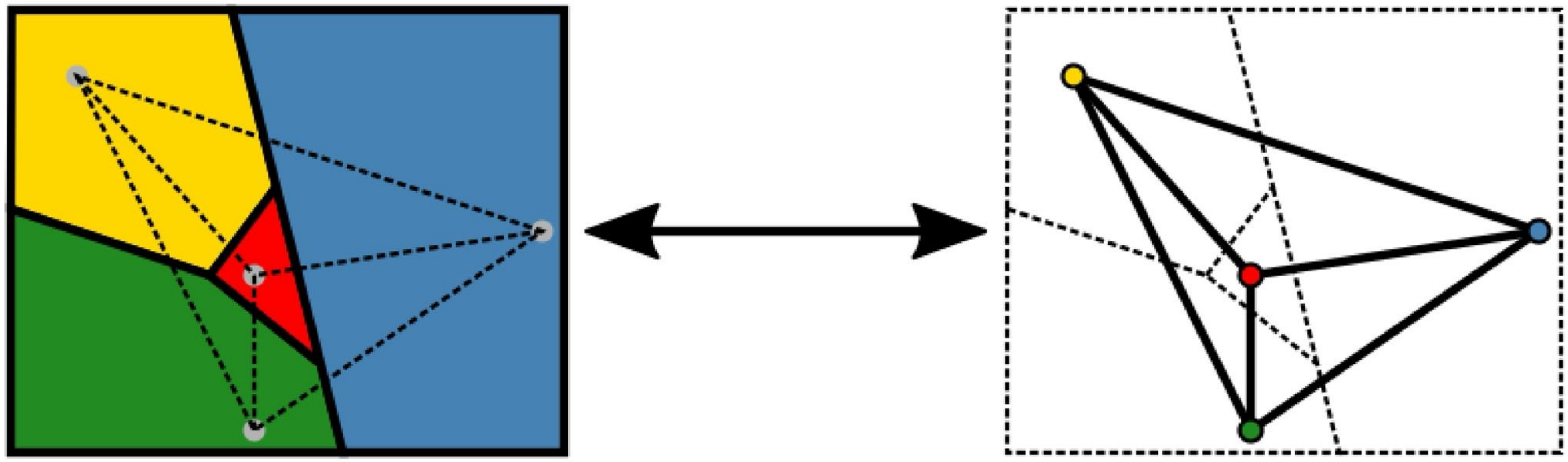
Four color theorem



Can any map be colored with just four colors
so that no two neighboring areas share the same color?

Simplification of the Problem



- represent regions as points
- represent relationships between adjacent regions as lines

Is every planar map four-colorable?

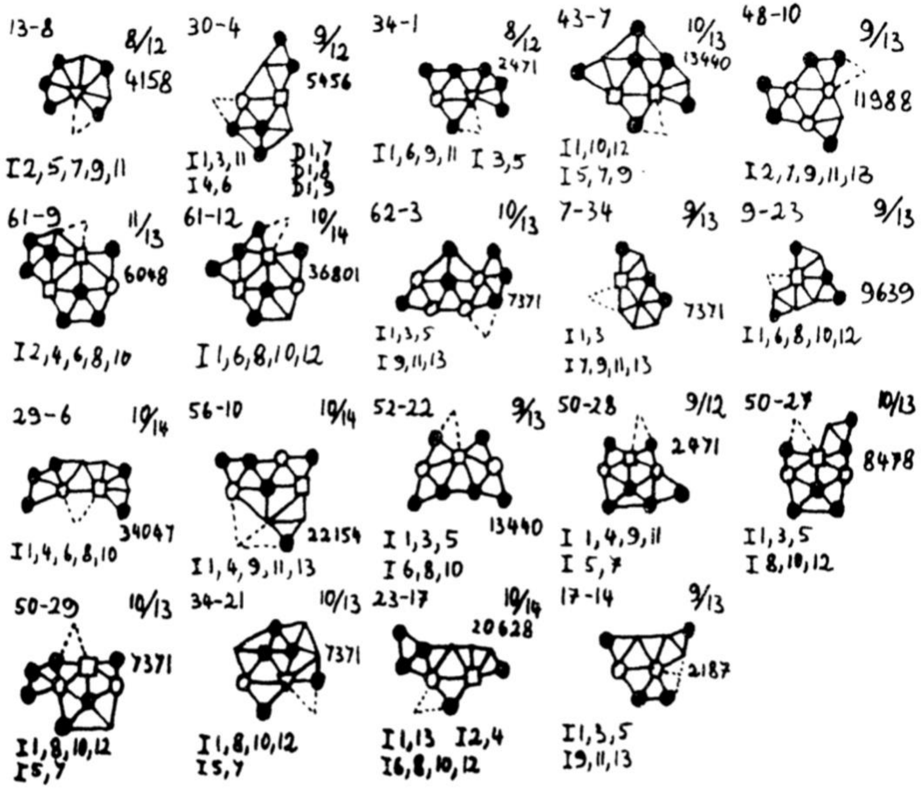
History

Early Proof Attempts

- Francis Guthrie first proposed the conjecture (1852)
- A proposed proof was given by Alfred Kempe (1879)
- Another proof was offered by Peter Guthrie Tait (1880)
- Kempe's proof was shown to be incorrect by Percy Heawood (1890)
  also proved the Five Color Theorem and further generalized the conjecture.
- Tait's proof was shown to be incorrect by Julius Petersen (1891)
- German mathematician Heinrich Heesch proposed and developed methods for solving mathematical proofs using computers (1950s~1960s)

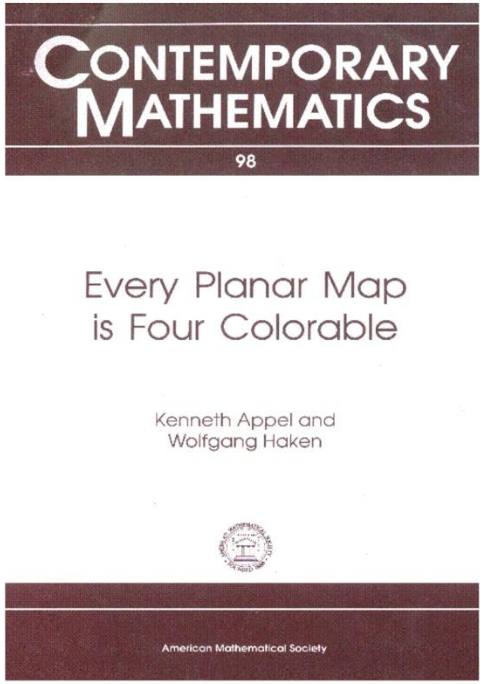
History

Proof by Computer (1976)



- The infinite number of possible maps was reduced to 1,834 reducible configurations.
- Configurations checked one by one by a computer, taking over a thousand hours.

Proof by Computer



Every Planar Map is Four-Colorable (1989) (741 pages)

- It may not be mathematically elegant, but it is still a valid proof.
- Is there a beautiful proof?

Human vs Machine

Human Mathematician

- Limited patience
- Prone to mistakes
- Needs sleep
- Can come up with **beautiful** proofs

Computer

- Infinite patience
- Superhuman level of **mathematical precision**
- Works for (almost) free, doesn't need to rest
- Fast but depends on algorithms

Potential

- In modern mathematics, detailed verification of research results requires the time and effort of outstanding mathematicians
- Controversies exist around proofs like the ABC Conjecture

Moreover

- Computers even can help detect errors in proofs
- Computers might even solve mathematical problems on their own someday

How Can a Computer Solve a Math Problem?

Propositional Logic

A: x is a natural number $(x \in \mathbb{N})$
B: x is greater than 5 $(x > 5)$

$A \wedge B$: x is a natural number which is greater than 5.

- $x = 6, 7, 8, \cdots$

$A \vee B$: x is a natural number or is greater than 5.

- $x = 1, 2, 3, \cdots$ or $5.01, 5.005, 2\pi, \cdots$

Natural Deduction

Inference Rules

Premises, conclusion

- $\dfrac{P \quad P \Rightarrow Q}{Q}$ (Modus Ponens)
- $\dfrac{P \quad Q}{P \land Q}$ ($\land$-Introduction)
- $\dfrac{P \land Q}{Q}$ ($\land$-Elimination Left)
- $\dfrac{P \quad \neg P}{\bot}$ (Law of Excluded Middle)

Proof Tree

- Construct a tree-shaped proof based on inference rules
- Analogous to how human solve math problems

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A}\,{}^{1} \qquad \dfrac{\dfrac{}{(A \to B) \land (B \to C)}\,{}^{2}}{A \to B}}{B} \qquad \dfrac{\dfrac{}{(A \to B) \land (B \to C)}\,{}^{2}}{B \to C}}{C}}{A \to C}\,{}^{1}}{(A \to B) \land (B \to C) \to (A \to C)}\,{}^{2}$$

How to Automate It?

- The series of processes must be formalized so that a computer can execute them.
- Construct a formal language for mathematical proof writing, based on inference rules.
- Automate the process using Large Language Model(LLM), proof search algorithm

Formal Proof Assistants

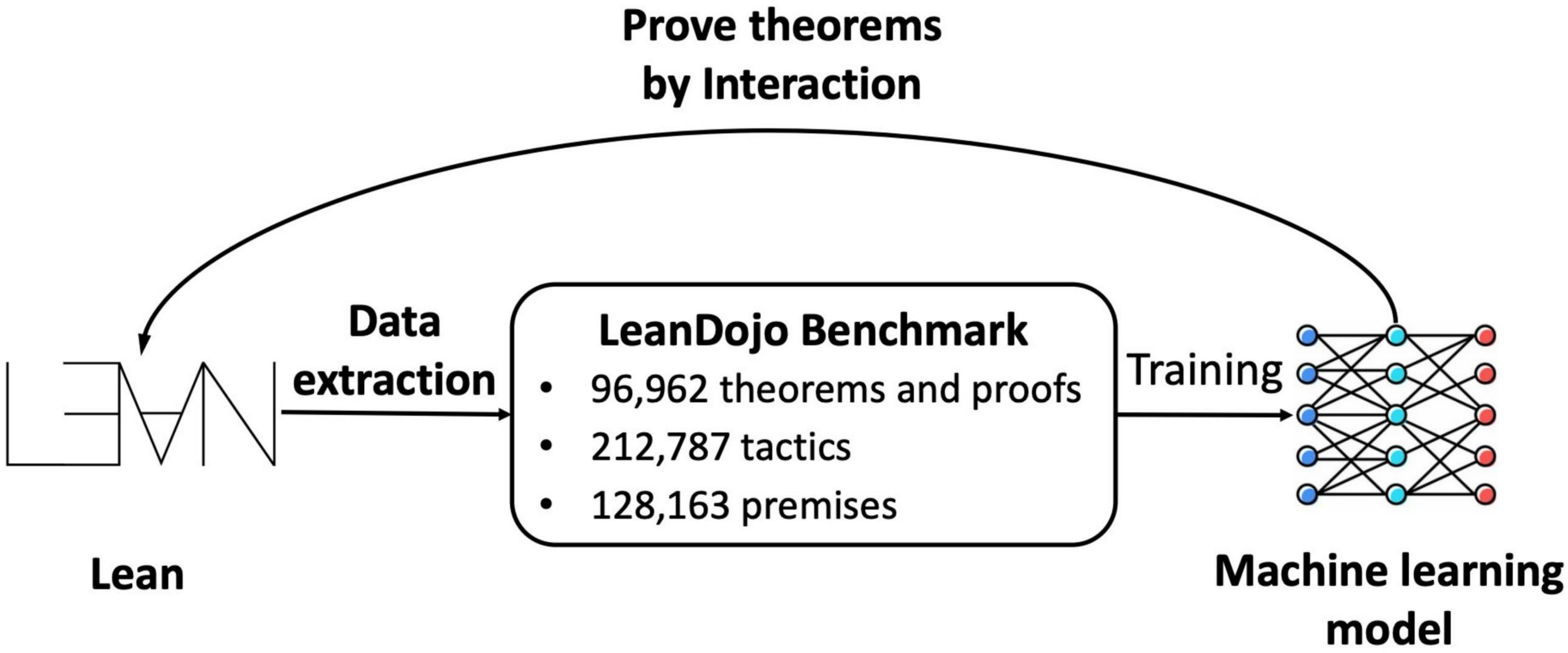
- Write and check mathematical proofs interactively
- Ensure mathematical rigor

Formal Proof Assistants

```
theorem le.antisymm : ∀ {a b : ℤ}, a ≤ b → b ≤ a → a = b :=
take a b : ℤ, assume (H₁ : a ≤ b) (H₂ : b ≤ a),
obtain (n : ℕ) (Hn : a + n = b), from le.elim H₁,
obtain (m : ℕ) (Hm : b + m = a), from le.elim H₂,
have H₃ : a + of_nat (n + m) = a + 0, from
... -- suppressed rest of the proof due to space limitations
have H₆ : n = 0, from nat.eq_zero_of_add_eq_zero_right H₅,
show a = b, from
  calc
    a = a + 0      : add_zero
      ... = a + n  : H₆
      ... = b      : Hn
```

The Lean Theorem Prover (System Description) (CADE 2015)

Large Language Models



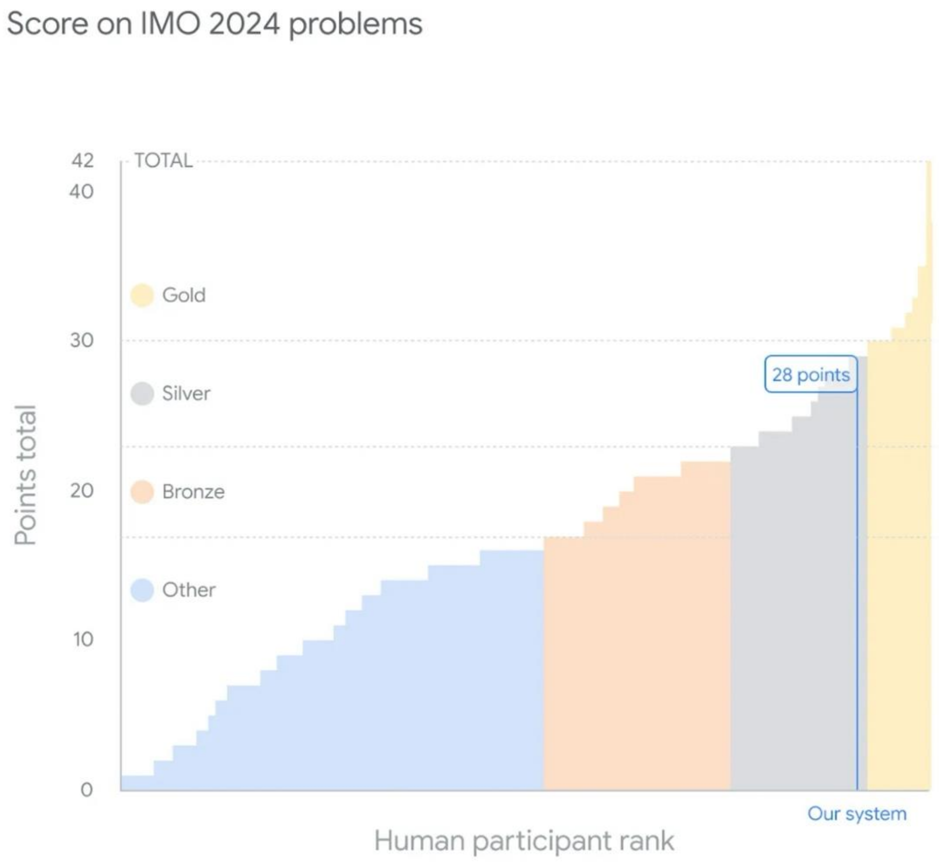
Leandojo: Theorem proving with retrieval-augmented language models, 2023.

IMO Grand Challenge

The challenge: build an AI that can win a gold medal in the IMO (2019)

Current Status

AI achieved a silver-medal standard in solving International Mathematical Olympiad problems (AlphaProof and AlphaGeometry teams, July 25, 2024).



The Future

- Mathematics
- Mathematical logic
- Programming Languages(PL) semantics and syntax
- Machine learning techniques like large language models (Automating proof generation or formalizing natural language proofs)

Moreover...

- Deep understanding of the human brain

Thank you