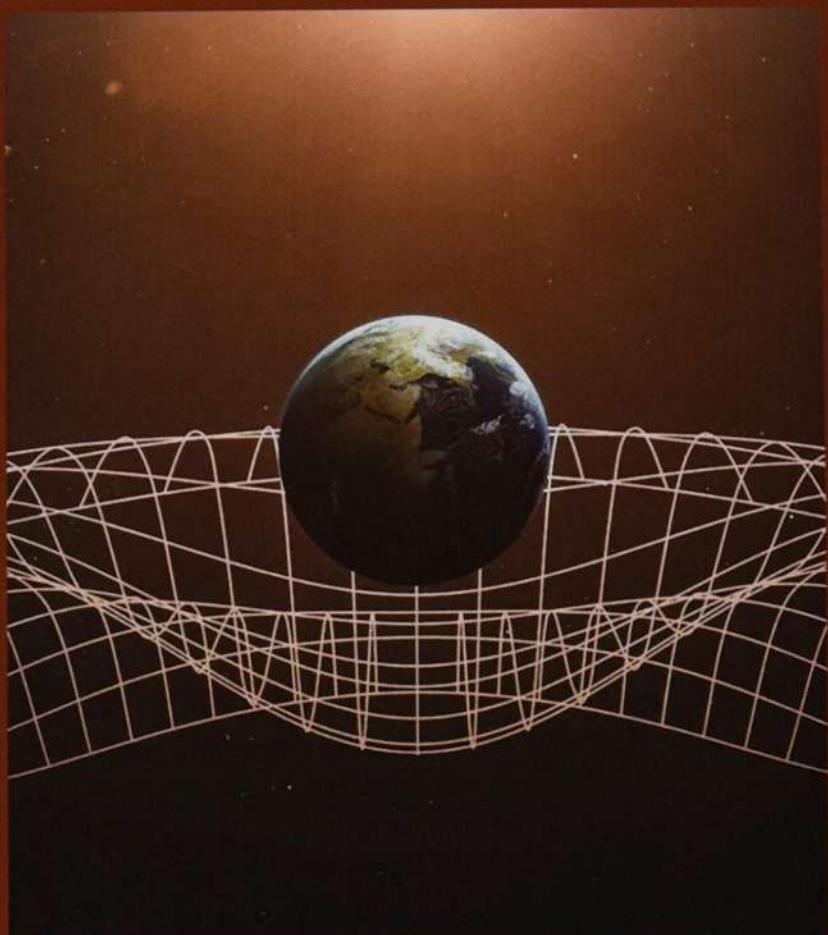


An Introduction to General Relativity  
**S P A C E T I M E**  
and  
**G E O M E T R Y**



S e a n   M .   C a r r o l l

## **Spacetime and Geometry**

### An Introduction to General Relativity

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*Spacetime and Geometry* is an introductory textbook on general relativity, specifically aimed at students. Using a lucid style, Carroll first covers the foundations of the theory and mathematical formalism, providing an approachable introduction to what can often be an intimidating subject. Three major applications of general relativity are then discussed: black holes, perturbation theory and gravitational waves, and cosmology. Students will learn the origin of how spacetime curves (the Einstein equation) and how matter moves through it (the geodesic equation). They will learn what black holes really are, how gravitational waves are generated and detected, and the modern view of the expansion of the universe. A brief introduction to quantum field theory in curved spacetime is also included. A student familiar with this book will be ready to tackle research-level problems in gravitational physics.

**Sean M. Carroll** is Research Professor of Physics at the California Institute of Technology. His research focuses on general relativity, cosmology, field theory, statistical mechanics, and quantum mechanics. He is the recipient of numerous awards, including the Gemant Award from the American Institute of Physics, the Winton Science Book Prize from the Royal Society, a Guggenheim fellowship, and teaching awards from MIT and the University of Chicago.

# **Spacetime and Geometry**

## An Introduction to General Relativity

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SEAN M. CARROLL



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"For if each Star is little more a mathematical Point, located upon the Hemisphere of Heaven by Right Ascension and Declination, then all the Stars, taken together, tho' innumerable, must like any other set of points, in turn represent some single gigantick Equation, to the mind of God as straightforward as, say, the Equation of a Sphere,—to us unreadable, incalculable. A lonely, uncompensated, perhaps even impossible Task,—yet some of us must ever be seeking, I suppose."

—Thomas Pynchon, *Mason & Dixon*

# Preface

General relativity is the most beautiful physical theory ever invented. It describes one of the most pervasive features of the world we experience—gravitation—in terms of an elegant mathematical structure—the differential geometry of curved spacetime—leading to unambiguous predictions that have received spectacular experimental confirmation. Consequences of general relativity, from the big bang to black holes, often get young people first interested in physics, and it is an unalloyed joy to finally reach the point in one's studies where these phenomena may be understood at a rigorous quantitative level. If you are contemplating reading this book, that point is here.

In recent decades, general relativity (GR) has become an integral and indispensable part of modern physics. For a long time after it was proposed by Einstein in 1916, GR was counted as a shining achievement that lay somewhat outside the mainstream of interesting research. Increasingly, however, contemporary students in a variety of specialties are finding it necessary to study Einstein's theory. In addition to being an active research area in its own right, GR is part of the standard syllabus for anyone interested in astrophysics, cosmology, string theory, and even particle physics. This is not to slight the more pragmatic uses of GR, including the workings of the Global Positioning System (GPS) satellite network.

There is no shortage of books on GR, and many of them are excellent. Indeed, approximately thirty years ago witnessed the appearance of no fewer than three books in the subject, each of which has become a classic in its own right: those by Weinberg (1972), Misner, Thorne, and Wheeler (1973), and Hawking and Ellis (1975). Each of these books is suffused with a strongly-held point of view advocated by the authors. This has led to a love-hate relationship between these works and their readers; in each case, it takes little effort to find students who will declare them to be the best textbook ever written, or other students who find them completely unpalatable. For the individuals in question, these judgments may very well be correct; there are many different ways to approach this subject.

The present book has a single purpose: to provide a clear introduction to general relativity, suitable for graduate students or advanced undergraduates. I have attempted to include enough material so that almost any one-semester introductory course on GR can find the appropriate subjects covered in the text, but not too much more than that. In particular, I have tried to resist the temptation to write a comprehensive reference book. The only goal of this book is to teach you GR.

An intentional effort has been made to prefer the conventional over the idiosyncratic. If I can be accused of any particular ideological bias, it would be a

tendency to think of general relativity as a field theory, a point of view that helps one to appreciate the connections among GR, particle physics, and string theory. At the same time, there are a number of exciting astrophysical applications of GR (black holes, gravitational lensing, the production and detection of gravitational waves, the early universe, the late universe, the cosmological constant), and I have endeavored to include at least enough background discussion of these issues to prepare students to tackle the current literature.

The primary question facing any introductory treatment of general relativity is the level of mathematical rigor at which to operate. There is no uniquely proper solution, as different students will respond with different levels of understanding and enthusiasm to different approaches. Recognizing this, I have tried to provide something for everyone. I have not shied away from detailed formalism, but have also attempted to include concrete examples and informal discussion of the concepts under consideration. Much of the most mathematical material has been relegated to the Appendices. Some of the material in the Appendices is actually an integral part of the course (for example, the discussion of conformal diagrams), but an individual reader or instructor can decide just when it is appropriate to delve into them; signposts are included in the body of the text.

Surprisingly, there are very few formal prerequisites for learning general relativity; most of the material is developed as we go along. Certainly no prior exposure to Riemannian geometry is assumed, nor would it necessarily be helpful. It would be nice to have already studied some special relativity; although a discussion is included in Chapter 1, its purpose is more to review the basics and introduce some notation, rather than to provide a self-contained introduction. Beyond that, some exposure to electromagnetism, Lagrangian mechanics, and linear algebra might be useful, but the essentials are included here.

The structure of the book should be clear. The first chapter is a review of special relativity and basic tensor algebra, including a brief discussion of classical field theory. The next two chapters introduce manifolds and curvature in some detail; some motivational physics is included, but building a mathematical framework is the primary goal. General relativity proper is introduced in Chapter 4, along with some discussion of alternative theories. The next four chapters discuss the three major applications of GR: black holes (two chapters), perturbation theory and gravitational waves, and cosmology. Each of these subjects has witnessed an explosion of research in recent years, so the discussions here will be necessarily introductory, but I have tried to emphasize issues of relevance to current work. These three applications can be covered in any order, although there are interdependencies highlighted in the text. Discussions of experimental tests are sprinkled through these chapters. Chapter 9 is a brief introduction to quantum field theory in curved spacetime; this is not a necessary part of a first look at GR, but has become increasingly important to work in quantum gravity and cosmology, and therefore deserves some mention. On the other hand, a few topics are scandalously neglected; the initial-value problem and cosmological perturbation theory come to mind, but there are others. Fortunately there is no shortage of other resources. The Appendices serve various purposes: There are discussions of

technical points that were avoided in the body of the book, crucial concepts that could have been put in various places, and extra topics that are useful but outside the main development.

Since the goal of the book is pedagogy rather than originality, I have often leaned heavily on other books (listed in the bibliography) when their expositions seemed perfectly sensible to me. When this leaning was especially heavy, I have indicated it in the text itself. It will be clear that a primary resource was the book by Wald (1984), which has become a standard reference in the field; readers of this book will hopefully be well-prepared to jump into the more advanced sections of Wald's book.

This book grew out of a set of lecture notes that were prepared when I taught a course on GR at MIT. These notes are available on the web for free, and will continue to be so; they will be linked to the website listed below. Perhaps a little over half of the material here is contained in the notes, although the advantages of owning the book (several copies, even) should go without saying.

Countless people have contributed greatly both to my own understanding of general relativity and to this book in particular—too many to acknowledge with any hope of completeness. Some people, however, deserve special mention. Ted Pyne learned the subject along with me, taught me a great deal, and collaborated with me the first time we taught a GR course, as a seminar in the astronomy department at Harvard; parts of this book are based on our mutual notes. Nick Warner taught the course at MIT from which I first learned GR, and his lectures were certainly a very heavy influence on what appears here. Neil Cornish was kind enough to provide a wealth of exercises, many of which have been included at the end of each chapter. And among the many people who have read parts of the manuscript and offered suggestions, Sanaz Arkani-Hamed was kind enough to go through the entire thing in great detail.

I would also like to thank everyone who either commented in person or by email on different parts of the book; these include Tigran Avazian, Teodora Be-loreshka, Ed Bertschinger, Patrick Brady, Peter Brown, Jennifer Chen, Michele Ferraz Figueiró, Eanna Flanagan, Jacques Fric, Ygor Geurts, Marco Godina, Monica Guica, Jim Hartle, Tamás Hauer, Daniel Holz, Ted Jacobson, Akash Kansagra, Chuck Keeton, Arthur Kosowsky, Eugene Lim, Jorma Louko, Robert A. McNees, Hayri Mutluay, Simon Ross, Itai Seggev, Robert Wald, and Barton Zwiebach. Apologies are due to anyone I may have neglected to mention. And along the way I was fortunate to be the recipient of wisdom and perspective from numerous people, including Shadi Bartsch, George Field, Deryn Fogg, Ilana Harrus, Gretchen Helfrich, Mari Ruti, Maria Spiropulu, Mark Trodden, and of course my family. (This wisdom often came in the form, “What were you thinking?”) Finally, I would like to thank the students in my GR classes, on whom the strategies deployed here were first tested, and express my gratitude to my students and collaborators, for excusing my book-related absences when I should have been doing research.

My friends who have written textbooks themselves tell me that the first printing of a book will sometimes contain mistakes. In the unlikely event that this happens

here, there will be a list of errata kept at the website for the book:

<http://spacetimeandgeometry.net/>

The website will also contain other relevant links of interest to readers.

During the time I was working on this book, I was supported by the National Science Foundation, the Department of Energy, the Alfred P. Sloan Foundation, and the David and Lucile Packard Foundation.

Sean Carroll  
Chicago, Illinois  
June 2003

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# Special Relativity and Flat Spacetime

## 1.1 ■ PRELUDE

General relativity (GR) is Einstein's theory of space, time, and gravitation. At heart it is a very simple subject (compared, for example, to anything involving quantum mechanics). The essential idea is perfectly straightforward: while most forces of nature are represented by fields defined on spacetime (such as the electromagnetic field, or the short-range fields characteristic of subnuclear forces), gravity is inherent in spacetime itself. In particular, what we experience as "gravity" is a manifestation of the *curvature* of spacetime.

Our task, then, is clear. We need to understand spacetime, we need to understand curvature, and we need to understand how curvature becomes gravity. Roughly, the first two chapters of this book are devoted to an exploration of spacetime, the third is about curvature, and the fourth explains the relationship between curvature and gravity, before we get into applications of the theory. However, let's indulge ourselves with a short preview of what is to come, which will perhaps motivate the initial steps of our journey.

GR is a theory of gravity, so we can begin by remembering our previous theory of gravity, that of Newton. There are two basic elements: an equation for the gravitational field as influenced by matter, and an equation for the response of matter to this field. The conventional Newtonian statement of these rules is in terms of forces between particles; the force between two objects of masses  $M$  and  $m$  separated by a vector  $\mathbf{r} = r\mathbf{e}_{(r)}$  is the famous inverse-square law,

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_{(r)}, \quad (1.1)$$

and this force acts on a particle of mass  $m$  to give it an acceleration according to Newton's second law,

$$\mathbf{F} = m\mathbf{a}. \quad (1.2)$$

Equivalently, we could use the language of the gravitational potential  $\Phi$ ; the potential is related to the mass density  $\rho$  by Poisson's equation,

$$\nabla^2\Phi = 4\pi G\rho, \quad (1.3)$$

and the acceleration is given by the gradient of the potential,

$$\mathbf{a} = -\nabla\Phi. \quad (1.4)$$

Either (1.1) and (1.2), or (1.3) and (1.4), serve to define Newtonian gravity. To define GR, we need to replace each of them by statements about the curvature of spacetime.

The hard part is the equation governing the response of spacetime curvature to the presence of matter and energy. We will eventually find what we want in the form of Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (1.5)$$

This looks more forbidding than it should, largely because of those Greek subscripts. In fact this is simply an equation between  $4 \times 4$  matrices, and the subscripts label elements of each matrix. The expression on the left-hand side is a measure of the curvature of spacetime, while the right-hand side measures the energy and momentum of matter, so this equation relates energy to curvature, as promised. But we will defer until later a detailed understanding of the inner workings of Einstein's equation.

The response of matter to spacetime curvature is somewhat easier to grasp: Free particles move along paths of “shortest possible distance,” or geodesics. In other words, particles try their best to move on straight lines, but in a curved spacetime there might not be any straight lines (in the sense we are familiar with from Euclidean geometry), so they do the next best thing. Their parameterized paths  $x^\mu(\lambda)$  obey the geodesic equation:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0. \quad (1.6)$$

At this point you aren't expected to understand (1.6) any more than (1.5); but soon enough it will all make sense.

As we will discuss later, the universal nature of geodesic motion is an extremely profound feature of GR. This universality is the origin of our claim that gravity is not actually a “force,” but a feature of spacetime. A charged particle in an electric field feels an acceleration, which deflects it from straight-line motion; in contrast, a particle in a gravitational field moves along a path that is the closest thing there is to a straight line. Such particles do not feel acceleration; they are freely falling. Once we become more familiar with the spirit of GR, it will make perfect sense to think of a ball flying through the air as being more truly “unaccelerated” than one sitting on a table; the one sitting on a table is being deflected away from the geodesic it would like to be on (which is why we feel a force on our feet as we stand on Earth).

The basic concept underlying our description of spacetime curvature will be that of the metric tensor, typically denoted by  $g_{\mu\nu}$ . The metric encodes the geometry of a space by expressing deviations from Pythagoras's theorem,  $(\Delta l)^2 = (\Delta x)^2 + (\Delta y)^2$  (where  $\Delta l$  is the distance between two points defined on a Cartesian grid with coordinate separations  $\Delta x$  and  $\Delta y$ ). This familiar formula is valid only in conventional Euclidean geometry, where it is implicitly assumed that space is flat. In the presence of curvature our deeply ingrained notions of ge-

ometry will begin to fail, and we can characterize the amount of curvature by keeping track of how Pythagoras's relation is altered. This information is contained in the metric tensor. From the metric we will derive the Riemann curvature tensor, used to define Einstein's equation, and also the geodesic equation. Setting up this mathematical apparatus is the subject of the next several chapters.

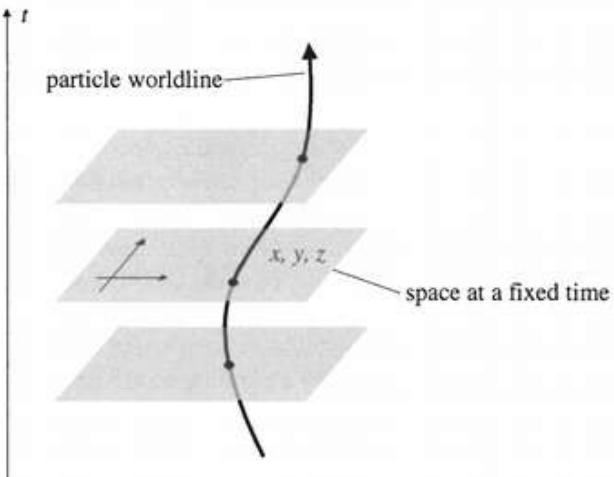
Despite the need to introduce a certain amount of formalism to discuss curvature in a quantitative way, the essential notion of GR ("gravity is the curvature of spacetime") is quite simple. So why does GR have, at least in some benighted circles, a reputation for difficulty or even abstruseness? Because the elegant truths of Einstein's theory are obscured by the accumulation of certain pre-relativity notions which, although very useful, must first be discarded in order to appreciate the world according to GR. Specifically, we live in a world in which spacetime curvature is very small, and particles are for the most part moving quite slowly compared to the speed of light. Consequently, the mechanics of Galileo and Newton comes very naturally to us, even though it is only an approximation to the deeper story.

So we will set about learning the deeper story by gradually stripping away the layers of useful but misleading Newtonian intuition. The first step, which is the subject of this chapter, will be to explore special relativity (SR), the theory of spacetime in the absence of gravity (curvature). Hopefully this is mostly review, as it will proceed somewhat rapidly. The point will be both to recall what SR is all about, and to introduce tensors and related concepts that will be crucial later on, without the extra complications of curvature on top of everything else. Therefore, for this chapter we will always be working in flat spacetime, and furthermore we will only use inertial (Cartesian-like) coordinates. Needless to say it is possible to do SR in any coordinate system you like, but it turns out that introducing the necessary tools for doing so would take us halfway to curved spaces anyway, so we will put that off for a while.

## 1.2 ■ SPACE AND TIME, SEPARATELY AND TOGETHER

A purely cold-blooded approach to GR would reverse the order of Chapter 2 (Manifolds) and Chapter 1 (Special Relativity and Flat Spacetime). A *manifold* is the kind of mathematical structure used to describe spacetime, while *special relativity* is a model that invokes a particular kind of spacetime (one with no curvature, and hence no gravity). However, if you are reading this book you presumably have at least some familiarity with special relativity (SR), while you may not know anything about manifolds. So our first step will be to explore the relatively familiar territory of SR, taking advantage of this opportunity to introduce concepts and notation that will be crucial to later developments.

Special relativity is a theory of the structure of spacetime, the background on which particles and fields evolve. SR serves as a replacement for Newtonian mechanics, which also is a theory of the structure of spacetime. In either case, we can distinguish between this basic structure and the various dynamical laws govern-



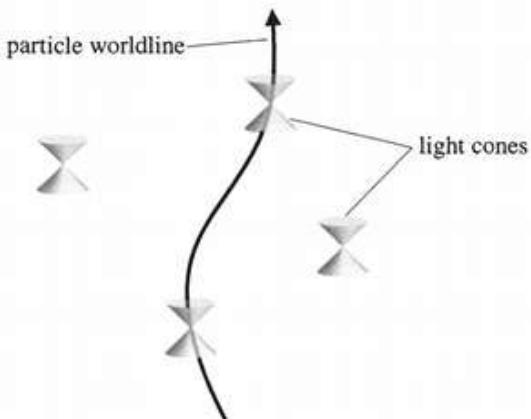
**FIGURE 1.1** In Newtonian spacetime there is an absolute slicing into distinct copies of space at different moments in time. Particle worldlines are constrained to move forward in time, but can travel through space at any velocity; there is universal agreement on the question of whether two events at different points in space occur at the same moment of time.

ing specific systems: Newtonian gravity is an example of a dynamical system set within the context of Newtonian mechanics, while Maxwell's electromagnetism is a dynamical system operating within the context of special relativity.

**Spacetime** is a four-dimensional set, with elements labeled by three dimensions of space and one of time. (We'll do a more rigorous job with the definitions in the next chapter.) An individual point in spacetime is called an **event**. The path of a particle is a curve through spacetime, a parameterized one-dimensional set of events, called the **worldline**. Such a description applies equally to SR and Newtonian mechanics. In either case, it seems clear that "time" is treated somewhat differently than "space"; in particular, particles always travel forward in time, whereas they are free to move back and forth in space.

There is an important difference, however, between the set of allowed paths that particles can take in SR and those in Newton's theory. In Newtonian mechanics, there is a basic division of spacetime into well-defined slices of "all of space at a fixed moment in time." The notion of *simultaneity*, when two events occur at the same time, is unambiguously defined. Trajectories of particles will move ever forward in time, but are otherwise unconstrained; in particular, there is no limit on the relative velocity of two such particles.

In SR the situation is dramatically altered: in particular, *there is no well-defined notion of two separated events occurring "at the same time."* That is not to say that spacetime is completely structureless. Rather, at any event we can define a **light cone**, which is the locus of paths through spacetime that could conceivably be taken by light rays passing through this event. The absolute division, in Newtonian



**FIGURE 1.2** In special relativity there is no absolute notion of “all of space at one moment in time.” Instead, there is a rule that particles always travel at less than or equal to the speed of light. We can therefore define light cones at every event, which locally describe the set of allowed trajectories. For two events that are outside each others’ light cones, there is no universal notion of which event occurred earlier in time.

mechanics, of spacetime into unique slices of space parameterized by time, is replaced by a rule that says that physical particles cannot travel faster than light, and consequently move along paths that always remain inside these light cones.

The absence of a preferred time-slicing in SR is at the heart of why the notion of spacetime is more fundamental in this context than in Newtonian mechanics. Of course we can choose specific coordinate systems in spacetime, and once we do, it makes sense to speak of separated events occurring at the same value of the time coordinate in this particular system; but there will also be other possible coordinates, related to the first by “rotating” space and time into each other. This phenomenon is a natural generalization of rotations in Euclidean geometry, to which we now turn.

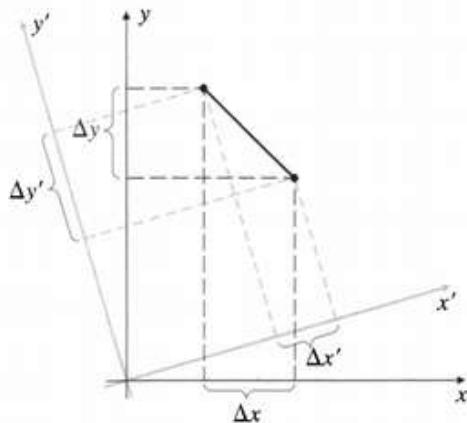
Consider a garden-variety two-dimensional plane. It is typically convenient to label the points on such a plane by introducing coordinates, for example by defining orthogonal  $x$  and  $y$  axes and projecting each point onto these axes in the usual way. However, it is clear that most of the interesting geometrical facts about the plane are independent of our choice of coordinates; there aren’t any preferred directions. As a simple example, we can consider the distance between two points, given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2. \quad (1.7)$$

In a different Cartesian coordinate system, defined by  $x'$  and  $y'$  axes that are rotated with respect to the originals, the formula for the distance is unaltered:

$$(\Delta s)^2 = (\Delta x')^2 + (\Delta y')^2. \quad (1.8)$$

We therefore say that the distance is invariant under such changes of coordinates.



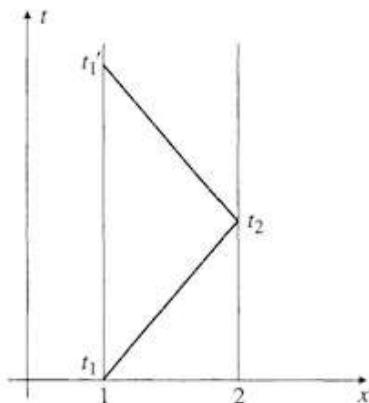
**FIGURE 1.3** Two-dimensional Euclidean space, with two different coordinate systems. Notions such as “the distance between two points” are independent of the coordinate system chosen.

This is why it is useful to think of the plane as an intrinsically two-dimensional space, rather than as two fundamentally distinct one-dimensional spaces brought arbitrarily together: Although we use two distinct numbers to label each point, the numbers are not the essence of the geometry, since we can rotate axes into each other while leaving distances unchanged. In Newtonian physics this is not the case with space and time; there is no useful notion of rotating space and time into each other. Rather, the notion of “all of space at a single moment in time” has a meaning independent of coordinates.

SR is a different story. Let us consider coordinates  $(t, x, y, z)$  on spacetime, set up in the following way. The spatial coordinates  $(x, y, z)$  comprise a standard Cartesian system, constructed for example by welding together rigid rods that meet at right angles. The rods must be moving freely, unaccelerated. The time coordinate is defined by a set of clocks, which are not moving with respect to the spatial coordinates. (Since this is a thought experiment, we can imagine that the rods are infinitely long and there is one clock at every point in space.) The clocks are synchronized in the following sense. Imagine that we send a beam of light from point 1 in space to point 2, in a straight line at a constant velocity  $c$ , and then immediately back to 1 (at velocity  $-c$ ). Then the time on the coordinate clock when the light beam reaches point 2, which we label  $t_2$ , should be halfway between the time on the coordinate clock when the beam left point 1 ( $t_1$ ) and the time on that same clock when it returned ( $t'_1$ ):

$$t_2 = \frac{1}{2}(t'_1 + t_1). \quad (1.9)$$

The coordinate system thus constructed is an **inertial frame**, or simply “inertial coordinates.” These coordinates are the natural generalization to spacetime of Cartesian (orthonormal) coordinates in space. (The reason behind the careful



**FIGURE 1.4** Synchronizing clocks in an inertial coordinate system. The clocks are synchronized if the time  $t_2$  is halfway between  $t_1$  and  $t'_1$  when we bounce a beam of light from point 1 to point 2 and back.

construction is so that we only make comparisons *locally*; never, for example, comparing two far-away clocks to each other at the same time. This kind of care will be even more necessary once we go to general relativity, where there will not be any way to construct inertial coordinates throughout spacetime.)

We can construct any number of inertial frames via this procedure, differing from the first one by an offset in initial position and time, angle, and (constant) velocity. In a Newtonian world, the new coordinates  $(t', x', y', z')$  would have the feature that  $t' = t + \text{constant}$ , independent of spatial coordinates. That is, there is an absolute notion of “two events occurring simultaneously, that is, at the same time.” But in SR this isn’t true; in general the three-dimensional “spaces” defined by  $t = \text{constant}$  will differ from those defined by  $t' = \text{constant}$ .

However, we have not descended completely into chaos. Consider, without any motivation for the moment, what we will call the **spacetime interval** between two events:

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2. \quad (1.10)$$

(Notice that it can be positive, negative, or zero even for two nonidentical points.) Here,  $c$  is some fixed conversion factor between space and time, that is, a fixed velocity. As an empirical matter, it turns out that electromagnetic waves propagate in vacuum at this velocity  $c$ , which we therefore refer to as “the speed of light.” The important thing, however, is not that photons happen to travel at that speed, but that there exists a  $c$  such that *the spacetime interval is invariant under changes of inertial coordinates*. In other words, if we set up a new inertial frame  $(t', x', y', z')$ , the interval will be of the same form:

$$(\Delta s)^2 = -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2. \quad (1.11)$$

This is why it makes sense to think of SR as a theory of four-dimensional spacetime, known as **Minkowski space**. (This is a special case of a four-dimensional manifold, which we will deal with in detail later.) As we shall see, the coordinate transformations that we have implicitly defined do, in a sense, rotate space and time into each other. There is no absolute notion of “simultaneous events”; whether two things occur at the same time depends on the coordinates used. Therefore, the division of Minkowski space into space and time is a choice we make for our own purposes, not something intrinsic to the situation.

Almost all of the “paradoxes” associated with SR result from a stubborn persistence of the Newtonian notions of a unique time coordinate and the existence of “space at a single moment in time.” By thinking in terms of spacetime rather than space and time together, these paradoxes tend to disappear.

Let’s introduce some convenient notation. Coordinates on spacetime will be denoted by letters with Greek superscript indices running from 0 to 3, with 0 generally denoting the time coordinate. Thus,

$$\begin{aligned} x^\mu : \quad & x^0 = ct \\ & x^1 = x \\ & x^2 = y \\ & x^3 = z. \end{aligned} \tag{1.12}$$

(Don’t start thinking of the superscripts as exponents.) Furthermore, for the sake of simplicity we will choose units in which

$$c = 1; \tag{1.13}$$

we will therefore leave out factors of  $c$  in all subsequent formulae. Empirically we know that  $c$  is  $3 \times 10^8$  meters per second; thus, we are working in units where 1 second equals  $3 \times 10^8$  meters. Sometimes it will be useful to refer to the space and time components of  $x^\mu$  separately, so we will use Latin superscripts to stand for the space components alone:

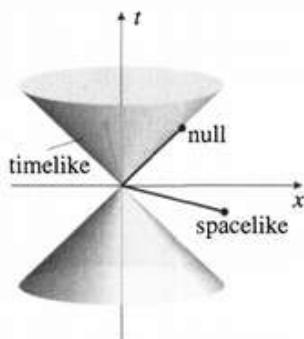
$$\begin{aligned} x^i : \quad & x^1 = x \\ & x^2 = y \\ & x^3 = z. \end{aligned} \tag{1.14}$$

It is also convenient to write the spacetime interval in a more compact form. We therefore introduce a  $4 \times 4$  matrix, the **metric**, which we write using two lower indices:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.15}$$

(Some references, especially field theory books, define the metric with the opposite sign, so be careful.) We then have the nice formula

$$(\Delta s)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu. \tag{1.16}$$



**FIGURE 1.5** A light cone, portrayed on a spacetime diagram. Points that are spacelike-, null-, and timelike-separated from the origin are indicated.

This formula introduces the **summation convention**, in which indices appearing both as superscripts and subscripts are summed over. We call such labels **dummy indices**; it is important to remember that they are summed over all possible values, rather than taking any specific one. (It will always turn out to be the case that dummy indices occur strictly in pairs, with one “upstairs” and one “downstairs.”) More on this later.) The content of (1.16) is therefore exactly the same as (1.10).

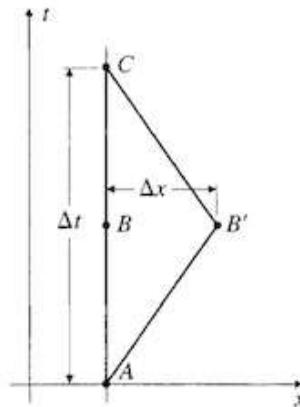
An extremely useful tool is the **spacetime diagram**, so let’s consider Minkowski space from this point of view. We can begin by portraying the initial  $t$  and  $x$  axes at right angles, and suppressing the  $y$  and  $z$  axes. (“Right angles” as drawn on a spacetime diagram don’t necessarily imply “orthogonal in spacetime,” although that turns out to be true for the  $t$  and  $x$  axes in this case.) It is enlightening to consider the paths corresponding to travel at the speed  $c = 1$ , given by  $x = \pm t$ . A set of points that are all connected to a single event by straight lines moving at the speed of light is the **light cone**, since if we imagine including one more spatial coordinate, the two diagonal lines get completed into a cone. Light cones are naturally divided into future and past; the set of all points inside the future and past light cones of a point  $p$  are called **timelike separated** from  $p$ , while those outside the light cones are **spacelike separated** and those on the cones are **null separated**. Referring back to (1.10), we see that the interval between timelike separated points is negative, between spacelike separated points is positive, and between null separated points is zero. (The interval is defined to be  $(\Delta s)^2$ , not the square root of this quantity.)

The fact that the interval is negative for a timelike line (on which a slower-than-light particle will actually move) is annoying, so we define the **proper time**  $\tau$  to satisfy

$$(\Delta\tau)^2 = -(\Delta s)^2 = -\eta_{\mu\nu}\Delta x^\mu\Delta x^\nu. \quad (1.17)$$

A crucial feature of the spacetime interval is that *the proper time between two events measures the time elapsed as seen by an observer moving on a straight path between the events*. This is easily seen in the very special case that the two events have the same spatial coordinates, and are only separated in time; this corresponds to the observer traveling between the events being at rest in the coordinate system used. Then  $(\Delta\tau)^2 = -\eta_{\mu\nu}\Delta x^\mu\Delta x^\nu = (\Delta t)^2$ , so  $\Delta\tau = \Delta t$ , and of course we defined  $t$  as the time measured by a clock located at a fixed spatial position. But the spacetime interval is invariant under changes of inertial frame; the proper time (1.17) between two fixed events will be the same when evaluated in an inertial frame where the observer is moving as it is in the frame where the observer is at rest.

A crucial fact is that, for more general trajectories, the proper time and coordinate time are different (although the proper time is always that measured by the clock carried by an observer along the trajectory). Consider two trajectories between events  $A$  and  $C$ , one a straight line passing through a halfway point marked  $B$ , and another traveled by an observer moving away from  $A$  at a constant velocity  $v = dx/dt$  to a point  $B'$  and then back at a constant velocity  $-v$  to intersect at



**FIGURE 1.6** The twin paradox. A traveler on the straight path through spacetime  $ABC$  will age more than someone on the nonstraight path  $AB'C$ . Since proper time is a measure of distance traveled through spacetime, this should come as no surprise. (The only surprise might be that the straight path is the one of *maximum* proper time; this can be traced to the minus sign for the timelike component of the metric.)

the event  $C$ . Choose inertial coordinates such that the straight trajectory describes a motionless particle, with event  $A$  located at coordinates  $(t, x) = (0, 0)$  and  $C$  located at  $(\Delta t, 0)$ . The two paths then describe an isosceles triangle in spacetime;  $B$  has coordinates  $(\frac{1}{2}\Delta t, 0)$  and  $B'$  has coordinates  $(\frac{1}{2}\Delta t, \Delta x)$ , with  $\Delta x = \frac{1}{2}v\Delta t$ . Clearly,  $\Delta\tau_{AB} = \frac{1}{2}\Delta t$ , but

$$\begin{aligned}\Delta\tau_{AB'} &= \sqrt{(\frac{1}{2}\Delta t)^2 - (\Delta x)^2} \\ &= \frac{1}{2}\sqrt{1-v^2}\Delta t.\end{aligned}\quad (1.18)$$

It should be obvious that  $\Delta\tau_{BC} = \Delta\tau_{AB}$  and  $\Delta\tau_{B'C} = \Delta\tau_{AB'}$ . Thus, the observer on the straight-line trip from event  $A$  to  $C$  experiences an elapsed time of  $\Delta\tau_{ABC} = \Delta t$ , whereas the one who traveled out and returned experiences

$$\Delta\tau_{AB'C} = \sqrt{1-v^2}\Delta t < \Delta t. \quad (1.19)$$

Even though the two observers begin and end at the same points in spacetime, they have aged different amounts. This is the famous “twin paradox,” the unfortunate scene of all sorts of misunderstandings and tortured explanations. The truth is straightforward: a nonstraight path in spacetime has a different interval than a straight path, just as a nonstraight path in space has a different length than a straight one. This isn’t as trivial as it sounds, of course; the profound insight is the way in which “elapsed time along a worldline” is related to the interval traversed through spacetime. In a Newtonian world, the coordinate  $t$  represents a universal flow of time throughout all of spacetime; in relativity,  $t$  is just a convenient coordinate, and the elapsed time depends on the path along which you travel. An

important distinction is that the nonstraight path has a *shorter* proper time. In space, the shortest distance between two points is a straight line; in spacetime, the longest proper time between two events is a straight trajectory.

Not all trajectories are nice enough to be constructed from pieces of straight lines. In more general circumstances it is useful to introduce the infinitesimal interval, or **line element**:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad (1.20)$$

for infinitesimal coordinate displacements  $dx^\mu$ . (We are being quite informal here, but we'll make amends later on.) From this definition it is tempting to take the square root and integrate along a path to obtain a finite interval, but it is somewhat unclear what  $\int \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu}$  is supposed to mean. Instead we consider a path through spacetime as a parameterized curve,  $x^\mu(\lambda)$ . Note that, unlike conventional practice in Newtonian mechanics, the parameter  $\lambda$  is not necessarily identified with the time coordinate. We can then calculate the derivatives  $dx^\mu/d\lambda$ , and write the path length along a spacelike curve (one whose infinitesimal intervals are spacelike) as

$$\Delta s = \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda, \quad (1.21)$$

where the integral is taken over the path. For timelike paths we use the proper time

$$\Delta\tau = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda, \quad (1.22)$$

which will be positive. (For null paths the interval is simply zero.) Of course we may consider paths that are timelike in some places and spacelike in others, but fortunately it is seldom necessary since the paths of physical particles never change their character (massive particles move on timelike paths, massless particles move on null paths). Once again,  $\Delta\tau$  really is the time measured by an observer moving along the trajectory.

The notion of *acceleration* in special relativity has a bad reputation, for no good reason. Of course we were careful, in setting up inertial coordinates, to make sure that particles at rest in such coordinates are unaccelerated. However, once we've set up such coordinates, we are free to consider any sort of trajectories for physical particles, whether accelerated or not. In particular, there is no truth to the rumor that SR is unable to deal with accelerated trajectories, and general relativity must be invoked. General relativity becomes relevant in the presence of gravity, when spacetime becomes curved. Any processes in flat spacetime are described within the context of special relativity; in particular, expressions such as (1.22) are perfectly general.

### 1.3 ■ LORENTZ TRANSFORMATIONS

We can now consider coordinate transformations in spacetime at a somewhat more abstract level than before. We are interested in a formal description of how to relate the various inertial frames constructed via the procedure outlined above; that is, coordinate systems that leave the interval (1.16) invariant. One simple variety are the **translations**, which merely shift the coordinates (in space or time):

$$x^\mu \rightarrow x^{\mu'} = \delta_\mu^{\mu'}(x^\mu + a^\mu), \quad (1.23)$$

where  $a^\mu$  is a set of four fixed numbers and  $\delta_\mu^{\mu'}$  is the four-dimensional version of the traditional Kronecker delta symbol:

$$\delta_\mu^{\mu'} = \begin{cases} 1 & \text{when } \mu' = \mu, \\ 0 & \text{when } \mu' \neq \mu. \end{cases} \quad (1.24)$$

Notice that we put the prime on the index, not on the  $x$ . The reason for this should become more clear once we start dealing with vectors and tensors; the notation serves to remind us that the geometrical object is the same, but its components are resolved with respect to a different coordinate system. Translations leave the differences  $\Delta x^\mu$  unchanged, so it is not remarkable that the interval is unchanged. The other relevant transformations include spatial **rotations** and offsets by a constant velocity vector, or **boosts**; these are linear transformations, described by multiplying  $x^\mu$  by a (spacetime-independent) matrix:

$$x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu, \quad (1.25)$$

or, in more conventional matrix notation,

$$x' = \Lambda x. \quad (1.26)$$

(We will generally use indices, rather than matrix notation, but right now we have an interest in relating our discussion to certain other familiar notions usually described by matrices.) These transformations do not leave the differences  $\Delta x^\mu$  unchanged, but multiply them also by the matrix  $\Lambda$ . What kind of matrices will leave the interval invariant? Sticking with the matrix notation, what we would like is

$$\begin{aligned} (\Delta s)^2 &= (\Delta x)^T \eta (\Delta x) = (\Delta x')^T \eta (\Delta x') \\ &= (\Delta x)^T \Lambda^T \eta \Lambda (\Delta x), \end{aligned} \quad (1.27)$$

and therefore

$$\eta = \Lambda^T \eta \Lambda, \quad (1.28)$$

or

$$\eta_{\rho\sigma} = \Lambda^{\mu'}{}_\rho \eta_{\mu'\nu'} \Lambda^{\nu'}{}_\sigma = \Lambda^{\mu'}{}_\rho \Lambda^{\nu'}{}_\sigma \eta_{\mu'\nu'}. \quad (1.29)$$

(In matrix notation the order matters, while in index notation it is irrelevant.) We want to find the matrices  $\Lambda^{\mu'}{}_\nu$  such that the components of the matrix  $\eta_{\mu'\nu'}$  are the same as those of  $\eta_{\rho\sigma}$ ; that is what it means for the interval to be invariant under these transformations.

The matrices that satisfy (1.28) are known as the **Lorentz transformations**; the set of them forms a group under matrix multiplication, known as the **Lorentz group**. There is a close analogy between this group and SO(3), the rotation group in three-dimensional space. The rotation group can be thought of as  $3 \times 3$  matrices  $R$  that satisfy  $R^T R = 1$ , where  $1$  is the  $3 \times 3$  identity matrix. Such matrices are called *orthogonal*, and the  $3 \times 3$  ones form the group O(3). This includes not only rotations but also reversals of orientation of the spatial axes (parity transformations). Sometimes we choose to exclude parity transformations by also demanding that the matrices have unit determinant,  $|R| = 1$ ; such matrices are called *special*, and the resulting group is SO(3). The orthogonality condition can be made to look more like (1.28) if we write it as

$$1 = R^T 1 R. \quad (1.30)$$

So the difference between the rotation group O(3) and the Lorentz group is the replacement of  $1$ , a  $3 \times 3$  diagonal matrix with all entries equal to  $+1$ , by  $\eta$ , a  $4 \times 4$  diagonal matrix with one entry equal to  $-1$  and the rest equal to  $+1$ . The Lorentz group is therefore often referred to as O(3,1). It includes not only boosts and rotations, but discrete reversals of the time direction as well as parity transformations. As before we can demand that  $|\Lambda| = 1$ , leaving the “proper Lorentz group” SO(3,1). However, this does not leave us with what we really want, which is the set of continuous Lorentz transformations (those connected smoothly to the identity), since a combination of a time reversal and a parity reversal would have unit determinant. From the  $(\rho, \sigma) = (0, 0)$  component of (1.29) we can easily show that  $|\Lambda^0{}_0| \geq 1$ , with negative values corresponding to time reversals. We can therefore demand at last that  $\Lambda^0{}_0 \geq 1$  (in addition to  $|\Lambda| = 1$ ), leaving the “proper orthochronous” or “restricted” Lorentz group. Sometimes this is denoted by something like SO(3, 1) $^\dagger$ , but usually we will not bother to make this distinction explicitly. Note that the  $3 \times 3$  identity matrix is simply the metric for ordinary flat space. Such a metric, in which all of the eigenvalues are positive, is called **Euclidean**, while those such as (1.15), which feature a single minus sign, are called **Lorentzian**.

It is straightforward to write down explicit expressions for simple Lorentz transformations. A familiar rotation in the  $x$ - $y$  plane is:

$$\Lambda^{\mu'}{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.31)$$

The rotation angle  $\theta$  is a periodic variable with period  $2\pi$ . The boosts may be thought of as “rotations between space and time directions.” An example is given by a boost in the  $x$ -direction:

$$\Lambda^{\mu'}_{\nu} = \begin{pmatrix} \cosh \phi & -\sinh \phi & 0 & 0 \\ -\sinh \phi & \cosh \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.32)$$

The boost parameter  $\phi$ , unlike the rotation angle, is defined from  $-\infty$  to  $\infty$ . A general transformation can be obtained by multiplying the individual transformations; the explicit expression for this six-parameter matrix (three boosts, three rotations) is not pretty, or sufficiently useful to bother writing down. In general Lorentz transformations will not commute, so the Lorentz group is nonabelian. The set of both translations and Lorentz transformations is a ten-parameter non-abelian group, the **Poincaré group**.

You should not be surprised to learn that the boosts correspond to changing coordinates by moving to a frame that travels at a constant velocity, but let’s see it more explicitly. (Don’t confuse “boosting” with “accelerating.” The difference between boosting to a different reference frame and accelerating an object is the same as the difference between rotating to a different coordinate system and setting an object spinning.) For the transformation given by (1.32), the transformed coordinates  $t'$  and  $x'$  will be given by

$$\begin{aligned} t' &= t \cosh \phi - x \sinh \phi \\ x' &= -t \sinh \phi + x \cosh \phi. \end{aligned} \quad (1.33)$$

From this we see that the point defined by  $x' = 0$  is moving; it has a velocity

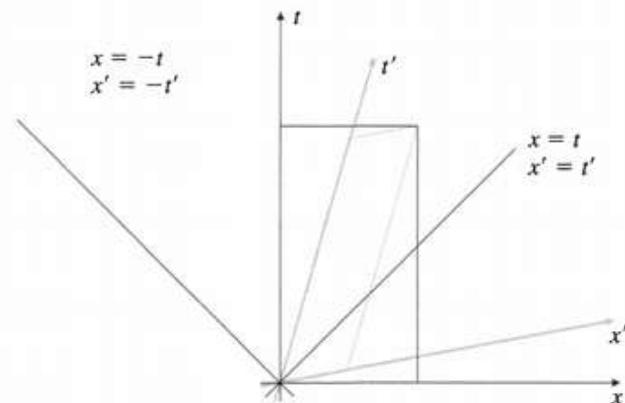
$$v = \frac{x}{t} = \frac{\sinh \phi}{\cosh \phi} = \tanh \phi. \quad (1.34)$$

To translate into more pedestrian notation, we can replace  $\phi = \tanh^{-1} v$  to obtain

$$\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt), \end{aligned} \quad (1.35)$$

where  $\gamma = 1/\sqrt{1-v^2}$ . So indeed, our abstract approach has recovered the conventional expressions for Lorentz transformations. Applying these formulae leads to time dilation, length contraction, and so forth.

It’s illuminating to consider Lorentz transformations in the context of space-time diagrams. According to (1.33), under a boost in the  $x$ - $t$  plane the  $x'$  axis ( $t' = 0$ ) is given by  $t = x \tanh \phi$ , while the  $t'$  axis ( $x' = 0$ ) is given by  $t = x/\tanh \phi$ . We therefore see that the space and time axes are rotated into each other, although they scissor together instead of remaining orthogonal in the traditional Euclidean sense. (As we shall see, the axes do in fact remain orthogonal in



**FIGURE 1.7** A Lorentz transformation relates the  $\{t', x'\}$  coordinates to the  $\{t, x\}$  coordinates. Note that light cones are unchanged.

the Lorentzian sense; that's the implication of the metric remaining invariant under boosts.) This should come as no surprise, since if spacetime behaved just like a four-dimensional version of space the world would be a very different place. We see quite vividly the distinction between this situation and the Newtonian world; in SR, it is impossible to say (in a coordinate-independent way) whether a point that is spacelike separated from  $p$  is in the future of  $p$ , the past of  $p$ , or "at the same time."

Note also that the paths defined by  $x' = \pm t'$  are precisely the same as those defined by  $x = \pm t$ ; these trajectories are left invariant under boosts along the  $x$ -axis. Of course we know that light travels at this speed; we have therefore found that the speed of light is the same in any inertial frame.

## 1.4 ■ VECTORS

To probe the structure of Minkowski space in more detail, it is necessary to introduce the concepts of vectors and tensors. We will start with vectors, which should be familiar. Of course, in spacetime vectors are four-dimensional, and are often referred to as **four-vectors**. This turns out to make quite a bit of difference—for example, there is no such thing as a cross product between two four-vectors.

Beyond the simple fact of dimensionality, the most important thing to emphasize is that each vector is located at a given point in spacetime. You may be used to thinking of vectors as stretching from one point to another in space, and even of "free" vectors that you can slide carelessly from point to point. These are not useful concepts outside the context of flat spaces; once we introduce curvature, we lose the ability to draw preferred curves from one point to another, or to move vectors uniquely around a manifold. Rather, to each point  $p$  in spacetime we associate the set of all possible vectors located at that point; this set is known as the **tangent space** at  $p$ , or  $T_p$ . The name is inspired by thinking of the set of

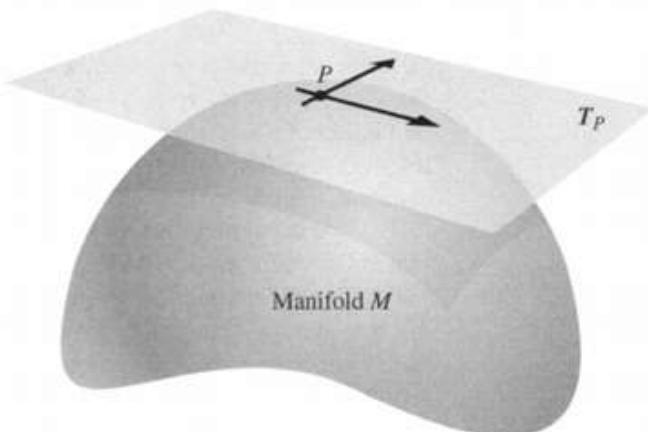
vectors attached to a point on a simple curved two-dimensional space as comprising a plane tangent to the point. (This picture relies on an embedding of the manifold and the tangent space in a higher-dimensional external space, which we won't generally have or need.) Inspiration aside, it is important to think of these vectors as being located at a single point, rather than stretching from one point to another (although this won't stop us from drawing them as arrows on spacetime diagrams).

In Chapter 2 we will relate the tangent space at each point to things we can construct from the spacetime itself. For right now, just think of  $T_p$  as an abstract vector space for each point in spacetime. A **(real) vector space** is a collection of objects (vectors) that can be added together and multiplied by real numbers in a linear way. Thus, for any two vectors  $V$  and  $W$  and real numbers  $a$  and  $b$ , we have

$$(a + b)(V + W) = aV + bV + aW + bW. \quad (1.36)$$

Every vector space has an origin, that is, a zero vector that functions as an identity element under vector addition. In many vector spaces there are additional operations such as taking an inner (dot) product, but this is extra structure over and above the elementary concept of a vector space.

A vector is a perfectly well-defined geometric object, as is a **vector field**, defined as a set of vectors with exactly one at each point in spacetime. [The set of all the tangent spaces of an  $n$ -dimensional manifold  $M$  can be assembled into a  $2n$ -dimensional manifold called the **tangent bundle**,  $T(M)$ . It is a specific example of a "fiber bundle," which is endowed with some extra mathematical structure; we won't need the details for our present purposes.] Nevertheless it is often useful to decompose vectors into components with respect to some set of basis vectors. A **basis** is any set of vectors which both spans the vector space (any vector is a linear combination of basis vectors) and is linearly independent (no vector in the basis



**FIGURE 1.8** A suggestive drawing of the tangent space  $T_p$ , the space of all vectors at the point  $p$ .

is a linear combination of other basis vectors). For any given vector space, there will be an infinite number of possible bases we could choose, but each basis will consist of the same number of vectors, known as the **dimension** of the space. (For a tangent space associated with a point in Minkowski space, the dimension is, of course, four.)

Let us imagine that at each tangent space we set up a basis of four vectors  $\hat{e}_{(\mu)}$ , with  $\mu \in \{0, 1, 2, 3\}$  as usual. In fact let us say that each basis is “adapted to the coordinates  $x^\mu$ ”—that is, the basis vector  $\hat{e}_{(1)}$  is what we would normally think of pointing along the  $x$ -axis. It is by no means necessary that we choose a basis adapted to any coordinate system at all, although it is often convenient. (As before, we really could be more precise here, but later on we will repeat the discussion at an excruciating level of precision, so some sloppiness now is forgivable.) Then any abstract vector  $A$  can be written as a linear combination of basis vectors:

$$A = A^\mu \hat{e}_{(\mu)}. \quad (1.37)$$

The coefficients  $A^\mu$  are the **components** of the vector  $A$ . More often than not we will forget the basis entirely and refer somewhat loosely to “the vector  $A^\mu$ ,” but keep in mind that this is shorthand. The real vector is an abstract geometrical entity, while the components are just the coefficients of the basis vectors in some convenient basis. (Since we will usually suppress the explicit basis vectors, the indices usually will label components of vectors and tensors. This is why there are parentheses around the indices on the basis vectors, to remind us that this is a collection of vectors, not components of a single vector.)

A standard example of a vector in spacetime is the tangent vector to a curve. A parameterized curve or path through spacetime is specified by the coordinates as a function of the parameter, for example,  $x^\mu(\lambda)$ . The tangent vector  $V(\lambda)$  has components

$$V^\mu = \frac{dx^\mu}{d\lambda}. \quad (1.38)$$

The entire vector is  $V = V^\mu \hat{e}_{(\mu)}$ . Under a Lorentz transformation the coordinates  $x^\mu$  change according to (1.25), while the parameterization  $\lambda$  is unaltered; we can therefore deduce that the components of the tangent vector must change as

$$V^\mu \rightarrow V^{\mu'} = \Lambda^{\mu'}_{\ \nu} V^\nu. \quad (1.39)$$

However, the vector  $V$  itself (as opposed to its components in some coordinate system) is invariant under Lorentz transformations. We can use this fact to derive the transformation properties of the basis vectors. Let us refer to the set of basis vectors in the transformed coordinate system as  $\hat{e}_{(\nu')}$ . Since the vector is invariant, we have

$$V = V^\mu \hat{e}_{(\mu)} = V^{\nu'} \hat{e}_{(\nu')} = \Lambda^{\nu'}_{\ \mu} V^\mu \hat{e}_{(\nu')}. \quad (1.40)$$

But this relation must hold no matter what the numerical values of the components  $V^\mu$  are. We can therefore say

$$\hat{e}_{(\mu)} = \Lambda^{\nu'}{}_\mu \hat{e}_{(\nu')}. \quad (1.41)$$

To get the new basis  $\hat{e}_{(\nu')}$  in terms of the old one  $\hat{e}_{(\mu)}$ , we should multiply by the inverse of the Lorentz transformation  $\Lambda^{\nu'}{}_\mu$ . But the inverse of a Lorentz transformation from the unprimed to the primed coordinates is also a Lorentz transformation, this time from the primed to the unprimed systems. We will therefore introduce a somewhat subtle notation, by using the same symbol for both matrices, just with primed and unprimed indices switched. That is, the Lorentz transformation specified by  $\Lambda^{\mu'}{}_\nu$  has an inverse transformation written as  $\Lambda^\sigma{}_{\sigma'}$ . Operationally this implies

$$\Lambda^\mu{}_\nu \Lambda^{\nu'}{}_\rho = \delta^\mu_\rho, \quad \Lambda^{\sigma'}{}_\lambda \Lambda^\lambda{}_{\tau'} = \delta^{\sigma'}_{\tau'}. \quad (1.42)$$

From (1.41) we then obtain the transformation rule for basis vectors:

$$\hat{e}_{(\nu')} = \Lambda^\mu{}_\nu \hat{e}_{(\mu)}. \quad (1.43)$$

Therefore the set of basis vectors transforms via the inverse Lorentz transformation of the coordinates or vector components.

Let's pause a moment to take all this in. We introduced coordinates labeled by upper indices, which transformed in a certain way under Lorentz transformations. We then considered vector components that also were written with upper indices, which made sense since they transformed in the same way as the coordinate functions. (In a fixed coordinate system, each of the four coordinates  $x^\mu$  can be thought of as a function on spacetime, as can each of the four components of a vector field.) The basis vectors associated with the coordinate system transformed via the inverse matrix, and were labeled by a lower index. This notation ensured that the invariant object constructed by summing over the components and basis vectors was left unchanged by the transformation, just as we would wish. It's probably not giving too much away to say that this will continue to be the case for tensors, which may have multiple indices.

## 1.5 ■ DUAL VECTORS (ONE-FORMS)

Once we have set up a vector space, we can define another associated vector space (of equal dimension) known as the **dual vector space**. The dual space is usually denoted by an asterisk, so that the dual space to the tangent space  $T_p$ , called the **cotangent space**, is denoted  $T_p^*$ . The dual space is the space of all linear maps from the original vector space to the real numbers; in math lingo, if  $\omega \in T_p^*$  is a dual vector, then it acts as a map such that

$$\omega(aV + bW) = a\omega(V) + b\omega(W) \in \mathbf{R}, \quad (1.44)$$

where  $V, W$  are vectors and  $a, b$  are real numbers. The nice thing about these maps is that they form a vector space themselves; thus, if  $\omega$  and  $\eta$  are dual vectors, we have

$$(a\omega + b\eta)(V) = a\omega(V) + b\eta(V). \quad (1.45)$$

To make this construction somewhat more concrete, we can introduce a set of basis dual vectors  $\hat{\theta}^{(v)}$  by demanding

$$\hat{\theta}^{(v)}(\hat{e}_{(\mu)}) = \delta_{\mu}^v. \quad (1.46)$$

Then every dual vector can be written in terms of its components, which we label with lower indices:

$$\omega = \omega_{\mu} \hat{\theta}^{(\mu)}. \quad (1.47)$$

Usually, we will simply write  $\omega_{\mu}$ , in perfect analogy with vectors, to stand for the entire dual vector. In fact, you will sometimes see elements of  $T_p$  (what we have called vectors) referred to as **contravariant vectors**, and elements of  $T_p^*$  (what we have called dual vectors) referred to as **covariant vectors**, although in this day and age these terms sound a little dated. If you just refer to ordinary vectors as vectors with upper indices and dual vectors as vectors with lower indices, nobody should be offended. Another name for dual vectors is **one-forms**, a somewhat mysterious designation that will become clearer in Chapter 2.

The component notation leads to a simple way of writing the action of a dual vector on a vector:

$$\begin{aligned} \omega(V) &= \omega_{\mu} \hat{\theta}^{(\mu)}(V^v \hat{e}_{(v)}) \\ &= \omega_{\mu} V^v \hat{\theta}^{(\mu)}(\hat{e}_{(v)}) \\ &= \omega_{\mu} V^v \delta_{\nu}^{\mu} \\ &= \omega_{\mu} V^{\mu} \in \mathbf{R}. \end{aligned} \quad (1.48)$$

This is why it is rarely necessary to write the basis vectors and dual vectors explicitly; the components do all of the work. The form of (1.48) also suggests that we can think of vectors as linear maps on dual vectors, by defining

$$V(\omega) \equiv \omega(V) = \omega_{\mu} V^{\mu}. \quad (1.49)$$

Therefore, the dual space to the dual vector space is the original vector space itself.

Of course in spacetime we will be interested not in a single vector space, but in fields of vectors and dual vectors. [The set of all cotangent spaces over  $M$  can be combined into the **cotangent bundle**,  $T^*(M)$ .] In that case the action of a dual vector field on a vector field is not a single number, but a **scalar** (function) on spacetime. A scalar is a quantity without indices, which is unchanged under

Lorentz transformations; it is a coordinate-independent map from spacetime to the real numbers.

We can use the same arguments that we earlier used for vectors (that geometrical objects are independent of coordinates, even if their components are not) to derive the transformation properties of dual vectors. The answers are, for the components,

$$\omega_{\mu'} = \Lambda^{\nu}_{\mu'} \omega_{\nu}, \quad (1.50)$$

and for basis dual vectors,

$$\hat{\theta}^{(\rho')} = \Lambda^{\rho'}_{\sigma} \hat{\theta}^{(\sigma)}. \quad (1.51)$$

This is just what we would expect from index placement; the components of a dual vector transform under the inverse transformation of those of a vector. Note that this ensures that the scalar (1.48) is invariant under Lorentz transformations, just as it should be.

In spacetime the simplest example of a dual vector is the **gradient** of a scalar function, the set of partial derivatives with respect to the spacetime coordinates, which we denote by a lowercase d:

$$d\phi = \frac{\partial \phi}{\partial x^{\mu}} \hat{\theta}^{(\mu)}. \quad (1.52)$$

The conventional chain rule used to transform partial derivatives amounts in this case to the transformation rule of components of dual vectors:

$$\begin{aligned} \frac{\partial \phi}{\partial x^{\mu'}} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial \phi}{\partial x^{\mu}} \\ &= \Lambda^{\mu}_{\mu'} \frac{\partial \phi}{\partial x^{\mu}}, \end{aligned} \quad (1.53)$$

where we have used (1.25) to relate the Lorentz transformation to the coordinates. The fact that the gradient is a dual vector leads to the following shorthand notations for partial derivatives:

$$\frac{\partial \phi}{\partial x^{\mu}} = \partial_{\mu} \phi = \phi_{,\mu}. \quad (1.54)$$

So,  $x^{\mu}$  has an upper index, but when it is in the denominator of a derivative it implies a lower index on the resulting object. In this book we will generally use  $\partial_{\mu}$  rather than the comma notation. Note that the gradient does in fact act in a natural way on the example we gave above of a vector, the tangent vector to a curve. The result is an ordinary derivative of the function along the curve:

$$\partial_{\mu} \phi \frac{\partial x^{\mu}}{\partial \lambda} = \frac{d\phi}{d\lambda}. \quad (1.55)$$

## 1.6 ■ TENSORS

A straightforward generalization of vectors and dual vectors is the notion of a **tensor**. Just as a dual vector is a linear map from vectors to  $\mathbf{R}$ , a tensor  $T$  of type (or rank)  $(k, l)$  is a multilinear map from a collection of dual vectors and vectors to  $\mathbf{R}$ :

$$T : T_p^* \times_{(k \text{ times})} \cdots \times T_p^* \times T_p \times_{(l \text{ times})} \cdots \times T_p \rightarrow \mathbf{R}. \quad (1.56)$$

Here, “ $\times$ ” denotes the Cartesian product, so that for example  $T_p \times T_p$  is the space of ordered pairs of vectors. Multilinearity means that the tensor acts linearly in each of its arguments; for instance, for a tensor of type  $(1, 1)$ , we have

$$\begin{aligned} T(a\omega + b\eta, cV + dW) &= acT(\omega, V) \\ &\quad + adT(\omega, W) + bcT(\eta, V) + bdT(\eta, W). \end{aligned} \quad (1.57)$$

From this point of view, a scalar is a type  $(0, 0)$  tensor, a vector is a type  $(1, 0)$  tensor, and a dual vector is a type  $(0, 1)$  tensor.

The space of all tensors of a fixed type  $(k, l)$  forms a vector space; they can be added together and multiplied by real numbers. To construct a basis for this space, we need to define a new operation known as the **tensor product**, denoted by  $\otimes$ . If  $T$  is a  $(k, l)$  tensor and  $S$  is an  $(m, n)$  tensor, we define a  $(k+m, l+n)$  tensor  $T \otimes S$  by

$$\begin{aligned} T \otimes S(\omega^{(1)}, \dots, \omega^{(k)}, \dots, \omega^{(k+m)}, V^{(1)}, \dots, V^{(l)}, \dots, V^{(l+n)}) \\ = T(\omega^{(1)}, \dots, \omega^{(k)}, V^{(1)}, \dots, V^{(l)}) \\ \times S(\omega^{(k+1)}, \dots, \omega^{(k+m)}, V^{(l+1)}, \dots, V^{(l+n)}). \end{aligned} \quad (1.58)$$

Note that the  $\omega^{(i)}$  and  $V^{(i)}$  are distinct dual vectors and vectors, not components thereof. In other words, first act  $T$  on the appropriate set of dual vectors and vectors, and then act  $S$  on the remainder, and then multiply the answers. Note that, in general, tensor products do not commute:  $T \otimes S \neq S \otimes T$ .

It is now straightforward to construct a basis for the space of all  $(k, l)$  tensors, by taking tensor products of basis vectors and dual vectors; this basis will consist of all tensors of the form

$$\hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(v_1)} \otimes \cdots \otimes \hat{\theta}^{(v_l)}. \quad (1.59)$$

In a four-dimensional spacetime there will be  $4^{k+l}$  basis tensors in all. In component notation we then write our arbitrary tensor as

$$T = T^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_l} \hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(v_1)} \otimes \cdots \otimes \hat{\theta}^{(v_l)}. \quad (1.60)$$

Alternatively, we could define the components by acting the tensor on basis vectors and dual vectors:

$$T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = T(\hat{\theta}^{(\mu_1)}, \dots, \hat{\theta}^{(\mu_k)}, \hat{e}_{(\nu_1)}, \dots, \hat{e}_{(\nu_l)}). \quad (1.61)$$

You can check for yourself, using (1.46) and so forth, that these equations all hang together properly.

As with vectors, we will usually take the shortcut of denoting the tensor  $T$  by its components  $T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$ . The action of the tensors on a set of vectors and dual vectors follows the pattern established in (1.48):

$$T(\omega^{(1)}, \dots, \omega^{(k)}, V^{(1)}, \dots, V^{(l)}) = T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} \omega_{\mu_1}^{(1)} \dots \omega_{\mu_k}^{(k)} V^{(1)\nu_1} \dots V^{(l)\nu_l}. \quad (1.62)$$

A  $(k, l)$  tensor thus has  $k$  upper indices and  $l$  lower indices. The order of the indices is obviously important, since the tensor need not act in the same way on its various arguments.

Finally, the transformation of tensor components under Lorentz transformations can be derived by applying what we already know about the transformation of basis vectors and dual vectors. The answer is just what you would expect from index placement,

$$T^{\mu'_1 \dots \mu'_k}_{\nu'_1 \dots \nu'_l} = \Lambda^{\mu'_1}_{\mu_1} \dots \Lambda^{\mu'_k}_{\mu_k} \Lambda^{\nu_1}_{\nu'_1} \dots \Lambda^{\nu_l}_{\nu'_l} T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}. \quad (1.63)$$

Thus, each upper index gets transformed like a vector, and each lower index gets transformed like a dual vector.

Although we have defined tensors as linear maps from sets of vectors and dual vectors to  $\mathbf{R}$ , there is nothing that forces us to act on a full collection of arguments. Thus, a  $(1, 1)$  tensor also acts as a map from vectors to vectors:

$$T^\mu{}_\nu : V^\nu \rightarrow T^\mu{}_\nu V^\nu. \quad (1.64)$$

You can check for yourself that  $T^\mu{}_\nu V^\nu$  is a vector (that is, obeys the vector transformation law). Similarly, we can act one tensor on (all or part of) another tensor to obtain a third tensor. For example,

$$U^\mu{}_\nu = T^{\mu\rho}{}_\sigma S^\sigma{}_{\rho\nu} \quad (1.65)$$

is a perfectly good  $(1, 1)$  tensor.

You may be concerned that this introduction to tensors has been somewhat too brief, given the esoteric nature of the material. In fact, the notion of tensors does not require a great deal of effort to master; it's just a matter of keeping the indices straight, and the rules for manipulating them are very natural. Indeed, a number of books like to *define* tensors as collections of numbers transforming according to (1.63). While this is operationally useful, it tends to obscure the deeper meaning of tensors as geometrical entities with a life independent of any chosen coordinate

system. There is, however, one subtlety that we have glossed over. The notions of dual vectors and tensors and bases and linear maps belong to the realm of linear algebra, and are appropriate whenever we have an abstract vector space at hand. In the case of interest to us we have not just a vector space, but a vector space at each point in spacetime. More often than not we are interested in tensor fields, which can be thought of as tensor-valued functions on spacetime. Fortunately, none of the manipulations we defined above really care whether we are dealing with a single vector space or a collection of vector spaces, one for each event. We will be able to get away with simply calling things functions of  $x^\mu$  when appropriate. However, you should keep straight the logical independence of the notions we have introduced and their specific application to spacetime and relativity.

In spacetime, we have already seen some examples of tensors without calling them that. The most familiar example of a  $(0, 2)$  tensor is the metric,  $\eta_{\mu\nu}$ . The action of the metric on two vectors is so useful that it gets its own name, the **inner product** (or scalar product, or dot product):

$$\eta(V, W) = \eta_{\mu\nu} V^\mu W^\nu = V \cdot W. \quad (1.66)$$

Just as with the conventional Euclidean dot product, we will refer to two vectors whose inner product vanishes as **orthogonal**. Since the inner product is a scalar, it is left invariant under Lorentz transformations; therefore, the basis vectors of any Cartesian inertial frame, which are chosen to be orthogonal by definition, are still orthogonal after a Lorentz transformation (despite the “scissoring together” we noticed earlier). The **norm** of a vector is defined to be inner product of the vector with itself; unlike in Euclidean space, this number is not positive definite:

$$\text{if } \eta_{\mu\nu} V^\mu V^\nu \text{ is } \begin{cases} < 0, & V^\mu \text{ is timelike} \\ = 0, & V^\mu \text{ is lightlike or null} \\ > 0, & V^\mu \text{ is spacelike.} \end{cases}$$

(A vector can have zero norm without being the zero vector.) You will notice that the terminology is the same as that which we used earlier to classify the relationship between two points in spacetime; it’s no accident, of course, and we will go into more detail later.

Another tensor is the Kronecker delta  $\delta_\rho^\mu$ , of type  $(1, 1)$ . Thought of as a map from vectors to vectors (or one-forms to one-forms), the Kronecker delta is simply the identity map. We follow the example of many other references in placing the upper and lower indices in the same column for this unique tensor; purists might write  $\delta_\rho^\mu$  or  $\delta_\rho^{\mu}$ , but these would be numerically identical, and we shouldn’t get in trouble being careless in this one instance.

Related to the Kronecker delta and the metric is the **inverse metric**  $\eta^{\mu\nu}$ , a type  $(2, 0)$  tensor defined (unsurprisingly) as the “inverse” of the metric:

$$\eta^{\mu\nu} \eta_{\nu\rho} = \eta_{\rho\nu} \eta^{\nu\mu} = \delta_\rho^\mu. \quad (1.67)$$

(It’s the inverse metric since, when multiplied by the metric, it yields the identity map.) In fact, as you can check, the inverse metric has exactly the same compo-

nents as the metric itself. This is only true in flat space in Cartesian coordinates, and will fail to hold in more general situations. There is also the **Levi–Civita symbol**, a  $(0, 4)$  tensor:

$$\tilde{\epsilon}_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise.} \end{cases} \quad (1.68)$$

Here, a “permutation of 0123” is an ordering of the numbers 0, 1, 2, 3, which can be obtained by starting with 0123 and exchanging two of the digits; an even permutation is obtained by an even number of such exchanges, and an odd permutation is obtained by an odd number. Thus, for example,  $\tilde{\epsilon}_{0321} = -1$ . (The tilde on  $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ , and referring to it as a symbol rather than simply a tensor, derive from the fact that this object is actually not a tensor in more general geometries or coordinates; instead, it is something called a “tensor density.” It is straightforward enough to define a related object that is a tensor, which we will denote by  $\epsilon_{\mu\nu\rho\sigma}$  and call the “Levi–Civita tensor.” See Chapter 2 for a discussion.)

A remarkable property of the above tensors—the metric, the inverse metric, the Kronecker delta, and the Levi–Civita symbol—is that, even though they all transform according to the tensor transformation law (1.63), their components remain unchanged in *any* inertial coordinate system in flat spacetime. In some sense this makes them nongeneric examples of tensors, since most tensors do not have this property. In fact, these are the *only* tensors with this property, although we won’t prove it. The Kronecker delta is even more unusual, in that it has exactly the same components in any coordinate system in any spacetime. This makes sense from the definition of a tensor as a linear map; the Kronecker tensor can be thought of as the identity map from vectors to vectors (or from dual vectors to dual vectors), which clearly must have the same components regardless of coordinate system. Meanwhile, the metric and its inverse characterize the structure of spacetime, while the Levi–Civita symbol is secretly not a true tensor at all. We shall therefore have to treat these objects more carefully when we drop our assumption of flat spacetime.

A more typical example of a tensor is the **electromagnetic field strength tensor**. We all know that the electromagnetic fields are made up of the electric field vector  $E_i$  and the magnetic field vector  $B_i$ . (Remember that we use Latin indices for spacelike components 1, 2, 3.) Actually these are only “vectors” under rotations in space, not under the full Lorentz group. In fact they are components of a  $(0, 2)$  tensor  $F_{\mu\nu}$ , defined by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{\nu\mu}. \quad (1.69)$$

From this point of view it is easy to transform the electromagnetic fields in one reference frame to those in another, by application of (1.63). The unifying power of the tensor formalism is evident: rather than a collection of two vectors whose

relationship and transformation properties are rather mysterious, we have a single tensor field to describe all of electromagnetism. (On the other hand, don't get carried away; sometimes it's more convenient to work in a single coordinate system using the electric and magnetic field vectors.)

## 1.7 ■ MANIPULATING TENSORS

With these examples in hand we can now be a little more systematic about some properties of tensors. First consider the operation of **contraction**, which turns a  $(k, l)$  tensor into a  $(k - 1, l - 1)$  tensor. Contraction proceeds by summing over one upper and one lower index:

$$S^{\mu\rho}{}_{\sigma} = T^{\mu\nu\rho}{}_{\sigma\nu}. \quad (1.70)$$

You can check that the result is a well-defined tensor. It is only permissible to contract an upper index with a lower index (as opposed to two indices of the same type); otherwise the result would *not* be a well-defined tensor. (By well-defined tensor we mean either "transforming according to the tensor transformation law," or "defining a unique multilinear map from a set of vectors and dual vectors to the real numbers"; take your pick.) Note also that the order of the indices matters, so that you can get different tensors by contracting in different ways; thus,

$$T^{\mu\nu\rho}{}_{\sigma\nu} \neq T^{\mu\rho\nu}{}_{\sigma\nu} \quad (1.71)$$

in general.

The metric and inverse metric can be used to **raise and lower indices** on tensors. That is, given a tensor  $T^{\alpha\beta}{}_{\gamma\delta}$ , we can use the metric to define new tensors, which we choose to denote by the same letter  $T$ :

$$\begin{aligned} T^{\alpha\beta\mu}{}_{\delta} &= \eta^{\mu\gamma} T^{\alpha\beta}{}_{\gamma\delta}, \\ T_{\mu}{}^{\beta}{}_{\gamma\delta} &= \eta_{\mu\alpha} T^{\alpha\beta}{}_{\gamma\delta}, \\ T_{\mu\nu}{}^{\rho\sigma} &= \eta_{\mu\alpha} \eta_{\nu\beta} \eta^{\rho\gamma} \eta^{\sigma\delta} T^{\alpha\beta}{}_{\gamma\delta}, \end{aligned} \quad (1.72)$$

and so forth. Notice that raising and lowering does not change the position of an index relative to other indices, and also that free indices (which are *not* summed over) must be the same on both sides of an equation, while dummy indices (which *are* summed over) only appear on one side. As an example, we can turn vectors and dual vectors into each other by raising and lowering indices:

$$\begin{aligned} V_{\mu} &= \eta_{\mu\nu} V^{\nu} \\ \omega^{\mu} &= \eta^{\mu\nu} \omega_{\nu}. \end{aligned} \quad (1.73)$$

Because the metric and inverse metric are truly inverses of each other, we are free to raise and lower simultaneously a pair of indices being contracted over:

$$A^{\lambda} B_{\lambda} = \eta^{\lambda\rho} A_{\rho} \eta_{\lambda\sigma} B^{\sigma} = \delta_{\sigma}^{\rho} A_{\rho} B^{\sigma} = A_{\sigma} B^{\sigma}, \quad (1.74)$$

The ability to raise and lower indices with a metric explains why the gradient in three-dimensional flat Euclidean space is usually thought of as an ordinary vector, even though we have seen that it arises as a dual vector; in Euclidean space (where the metric is diagonal with all entries +1) a dual vector is turned into a vector with precisely the same components when we raise its index. You may then wonder why we have belabored the distinction at all. One simple reason, of course, is that in a Lorentzian spacetime the components are not equal:

$$\omega^\mu = (-\omega_0, \omega_1, \omega_2, \omega_3). \quad (1.75)$$

In a curved spacetime, where the form of the metric is generally more complicated, the difference is rather more dramatic. But there is a deeper reason, namely that tensors generally have a “natural” definition independent of the metric. Even though we will always have a metric available, it is helpful to be aware of the logical status of each mathematical object we introduce. The gradient, with its action on vectors, is perfectly well-defined regardless of any metric, whereas the “gradient with upper indices” is not. (As an example, we will eventually want to take variations of functionals with respect to the metric, and will therefore have to know exactly how the functional depends on the metric, something that is easily obscured by the index notation.)

Continuing our compilation of tensor jargon, we refer to a tensor as **symmetric** in any of its indices if it is unchanged under exchange of those indices. Thus, if

$$S_{\mu\nu\rho} = S_{\nu\mu\rho}, \quad (1.76)$$

we say that  $S_{\mu\nu\rho}$  is symmetric in its first two indices, while if

$$S_{\mu\nu\rho} = S_{\mu\rho\nu} = S_{\rho\mu\nu} = S_{\nu\mu\rho} = S_{\nu\rho\mu} = S_{\rho\nu\mu}, \quad (1.77)$$

we say that  $S_{\mu\nu\rho}$  is symmetric in all three of its indices. Similarly, a tensor is **antisymmetric** (or skew-symmetric) in any of its indices if it changes sign when those indices are exchanged; thus,

$$A_{\mu\nu\rho} = -A_{\rho\nu\mu} \quad (1.78)$$

means that  $A_{\mu\nu\rho}$  is antisymmetric in its first and third indices (or just “antisymmetric in  $\mu$  and  $\rho$ ”). If a tensor is (anti-) symmetric in all of its indices, we refer to it as simply (anti-) symmetric (sometimes with the redundant modifier “completely”). As examples, the metric  $\eta_{\mu\nu}$  and the inverse metric  $\eta^{\mu\nu}$  are symmetric, while the Levi-Civita symbol  $\tilde{\epsilon}_{\mu\nu\rho\sigma}$  and the electromagnetic field strength tensor  $F_{\mu\nu}$  are antisymmetric. (Check for yourself that if you raise or lower a set of indices that are symmetric or antisymmetric, they remain that way.) Notice that it makes no sense to exchange upper and lower indices with each other, so don’t succumb to the temptation to think of the Kronecker delta  $\delta_\beta^\alpha$  as symmetric. On the other hand, the fact that lowering an index on  $\delta_\beta^\alpha$  gives a symmetric tensor (in fact, the metric) means that the order of indices doesn’t really matter, which is why we don’t keep track of index placement for this one tensor.

Given any tensor, we can **symmetrize** (or antisymmetrize) any number of its upper or lower indices. To symmetrize, we take the sum of all permutations of the relevant indices and divide by the number of terms:

$$T_{(\mu_1\mu_2\cdots\mu_n)\rho}{}^\sigma = \frac{1}{n!} (T_{\mu_1\mu_2\cdots\mu_n\rho}{}^\sigma + \text{sum over permutations of indices } \mu_1 \cdots \mu_n), \quad (1.79)$$

while antisymmetrization comes from the alternating sum:

$$T_{[\mu_1\mu_2\cdots\mu_n]\rho}{}^\sigma = \frac{1}{n!} (T_{\mu_1\mu_2\cdots\mu_n\rho}{}^\sigma + \text{alternating sum over permutations of indices } \mu_1 \cdots \mu_n). \quad (1.80)$$

By “alternating sum” we mean that permutations that are the result of an odd number of exchanges are given a minus sign, thus:

$$T_{[\mu\nu\rho]\sigma} = \frac{1}{6} (T_{\mu\nu\rho\sigma} - T_{\mu\rho\nu\sigma} + T_{\rho\mu\nu\sigma} - T_{\nu\mu\rho\sigma} + T_{\nu\rho\mu\sigma} - T_{\rho\nu\mu\sigma}). \quad (1.81)$$

Notice that round/square brackets denote symmetrization/antisymmetrization. Furthermore, we may sometimes want to (anti-) symmetrize indices that are not next to each other, in which case we use vertical bars to denote indices not included in the sum:

$$T_{(\mu|\nu|\rho)} = \frac{1}{2} (T_{\mu\nu\rho} + T_{\rho\nu\mu}). \quad (1.82)$$

If we are contracting over a pair of upper indices that are symmetric on one tensor, only the symmetric part of the lower indices will contribute; thus,

$$X^{(\mu\nu)} Y_{\mu\nu} = X^{(\mu\nu)} Y_{(\mu\nu)}, \quad (1.83)$$

regardless of the symmetry properties of  $Y_{\mu\nu}$ . (Analogous statements hold for antisymmetric indices, or if it’s the lower indices that are symmetric to start with.) For any *two* indices, we can decompose a tensor into symmetric and antisymmetric parts,

$$T_{\mu\nu\rho\sigma} = T_{(\mu\nu)\rho\sigma} + T_{[\mu\nu]\rho\sigma}, \quad (1.84)$$

but this will not in general hold for three or more indices,

$$T_{\mu\nu\rho\sigma} \neq T_{(\mu\nu\rho)\sigma} + T_{[\mu\nu\rho]\sigma}, \quad (1.85)$$

because there are parts with mixed symmetry that are not specified by either the symmetric or antisymmetric pieces. Finally, some people use a convention in which the factor of  $1/n!$  is omitted. The one used here is a good one, since, for example, a symmetric tensor satisfies

$$S_{\mu_1 \dots \mu_n} = S_{(\mu_1 \dots \mu_n)}, \quad (1.86)$$

and likewise for antisymmetric tensors.

For a  $(1, 1)$  tensor  $X^{\mu}_{\nu}$ , the **trace** is a scalar, often denoted by leaving off the indices, which is simply the contraction:

$$X = X^{\lambda}_{\lambda}. \quad (1.87)$$

If we think of  $X^{\mu}_{\nu}$  as a matrix, this is just the sum of the diagonal components, so it makes sense. However, we will also use trace in the context of a  $(0, 2)$  tensor  $Y_{\mu\nu}$ , in which case it means that we should first raise an index ( $Y^{\mu}_{\nu} = g^{\mu\lambda} Y_{\lambda\nu}$ ) and then contract:

$$Y = Y^{\lambda}_{\lambda} = \eta^{\mu\nu} Y_{\mu\nu}. \quad (1.88)$$

(It must be this way, since we cannot sum over two lower indices.) Although this is the sum of the diagonal components of  $Y^{\mu}_{\nu}$ , it is certainly *not* the sum of the diagonal components of  $Y_{\mu\nu}$ ; we had to raise an index, which in general will change the numerical value of the components. For example, you might guess that the trace of the metric is  $-1 + 1 + 1 + 1 = 2$ , but it's not:

$$\eta^{\mu\nu} \eta_{\mu\nu} = \delta^{\mu}_{\mu} = 4. \quad (1.89)$$

(In  $n$  dimensions,  $\delta^{\mu}_{\mu} = n$ .) There is no reason to denote this trace by  $g$  (or  $\delta$ ), since it will always be the same number, even after we make the transition to curved spaces where the metric components are more complicated. Note that antisymmetric  $(0, 2)$  tensors are always traceless.

We have been careful so far to distinguish clearly between things that are always true (on a manifold with arbitrary metric) and things that are only true in Minkowski space in inertial coordinates. One of the most important distinctions arises with **partial derivatives**. If we are working in flat spacetime with inertial coordinates, then the partial derivative of a  $(k, l)$  tensor is a  $(k, l + 1)$  tensor; that is,

$$T_{\alpha}{}^{\mu}_{\nu} = \partial_{\alpha} R^{\mu}_{\nu} \quad (1.90)$$

transforms properly under Lorentz transformations. However, this will no longer be true in more general spacetimes, and we will have to define a covariant derivative to take the place of the partial derivative. Nevertheless, we can still use the fact that partial derivatives give us tensors in this special case, as long as we keep our wits about us. [The one exception to this warning is the partial derivative of a scalar,  $\partial_{\alpha}\phi$ , which is a perfectly good tensor (the gradient) in any spacetime.] Of course, if we fix a particular coordinate system, the partial derivative is a perfectly good operator, which we will use all the time; its failure is only that it doesn't transform in the same way as the tensors we will be using (or equivalently, that the map it defines is not coordinate-independent). One of the most

useful properties of partial derivatives is that they commute,

$$\partial_\mu \partial_\nu (\dots) = \partial_\nu \partial_\mu (\dots), \quad (1.91)$$

no matter what kind of object is being differentiated.

## 1.8 ■ MAXWELL'S EQUATIONS

We have now accumulated enough tensor know-how to illustrate some of these concepts using actual physics. Specifically, we will examine **Maxwell's equations** of electrodynamics. In 19th-century notation, these are

$$\begin{aligned} \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= \mathbf{J} \\ \nabla \cdot \mathbf{E} &= \rho \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned} \quad (1.92)$$

Here,  $\mathbf{E}$  and  $\mathbf{B}$  are the electric and magnetic field 3-vectors,  $\mathbf{J}$  is the current,  $\rho$  is the charge density, and  $\nabla \times$  and  $\nabla \cdot$  are the conventional curl and divergence. These equations are invariant under Lorentz transformations, of course; that's how the whole business got started. But they don't look obviously invariant; our tensor notation can fix that. Let's begin by writing these equations in component notation,

$$\begin{aligned} \bar{\epsilon}^{ijk} \partial_j B_k - \partial_0 E^i &= J^i \\ \partial_i E^i &= J^0 \\ \bar{\epsilon}^{ijk} \partial_j E_k + \partial_0 B^i &= 0 \\ \partial_i B^i &= 0. \end{aligned} \quad (1.93)$$

In these expressions, spatial indices have been raised and lowered with abandon, without any attempt to keep straight where the metric appears, because  $\delta_{ij}$  is the metric on flat 3-space, with  $\delta^{ij}$  its inverse (they are equal as matrices). We can therefore raise and lower indices at will, since the components don't change. Meanwhile, the three-dimensional Levi-Civita symbol  $\bar{\epsilon}^{ijk}$  is defined just as the four-dimensional one, although with one fewer index (normalized so that  $\bar{\epsilon}^{123} = \bar{\epsilon}_{123} = 1$ ). We have replaced the charge density by  $J^0$ ; this is legitimate because the density and current together form the **current 4-vector**,  $J^\mu = (\rho, J^x, J^y, J^z)$ .

From (1.93), and the definition (1.69) of the field strength tensor  $F_{\mu\nu}$ , it is easy to get a completely tensorial 20th-century version of Maxwell's equations. Begin by noting that we can express the field strength with upper indices as

$$\begin{aligned} F^{0i} &= E^i \\ F^{ij} &= \tilde{\epsilon}^{ijk} B_k. \end{aligned} \quad (1.94)$$

To check this, note for example that  $F^{01} = \eta^{00}\eta^{11}F_{01}$  and  $F^{12} = \tilde{\epsilon}^{123}B_3$ . Then the first two equations in (1.93) become

$$\begin{aligned} \partial_j F^{ij} - \partial_0 F^{0i} &= J^i \\ \partial_i F^{0i} &= J^0. \end{aligned} \quad (1.95)$$

Using the antisymmetry of  $F^{\mu\nu}$ , we see that these may be combined into the single tensor equation

$$\partial_\mu F^{\nu\mu} = J^\nu. \quad (1.96)$$

A similar line of reasoning, which is left as an exercise, reveals that the third and fourth equations in (1.93) can be written

$$\partial_{[\mu} F_{\nu\lambda]} = 0. \quad (1.97)$$

It's simple to verify that the antisymmetry of  $F_{\mu\nu}$  implies that (1.97) can be equivalently expressed as

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \quad (1.98)$$

The four traditional Maxwell equations are thus replaced by two, vividly demonstrating the economy of tensor notation. More importantly, however, both sides of equations (1.96) and (1.97) manifestly transform as tensors; therefore, if they are true in one inertial frame, they must be true in any Lorentz-transformed frame. This is why tensors are so useful in relativity—we often want to express relationships without recourse to any reference frame, and the quantities on each side of an equation must transform in the same way under changes of coordinates. As a matter of jargon, we will sometimes refer to quantities written in terms of tensors as **covariant** (which has nothing to do with “covariant” as opposed to “contravariant”). Thus, we say that (1.96) and (1.97) together serve as the covariant form of Maxwell’s equations, while (1.92) or (1.93) are noncovariant.

## 1.9 ■ ENERGY AND MOMENTUM

We’ve now gone over essentially everything there is to know about the care and feeding of tensors. In the next chapter we will look more carefully at the rigorous definitions of manifolds and tensors, but the basic mechanics have been pretty well covered. Before jumping to more abstract mathematics, let’s review how physics works in Minkowski spacetime.

Start with the worldline of a single particle. This is specified by a map  $\mathbf{R} \rightarrow M$ , where  $M$  is the manifold representing spacetime; we usually think of the path as

a parameterized curve  $x^\mu(\lambda)$ . As mentioned earlier, the tangent vector to this path is  $dx^\mu/d\lambda$  (note that it depends on the parameterization). An object of primary interest is the norm of the tangent vector, which serves to characterize the path; if the tangent vector is timelike/null/spacelike at some parameter value  $\lambda$ , we say that the path is timelike/null/spacelike at that point. This explains why the same words are used to classify vectors in the tangent space and intervals between two points—because a straight line connecting, say, two timelike separated points will itself be timelike at every point along the path.

Nevertheless, be aware of the sleight of hand being pulled here. The metric, as a  $(0, 2)$  tensor, is a machine that acts on two vectors (or two copies of the same vector) to produce a number. It is therefore very natural to classify tangent vectors according to the sign of their norm. But the interval between two points isn't something quite so natural; it depends on a specific choice of path (a "straight line") that connects the points, and this choice in turn depends on the fact that spacetime is flat (which allows a unique choice of straight line between the points).

Let's move from the consideration of paths in general to the paths of massive particles (which will always be timelike). Since the proper time is measured by a clock traveling on a timelike worldline, it is convenient to use  $\tau$  as the parameter along the path. That is, we use (1.22) to compute  $\tau(\lambda)$ , which (if  $\lambda$  is a good parameter in the first place) we can invert to obtain  $\lambda(\tau)$ , after which we can think of the path as  $x^\mu(\tau)$ . The tangent vector in this parameterization is known as the **four-velocity**,  $U^\mu$ :

$$U^\mu = \frac{dx^\mu}{d\tau}. \quad (1.99)$$

Since  $d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$ , the four-velocity is automatically normalized:

$$\eta_{\mu\nu}U^\mu U^\nu = -1. \quad (1.100)$$

This absolute normalization is a reflection of the fact that the four-velocity is not a velocity through space, which can of course take on different magnitudes, but a "velocity through spacetime," through which one always travels at the same rate. The norm of the four-velocity will always be negative, since we are only defining it for timelike trajectories. You could define an analogous vector for spacelike paths as well; for null paths the proper time vanishes, so  $\tau$  can't be used as a parameter, and you have to be more careful. In the rest frame of a particle, its four-velocity has components  $U^\mu = (1, 0, 0, 0)$ .

A related vector is the **momentum four-vector**, defined by

$$p^\mu = mU^\mu. \quad (1.101)$$

where  $m$  is the mass of the particle. The mass is a fixed quantity independent of inertial frame, what you may be used to thinking of as the “rest mass.” It turns out to be much more convenient to take this as the mass once and for all, rather than thinking of mass as depending on velocity. The **energy** of a particle is simply  $E = p^0$ , the timelike component of its momentum vector. Since it’s only one component of a four-vector, it is not invariant under Lorentz transformations; that’s to be expected, however, since the energy of a particle at rest is not the same as that of the same particle in motion. In the particle’s rest frame we have  $p^0 = m$ ; recalling that we have set  $c = 1$ , we see that we have found the equation that made Einstein a celebrity,  $E = mc^2$ . (The field equation of general relativity is actually more fundamental than this one, but  $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}$  doesn’t elicit the visceral reaction that you get from  $E = mc^2$ .) In a moving frame we can find the components of  $p^\mu$  by performing a Lorentz transformation; for a particle moving with three-velocity  $v = dx/dt$  along the  $x$  axis we have

$$p^\mu = (\gamma m, v\gamma m, 0, 0), \quad (1.102)$$

where  $\gamma = 1/\sqrt{1-v^2}$ . For small  $v$ , this gives  $p^0 = m + \frac{1}{2}mv^2$  (what we usually think of as rest energy plus kinetic energy) and  $p^1 = mv$  (what we usually think of as Newtonian momentum). Outside this approximation, we can simply write

$$p_\mu p^\mu = -m^2, \quad (1.103)$$

or

$$E = \sqrt{m^2 + \mathbf{p}^2}, \quad (1.104)$$

where  $\mathbf{p}^2 = \delta_{ij} p^i p^j$ .

The centerpiece of pre-relativity physics is Newton’s Second Law, or  $\mathbf{f} = m\mathbf{a} = d\mathbf{p}/dt$ . An analogous equation should hold in SR, and the requirement that it be tensorial leads us directly to introduce a force four-vector  $f^\mu$  satisfying

$$f^\mu = m \frac{d^2}{d\tau^2} x^\mu(\tau) = \frac{d}{d\tau} p^\mu(\tau). \quad (1.105)$$

The simplest example of a force in Newtonian physics is the force due to gravity. In relativity, however, gravity is not described by a force, but rather by the curvature of spacetime itself. Instead, let us consider electromagnetism. The three-dimensional Lorentz force is given by  $\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ , where  $q$  is the charge on the particle. We would like a tensorial generalization of this equation. There turns out to be a unique answer:

$$f^\mu = -q U^\lambda F_\lambda{}^\mu. \quad (1.106)$$

You can check for yourself that this reduces to the Newtonian version in the limit of small velocities. Notice how the requirement that the equation be tensorial,

which is one way of guaranteeing Lorentz invariance, severely restricts the possible expressions we can get. This is an example of a very general phenomenon, in which a small number of an apparently endless variety of possible physical laws are picked out by the demands of symmetry.

Although  $p^\mu$  provides a complete description of the energy and momentum of an individual particle, we often need to deal with extended systems comprised of huge numbers of particles. Rather than specify the individual momentum vectors of each particle, we instead describe the system as a **fluid**—a continuum characterized by macroscopic quantities such as density, pressure, entropy, viscosity, and so on. Although such a fluid may be composed of many individual particles with different four-velocities, the fluid itself has an overall four-velocity field. Just think of everyday fluids like air or water, where it makes sense to define a velocity for each individual fluid element even though nearby molecules may have appreciable relative velocities.

A single momentum four-vector field is insufficient to describe the energy and momentum of a fluid; we must go further and define the **energy-momentum tensor** (sometimes called the stress-energy tensor),  $T^{\mu\nu}$ . This symmetric  $(2,0)$  tensor tells us all we need to know about the energy-like aspects of a system: energy density, pressure, stress, and so forth. A general definition of  $T^{\mu\nu}$  is “the flux of four-momentum  $p^\mu$  across a surface of constant  $x^\nu$ .” In fact, this definition is not going to be incredibly useful; in Chapter 4 we will define the energy-momentum tensor in terms of a functional derivative of the action with respect to the metric, which will be a more algorithmic procedure for finding an explicit expression for  $T^{\mu\nu}$ . But the definition here does afford some physical insight. Consider an infinitesimal element of the fluid in its rest frame, where there are no bulk motions. Then  $T^{00}$ , the “flux of  $p^0$  (energy) in the  $x^0$  (time) direction,” is simply the rest-frame **energy density**  $\rho$ . Similarly, in this frame,  $T^{0i} = T^{i0}$  is the momentum density. The spatial components  $T^{ij}$  are the momentum flux, or the **stress**; they represent the forces between neighboring infinitesimal elements of the fluid. Off-diagonal terms in  $T^{ij}$  represent shearing terms, such as those due to viscosity. A diagonal term such as  $T^{11}$  gives the  $x$ -component of the force being exerted (per unit area) by a fluid element in the  $x$ -direction; this is what we think of as the  $x$ -component of the **pressure**,  $p_x$  (don’t confuse it with the momentum). The pressure has three components, given in the fluid rest frame (in inertial coordinates) by

$$p_i = T^{ii}. \quad (1.107)$$

There is no sum over  $i$ .

To make this more concrete, let’s start with the simple example of **dust**. (Cosmologists tend to use “matter” as a synonym for dust.) Dust may be defined in flat spacetime as a collection of particles at rest with respect to each other. The four-velocity field  $U^\mu(x)$  is clearly going to be the constant four-velocity of the individual particles. Indeed, its components will be the same at each point. Define the **number-flux four-vector** to be

$$N^\mu = n U^\mu, \quad (1.108)$$

where  $n$  is the number density of the particles as measured in their rest frame. (This doesn't sound coordinate-invariant, but it is; in any frame, the number density that would be measured if you were in the rest frame is a fixed quantity.) Then  $N^0$  is the number density of particles as measured in any other frame, while  $N^i$  is the flux of particles in the  $x^i$  direction. Let's now imagine that each of the particles has the same mass  $m$ . Then in the rest frame the energy density of the dust is given by

$$\rho = mn. \quad (1.109)$$

By definition, the energy density completely specifies the dust. But  $\rho$  only measures the energy density in the rest frame; what about other frames? We notice that both  $n$  and  $m$  are 0-components of four-vectors in their rest frame; specifically,  $N^\mu = (n, 0, 0, 0)$  and  $p^\mu = (m, 0, 0, 0)$ . Therefore  $\rho$  is the  $\mu = 0, \nu = 0$  component of the tensor  $p \otimes N$  as measured in its rest frame. We are therefore led to define the energy-momentum tensor for dust:

$$T_{\text{dust}}^{\mu\nu} = p^\mu N^\nu = mnU^\mu U^\nu = \rho U^\mu U^\nu, \quad (1.110)$$

where  $\rho$  is defined as the energy density in the rest frame. (Typically you don't just guess energy-momentum tensors by such a procedure; you derive them from equations of motion or an action principle.) Note that the pressure of the dust in any direction is zero; this should not be surprising, since pressure arises from the random motions of particles within the fluid, and we have defined dust to be free of such motions.

Dust is not sufficiently general to describe most of the interesting fluids that appear in general relativity; we only need a slight generalization, however, to arrive at the concept of a **perfect fluid**. A perfect fluid is one that can be completely specified by two quantities, the rest-frame energy density  $\rho$ , and an isotropic rest-frame pressure  $p$ . The single parameter  $p$  serves to specify the pressure in every direction. A consequence of isotropy is that  $T^{\mu\nu}$  is diagonal in its rest frame—there is no net flux of any component of momentum in an orthogonal direction. Furthermore, the nonzero spacelike components must all be equal,  $T^{11} = T^{22} = T^{33}$ . The only two independent numbers are therefore the energy density  $\rho = T^{00}$  and the pressure  $p = T^{ii}$ ; we don't need a subscript on  $p$ , since the pressure is equal in every direction. The energy-momentum tensor of a perfect fluid therefore takes the following form in its rest frame:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \quad (1.111)$$

(Remember that we are in flat spacetime; this will change when curvature is introduced.) We would like, of course, a formula that is good in any frame. For dust we had  $T^{\mu\nu} = \rho U^\mu U^\nu$ , so we might begin by guessing  $(\rho + p)U^\mu U^\nu$ , which

gives

$$\begin{pmatrix} \rho + p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (1.112)$$

This is not a very clever guess, to be honest. But by subtracting this guess from our desired answer, we see that what we need to add is

$$\begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \quad (1.113)$$

Fortunately, this has an obvious covariant generalization, namely  $p\eta^{\mu\nu}$ . Thus, the general form of the energy-momentum tensor for a perfect fluid is

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}. \quad (1.114)$$

It may seem that the procedure used to arrive at this formula was somewhat arbitrary, but we can have complete confidence in the result. Given that (1.111) should be the form of  $T^{\mu\nu}$  in the rest frame, and that (1.114) is a perfectly tensorial expression that reduces to (1.111) in the rest frame, we know that (1.114) must be the right expression in any frame.

The concept of a perfect fluid is general enough to describe a wide variety of physical forms of matter. To determine the evolution of such a fluid, we specify an equation of state relating the pressure to the energy density,  $p = p(\rho)$ . Dust is a special case for which  $p = 0$ , while an isotropic gas of photons has  $p = \frac{1}{3}\rho$ . A more exotic example is vacuum energy, for which the energy-momentum tensor is proportional to the metric,  $T^{\mu\nu} = -\rho_{\text{vac}}\eta^{\mu\nu}$ . By comparing to (1.114) we find that vacuum energy is a kind of perfect fluid for which  $p_{\text{vac}} = -\rho_{\text{vac}}$ . The notion of an energy density in vacuum is completely pointless in special relativity, since in nongravitational physics the absolute value of the energy doesn't matter, only the difference in energy between two states. In general relativity, however, all energy couples to gravity, so the possibility of a nonzero vacuum energy will become an important consideration, which we will discuss more fully in Chapter 4.

Besides being symmetric,  $T^{\mu\nu}$  has the even more important property of being *conserved*. In this context, conservation is expressed as the vanishing of the “divergence”:

$$\partial_\mu T^{\mu\nu} = 0. \quad (1.115)$$

This expression is a set of four equations, one for each value of  $\nu$ . The equation with  $\nu = 0$  corresponds to conservation of energy, while  $\partial_\mu T^{\mu k} = 0$  expresses

conservation of the  $k$ th component of the momentum. Let's apply this equation to a perfect fluid, for which we have

$$\partial_\mu T^{\mu\nu} = \partial_\mu(\rho + p)U^\mu U^\nu + (\rho + p)(U^\nu \partial_\mu U^\mu + U^\mu \partial_\mu U^\nu) + \partial^\nu p. \quad (1.116)$$

To analyze what this equation means, it is helpful to consider separately what happens when we project it into pieces along and orthogonal to the four-velocity field  $U^\mu$ . We first note that the normalization  $U_\nu U^\nu = -1$  implies the useful identity

$$U_\nu \partial_\mu U^\nu = \frac{1}{2} \partial_\mu(U_\nu U^\nu) = 0. \quad (1.117)$$

To project (1.116) along the four-velocity, simply contract it into  $U_\nu$ :

$$U_\nu \partial_\mu T^{\mu\nu} = -\partial_\mu(\rho U^\mu) - p \partial_\mu U^\mu. \quad (1.118)$$

Setting this to zero gives the relativistic equation of energy conservation for a perfect fluid. It will look more familiar in the nonrelativistic limit, in which

$$U^\mu = (1, v^i), \quad |v^i| \ll 1, \quad p \ll \rho. \quad (1.119)$$

The last condition makes sense, because pressure comes from the random motions of the individual particles, and in this limit these motions (as well as the bulk motion described by  $U^\mu$ ) are taken to be small. So in ordinary nonrelativistic language, (1.118) becomes

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1.120)$$

the continuity equation for the energy density. We next consider the part of (1.116) that is orthogonal to the four-velocity. To project a vector orthogonal to  $U^\mu$ , we multiply it by the projection tensor

$$P^\sigma_\nu = \delta^\sigma_\nu + U^\sigma U_\nu, \quad (1.121)$$

To convince yourself this does the trick, check that if we have a vector  $V_\parallel^\mu$ , parallel to  $U^\mu$ , and another vector  $W_\perp^\mu$ , perpendicular to  $U^\mu$ , the projection tensor will annihilate the parallel vector and preserve the orthogonal one:

$$\begin{aligned} P^\sigma_\nu V_\parallel^\nu &= 0 \\ P^\sigma_\nu W_\perp^\nu &= W_\perp^\sigma. \end{aligned} \quad (1.122)$$

Applied to  $\partial_\mu T^{\mu\nu}$ , we obtain

$$P^\sigma_\nu \partial_\mu T^{\mu\nu} = (\rho + p)U^\mu \partial_\mu U^\sigma + \partial^\sigma p + U^\sigma U^\mu \partial_\mu p. \quad (1.123)$$

In the nonrelativistic limit given by (1.119), setting the spatial components of this expression equal to zero yields

$$\rho [\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}] + \nabla p + \mathbf{v}(\partial_t p + \mathbf{v} \cdot \nabla p) = 0. \quad (1.124)$$

But notice that the last set of terms involve derivatives of  $p$  times the three-velocity  $\mathbf{v}$ , assumed to be small; these will therefore be negligible compared to the  $\nabla p$  term, and can be neglected. We are left with

$$\rho [\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}] = -\nabla p, \quad (1.125)$$

which is the Euler equation familiar from fluid mechanics.

## 1.10 ■ CLASSICAL FIELD THEORY

When we make the transition from special relativity to general relativity, the metric  $\eta_{\mu\nu}$  will be promoted to a dynamical tensor field,  $g_{\mu\nu}(x)$ . GR is thus a particular example of a classical field theory; we can build up some feeling for how such theories work by considering classical fields defined on flat spacetime. (We say classical field theory in contrast with quantum field theory, which is quite a different story; we will discuss it briefly in Chapter 9, but it is outside our main area of interest here.)

Let's begin with the familiar example of the classical mechanics of a single particle in one dimension with coordinate  $q(t)$ . We can derive the equations of motion for such a particle by using the “principle of least action”: we search for critical points (as a function of the trajectory) of an **action**  $S$ , written as

$$S = \int dt L(q, \dot{q}), \quad (1.126)$$

where the function  $L(q, \dot{q})$  is the **Lagrangian**. The Lagrangian in point-particle mechanics is typically of the form

$$L = K - V, \quad (1.127)$$

where  $K$  is the kinetic energy and  $V$  the potential energy. Following the calculus-of-variations procedure, which is described in any advanced textbook on classical mechanics, we show that critical points of the action [trajectories  $q(t)$  for which  $S$  remains stationary under small variations] are those that satisfy the **Euler-Lagrange equations**,

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial (\dot{q})} \right) = 0. \quad (1.128)$$

For example,  $L = \frac{1}{2}\dot{q}^2 - V(q)$  leads to

$$\ddot{q} = -\frac{dV}{dq}. \quad (1.129)$$

Field theory is a similar story, except that we replace the single coordinate  $q(t)$  by a set of spacetime-dependent **fields**,  $\Phi^i(x^\mu)$ , and the action  $S$  becomes a *functional* of these fields. A functional is simply a function of an infinite number of

variables, such as the values of a field in some region of spacetime. Functionals are often expressed as integrals. Each  $\Phi^i$  is a function on spacetime (at least in some coordinate system), and  $i$  is an index labeling our individual fields. For example, in electromagnetism (as we will see below) the fields are the four components of a one-form called the “vector potential,”  $A_\mu$ :

$$\Phi^i = \{A_0, A_1, A_2, A_3\}. \quad (1.130)$$

We’re being very lowbrow here, in thinking of a one-form field as four different functions rather than a single tensor object. This point of view makes sense so long as we stick to a fixed coordinate system, and it will make our calculations more straightforward.

In field theory, the Lagrangian can be expressed as an integral over space of a **Lagrange density**  $\mathcal{L}$ , which is a function of the fields  $\Phi^i$  and their spacetime derivatives  $\partial_\mu \Phi^i$ :

$$L = \int d^3x \mathcal{L}(\Phi^i, \partial_\mu \Phi^i). \quad (1.131)$$

So the action is

$$S = \int dt L = \int d^4x \mathcal{L}(\Phi^i, \partial_\mu \Phi^i). \quad (1.132)$$

The Lagrange density is a Lorentz scalar. We typically just say “Lagrangian” when we mean “Lagrange density.” It will most often be convenient to define a field theory by specifying the Lagrange density, from which all of the equations of motion can be readily derived.

We will use “natural units,” in which not only  $c = 1$  but also  $\hbar = k = 1$ , where  $\hbar = h/2\pi$ ,  $h$  is Planck’s constant, and  $k$  is Boltzmann’s constant. The objection might be raised that we shouldn’t involve  $\hbar$  in a purely classical discussion; but all we are doing here is choosing units, not determining physics. (The relevance of  $\hbar$  would appear if we were to quantize our field theory and obtain particles, but we won’t get that far right now.) In natural units we have

$$[\text{energy}] = [\text{mass}] = [(\text{length})^{-1}] = [(\text{time})^{-1}]. \quad (1.133)$$

We will most often use energy or mass as our fundamental unit. Since the action is an integral of  $L$  (with units of energy) over time, it is dimensionless:

$$[S] = [E][T] = M^0. \quad (1.134)$$

The volume element has units

$$[d^4x] = M^{-4}, \quad (1.135)$$

so to get a dimensionless action we require that the Lagrange density have units

$$[\mathcal{L}] = M^4. \quad (1.136)$$

The Euler–Lagrange equations come from requiring that the action be unchanged under small variations of the fields,

$$\Phi^i \rightarrow \Phi^i + \delta\Phi^i, \quad (1.137)$$

$$\partial_\mu \Phi^i \rightarrow \partial_\mu \Phi^i + \delta(\partial_\mu \Phi^i) = \partial_\mu \Phi^i + \partial_\mu (\delta\Phi^i). \quad (1.138)$$

The expression for the variation in  $\partial_\mu \Phi^i$  is simply the derivative of the variation of  $\Phi^i$ . Since  $\delta\Phi^i$  is assumed to be small, we may Taylor-expand the Lagrangian under this variation:

$$\begin{aligned} \mathcal{L}(\Phi^i, \partial_\mu \Phi^i) &\rightarrow \mathcal{L}(\Phi^i + \delta\Phi^i, \partial_\mu \Phi^i + \partial_\mu (\delta\Phi^i)) \\ &= \mathcal{L}(\Phi^i, \partial_\mu \Phi^i) + \frac{\partial \mathcal{L}}{\partial \Phi^i} \delta\Phi^i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^i)} \partial_\mu (\delta\Phi^i). \end{aligned} \quad (1.139)$$

Correspondingly, the action goes to  $S \rightarrow S + \delta S$ , with

$$\delta S = \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \Phi^i} \delta\Phi^i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^i)} \partial_\mu (\delta\Phi^i) \right]. \quad (1.140)$$

We would like to factor out  $\delta\Phi^i$  from the integrand, by integrating the second term by parts:

$$\begin{aligned} \int d^4x \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^i)} \partial_\mu (\delta\Phi^i) &= - \int d^4x \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^i)} \right) \delta\Phi^i \\ &\quad + \int d^4x \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^i)} \delta\Phi^i \right). \end{aligned} \quad (1.141)$$

The final term is a total derivative—the integral of something of the form  $\partial_\mu V^\mu$ —that can be converted to a surface term by Stokes's theorem (the four-dimensional version, that is; see Appendix E for a discussion). Since we are considering variational problems, we can choose to consider variations that vanish at the boundary (along with their derivatives). It is therefore traditional in such contexts to integrate by parts with complete impunity, always ignoring the boundary contributions. (Sometimes this is not okay, as in instanton calculations in Yang–Mills theory.)

We are therefore left with

$$\delta S = \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^i)} \right) \right] \delta\Phi^i. \quad (1.142)$$

The functional derivative  $\delta S/\delta\Phi^i$  of a functional  $S$  with respect to a function  $\Phi^i$  is defined to satisfy

$$\delta S = \int d^4x \frac{\delta S}{\delta \Phi^i} \delta\Phi^i, \quad (1.143)$$

when such an expression is valid. We can therefore express the notion that  $S$  is at a critical point by saying that the functional derivative vanishes. The final equations of motion for our field theory are thus:

$$\boxed{\frac{\delta S}{\delta \Phi^i} = \frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^i)} \right) = 0.} \quad (1.144)$$

These are known as the Euler–Lagrange equations for a field theory in flat spacetime.

The simplest example of a field is a real scalar field:

$$\phi(x^\mu) : (\text{spacetime}) \rightarrow \mathbf{R}. \quad (1.145)$$

Slightly more complicated examples would include complex scalar fields, or maps from spacetime to any vector space or even any manifold (sometimes called “non-linear sigma models”). Upon quantization, excitations of the field are observable as particles. Scalar fields give rise to spinless particles, while vector fields and other tensors give rise to higher-spin particles. If the field were complex instead of real, it would have two degrees of freedom rather than just one, which would be interpreted as a particle and a distinct antiparticle. Real fields are their own antiparticles. An example of a real scalar field would be the neutral  $\pi$ -meson.

So let’s consider the classical mechanics of a single real scalar field. It will have an energy density that is a local function of spacetime, and includes various contributions:

$$\begin{aligned} \text{kinetic energy : } & \frac{1}{2} \dot{\phi}^2 \\ \text{gradient energy : } & \frac{1}{2} (\nabla \phi)^2 \\ \text{potential energy : } & V(\phi). \end{aligned} \quad (1.146)$$

Actually, although the potential is a Lorentz-invariant function, the kinetic and gradient energies are not by themselves Lorentz-invariant; but we can combine them into a manifestly Lorentz-invariant form:

$$-\frac{1}{2} \eta^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2. \quad (1.147)$$

[The combination  $\eta^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi)$  is often abbreviated as  $(\partial \phi)^2$ .] So a reasonable choice of Lagrangian for our single real scalar field, analogous to  $L = K - V$  in the point-particle case, would be

$$\mathcal{L} = -\frac{1}{2} \eta^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - V(\phi). \quad (1.148)$$

This generalizes “kinetic minus potential energy” to “kinetic minus gradient minus potential energy density.” Note that since  $[\mathcal{L}] = M^4$ , we must have  $[V] = M^4$ . Also, since  $[\partial_\mu] = [\partial/\partial x^\mu] = M^1$ , we have

$$[\phi] = M^1. \quad (1.149)$$

For the Lagrangian (1.148) we have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{dV}{d\phi}, \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = -\eta^{\mu\nu} \partial_\nu \phi. \quad (1.150)$$

The second of these equations is a little tricky, so let's go through it slowly. When differentiating the Lagrangian, the trick is to make sure that the index placement is "compatible" (so that if you have a lower index on the thing being differentiated with respect to, you should have only lower indices when the same kind of object appears in the thing being differentiated), and also that the indices are strictly different. The first of these is already satisfied in our example, since we are differentiating a function of  $\partial_\mu \phi$  with respect to  $\partial_\mu \phi$ . Later on, we will need to be more careful. To fulfill the second, we simply relabel dummy indices:

$$\eta^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) = \eta^{\rho\sigma} (\partial_\rho \phi) (\partial_\sigma \phi). \quad (1.151)$$

Then we can use the general rule, for any object with one index such as  $V_\mu$ , that

$$\frac{\partial V_\alpha}{\partial V_\beta} = \delta_\alpha^\beta \quad (1.152)$$

because each component of  $V_\alpha$  is treated as a distinct variable. So we have

$$\begin{aligned} \frac{\partial}{\partial (\partial_\mu \phi)} [\eta^{\rho\sigma} (\partial_\rho \phi) (\partial_\sigma \phi)] &= \eta^{\rho\sigma} [\delta_\rho^\mu (\partial_\sigma \phi) + (\partial_\rho \phi) \delta_\sigma^\mu] \\ &= \eta^{\mu\sigma} (\partial_\sigma \phi) + \eta^{\rho\mu} (\partial_\rho \phi) = 2\eta^{\mu\nu} \partial_\nu \phi. \end{aligned} \quad (1.153)$$

This leads to the second expression in (1.150).

Putting (1.150) into (1.144) leads to the equation of motion

$$\square \phi - \frac{dV}{d\phi} = 0, \quad (1.154)$$

where  $\square \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$  is known as the **d'Alembertian**. Note that our metric sign convention  $(-+++)$  comes into this equation; with the alternative  $(+-+-)$  convention the sign would have been switched. In flat spacetime (1.154) is equivalent to

$$\ddot{\phi} - \nabla^2 \phi + \frac{dV}{d\phi} = 0. \quad (1.155)$$

A popular choice for the potential  $V$  is that of a simple harmonic oscillator,  $V(\phi) = \frac{1}{2}m^2\phi^2$ . The parameter  $m$  is called the mass of the field, and you should notice that the units work out correctly. You may be wondering how a field can have mass. When we quantize the field we find that momentum eigenstates are collections of particles, each with mass  $m$ . At the classical level, we think of "mass" as simply a convenient characterization of the field dynamics. Then our

equation of motion is

$$\square \phi - m^2 \phi = 0, \quad (1.156)$$

the famous **Klein–Gordon equation**. This is a linear differential equation, so the sum of two solutions is a solution; a complete set of solutions (in the form of plane waves) is easy to find, as you can check for yourself.

A slightly more elaborate example of a field theory is provided by electromagnetism. We mentioned that the relevant field is the **vector potential**  $A_\mu$ ; the timelike component  $A_0$  can be identified with the electrostatic potential  $\Phi$ , and the spacelike components with the traditional vector potential  $\mathbf{A}$  (in terms of which the magnetic field is given by  $\mathbf{B} = \nabla \times \mathbf{A}$ ). The field strength tensor, with components given by (1.69), is related to the vector potential by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1.157)$$

From this definition we see that the field strength tensor has the important property of **gauge invariance**: when we perform a **gauge transformation** on the vector potential,

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda(x), \quad (1.158)$$

the field strength tensor is left unchanged:

$$F_{\mu\nu} \rightarrow F_{\mu\nu} + \partial_\mu \partial_\nu \lambda - \partial_\nu \partial_\mu \lambda = F_{\mu\nu}. \quad (1.159)$$

The last equality follows from the fact that partial derivatives commute,  $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ . Gauge invariance is a symmetry that is fundamental to our understanding of electromagnetism, and all observable quantities must be gauge-invariant. Thus, while the dynamical field of the theory (with respect to which we vary the action to derive equations of motion) is  $A_\mu$ , physical quantities will generally be expressed in terms of  $F_{\mu\nu}$ .

We already know that the dynamical equations of electromagnetism are Maxwell's equations, (1.96) and (1.97). Given the definition of the field strength tensor in terms of the vector potential, (1.97) is actually automatic:

$$\partial_{[\mu} F_{\nu\sigma]} = \partial_{[\mu} \partial_\nu A_{\sigma]} - \partial_{[\mu} \partial_\sigma A_{\nu]} = 0, \quad (1.160)$$

again because partial derivatives commute. On the other hand, (1.96) is equivalent to Euler–Lagrange equations of the form

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right) = 0, \quad (1.161)$$

if we presciently choose the Lagrangian to be

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu. \quad (1.162)$$

For this choice, the first term in the Euler–Lagrange equation is straightforward:

$$\frac{\partial \mathcal{L}}{\partial A_v} = \delta_\mu^v J^\mu = J^v. \quad (1.163)$$

The second term is trickier. First we write  $F_{\mu\nu}F^{\mu\nu}$  as

$$F_{\mu\nu}F^{\mu\nu} = F_{\alpha\beta}F^{\alpha\beta} = \eta^{\alpha\rho}\eta^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma}. \quad (1.164)$$

We want to work with lower indices on  $F_{\mu\nu}$ , since we are differentiating with respect to  $\partial_\mu A_v$ , which has lower indices. Likewise we change the dummy indices on  $F_{\mu\nu}F^{\mu\nu}$ , since we want to have different indices on the thing being differentiated and the thing we are differentiating with respect to. Once you get familiar with this stuff it will become second nature and you won't need nearly so many steps. This lets us write

$$\frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_v)} = \eta^{\alpha\rho}\eta^{\beta\sigma} \left[ \left( \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_v)} \right) F_{\rho\sigma} + F_{\alpha\beta} \left( \frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_v)} \right) \right]. \quad (1.165)$$

Then, since  $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ , we have

$$\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_v)} = \delta_\alpha^\mu \delta_\beta^v - \delta_\beta^\mu \delta_\alpha^v. \quad (1.166)$$

Combining (1.166) with (1.165) yields

$$\begin{aligned} \frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_v)} &= \eta^{\alpha\rho}\eta^{\beta\sigma} \left[ (\delta_\alpha^\mu \delta_\beta^v - \delta_\beta^\mu \delta_\alpha^v)F_{\rho\sigma} + (\delta_\rho^\mu \delta_\sigma^v - \delta_\sigma^\mu \delta_\rho^v)F_{\alpha\beta} \right] \\ &= (\eta^{\mu\rho}\eta^{v\sigma} - \eta^{v\rho}\eta^{\mu\sigma})F_{\rho\sigma} + (\eta^{\alpha\mu}\eta^{\beta v} - \eta^{\alpha v}\eta^{\beta\mu})F_{\alpha\beta} \\ &= F^{\mu\nu} - F^{v\mu} + F^{\mu v} - F^{v\nu} \\ &= 4F^{\mu\nu}, \end{aligned} \quad (1.167)$$

so

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_v)} = -F^{\mu\nu}. \quad (1.168)$$

Then sticking (1.163) and (1.168) into (1.161) yields precisely (1.96):

$$\partial_\mu F^{\nu\mu} = J^\nu. \quad (1.169)$$

Note that we switched the order of the indices on  $F^{\mu\nu}$  in order to save ourselves from an unpleasant minus sign.

You may wonder what the purpose of introducing a Lagrangian formulation is, if we were able to invent the equations of motion before we ever knew the Lagrangian (as Maxwell did for his equations). There are a number of reasons,

starting with the basic simplicity of positing a single scalar function of spacetime, the Lagrange density, rather than a number of (perhaps tensor-valued) equations of motion. Another reason is the ease with which symmetries are implemented; demanding that the action be invariant under a symmetry ensures that the dynamics respects the symmetry as well. Finally, as we will see in Chapter 4, the action leads via a direct procedure (involving varying with respect to the metric itself) to a unique energy-momentum tensor. Applying this procedure to (1.148) leads straight to the energy-momentum tensor for a scalar field theory,

$$T_{\text{scalar}}^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\sigma}\partial_\lambda\phi\partial_\sigma\phi - \eta^{\mu\nu}\left[\frac{1}{2}\eta^{\lambda\sigma}\partial_\lambda\phi\partial_\sigma\phi + V(\phi)\right]. \quad (1.170)$$

Similarly, from (1.162) we can derive the energy-momentum tensor for electromagnetism,

$$T_{\text{EM}}^{\mu\nu} = F^{\mu\lambda}F^\nu{}_\lambda - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}. \quad (1.171)$$

Using the appropriate equations of motion, you can show that these energy-momentum tensors are conserved,  $\partial_\mu T^{\mu\nu} = 0$  (and will be asked to do so in the Exercises).

The two examples we have considered—scalar field theory and electromagnetism—are paradigms for much of our current understanding of nature. The Standard Model of particle physics consists of three types of fields: gauge fields, Higgs fields, and fermions. The gauge fields describe the “forces” of nature, including the strong and weak nuclear forces in addition to electromagnetism. The gauge fields giving rise to the nuclear forces are described by one-form potentials, just as in electromagnetism; the difference is that they are matrix-valued rather than ordinary one-forms, and the symmetry groups corresponding to gauge transformations are therefore noncommutative (nonabelian) symmetries. The Higgs fields are scalar fields much as we have described, although they are also matrix-valued. The fermions include leptons (such as electrons and neutrinos) and quarks, and are not described by any of the tensor fields we have discussed here, but rather by a different kind of field called a **spinor**. We won’t get around to discussing spinors in this book, but they play a crucial role in particle physics and their coupling to gravity is interesting and subtle. Upon quantization, these fields give rise to particles of different spins; gauge fields are spin-1, scalar fields are spin-0, and the Standard Model fermions are spin- $\frac{1}{2}$ .

Before concluding this chapter, let’s ask an embarrassingly simple question: Why should we consider one classical field theory rather than some other one? More concretely, let’s say that we have discovered some particle in nature, and we know what kind of field we want to use to describe it; how should we pick the Lagrangian for this field? For example, when we wrote down our scalar-field Lagrangian (1.148), why didn’t we include a term of the form

$$\mathcal{L}' = \lambda\phi^2\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi), \quad (1.172)$$

where  $\lambda$  is a coupling constant? Ultimately, of course, we work by trial and error and try to fit the data given to us by experiment. In classical field theory, there's not much more we could do; generally we would start with a simple Lagrangian, and perhaps make it more complicated if the first try failed to agree with the data. But quantum field theory actually provides some simple guidelines, and since we use classical field theory as an approximation to some underlying quantum theory, it makes sense to take advantage of these principles. To make a long story short, quantum field theory allows "virtual" processes at arbitrarily high energies to contribute to what we observe at low energies. Fortunately, the effect of these processes can be summarized in a low-energy **effective field theory**. In the effective theory, which is what we actually observe, the result of high-energy processes is simply to "renormalize" the coupling constants of our theory. Consider an arbitrary coupling constant, which we can express as a parameter  $\mu$  (with dimensions of mass) raised to some power,  $\lambda = \mu^q$  (unless  $\lambda$  is dimensionless, in which case the discussion becomes more subtle). Very roughly speaking, *the effect of high-energy processes will be to make  $\mu$  very large*. Slightly more specifically,  $\mu$  will be pushed up to a scale at which new physics kicks in, whatever that may be. Therefore, potential higher-order terms we might think of adding to a Lagrangian are suppressed, because they are multiplied by coupling constants that are very small. For (1.172), for example, we must have  $\lambda = \mu^{-2}$ , so  $\lambda$  will be tiny (because  $\mu$  will be big). Only the lowest-order terms we can put in our Lagrangian will come with dimensionless couplings (or ones with units of mass to a positive power), so we only need bother with those at low energies. This feature of field theory allows for a dramatic simplification in considering all of the models we might want to examine.

As mentioned at the beginning of this section, general relativity itself is a classical field theory, in which the dynamical field is the metric tensor. It is nevertheless fair to think of GR as somehow different; for the most part other classical field theories rely on the existence of a pre-existing spacetime geometry, whereas in GR the geometry is determined by the equations of motion. (There are exceptions to this idea, called topological field theories, in which the metric makes no appearance.) Our task in the next few chapters is to explore the nature of curved geometries as characterized by the spacetime metric, before moving in Chapter 4 to putting these notions to work in constructing a theory of gravitation.

## 1.11 ■ EXERCISES

1. Consider an inertial frame  $S$  with coordinates  $x^\mu = (t, x, y, z)$ , and a frame  $S'$  with coordinates  $x'^\mu$  related to  $S$  by a boost with velocity parameter  $v$  along the  $y$ -axis. Imagine we have a wall at rest in  $S'$ , lying along the line  $x' = -y'$ . From the point of view of  $S$ , what is the relationship between the incident angle of a ball hitting the wall (traveling in the  $x$ - $y$  plane) and the reflected angle? What about the velocity before and after?

2. Imagine that space (not spacetime) is actually a finite box, or in more sophisticated terms, a three-torus, of size  $L$ . By this we mean that there is a coordinate system  $x^\mu = (t, x, y, z)$  such that every point with coordinates  $(t, x, y, z)$  is *identified* with every point with coordinates  $(t, x + L, y, z)$ ,  $(t, x, y + L, z)$ , and  $(t, x, y, z + L)$ . Note that the time coordinate is the same. Now consider two observers; observer  $A$  is at rest in this coordinate system (constant spatial coordinates), while observer  $B$  moves in the  $x$ -direction with constant velocity  $v$ .  $A$  and  $B$  begin at the same event, and while  $A$  remains still,  $B$  moves once around the universe and comes back to intersect the worldline of  $A$  without ever having to accelerate (since the universe is periodic). What are the relative proper times experienced in this interval by  $A$  and  $B$ ? Is this consistent with your understanding of Lorentz invariance?
3. Three events,  $A, B, C$ , are seen by observer  $\mathcal{O}$  to occur in the order  $ABC$ . Another observer,  $\tilde{\mathcal{O}}$ , sees the events to occur in the order  $CBA$ . Is it possible that a third observer sees the events in the order  $ACB$ ? Support your conclusion by drawing a spacetime diagram.
4. Projection effects can trick you into thinking that an astrophysical object is moving “superluminally.” Consider a quasar that ejects gas with speed  $v$  at an angle  $\theta$  with respect to the line-of-sight of the observer. Projected onto the sky, the gas appears to travel perpendicular to the line of sight with angular speed  $v_{\text{app}}/D$ , where  $D$  is the distance to the quasar and  $v_{\text{app}}$  is the apparent speed. Derive an expression for  $v_{\text{app}}$  in terms of  $v$  and  $\theta$ . Show that, for appropriate values of  $v$  and  $\theta$ ,  $v_{\text{app}}$  can be greater than 1.
5. Particle physicists are so used to setting  $c = 1$  that they measure mass in units of energy. In particular, they tend to use electron volts ( $1 \text{ eV} = 1.6 \times 10^{-12} \text{ erg} = 1.8 \times 10^{-33} \text{ g}$ ), or, more commonly, keV, MeV, and GeV ( $10^3 \text{ eV}$ ,  $10^6 \text{ eV}$ , and  $10^9 \text{ eV}$ , respectively). The muon has been measured to have a mass of  $0.106 \text{ GeV}$  and a rest frame lifetime of  $2.19 \times 10^{-6}$  seconds. Imagine that such a muon is moving in the circular storage ring of a particle accelerator, 1 kilometer in diameter, such that the muon’s total energy is  $1000 \text{ GeV}$ . How long would it appear to live from the experimenter’s point of view? How many radians would it travel around the ring?
6. In Euclidean three-space, let  $p$  be the point with coordinates  $(x, y, z) = (1, 0, -1)$ . Consider the following curves that pass through  $p$ :

$$x^i(\lambda) = (\lambda, (\lambda - 1)^2, -\lambda)$$

$$x^i(\mu) = (\cos \mu, \sin \mu, \mu - 1)$$

$$x^i(\sigma) = (\sigma^2, \sigma^3 + \sigma^2, \sigma).$$

- (a) Calculate the components of the tangent vectors to these curves at  $p$  in the coordinate basis  $\{\partial_x, \partial_y, \partial_z\}$ .
- (b) Let  $f = x^2 + y^2 - yz$ . Calculate  $df/d\lambda$ ,  $df/d\mu$  and  $df/d\sigma$ .
7. Imagine we have a tensor  $X^{\mu\nu}$  and a vector  $V^\mu$ , with components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \quad V^\mu = (-1, 2, 0, -2).$$

Find the components of:

- (a)  $X^\mu{}_\nu$
- (b)  $X_\mu{}^\nu$
- (c)  $X^{(\mu\nu)}$
- (d)  $X_{[\mu\nu]}$
- (e)  $X^\lambda{}_\lambda$
- (f)  $V^\mu V_\mu$
- (g)  $V_\mu X^{\mu\nu}$

8. If  $\partial_\nu T^{\mu\nu} = Q^\mu$ , what physically does the spatial vector  $Q^i$  represent? Use the dust energy momentum tensor to make your case.
9. For a system of discrete point particles the energy-momentum tensor takes the form

$$T_{\mu\nu} = \sum_a \frac{p_\mu^{(a)} p_\nu^{(a)}}{p^{0(a)}} \delta^{(3)}(\mathbf{x} - \mathbf{x}^{(a)}), \quad (1.173)$$

where the index  $a$  labels the different particles. Show that, for a dense collection of particles with isotropically distributed velocities, we can smooth over the individual particle worldlines to obtain the perfect-fluid energy-momentum tensor (1.114).

10. Using the tensor transformation law applied to  $F_{\mu\nu}$ , show how the electric and magnetic field 3-vectors  $\mathbf{E}$  and  $\mathbf{B}$  transform under
  - (a) a rotation about the  $y$ -axis,
  - (b) a boost along the  $z$ -axis.
11. Verify that (1.98) is indeed equivalent to (1.97), and that they are both equivalent to the last two equations in (1.93).
12. Consider the two field theories we explicitly discussed, Maxwell's electromagnetism (let  $J^\mu = 0$ ) and the scalar field theory defined by (1.148).
  - (a) Express the components of the energy-momentum tensors of each theory in three-vector notation, using the divergence, gradient, curl, electric, and magnetic fields, and an overdot to denote time derivatives.
  - (b) Using the equations of motion, verify (in any notation you like) that the energy-momentum tensors are conserved.
13. Consider adding to the Lagrangian for electromagnetism an additional term of the form  $\mathcal{L}' = \tilde{\epsilon}_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$ .
  - (a) Express  $\mathcal{L}'$  in terms of  $\mathbf{E}$  and  $\mathbf{B}$ .
  - (b) Show that including  $\mathcal{L}'$  does not affect Maxwell's equations. Can you think of a deep reason for this?

## CHAPTER

## 2

## Manifolds

## 2.1 ■ GRAVITY AS GEOMETRY

Gravity is special. In the context of general relativity, we ascribe this specialness to the fact that the dynamical field giving rise to gravitation is the metric tensor describing the curvature of spacetime itself, rather than some additional field propagating through spacetime; this was Einstein's profound insight. The physical principle that led him to this idea was the *universality* of the gravitational interaction, as formalized by the **Principle of Equivalence**. Let's see how this physical principle leads us to the mathematical strategy of describing gravity as the geometry of a curved manifold.

The Principle of Equivalence comes in a variety of forms, the first of which is the **Weak Equivalence Principle**, or WEP. The WEP states that the inertial mass and gravitational mass of any object are equal. To see what this means, think about Newtonian mechanics. The Second Law relates the force exerted on an object to the acceleration it undergoes, setting them proportional to each other with the constant of proportionality being the inertial mass  $m_i$ :

$$\mathbf{F} = m_i \mathbf{a}. \quad (2.1)$$

The inertial mass clearly has a universal character, related to the resistance you feel when you try to push on the object; it takes the same value no matter what kind of force is being exerted. We also have Newton's law of gravitation, which can be thought of as stating that the gravitational force exerted on an object is proportional to the gradient of a scalar field  $\Phi$ , known as the gravitational potential. The constant of proportionality in this case is called the gravitational mass  $m_g$ :

$$\mathbf{F}_g = -m_g \nabla \Phi. \quad (2.2)$$

On the face of it,  $m_g$  has a very different character than  $m_i$ ; it is a quantity specific to the gravitational force. If you like,  $m_g/m_i$  can be thought of as the "gravitational charge" of the body. Nevertheless, Galileo long ago showed (apocryphally by dropping weights off of the Leaning Tower of Pisa, actually by rolling balls down inclined planes) that the response of matter to gravitation is universal—every object falls at the same rate in a gravitational field, independent of the composition of the object. In Newtonian mechanics this translates into the WEP, which is simply

$$m_i = m_g \quad (2.3)$$

for any object. An immediate consequence is that the behavior of freely-falling test particles is universal, independent of their mass (or any other qualities they may have); in fact, we have

$$\mathbf{a} = -\nabla\Phi. \quad (2.4)$$

Experimentally, the independence of the acceleration due to gravity on the composition of the falling object has been verified to extremely high precision by the Eötvös experiment and its modern successors.

This suggests an equivalent formulation of the WEP: there exists a preferred class of trajectories through spacetime, known as *inertial* (or “freely-falling”) trajectories, on which unaccelerated particles travel—where unaccelerated means “subject only to gravity.” Clearly this is not true for other forces, such as electromagnetism. In the presence of an electric field, particles with opposite charges will move on quite different trajectories. Every particle, on the other hand, has an identical gravitational charge.

The universality of gravitation, as implied by the WEP, can be stated in another, more popular, form. Imagine that we consider a physicist in a tightly sealed box, unable to observe the outside world, who is doing experiments involving the motion of test particles, for example to measure the local gravitational field. Of course she would obtain different answers if the box were sitting on the moon or on Jupiter than she would on Earth. But the answers would also be different if the box were accelerating at a constant rate; this would change the acceleration of the freely-falling particles with respect to the box. The WEP implies that there is no way to disentangle the effects of a gravitational field from those of being in a uniformly accelerating frame, simply by observing the behavior of freely-falling particles. This follows from the universality of gravitation; in electrodynamics, in contrast, it would be possible to distinguish between uniform acceleration and an electromagnetic field, by observing the behavior of particles with different charges. But with gravity it is impossible, since the “charge” is necessarily proportional to the (inertial) mass.

To be careful, we should limit our claims about the impossibility of distinguishing gravity from uniform acceleration by restricting our attention to “small enough regions of spacetime.” If the sealed box were sufficiently big, the gravitational field would change from place to place in an observable way, while the effect of acceleration would always be in the same direction. In a rocket ship or elevator, the particles would always fall straight down. In a very big box in a gravitational field, however, the particles would move toward the center of the Earth, for example, which would be a different direction for widely separated experiments. The WEP can therefore be stated as follows: *The motion of freely-falling particles are the same in a gravitational field and a uniformly accelerated frame, in small enough regions of spacetime.* In larger regions of spacetime there will be inhomogeneities in the gravitational field, which will lead to tidal forces, which can be detected.

After the advent of special relativity, the concept of mass lost some of its uniqueness, as it became clear that mass was simply a manifestation of energy

and momentum (as we have seen in Chapter 1). It was therefore natural for Einstein to think about generalizing the WEP to something more inclusive. His idea was simply that there should be no way whatsoever for the physicist in the box to distinguish between uniform acceleration and an external gravitational field, no matter what experiments she did (not only by dropping test particles). This reasonable extrapolation became what is now known as the **Einstein Equivalence Principle**, or EEP: *In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments.*

In fact, it is hard to imagine theories that respect the WEP but violate the EEP. Consider a hydrogen atom, a bound state of a proton and an electron. Its mass is actually less than the sum of the masses of the proton and electron considered individually, because there is a negative binding energy—you have to put energy into the atom to separate the proton and electron. According to the WEP, the gravitational mass of the hydrogen atom is therefore less than the sum of the masses of its constituents; the gravitational field couples to electromagnetism (which holds the atom together) in exactly the right way to make the gravitational mass come out right. This means that not only must gravity couple to rest mass universally, but also to all forms of energy and momentum—which is practically the claim of the EEP. It is possible to come up with counterexamples, however; for example, we could imagine a theory of gravity in which freely falling particles began to rotate as they moved through a gravitational field. Then they could fall along the same paths as they would in an accelerated frame (thereby satisfying the WEP), but you could nevertheless detect the existence of the gravitational field (in violation of the EEP). Such theories seem contrived, but there is no law of nature that forbids them.

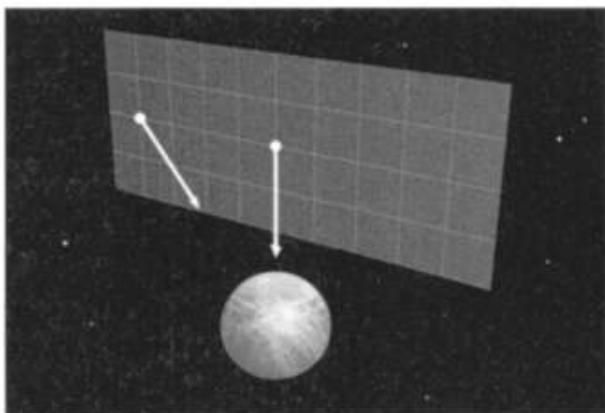
Sometimes a distinction is drawn between “gravitational laws of physics” and “nongravitational laws of physics,” and the EEP is defined to apply only to the latter. Then the Strong Equivalence Principle (SEP) is defined to include all of the laws of physics, gravitational and otherwise. A theory that violated the SEP but not the EEP would be one in which the *gravitational* binding energy did not contribute equally to the inertial and gravitational mass of a body; thus, for example, test particles with appreciable self-gravity (to the extent that such a concept makes sense) could fall along different trajectories than lighter particles.

It is the EEP that implies (or at least suggests) that we should attribute the action of gravity to the curvature of spacetime. Remember that in special relativity a prominent role is played by inertial frames—while it is not possible to single out some frame of reference as uniquely “at rest,” it is possible to single out a family of frames that are “unaccelerated” (inertial). The acceleration of a charged particle in an electromagnetic field is therefore uniquely defined with respect to these frames. The EEP, on the other hand, implies that gravity is inescapable—there is no such thing as a “gravitationally neutral object” with respect to which we can measure the acceleration due to gravity. It follows that the acceleration due to gravity is not something that can be reliably defined, and therefore is of little use.

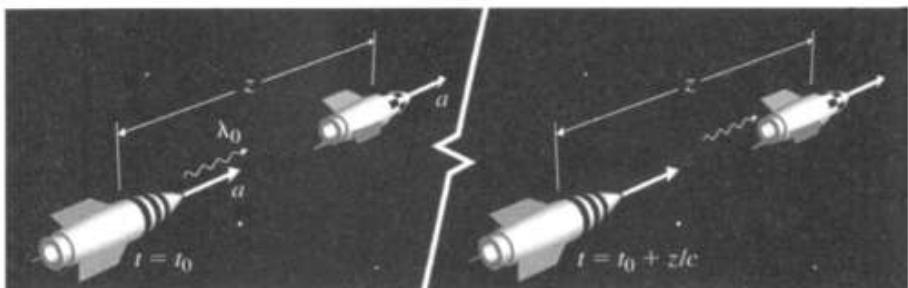
Instead, it makes more sense to *define* “unaccelerated” as “freely falling,” and that is what we shall do. From here we are led to the idea that gravity is not a “force”—a force is something that leads to acceleration, and our definition of zero acceleration is “moving freely in the presence of whatever gravitational field happens to be around.”

This seemingly innocuous step has profound implications for the nature of spacetime. In SR, we have a procedure for starting at some point and constructing an inertial frame that stretches throughout spacetime, by joining together rigid rods and attaching clocks to them. But, again due to inhomogeneities in the gravitational field, this is no longer possible. If we start in some freely-falling state and build a large structure out of rigid rods, at some distance away freely-falling objects will look like they are accelerating with respect to this reference frame, as shown in Figure 2.1. The solution is to retain the notion of inertial frames, but to discard the hope that they can be uniquely extended throughout space and time. Instead we can define **locally inertial frames**, those that follow the motion of individual freely falling particles in small enough regions of spacetime. (Every time we say “small enough regions,” purists should imagine a limiting procedure in which we take the appropriate spacetime volume to zero.) This is the best we can do, but it forces us to give up a good deal. For example, we can no longer speak with confidence about the relative velocity of far-away objects, since the inertial reference frames appropriate to those objects are completely different from those appropriate to us.

Our job as physicists is to construct mathematical models of the world, and then test the predictions of such models against observations and experiments. Following the implications of the universality of gravitation has led us to give up on the idea of expressing gravity as a force propagating through spacetime,



**FIGURE 2.1** Failure of global frames. Since every particle feels the influence of gravity, we define “unaccelerating” as “freely falling.” As a consequence, it becomes impossible to define globally inertial coordinate systems by the procedure outlined in Chapter 1, since particles initially at rest will begin to move with respect to such a frame.



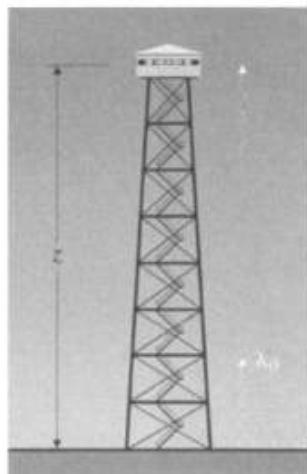
**FIGURE 2.2** The Doppler shift as measured by two rockets separated by a distance  $z$ , each feeling an acceleration  $a$ .

and indeed to give up on the idea of global reference frames stretching throughout spacetime. We therefore need to invoke a mathematical framework in which physical theories can be consistent with these conclusions. The solution will be to imagine that spacetime has a curved geometry, and that gravitation is a manifestation of this curvature. The appropriate mathematical structure used to describe curvature is that of a *differentiable manifold*: essentially, a kind of set that looks locally like flat space, but might have a very different global geometry. (Remember that the EEP can be stated as “the laws of physics reduce to those of special relativity in small regions of spacetime,” which matches well with the mathematical notion of a set that locally resembles flat space.)

We cannot prove that gravity should be thought of as the curvature of spacetime; instead we can propose the idea, derive its consequences, and see if the result is a reasonable fit to our experience of the world. Let’s set about doing just that.

Consider one of the celebrated predictions of the EEP, the gravitational redshift. Imagine two rockets, a distance  $z$  apart, each moving with some constant acceleration  $a$  in a region far away from any gravitational fields, as shown in Figure 2.2. At time  $t_0$  the trailing rocket emits a photon of wavelength  $\lambda_0$ . The rockets remain a constant distance apart, so the photon reaches the leading rocket after a time  $\Delta t = z/c$  in our background reference frame. (We assume  $\Delta v/c$  is small, so we only work to first order.) In this time the rockets will have picked up an additional velocity  $\Delta v = a\Delta t = az/c$ . Therefore, the photon reaching the leading rocket will be redshifted by the conventional Doppler effect, by an amount

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{az}{c^2}. \quad (2.5)$$



**FIGURE 2.3** Gravitational redshift on the surface of the Earth, as measured by observers at different elevations.

According to the EEP, the same thing should happen in a uniform gravitational field. So we imagine a tower of height  $z$  sitting on the surface of a planet, with  $a_g$  the strength of the gravitational field (what Newton would have called the “acceleration due to gravity”), as portrayed in Figure 2.3. We imagine that observers in the rocket at the top of the tower are able to detect photons emitted from the ground, but are otherwise unable to look outside and see that they are sitting on a

tower. In other words, they have no way of distinguishing this situation from that of the accelerating rockets. Therefore, the EEP allows us to conclude immediately that a photon emitted from the ground with wavelength  $\lambda_0$  will be redshifted by an amount

$$\frac{\Delta\lambda}{\lambda_0} = \frac{a_g z}{c^2}. \quad (2.6)$$

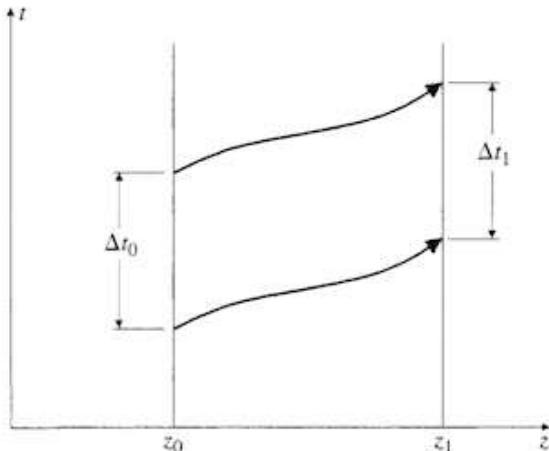
This is the famous gravitational redshift. Notice that it is a direct consequence of the EEP; the details of general relativity were not required.

The formula for the redshift is more often stated in terms of the Newtonian potential  $\Phi$ , where  $\mathbf{a}_g = \nabla\Phi$ . (The sign is changed with respect to the usual convention, since we are thinking of  $\mathbf{a}_g$  as the acceleration of the reference frame, not of a particle with respect to this reference frame.) A nonconstant gradient of  $\Phi$  is like a time-varying acceleration, and the equivalent net velocity is given by integrating over the time between emission and absorption of the photon. We then have

$$\begin{aligned} \frac{\Delta\lambda}{\lambda_0} &= \frac{1}{c} \int \nabla\Phi \, dt \\ &= \frac{1}{c^2} \int \partial_z \Phi \, dz \\ &= \Delta\Phi, \end{aligned} \quad (2.7)$$

where  $\Delta\Phi$  is the total change in the gravitational potential, and we have once again set  $c = 1$ . This simple formula for the gravitational redshift continues to be true in more general circumstances. Of course, by using the Newtonian potential at all, we are restricting our domain of validity to weak gravitational fields.

From the EEP we have argued in favor of a gravitational redshift; we may now use this phenomenon to provide further support for the idea that we should think of spacetime as curved. Consider the same experimental setup that we had before, now portrayed on the spacetime diagram in Figure 2.4. A physicist on the ground emits a beam of light with wavelength  $\lambda_0$  from a height  $z_0$ , which travels to the top of the tower at height  $z_1$ . The time between when the beginning of any single wavelength of the light is emitted and the end of that same wavelength is emitted is  $\Delta t_0 = \lambda_0/c$ , and the same time interval for the absorption is  $\Delta t_1 = \lambda_1/c$ , where time is measured by clocks located at the respective elevations. Since we imagine that the gravitational field is static, the paths through spacetime followed by the leading and trailing edge of the single wave must be precisely congruent. (They are represented by generic curved paths, since we do not pretend that we know just what the paths will be.) Simple geometry seems to imply that the times  $\Delta t_0$  and  $\Delta t_1$  must be the same. But of course they are not; the gravitational redshift implies that the elevated experimenters observe fewer wavelengths per second, so that  $\Delta t_1 > \Delta t_0$ . We can interpret this roughly as “the clock on the tower appears to run more quickly.” What went wrong? Simple geometry—the spacetime through which the photons traveled was curved.



**FIGURE 2.4** Spacetime diagram of the gravitational-redshift experiment portrayed in Figure 2.3. Spacetime paths beginning at different moments are congruent, but the time intervals as measured on the ground and on the tower are different, signaling a breakdown of Euclidean geometry.

We therefore would like to describe spacetime as a kind of mathematical structure that looks locally like Minkowski space, but may possess nontrivial curvature over extended regions. The kind of object that encompasses this notion is that of a manifold. In this chapter we will confine ourselves to understanding the concept of manifolds and the structures we may define on them, leaving the precise characterization of curvature for the next chapter.

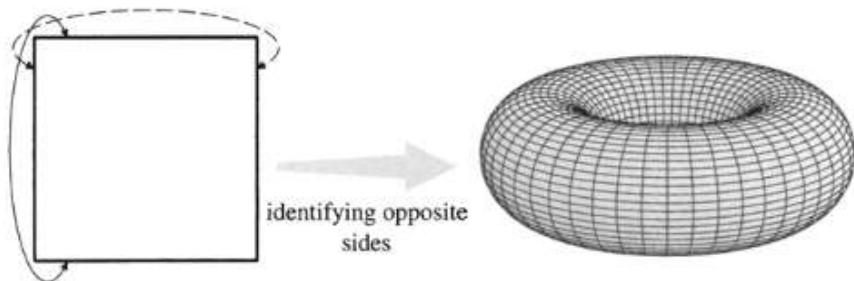
## 2.2 ■ WHAT IS A MANIFOLD?

Manifolds (or differentiable manifolds) are one of the most fundamental concepts in mathematics and physics. We are all used to the properties of  $n$ -dimensional Euclidean space,  $\mathbf{R}^n$ , the set of  $n$ -tuples  $(x^1, \dots, x^n)$ , often equipped with a flat positive-definite metric with components  $\delta_{ij}$ . Mathematicians have worked for many years to develop the theory of analysis in  $\mathbf{R}^n$ —differentiation, integration, properties of functions, and so on. But clearly there are other spaces (spheres, for example) which we intuitively think of as “curved” or perhaps topologically complicated, on which we would like to perform analogous operations.

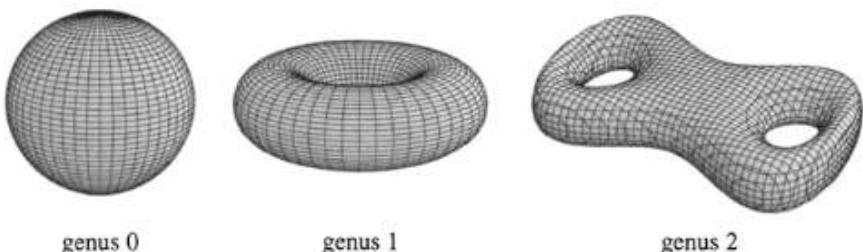
To address this problem we invent the notion of a manifold, which corresponds to a space that may be curved and have a complicated topology, but in local regions looks just like  $\mathbf{R}^n$ . Here by “looks like” we do not mean that the metric is the same, but only that more primitive notions like functions and coordinates work in a similar way. The entire manifold is constructed by smoothly sewing together these local regions. A crucial point is that the dimensionality  $n$  of the Euclidean spaces being used must be the same in every patch of the manifold; we then say

that the manifold is of dimension  $n$ . With this approach we can analyze functions on such a space by converting them (locally) to functions in a Euclidean space. Examples of manifolds include:

- $\mathbf{R}^n$  itself, including the line ( $\mathbf{R}$ ), the plane ( $\mathbf{R}^2$ ), and so on. This should be obvious, since  $\mathbf{R}^n$  looks like  $\mathbf{R}^n$  not only locally but globally.
- The  $n$ -sphere,  $S^n$ . This can be defined as the locus of all points some fixed distance from the origin in  $\mathbf{R}^{n+1}$ . The circle is of course  $S^1$ , and the two-sphere  $S^2$  is one of the most useful examples of a manifold. (The zero-sphere  $S^0$ , if you think about it, consists of two points. We say that  $S^0$  is a disconnected zero-dimensional manifold.) It's worth emphasizing that the definition of  $S^n$  in terms of an embedding in  $\mathbf{R}^{n+1}$  is simply a convenient shortcut; all of the manifolds we will discuss may be defined in their own right, without recourse to higher-dimensional flat spaces.
- The  $n$ -torus  $T^n$  results from taking an  $n$ -dimensional cube and identifying opposite sides. The two-torus  $T^2$  is a square with opposite sides identified, as shown in Figure 2.5. The surface of a doughnut is a familiar example.
- A Riemann surface of genus  $g$  is essentially a two-torus with  $g$  holes instead of just one, as shown in Figure 2.6.  $S^2$  may be thought of as a Riemann surface of genus zero. In technical terms (not really relevant to our present dis-



**FIGURE 2.5** The torus,  $T^2$ , constructed by identifying opposite sides of a square.

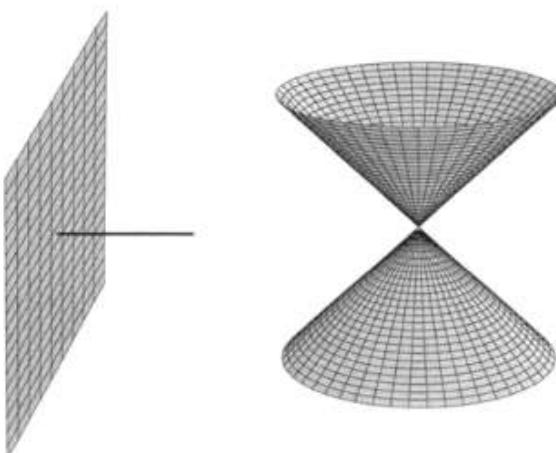


**FIGURE 2.6** Riemann surfaces of different genera (plural of "genus").

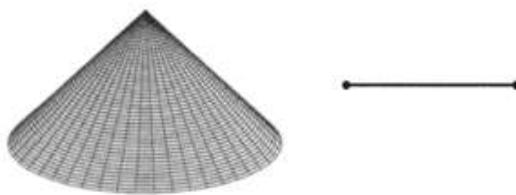
cussion), every “compact orientable boundaryless” two-dimensional manifold is a Riemann surface of some genus.

- More abstractly, a set of continuous transformations such as rotations in  $\mathbf{R}^n$  forms a manifold. Lie groups are manifolds that also have a group structure. So for example  $\text{SO}(2)$ , the set of rotations in two dimensions, is the same manifold as  $S^1$  (although in general group manifolds will be more complicated than spheres).
- The direct product of two manifolds is a manifold. That is, given manifolds  $M$  and  $M'$  of dimension  $n$  and  $n'$ , we can construct a manifold  $M \times M'$ , of dimension  $n + n'$ , consisting of ordered pairs  $(p, p')$  with  $p \in M$  and  $p' \in M'$ .

With all of these examples, the notion of a manifold may seem vacuous: what *isn't* a manifold? Plenty of things are not manifolds, because somewhere they do not look locally like  $\mathbf{R}^n$ . Examples include a one-dimensional line running into a two-dimensional plane, and two cones stuck together at their vertices, as portrayed in Figure 2.7. More subtle examples are shown in Figure 2.8. Consider for example a single (two-dimensional) cone. There is clearly a sense in which the cone looks locally like  $\mathbf{R}^2$ ; at the same time, there is just as clearly something singular about the vertex of the cone. This is where the word “differentiable” in “differentiable manifold” begins to play a role; as we will see when we develop the formal definition, the cone is perfectly smooth as a manifold, even though the curvature is not smooth at its vertex. (Other types of singularities are more severe, and will prevent us from thinking of certain spaces as manifolds, smooth or otherwise.) Another example is a line segment (with endpoints included). This



**FIGURE 2.7** Examples of spaces that are not manifolds: a line ending on a plane, and two cones intersecting at their vertices. In each case there is a point that does not look locally like a Euclidean space of fixed dimension.

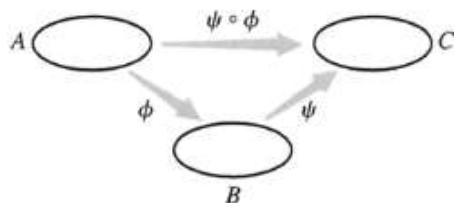


**FIGURE 2.8** Subtle examples. The single cone is a smooth manifold, even though the curvature is singular at its vertex. A line segment is not a manifold, but may be described by the more general notion of “manifold with boundary.”

certainly will not fit under the definition of manifolds we will develop, due to the endpoints. Nevertheless, we can extend the definition to include “manifolds with boundary,” of which the line segment is a paradigmatic example. A brief discussion of manifolds with boundary is in Appendix D.

These subtle cases should convince you of the need for a rigorous definition, which we now begin to construct; our discussion follows that of Wald (1984). The informal idea of a manifold is that of a space consisting of patches that look locally like  $\mathbf{R}^n$ , and are smoothly sewn together. We therefore need to formalize the notions of “looking locally like  $\mathbf{R}^n$ ” and “smoothly sewn together.” We require a number of preliminary definitions, most of which are fairly clear, but it’s nice to be complete. The most elementary notion is that of a **map** between two sets. (We assume you know what a set is, or think you do; we won’t need to be too precise.) Given two sets  $M$  and  $N$ , a map  $\phi : M \rightarrow N$  is a relationship that assigns, to each element of  $M$ , exactly one element of  $N$ . A map is therefore just a simple generalization of a function. Given two maps  $\phi : A \rightarrow B$  and  $\psi : B \rightarrow C$ , we define the **composition**  $\psi \circ \phi : A \rightarrow C$  by the operation  $(\psi \circ \phi)(a) = \psi(\phi(a))$ , as in Figure 2.9. So  $a \in A$ ,  $\phi(a) \in B$ , and thus  $(\psi \circ \phi)(a) \in C$ . The order in which the maps are written makes sense, since the one on the right acts first.

A map  $\phi$  is called **one-to-one** (or injective) if each element of  $N$  has at most one element of  $M$  mapped into it, and **onto** (or surjective) if each element of  $N$  has at least one element of  $M$  mapped into it. (If you think about it, better names for “one-to-one” would be “one-from-one” or for that matter “two-to-two.”) Consider functions  $\phi : \mathbf{R} \rightarrow \mathbf{R}$ . Then  $\phi(x) = e^x$  is one-to-one, but not onto;  $\phi(x) = x^3 - x$  is onto, but not one-to-one;  $\phi(x) = x^3$  is both; and  $\phi(x) = x^2$  is neither, as in Figure 2.10.



**FIGURE 2.9** The map  $\psi \circ \phi : A \rightarrow C$  is formed by composing  $\phi : A \rightarrow B$  and  $\psi : B \rightarrow C$ .

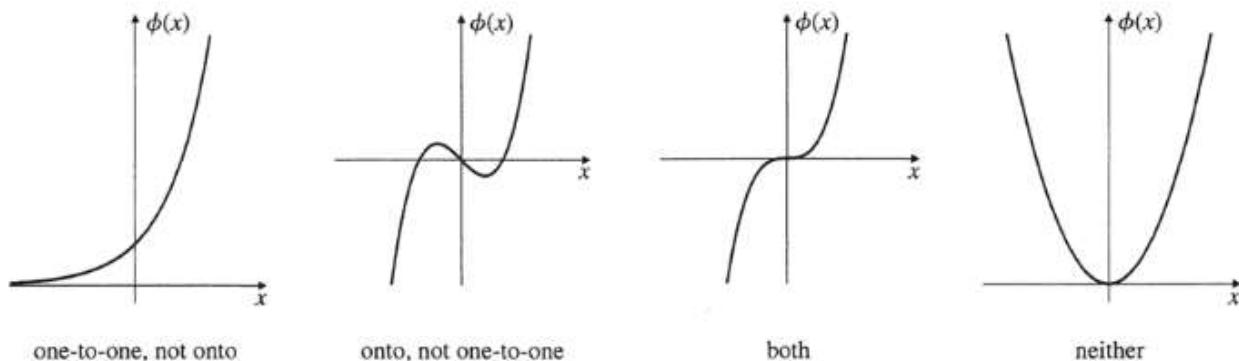


FIGURE 2.10 Types of maps.

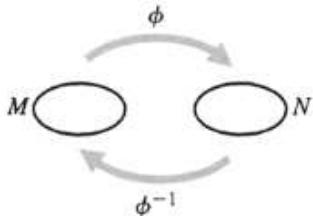


FIGURE 2.11 A map and its inverse.

The set  $M$  is known as the **domain** of the map  $\phi$ , and the set of points in  $N$  that  $M$  gets mapped into is called the **image** of  $\phi$ . For any subset  $U \subset N$ , the set of elements of  $M$  that get mapped to  $U$  is called the **preimage** of  $U$  under  $\phi$ , or  $\phi^{-1}(U)$ . A map that is both one-to-one and onto is known as **invertible** (or bijective). In this case we can define the **inverse map**  $\phi^{-1} : N \rightarrow M$  by  $(\phi^{-1} \circ \phi)(a) = a$ , as in Figure 2.11. Note that the same symbol  $\phi^{-1}$  is used for both the preimage and the inverse map, even though the former is always defined and the latter is only defined in some special cases.

The notion of **continuity** of a map is actually a very subtle one, the precise formulation of which we won't need. Instead we will assume you understand the concepts of continuity and differentiability as applied to ordinary functions, maps  $\phi : \mathbf{R} \rightarrow \mathbf{R}$ . It will then be useful to extend these notions to maps between more general Euclidean spaces,  $\phi : \mathbf{R}^m \rightarrow \mathbf{R}^n$ . A map from  $\mathbf{R}^m$  to  $\mathbf{R}^n$  takes an  $m$ -tuple  $(x^1, x^2, \dots, x^m)$  to an  $n$ -tuple  $(y^1, y^2, \dots, y^n)$ , and can therefore be thought of as a collection of  $n$  functions  $\phi^i$  of  $m$  variables:

$$\begin{aligned} y^1 &= \phi^1(x^1, x^2, \dots, x^m) \\ y^2 &= \phi^2(x^1, x^2, \dots, x^m) \\ &\vdots \\ y^n &= \phi^n(x^1, x^2, \dots, x^m). \end{aligned} \tag{2.8}$$

We will refer to any one of these functions as  $C^p$  if its  $p$ th derivative exists and is continuous, and refer to the entire map  $\phi : \mathbf{R}^m \rightarrow \mathbf{R}^n$  as  $C^p$  if each of its component functions are at least  $C^p$ . Thus a  $C^0$  map is continuous but not necessarily differentiable, while a  $C^\infty$  map is continuous and can be differentiated as many times as you like. Consider for example the function of one variable  $\phi(x) = |x^3|$ . This function is infinitely differentiable everywhere except at  $x = 0$ , where it is differentiable twice but not three times; we therefore say that it is  $C^2$ .  $C^\infty$  maps are sometimes called **smooth**.

We will call two sets  $M$  and  $N$  **diffeomorphic** if there exists a  $C^\infty$  map  $\phi : M \rightarrow N$  with a  $C^\infty$  inverse  $\phi^{-1} : N \rightarrow M$ ; the map  $\phi$  is then called a diffeomorphism. This is the best notion we have that two spaces are “the same” as manifolds. For example, when we said that  $\text{SO}(2)$  was the same manifold as  $S^1$ , we meant they were diffeomorphic. See Appendix B for more discussion.

These basic definitions may have been familiar to you, even if only vaguely remembered. We will now put them to use in the rigorous definition of a manifold. Unfortunately, a somewhat baroque procedure is required to formalize this relatively intuitive notion. We will first have to define the notion of an open set, on which we can put coordinate systems, and then sew the open sets together in an appropriate way.

We start with the notion of an **open ball**, which is the set of all points  $x$  in  $\mathbf{R}^n$  such that  $|x - y| < r$  for some fixed  $y \in \mathbf{R}^n$  and  $r \in \mathbf{R}$ , where  $|x - y| = [\sum_i (x^i - y^i)^2]^{1/2}$ . Note that this is a strict inequality—the open ball is the interior of an  $n$ -sphere of radius  $r$  centered at  $y$ , as shown in Figure 2.12. An **open set** in  $\mathbf{R}^n$  is a set constructed from an arbitrary (maybe infinite) union of open balls. In other words,  $V \subset \mathbf{R}^n$  is open if, for any  $y \in V$ , there is an open ball centered at  $y$  that is completely inside  $V$ . Roughly speaking, an open set is the interior of some  $(n - 1)$ -dimensional closed surface (or the union of several such interiors).

A **chart** or **coordinate system** consists of a subset  $U$  of a set  $M$ , along with a one-to-one map  $\phi : U \rightarrow \mathbf{R}^n$ , such that the image  $\phi(U)$  is open in  $\mathbf{R}^n$ , as in Figure 2.13. (Any map is onto its image, so the map  $\phi : U \rightarrow \phi(U)$  is invertible if it is one-to-one.) We then can say that  $U$  is an open set in  $M$ . A  $C^\infty$  **atlas** is an indexed collection of charts  $\{(U_\alpha, \phi_\alpha)\}$  that satisfies two conditions:

1. The union of the  $U_\alpha$  is equal to  $M$ ; that is, the  $U_\alpha$  cover  $M$ .
2. The charts are smoothly sewn together. More precisely, if two charts overlap,  $U_\alpha \cap U_\beta \neq \emptyset$ , then the map  $(\phi_\alpha \circ \phi_\beta^{-1})$  takes points in  $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbf{R}^n$  onto an open set  $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbf{R}^n$ , and all of these maps must be  $C^\infty$  where they are defined. This should be clearer from Figure 2.14, adapted from Wald (1984).

So a chart is what we normally think of as a coordinate system on some open set, and an atlas is a system of charts that are smoothly related on their overlaps.

At long last, then: a  $C^\infty$   $n$ -dimensional **manifold** (or  $n$ -manifold for short) is simply a set  $M$  along with a maximal atlas, one that contains every possible

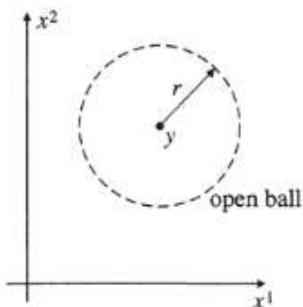


FIGURE 2.12 An open ball defined in  $\mathbf{R}^n$ .

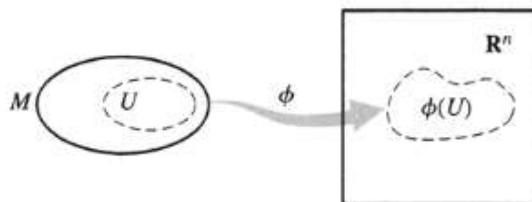


FIGURE 2.13 A coordinate chart covering an open subset  $U$  of  $M$ .

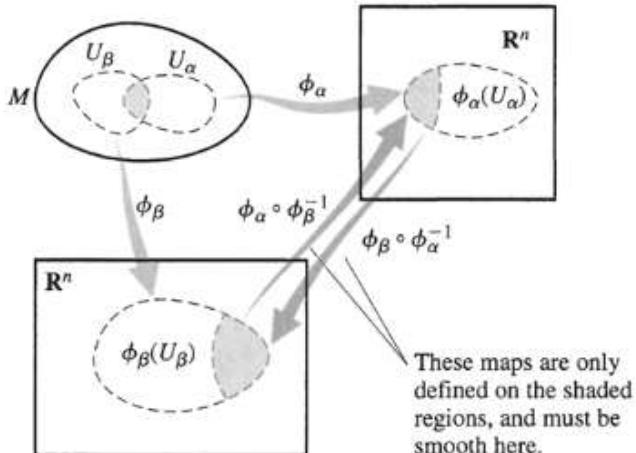
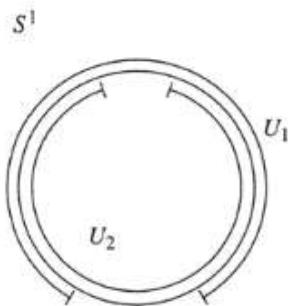


FIGURE 2.14 Overlapping coordinate charts.

compatible chart. We can also replace  $C^\infty$  by  $C^p$  in all the above definitions. For our purposes the degree of differentiability of a manifold is not crucial; we will always assume that any manifold is as differentiable as necessary for the application under consideration. The requirement that the atlas be maximal is so that two equivalent spaces equipped with different atlases don't count as different manifolds. This definition captures in formal terms our notion of a set that looks locally like  $\mathbb{R}^n$ . Of course we will rarely have to make use of the full power of the definition, but precision is its own reward.

One nice thing about our definition is that it does not rely on an embedding of the manifold in some higher-dimensional Euclidean space. In fact, any  $n$ -dimensional manifold can be embedded in  $\mathbb{R}^{2n}$  (Whitney's embedding theorem), and sometimes we will make use of this fact, such as in our definition of the sphere above. (A Klein bottle is an example of a 2-manifold that cannot be embedded in  $\mathbb{R}^3$ , although it can be embedded in  $\mathbb{R}^4$ .) But it is important to recognize that the manifold has an individual existence independent of any embedding. It is not necessary to believe, for example, that four-dimensional spacetime is stuck in some larger space. On the other hand, it might be; we really don't know. Recent advances in string theory have led to the suggestion that our visible universe is actually a “brane” (generalization of “membrane”) inside a higher-dimensional space. But as far as classical GR is concerned, the four-dimensional view is perfectly adequate.

Why was it necessary to be so finicky about charts and their overlaps, rather than just covering every manifold with a single chart? Because most manifolds cannot be covered with just one chart. Consider the simplest example,  $S^1$ . There is a conventional coordinate system,  $\theta : S^1 \rightarrow \mathbb{R}$ , where  $\theta = 0$  at the top of the circle and wraps around to  $2\pi$ . However, in the definition of a chart we have required that the image  $\theta(S^1)$  be open in  $\mathbb{R}$ . If we include either  $\theta = 0$  or  $\theta = 2\pi$ , we have a closed interval rather than an open one; if we exclude both points, we



**FIGURE 2.15** Two coordinate charts, which together cover  $S^1$ .

haven't covered the whole circle. So we need at least two charts, as shown in Figure 2.15.

A somewhat more complicated example is provided by  $S^2$ , where once again no single chart will cover the manifold. A Mercator projection, traditionally used for world maps, misses both the North and South poles (as well as the International Date Line, which involves the same problem with  $\theta$  that we found for  $S^1$ .) Let's take  $S^2$  to be the set of points in  $\mathbf{R}^3$  defined by  $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$ . We can construct a chart from an open set  $U_1$ , defined to be the sphere minus the north pole, via stereographic projection, illustrated in Figure 2.16. Thus, we draw a straight line from the north pole to the plane defined by  $x^3 = -1$ , and assign to the point on  $S^2$  intercepted by the line the Cartesian coordinates  $(y^1, y^2)$  of the appropriate point on the plane. Explicitly, the map is given by

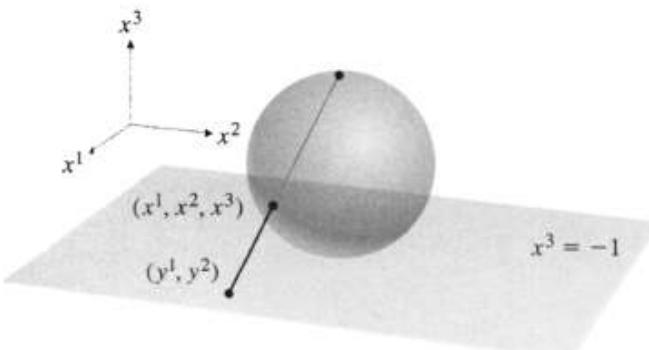
$$\phi_1(x^1, x^2, x^3) \equiv (y^1, y^2) = \left( \frac{2x^1}{1-x^3}, \frac{2x^2}{1-x^3} \right). \quad (2.9)$$

Check this for yourself. Another chart  $(U_2, \phi_2)$  is obtained by projecting from the south pole to the plane defined by  $x^3 = +1$ . The resulting coordinates cover the sphere minus the south pole, and are given by

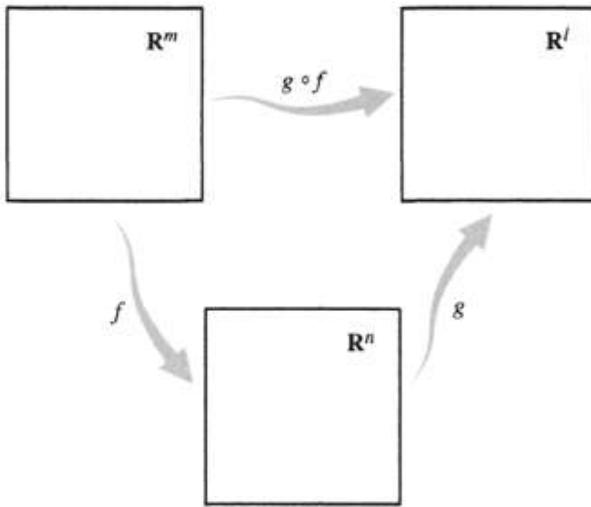
$$\phi_2(x^1, x^2, x^3) \equiv (z^1, z^2) = \left( \frac{2x^1}{1+x^3}, \frac{2x^2}{1+x^3} \right). \quad (2.10)$$

Together, these two charts cover the entire manifold, and they overlap in the region  $-1 < x^3 < +1$ . Another thing you can check is that the composition  $\phi_2 \circ \phi_1^{-1}$  is given by

$$z^i = \frac{4y^i}{[(y^1)^2 + (y^2)^2]}, \quad (2.11)$$



**FIGURE 2.16** Defining a stereographic coordinate chart on  $S^2$  by projecting from the north pole down to a plane tangent to the south pole. Such a chart covers all of the sphere except for the north pole itself.



**FIGURE 2.17** The chain rule relates the partial derivatives of  $g \circ f$  to those of  $g$  and  $f$ .

and is  $C^\infty$  in the region of overlap. As long as we restrict our attention to this region, (2.11) is just what we normally think of as a change of coordinates.

We therefore see the necessity of charts and atlases: Many manifolds cannot be covered with a single coordinate system. Nevertheless, it is often convenient to work with a single chart, and just keep track of the set of points that aren't included.

One piece of conventional calculus that we will need later is the **chain rule**. Let us imagine that we have maps  $f : \mathbf{R}^m \rightarrow \mathbf{R}^n$  and  $g : \mathbf{R}^n \rightarrow \mathbf{R}^l$ , and therefore the composition  $(g \circ f) : \mathbf{R}^m \rightarrow \mathbf{R}^l$ , as shown in Figure 2.17. We can label points in each space in terms of components:  $x^a$  on  $\mathbf{R}^m$ ,  $y^b$  on  $\mathbf{R}^n$ , and  $z^c$  on  $\mathbf{R}^l$ , where the indices range over the appropriate values. The chain rule relates the partial derivatives of the composition to the partial derivatives of the individual maps:

$$\frac{\partial}{\partial x^a} (g \circ f)^c = \sum_b \frac{\partial f^b}{\partial x^a} \frac{\partial g^c}{\partial y^b}. \quad (2.12)$$

This is usually abbreviated to

$$\frac{\partial}{\partial x^a} = \sum_b \frac{\partial y^b}{\partial x^a} \frac{\partial}{\partial y^b}. \quad (2.13)$$

There is nothing illegal or immoral about using this shorthand form of the chain rule, but you should be able to visualize the maps that underlie the construction. Recall that when  $m = n$ , the determinant of the matrix  $\partial y^b / \partial x^a$  is called the **Jacobian** of the map, and the map is invertible whenever the Jacobian is nonzero.

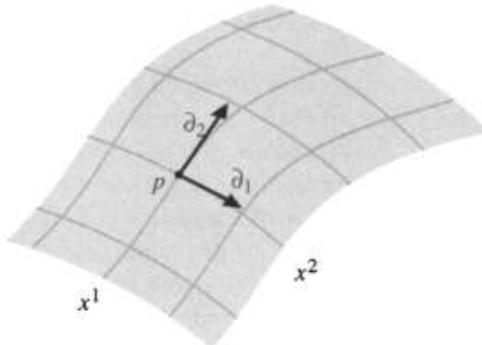
### 2.3 ■ VECTORS AGAIN

Having constructed this groundwork, we can now proceed to introduce various kinds of structure on manifolds. We begin with vectors and tangent spaces. In our discussion of special relativity we were intentionally vague about the definition of vectors and their relationship to the spacetime. One point we stressed was the notion of a tangent space—the set of all vectors at a single point in spacetime. The reason for this emphasis was to remove from your minds the idea that a vector stretches from one point on the manifold to another, but instead is just an object associated with a single point. What is temporarily lost by adopting this view is a way to make sense of statements like “the vector points in the  $x$  direction”—if the tangent space is merely an abstract vector space associated with each point, it’s hard to know what this should mean. Now it’s time to fix the problem.

Let’s imagine that we wanted to construct the tangent space at a point  $p$  in a manifold  $M$ , using only things that are intrinsic to  $M$  (no embeddings in higher-dimensional spaces). A first guess might be to use our intuitive knowledge that there are objects called “tangent vectors to curves,” which belong in the tangent space. We might therefore consider the set of all parameterized curves through  $p$ —that is, the space of all (nondegenerate) maps  $\gamma : \mathbf{R} \rightarrow M$ , such that  $p$  is in the image of  $\gamma$ . The temptation is to define the tangent space as simply the space of all tangent vectors to these curves at the point  $p$ . But this is obviously cheating; the tangent space  $T_p$  is supposed to be the space of vectors at  $p$ , and before we have defined this we don’t have an independent notion of what “the tangent vector to a curve” is supposed to mean. In some coordinate system  $x^\mu$  any curve through  $p$  defines an element of  $\mathbf{R}^n$  specified by the  $n$  real numbers  $dx^\mu/d\lambda$  (where  $\lambda$  is the parameter along the curve), but this map is clearly coordinate-dependent, which is not what we want.

Nevertheless we are on the right track, we just have to make things independent of coordinates. To this end we define  $\mathcal{F}$  to be the space of all smooth functions on  $M$  (that is,  $C^\infty$  maps  $f : M \rightarrow \mathbf{R}$ ). Then we notice that each curve through  $p$  defines an operator on this space, the directional derivative, which maps  $f \rightarrow df/d\lambda$  (at  $p$ ). We will make the following claim: *the tangent space  $T_p$  can be identified with the space of directional derivative operators along curves through  $p$ .* To establish this idea we must demonstrate two things: first, that the space of directional derivatives is a vector space, and second that it is the vector space we want (it has the same dimensionality as  $M$ , yields a natural idea of a vector pointing along a certain direction, and so on).

The first claim, that directional derivatives form a vector space, seems straightforward enough. Imagine two operators  $d/d\lambda$  and  $d/d\eta$  representing derivatives along two curves  $x^\mu(\lambda)$  and  $x^\mu(\eta)$  through  $p$ . There is no problem adding these and scaling by real numbers, to obtain a new operator  $a(d/d\lambda) + b(d/d\eta)$ . It is not immediately obvious, however, that the space closes; in other words, that the resulting operator is itself a derivative operator. A good derivative operator is one that acts linearly on functions, and obeys the conventional Leibniz (product) rule on products of functions. Our new operator is manifestly linear, so we need to



**FIGURE 2.18** Partial derivatives define directional derivatives along curves that keep all of the other coordinates constant.

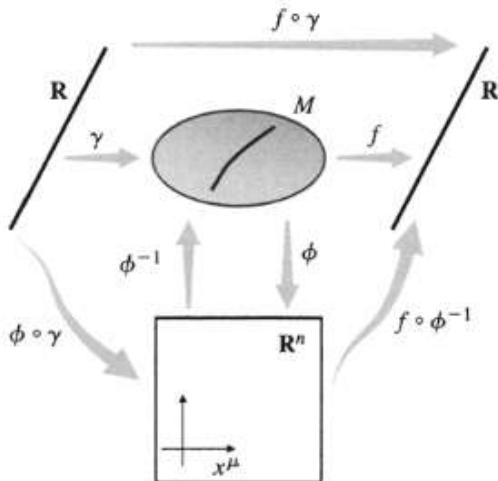
verify that it obeys the Leibniz rule. We have

$$\begin{aligned} \left( a \frac{d}{d\lambda} + b \frac{d}{d\eta} \right) (fg) &= af \frac{dg}{d\lambda} + ag \frac{df}{d\lambda} + bf \frac{dg}{d\eta} + bg \frac{df}{d\eta} \\ &= \left( a \frac{df}{d\lambda} + b \frac{df}{d\eta} \right) g + \left( a \frac{dg}{d\lambda} + b \frac{dg}{d\eta} \right) f. \end{aligned} \quad (2.14)$$

As we had hoped, the product rule is satisfied, and the set of directional derivatives is therefore a vector space.

Is it the vector space that we would like to identify with the tangent space? The easiest way to become convinced is to find a basis for the space. Consider again a coordinate chart with coordinates  $x^\mu$ . Then there is an obvious set of  $n$  directional derivatives at  $p$ , namely the partial derivatives  $\partial_\mu$  at  $p$ , as shown in Figure 2.18. Note that this is really the *definition* of the partial derivative with respect to  $x^\mu$ : the directional derivative along a curve defined by  $x^\nu = \text{constant}$  for all  $\nu \neq \mu$ , parameterized by  $x^\mu$  itself. We are now going to claim that the partial derivative operators  $\{\partial_\mu\}$  at  $p$  form a basis for the tangent space  $T_p$ . (It follows immediately that  $T_p$  is  $n$ -dimensional, since that is the number of basis vectors.) To see this we will show that any directional derivative can be decomposed into a sum of real numbers times partial derivatives. This will just be the familiar expression for the components of a tangent vector, but it's nice to see it from the big-machinery approach. Consider an  $n$ -manifold  $M$ , a coordinate chart  $\phi : M \rightarrow \mathbf{R}^n$ , a curve  $\gamma : \mathbf{R} \rightarrow M$ , and a function  $f : M \rightarrow \mathbf{R}$ . This leads to the tangle of maps shown in Figure 2.19. If  $\lambda$  is the parameter along  $\gamma$ , we want to express the vector/operator  $d/d\lambda$  in terms of the partials  $\partial_\mu$ . Using the chain rule (2.12), we have

$$\begin{aligned} \frac{d}{d\lambda} f &= \frac{d}{d\lambda} (f \circ \gamma) \\ &= \frac{d}{d\lambda} [(f \circ \phi^{-1}) \circ (\phi \circ \gamma)] \end{aligned}$$



**FIGURE 2.19** Decomposing the tangent vector to a curve  $\gamma : \mathbf{R} \rightarrow M$  in terms of partial derivatives with respect to coordinates on  $M$ .

$$\begin{aligned} &= \frac{d(\phi \circ \gamma)^\mu}{d\lambda} \frac{\partial(f \circ \phi^{-1})}{\partial x^\mu} \\ &= \frac{dx^\mu}{d\lambda} \partial_\mu f. \end{aligned} \quad (2.15)$$

The first line simply takes the informal expression on the left-hand side and rewrites it as an honest derivative of the function  $(f \circ \gamma) : \mathbf{R} \rightarrow \mathbf{R}$ . The second line just comes from the definition of the inverse map  $\phi^{-1}$  (and associativity of the operation of composition). The third line is the formal chain rule (2.12), and the last line is a return to the informal notation of the start. Since the function  $f$  was arbitrary, we have

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu. \quad (2.16)$$

Thus, the partials  $\{\partial_\mu\}$  do indeed represent a good basis for the vector space of directional derivatives, which we can therefore safely identify with the tangent space.

Of course, the vector represented by  $d/d\lambda$  is one we already know; it's the tangent vector to the curve with parameter  $\lambda$ . Thus (2.16) can be thought of as a restatement of equation (1.38), where we claimed that the components of the tangent vector were simply  $dx^\mu/d\lambda$ . The only difference is that we are working on an arbitrary manifold, and we have specified our basis vectors to be  $\hat{e}_{(\mu)} = \partial_\mu$ .

This particular basis ( $\hat{e}_{(\mu)} = \partial_\mu$ ) is known as a **coordinate basis** for  $T_p$ ; it is the formalization of the notion of setting up the basis vectors to point along the coordinate axes. There is no reason why we are limited to coordinate bases when we consider tangent vectors. For example, the coordinate basis vectors are typically not normalized to unity, nor orthogonal to each other, as we shall see

shortly. This is not a situation we can define away; on a curved manifold, a coordinate basis will *never* be orthonormal throughout a neighborhood of any point where the curvature does not vanish. Of course we can define noncoordinate orthonormal bases, for example by giving their components in a coordinate basis, and sometimes this technique is useful. However, coordinate bases are very simple and natural, and we will use them almost exclusively throughout the book; for a look at orthonormal bases, see Appendix J. (It is standard in the study of vector analysis in three-dimensional Euclidean space to choose orthonormal bases rather than coordinate bases; you should therefore be careful when applying formulae from GR texts to the study of non-Cartesian coordinates in flat space.)

One of the advantages of the rather abstract point of view we have taken toward vectors is that the transformation law under changes of coordinates is immediate. Since the basis vectors are  $\hat{e}_{(\mu)} = \partial_\mu$ , the basis vectors in some new coordinate system  $x^{\mu'}$  are given by the chain rule (2.13) as

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu. \quad (2.17)$$

We can get the transformation law for vector components by the same technique used in flat space, demanding that the vector  $V = V^\mu \partial_\mu$  be unchanged by a change of basis. We have

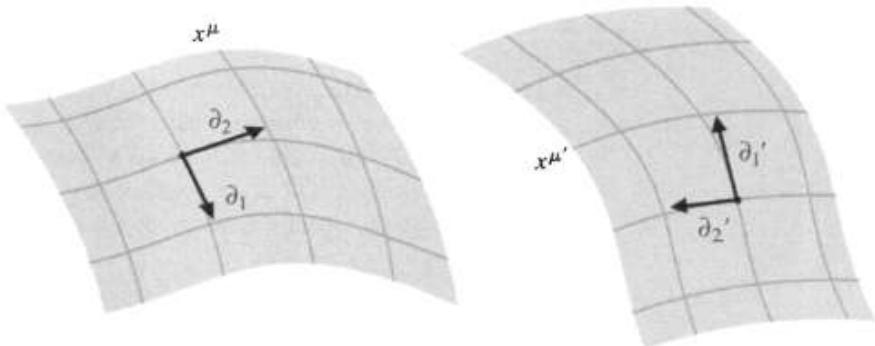
$$\begin{aligned} V^\mu \partial_\mu &= V^{\mu'} \partial_{\mu'} \\ &= V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu, \end{aligned} \quad (2.18)$$

and hence, since the matrix  $\partial x^{\mu'}/\partial x^\mu$  is the inverse of the matrix  $\partial x^\mu/\partial x^{\mu'}$ ,

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu. \quad (2.19)$$

Since the basis vectors are usually not written explicitly, the rule (2.19) for transforming components is what we call the “vector transformation law.” We notice that it is compatible with the transformation of vector components in special relativity under Lorentz transformations,  $V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu$ , since a Lorentz transformation is a special kind of coordinate transformation, with  $x^{\mu'} = \Lambda^{\mu'}{}_\mu x^\mu$ . But (2.19) is much more general, as it encompasses the behavior of vectors under arbitrary changes of coordinates (and therefore bases), not just linear transformations. As usual, we are trying to emphasize a somewhat subtle ontological distinction—in principle, tensor components need not change when we change coordinates, they change when we change the basis in the tangent space, but we have decided to use the coordinates to define our basis. Therefore a change of coordinates induces a change of basis, as indicated in Figure 2.20.

Since a vector at a point can be thought of as a directional derivative operator along a path through that point, it should be clear that a vector *field* defines a map from smooth functions to smooth functions all over the manifold, by taking a



**FIGURE 2.20** A change of coordinates  $x^\mu \rightarrow x^{\mu'}$  induces a change of basis in the tangent space.

derivative at each point. Given two vector fields  $X$  and  $Y$ , we can therefore define their **commutator**  $[X, Y]$  by its action on a function  $f(x^\mu)$ :

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)). \quad (2.20)$$

The virtue of the abstract point of view is that, clearly, this operator is independent of coordinates. In fact, the commutator of two vector fields is itself a vector field: if  $f$  and  $g$  are functions and  $a$  and  $b$  are real numbers, the commutator is linear,

$$[X, Y](af + bg) = a[X, Y](f) + b[X, Y](g), \quad (2.21)$$

and obeys the Leibniz rule,

$$[X, Y](fg) = f[X, Y](g) + g[X, Y](f). \quad (2.22)$$

Both properties are straightforward to check, which is a useful exercise to do. An equally interesting exercise is to derive an explicit expression for the components of the vector field  $[X, Y]^\mu$ , which turns out to be

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu. \quad (2.23)$$

By construction this is a well-defined tensor; but you should be slightly worried by the appearance of the partial derivatives, since partial derivatives of vectors are not well-defined tensors (as we discuss in the next section). Yet another fascinating exercise is to perform explicitly a coordinate transformation on the expression (2.23), to verify that all potentially nontensorial pieces cancel and the result transforms like a vector field. The commutator is a special case of the Lie derivative, discussed in Appendix B; it is sometimes referred to as the **Lie bracket**. Note that since partials commute, the commutator of the vector fields given by the partial derivatives of coordinate functions,  $\{\partial_\mu\}$ , always vanishes.

## 2.4 ■ TENSORS AGAIN

Having explored the world of vectors, we continue to retrace the steps we took in flat space, and now consider dual vectors (one-forms). Once again the cotangent space  $T_p^*$  can be thought of as the set of linear maps  $\omega : T_p \rightarrow \mathbf{R}$ . The canonical example of a one-form is the gradient of a function  $f$ , denoted  $df$ , as in (1.52). Its action on a vector  $d/d\lambda$  is exactly the directional derivative of the function:

$$df \left( \frac{d}{d\lambda} \right) = \frac{df}{d\lambda}. \quad (2.24)$$

It's tempting to ask, "why shouldn't the function  $f$  itself be considered the one-form, and  $df/d\lambda$  its action?" The point is that a one-form, like a vector, exists only at the point it is defined, and does not depend on information at other points on  $M$ . If you know a function in some neighborhood of a point, you can take its derivative, but not just from knowing its value at the point; the gradient, on the other hand, encodes precisely the information necessary to take the directional derivative along any curve through  $p$ , fulfilling its role as a dual vector.

You may have noticed that we defined vectors using structures intrinsic to the manifold (directional derivatives along curves), and used that definition to define one-forms in terms of the dual vector space. This might lead to the impression that vectors are somehow more fundamental; in fact, however, we could just as well have begun with an intrinsic definition of one-forms and used that to define vectors as the dual space. Roughly speaking, the space of one-forms at  $p$  is equivalent to the space of all functions that vanish at  $p$  and have the same second partial derivatives. In fact, doing it that way is more fundamental, if anything, since we can provide intrinsic definitions of all  $q$ -forms (totally antisymmetric tensors with  $q$  lower indices), which we will discuss in Section 2.9 (although we will not delve into the specifics of the intrinsic definitions).

Just as the partial derivatives along coordinate axes provide a natural basis for the tangent space, the gradients of the coordinate functions  $x^\mu$  provide a natural basis for the cotangent space. Recall that in flat space we constructed a basis for  $T_p^*$  by demanding that  $\theta^{(\mu)}(\hat{e}_{(v)}) = \delta_v^\mu$ . Continuing the same philosophy on an arbitrary manifold, we find that (2.24) leads to

$$dx^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta_\nu^\mu. \quad (2.25)$$

Therefore the gradients  $\{dx^\mu\}$  are an appropriate set of basis one-forms; an arbitrary one-form is expanded into components as  $\omega = \omega_\mu dx^\mu$ .

The transformation properties of basis dual vectors and components follow from what is by now the usual procedure. We obtain, for basis one-forms,

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu \quad (2.26)$$

and for components,

$$\omega_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \omega_{\mu}. \quad (2.27)$$

We will usually write the components  $\omega_{\mu}$  when we speak about a one-form  $\omega$ .

Just as in flat space, a  $(k, l)$  tensor is a multilinear map from a collection of  $k$  dual vectors and  $l$  vectors to  $\mathbf{R}$ . Its components in a coordinate basis can be obtained by acting the tensor on basis one-forms and vectors,

$$T^{\mu_1 \dots \mu_k}{}_{v_1 \dots v_l} = T(dx^{\mu_1}, \dots, dx^{\mu_k}, \partial_{v_1}, \dots, \partial_{v_l}). \quad (2.28)$$

This is equivalent to the expansion

$$T = T^{\mu_1 \dots \mu_k}{}_{v_1 \dots v_l} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes dx^{v_1} \otimes \dots \otimes dx^{v_l}. \quad (2.29)$$

The transformation law for general tensors follows the same pattern of replacing the Lorentz transformation matrix used in flat space with a matrix representing more general coordinate transformations:

$$T^{\mu'_1 \dots \mu'_k}{}_{v'_1 \dots v'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \dots \frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{v'_1}}{\partial x^{v_1}} \dots \frac{\partial x^{v'_l}}{\partial x^{v_l}} T^{\mu_1 \dots \mu_k}{}_{v_1 \dots v_l}. \quad (2.30)$$

This tensor transformation law is straightforward to remember, since there really isn't anything else it could be, given the placement of indices.

Actually, however, it is often easier to transform a tensor by taking the identity of basis vectors and one-forms as partial derivatives and gradients at face value, and simply substituting in the coordinate transformation. As an example, consider a symmetric  $(0, 2)$  tensor  $S$  on a two-dimensional manifold, whose components in a coordinate system ( $x^1 = x, x^2 = y$ ) are given by

$$S_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}. \quad (2.31)$$

This can be written equivalently as

$$\begin{aligned} S &= S_{\mu\nu} (dx^\mu \otimes dx^\nu) \\ &= (dx)^2 + x^2 (dy)^2, \end{aligned} \quad (2.32)$$

where in the last line the tensor product symbols are suppressed for brevity (as will become our custom). Now consider new coordinates

$$\begin{aligned} x' &= \frac{2x}{y} \\ y' &= \frac{y}{2} \end{aligned} \quad (2.33)$$

(valid, for example, when  $x > 0, y > 0$ ). These can be immediately inverted to obtain

$$\begin{aligned}x &= x'y' \\y &= 2y'.\end{aligned}\tag{2.34}$$

Instead of using the tensor transformation law, we can simply use the fact that we know how to take derivatives to express  $dx^\mu$  in terms of  $dx^{\mu'}$ . We have

$$\begin{aligned}dx &= y'dx' + x'dy' \\dy &= 2dy'.\end{aligned}\tag{2.35}$$

We need only plug these expressions directly into (2.32) to obtain (remembering that tensor products don't commute, so  $dx'dy' \neq dy'dx'$ ):

$$S = (y')^2(dx')^2 + x'y'(dx'dy' + dy'dx') + [(x')^2 + 4(x'y')^2](dy')^2,\tag{2.36}$$

or

$$S_{\mu'\nu'} = \begin{pmatrix} (y')^2 & x'y' \\ x'y' & (x')^2 + 4(x'y')^2 \end{pmatrix}.\tag{2.37}$$

Notice that it is still symmetric. We did not use the transformation law (2.30) directly, but doing so would have yielded the same result, as you can check.

For the most part the various tensor operations we defined in flat space are unaltered in a more general setting: contraction, symmetrization, and so on. There are three important exceptions: partial derivatives, the metric, and the Levi–Civita tensor. Let's look at the partial derivative first.

Unfortunately, the partial derivative of a tensor is not, in general, a new tensor. The gradient, which is the partial derivative of a scalar, is an honest  $(0, 1)$  tensor, as we have seen. But the partial derivative of higher-rank tensors is not tensorial, as we can see by considering the partial derivative of a one-form,  $\partial_\mu W_\nu$ , and changing to a new coordinate system:

$$\begin{aligned}\frac{\partial}{\partial x^{\mu'}} W_{\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\nu}{\partial x^{\nu'}} W_\nu \right) \\&= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \left( \frac{\partial}{\partial x^\mu} W_\nu \right) + W_\nu \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}}.\end{aligned}\tag{2.38}$$

The second term in the last line should not be there if  $\partial_\mu W_\nu$  were to transform as a  $(0, 2)$  tensor. As you can see, it arises because the derivative of the transformation matrix does not vanish, as it did for Lorentz transformations in flat space.

Differentiation is obviously an important tool in physics, so we will have to invent new tensorial operations to take the place of the partial derivative. In fact we will invent several: the exterior derivative, the covariant derivative, and the Lie derivative.

## 2.5 ■ THE METRIC

The metric tensor is such an important object in curved space that it is given a new symbol,  $g_{\mu\nu}$  (while  $\eta_{\mu\nu}$  is reserved specifically for the Minkowski metric). There are few restrictions on the components of  $g_{\mu\nu}$ , other than that it be a symmetric (0, 2) tensor. It is usually, though not always, taken to be nondegenerate, meaning that the determinant  $g = |g_{\mu\nu}|$  doesn't vanish. This allows us to define the inverse metric  $g^{\mu\nu}$  via

$$g^{\mu\nu} g_{\nu\sigma} = g_{\lambda\sigma} g^{\lambda\mu} = \delta_\sigma^\mu. \quad (2.39)$$

The symmetry of  $g_{\mu\nu}$  implies that  $g^{\mu\nu}$  is also symmetric. Just as in special relativity, the metric and its inverse may be used to raise and lower indices on tensors. You may be familiar with the notion of a "metric" used in the study of topology, where we also demand that the metric be positive-definite (no negative eigenvalues). The metric we use in general relativity cannot be used to define a topology, but it will have other uses.

It will take some time to fully appreciate the role of the metric in all of its glory, but for purposes of inspiration [following Sachs and Wu (1977)] we can list the various uses to which  $g_{\mu\nu}$  will be put: (1) the metric supplies a notion of "past" and "future"; (2) the metric allows the computation of path length and proper time; (3) the metric determines the "shortest distance" between two points, and therefore the motion of test particles; (4) the metric replaces the Newtonian gravitational field  $\phi$ ; (5) the metric provides a notion of locally inertial frames and therefore a sense of "no rotation"; (6) the metric determines causality, by defining the speed of light faster than which no signal can travel; (7) the metric replaces the traditional Euclidean three-dimensional dot product of Newtonian mechanics. Obviously these ideas are not all completely independent, but we get some sense of the importance of this tensor.

In our discussion of path lengths in special relativity we (somewhat handwavingly) introduced the line element as  $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ , which was used to get the length of a path. Of course now that we know that  $dx^\mu$  is really a basis dual vector, it becomes natural to use the terms "metric" and "line element" interchangeably, and write

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (2.40)$$

To be perfectly consistent we should write this as "g," and sometimes will, but more often than not  $g$  is used for the determinant  $|g_{\mu\nu}|$ . For example, we know that the Euclidean line element in a three-dimensional space with Cartesian coordinates is

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2. \quad (2.41)$$

We can now change to any coordinate system we choose. For example, in spherical coordinates we have

$$\begin{aligned}x &= r \sin \theta \cos \phi \\y &= r \sin \theta \sin \phi \\z &= r \cos \theta,\end{aligned}\tag{2.42}$$

which leads directly to

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.\tag{2.43}$$

Obviously the components of the metric look different than those in Cartesian coordinates, but all of the properties of the space remain unaltered.

Most references are not sufficiently picky to distinguish between “ $dx$ ,” the informal notion of an infinitesimal displacement, and “ $dx$ ,” the rigorous notion of a basis one-form given by the gradient of a coordinate function. (They also tend to neglect the fact that tensor products don’t commute, and write expressions like  $dx dy + dy dx$  as  $2dx dy$ ; it should be clear what is meant from the context.) In fact our notation “ $ds^2$ ” does not refer to the differential of anything, or the square of anything; it’s just conventional shorthand for the metric tensor, a multilinear map from two vectors to the real numbers. Thus, we have a set of equivalent expressions for the inner product of two vectors  $V^\mu$  and  $W^\nu$ :

$$g_{\mu\nu} V^\mu W^\nu = g(V, W) = ds^2(V, W).\tag{2.44}$$

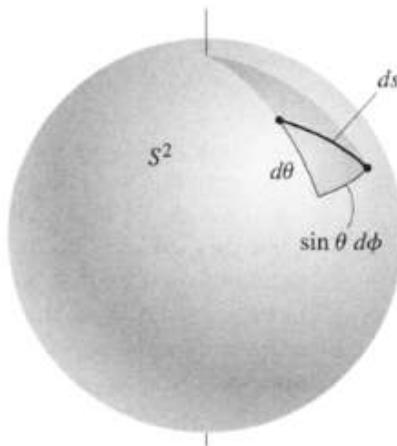
Meanwhile, “ $(dx)^2$ ” refers specifically to the honest  $(0, 2)$  tensor  $dx \otimes dx$ .

A good example of a non-Euclidean manifold is the two-sphere, which can be thought of as the locus of points in  $\mathbf{R}^3$  at distance 1 from the origin. The metric in the  $(\theta, \phi)$  coordinate system can be derived by setting  $r = 1$  and  $dr = 0$  in (2.43):

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2.\tag{2.45}$$

This is completely consistent with the interpretation of  $ds$  as an infinitesimal length, as illustrated in Figure 2.21. Anyone paying attention should at this point be asking, “What in the world does it mean to set  $dr = 0$ ? We know that  $dr$  is a well-defined nonvanishing one-form field.” As occasionally happens, we are using sloppy language to motivate a step that is actually quite legitimate; see Appendix A for a discussion of how submanifolds inherit metrics from the spaces in which they are embedded.

As we shall see, the metric tensor contains all the information we need to describe the curvature of the manifold (at least in what is called Riemannian geometry; we will get into some of the subtleties in the next chapter). In Minkowski space we can choose coordinates in which the components of the metric are constant; but it should be clear that the existence of curvature is more subtle than having the metric depend on the coordinates, since in the example above we showed how the metric in flat Euclidean space in spherical coordinates is a function of  $r$  and  $\theta$ . Later, we shall see that constancy of the metric components is sufficient for a space to be flat, and in fact there always exists a coordinate system on any



**FIGURE 2.21** The line element on a two-dimensional sphere.

flat space in which the metric is constant. But we might not know how to find such a coordinate system, and there are many ways for a space to deviate from flatness; we will therefore want a more precise characterization of the curvature, which will be introduced later.

A useful characterization of the metric is obtained by putting  $g_{\mu\nu}$  into its **canonical form**. In this form the metric components become

$$g_{\mu\nu} = \text{diag}(-1, -1, \dots, -1, +1, +1, \dots, +1, 0, 0, \dots, 0), \quad (2.46)$$

where “diag” means a diagonal matrix with the given elements. The **signature** of the metric is the number of both positive and negative eigenvalues; we speak of “a metric with signature minus-plus-plus-plus” for Minkowski space, for example. If any of the eigenvalues are zero, the metric is “degenerate,” and the inverse metric will not exist; if the metric is continuous and nondegenerate, its signature will be the same at every point. We will always deal with continuous, nondegenerate metrics. If all of the signs are positive, the metric is called **Euclidean** or **Riemannian** (or just positive definite), while if there is a single minus it is called **Lorentzian** or **pseudo-Riemannian**, and any metric with some +1’s and some -1’s is called indefinite. (So the word Euclidean sometimes means that the space is flat, and sometimes doesn’t, but it always means that the canonical form is strictly positive; the terminology is unfortunate but standard.) The spacetimes of interest in general relativity have Lorentzian metrics.

We haven’t yet demonstrated that it is always possible to put the metric into canonical form. In fact it is always possible to do so at some point  $p \in M$ , but in general it will only be possible at that single point, not in any neighborhood of  $p$ . Actually we can do slightly better than this; it turns out that at any point  $p$  there exists a coordinate system  $x^\mu$  in which  $g_{\hat{\mu}\hat{\nu}}$  takes its canonical form and the first derivatives  $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}$  all vanish (while the second derivatives  $\partial_{\hat{\rho}} \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}$  cannot be

made to all vanish):

$$g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}}, \quad \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0. \quad (2.47)$$

Such coordinates are known as **locally inertial coordinates**, and the associated basis vectors constitute a **local Lorentz frame**; we often put hats on the indices when we are in these special coordinates. Notice that in locally inertial coordinates the metric at  $p$  looks like that of flat space to first order. This is the rigorous notion of the idea that “small enough regions of spacetime look like flat (Minkowski) space.” Also, there is no difficulty in simultaneously constructing sets of *basis vectors* at every point in  $M$  such that the metric takes its canonical form; the problem is that in general there will not be a *coordinate system* from which this basis can be derived. Bases of this sort are discussed in Appendix J.

We will delay a discussion of how to construct locally inertial coordinates until Chapter 3. It is useful, however, to sketch a proof of their existence for the specific case of a Lorentzian metric in four dimensions. The idea is to consider the transformation law for the metric

$$g_{\hat{\mu}\hat{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}} g_{\mu\nu}, \quad (2.48)$$

and expand both sides in Taylor series in the sought-after coordinates  $x^{\hat{\mu}}$ . The expansion of the old coordinates  $x^{\mu}$  looks like

$$\begin{aligned} x^{\mu} &= \left( \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \right)_p x^{\hat{\mu}} + \frac{1}{2} \left( \frac{\partial^2 x^{\mu}}{\partial x^{\hat{\mu}_1} \partial x^{\hat{\mu}_2}} \right)_p x^{\hat{\mu}_1} x^{\hat{\mu}_2} \\ &\quad + \frac{1}{6} \left( \frac{\partial^3 x^{\mu}}{\partial x^{\hat{\mu}_1} \partial x^{\hat{\mu}_2} \partial x^{\hat{\mu}_3}} \right)_p x^{\hat{\mu}_1} x^{\hat{\mu}_2} x^{\hat{\mu}_3} + \dots, \end{aligned} \quad (2.49)$$

with the other expansions proceeding along the same lines. [For simplicity we have set  $x^{\mu}(p) = x^{\hat{\mu}}(p) = 0$ .] Then, using some extremely schematic notation, the expansion of (2.48) to second order is

$$\begin{aligned} (\hat{g})_p + (\hat{\partial} \hat{g})_p \hat{x} + (\hat{\partial} \hat{\partial} \hat{g})_p \hat{x} \hat{x} \\ = \left( \frac{\partial x}{\partial \hat{x}} \frac{\partial x}{\partial \hat{x}} g \right)_p + \left( \frac{\partial x}{\partial \hat{x}} \frac{\partial^2 x}{\partial \hat{x} \partial \hat{x}} g + \frac{\partial x}{\partial \hat{x}} \frac{\partial x}{\partial \hat{x}} \hat{\partial} g \right)_p \hat{x} \\ + \left( \frac{\partial x}{\partial \hat{x}} \frac{\partial^3 x}{\partial \hat{x} \partial \hat{x} \partial \hat{x}} g + \frac{\partial^2 x}{\partial \hat{x} \partial \hat{x}} \frac{\partial^2 x}{\partial \hat{x} \partial \hat{x}} g + \frac{\partial x}{\partial \hat{x}} \frac{\partial^2 x}{\partial \hat{x} \partial \hat{x}} \hat{\partial} g + \frac{\partial x}{\partial \hat{x}} \frac{\partial x}{\partial \hat{x}} \hat{\partial} \hat{\partial} g \right)_p \hat{x} \hat{x}. \end{aligned} \quad (2.50)$$

We can set terms of equal order in  $\hat{x}$  on each side equal to each other. Therefore, the components  $g_{\hat{\mu}\hat{\nu}}(p)$ , 10 numbers in all (to describe a symmetric two-index tensor), are determined by the matrix  $(\partial x^{\mu}/\partial x^{\hat{\mu}})_p$ . This is a  $4 \times 4$

matrix with no constraints; thus, we are free to choose 16 numbers. Clearly this is enough freedom to put the 10 numbers of  $g_{\hat{\mu}\hat{\nu}}(p)$  into canonical form, at least as far as having enough degrees of freedom is concerned. (In fact there are some limitations—if you go through the procedure carefully, you find for example that you cannot change the signature.) The six remaining degrees of freedom can be interpreted as exactly the six parameters of the Lorentz group; we know that these leave the canonical form unchanged. At first order we have the derivatives  $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p)$ , four derivatives of ten components for a total of 40 numbers. But looking at the right-hand side of (2.50) we see that we now have the additional freedom to choose  $(\partial^2 x^\mu / \partial x^{\hat{\mu}1} \partial x^{\hat{\mu}2})_p$ . In this set of numbers there are 10 independent choices of the indices  $\hat{\mu}_1$  and  $\hat{\mu}_2$  (it's symmetric, since partial derivatives commute) and four choices of  $\mu$ , for a total of 40 degrees of freedom. This is precisely the number of choices we need to determine all of the first derivatives of the metric, which we can therefore set to zero. At second order, however, we are concerned with  $\partial_{\hat{\rho}} \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p)$ ; this is symmetric in  $\hat{\rho}$  and  $\hat{\sigma}$  as well as  $\hat{\mu}$  and  $\hat{\nu}$ , for a total of  $10 \times 10 = 100$  numbers. Our ability to make additional choices is contained in  $(\partial^3 x^\mu / \partial x^{\hat{\mu}1} \partial x^{\hat{\mu}2} \partial x^{\hat{\mu}3})_p$ . This is symmetric in the three lower indices, which gives 20 possibilities, times four for the upper index gives us 80 degrees of freedom—20 fewer than we require to set the second derivatives of the metric to zero. So in fact we cannot make the second derivatives vanish; the deviation from flatness must therefore be measured by the 20 degrees of freedom representing the second derivatives of the metric tensor field. We will see later how this comes about, when we characterize curvature using the Riemann tensor, which will turn out to have 20 independent components in four dimensions.

Locally inertial coordinates are unbelievably useful. Best of all, their usefulness does not generally require that we actually do the work of constructing such coordinates (although we will give a recipe for doing so in the next chapter), but simply that we know that they do exist. The usual trick is to take a question of physical interest, answer it in the context of locally inertial coordinates, and then express that answer in a coordinate-independent form. Take a very simple example, featuring an observer with four-velocity  $U^{\hat{\mu}}$  and a rocket flying past with four-velocity  $V^{\hat{\mu}}$ . What does the observer measure as the ordinary three-velocity of the rocket? In special relativity the answer is straightforward. Work in inertial coordinates (globally, not just locally) such that the observer is in the rest frame and the rocket is moving along the  $x$ -axis. Then the four-velocity of the observer is  $U^{\hat{\mu}} = (1, 0, 0, 0)$  and the four-velocity of the rocket is  $V^{\hat{\mu}} = (\gamma, v\gamma, 0, 0)$ , where  $v$  is the three-velocity and  $\gamma = 1/\sqrt{1 - v^2}$ , so that  $v = \sqrt{1 - \gamma^{-2}}$ . Since we are in flat spacetime (for the moment), we have

$$\gamma = -\eta_{\hat{\mu}\hat{\nu}} U^{\hat{\mu}} V^{\hat{\nu}} = -U_{\hat{\mu}} V^{\hat{\mu}}, \quad (2.51)$$

since  $\eta_{00} = -1$ . The flat-spacetime answer would therefore be

$$v = \sqrt{1 - (U_{\hat{\mu}} V^{\hat{\mu}})^{-2}}. \quad (2.52)$$

Now we can go back to curved spacetime, where the metric is no longer flat. But at the point where the measurement is being done, we are free to use locally inertial coordinates, in which case the components of  $g_{\hat{\mu}\hat{\nu}}$  are precisely those of  $\eta_{\hat{\mu}\hat{\nu}}$ . So (2.52) is still true in curved spacetime in this particular coordinate system. But (2.52) is a completely tensorial equation, which doesn't care what coordinate system we are in; therefore it is true in complete generality. This kind of procedure will prove its value over and over.

## 2.6 ■ AN EXPANDING UNIVERSE

A simple example of a nontrivial Lorentzian geometry is provided by a four-dimensional cosmological spacetime with metric

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2]. \quad (2.53)$$

This describes a universe for which “space at a fixed moment of time” is a flat three-dimensional Euclidean space, which is expanding as a function of time. Worldlines that remain at constant spatial coordinates  $x^i$  are said to be comoving; similarly, we denote a region of space that expands along with boundaries defined by fixed spatial coordinates as a “comoving volume.” Since the metric describes (distance)<sup>2</sup>, the relative distance between comoving points is growing as  $a(t)$  in this spacetime; the function  $a$  is called the scale factor. This is a special case of a Robertson–Walker metric, one in which spatial slices are geometrically flat; there are other cases for which spatial slices are curved (as we will discuss in Chapter 8). But our interest right now is not in where this metric came from, but in using it as a playground to illustrate some of the ideas we have developed.

Typical solutions for the scale factor are power laws,

$$a(t) = t^q, \quad 0 < q < 1. \quad (2.54)$$

Actually there are all sorts of solutions, but these are some particularly simple and relevant ones. A matter-dominated flat universe satisfies  $q = \frac{2}{3}$ , while a radiation-dominated flat universe satisfies  $q = \frac{1}{2}$ . An obvious feature is that the scale factor goes to zero as  $t \rightarrow 0$ , and along with it the spatial components of the metric. This is a coordinate-dependent statement, and in principle there might be another coordinate system in which everything looks finite; in this case, however,  $t = 0$  represents a true singularity of the geometry (the “Big Bang”), and should be excluded from the manifold. The range of the  $t$  coordinate is therefore

$$0 < t < \infty. \quad (2.55)$$

Our spacetime comes to an end at  $t = 0$ .

Light cones in this curved geometry are defined by null paths, those for which  $ds^2 = 0$ . We can draw a spacetime diagram by considering null paths for which

$y$  and  $z$  are held constant; then

$$0 = -dt^2 + t^{2q} dx^2, \quad (2.56)$$

which implies

$$\frac{dx}{dt} = \pm t^{-q}. \quad (2.57)$$

You might worry that, after all that fuss about  $dx^\mu$  being a basis one-form and not a differential, we have sloppily “divided by  $dt^2$ ” to go from (2.56) to (2.57). The truth is much more respectable. What we actually did was to take the  $(0, 2)$  tensor defined by (2.56), which takes two vectors and returns a real number, and act it on two copies of the vector  $V = (dx^\mu/d\lambda)\partial_\mu$ , the tangent vector to a curve  $x^\mu(\lambda)$ . Consider just the  $dt^2$  piece acting on  $V$ :

$$dt^2(V, V) \equiv (dt \otimes dt)(V, V) = dt(V) \cdot dt(V), \quad (2.58)$$

where the notation  $dt(V)$  refers to a real number that we compute as

$$\begin{aligned} dt(V) &= dt \left( \frac{dx^\mu}{d\lambda} \partial_\mu \right) \\ &= \frac{dx^\mu}{d\lambda} dt(\partial_\mu) \\ &= \frac{dx^\mu}{d\lambda} \frac{\partial t}{\partial x^\mu} \\ &= \frac{dt}{d\lambda}, \end{aligned} \quad (2.59)$$

where in the third line we have invoked (2.25). Following the same procedure with  $dx^2$ , we find that (2.56) implies

$$0 = -\left(\frac{dt}{d\lambda}\right)^2 + t^{2q} \left(\frac{dx}{d\lambda}\right)^2, \quad (2.60)$$

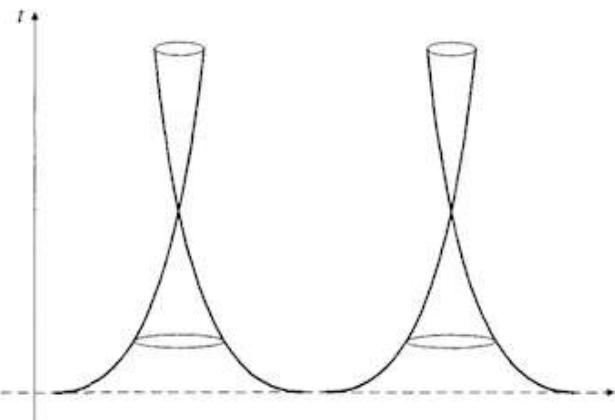
from which (2.57) follows via the one-dimensional chain rule,

$$\frac{dx}{dt} = \frac{dx}{d\lambda} \frac{d\lambda}{dt}. \quad (2.61)$$

The lesson should be clear: expressions such as (2.56) describe well-defined tensors, but manipulation of the basis one-forms as if they were simply “differentials” does get you the right answer. (At least, most of the time; it’s a good idea to keep the more formal definitions in mind.)

We can solve (2.57) to obtain

$$t = (1-q)^{1/(1-q)} (\pm x - x_0)^{1/(1-q)}, \quad (2.62)$$



**FIGURE 2.22** Spacetime diagram for a flat Robertson–Walker universe with  $a(t) \propto t^q$ , for  $0 < q < 1$ . The dashed line at the bottom of the figure represents the singularity at  $t = 0$ . Since light cones are tangent to the singularity, the pasts of two points may be nonoverlapping.

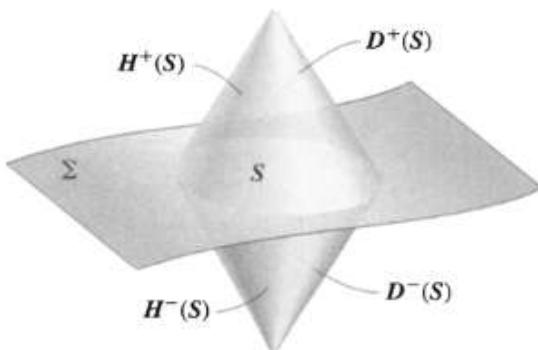
where  $x_0$  is a constant of integration. These curves define the light cones of our expanding universe, as plotted in Figure 2.22. Since we have assumed  $0 < q < 1$ , the light cones are tangent to the singularity at  $t = 0$ . A crucial feature of this geometry is that the light cones of two points need not intersect in the past; this is in contrast to Minkowski space, for which the light cones of any two points always intersect in both the past and future. We say that every event defines an “horizon,” outside of which there exist worldlines that can have had no influence on what happens at that event. This is because, since nothing can travel faster than light, each point can only be influenced by events that are either on, or in the interior of, its past light cone (indeed, we refer to the past light cone plus its interior as simply “the past” of an event). Two events outside each others’ horizons are said to be “out of causal contact.” These notions will be explored more carefully in the next section, as well as in Chapters 4 and 8.

## 2.7 ■ CAUSALITY

Many physical questions can be cast as an initial-value problem: given the state of a system at some moment in time, what will be the state at some later time? The fact that such questions have definite answers is due to causality, the idea that future events can be understood as consequences of initial conditions plus the laws of physics. Initial-value problems are as common in GR as in Newtonian physics or special relativity; however, the dynamical nature of the spacetime background introduces new ways in which an initial-value formulation could break down. Here we very briefly introduce some of the concepts used in understanding how causality works in GR.

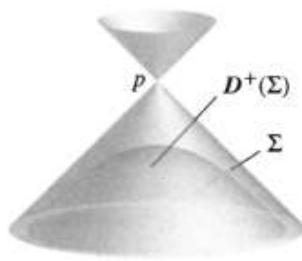
We will look at the problem of evolving matter fields on a fixed background spacetime, rather than the evolution of the metric itself. Our guiding principle will be that no signals can travel faster than the speed of light; therefore information will only flow along timelike or null trajectories (not necessarily geodesics). Since it is sometimes useful to distinguish between purely timelike paths and ones that are merely non-spacelike, we define a **causal curve** to be one which is timelike or null everywhere. Then, given any subset  $S$  of a manifold  $M$ , we define the **causal future** of  $S$ , denoted  $J^+(S)$ , to be the set of points that can be reached from  $S$  by following a future-directed causal curve; the **chronological future**  $I^+(S)$  is the set of points that can be reached by following a future-directed timelike curve. Note that a curve of zero length is causal but not chronological; therefore, a point  $p$  will always be in its own causal future  $J^+(p)$ , but not necessarily in its own chronological future  $I^+(p)$  (although it could be, as we mention below). The causal past  $J^-$  and chronological past  $I^-$  are defined analogously.

A subset  $S \subset M$  is called **achronal** if no two points in  $S$  are connected by a timelike curve; for example, any edgeless spacelike hypersurface in Minkowski spacetime is achronal. Given a closed achronal set  $S$ , we define the **future domain of dependence** of  $S$ , denoted  $D^+(S)$ , as the set of all points  $p$  such that every past-moving inextendible causal curve through  $p$  must intersect  $S$ . (Inextendible just means that the curve goes on forever, not ending at some finite point; closed means that the complement of the set is an open set.) Elements of  $S$  themselves are elements of  $D^+(S)$ . The past domain of dependence  $D^-(S)$  is defined by replacing future with past. Generally speaking, some points in  $M$  will be in one of the domains of dependence, and some will be outside; we define the boundary of  $D^+(S)$  to be the **future Cauchy horizon**  $H^+(S)$ , and likewise the boundary of  $D^-(S)$  to be the past Cauchy horizon  $H^-(S)$ . You can convince yourself that they are both null surfaces. The domains of dependence and Cauchy horizons are illustrated in Figure 2.23, in which  $S$  is taken to be a connected subset of an achronal surface  $\Sigma$ .



**FIGURE 2.23** A connected subset  $S$  of a spacelike surface  $\Sigma$ , along with its causal structure.  $D^\pm(S)$  denotes the future/past domain of dependence of  $S$ , and  $H^\pm(S)$  the future/past Cauchy horizon.

The usefulness of these definitions should be apparent; if nothing moves faster than light, signals cannot propagate outside the light cone of any point  $p$ . Therefore, if every curve that remains inside this light cone must intersect  $S$ , then information specified on  $S$  should be sufficient to predict what the situation is at  $p$ ; that is, initial data for matter fields given on  $S$  can be used to solve for the value of the fields at  $p$ . The set of all points for which we can predict what happens by knowing what happens on  $S$  is the union  $D(S) = D^+(S) \cup D^-(S)$ , called simply the domain of dependence. A closed achronal surface  $\Sigma$  is said to be a **Cauchy surface** if the domain of dependence  $D(\Sigma)$  is the entire manifold; from information given on a Cauchy surface, we can predict what happens throughout all of spacetime. If a spacetime has a Cauchy surface (which it may not), it is said to be **globally hyperbolic**.



**FIGURE 2.24** The surface  $\Sigma$  is everywhere spacelike but lies in the past of the past light cone of the point  $p$ ; its domain of dependence is not all of the spacetime.

Any set  $\Sigma$  that is closed, achronal, and has no edge, is called a **partial Cauchy surface**. A partial Cauchy surface can fail to be an actual Cauchy surface either through its own fault, or through a fault of the spacetime. One possibility is that we have just chosen a “bad” hypersurface (although it is hard to give a general prescription for when a hypersurface is bad in this sense). Consider Minkowski space, and an edgeless spacelike hypersurface  $\Sigma$ , which remains to the past of the light cone of some point, as in Figure 2.24. In this case  $\Sigma$  is an achronal surface, but it is clear that  $D^+(\Sigma)$  ends at the light cone, and we cannot use information on  $\Sigma$  to predict what happens throughout Minkowski space. Of course, there are other surfaces we could have picked for which the domain of dependence would have been the entire manifold, so this doesn’t worry us too much.

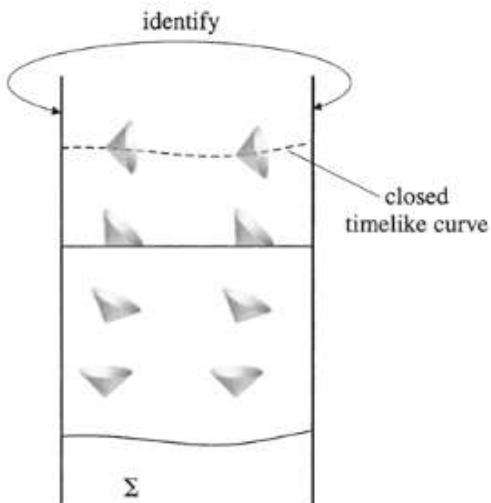
A somewhat more nontrivial way for a Cauchy horizon to arise is through the appearance of **closed timelike curves**. In Newtonian physics, causality is enforced by the relentless forward march of an absolute notion of time. In special relativity things are even more restrictive; not only must you move forward in time, but the speed of light provides a limit on how swiftly you may move through space (you must stay within your forward light cone). In general relativity it remains true that you must stay within your forward light cone; however, this becomes strictly a local notion, as globally the curvature of spacetime might “tilt” light cones from one place to another. It becomes possible in principle for light cones to be sufficiently distorted that an observer can move on a forward-directed path that is everywhere timelike and yet intersects itself at a point in its “past”—this is a closed timelike curve.

As a simple example, consider a two-dimensional geometry with coordinates  $(t, x)$ , such that points with coordinates  $(t, x)$  and  $(t, x + 1)$  are identified. The topology is thus  $\mathbf{R} \times S^1$ . We take the metric to be

$$ds^2 = -\cos(\lambda)dt^2 - \sin(\lambda)[dt\,dx + dx\,dt] + \cos(\lambda)dx^2, \quad (2.63)$$

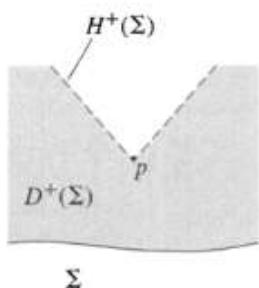
where

$$\lambda = \cot^{-1} t, \quad (2.64)$$



**FIGURE 2.25** A cylindrical spacetime with closed timelike curves. The light cones progressively tilt, such that the domain of dependence of the surface  $\Sigma$  fills the lower part of the spacetime, but comes to an end when the closed timelike curves come into existence.

which goes from  $\lambda(t = -\infty) = 0$  to  $\lambda(t = \infty) = \pi$ . This metric doesn't represent any special famous solution to general relativity, it was just cooked up to provide an interesting example of closed timelike curves; but there is a well-known example known as Misner space, with similar properties. In the spacetime defined by (2.63), the light cones progressively tilt as you go forward in time, as shown in Figure 2.25. For  $t < 0$ , the light cones point forward, and causality is maintained. Once  $t > 0$ , however,  $x$  becomes the timelike coordinate, and it is possible to travel on a timelike trajectory that wraps around the  $S^1$  and comes back to itself; this is a closed timelike curve. If we had specified a surface  $\Sigma$  to this past of this point, then none of the points in the region containing closed timelike curves are in the domain of dependence of  $\Sigma$ , since the closed timelike curves themselves do not intersect  $\Sigma$ . There is thus necessarily a Cauchy horizon at the surface  $t = 0$ . This is obviously a worse problem than the previous one, since a well-defined initial value problem does not seem to exist in this spacetime.



**FIGURE 2.26** A singularity at  $p$  removes any points in its future from the domain of dependence of a surface  $\Sigma$  in its past.

A final example is provided by the existence of singularities, points that are not in the manifold even though they can be reached by traveling along a geodesic for a finite distance. Typically these occur when the curvature becomes infinite at some point; if this happens, the point can no longer be said to be part of the spacetime. Such an occurrence can lead to the emergence of a Cauchy horizon, as depicted in Figure 2.26—a point  $p$ , which is in the future of a singularity, cannot be in the domain of dependence of a hypersurface to the past of the singularity, because there will be curves from  $p$  that simply end at the singularity.

These obstacles can also arise in the initial value problem for GR, when we try to evolve the metric itself from initial data. However, they are of different degrees of troublesomeness. The possibility of picking a “bad” initial hypersurface does not arise very often, especially since most solutions are found globally (by solving Einstein’s equation throughout spacetime). The one situation in which you have to be careful is in numerical solution of Einstein’s equation, where a bad choice of hypersurface can lead to numerical difficulties, even if in principle a complete solution exists. Closed timelike curves seem to be something that GR works hard to avoid—there are certainly solutions that contain them, but evolution from generic initial data does not usually produce them. Singularities, on the other hand, are practically unavoidable. The simple fact that the gravitational force is always attractive tends to pull matter together, increasing the curvature, and generally leading to some sort of singularity. Apparently we must learn to live with this, although there is some hope that a well-defined theory of quantum gravity will eliminate (or at least teach us how to deal with) the singularities of classical GR.

## 2.8 ■ TENSOR DENSITIES

Tensors possess a compelling beauty and simplicity, but there are times when it is useful to consider nontensorial objects. Recall that in Chapter 1 we introduced the completely antisymmetric Levi–Civita symbol, defined as

$$\tilde{\epsilon}_{\mu_1 \mu_2 \dots \mu_n} = \begin{cases} +1 & \text{if } \mu_1 \mu_2 \dots \mu_n \text{ is an even permutation of } 01 \dots (n-1), \\ -1 & \text{if } \mu_1 \mu_2 \dots \mu_n \text{ is an odd permutation of } 01 \dots (n-1), \\ 0 & \text{otherwise.} \end{cases} \quad (2.65)$$

By definition, the Levi–Civita symbol has the components specified above in *any coordinate system* (at least, in any right-handed coordinate system; switching the handedness multiplies the components of  $\tilde{\epsilon}_{\mu_1 \mu_2 \dots \mu_n}$  by an overall minus sign). This is called a “symbol,” of course, because it is not a tensor; it is defined not to change under coordinate transformations. We were only able to treat it as a tensor in inertial coordinates in flat spacetime, since Lorentz transformations would have left the components invariant anyway. Its behavior can be related to that of an ordinary tensor by first noting that, given any  $n \times n$  matrix  $M^{\mu}_{\mu'}$ , the determinant  $|M|$  obeys

$$\tilde{\epsilon}_{\mu'_1 \mu'_2 \dots \mu'_n} |M| = \tilde{\epsilon}_{\mu_1 \mu_2 \dots \mu_n} M^{\mu_1}_{\mu'_1} M^{\mu_2}_{\mu'_2} \dots M^{\mu_n}_{\mu'_n}. \quad (2.66)$$

This is just a streamlined expression for the determinant of any matrix, completely equivalent to the usual formula in terms of matrices of cofactors. (You can check it for yourself for  $2 \times 2$  or  $3 \times 3$  matrices.) It follows that, setting  $M^{\mu}_{\mu'} = \partial x^{\mu} / \partial x^{\mu'}$ , we have

$$\tilde{\epsilon}_{\mu'_1 \mu'_2 \dots \mu'_n} = \left| \frac{\partial x^{\mu'}}{\partial x^\mu} \right| \tilde{\epsilon}_{\mu_1 \mu_2 \dots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}} \frac{\partial x^{\mu_2}}{\partial x^{\mu'_2}} \dots \frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}}, \quad (2.67)$$

where we have also used the facts that the matrix  $\partial x^{\mu'}/\partial x^\mu$  is the inverse of  $\partial x^\mu/\partial x^{\mu'}$ , and that the determinant of an inverse matrix is the inverse of the determinant,  $|M^{-1}| = |M|^{-1}$ . So the Levi–Civita symbol transforms in a way close to the tensor transformation law, except for the determinant out front. Objects transforming in this way are known as **tensor densities**. Another example is given by the determinant of the metric,  $g = |g_{\mu\nu}|$ . It's easy to check, by taking the determinant of both sides of (2.48), that under a coordinate transformation we get

$$g(x^{\mu'}) = \left| \frac{\partial x^{\mu'}}{\partial x^\mu} \right|^{-2} g(x^\mu). \quad (2.68)$$

Therefore  $g$  is also not a tensor; it transforms in a way similar to the Levi–Civita symbol, except that the Jacobian is raised to the  $-2$  power. The power to which the Jacobian is raised is known as the **weight** of the tensor density; the Levi–Civita symbol is a density of weight 1, while  $g$  is a (scalar) density of weight  $-2$ .

However, we don't like tensor densities as much as we like tensors. There is a simple way to convert a density into an honest tensor—multiply by  $|g|^{w/2}$ , where  $w$  is the weight of the density (the absolute value signs are there because  $g < 0$  for Lorentzian metrics). The result will transform according to the tensor transformation law. Therefore, for example, we can define the **Levi–Civita tensor** as

$$\epsilon_{\mu_1 \mu_2 \dots \mu_n} = \sqrt{|g|} \tilde{\epsilon}_{\mu_1 \mu_2 \dots \mu_n}. \quad (2.69)$$

Since this is a real tensor, we can raise indices and so on. Sometimes people define a version of the Levi–Civita symbol with upper indices,  $\tilde{\epsilon}^{\mu_1 \mu_2 \dots \mu_n}$ , whose components are numerically equal to  $\text{sgn}(g)\tilde{\epsilon}_{\mu_1 \mu_2 \dots \mu_n}$ , where  $\text{sgn}(g)$  is the sign of the metric determinant. This turns out to be a density of weight  $-1$ , and is related to the tensor with upper indices (obtained by using  $g^{\mu\nu}$  to raise indices on  $\epsilon_{\mu_1 \mu_2 \dots \mu_n}$ ) by

$$\epsilon^{\mu_1 \mu_2 \dots \mu_n} = \frac{1}{\sqrt{|g|}} \tilde{\epsilon}^{\mu_1 \mu_2 \dots \mu_n}. \quad (2.70)$$

Something you often end up doing is contracting  $p$  indices on  $\epsilon^{\mu_1 \mu_2 \dots \mu_n}$  with  $\epsilon_{\mu_1 \mu_2 \dots \mu_n}$ ; the result can be expressed in terms of an antisymmetrized product of Kronecker deltas as

$$\epsilon^{\mu_1 \mu_2 \dots \mu_p \alpha_1 \dots \alpha_{n-p}} \epsilon_{\mu_1 \mu_2 \dots \mu_p \beta_1 \dots \beta_{n-p}} = (-1)^s p!(n-p)! \delta_{\beta_1}^{[\alpha_1} \dots \delta_{\beta_{n-p}}^{\alpha_{n-p}]}, \quad (2.71)$$

where  $s$  is the number of negative eigenvalues of the metric (for Lorentzian signature with our conventions,  $s = 1$ ). The most common example is  $p = n - 1$ ,

for which we have

$$\epsilon^{\mu_1 \mu_2 \cdots \mu_{n-1} \alpha} \epsilon_{\mu_1 \mu_2 \cdots \mu_{n-1} \beta} = (-1)^s (n-1)! \delta_\beta^\alpha. \quad (2.72)$$

## 2.9 ■ DIFFERENTIAL FORMS

Let us now introduce a special class of tensors, known as **differential forms** (or just forms). A differential  $p$ -form is simply a  $(0, p)$  tensor that is completely antisymmetric. Thus, scalars are automatically 0-forms, and dual vectors are automatically one-forms (thus explaining this terminology from before). We also have the 4-form  $\epsilon_{\mu\nu\rho\sigma}$ . The space of all  $p$ -forms is denoted  $\Lambda^p$ , and the space of all  $p$ -form fields over a manifold  $M$  is denoted  $\Lambda^p(M)$ . A semi-straightforward exercise in combinatorics reveals that the number of linearly independent  $p$ -forms on an  $n$ -dimensional vector space is  $n!/(p!(n-p)!)$ . So at a point on a four-dimensional spacetime there is one linearly independent 0-form, four 1-forms, six 2-forms, four 3-forms, and one 4-form. There are no  $p$ -forms for  $p > n$ , since all of the components will automatically be zero by antisymmetry.

Why should we care about differential forms? This question is hard to answer without some more work, but the basic idea is that forms can be both differentiated and integrated, without the help of any additional geometric structure. We will glance briefly at both of these operations.

Given a  $p$ -form  $A$  and a  $q$ -form  $B$ , we can form a  $(p+q)$ -form known as the **wedge product**  $A \wedge B$  by taking the antisymmetrized tensor product:

$$(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p! q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}. \quad (2.73)$$

Thus, for example, the wedge product of two 1-forms is

$$(A \wedge B)_{\mu\nu} = 2A_{[\mu} B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu. \quad (2.74)$$

Note that

$$A \wedge B = (-1)^{pq} B \wedge A, \quad (2.75)$$

so you can alter the order of a wedge product if you are careful with signs. We are free to suppress indices when using forms, since we know that all of the indices are downstairs and the tensors are completely antisymmetric.

The **exterior derivative**  $d$  allows us to differentiate  $p$ -form fields to obtain  $(p+1)$ -form fields. It is defined as an appropriately normalized, antisymmetrized partial derivative:

$$(dA)_{\mu_1 \cdots \mu_{p+1}} = (p+1) \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}].} \quad (2.76)$$

The simplest example is the gradient, which is the exterior derivative of a 0-form:

$$(d\phi)_\mu = \partial_\mu \phi. \quad (2.77)$$

Exterior derivatives obey a modified version of the Leibniz rule when applied to the product of a  $p$ -form  $\omega$  and a  $q$ -form  $\eta$ :

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge (d\eta). \quad (2.78)$$

You are encouraged to prove this yourself.

The reason why the exterior derivative deserves special attention is that *it is a tensor*, even in curved spacetimes, unlike its cousin the partial derivative. For  $p = 1$  we can see this from the transformation law for the partial derivative of a one form, (2.38); the offending nontensorial term can be written

$$W_v \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}} = W_v \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\nu'}}. \quad (2.79)$$

This expression is symmetric in  $\mu'$  and  $\nu'$ , since partial derivatives commute. But the exterior derivative is defined to be the antisymmetrized partial derivative, so this term vanishes (the antisymmetric part of a symmetric expression is zero). We are then left with the correct tensor transformation law; extension to arbitrary  $p$  is straightforward. So the exterior derivative is a legitimate tensor operator; it is not, however, an adequate substitute for the partial derivative, since it is only defined on forms. In the next chapter we will define a covariant derivative, which is closer to what we might think of as the extension of the partial derivative to arbitrary manifolds.

Another interesting fact about exterior differentiation is that, for any form  $A$ ,

$$d(dA) = 0, \quad (2.80)$$

which is often written  $d^2 = 0$ . This identity is a consequence of the definition of  $d$  and the fact that partial derivatives commute,  $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$  (acting on anything). This leads us to the following mathematical aside, just for fun. We define a  $p$ -form  $A$  to be **closed** if  $dA = 0$ , and **exact** if  $A = dB$  for some  $(p-1)$ -form  $B$ . Obviously, all exact forms are closed, but the converse is not necessarily true. On a manifold  $M$ , closed  $p$ -forms comprise a vector space  $Z^p(M)$ , and exact forms comprise a vector space  $B^p(M)$ . Define a new vector space, consisting of elements called cohomology classes, as the closed forms modulo the exact forms:

$$H^p(M) = \frac{Z^p(M)}{B^p(M)}. \quad (2.81)$$

That is, two closed forms [elements of  $Z^p(M)$ ] define the same cohomology class [elements of  $H^p(M)$ ] if they differ by an exact form [an element of  $B^p(M)$ ]. Miraculously, the dimensionality of the cohomology spaces  $H^p(M)$  depends only on the topology of the manifold  $M$ . Minkowski space is topologically equivalent to  $\mathbf{R}^4$ , which is uninteresting, so that all of the  $H^p(M)$  vanish for  $p > 0$ ; for  $p = 0$  we have  $H^0(M) = \mathbf{R}$ . Therefore in Minkowski space all closed forms are exact except for zero-forms; zero-forms can't be exact since there are no  $-1$ -

forms for them to be the exterior derivative of. It is striking that information about the topology can be extracted in this way, which essentially involves the solutions to differential equations.

The final operation on differential forms we will introduce is **Hodge duality**. We define the *Hodge star operator* on an  $n$ -dimensional manifold as a map from  $p$ -forms to  $(n-p)$ -forms,

$$(*A)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \dots \nu_p} \mu_1 \dots \mu_{n-p} A_{\nu_1 \dots \nu_p}, \quad (2.82)$$

mapping  $A$  to “ $A$  dual.” Unlike our other operations on forms, the Hodge dual does depend on the metric of the manifold [which should be obvious, since we had to raise some indices on the Levi–Civita tensor in order to define (2.82)]. Applying the Hodge star twice returns either plus or minus the original form:

$$**A = (-1)^{s+p(n-p)} A, \quad (2.83)$$

where  $s$  is the number of minus signs in the eigenvalues of the metric.

Two facts on the Hodge dual: First, “duality” in the sense of Hodge is distinct from the relationship between vectors and dual vectors. The idea of “duality” is that of a transformation from one space to another with the property that doing the transformation twice gets you back to the original space. It should be clear that this holds true for both the duality between vectors and one-forms, and the Hodge duality between  $p$ -forms and  $(n-p)$ -forms. A requirement of dualities between vector spaces is that the original and transformed spaces have the same dimensionality; this is true of the spaces of  $p$ - and  $(n-p)$ -forms.

The second fact concerns differential forms in three-dimensional Euclidean space. The Hodge dual of the wedge product of two 1-forms gives another 1-form:

$$*(U \wedge V)_i = \epsilon_i^{jk} U_j V_k. \quad (2.84)$$

(All of the prefactors cancel.) Since 1-forms in Euclidean space are just like vectors, we have a map from two vectors to a single vector. You should convince yourself that this is just the conventional cross product, and that the appearance of the Levi–Civita tensor explains why the cross product changes sign under parity (interchange of two coordinates, or equivalently basis vectors). This is why the cross product only exists in three dimensions—because only in three dimensions do we have an interesting map from two dual vectors to a third dual vector.

Electrodynamics provides an especially compelling example of the use of differential forms. From the definition of the exterior derivative, it is clear that equation (1.97) can be concisely expressed as closure of the two-form  $F_{\mu\nu}$ :

$$dF = 0. \quad (2.85)$$

Does this mean that  $F$  is also exact? Yes; as we’ve noted, Minkowski space is topologically trivial, so all closed forms are exact. There must therefore be a one-

form  $A_\mu$  such that

$$F = dA. \quad (2.86)$$

This one-form is the familiar **vector potential** of electromagnetism, with the 0 component given by the scalar potential,  $A_0 = \Phi$ , as we discussed in Chapter 1. Gauge invariance is expressed by the observation that the theory is invariant under  $A \rightarrow A + d\lambda$  for some scalar (zero-form)  $\lambda$ , and this is also immediate from the relation (2.86). The other one of Maxwell's equations, (1.96), can be expressed as an equation between three-forms:

$$d(*F) = *J, \quad (2.87)$$

where the current one-form  $J$  is just the current four-vector with index lowered. Filling in the details is left for you, as good practice converting from differential-form notation to ordinary index notation.

Hodge duality is intimately related to a fascinating feature of certain field theories: duality between strong and weak coupling. It's hard not to notice that the equations (2.85) and (2.87) look very similar. Indeed, if we set  $J_\mu = 0$ , the equations are invariant under the "duality transformations"

$$\begin{aligned} F &\rightarrow *F, \\ *F &\rightarrow -F. \end{aligned} \quad (2.88)$$

We therefore say that the vacuum Maxwell's equations are duality invariant, while the invariance is spoiled in the presence of charges. We might imagine that magnetic as well as electric monopoles existed in nature; then we could add a magnetic current term  $*J_M$  to the right-hand side of (2.85), and the equations would be invariant under duality transformations plus the additional replacement  $J \leftrightarrow J_M$ . (Of course a nonzero right-hand side to (2.85) is inconsistent with  $F = dA$ , so this idea only works if  $A_\mu$  is not a fundamental variable.) Dirac considered the idea of magnetic monopoles and showed that a necessary condition for their existence is that the fundamental monopole charge be inversely proportional to the fundamental electric charge. Now, the fundamental electric charge is a small number; electrodynamics is *weakly coupled*, which is why perturbation theory is so remarkably successful in quantum electrodynamics (QED). But Dirac's condition on magnetic charges implies that a duality transformation takes a theory of weakly coupled electric charges to a theory of strongly coupled magnetic monopoles (and vice-versa). Unfortunately monopoles don't fit easily into ordinary electromagnetism, so these ideas aren't directly applicable; but some sort of duality symmetry may exist in certain theories (such as supersymmetric nonabelian gauge theories). If it did, we would have the opportunity to analyze a theory that looked strongly coupled (and therefore hard to solve) by looking at the weakly coupled dual version; this is exactly what happens in certain theories. The hope is that these techniques will allow us to explore various phenomena that we know exist in strongly coupled quantum field theories, such as confinement of quarks in hadrons.

## 2.10 ■ INTEGRATION

An important appearance of both tensor densities and differential forms is in integration on manifolds. You have probably been exposed to the fact that in ordinary calculus on  $\mathbf{R}^n$  the volume element  $d^n x$  picks up a factor of the Jacobian under change of coordinates:

$$d^n x' = \left| \frac{\partial x^{\mu'}}{\partial x^\mu} \right| d^n x. \quad (2.89)$$

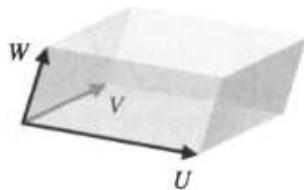
There is actually a beautiful explanation of this formula from the point of view of differential forms, which arises from the following fact: *on an  $n$ -dimensional manifold  $M$ , the integrand is properly understood as an  $n$ -form*. In other words, an integral over an  $n$ -dimensional region  $\Sigma \subset M$  is a map from an  $n$ -form field  $\omega$  to the real numbers:

$$\int_{\Sigma} : \omega \rightarrow \mathbf{R}. \quad (2.90)$$

Such a statement may seem strange, but it certainly looks familiar in the context of line integrals. In one dimension any one-form can be written  $\omega = \omega(x)dx$ , where the first  $\omega$  is a one-form and  $\omega(x)$  denotes the (single) component function. And indeed, we write integrals in one dimension as  $\int \omega(x)dx$ ; you may be used to thinking of the symbol  $dx$  as an infinitesimal distance, but it is more properly a differential form.

To make this more clear, consider more than one dimension. If we are claiming that the integrand is an  $n$ -form, we need to explain in what sense it is antisymmetric, and for that matter why it is a  $(0, n)$  tensor (a linear map from a set of  $n$  vectors to  $\mathbf{R}$ ) at all. We all agree that integrals can be written as  $\int f(x) d\mu$ , where  $f(x)$  is a scalar function on the manifold and  $d\mu$  is the volume element, or measure. The role of the volume element is to assign to every (infinitesimal) region an (infinitesimal) real number, namely the volume of that region. A nice feature of infinitesimal regions (as opposed to ones of finite size) is that they can be taken to be rectangular parallelepipeds—in the presence of curvature we have no clear sense of what a “rectangular parallelepiped” is supposed to mean, but the effects of curvature can be neglected when we work in infinitesimal regions. Clearly we are not being rigorous here, but our present purpose is exclusively motivational.

As shown in Figure 2.27 (in which we take our manifold to be three-dimensional for purposes of illustration), a parallelepiped is specified by  $n$  vectors that define its edges. Our volume element, then, should be a map from  $n$  vectors to the real numbers:  $d\mu(U, V, W) \in \mathbf{R}$ . (Actually it should be a map from infinitesimal vectors to infinitesimal numbers, but such a map also will take finite vectors to finite numbers.) It’s also clear that it should be linearly scalable by real numbers; if we change the length of any of the defining vectors, the volume changes accordingly:  $d\mu(aU, bV, cW) = abc d\mu(U, V, W)$ . Linearity with respect to adding vectors is not so obvious, but you can convince yourself by drawing pictures.



**FIGURE 2.27** An infinitesimal  $n$ -dimensional region, represented as a parallelepiped, is defined by an ordered set of  $n$  vectors, shown here as  $U$ ,  $V$ , and  $W$ .

Therefore our volume element is an honest  $(0, n)$  tensor. Why antisymmetric? Because we are defining an oriented element; if two of the vectors are interchanged we should get a volume of the same magnitude but opposite sign. (If this is not obvious, you should at least convince yourself that the volume should vanish when two vectors are collinear.) Thus, volume elements in  $n$  dimensions are in a very real sense  $n$ -forms.

To actually do calculations, we need to make these ideas more concrete, which turns out to be straightforward. The essential insight is to identify the naive volume element  $d^n x$  as an antisymmetric tensor density constructed with wedge products:

$$d^n x = dx^0 \wedge \cdots \wedge dx^{n-1}. \quad (2.91)$$

The expression on the right-hand side can be misleading, because it looks like a tensor (an  $n$ -form, actually) but is really a density. Certainly if we have two functions  $f$  and  $g$  on  $M$ , then  $df$  and  $dg$  are one-forms, and  $df \wedge dg$  is a two-form. But the functions appearing in (2.91) are the coordinate functions themselves, so when we change coordinates we replace the one-forms  $dx^\mu$  with a new set  $dx^{\mu'}$ . You see the funny business—ordinarily a coordinate transformation changes components, but not one-forms themselves. The right-hand side of (2.91) is a coordinate-dependent object (a tensor density, to be precise) which, in the  $x^\mu$  coordinate system, acts like  $dx^0 \wedge \cdots \wedge dx^{n-1}$ . Let's see this in action. First notice that the definition of the wedge product allows us to write

$$dx^0 \wedge \cdots \wedge dx^{n-1} = \frac{1}{n!} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}, \quad (2.92)$$

since both the wedge product and the Levi-Civita symbol are completely antisymmetric. (The factor of  $1/n!$  takes care of the overcounting introduced by summing over permutations of the indices.) Under a coordinate transformation  $\tilde{\epsilon}_{\mu_1 \cdots \mu_n}$  stays the same, while the one-forms change according to (2.26), leading to

$$\begin{aligned} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} &= \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}} dx^{\mu'_1} \wedge \cdots \wedge dx^{\mu'_n} \\ &= \left| \frac{\partial x^\mu}{\partial x^{\mu'}} \right| \tilde{\epsilon}_{\mu'_1 \cdots \mu'_n} dx^{\mu'_1} \wedge \cdots \wedge dx^{\mu'_n}. \end{aligned} \quad (2.93)$$

Multiplying by the Jacobian on both sides and using (2.91) and (2.92) recovers (2.89).

It is clear that the naive volume element  $d^n x$  transforms as a density, not a tensor, but it is straightforward to construct an invariant volume element by multiplying by  $\sqrt{|g|}$ :

$$\sqrt{|g'|} dx^{0'} \wedge \cdots \wedge dx^{(n-1)'} = \sqrt{|g|} dx^0 \wedge \cdots \wedge dx^{n-1}, \quad (2.94)$$

which is of course just  $(n!)^{-1} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$ . In the interest of simplicity we will usually write the volume element as  $\sqrt{|g|} d^n x$ , rather than as the explicit

wedge product:

$$\sqrt{|g|} d^n x \equiv \sqrt{|g|} dx^0 \wedge \cdots \wedge dx^{n-1}; \quad (2.95)$$

it will be enough to keep in mind that it's supposed to be an  $n$ -form. In fact, the volume element is no more or less than the Levi–Civita tensor  $\epsilon_{\mu_1 \dots \mu_n}$ ; restoring the explicit basis one-forms, we see

$$\begin{aligned} \epsilon &\equiv \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_n} \\ &= \frac{1}{n!} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \\ &= \frac{1}{n!} \sqrt{|g|} \tilde{\epsilon}_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \\ &= \sqrt{|g|} dx^0 \wedge \cdots \wedge dx^{n-1} \\ &\equiv \sqrt{|g|} d^n x. \end{aligned} \quad (2.96)$$

Notice that the combinatorial factors introduced by the epsilon tensor precisely cancel those from switching from tensor products to wedge products, which is only allowed because the epsilon tensor automatically antisymmetrizes.

The punch line, then, is simple: the integral  $I$  of a scalar function  $\phi$  over an  $n$ -manifold is written as

$$I = \int \phi(x) \sqrt{|g|} d^n x,$$

(2.97)

Given explicit forms for  $\phi(x)$  and  $\sqrt{|g|}$ , such an integral can be directly evaluated by the usual methods of multivariable calculus. The metric determinant serves to automatically take care of the correct transformation properties. You will sometimes see the more abstract notation

$$I = \int \phi(x) \epsilon : \quad (2.98)$$

given (2.96), these two versions convey the same content.

## 2.11 ■ EXERCISES

- Just because a manifold is topologically nontrivial doesn't necessarily mean it can't be covered with a single chart. In contrast to the circle  $S^1$ , show that the infinite cylinder  $\mathbf{R} \times S^1$  can be covered with just one chart, by explicitly constructing the map.
- By clever choice of coordinate charts, can we make  $\mathbf{R}^2$  look like a one-dimensional manifold? Can we make  $\mathbf{R}^1$  look like a two-dimensional manifold? If so, explicitly construct an appropriate atlas, and if not, explain why not. The point of this problem

is to provoke you to think deeply about what a manifold is; it can't be answered rigorously without going into more details about topological spaces. In particular, you might have to forget that you already know a definition of "open set" in the original  $\mathbf{R}^2$  or  $\mathbf{R}^1$ , and define them as being appropriately inherited from the  $\mathbf{R}^1$  or  $\mathbf{R}^2$  to which they are being mapped.

3. Show that the two-dimensional torus  $T^2$  is a manifold, by explicitly constructing an appropriate atlas. (Not a maximal one, obviously.)
4. Verify the claims made about the commutator of two vector fields at the end of Section 2.3 (linearity, Leibniz, component formula, transformation as a vector field).
5. Give an example of two linearly independent, nowhere-vanishing vector fields in  $\mathbf{R}^2$  whose commutator does not vanish. Notice that these fields provide a basis for the tangent space at each point, but it cannot be a coordinate basis since the commutator doesn't vanish.
6. Consider  $\mathbf{R}^3$  as a manifold with the flat Euclidean metric, and coordinates  $\{x, y, z\}$ . Introduce spherical polar coordinates  $\{r, \theta, \phi\}$  related to  $\{x, y, z\}$  by

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta, \end{aligned} \tag{2.99}$$

so that the metric takes the form

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \tag{2.100}$$

- (a) A particle moves along a parameterized curve given by

$$x(\lambda) = \cos \lambda, \quad y(\lambda) = \sin \lambda, \quad z(\lambda) = \lambda. \tag{2.101}$$

Express the path of the curve in the  $\{r, \theta, \phi\}$  system.

- (b) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems.

7. Prolate spheroidal coordinates can be used to simplify the Kepler problem in celestial mechanics. They are related to the usual cartesian coordinates  $(x, y, z)$  of Euclidean three-space by

$$\begin{aligned} x &= \sinh \chi \sin \theta \cos \phi, \\ y &= \sinh \chi \sin \theta \sin \phi, \\ z &= \cosh \chi \cos \theta. \end{aligned}$$

Restrict your attention to the plane  $y = 0$  and answer the following questions.

- (a) What is the coordinate transformation matrix  $\partial x^\mu / \partial x^{\nu'}$  relating  $(x, z)$  to  $(\chi, \theta)$ ?  
(b) What does the line element  $ds^2$  look like in prolate spheroidal coordinates?

8. Verify (2.78): for the exterior derivative of a product of a  $p$ -form  $\omega$  and a  $q$ -form  $\eta$ , we have

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge (d\eta). \tag{2.102}$$

9. In Minkowski space, suppose  $*F = q \sin \theta \, d\theta \wedge d\phi$ .
- Evaluate  $d * F = *J$ .
  - What is the two-form  $F$  equal to?
  - What are the electric and magnetic fields equal to for this solution?
  - Evaluate  $\int_V d*F$ , where  $V$  is a ball of radius  $R$  in Euclidean three space at a fixed moment of time.
10. Consider Maxwell's equations,  $dF = 0$ ,  $d*F = *J$ , in 2-dimensional spacetime. Explain why one of the two sets of equations can be discarded. Show that the electromagnetic field can be expressed in terms of a scalar field. Write out the field equations for this scalar field in component form.
11. There are a lot of motivational words attached here to what is a very simple problem; don't get too distracted. In ordinary electromagnetism with point particles, the part of the action which represents the coupling of the gauge-potential one-form  $A^{(1)}$  to a charged particle can be written  $S = \int_{\gamma} A^{(1)}$ , where  $\gamma$  is the particle worldline. (The superscript on  $A^{(1)}$  is just to remind you that it is a one-form.) For this problem you will consider a theory related to ordinary electromagnetism, but this time in 11 space-time dimensions, with a three-form gauge potential  $A^{(3)}$  and four-form field strength  $F^{(4)} = dA^{(3)}$ . Note that the field strength is invariant under a gauge transformation  $A^{(3)} \rightarrow A^{(3)} + d\lambda^{(2)}$  for any two-form  $\lambda^{(2)}$ .
- What would be the number of spatial dimensions of an object to which this gauge field would naturally couple (for example, ordinary E+M couples to zero-dimensional objects—point particles)?
  - The electric charge of an ordinary electron is given by the integral of the dual of the two-form gauge field strength over a two-sphere surrounding the particle. How would you define the "charge" of the object to which  $A^{(3)}$  couples? Argue that it is conserved if  $d * F^{(4)} = 0$ .
  - Imagine there is a "dual gauge potential"  $\tilde{A}$  that satisfies  $d(\tilde{A}) = *F^{(4)}$ . To what dimensionality object does it naturally couple?
  - The action for the gauge field itself (as opposed to its coupling to other things) will be an integral over the entire 11-dimensional spacetime. What are the terms that would be allowed in such an action that are invariant under "local" gauge transformations, for instance, gauge transformations specified by a two-form  $\lambda^{(2)}$  that vanishes at infinity? Restrict yourself to terms of first, second, or third order in  $A^{(3)}$  and its first derivatives (no second derivatives, no higher-order terms). You may use the exterior derivative, wedge product, and Hodge dual, but not any explicit appearance of the metric.

More background: "Supersymmetry" is a hypothetical symmetry relating bosons (particles with integral spin) and fermions (particles with spin  $\frac{1}{2}, \frac{3}{2}$ , etc.). An interesting feature is that supersymmetric theories are only well-defined in 11 dimensions or less—in larger numbers of dimensions, supersymmetry would require the existence of particles with spins greater than 2, which cannot be consistently quantized. Eleven-dimensional supersymmetry is a unique theory, which naturally includes a three-form gauge potential (not to mention gravity). Recent work has shown that it also includes the various higher-dimensional objects alluded to in this problem (although we've cut some corners here). This theory turns out to be a well-defined limit of something called *M*-theory, which has as other limits various 10-dimensional superstring theories.

## CHAPTER

## 3

## Curvature

## 3.1 ■ OVERVIEW

We all know what curvature means, at least informally, and in the first two chapters of this book we have felt free to refer on occasion to the concept of curvature without giving it a careful definition. Clearly curvature depends somehow on the metric, which defines the geometry of our manifold; but it is not immediately clear how we should attribute curvature to any given metric (since, as we have seen, even the metric of a flat space can look arbitrarily complicated in a sufficiently extravagant coordinate system). As is often the case in mathematics, we require quite a bit of care to formalize our intuition about a concept into a usable mathematical structure; formalizing what we think of as “curvature” is the subject of this chapter.

The techniques we are about to develop are absolutely crucial to the subject; it is safe to say that there is a higher density of useful formulas per page in this chapter than in any of the others. Let’s quickly summarize the most important ones, to provide a roadmap for the formalism to come.

All the ways in which curvature manifests itself rely on something called a “connection,” which gives us a way of relating vectors in the tangent spaces of nearby points. There is a unique connection that we can construct from the metric, and it is encapsulated in an object called the *Christoffel symbol*, given by

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\lambda\sigma}(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}). \quad (3.1)$$

The notation makes  $\Gamma_{\mu\nu}^{\lambda}$  look like a tensor, but in fact it is not; this is why we call it an “object” or “symbol.” The fundamental use of a connection is to take a covariant derivative  $\nabla_{\mu}$  (a generalization of the partial derivative); the covariant derivative of a vector field  $V^{\nu}$  is given by

$$\nabla_{\mu}V^{\nu} = \partial_{\mu}V^{\nu} + \Gamma_{\mu\sigma}^{\nu}V^{\sigma}, \quad (3.2)$$

and covariant derivatives of other sorts of tensors are given by similar expressions. The connection also appears in the definition of *geodesics* (a generalization of the notion of a straight line). A parameterized curve  $x^{\mu}(\lambda)$  is a geodesic if it obeys

$$\frac{d^2x^{\mu}}{d\lambda^2} + \Gamma_{\rho\sigma}^{\mu}\frac{dx^{\rho}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0, \quad (3.3)$$

known as the geodesic equation.

Finally, the technical expression of curvature is contained in the Riemann tensor, a  $(1, 3)$  tensor obtained from the connection by

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \quad (3.4)$$

Everything we want to know about the curvature of a manifold is given to us by the Riemann tensor; it will vanish if and only if the metric is perfectly flat. Einstein's equation of general relativity relates certain components of this tensor to the energy-momentum tensor.

These four equations are all of primary importance in the study of curved manifolds. We will now see how they arise from a careful consideration of how familiar notions of geometry in flat space adapt to this more general context.

### 3.2 ■ COVARIANT DERIVATIVES

In our discussion of manifolds, it became clear that there were various notions we could talk about as soon as the manifold was defined: we could define functions, take their derivatives, consider parameterized paths, set up tensors, and so on. Other concepts, such as the volume of a region or the length of a path, required some additional piece of structure, namely the introduction of a metric. It would be natural to think of the notion of curvature as something that depends exclusively on the metric. In a more careful treatment, however, we find that curvature depends on a connection, and connections may or may not depend on the metric. Nevertheless, we will also show how the existence of a metric implies a certain unique connection, whose curvature may be thought of as that of the metric. This is the connection used in general relativity, so in this particular context it is legitimate to think of curvature as characterizing the metric, without introducing any additional structures.

The connection becomes necessary when we attempt to address the problem of the partial derivative not being a good tensor operator. What we would like is a covariant derivative, that is, an operator that reduces to the partial derivative in flat space with inertial coordinates, but transforms as a tensor on an arbitrary manifold. It is conventional to spend a certain amount of time motivating the introduction of a covariant derivative, but in fact the need is obvious; equations such as  $\partial_\mu T^{\mu\nu} = 0$  must be generalized to curved space somehow. So let's agree that a covariant derivative would be a good thing to have, and go about setting it up.

In flat space in inertial coordinates, the partial derivative operator  $\partial_\mu$  is a map from  $(k, l)$  tensor fields to  $(k, l+1)$  tensor fields, which acts linearly on its arguments and obeys the Leibniz rule on tensor products. All of this continues to be true in the more general situation we would now like to consider, but the map provided by the partial derivative depends on the coordinate system used. We would therefore like to define a **covariant derivative** operator  $\nabla$  to perform the functions of the partial derivative, but in a way independent of coordinates. Rather than simply postulating the answer (which would be perfectly acceptable), let's

motivate it by thinking carefully about what properties a covariant generalization of the partial derivative *should* have—mathematical structures are, after all, invented by human beings, not found lying on sidewalks. We begin by requiring that  $\nabla$  be a map from  $(k, l)$  tensor fields, to  $(k, l + 1)$  tensor fields which has these two properties:

1. linearity:  $\nabla(T + S) = \nabla T + \nabla S$ ;
2. Leibniz (product) rule:  $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$ .

If  $\nabla$  is going to obey the Leibniz rule, it can always be written as the partial derivative plus some linear transformation. That is, to take the covariant derivative we first take the partial derivative, and then apply a correction to make the result covariant. [We aren't going to prove this reasonable-sounding statement; see Wald (1984) if you are interested.] Let's consider what this means for the covariant derivative of a vector  $V^\nu$ . It means that, for each direction  $\mu$ , the covariant derivative  $\nabla_\mu$  will be given by the partial derivative  $\partial_\mu$  plus a correction specified by a set of  $n$  matrices  $(\Gamma_\mu)^\rho_\sigma$  (one  $n \times n$  matrix, where  $n$  is the dimensionality of the manifold, for each  $\mu$ ). In fact the parentheses are usually dropped and we write these matrices, known as the **connection coefficients**, with haphazard index placement as  $\Gamma^\rho_{\mu\sigma}$ . We therefore have

$$\boxed{\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\lambda}^\nu V^\lambda.} \quad (3.5)$$

Notice that in the second term the index originally on  $V$  has moved to the  $\Gamma$ , and a new index is summed over. If this is the expression for the covariant derivative of a vector in terms of the partial derivative, we should be able to determine the transformation properties of  $\Gamma_{\mu\lambda}^\nu$  by demanding that the left-hand side be a  $(1, 1)$  tensor. That is, we want the transformation law to be

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu. \quad (3.6)$$

Let's look at the left side first; we can expand it using (3.5) and then transform the parts that we understand (which is everything except  $\Gamma_{\mu'\lambda'}^{\nu'}$ ):

$$\begin{aligned} \nabla_{\mu'} V^{\nu'} &= \partial_{\mu'} V^{\nu'} + \Gamma_{\mu'\lambda'}^{\nu'} V^{\lambda'} \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma_{\mu'\lambda'}^{\nu'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda. \end{aligned} \quad (3.7)$$

On the right-hand side we can also expand  $\nabla_\mu V^\nu$ :

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma_{\mu\lambda}^\nu V^\lambda. \quad (3.8)$$

These last two expressions are to be equated; the first terms in each are identical and therefore cancel, so we have

$$\Gamma_{\mu'\lambda'}^{\nu'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\lambda \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\lambda} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma_{\mu\lambda}^\nu V^\lambda, \quad (3.9)$$

where we have changed a dummy index from  $\nu$  to  $\lambda$ . This equation must be true for any vector  $V^\lambda$ , so we can eliminate that on both sides. Then the connection coefficients in the primed coordinates may be isolated by multiplying by  $\partial x^\lambda / \partial x^{\sigma'}$  and relabeling  $\sigma' \rightarrow \lambda'$ . The result is

$$\Gamma_{\mu'\lambda'}^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma_{\mu\lambda}^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \quad (3.10)$$

This is not, of course, the tensor transformation law; the second term on the right spoils it. That's okay, because *the connection coefficients are not the components of a tensor*. They are purposefully constructed to be nontensorial, but in such a way that the combination (3.5) transforms as a tensor—the extra terms in the transformation of the partials and the  $\Gamma$ 's exactly cancel. This is why we are not so careful about index placement on the connection coefficients; they are not a tensor, and therefore you should try not to raise and lower their indices.

What about the covariant derivatives of other sorts of tensors? By similar reasoning to that used for vectors, the covariant derivative of a one-form can also be expressed as a partial derivative plus some linear transformation. But there is no reason as yet that the matrices representing this transformation should be related to the coefficients  $\Gamma_{\mu\lambda}^\nu$ . In general we could write something like

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \tilde{\Gamma}_{\mu\nu}^\lambda \omega_\lambda, \quad (3.11)$$

where  $\tilde{\Gamma}_{\mu\nu}^\lambda$  is a new set of matrices for each  $\mu$ . Pay attention to where all of the various indices go. It is straightforward to derive that the transformation properties of  $\tilde{\Gamma}$  must be similar to those of  $\Gamma$ , since otherwise  $\nabla_\mu \omega_\nu$  wouldn't transform as a tensor, but otherwise no relationship has been established. To do so, we need to introduce two new properties that we would like our covariant derivative to have, in addition to the two above:

3. commutes with contractions:  $\nabla_\mu (T^\lambda{}_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$ ,
4. reduces to the partial derivative on scalars:  $\nabla_\mu \phi = \partial_\mu \phi$ .

There is no way to “derive” these properties; we are simply demanding that they be true as part of the definition of a covariant derivative. Note that property 3 is equivalent to saying that the Kronecker delta (the identity map) is covariantly constant,  $\nabla_\mu \delta_\sigma^\lambda = 0$ ; this is certainly a reasonable thing to ask.

Let's see what these new properties imply. Given some one-form field  $\omega_\mu$  and vector field  $V^\mu$ , we can take the covariant derivative of the scalar defined by  $\omega_\lambda V^\lambda$  to get

$$\begin{aligned}\nabla_\mu(\omega_\lambda V^\lambda) &= (\nabla_\mu \omega_\lambda)V^\lambda + \omega_\lambda(\nabla_\mu V^\lambda) \\ &= (\partial_\mu \omega_\lambda)V^\lambda + \tilde{\Gamma}_{\mu\lambda}^\sigma \omega_\sigma V^\lambda + \omega_\lambda(\partial_\mu V^\lambda) + \omega_\lambda \Gamma_{\mu\rho}^\lambda V^\rho.\end{aligned}\quad (3.12)$$

But since  $\omega_\lambda V^\lambda$  is a scalar, this must also be given by the partial derivative:

$$\begin{aligned}\nabla_\mu(\omega_\lambda V^\lambda) &= \partial_\mu(\omega_\lambda V^\lambda) \\ &= (\partial_\mu \omega_\lambda)V^\lambda + \omega_\lambda(\partial_\mu V^\lambda).\end{aligned}\quad (3.13)$$

This can only be true if the terms in (3.12) with connection coefficients cancel each other; that is, rearranging dummy indices, we must have

$$0 = \tilde{\Gamma}_{\mu\lambda}^\sigma \omega_\sigma V^\lambda + \Gamma_{\mu\lambda}^\sigma \omega_\sigma V^\lambda. \quad (3.14)$$

But both  $\omega_\sigma$  and  $V^\lambda$  are completely arbitrary, so

$$\tilde{\Gamma}_{\mu\lambda}^\sigma = -\Gamma_{\mu\lambda}^\sigma. \quad (3.15)$$

The two extra conditions we have imposed therefore allow us to express the covariant derivative of a one-form using the same connection coefficients as were used for the vector, but now with a minus sign (and indices matched up somewhat differently):

$$\boxed{\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma_{\mu\nu}^\lambda \omega_\lambda.} \quad (3.16)$$

It should come as no surprise that the connection coefficients encode all the information necessary to take the covariant derivative of a tensor of arbitrary rank. The formula is quite straightforward; for each upper index you introduce a term with a single  $+\Gamma$ , and for each lower index a term with a single  $-\Gamma$ :

$$\begin{aligned}\nabla_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} &= \partial_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} \\ &\quad + \Gamma_{\sigma\lambda}^{\mu_1} T^{\lambda \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} + \Gamma_{\sigma\lambda}^{\mu_2} T^{\mu_1 \lambda \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} + \cdots \\ &\quad - \Gamma_{\sigma\nu_1}^{\lambda} T^{\mu_1 \mu_2 \cdots \mu_k}_{\lambda \nu_2 \cdots \nu_l} - \Gamma_{\sigma\nu_2}^{\lambda} T^{\mu_1 \mu_2 \cdots \mu_k}_{\nu_1 \lambda \cdots \nu_l} - \cdots.\end{aligned}\quad (3.17)$$

This is the general expression for the covariant derivative. You can check it yourself; it comes from the set of axioms we have established, and the usual requirements that tensors of various sorts be coordinate-independent entities. Sometimes an alternative notation is used; just as commas are used for partial derivatives, semicolons are used for covariant ones:

$$\nabla_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} \equiv T^{\mu_1 \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l ; \sigma}. \quad (3.18)$$

Once again, in this book we will stick to “ $\nabla_\sigma$ .”

To define a covariant derivative, then, we need to put a connection on our manifold, which is specified in some coordinate system by a set of coefficients  $\Gamma_{\mu\nu}^\lambda$  ( $n^3 = 64$  independent components in  $n = 4$  dimensions) that transform according to (3.10). (The name *connection* comes from the fact that it is used to transport vectors from one tangent space to another, as we will soon see; it is sometimes used to refer to the operator  $\nabla$ , and sometimes to the coefficients  $\Gamma_{\mu\nu}^\lambda$ .) Evidently, we could define a large number of connections on any manifold, and each of them implies a distinct notion of covariant differentiation. In general relativity this freedom is not a big concern, because it turns out that every metric defines a unique connection, which is the one used in GR. Let's see how that works.

The first thing to notice is that the difference of two connections is a tensor. Imagine we have defined two different kinds of covariant derivative,  $\nabla_\mu$  and  $\widehat{\nabla}_\mu$ , with associated connection coefficients  $\Gamma_{\mu\nu}^\lambda$  and  $\widehat{\Gamma}_{\mu\nu}^\lambda$ . Then the difference

$$S^\lambda{}_{\mu\nu} = \Gamma_{\mu\nu}^\lambda - \widehat{\Gamma}_{\mu\nu}^\lambda \quad (3.19)$$

is a  $(1, 2)$  tensor. (Notice that we had to choose a convention for index placement.) We could show this by brute force, plugging in the transformation laws for the connection coefficients, but let's be a little more slick. Given an arbitrary vector field  $V^\lambda$ , we know that both  $\nabla_\mu V^\lambda$  and  $\widehat{\nabla}_\mu V^\lambda$  are tensors, so their difference must also be. This difference is simply

$$\begin{aligned} \nabla_\mu V^\lambda - \widehat{\nabla}_\mu V^\lambda &= \partial_\mu V^\lambda + \Gamma_{\mu\nu}^\lambda V^\nu - \partial_\mu V^\lambda - \widehat{\Gamma}_{\mu\nu}^\lambda V^\nu \\ &= S^\lambda{}_{\mu\nu} V^\nu. \end{aligned} \quad (3.20)$$

Since  $V^\lambda$  was arbitrary, and the left-hand side is a tensor,  $S^\lambda{}_{\mu\nu}$  must be a tensor. As a trivial consequence, we learn that any set of connection coefficients can be expressed as some fiducial connection plus a tensorial correction,

$$\Gamma_{\mu\nu}^\lambda = \widehat{\Gamma}_{\mu\nu}^\lambda + S^\lambda{}_{\mu\nu}. \quad (3.21)$$

Next notice that, given a connection specified by  $\Gamma_{\mu\nu}^\lambda$ , we can immediately form another connection simply by permuting the lower indices. That is, the set of coefficients  $\Gamma_{v\mu}^\lambda$  will also transform according to (3.10) (since the partial derivatives appearing in the last term can be commuted), so they determine a distinct connection. There is thus a tensor we can associate with any given connection, known as the **torsion tensor**, defined by

$$T^\lambda{}_{\mu\nu} = \Gamma_{\mu\nu}^\lambda - \Gamma_{v\mu}^\lambda = 2\Gamma_{[\mu\nu]}^\lambda. \quad (3.22)$$

It is clear that the torsion is antisymmetric in its lower indices, and a connection that is symmetric in its lower indices is known as “torsion-free.”

We can now define a unique connection on a manifold with a metric  $g_{\mu\nu}$  by introducing two additional properties:

- torsion-free:  $\Gamma_{\mu\nu}^\lambda = \Gamma_{(\mu\nu)}^\lambda$ .
- metric compatibility:  $\nabla_\rho g_{\mu\nu} = 0$ .

A connection is **metric compatible** if the covariant derivative of the metric with respect to that connection is everywhere zero. This implies a couple of nice properties. First, it's easy to show that both the Levi-Civita tensor and the inverse metric also have zero covariant derivative,

$$\begin{aligned}\nabla_\lambda \epsilon_{\mu\nu\rho\sigma} &= 0 \\ \nabla_\rho g^{\mu\nu} &= 0.\end{aligned}\tag{3.23}$$

Second, a metric-compatible covariant derivative commutes with raising and lowering of indices. Thus, for some vector field  $V^\lambda$ ,

$$g_{\mu\lambda} \nabla_\rho V^\lambda = \nabla_\rho (g_{\mu\lambda} V^\lambda) = \nabla_\rho V_\mu.\tag{3.24}$$

With non-metric-compatible connections we would have to be very careful about index placement when taking a covariant derivative.

Our claim is therefore that there is exactly one torsion-free connection on a given manifold that is compatible with some given metric on that manifold. We do not want to make these two requirements part of the definition of a covariant derivative; they simply single out one of the many possible ones.

We can demonstrate both existence and uniqueness by deriving a manifestly unique expression for the connection coefficients in terms of the metric. To accomplish this, we expand out the equation of metric compatibility for three different permutations of the indices:

$$\begin{aligned}\nabla_\rho g_{\mu\nu} &= \partial_\rho g_{\mu\nu} - \Gamma_{\rho\mu}^\lambda g_{\lambda\nu} - \Gamma_{\rho\nu}^\lambda g_{\mu\lambda} = 0 \\ \nabla_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma_{\mu\nu}^\lambda g_{\lambda\rho} - \Gamma_{\mu\rho}^\lambda g_{\nu\lambda} = 0 \\ \nabla_\nu g_{\rho\mu} &= \partial_\nu g_{\rho\mu} - \Gamma_{\nu\rho}^\lambda g_{\lambda\mu} - \Gamma_{\nu\mu}^\lambda g_{\rho\lambda} = 0.\end{aligned}\tag{3.25}$$

We subtract the second and third of these from the first, and use the symmetry of the connection to obtain

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma_{\mu\nu}^\lambda g_{\lambda\rho} = 0.\tag{3.26}$$

It is straightforward to solve this for the connection by multiplying by  $g^{\sigma\rho}$ . The result is

$$\boxed{\Gamma_{\mu\nu}^\sigma = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})}.\tag{3.27}$$

This formula is one of the most important in this subject; commit it to memory. Of course, we have only proved that if a metric-compatible and torsion-free connection exists, it must be of the form (3.27); you can check for yourself that the right-hand side of (3.27) transforms like a connection.

This connection we have derived from the metric is the one on which conventional general relativity is based. It is known by different names: sometimes the **Christoffel** connection, sometimes the **Levi-Civita** connection, sometimes the **Riemannian** connection. The associated connection coefficients are sometimes called **Christoffel symbols** and written as  $\{\sigma_{\mu\nu}^{\rho}\}$ ; we will sometimes call them Christoffel symbols, but we won't use the funny notation. The study of manifolds with metrics and their associated connections is called Riemannian geometry, or sometimes pseudo-Riemannian when the metric has a Lorentzian signature.

Before putting our covariant derivatives to work, we should mention some miscellaneous properties. First, note that in ordinary flat space there is an implicit connection we use all the time—the Christoffel connection constructed from the flat metric. The coefficients of the Christoffel connection in flat space vanish in Cartesian coordinates, but not in curvilinear coordinate systems. Consider for example the plane in polar coordinates, with metric

$$ds^2 = dr^2 + r^2 d\theta^2. \quad (3.28)$$

The nonzero components of the inverse metric are readily found to be  $g^{rr} = 1$  and  $g^{\theta\theta} = r^{-2}$ . Notice that we use  $r$  and  $\theta$  as indices in an obvious notation. We can compute a typical connection coefficient:

$$\begin{aligned} \Gamma_{rr}^r &= \frac{1}{2} g^{r\rho} (\partial_r g_{r\rho} + \partial_\rho g_{rr} - \partial_\rho g_{rr}) \\ &= \frac{1}{2} g^{rr} (\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}) \\ &\quad + \frac{1}{2} g^{r\theta} (\partial_r g_{r\theta} + \partial_r g_{\theta r} - \partial_\theta g_{rr}) \\ &= \frac{1}{2}(1)(0+0-0) + \frac{1}{2}(0)(0+0-0) \\ &= 0. \end{aligned} \quad (3.29)$$

Sadly, it vanishes. But not all of them do:

$$\begin{aligned} \Gamma_{\theta\theta}^r &= \frac{1}{2} g^{r\rho} (\partial_\theta g_{\theta\rho} + \partial_\theta g_{\rho\theta} - \partial_\rho g_{\theta\theta}) \\ &= \frac{1}{2} g^{rr} (\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}) \\ &= \frac{1}{2}(1)(0+0-2r) \\ &= -r. \end{aligned} \quad (3.30)$$

Continuing to turn the crank, we eventually find

$$\begin{aligned} \Gamma_{\theta r}^r &= \Gamma_{r\theta}^r = 0 \\ \Gamma_{rr}^\theta &= 0 \\ \Gamma_{r\theta}^\theta &= \Gamma_{\theta r}^\theta = \frac{1}{r} \\ \Gamma_{\theta\theta}^\theta &= 0. \end{aligned} \quad (3.31)$$

From these and similar expressions, we can derive formulas for the divergence, gradient, and curl in curvilinear coordinate systems.

Contriwise, even in a curved space it is still possible to make the Christoffel symbols vanish at any one point. This is because, as we argued in the last chapter, we can always make the first derivative of the metric vanish at a point; so by (3.27) the connection coefficients derived from this metric will also vanish. Of course this can only be established at a point, not in some neighborhood of the point. We will discuss this more fully in Section 3.4.

Another useful property is that the formula for the divergence of a vector (with respect to the Christoffel connection) has a simplified form. The covariant divergence of  $V^\mu$  is given by

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma_{\mu\lambda}^\mu V^\lambda. \quad (3.32)$$

It is straightforward to show that the Christoffel connection satisfies

$$\Gamma_{\mu\lambda}^\mu = \frac{1}{\sqrt{|g|}} \partial_\lambda \sqrt{|g|}, \quad (3.33)$$

and we therefore obtain

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} V^\mu). \quad (3.34)$$

There are also formulas for the divergences of higher-rank tensors, but they are generally not such a great simplification.

We use the Christoffel covariant derivative in the curved-space version of Stokes's theorem (see Appendix E). If  $V^\mu$  is a vector field over a region  $\Sigma$  with boundary  $\partial\Sigma$ , Stokes's theorem is

$$\boxed{\int_\Sigma \nabla_\mu V^\mu \sqrt{|g|} d^n x = \int_{\partial\Sigma} n_\mu V^\mu \sqrt{|\gamma|} d^{n-1} x,} \quad (3.35)$$

where  $n_\mu$  is normal to  $\partial\Sigma$ , and  $\gamma_{ij}$  is the induced metric on  $\partial\Sigma$ . If the connection weren't metric-compatible or torsion-free, there would be additional terms in this equation.

The last thing we need to mention is that converting partial derivatives into covariant derivatives is not always necessary in order to construct well-defined tensors; in particular, the exterior derivative and the vector-field commutator are both well-defined in terms of partials, essentially because both involve an antisymmetrization that cancels the nontensorial piece of the partial derivative transformation law. The same feature implies that they could, on the other hand, be equally well-defined in terms of (torsion-free) covariant derivatives; the antisymmetrization causes the connection coefficient terms to vanish. Thus, if  $\nabla$  is the Christoffel connection,  $\omega_\mu$  is a one-form, and  $X^\mu$  and  $Y^\mu$  are vector fields, we

can write

$$(d\omega)_{\mu\nu} = 2\partial_{[\mu}\omega_{\nu]} = 2\nabla_{[\mu}\omega_{\nu]} \quad (3.36)$$

and

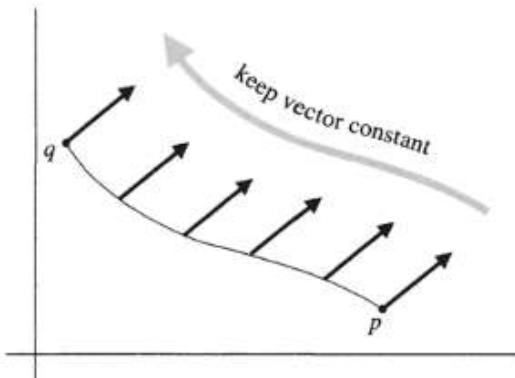
$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu = X^\lambda \nabla_\lambda Y^\mu - Y^\lambda \nabla_\lambda X^\mu. \quad (3.37)$$

If the connection is not torsion-free, the last equalities in these expressions are no longer true; the more fundamental definitions of the exterior derivative and the commutator are those in terms of the partial derivative.

Before moving on, let's review the process by which we have been adding structures to our mathematical constructs. We started with the basic notion of a set, which you were presumed to be familiar with (informally, if not rigorously). We introduced the concept of open subsets of our set; this is equivalent to introducing a topology, and promoted the set to a topological space. Then by demanding that each open set look like a region of  $\mathbf{R}^n$  (with  $n$  the same for each set) and that the coordinate charts be smoothly sewn together, the topological space became a manifold. A manifold is simultaneously a very flexible and powerful structure, and comes equipped naturally with a tangent bundle, tensor bundles of various ranks, the ability to take exterior derivatives, and so forth. We then proceeded to put a metric on the manifold, resulting in a manifold with metric (sometimes Riemannian manifold). Independently of the metric we found we could introduce a connection, allowing us to take covariant derivatives. Once we have a metric, however, there is automatically a unique torsion-free metric-compatible connection. Likewise we could introduce an independent volume form, although one is automatically determined by the metric. In principle there is nothing to stop us from introducing more than one connection, or volume form, or metric, on any given manifold. In general relativity we do have a physical metric, which determines volumes and the covariant derivative, and the independence of these notions is not a crucial feature.

### 3.3 ■ PARALLEL TRANSPORT AND GEODESICS

Now that we know how to take covariant derivatives, let's step back and put this in the context of differentiation more generally. We think of a derivative as a way of quantifying how fast something is changing. In the case of tensors, the crucial issue is “changing with respect to what?” An ordinary function defines a number at each point in spacetime, and it is straightforward to compare two different numbers, so we shouldn't be surprised that the partial derivative of functions remained valid on arbitrary manifolds. But a tensor is a map from vectors and dual vectors to the real numbers, and it's not clear how to compare such maps at different points in spacetime. Since we have successfully constructed a covariant derivative, can we think of it as somehow measuring the rate of change of tensors? The answer turns out to be yes: the covariant derivative quantifies the instantaneous rate of



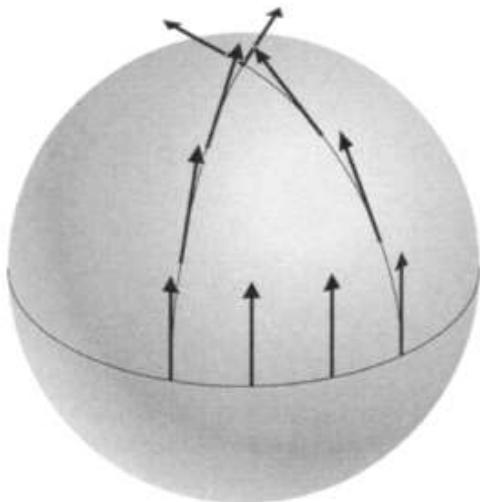
**FIGURE 3.1** In flat space, we can parallel transport a vector by simply keeping its Cartesian components constant.

change of a tensor field in comparison to what the tensor would be if it were “parallel transported.” In other words, a connection defines a specific way of keeping a tensor constant (along some path), on the basis of which we can compare nearby tensors.

It turns out that the concept of parallel transport is interesting in its own right, and worth spending some time thinking about. Recall that in flat space it is unnecessary to be very careful about the fact that vectors are elements of tangent spaces defined at individual points; it is actually very natural to compare vectors at different points (where by “compare” we mean add, subtract, take the dot product, and so on). The reason why it is natural is because it makes sense, in flat space, to move a vector from one point to another while keeping it constant, as illustrated in Figure 3.1. Then, once we get the vector from one point to another, we can do the usual operations allowed in a vector space.

This concept of moving a vector along a path, keeping constant all the while, is known as *parallel transport*. Parallel transport requires a connection to be well-defined; the intuitive manipulation of vectors in flat space makes implicit use of the Christoffel connection on this space. The crucial difference between flat and curved spaces is that, in a curved space, *the result of parallel transporting a vector from one point to another will depend on the path taken between the points*. Without yet assembling the complete mechanism of parallel transport, we can use our intuition about the two-sphere to see that this is the case. Start with a vector on the equator, pointing along a line of constant longitude. Parallel transport it up to the north pole along a line of longitude in the obvious way. Then take the original vector, parallel transport it along the equator by an angle  $\theta$ , and then move it up to the north pole as before. As shown in Figure 3.2, it should be clear that the vector, parallel transported along two paths, arrived at the same destination with two different values (rotated by  $\theta$ ).

It therefore appears as if there is no natural way to uniquely move a vector from one tangent space to another; we can always parallel-transport it, but the result depends on the path, and there is no natural choice of which path to take.



**FIGURE 3.2** Parallel transport on a two-sphere. On a curved manifold, the result of parallel transport can depend on the path taken.

Unlike some of the problems we have encountered, *there is no solution to this one*—we simply must learn to live with the fact that two vectors can only be compared in a natural way if they are elements of the same tangent space. For example, two particles passing by each other have a well-defined relative velocity, which cannot be greater than the speed of light. But two particles at different points on a curved manifold do not have any well-defined notion of relative velocity—the concept simply makes no sense. Of course, in certain special situations it is still useful to talk as if it did make sense, but occasional usefulness is not a substitute for rigorous definition. In cosmology, for example, the light from distant galaxies is redshifted with respect to the frequencies we would observe from a nearby stationary source. Since this phenomenon bears such a close resemblance to the conventional Doppler effect due to relative motion, we are very tempted to say that the galaxies are “receding away from us” at a speed defined by their redshift. At a rigorous level this is nonsense, what Wittgenstein would call a “grammatical mistake”—the galaxies are not receding, since the notion of their velocity with respect to us is not well-defined. What is actually happening is that the metric of spacetime between us and the galaxies has changed (the universe has expanded) along the path of the photon from here to there, leading to an increase in the wavelength of the light. As an example of how you can go wrong, naive application of the Doppler formula to the redshift of galaxies implies that some of them are receding faster than light, in apparent contradiction with relativity. The resolution of this apparent paradox is simply that the very notion of their recession should not be taken literally.

Enough about what we cannot do; let’s see what we can. Parallel transport is supposed to be the curved-space generalization of the concept of “keeping the vector constant” as we move it along a path; similarly for a tensor of arbitrary rank.

Given a curve  $x^\mu(\lambda)$ , the requirement of constancy of a tensor  $T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l}$  along this curve in flat space is simply that the components be constant:

$$\frac{d}{d\lambda} T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} = \frac{dx^\mu}{d\lambda} \frac{\partial}{\partial x^\mu} T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} = 0.$$

To make this properly tensorial we simply replace this partial derivative by a covariant one, and define the **directional covariant derivative** to be

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu. \quad (3.38)$$

This is a map, defined only along the path, from  $(k, l)$  tensors to  $(k, l)$  tensors. We then define **parallel transport** of the tensor  $T$  along the path  $x^\mu(\lambda)$  to be the requirement that the covariant derivative of  $T$  along the path vanishes:

$$\left( \frac{D}{d\lambda} T \right)^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} \equiv \frac{dx^\sigma}{d\lambda} \nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} = 0. \quad (3.39)$$

This is a well-defined tensor equation (since both the tangent vector  $dx^\mu/d\lambda$  and the covariant derivative  $\nabla T$  are tensors), known as the **equation of parallel transport**. For a vector it takes the form

$$\frac{d}{d\lambda} V^\mu + \Gamma_{\sigma\rho}^\mu \frac{dx^\sigma}{d\lambda} V^\rho = 0. \quad (3.40)$$

We can look at the parallel transport equation as a first-order differential equation defining an initial-value problem: given a tensor at some point along the path, there will be a unique continuation of the tensor to other points along the path such that the continuation solves (3.39). We say that such a tensor is parallel-transported.

The notion of parallel transport is obviously dependent on the connection, and different connections lead to different answers. If the connection is metric-compatible, the metric is always parallel transported with respect to it:

$$\frac{D}{d\lambda} g_{\mu\nu} = \frac{dx^\sigma}{d\lambda} \nabla_\sigma g_{\mu\nu} = 0. \quad (3.41)$$

It follows that the inner product of two parallel-transported vectors is preserved. That is, if  $V^\mu$  and  $W^\nu$  are parallel-transported along a curve  $x^\sigma(\lambda)$ , we have

$$\begin{aligned} \frac{D}{d\lambda} (g_{\mu\nu} V^\mu W^\nu) &= \left( \frac{D}{d\lambda} g_{\mu\nu} \right) V^\mu W^\nu + g_{\mu\nu} \left( \frac{D}{d\lambda} V^\mu \right) W^\nu + g_{\mu\nu} V^\mu \left( \frac{D}{d\lambda} W^\nu \right) \\ &= 0. \end{aligned} \quad (3.42)$$

This means that parallel transport with respect to a metric-compatible connection preserves the norm of vectors, the sense of orthogonality, and so on.

With parallel transport defined, the next logical step is to discuss geodesics. A geodesic is the curved-space generalization of the notion of a straight line in Euclidean space. We all know what a straight line is: it's the path of shortest distance

between two points. But there is an equally good definition—a straight line is a path that parallel-transports its own tangent vector. It will turn out that these two concepts coincide if and only if the connection is the Christoffel connection.

We'll start with the second definition (a geodesic is a curve along which the tangent vector is parallel-transported), since it is computationally much more straightforward. The tangent vector to a path  $x^\mu(\lambda)$  is  $dx^\mu/d\lambda$ . The condition that it be parallel transported is thus

$$\frac{D}{d\lambda} \frac{dx^\mu}{d\lambda} = 0, \quad (3.43)$$

or alternatively

$$\boxed{\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0.} \quad (3.44)$$

This is the **geodesic equation**, another one you should memorize. We can easily see that it reproduces the usual notion of straight lines if the connection coefficients are the Christoffel symbols in Euclidean space; in that case we can choose Cartesian coordinates in which  $\Gamma_{\rho\sigma}^\mu = 0$ , and the geodesic equation is just  $d^2x^\mu/d\lambda^2 = 0$ , which is the equation for a straight line.

That was embarrassingly simple; let's turn to the more nontrivial case of the shortest-distance definition. As we know, various subtleties are involved in the definition of distance in a Lorentzian spacetime; for null paths the distance is zero, for timelike paths it's more convenient to use the proper time. So in the name of simplicity let's do the calculation just for a timelike path—the resulting equation will turn out to be good for any path, so we are not losing any generality. We therefore consider the proper time functional,

$$\tau = \int \left( -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda, \quad (3.45)$$

where the integral is over the path. To search for shortest-distance paths, we could do the usual calculus-of-variations treatment to seek critical points of this functional. They will turn out to be curves of *maximum* proper time, consistent with our discussion of the twin paradox in Chapter 1. However, we can simplify the algebra by means of a trick. The integral (3.45) is of the form  $\tau = \int \sqrt{-f} d\lambda$ , where  $f = g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda)$ . The variation looks like

$$\begin{aligned} \delta\tau &= \int \delta \sqrt{-f} d\lambda \\ &= - \int \frac{1}{2} (-f)^{-1/2} \delta f d\lambda. \end{aligned} \quad (3.46)$$

It makes things easier if we now specify that our parameter is the proper time  $\tau$  itself, rather than the arbitrary parameter  $\lambda$ , so that the tangent vector is the

four-velocity  $U^\mu$ . This fixes the value of  $f$ ,

$$f = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = g_{\mu\nu} U^\mu U^\nu = -1. \quad (3.47)$$

From (3.46) we then have

$$\delta\tau = -\frac{1}{2} \int \delta f \, d\tau. \quad (3.48)$$

Stationary points of (3.45)—paths for which  $\delta\tau = 0$ —are therefore equivalent to stationary points (with fixed parameterization) of the simpler integral

$$I = \frac{1}{2} \int f \, d\tau = \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \, d\tau. \quad (3.49)$$

(The  $\frac{1}{2}$  is by no means necessary, but will make things nicer later on.) Taking variations of this expression is thus a shortcut to finding shortest-distance paths, one that we will wisely follow.

Stationary points of  $I$  will of course obey the Euler–Lagrange equations (1.128), but evaluating them involves repeated application of the chain rule, and it is just as simple to directly consider the change in the integral under infinitesimal variations of the path,

$$\begin{aligned} x^\mu &\rightarrow x^\mu + \delta x^\mu \\ g_{\mu\nu} &\rightarrow g_{\mu\nu} + (\partial_\sigma g_{\mu\nu}) \delta x^\sigma. \end{aligned} \quad (3.50)$$

The second line comes from Taylor expansion in curved spacetime, which as you can see, uses the partial derivative, not the covariant derivative. This is because we are simply thinking of the components  $g_{\mu\nu}$  as functions on spacetime in some specific coordinate system. Plugging this into (3.49) and keeping only terms first-order in  $\delta x^\mu$ , we get

$$\delta I = \frac{1}{2} \int \left[ \partial_\sigma g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \delta x^\sigma + g_{\mu\nu} \frac{d(\delta x^\mu)}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d(\delta x^\nu)}{d\tau} \right] d\tau. \quad (3.51)$$

The last two terms can be integrated by parts; for example,

$$\begin{aligned} \frac{1}{2} \int \left[ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d(\delta x^\nu)}{d\tau} \right] d\tau &= -\frac{1}{2} \int \left[ g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \frac{dg_{\mu\nu}}{d\tau} \frac{dx^\mu}{d\tau} \right] \delta x^\nu d\tau \\ &= -\frac{1}{2} \int \left[ g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \partial_\sigma g_{\mu\nu} \frac{dx^\sigma}{d\tau} \frac{dx^\mu}{d\tau} \right] \delta x^\nu d\tau, \end{aligned} \quad (3.52)$$

where we have neglected boundary terms, which vanish because we take our variation  $\delta x^\mu$  to vanish at the endpoints of the path. In the second line we have used

the chain rule on the derivative of  $g_{\mu\nu}$ . The variation (3.51) then becomes, after rearranging some dummy indices,

$$\delta I = - \int \left[ g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right] \delta x^\sigma d\tau. \quad (3.53)$$

Since we are searching for stationary points, we want  $\delta I$  to vanish for any variation  $\delta x^\sigma$ ; this implies

$$g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \quad (3.54)$$

and multiplying by the inverse metric  $g^{\rho\sigma}$  finally leads to

$$\frac{d^2 x^\rho}{d\tau^2} + \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \quad (3.55)$$

We see that this is precisely the geodesic equation (3.40), but with the specific choice of Christoffel connection (3.27). Thus, on a manifold with metric, extremals of the length functional are curves that parallel transport their tangent vector with respect to the Christoffel connection associated with that metric. It doesn't matter if any other connection is defined on the same manifold. Of course, in GR the Christoffel connection is the only one used, so the two notions are the same.

The variational principle provides a convenient way to actually calculate the Christoffel symbols for a given metric. Rather than simply plugging into (3.27), it is often less work to explicitly vary the integral (3.49), with the metric of interest substituted in for  $g_{\mu\nu}$ . An example of this procedure is shown in Section 3.5.

### 3.4 ■ PROPERTIES OF GEODESICS

The primary usefulness of geodesics in general relativity is that they are the paths followed by unaccelerated test particles. A **test particle** is a body that does not itself influence the geometry through which it moves—never perfectly true, but often an excellent approximation. This concept allows us to explore, for example, the properties of the gravitational field around the Sun, without worrying about the field of the planet whose motion we are considering. The geodesic equation can be thought of as the generalization of Newton's law  $\mathbf{f} = m\mathbf{a}$ , for the case  $\mathbf{f} = 0$ , to curved spacetime. It is also possible to introduce forces by adding terms to the right-hand side; in fact, looking back to the expression (1.106) for the Lorentz force in special relativity, it is natural to guess that

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = \frac{q}{m} F^\mu_\nu \frac{dx^\nu}{d\tau}. \quad (3.56)$$

We will talk about this more later, but in fact your guess would be correct.

We should say some more careful words about the parameterization of a geodesic path. When we presented the geodesic equation as the requirement that the tangent vector be parallel-transported, (3.44), we parameterized our path with some parameter  $\lambda$ , whereas when we found the formula (3.55) for the extremal of the spacetime interval, we wound up with a very specific parameterization, the proper time. Of course from the form of (3.55) it is clear that a transformation,

$$\tau \rightarrow \lambda = a\tau + b, \quad (3.57)$$

for some constants  $a$  and  $b$ , leaves the equation invariant. Any parameter related to the proper time in this way is called an **affine parameter**, and is just as good as the proper time for parameterizing a geodesic. What was hidden in our derivation of (3.44) was that *the demand that the tangent vector be parallel-transported actually constrains the parameterization of the curve*, specifically to one related to the proper time by (3.57). In other words, if you start at some point and with some initial direction, and then construct a curve by beginning to walk in that direction and keeping your tangent vector parallel transported, you will not only define a path in the manifold but also (up to linear transformations) define the parameter along the path.

Of course, there is nothing to stop you from using any other parameterization you like, but then (3.44) will not be satisfied. More generally you will satisfy an equation of the form

$$\frac{d^2x^\mu}{d\alpha^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\alpha} \frac{dx^\sigma}{d\alpha} = f(\alpha) \frac{dx^\mu}{d\alpha}, \quad (3.58)$$

for some parameter  $\alpha(\lambda)$ , where  $f(\alpha)$  is related to the affine parameter by

$$f(\alpha) = - \left( \frac{d^2\alpha}{d\lambda^2} \right) \left( \frac{d\alpha}{d\lambda} \right)^{-2}. \quad (3.59)$$

Conversely, if (3.58) is satisfied along a curve you can always find an affine parameter  $\lambda(\alpha)$  for which the geodesic equation (3.44) will be satisfied.

For timelike paths, we can write the geodesic equation in terms of the four-velocity  $U^\mu = dx^\mu/d\tau$  as

$$U^\lambda \nabla_\lambda U^\mu = 0. \quad (3.60)$$

Similarly, in terms of the four-momentum  $p^\mu = mU^\mu$ , the geodesic equation is simply

$$p^\lambda \nabla_\lambda p^\mu = 0. \quad (3.61)$$

This relation expresses the idea that freely-falling particles keep moving in the direction in which their momenta are pointing.

For null paths, the proper time vanishes and  $\tau$  is not an appropriate affine parameter. Nevertheless, it is still perfectly well-defined to ask whether a parameter-

ized path  $x^\mu(\lambda)$  satisfies the geodesic equation (3.44). If a null path is a geodesic for some parameter  $\lambda$ , it will also be a geodesic for any other affine parameter of the form  $a\lambda + b$ . However, there is no preferred choice among these parameters like the proper time is for timelike paths. Once we choose a parameter at some point along the path, of course, there is a unique continuation to the rest of the path if we want to solve the geodesic equation. It is often convenient to choose the normalization of the affine parameter  $\lambda$  along a null geodesic such that  $dx^\mu/d\lambda$  is equal to the momentum four-vector:

$$p^\mu = \frac{dx^\mu}{d\lambda}. \quad (3.62)$$

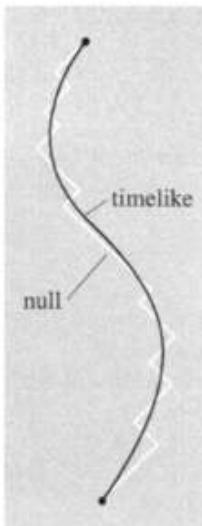
This is in contrast to timelike paths, where  $dx^\mu/d\tau$  is the momentum per unit mass. Then an observer with four-velocity  $U^\mu$  measures the energy of the particle (or equivalently the frequency, since we are setting  $\hbar = 1$ ) to be

$$E = -p_\mu U^\mu. \quad (3.63)$$

This expression always tells us the energy of a particle with momentum  $p^\mu$  as measured by an observer with four-velocity  $U^\mu$ , whether  $p^\mu$  is null or timelike; you can check it by going to locally inertial coordinates. (A caveat: this expression for  $E$  does not include potential energy, only the intrinsic energy from motion and inertia. In a general spacetime there will not be a well-defined notion of gravitational potential energy, although in special cases it does exist.)

An important property of geodesics in a spacetime with Lorentzian metric is that the character (timelike/null/spacelike) of the geodesic, relative to a metric-compatible connection, never changes. This is simply because parallel transport preserves inner products, and the character is determined by the inner product of the tangent vector with itself. This is why we were consistent to consider purely timelike paths when we derived (3.55); for spacelike paths we would have derived the same equation, since the only difference is an overall minus sign in the final answer.

Let's now explain the earlier remark that timelike geodesics are maxima of the proper time. The reason we know this is true is that, given any timelike curve (geodesic or not), we can approximate it to arbitrary accuracy by a null curve. To do this all we have to do is to consider "jagged" null curves that follow the timelike one, as portrayed in Figure 3.3. As we increase the number of sharp corners, the null curve comes closer and closer to the timelike curve while still having zero path length. Timelike geodesics cannot therefore be curves of minimum proper time, since they are always infinitesimally close to curves of less proper time (zero, in fact); actually they maximize the proper time. This is how you can remember which twin in the twin paradox ages more—the one who stays home is basically on a geodesic, and therefore experiences more proper time. Of course even this is being a little cavalier; actually every time we say "maximize" or "minimize" we should add the modifier "locally." Often the case is that between two points on a manifold there is more than one geodesic. For instance, on  $S^2$  we can



**FIGURE 3.3** We can always approximate a timelike path by a sequence of null paths with a total path length of zero. Hence, timelike geodesics must be maxima of the proper time rather than minima.

draw a great circle through any two points, and imagine traveling between them either the short way or the long way around. One of these is obviously longer than the other, although both are stationary points of the length functional.

Geodesics provide a convenient way of mapping the tangent space  $T_p$  of a point  $p$  to a region of the manifold that contains  $p$ , called the **exponential map**. This map in turn defines a set of coordinates for this region that are automatically the locally inertial coordinates discussed in Section 2.5 [coordinates  $x^{\hat{\mu}}$  around a point  $p$  such that  $g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}}$  and  $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0$ ]. We begin by noticing that any vector  $k \in T_p$  defines a unique geodesic passing through it, for which  $k$  is the tangent vector at  $p$ , and  $\lambda(p) = 0$ :

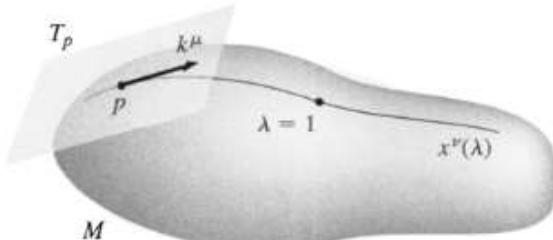
$$\frac{dx^\mu}{d\lambda}(\lambda = 0) = k^\mu. \quad (3.64)$$

Uniqueness follows from the fact that the geodesic equation is a second-order differential equation, and specifying initial data in the form  $x^\mu(p)$  and  $k^\mu = (dx^\mu/d\lambda)(p)$  completely determines a solution. On this geodesic there will be a unique point in  $M$  for which  $\lambda = 1$ . The exponential map at  $p$ ,  $\exp_p : T_p \rightarrow M$ , is then defined as

$$\exp_p(k) = x^\nu(\lambda = 1), \quad (3.65)$$

where  $x^\nu(\lambda)$  solves the geodesic equation subject to (3.64), as shown in Figure 3.4.

For some set of tangent vectors  $k^\mu$  near the zero vector, this map will be well-defined, and in fact invertible. Depending on the geometry, however, different geodesics emanating from a single point may eventually cross, at which point  $\exp_p : T_p \rightarrow M$  is no longer one-to-one. Furthermore, the range of the exponential map is not necessarily the whole manifold, and the domain is not necessarily the whole tangent space. The range can fail to be all of  $M$  simply because there can be two points that are not connected by any geodesic. An example is given by anti-de Sitter space, discussed in Chapter 8. The domain can fail to be all of  $T_p$  because a geodesic may run into a singularity, which we think of as “the edge of the manifold.” Manifolds that have such singularities are known as **geodesically incomplete**. In a more careful discussion it would actually be the



**FIGURE 3.4** The exponential map takes a vector in  $T_p$  to a point in  $M$  that lies at unit affine parameter along the geodesic to which the vector is tangent.

other way around: the best way we have of defining a singularity is as a place where geodesics appear to “end,” after we remove trivial cases in which a part of the manifold is artificially excluded by hand. See Wald (1984) or Hawking and Ellis (1973). This problem is not merely technical; the singularity theorems of Hawking and Penrose state that, for certain matter content, spacetimes in general relativity are almost guaranteed to be geodesically incomplete. As examples, two of the most useful spacetimes in GR—the Schwarzschild solution describing black holes and the Friedmann–Robertson–Walker solutions describing homogeneous, isotropic cosmologies—both feature important singularities; these will be discussed in later chapters.

We now use the exponential map to construct locally inertial coordinates. The easy part is to find basis vectors  $\{\hat{e}_{(\hat{\mu})}\}$  for  $T_p$  such that the components of the metric are those of the canonical form:

$$g_{\hat{\mu}\hat{\nu}} = g(\hat{e}_{(\hat{\mu})}, \hat{e}_{(\hat{\nu})}) = \eta_{\hat{\mu}\hat{\nu}}. \quad (3.66)$$

Here  $g( , )$  denotes the metric, thought of as a multilinear map from  $T_p \times T_p$  to  $\mathbf{R}$ . And the hats have different meanings: over  $e$  they remind us that it’s a basis vector, and over the indices they remind us that we are in locally inertial coordinates (as we shall see). This is easy because it’s just linear algebra, not yet referring to coordinates; starting with any set of components for  $g_{\mu\nu}$ , we can always diagonalize this matrix and then rescale our basis vectors to satisfy (3.66). The hard part, we would expect, is finding a coordinate system  $x^{\hat{\mu}}$  for which the basis vectors  $\{\hat{e}_{(\hat{\mu})}\}$  comprise the coordinate basis,  $\hat{e}_{(\hat{\mu})} = \partial_{\hat{\mu}}$ , and such that the first partial derivatives of  $g_{\hat{\mu}\hat{\nu}}$  vanish. In fact, however, the exponential map achieves this automatically. For any point  $q$  sufficiently close to  $p$ , there is a unique geodesic path connecting  $p$  to  $q$ , and a unique parameterization  $\lambda$  such that  $\lambda(p) = 0$  and  $\lambda(q) = 1$ . At  $p$  the tangent vector  $k$  to this geodesic can be written as a linear combination of our basis vectors,  $k = k^{\hat{\mu}} \hat{e}_{(\hat{\mu})}$ . We define the sought-after coordinates  $x^{\hat{\mu}}$  simply to be these components:  $x^{\hat{\mu}}(q) = k^{\hat{\mu}}$ . In other words, we have defined the coordinates  $x^{\hat{\mu}}(q)$  to be the components (with respect to our normalized basis  $\{\hat{e}_{(\hat{\mu})}\}$ ) of the tangent vector  $k$  that gets mapped to  $q$  by  $\exp_p$ . Coordinates constructed in this way are known as **Riemann normal coordinates** at  $p$ .

We still need to verify that these Riemann normal coordinates satisfy  $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0$ . Note that a ray in the tangent space (a parameterized set of vectors of the form  $\lambda k^{\hat{\mu}}$ , for some fixed vector  $k^{\hat{\mu}}$ ) gets mapped to a geodesic by the exponential map. Therefore, in Riemann normal coordinates, a curve  $x^{\hat{\mu}}(\lambda)$  of the form

$$x^{\hat{\mu}}(\lambda) = \lambda k^{\hat{\mu}} \quad (3.67)$$

will solve the geodesic equation. Indeed, *any* geodesic through  $p$  may be expressed this way, for some appropriate vector  $k^{\hat{\mu}}$ . We therefore have

$$\frac{d^2 x^{\hat{\mu}}}{d\lambda^2} = 0 \quad (3.68)$$

along any geodesic through  $p$  in this coordinate system. But, by the geodesic equation, we also have

$$\frac{d^2x^{\hat{\mu}}}{d\lambda^2}(p) = -\Gamma_{\hat{\rho}\hat{\sigma}}^{\hat{\mu}}(p)k^{\hat{\rho}}k^{\hat{\sigma}}, \quad (3.69)$$

where  $k^{\hat{\rho}} = (dx^{\hat{\rho}}/d\lambda)(p)$ . Since this holds for arbitrary  $k^{\hat{\rho}}$ , we conclude that

$$\Gamma_{\hat{\rho}\hat{\sigma}}^{\hat{\mu}}(p) = 0. \quad (3.70)$$

Now apply metric compatibility:

$$\begin{aligned} 0 &= \nabla_{\hat{\sigma}}g_{\hat{\mu}\hat{\nu}} \\ &= \partial_{\hat{\sigma}}g_{\hat{\mu}\hat{\nu}} - \Gamma_{\hat{\sigma}\hat{\mu}}^{\hat{\lambda}}g_{\hat{\lambda}\hat{\nu}} - \Gamma_{\hat{\sigma}\hat{\nu}}^{\hat{\lambda}}g_{\hat{\mu}\hat{\lambda}} \\ &= \partial_{\hat{\sigma}}g_{\hat{\mu}\hat{\nu}}, \end{aligned} \quad (3.71)$$

where all quantities are evaluated at  $p$ . We see that Riemann normal coordinates provide a realization of the locally inertial coordinates discussed in Section 2.5. They are not unique; there are an infinite number of non-Riemann-normal coordinate systems in which  $g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}}$  and  $\partial_{\hat{\sigma}}g_{\hat{\mu}\hat{\nu}}(p) = 0$ , but in an expansion around  $p$  they will differ from the Riemann normal coordinates only at third order in  $x^{\hat{\mu}}$ .

### 3.5 ■ THE EXPANDING UNIVERSE REVISITED

Let's put some of the technology we have developed to work in understanding a simple metric. Recall the expanding-universe metric we studied in Chapter 2,

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] \\ &= -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \end{aligned} \quad (3.72)$$

This metric describes a universe consisting of flat spatial sections expanding as a function of time, with the relative distance between particles at fixed spatial coordinates growing proportionally to the scale factor  $a(t)$ .

Faced with a metric, the first thing we do is to calculate the Christoffel symbols. As mentioned at the end of Section 3.3, the easiest technique for doing so is actually to vary explicitly an integral of the form (3.49). Plugging in the metric under consideration, we have

$$I = \frac{1}{2} \int \left[ -\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \right] d\tau. \quad (3.73)$$

The technique is to consider variations  $x^\mu \rightarrow x^\mu + \delta x^\mu$  and demand that  $\delta I$  vanish. We get  $n$  equations on an  $n$ -dimensional manifold (in this case  $n = 4$ ), one for each  $\mu$ ; each equation corresponds to a component of the geodesic equation

(3.44). In the equation derived by varying with respect to  $x^\mu$ , the coefficient of  $(dx^\rho/d\tau)(dx^\sigma/d\tau)$  will be  $\Gamma_{\rho\sigma}^\mu$ .

For the metric (3.72), we need to consider separately variations with respect to  $x^0 = t$  and one of the  $x^i$ 's (it doesn't matter which one, since the results for each spacelike direction will be equivalent). Let's start with  $t \rightarrow t + \delta t$ . The nontrivial time dependence comes from the scale factor, for which, to first order,

$$a(t + \delta t) = a(t) + \dot{a}\delta t, \quad (3.74)$$

where  $\dot{a} = da/dt$ . We therefore have

$$\begin{aligned} \delta I &= \frac{1}{2} \int \left[ -2 \frac{dt}{d\tau} \frac{d(\delta t)}{d\tau} + 2a\dot{a}\delta_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \delta t \right] d\tau \\ &= \int \left[ \frac{d^2 t}{d\tau^2} + a\dot{a}\delta_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \right] \delta t d\tau, \end{aligned} \quad (3.75)$$

where we have dropped a boundary term after integrating by parts (as always). Setting the coefficient of  $\delta t$  equal to zero implies

$$\frac{d^2 t}{d\tau^2} + a\dot{a}\delta_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0, \quad (3.76)$$

which is supposed to be equivalent to the geodesic equation (with  $\mu = 0$ )

$$\frac{d^2 x^0}{d\tau^2} + \Gamma_{\rho\sigma}^0 \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0. \quad (3.77)$$

Comparison of these two equations implies

$$\begin{aligned} \Gamma_{00}^0 &= 0, \\ \Gamma_{i0}^0 &= \Gamma_{0i}^0 = 0, \\ \Gamma_{ij}^0 &= a\dot{a} \delta_{ij}. \end{aligned} \quad (3.78)$$

We can repeat this procedure for a spatial coordinate,  $x^i \rightarrow x^i + \delta x^i$ . The variation is then

$$\begin{aligned} \delta I &= \frac{1}{2} \int a^2 \left( 2\delta_{ij} \frac{dx^i}{d\tau} \frac{d(\delta x^j)}{d\tau} \right) d\tau \\ &= - \int \left( a^2 \frac{d^2 x^i}{d\tau^2} + 2a \frac{da}{d\tau} \frac{dx^i}{d\tau} \right) \delta_{ij} \delta x^j d\tau. \end{aligned} \quad (3.79)$$

We can express  $da/d\tau$  in terms of  $\dot{a}$  by using the chain rule,

$$\frac{da}{d\tau} = \dot{a} \frac{dt}{d\tau}. \quad (3.80)$$

Then setting the coefficient of  $\delta x^j$  equal to zero in (3.79) implies

$$\frac{d^2x^i}{d\tau^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau} = 0. \quad (3.81)$$

Comparing to the geodesic equation, we find that the Christoffel symbols must satisfy

$$\Gamma_{\rho\sigma}^i \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau}. \quad (3.82)$$

The Christoffel symbols are therefore given by

$$\begin{aligned} \Gamma_{00}^i &= 0 \\ \Gamma_{j0}^i &= \Gamma_{0j}^i = \frac{\dot{a}}{a}\delta_j^i \\ \Gamma_{jk}^i &= 0. \end{aligned} \quad (3.83)$$

Together, (3.78) and (3.83) are all of the connection coefficients for the metric (3.72). These are, of course, necessary both for studying geodesics of the space-time and for taking covariant derivatives; in fact, (3.76) and (3.81) together are the geodesic equation. Let's put this to work by solving for null geodesics, those followed by massless particles such as photons, for which we have to use  $\lambda$  rather than  $\tau$  as a parameter. Without loss of generality we can consider paths along the  $x$ -direction, for which  $x^\mu(\lambda) = \{t(\lambda), x(\lambda), 0, 0\}$ . It is trivial to solve for null paths of this sort, using  $ds^2 = 0$ . We have

$$0 = -dt^2 + a^2(t)dx^2, \quad (3.84)$$

which implies

$$\frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda}. \quad (3.85)$$

In Section 2.6 we solved this for  $a = t^q$ , but here we will remain more general. Also, we have chosen to consider paths moving in the positive  $x$ -direction, which determines the sign of  $dx/d\lambda$ . We must distinguish, however, between “null paths” and “null geodesics”: the latter are a much more restrictive class, and to show that these paths are geodesics, we need to solve for the coordinates  $t$  and  $x$  in terms of the parameter  $\lambda$ .

Let's solve for  $dt/d\lambda$ , which will turn out to be the quantity in which we are most interested. Plugging the null condition (3.85) into the  $\mu = 0$  component of the geodesic equation (3.76), and remembering to replace  $\tau \rightarrow \lambda$ , we get

$$\frac{d^2t}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 = 0. \quad (3.86)$$

It is straightforward to verify that this is solved by

$$\frac{dt}{d\lambda} = \frac{\omega_0}{a}, \quad (3.87)$$

where  $\omega_0$  is a constant. For a given  $a(t)$ , this could be instantly integrated to yield  $t(\lambda)$ . But more interesting is to consider the energy  $E$  of the photon as it would be measured by a comoving observer (one at fixed spatial coordinates), who would have four-velocity

$$U^\mu = (1, 0, 0, 0). \quad (3.88)$$

Don't get tricked into thinking that the timelike component of the four-velocity of a particle at rest will always equal unity; we need to satisfy the normalization condition  $g_{\mu\nu}U^\mu U^\nu = -1$ , which in the rest frame ( $U^i = 0$ ) implies  $U^0 = \sqrt{-g_{00}}$ . According to (3.63), and using  $p^\mu = dx^\mu/d\lambda$ , we have

$$\begin{aligned} E &= -p_\mu U^\mu \\ &= -g_{00} \frac{dx^0}{d\lambda} U^0 \\ &= \frac{\omega_0}{a}. \end{aligned} \quad (3.89)$$

We see why the notation  $\omega_0$  was chosen for the constant of proportionality in (3.87):  $\omega_0$  is simply the frequency of the photon when  $a = 1$ . Recall that  $E = \hbar\omega$ , and we are using units in which  $\hbar = 1$ .

We have uncovered a profound phenomenon: the **cosmological redshift**. A photon emitted with energy  $E_1$  at scale factor  $a_1$  and observed with energy  $E_2$  at scale factor  $a_2$  will have

$$\frac{E_2}{E_1} = \frac{a_1}{a_2}. \quad (3.90)$$

This is called a “redshift” because the wavelength of the photon is inversely proportional to the frequency, and in an expanding universe the wavelength therefore grows with time. As a practical matter this provides an easy way to measure the change in the scale factor between us and distant galaxies, and also serves as a proxy for the distance: since the universe has been monotonically expanding, a greater redshift implies a greater distance. In conventional notation, the amount of redshift is denoted by

$$z = \frac{\omega_1 - \omega_2}{\omega_2} = \frac{a_2}{a_1} - 1, \quad (3.91)$$

so that  $z$  vanishes if there has been no expansion, for instance, if the emitter and observer are so close that there hasn't been enough time for the universe to appreciably expand.

As mentioned in Section 3.3, the cosmological redshift is *not* a Doppler shift (despite the understandable temptation to refer to the “velocity” of receding galaxies). Now we can understand this statement quantitatively. You might imagine that, as far as the behavior of emitted photons is concerned, there is little difference between two galaxies physically moving apart in a flat spacetime and two galaxies at fixed comoving coordinates in an expanding spacetime. But let’s consider a specific (unrealistic, but educational) example. Start with flat spacetime, and imagine that our two galaxies are initially not moving apart, but are at rest in some globally inertial coordinate system. One emits a photon toward the other; while the photon is traveling, we quickly move the two galaxies apart until they are twice their original separation, then leave them stationary at that distance; and then the photon is absorbed by the second galaxy. Clearly there will be no Doppler shift, since the galaxies were at rest both at emission and absorption. Now consider the analogous phenomenon in an expanding spacetime, with the galaxies stuck at fixed comoving coordinates. We begin with the scale factor constant (the universe is not expanding). One galaxy emits a photon, and we imagine that during the photon’s journey the universe starts expanding until the scale factor is twice its original size, and then stops expanding before the photon is absorbed. In this case there certainly will be a redshift, despite the fact that there was no “relative motion” (an ill-defined concept in any case) at either absorption or emission; the photon’s wavelength will have doubled as the scale factor doubled, so we observe a redshift  $z = 1$ . This demonstrates the conceptual distinction between the cosmological redshift and the conventional Doppler effect.

Beyond the geodesic equation, covariant derivatives will play a role in generalizing laws of physics from the flat spacetime of special relativity to the curved geometry of general relativity. As we will discuss in more detail in the next chapter, a simple rule of thumb is simply to replace all partial derivatives by covariant derivatives, and all appearances of the flat spacetime metric  $\eta_{\mu\nu}$  by the curved metric  $g_{\mu\nu}$ . For example, the energy-momentum conservation equation of special relativity,  $\partial_\mu T^{\mu\nu} = 0$ , where  $T^{\mu\nu}$  is the energy-momentum tensor, becomes

$$\nabla_\mu T^{\mu\nu} = 0. \quad (3.92)$$

In cosmology, we typically model the matter filling the universe as a perfect fluid; the corresponding energy-momentum tensor comes from generalizing (1.114) to curved spacetime,

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu}. \quad (3.93)$$

Recall that  $\rho$  is the energy density,  $p$  is the pressure, and  $U^\mu$  is the four-velocity of the fluid. For the metric (3.72) the components of the inverse metric are

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^{-2} & & \\ & & a^{-2} & \\ & & & a^{-2} \end{pmatrix}. \quad (3.94)$$

We can take the fluid to be in its rest frame in these coordinates, so that the components of the four-velocity are  $U^\mu = (1, 0, 0, 0)$ . In fact the fluid would have to be in its rest frame for this particular metric to solve Einstein's equation, as we will later see. The energy-momentum tensor therefore takes the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & a^{-2}p & & \\ & & a^{-2}p & \\ & & & a^{-2}p \end{pmatrix}. \quad (3.95)$$

Note that these components are specific to the metric (3.72), and will generally look different for other metrics.

Let's see what the energy-momentum conservation equation  $\nabla_\mu T^{\mu\nu} = 0$  implies for a perfect fluid in an expanding universe. The rule for covariant derivatives (3.17) implies

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma_{\mu\lambda}^\mu T^{\lambda\nu} + \Gamma_{\nu\lambda}^\nu T^{\mu\lambda} = 0. \quad (3.96)$$

This equation has four components, one for each  $\mu$ , although the three  $\mu = i \in \{1, 2, 3\}$  are equivalent. Let's first look at the  $\nu = 0$  component, piece by piece. The first term is straightforward,

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} = \dot{\rho}. \quad (3.97)$$

The second term is

$$\Gamma_{\mu\lambda}^\mu T^{\lambda 0} = \Gamma_{\mu 0}^\mu T^{00} = 3\frac{\dot{a}}{a}\rho, \quad (3.98)$$

and the third term is

$$\Gamma_{\mu\lambda}^0 T^{\mu\lambda} = \Gamma_{00}^0 T^{00} + \Gamma_{11}^0 T^{11} + \Gamma_{22}^0 T^{22} + \Gamma_{33}^0 T^{33} = 3\frac{\dot{a}}{a}p. \quad (3.99)$$

In each of these sets of equations, we have first invoked the fact that  $T^{\mu\nu}$  is diagonal, and then used the explicit formulae for the energy-momentum tensor and the connection coefficients in this metric. All together, then, we find

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p). \quad (3.100)$$

Now let's look at one of the spatial components, choosing  $\nu = 1$  for definiteness. Once again working piece by piece, we have for the first term in (3.96),

$$\partial_\mu T^{\mu 1} = \partial_1 T^{11} = a^{-2} \partial_x p. \quad (3.101)$$

The second and third terms are

$$\Gamma_{\mu\lambda}^\mu T^{\lambda 1} = \Gamma_{\mu 1}^\mu T^{11} = 0, \quad (3.102)$$

and

$$\Gamma_{\mu\lambda}^1 T^{\mu\lambda} = \Gamma_{00}^1 T^{00} + \Gamma_{11}^1 T^{11} + \Gamma_{22}^1 T^{22} + \Gamma_{33}^1 T^{33} = 0. \quad (3.103)$$

Equivalent results will hold for  $v = 2$  and  $v = 3$ . So the spatial components of the energy-momentum conservation equation simply amount to

$$\partial_i p = 0. \quad (3.104)$$

It is illuminating to compare these results to those we would obtain in Minkowski spacetime, which can be found by simply setting  $a = 1$ ,  $\dot{a} = 0$ . The pressure-gradient equation (3.104) is unaffected, so there is no effect of curvature on the spatial components: for a fluid that is motionless as measured by a comoving observer, the pressure must be constant throughout space. For the timelike component, on the other hand, the expansion of the universe introduces a nonzero right-hand side to (3.100). To understand the consequences of this new feature, let us consider equations of state of the form

$$p = w\rho, \quad (3.105)$$

where  $w$  is some constant. Then (3.100) becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}, \quad (3.106)$$

which can be solved to yield

$$\rho \propto a^{-3(1+w)}. \quad (3.107)$$

In Chapter 1 we mentioned three kinds of perfect fluid with equations of state of the form (3.105): dust, with  $w = 0$ ; radiation, with  $w = \frac{1}{3}$ ; and vacuum, with  $w = -1$ . A set of nonrelativistic, noninteracting particles behaves like dust; a set of photons or other massless particles behaves like radiation; and a nonzero constant energy density throughout spacetime acts like vacuum. From (3.107) we see that the equation of state determines how the energy density evolves as the universe expands:

matter	$p = 0$	$\rho \propto a^{-3}$
radiation	$p = \frac{1}{3}\rho$	$\rho \propto a^{-4}$
vacuum	$p = -\rho$	$\rho = \text{constant}$

(3.108)

We will explore these behaviors more thoroughly in Chapter 8; for right now let's simply note that they make sense. For dust, the energy density comes from the rest mass of each particle; if all the particles have mass  $m$ , the energy density is simply  $\rho = nm$ , where  $n$  is the number density. Since the number density goes down as  $a^{-3}$  (the physical volume of a comoving region goes up, while the

total number of particles stays constant), while the masses remain unchanged, we expect that the energy density obeys  $\rho \propto a^{-3}$ . For radiation, meanwhile, the energy of each particle (such as a photon) redshifts away as  $a^{-1}$  as the universe expands; since the number density still obeys  $n \propto a^{-3}$ , we expect that  $\rho \propto a^{-4}$ . Finally, the vacuum energy density is an intrinsic and unchanging amount of energy in any physical volume; it doesn't redshift at all as the universe expands, so we get  $\rho = \text{constant}$ .

This example brings to life the differences between flat and curved spacetimes. For example, consider what we might be tempted to call the “energy,” the integral over space of the energy density:  $E = \int \rho a^3 d^3x$ , where the boundaries are at fixed comoving coordinates, so the region expands along with the universe, and the factor of  $a^3$  comes from the square root of the determinant of the spatial metric  $a^2 \delta_{ij}$ . This number is clearly not conserved in general. For dust, since  $\rho \propto a^{-3}$ ,  $E$  remains constant as the universe expands; but for radiation it decreases, and for vacuum energy it increases. This is upsetting, since conservation of energy is one of the more cherished principles of physics. What has happened? One way of thinking about this is from the viewpoint of Noether’s theorem, which states that every symmetry implies a conserved quantity. Energy is the conserved quantity that derives from invariance under time translations. Clearly, in an expanding universe, the energy-momentum tensor is defined on a background that is changing with time; therefore there is no reason to believe that the energy should be conserved. (“There is no timelike Killing vector,” in the language to be introduced in Section 3.8.) Nevertheless, we continue to refer to  $\nabla_\mu T^{\mu\nu} = 0$  as the energy-momentum conservation equation. It conveys the idea that there is a definite law obeyed by the energy-momentum tensor, even if there is no integral corresponding to a conserved energy. The transition from flat to curved spacetime induces the additional Christoffel-symbol terms in (3.96); these terms serve, roughly speaking, to allow transfer of energy between the matter fields (comprising  $T^{\mu\nu}$ ) and the gravitational field. This notion is not very formal, however, and you shouldn’t push it too far—it turns out to be difficult to associate a local energy density to the gravitational field, although it is possible in certain circumstances.

Of course there is also a notion of time-translation invariance that refers not to the background spacetime, but to the theory itself (that is, to the equations that define the theory rather than a specific solution to them). We haven’t yet developed the dynamical equations of general relativity, but they will turn out to be invariant under time translations, as well as under any other sort of coordinate transformations, as indeed they must be. This general coordinate invariance leads to a set of constraints on allowed configurations of the theory, and generally requires a more subtle analysis.

In the end, you should come to accept that there is a profound difference between flat and curved spacetimes, and some of our favorite notions from flat-spacetime physics will be seriously modified in this more general context. This is not a sign of any flaw in general relativity, but a natural consequence of discarding the rigid spacetime geometry we learn to take for granted.

### 3.6 ■ THE RIEMANN CURVATURE TENSOR

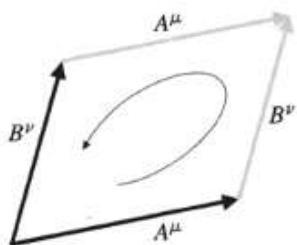
Having set up the machinery of covariant derivatives and parallel transport, we are at last prepared to discuss curvature proper. The curvature is quantified by the Riemann tensor, which is derived from the connection. The idea behind this measure of curvature is that we know what we mean by “flatness” of a connection—the conventional (and usually implicit) Christoffel connection associated with a Euclidean or Minkowskian metric has a number of properties that can be thought of as different manifestations of flatness. These include the fact that parallel transport around a closed loop leaves a vector unchanged, that covariant derivatives of tensors commute, and that initially parallel geodesics remain parallel. As we shall see, the Riemann tensor arises when we study how any of these properties are altered in more general contexts.

We have already argued, using the two-sphere as an example, that parallel transport of a vector around a closed loop in a curved space will lead to a transformation of the vector. The resulting transformation depends on the total curvature enclosed by the loop; it would be more useful to have a local description of the curvature at each point, which is what the Riemann tensor is supposed to provide. One conventional way to introduce the Riemann tensor, therefore, is to consider parallel transport around an infinitesimal loop. We are not going to do that here, but take a more direct route. Nevertheless, even without working through the details, it is possible to see what form the answer should take. Since spacetime looks flat in sufficiently small regions, our loop will be specified by two (infinitesimal) vectors  $A^\mu$  and  $B^\nu$ . We imagine parallel transporting a vector  $V^\mu$  by first moving it in the direction of  $A^\mu$ , then along  $B^\nu$ , then backward along  $A^\mu$  and  $B^\nu$  to return to the starting point, as shown in Figure 3.5. We know the action of parallel transport is independent of coordinates, so there should be some tensor that tells us how the vector changes when it comes back to its starting point; it will be a linear transformation on a vector, and therefore involve one upper and one lower index. But it will also depend on the two vectors  $A$  and  $B$  that define the loop; therefore there should be two additional lower indices to contract with  $A^\mu$  and  $B^\nu$ . Furthermore, the tensor should be antisymmetric in these two indices, since interchanging the vectors corresponds to traversing the loop in the opposite direction, and should give the inverse of the original answer. This is consistent with the fact that the transformation should vanish if  $A$  and  $B$  are the same vector. We therefore expect that the expression for the change  $\delta V^\rho$  experienced by this vector when parallel transported around the loop should be of the form

$$\delta V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma A^\mu B^\nu, \quad (3.109)$$

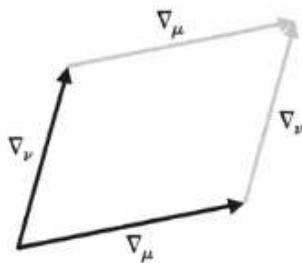
where  $R^\rho_{\sigma\mu\nu}$  is a  $(1, 3)$  tensor known as the **Riemann tensor** (or simply curvature tensor). It is antisymmetric in the last two indices:

$$R^\rho_{\sigma\mu\nu} = -R^\rho_{\sigma\nu\mu}. \quad (3.110)$$



**FIGURE 3.5** An infinitesimal loop defined by two vectors  $A^\mu$  and  $B^\nu$ .

Of course, if (3.109) is taken as a definition of the Riemann tensor, a convention needs to be chosen for the ordering of the indices. There is no agreement at all on what this convention should be, so be careful.



**FIGURE 3.6** The commutator of two covariant derivatives.

Knowing what we do about parallel transport, we could very carefully perform the necessary manipulations to see what happens to the vector under this operation, and the result would be a formula for the curvature tensor in terms of the connection coefficients. It is much quicker, however, to consider a related operation, the commutator of two covariant derivatives. The relationship between this and parallel transport around a loop should be evident; the covariant derivative of a tensor in a certain direction measures how much the tensor changes relative to what it would have been if it had been parallel transported, since the covariant derivative of a tensor in a direction along which it is parallel transported is zero. The commutator of two covariant derivatives, then, measures the difference between parallel transporting the tensor first one way and then the other, versus the opposite ordering, as shown in Figure 3.6.

The actual computation is very straightforward. Considering a vector field  $V^\rho$ , we take

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]V^\rho &= \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \\ &= \partial_\mu (\nabla_\nu V^\rho) - \Gamma_{\mu\nu}^\lambda \nabla_\lambda V^\rho + \Gamma_{\mu\sigma}^\rho \nabla_\nu V^\sigma - (\mu \leftrightarrow \nu) \\ &= \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma_{\nu\sigma}^\rho) V^\sigma + \Gamma_{\nu\sigma}^\rho \partial_\mu V^\sigma - \Gamma_{\mu\nu}^\lambda \partial_\lambda V^\rho - \Gamma_{\mu\nu}^\lambda \Gamma_{\lambda\sigma}^\rho V^\sigma \\ &\quad + \Gamma_{\mu\sigma}^\rho \partial_\nu V^\sigma + \Gamma_{\mu\sigma}^\rho \Gamma_{\nu\lambda}^\sigma V^\lambda - (\mu \leftrightarrow \nu) \\ &= (\partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda) V^\sigma - 2\Gamma_{[\mu\nu]}^\lambda \nabla_\lambda V^\rho. \end{aligned} \tag{3.111}$$

In the last step we have relabeled some dummy indices and eliminated some terms that cancel when antisymmetrized. We recognize that the antisymmetrized connection coefficients in the last term are simply one-half times the torsion tensor, and that the left hand side is manifestly a tensor; therefore the expression in parentheses must be a tensor itself. We write

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho, \tag{3.112}$$

where the Riemann tensor is identified as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda. \tag{3.113}$$

Notice a number of things about the derivation of this expression:

- Of course we have not demonstrated that (3.113) is actually the same tensor that appeared in (3.109), but in fact it's true. You are asked to show this in the Exercises.
- It is perhaps surprising that the commutator  $[\nabla_\mu, \nabla_\nu]$ , which appears to be a differential operator, has an action on vector fields that (in the absence of

torsion, at any rate) is a simple multiplicative transformation. The Riemann tensor measures that part of the commutator of covariant derivatives that is proportional to the vector field, while the torsion tensor measures the part that is proportional to the covariant derivative of the vector field; the second derivative doesn't enter at all.

- Notice that the expression (3.113) is constructed from nontensorial elements; you can check that the transformation laws all work out to make this particular combination a legitimate tensor.
- The antisymmetry of  $R^\rho_{\sigma\mu\nu}$  in its last two indices is immediate from this formula and its derivation.
- We constructed the curvature tensor completely from the connection (no mention of the metric was made). We were sufficiently careful that the above expression is true for any connection, whether or not it is metric compatible or torsion free.
- Using what are by now our usual methods, the action of  $[\nabla_\rho, \nabla_\sigma]$  can be computed on a tensor of arbitrary rank. The answer is

$$\begin{aligned} [\nabla_\rho, \nabla_\sigma] X^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} &= -T^\lambda_{\rho\sigma} \nabla_\lambda X^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} \\ &\quad + R^{\mu_1}_{\lambda\rho\sigma} X^{\lambda\mu_2 \dots \mu_k}_{\nu_1 \dots \nu_l} + R^{\mu_2}_{\lambda\rho\sigma} X^{\mu_1\lambda \dots \mu_k}_{\nu_1 \dots \nu_l} + \dots \\ &\quad - R^\lambda_{\nu_1\rho\sigma} X^{\mu_1 \dots \mu_k}_{\lambda\nu_2 \dots \nu_l} - R^\lambda_{\nu_2\rho\sigma} X^{\mu_1 \dots \mu_k}_{\nu_1\lambda \dots \nu_l} - \dots \end{aligned} \quad (3.114)$$

Both the torsion tensor and the Riemann tensor, thought of as multilinear maps, have elegant expressions in terms of the vector-field commutator. Thinking of the torsion as a map from two vector fields to a third vector field, we have

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad (3.115)$$

and thinking of the Riemann tensor as a map from three vector fields to a fourth one, we have (in funny-looking but standard notation)

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (3.116)$$

In these expressions, the notation  $\nabla_X$  refers to the covariant derivative along the vector field  $X$ ; in components,  $\nabla_X = X^\mu \nabla_\mu$ . So, for example, (3.116) is equivalent to

$$\begin{aligned} R^\rho_{\sigma\mu\nu} X^\mu Y^\nu Z^\sigma &= X^\lambda \nabla_\lambda (Y^\eta \nabla_\eta Z^\rho) - Y^\lambda \nabla_\lambda (X^\eta \nabla_\eta Z^\rho) \\ &\quad - (X^\lambda \partial_\lambda Y^\eta - Y^\lambda \partial_\lambda X^\eta) \nabla_\eta Z^\rho, \end{aligned} \quad (3.117)$$

which you can check is equivalent to (3.113). Note that the two vectors  $X$  and  $Y$  in (3.116) correspond to the last two indices in the component form of the Riemann

tensor. The last term in (3.116), involving the commutator  $[X, Y]$ , vanishes when  $X$  and  $Y$  are taken to be the coordinate basis vector fields (since  $[\partial_\mu, \partial_\nu] = 0$ ), which is why this term did not arise when we originally took the commutator of two covariant derivatives. We will not use this notation extensively, but you might see it in the literature, so you should be able to decode it.

Having defined the curvature tensor as something that characterizes the connection, let us now admit that in GR we are most concerned with the Christoffel connection. In this case the connection is derived from the metric, and the associated curvature may be thought of as that of the metric itself. This identification allows us to finally make sense of our informal notion that spaces for which the metric looks Euclidean or Minkowskian are flat. In fact it works both ways:

- If a coordinate system exists in which the components of the metric are constant, the Riemann tensor will vanish.
- If the Riemann tensor vanishes, we can always construct a coordinate system in which the metric components are constant.

Technically, these statements should be restricted to a region of the manifold that is simply-connected (all loops in the region can be smoothly deformed to a point without leaving the region); we will implicitly assume this condition below.

The first of these is easy to show. If we are in some coordinate system such that  $\partial_\sigma g_{\mu\nu} = 0$  everywhere, not just at a point, then  $\Gamma_{\mu\nu}^\rho = 0$  and  $\partial_\sigma \Gamma_{\mu\nu}^\rho = 0$ ; thus  $R^\rho{}_{\sigma\mu\nu} = 0$  by (3.113). But this is a tensor equation, and if it is true in one coordinate system it must be true in any coordinate system. Therefore, the statement that the Riemann tensor vanishes is a necessary condition for it to be possible to find coordinates in which the components of  $g_{\mu\nu}$  are constant everywhere.

The second claim, that  $R^\rho{}_{\sigma\mu\nu} = 0$  everywhere implies we can find a coordinate system in which the metric components are constant everywhere, is harder to prove (but not very hard). Consider as a warm-up a one-form  $\omega = \omega_\mu dx^\mu$ , defined at some point  $p$ . For any path  $x^\mu(\lambda)$  that includes  $p$ , we can construct a unique one-form field along the path by demanding that  $\omega_\mu$  be parallel-transported:

$$\frac{dx^\mu}{d\lambda} \nabla_\mu \omega_\nu = 0. \quad (3.118)$$

In general, if we performed this procedure for distinct paths that started at  $p$  and passed through some other point  $q$ , the value of  $\omega_\mu$  at  $q$  would depend on the path. However, if the Riemann tensor vanishes everywhere, the parallel-transport will be path-independent, and we can define a unique one-form field throughout the manifold. Therefore (3.118) must be true for arbitrary  $dx^\mu/d\lambda$ ; this can only be true if  $\omega_\mu$  is covariantly constant:

$$\nabla_\mu \omega_\nu = 0. \quad (3.119)$$

On an arbitrary manifold there will be no solutions to this equation; it is only possible here because we are assuming that the curvature vanishes. We can take

the antisymmetric part of (3.119), and from (3.36) we know that this is just the exterior derivative:

$$\nabla_{[\mu}\omega_{\nu]} = \partial_{[\mu}\omega_{\nu]} = 0, \quad (3.120)$$

or, in index-free notation,

$$d\omega = 0. \quad (3.121)$$

In other words,  $\omega$  is closed. It is also exact (there exists a scalar function  $\alpha$  such that  $\omega = d\alpha$ ), since we have restricted the topology of the region in which we are working. In components we have

$$\omega_\mu = \partial_\mu \alpha. \quad (3.122)$$

There is nothing special about the one-form  $\omega$ , so we can repeat this procedure with a set of one-forms  $\hat{\theta}^{(a)}$ , where  $a \in \{1 \dots n\}$  on an  $n$ -dimensional manifold. We may choose our one-forms to comprise a normalized basis for the dual space  $T_p^*$ , such that the components of the metric with respect to this basis are those of the canonical form; in other words,

$$ds^2(p) = \eta_{ab} \hat{\theta}^{(a)} \otimes \hat{\theta}^{(b)}. \quad (3.123)$$

Here we use  $\eta_{ab}$  in a generalized sense, as a matrix with either +1 or -1 for each diagonal element and zeroes elsewhere. The actual arrangement of the +1's and -1's depends on the canonical form of the metric, but is irrelevant for the present argument. Now let us parallel transport the entire set of basis forms all over the manifold; the vanishing of the Riemann tensor ensures that the result will be independent of the path taken. Since the metric is always automatically parallel-transported with respect to a metric-compatible connection, the metric components will remain unchanged,

$$ds^2(\text{anywhere}) = \eta_{ab} \hat{\theta}^{(a)} \otimes \hat{\theta}^{(b)}. \quad (3.124)$$

We therefore have specified a set of one-form fields, which everywhere define a basis in which the metric components are constant. This is completely unimpressive; it can be done on any manifold, regardless of what the curvature is. What we would like to show is that this is a *coordinate* basis (which will only be possible if the curvature vanishes). However, by the same arguments that led to (3.122), we know that all of the  $\hat{\theta}^{(a)}$ 's are exact forms, so that there exists a set of functions  $y^a$  such that the one-form fields are their gradients,

$$\hat{\theta}^{(a)} = dy^a. \quad (3.125)$$

These  $n$  functions are precisely the sought-after coordinates; all over the manifold the metric takes the form

$$ds^2 = \eta_{ab} dy^a dy^b. \quad (3.126)$$

At this point you are welcome to switch from using  $a, b$  as indices to  $\mu, \nu$  if you prefer.

We have thus verified that the Riemann tensor provides us with an answer to the question of whether some horrible-looking metric is secretly that of flat space in a perverse coordinate system. If we calculate the Riemann tensor of such a metric and find that it vanishes, we know that the metric is flat; if it doesn't vanish, there is curvature.

### 3.7 ■ PROPERTIES OF THE RIEMANN TENSOR

The Riemann tensor, with four indices, naively has  $n^4$  independent components in an  $n$ -dimensional space. In fact the antisymmetry property (3.110) means that there are only  $n(n - 1)/2$  independent values these last two indices can take on, leaving us with  $n^3(n - 1)/2$  independent components. When we consider the Christoffel connection, however, a number of other symmetries reduce the number of independent components further. Let's consider these now.

The simplest way to derive these additional symmetries is to examine the Riemann tensor with all lower indices,

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R_{\sigma\mu\nu}^{\lambda}. \quad (3.127)$$

Let us further consider the components of this tensor in locally inertial coordinates  $x^\mu$  established at a point  $p$ . Then the Christoffel symbols themselves will vanish, although their derivatives will not. We therefore have

$$\begin{aligned} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}(p) &= g_{\hat{\rho}\hat{\lambda}}(\partial_{\hat{\mu}}\Gamma_{\hat{\nu}\hat{\sigma}}^{\hat{\lambda}} - \partial_{\hat{\nu}}\Gamma_{\hat{\mu}\hat{\sigma}}^{\hat{\lambda}}) \\ &= \frac{1}{2}g_{\hat{\rho}\hat{\lambda}}g^{\hat{\lambda}\hat{\tau}}(\partial_{\hat{\mu}}\partial_{\hat{\nu}}g_{\hat{\sigma}\hat{\tau}} + \partial_{\hat{\mu}}\partial_{\hat{\sigma}}g_{\hat{\nu}\hat{\tau}} - \partial_{\hat{\mu}}\partial_{\hat{\tau}}g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\partial_{\hat{\mu}}g_{\hat{\sigma}\hat{\tau}} \\ &\quad - \partial_{\hat{\nu}}\partial_{\hat{\sigma}}g_{\hat{\tau}\hat{\mu}} + \partial_{\hat{\nu}}\partial_{\hat{\tau}}g_{\hat{\mu}\hat{\sigma}}) \\ &= \frac{1}{2}(\partial_{\hat{\mu}}\partial_{\hat{\sigma}}g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\rho}}g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\partial_{\hat{\sigma}}g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\nu}}\partial_{\hat{\rho}}g_{\hat{\mu}\hat{\sigma}}). \end{aligned} \quad (3.128)$$

In the first line we have used  $\Gamma_{\hat{\mu}\hat{\nu}}^{\hat{\tau}}(p) = 0$ , in the second line we have used  $\partial_{\hat{\mu}}g_{\hat{\lambda}\hat{\tau}} = 0$  in Riemann normal coordinates, and the fact that partials commute in the third line. From this expression we can notice immediately three properties of  $R_{\rho\sigma\mu\nu}$ : it is antisymmetric in its first two indices,

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}, \quad (3.129)$$

it is antisymmetric in its last two indices [which we already knew from (3.110)],

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}, \quad (3.130)$$

and it is invariant under interchange of the first pair of indices with the second:

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}. \quad (3.131)$$

With a little more work, which is left to your imagination, we can see that the sum of cyclic permutations of the last three indices vanishes:

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0. \quad (3.132)$$

Given (3.130), it's easy to see that this last property is equivalent to the vanishing of the antisymmetric part of the last three indices:

$$R_{\rho[\sigma\mu\nu]} = 0. \quad (3.133)$$

All of these properties have been derived in a special coordinate system, but they are all tensor equations; therefore they will be true in any coordinates (so we haven't bothered with hats on the indices). Not all of them are independent; with some effort, you can show that (3.129), (3.130), and (3.133) together imply (3.131). The logical interdependence of the equations is usually less important than the fact that they are true.

Given these relationships between the different components of the Riemann tensor, how many independent quantities remain? Let's begin with the facts that  $R_{\rho\sigma\mu\nu}$  is antisymmetric in the first two indices, antisymmetric in the last two indices, and symmetric under interchange of these two pairs. This means that we can think of it as a symmetric matrix  $R_{[\rho\sigma][\mu\nu]}$ , where the pairs  $\rho\sigma$  and  $\mu\nu$  are thought of as individual indices. An  $m \times m$  symmetric matrix has  $m(m+1)/2$  independent components, while an  $n \times n$  antisymmetric matrix has  $n(n-1)/2$  independent components. We therefore have

$$\frac{1}{2} \left[ \frac{1}{2}n(n-1) \right] \left[ \frac{1}{2}n(n-1) + 1 \right] = \frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n) \quad (3.134)$$

independent components. We still have to deal with the additional symmetry (3.133). An immediate consequence of (3.133) is that the totally antisymmetric part of the Riemann tensor vanishes,

$$R_{\{\rho\sigma\mu\nu\}} = 0. \quad (3.135)$$

In fact, this equation plus the other symmetries (3.129), (3.130), and (3.131), are enough to imply (3.133), as can be easily shown by expanding (3.135) and manipulating the resulting terms. Therefore imposing the additional constraint of (3.135) is equivalent to imposing (3.133), once the other symmetries have been accounted for. How many independent restrictions does this represent? Let us imagine decomposing

$$R_{\rho\sigma\mu\nu} = X_{\rho\sigma\mu\nu} + R_{\{\rho\sigma\mu\nu\}}. \quad (3.136)$$

It is easy to see that any totally antisymmetric 4-index tensor is automatically antisymmetric in its first and last indices, and symmetric under interchange of the two pairs. Therefore these properties are independent restrictions on  $X_{\rho\sigma\mu\nu}$ , unrelated to the requirement (3.135). Now a totally antisymmetric 4-index tensor has  $n(n-1)(n-2)(n-3)/4!$  terms, and therefore (3.135) reduces the number of independent components by this amount. We are left with

$$\frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n) - \frac{1}{24}n(n-1)(n-2)(n-3) = \frac{1}{12}n^2(n^2 - 1) \quad (3.137)$$

independent components of the Riemann tensor.

In four dimensions, therefore, the Riemann tensor has 20 independent components. (In one dimension it has none.) These twenty functions are precisely the 20 degrees of freedom in the second derivatives of the metric that we could not set to zero by a clever choice of coordinates when we first discussed locally inertial coordinates in Chapter 2. This should reinforce your confidence that the Riemann tensor is an appropriate measure of curvature.

In addition to the algebraic symmetries of the Riemann tensor (which constrain the number of independent components at any point), it obeys a differential identity, which constrains its relative values at different points. Consider the covariant derivative of the Riemann tensor, evaluated in locally inertial coordinates:

$$\begin{aligned} \nabla_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} &= \partial_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} \\ &= \frac{1}{2} \partial_{\hat{\lambda}} (\partial_{\hat{\mu}} \partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\mu}} \partial_{\hat{\rho}} g_{\hat{\sigma}\hat{\nu}} - \partial_{\hat{\nu}} \partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\nu}} \partial_{\hat{\rho}} g_{\hat{\sigma}\hat{\mu}}). \end{aligned} \quad (3.138)$$

It may seem illegitimate to take the derivative of an expression that is only true at a point, but the terms we are neglecting are all proportional to  $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}$ , and therefore vanish. We would like to consider the sum of cyclic permutations of the first three indices:

$$\begin{aligned} \nabla_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} + \nabla_{\hat{\rho}} R_{\hat{\sigma}\hat{\lambda}\hat{\mu}\hat{\nu}} + \nabla_{\hat{\sigma}} R_{\hat{\lambda}\hat{\rho}\hat{\mu}\hat{\nu}} \\ &= \frac{1}{2} (\partial_{\hat{\lambda}} \partial_{\hat{\mu}} \partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\lambda}} \partial_{\hat{\mu}} \partial_{\hat{\rho}} g_{\hat{\sigma}\hat{\nu}} - \partial_{\hat{\lambda}} \partial_{\hat{\nu}} \partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\lambda}} \partial_{\hat{\nu}} \partial_{\hat{\rho}} g_{\hat{\sigma}\hat{\mu}} \\ &\quad + \partial_{\hat{\rho}} \partial_{\hat{\mu}} \partial_{\hat{\sigma}} g_{\hat{\lambda}\hat{\nu}} - \partial_{\hat{\rho}} \partial_{\hat{\mu}} \partial_{\hat{\sigma}} g_{\hat{\lambda}\hat{\nu}} - \partial_{\hat{\rho}} \partial_{\hat{\nu}} \partial_{\hat{\sigma}} g_{\hat{\lambda}\hat{\mu}} + \partial_{\hat{\rho}} \partial_{\hat{\nu}} \partial_{\hat{\sigma}} g_{\hat{\lambda}\hat{\mu}} \\ &\quad + \partial_{\hat{\sigma}} \partial_{\hat{\mu}} \partial_{\hat{\rho}} g_{\hat{\lambda}\hat{\nu}} - \partial_{\hat{\sigma}} \partial_{\hat{\mu}} \partial_{\hat{\lambda}} g_{\hat{\nu}\hat{\rho}} - \partial_{\hat{\sigma}} \partial_{\hat{\nu}} \partial_{\hat{\rho}} g_{\hat{\lambda}\hat{\mu}} + \partial_{\hat{\sigma}} \partial_{\hat{\nu}} \partial_{\hat{\lambda}} g_{\hat{\mu}\hat{\rho}}) \\ &= 0. \end{aligned} \quad (3.139)$$

Once again, since this is an equation between tensors it is true in any coordinate system, even though we derived it in a particular one. We recognize by now that the antisymmetry  $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$  allows us to write this result as

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0. \quad (3.140)$$

This is known as the **Bianchi identity**. For a general connection there would be additional terms involving the torsion tensor. It is closely related to the Jacobi

identity, since (recalling the definition of the Riemann tensor in terms of the commutator of covariant derivatives) it expresses

$$[[\nabla_\lambda, \nabla_\rho], \nabla_\sigma] + [[\nabla_\rho, \nabla_\sigma], \nabla_\lambda] + [[\nabla_\sigma, \nabla_\lambda], \nabla_\rho] = 0. \quad (3.141)$$

The Riemann tensor has four indices. At times it is useful to express a tensor as a sum of various pieces that are individually easier to handle and may have direct physical interpretations. The trick is to do this in a coordinate-invariant way. For example, we could decompose the Riemann tensor into  $R^\rho_{\sigma ij}$  and  $R^\rho_{\sigma i0}$ , from which we could reconstruct the entire tensor (since  $R^\rho_{\sigma 00}$  vanishes). But clearly this decomposition is not invariant under change of basis; we want to find a decomposition that is preserved when we change coordinates. What we are really doing is considering representations of the Lorentz group. We have two fundamental tricks at our disposal: taking contractions, and taking symmetric/antisymmetric parts. For example, given an arbitrary  $(0, 2)$  tensor  $X_{\mu\nu}$ , we can decompose it into its symmetric and antisymmetric pieces,

$$X_{\mu\nu} = X_{(\mu\nu)} + X_{[\mu\nu]}, \quad (3.142)$$

and the symmetric part can be further decomposed into its trace  $X = g^{\mu\nu}X_{(\mu\nu)}$  and a trace-free part  $\widehat{X}_{\mu\nu} = X_{(\mu\nu)} - \frac{1}{n}Xg_{\mu\nu}$ , so that

$$X_{\mu\nu} = \frac{1}{n}Xg_{\mu\nu} + \widehat{X}_{\mu\nu} + X_{[\mu\nu]}. \quad (3.143)$$

(Note that  $X_{[\mu\nu]}$  is automatically traceless.) When we change coordinates, the different pieces  $Xg_{\mu\nu}$ ,  $\widehat{X}_{\mu\nu}$ , and  $X_{[\mu\nu]}$  are rotated into themselves, not into each other; we say that they define “invariant subspaces” of the space of  $(0, 2)$  tensors. For more complicated tensors the equivalent decomposition might not be so simple.

For the Riemann tensor, our first step is to take a contraction to form the **Ricci tensor**:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}. \quad (3.144)$$

For the curvature tensor formed from an arbitrary (not necessarily Christoffel) connection, there are a number of independent contractions to take. Our primary concern is with the Christoffel connection, for which (3.144) is the only independent contraction; all others either vanish, or are related to this one. The Ricci tensor associated with the Christoffel connection is automatically symmetric,

$$R_{\mu\nu} = R_{\nu\mu}, \quad (3.145)$$

as a consequence of the symmetries of the Riemann tensor. The trace of the Ricci tensor is the **Ricci scalar** (or **curvature scalar**):

$$R = R^\mu_\mu = g^{\mu\nu} R_{\mu\nu}. \quad (3.146)$$

We could also form the trace-free part  $\widehat{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu}$ , but this turns out not to be especially useful; it is more common to express things in terms of  $R_{\mu\nu}$  and  $R$ .

The Ricci tensor and scalar contain all of the information about traces of the Riemann tensor, leaving us the trace-free parts. These are captured by the **Weyl tensor**, which is basically the Riemann tensor with all of its contractions removed. It is given in  $n$  dimensions by

$$\begin{aligned} C_{\rho\sigma\mu\nu} &= R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)} (g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}) \\ &\quad + \frac{2}{(n-1)(n-2)} g_{\rho[\mu} g_{\nu]\sigma} R \end{aligned} \quad (3.147)$$

This messy formula is designed so that all possible contractions of  $C_{\rho\sigma\mu\nu}$  vanish, while it retains the symmetries of the Riemann tensor:

$$\begin{aligned} C_{\rho\sigma\mu\nu} &= C_{[\rho\sigma][\mu\nu]}, \\ C_{\rho\sigma\mu\nu} &= C_{\mu\nu\rho\sigma}, \\ C_{\rho[\sigma\mu\nu]} &= 0. \end{aligned} \quad (3.148)$$

The Weyl tensor is only defined in three or more dimensions, and in three dimensions it vanishes identically. One of the most important properties of the Weyl tensor is that it is invariant under conformal transformations (discussed in Appendix G). This means that if you compute  $C^\rho_{\sigma\mu\nu}$  (note that the first index is upstairs) for some metric  $g_{\mu\nu}$ , and then compute it again for a metric given by  $\omega^2(x)g_{\mu\nu}$ , where  $\omega(x)$  is an arbitrary nonvanishing function of spacetime, you get the same answer. For this reason it is often known as the *conformal tensor*.

An especially useful form of the Bianchi identity comes from contracting twice on (3.139):

$$\begin{aligned} 0 &= g^{\nu\sigma} g^{\mu\lambda} (\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu}) \\ &= \nabla^\mu R_{\rho\mu} - \nabla_\rho R + \nabla^\nu R_{\rho\nu}, \end{aligned} \quad (3.149)$$

or

$$\nabla^\mu R_{\rho\mu} = \frac{1}{2} \nabla_\rho R. \quad (3.150)$$

Notice that, unlike the partial derivative, it makes sense to raise an index on the covariant derivative, due to metric compatibility. We define the **Einstein tensor**

as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \quad (3.151)$$

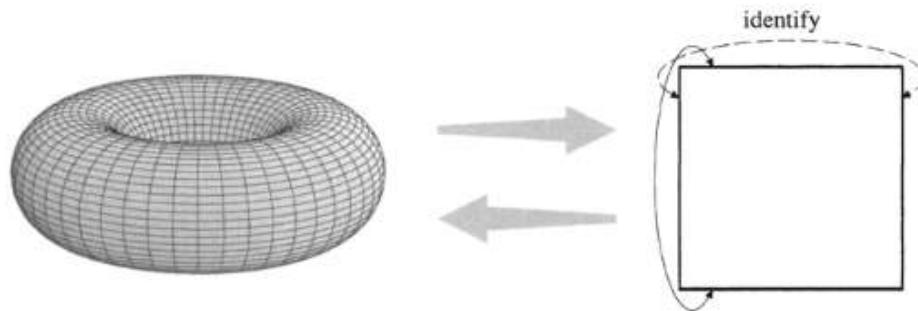
In four dimensions the Einstein tensor can be thought of as a trace-reversed version of the Ricci tensor. We then see that the twice-contracted Bianchi identity (3.150) is equivalent to

$$\nabla^\mu G_{\mu\nu} = 0. \quad (3.152)$$

The Einstein tensor, which is symmetric due to the symmetry of the Ricci tensor and the metric, will be of great importance in general relativity.

We should pause at this point to contrast the formalism we have developed with our intuitive notion of curvature. Our intuition is unfortunately contaminated by the fact that we are used to thinking about one- and two-dimensional spaces embedded in the (almost) Euclidean space in which we live. We think, for example, of a straight line as having no curvature, while a circle ( $S^1$ ) is curved. However, according to (3.137), in one, two, three, and four dimensions there are 0, 1, 6 and 20 independent components of the Riemann tensor, respectively. (Everything we say about the curvature in these examples refers to the curvature associated with the Christoffel connection, and therefore the metric.) Therefore it is impossible for a one-dimensional space such as  $S^1$  to have any curvature as we have defined it. The apparent contradiction stems from the fact that our intuitive notion of curvature depends on the extrinsic geometry of the manifold, which characterizes how a space is embedded in some larger space, while the Riemann curvature is a property of the intrinsic geometry of a space, which could be measured by observers confined to the manifold. Beings that lived on a circle and had no access to the larger world would necessarily think that they lived in a flat geometry—for example, there is no possibility of a nondegenerate infinitesimal loop around which we could parallel-transport a vector and have it come back rotated from its original position. Extrinsic curvature, discussed in Appendix D, is occasionally useful in GR when we wish to describe submanifolds of spacetime; but most often we are interested in the intrinsic geometry of spacetime itself, which does not rely on any embeddings.

We can illustrate the intrinsic/extrinsic difference further with an example from two dimensions, where the curvature has one independent component. In fact, all of the information about the curvature is contained in the single component of the Ricci scalar. Consider a torus, portrayed in Figure 3.7, which can be thought of as a square region of the plane with opposite sides identified (topologically,  $S^1 \times S^1$ ). Although a torus embedded in three dimensions looks curved from our point of view, it should be clear that we can put a metric on the torus whose components are constant in an appropriate coordinate system—simply unroll it and use the Euclidean metric of the plane,  $ds^2 = dx^2 + dy^2$ . In this metric, the torus is flat.



**FIGURE 3.7** A torus thought of as a square in flat space with opposite sides identified.

There is also nothing to stop us from introducing a different metric in which the torus is not flat, but the point we are trying to emphasize is that it can be made flat in some metric. Every time we embed a manifold in a larger space, the manifold inherits an “induced metric” from the background in which it is embedded, as discussed in the Appendix A. Our point here is that a torus embedded in a flat three-dimensional Euclidean space will have an induced metric that is curved, but we can nevertheless choose to put a different metric on it so that the intrinsic geometry is flat.

Let’s turn to a simple example where the curvature does not vanish. We have already talked about the two-sphere  $S^2$ , with metric

$$ds^2 = a^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (3.153)$$

where  $a$  is the radius of the sphere. It will actually be the radius if our sphere is embedded in  $\mathbf{R}^3$ , but we can call it the radius even in the absence of any embedding. Two-dimensional people living on the sphere could calculate  $a$  by measuring the area of the sphere, dividing by  $4\pi$ , and taking the square root; using the word “radius” to refer to this quantity is merely a convenience. We should also point out that the notion of a sphere is sometimes used in the weaker topological sense, without any particular metric being assumed; the metric we are using is called the *round metric*. Without going through the details, the nonzero connection coefficients for (3.153) are

$$\begin{aligned}\Gamma_{\phi\phi}^\theta &= -\sin \theta \cos \theta \\ \Gamma_{\theta\phi}^\phi &= \Gamma_{\phi\theta}^\phi = \cot \theta.\end{aligned}\quad (3.154)$$

Let’s compute a promising component of the Riemann tensor:

$$\begin{aligned}R^\theta_{\phi\theta\phi} &= \partial_\theta \Gamma_{\phi\phi}^\theta - \partial_\phi \Gamma_{\theta\phi}^\theta + \Gamma_{\theta\lambda}^\theta \Gamma_{\phi\phi}^\lambda - \Gamma_{\phi\lambda}^\theta \Gamma_{\theta\phi}^\lambda \\ &= (\sin^2 \theta - \cos^2 \theta) - (0) + (0) - (-\sin \theta \cos \theta)(\cot \theta) \\ &= \sin^2 \theta.\end{aligned}\quad (3.155)$$

The notation is obviously imperfect, since the Greek letter  $\lambda$  is a dummy index that is summed over, while the Greek letters  $\theta$  and  $\phi$  represent specific coordinates. Lowering an index, we have

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta\lambda} R^{\lambda}_{\phi\theta\phi} \\ &= g_{\theta\theta} R^{\theta}_{\phi\theta\phi} \\ &= a^2 \sin^2 \theta. \end{aligned} \quad (3.156)$$

It is easy to check that all of the components of the Riemann tensor either vanish or are related to this one by symmetry. We can go on to compute the Ricci tensor via  $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$ . We obtain

$$\begin{aligned} R_{\theta\theta} &= g^{\phi\phi} R_{\phi\theta\theta\phi} = 1 \\ R_{\theta\phi} &= R_{\phi\theta} = 0 \\ R_{\phi\phi} &= g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2 \theta. \end{aligned} \quad (3.157)$$

The Ricci scalar is similarly straightforward:

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{a^2}. \quad (3.158)$$

Therefore the Ricci scalar, which for a two-dimensional manifold completely characterizes the curvature, is a constant over the two-sphere. If we had perturbed the metric (corresponding physically to bumps on the sphere), this would no longer have been the case. Note that the scalar curvature decreases as the radius of the sphere increases. Even in more general contexts, we will sometimes refer to the “radius of curvature” of a manifold as providing a length scale over which the curvature varies; the larger the radius of curvature, the smaller the curvature itself.

### 3.8 ■ SYMMETRIES AND KILLING VECTORS

The real world is a messy place, and we have no hope of finding a metric that describes our actual universe, or even any small part thereof, with perfect precision. Instead, we model spacetime via various approximations appropriate to the physical situation being studied. For example, the geometry outside a star or planet may be approximated, to some order of precision, as being spherically symmetric, even if the real situation includes small deviations from this symmetry—these may be added in later as perturbations.

General relativity is no different from other fields of physics, then, in being especially interested in solutions with symmetry. In fact, such properties may be even more crucial in GR than in, say, electromagnetism, since the nonlinear nature of Einstein’s equation (discussed in the next chapter) makes it hard to find any exact solutions at all. In the context of curved spacetime, however, we need to be

more careful than usual about what exactly is meant by “symmetry.” In this section we develop some useful tools for studying symmetry; a deeper investigation can be found in Appendix B.

We think of a manifold  $M$  as possessing a symmetry if the geometry is invariant under a certain transformation that maps  $M$  to itself; that is, if the metric is the same, in some sense, from one point to another. In fact different tensor fields may possess different symmetries; symmetries of the metric are called **isometries**. Sometimes the existence of isometries is obvious; consider, for example, four-dimensional Minkowski space,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2. \quad (3.159)$$

We know of several isometries of this space; these include translations ( $x^\mu \rightarrow x^\mu + a^\mu$ , with  $a^\mu$  fixed) and Lorentz transformations ( $x^\mu \rightarrow \Lambda^\mu_\nu x^\nu$ , with  $\Lambda^\mu_\nu$  a Lorentz-transformation matrix). The fact that the metric is invariant under translations is made immediately apparent by the simple fact that the metric coefficients  $\eta_{\mu\nu}$  are independent of the individual coordinate functions  $x^\mu$ . Indeed, whenever  $\partial_{\sigma_*} g_{\mu\nu} = 0$  for some fixed  $\sigma_*$  (but for all  $\mu$  and  $\nu$ ), there will be a symmetry under translations along  $x^{\sigma_*}$ :

$$\partial_{\sigma_*} g_{\mu\nu} = 0 \quad \Rightarrow \quad x^{\sigma_*} \rightarrow x^{\sigma_*} + a^{\sigma_*} \text{ is a symmetry.} \quad (3.160)$$

The careful reader will have noticed that we still haven’t precisely defined what we mean by symmetry; roughly we imagine that the metric is invariant under some transformation, but the precise meaning is only developed in Appendix B. Also, the implication arrow in (3.160) only goes one way, and it would be nice to have a clean criterion for deciding when a given transformation counts as a symmetry; this will come soon.

Isometries of the form (3.160) have immediate consequences for the motion of test particles as described by the geodesic equation. Recall from (3.61) that the geodesic equation can be written in terms of the four-momentum  $p^\mu = mU^\mu$  (valid for timelike paths, at least) as

$$p^\lambda \nabla_\lambda p^\mu = 0. \quad (3.161)$$

By metric compatibility we are free to lower the index  $\mu$ , and then we may expand the covariant derivative to obtain

$$p^\lambda \partial_\lambda p_\mu - \Gamma^\sigma_{\lambda\mu} p^\lambda p_\sigma = 0. \quad (3.162)$$

The first term tells us how the momentum components change along the path,

$$p^\lambda \partial_\lambda p_\mu = m \frac{dx^\lambda}{d\tau} \partial_\lambda p_\mu = m \frac{dp_\mu}{d\tau}. \quad (3.163)$$

while the second term is

$$\Gamma_{\lambda\mu}^{\sigma} p^{\lambda} p_{\sigma} = \frac{1}{2} g^{\sigma\nu} (\partial_{\lambda} g_{\mu\nu} + \partial_{\mu} g_{\nu\lambda} - \partial_{\nu} g_{\lambda\mu}) p^{\lambda} p_{\sigma} \quad (3.164)$$

$$= \frac{1}{2} (\partial_{\lambda} g_{\mu\nu} + \partial_{\mu} g_{\nu\lambda} - \partial_{\nu} g_{\lambda\mu}) p^{\lambda} p^{\nu} \quad (3.165)$$

$$= \frac{1}{2} (\partial_{\mu} g_{\nu\lambda}) p^{\lambda} p^{\nu}, \quad (3.166)$$

where we have used the symmetry of  $p^{\lambda} p^{\nu}$  to go from the second line to the third. So, without yet making any assumptions about symmetry, we see that the geodesic equation can be written as

$$m \frac{dp_{\mu}}{d\tau} = \frac{1}{2} (\partial_{\mu} g_{\nu\lambda}) p^{\lambda} p^{\nu}. \quad (3.167)$$

Therefore, if all of the metric coefficients are independent of the coordinate  $x^{\sigma_*}$ , we find that this isometry implies that the momentum component  $p_{\sigma_*}$  is a conserved quantity of the motion:

$$\partial_{\sigma_*} g_{\mu\nu} = 0 \Rightarrow \frac{dp_{\sigma_*}}{d\tau} = 0. \quad (3.168)$$

This will hold along any geodesic, even though we only derived it for timelike ones. The conserved quantities implied by isometries are extremely useful in studying the motion of test particles in curved backgrounds.

Of course, even though independence of the metric components on one or more coordinates implies the existence of isometries, the converse does not necessarily hold. Symmetry under Lorentz transformations, for example, is not manifest as independence of  $\eta_{\mu\nu}$  on any coordinates; indeed, in four dimensions, there are four types of translations and six types of Lorentz transformations, for a total of ten, which is obviously larger than the number of dimensions the metric could possibly be independent of. What is more, it would be simple enough to transform to a complicated coordinate system where not even the translational symmetries were obvious. Such a coordinate transformation would change the metric components, but not the underlying geometry, which is what the symmetry is really characterizing. Clearly a more systematic procedure is called for.

We can develop such a procedure by casting the right-hand equation of (3.168), expressing constancy of one of the components of the momentum, in a more manifestly covariant form. If  $x^{\sigma_*}$  is the coordinate which  $g_{\mu\nu}$  is independent of, let us consider the vector  $\partial_{\sigma_*}$ , which we label as  $K$ :

$$K = \partial_{\sigma_*}, \quad (3.169)$$

which is equivalent in component notation to

$$K^{\mu} = (\partial_{\sigma_*})^{\mu} = \delta_{\sigma_*}^{\mu}. \quad (3.170)$$

We say that the vector  $K^{\mu}$  generates the isometry; this means that the transformation under which the geometry is invariant is expressed infinitesimally as a motion

in the direction of  $K^\mu$ . Again, the notion is developed more fully in Appendix B. In terms of this vector, the noncovariant-looking quantity  $p_{\sigma_*}$  is simply

$$p_{\sigma_*} = K^\nu p_\nu = K_\nu p^\nu. \quad (3.171)$$

Meanwhile, the constancy of this (scalar) quantity along the path is equivalent to the statement that its directional derivative along the geodesic vanishes:

$$\frac{dp_{\sigma_*}}{d\tau} = 0 \quad \leftrightarrow \quad p^\mu \nabla_\mu (K_\nu p^\nu) = 0. \quad (3.172)$$

Expanding the expression on the right, we obtain

$$\begin{aligned} p^\mu \nabla_\mu (K_\nu p^\nu) &= p^\mu K_\nu \nabla_\mu p^\nu + p^\mu p^\nu \nabla_\mu K_\nu \\ &= p^\mu p^\nu \nabla_\mu K_\nu \\ &= p^\mu p^\nu \nabla_{(\mu} K_{\nu)}, \end{aligned} \quad (3.173)$$

where in the second line we have invoked the geodesic equation ( $p^\mu \nabla_\mu p^\nu = 0$ ). In the third line we have used the fact that  $p^\mu p^\nu$  is automatically symmetric in  $\mu$  and  $\nu$ , so only the symmetric part of  $\nabla_\mu K_\nu$  could possibly contribute. We therefore conclude that any vector  $K_\mu$  that satisfies  $\nabla_{(\mu} K_{\nu)} = 0$  implies that  $K_\nu p^\nu$  is conserved along a geodesic trajectory:

$$\boxed{\nabla_{(\mu} K_{\nu)} = 0 \quad \Rightarrow \quad p^\mu \nabla_\mu (K_\nu p^\nu) = 0.} \quad (3.174)$$

The equation on the left is known as **Killing's equation**, and vector fields that satisfy it are known as **Killing vector fields** (or simply Killing vectors). You can verify for yourself that, if the metric is independent of some coordinate  $x^{\sigma_*}$ , the vector  $\partial_{\sigma_*}$  will satisfy Killing's equation. In fact, if a vector  $K^\mu$  satisfies Killing's equation, it will always be possible to find a coordinate system in which  $K = \partial_{\sigma_*}$ ; but in general we cannot find coordinates in which all the Killing vectors are simultaneously of this form, nor is this form necessary for the vector to satisfy Killing's equation.

As we investigate in Appendix B, Killing vector fields on a manifold are in one-to-one correspondence with continuous symmetries of the metric on that manifold. Every Killing vector implies the existence of conserved quantities associated with geodesic motion. This can be understood physically: by definition the metric is unchanging along the direction of the Killing vector. Loosely speaking, therefore, a free particle will not feel any forces in this direction, and the component of its momentum in that direction will consequently be conserved. In fact, the same kind of logic by which we showed that  $K_\nu p^\nu$  is conserved along a geodesic if  $\nabla_{(\mu} K_{\nu)} = 0$  generalizes to additional indices: a **Killing tensor** is a symmetric  $l$ -index tensor  $K_{v_1 \dots v_l}$  that satisfies the obvious generalization of Killing's equation, and correspondingly leads to conserved quantities by contracting with  $l$  copies of

the momentum:

$$\nabla_{(\mu} K_{v_1 \dots v_l)} = 0 \quad \Rightarrow \quad p^\mu \nabla_\mu (K_{v_1 \dots v_l} p^{v_1} \dots p^{v_l}) = 0. \quad (3.175)$$

Simple examples of Killing tensors are the metric itself, and symmetrized tensor products of Killing vectors. Killing tensors are not related in a simple way to symmetries of the spacetime, but they will simplify our analysis of rotating black holes and expanding universes.

Derivatives of Killing vectors can be related to the Riemann tensor by

$$\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu, \quad (3.176)$$

as you are asked to prove in the exercises. Contracting this expression yields

$$\nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu. \quad (3.177)$$

These relations, along with the Bianchi identity and Killing's equation, suffice to show that the directional derivative of the Ricci scalar along a Killing vector field will vanish,

$$K^\lambda \nabla_\lambda R = 0. \quad (3.178)$$

This last fact is another reflection of the idea that the geometry is not changing along a Killing vector field.

Besides leading to conserved quantities for the motion of individual particles, the existence of a timelike Killing vector allows us to define a conserved energy for the entire spacetime. Given a Killing vector  $K_\nu$  and a conserved energy-momentum tensor  $T_{\mu\nu}$ , we can construct a current

$$J_T^\mu = K_\nu T^{\mu\nu} \quad (3.179)$$

that is automatically conserved,

$$\begin{aligned} \nabla_\mu J_T^\mu &= (\nabla_\mu K_\nu) T^{\mu\nu} + K_\nu (\nabla_\mu T^{\mu\nu}) \\ &= 0. \end{aligned} \quad (3.180)$$

The first term vanishes by virtue of Killing's equation (since the symmetry of the upper indices serves to automatically symmetrize the lower indices), and the second term vanishes by conservation of  $T_{\mu\nu}$ . If  $K_\nu$  is timelike, we can integrate over a spacelike hypersurface  $\Sigma$  to define the total energy,

$$E_T = \int_{\Sigma} J_T^\mu n_\mu \sqrt{\gamma} d^3x, \quad (3.181)$$

where  $\gamma_{ij}$  is the induced metric on  $\Sigma$  and  $n_\mu$  is the normal vector to  $\Sigma$ . In Appendix E we discuss integration over hypersurfaces, and in particular Stokes's theorem; as explained there,  $E_T$  will be the same when integrated over any spacelike

hypersurface, and is therefore conserved. This result fits nicely with our discussion in Section 3.5, where we found that the total energy is not typically conserved in an expanding universe; expansion means that the metric is changing with time, so there is no isometry in this direction. When there is a timelike Killing vector, we can write the metric in a form where it is independent of the timelike coordinate, and Noether's theorem implies a conserved energy. Similarly, spacelike Killing vectors may be used to construct conserved momenta (or angular momenta).

Although it may or may not be simple to actually solve Killing's equation in any given spacetime, it is frequently possible to write down some Killing vectors by inspection. (Of course a generic metric has no Killing vectors at all, but to keep things simple we often deal with metrics with high degrees of symmetry.) For example, in  $\mathbf{R}^3$  with metric  $ds^2 = dx^2 + dy^2 + dz^2$ , independence of the metric components with respect to  $x$ ,  $y$ , and  $z$  immediately yields three Killing vectors:

$$\begin{aligned} X^\mu &= (1, 0, 0) \\ Y^\mu &= (0, 1, 0) \\ Z^\mu &= (0, 0, 1). \end{aligned} \quad (3.182)$$

These clearly represent the three translations. There are also three rotational symmetries in  $\mathbf{R}^3$ , which are not quite as simple. To find them, imagine first going to polar coordinates,

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta, \end{aligned} \quad (3.183)$$

where the metric takes the form

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (3.184)$$

Now the metric (the *same* metric, just in a different coordinate system) is manifestly independent of  $\phi$ . We therefore know that  $R = \partial_\phi$  is a Killing vector. Transforming back to Cartesian coordinates, this becomes

$$R = -y\partial_x + x\partial_y. \quad (3.185)$$

The Cartesian components  $R^\mu$  are therefore  $(-y, x, 0)$ . Since this represents a rotation about the  $z$ -axis, it is straightforward to guess the components of all three rotational Killing vectors:

$$\begin{aligned} R^\mu &= (-y, x, 0) \\ S^\mu &= (z, 0, -x) \\ T^\mu &= (0, -z, y), \end{aligned} \quad (3.186)$$

representing rotations about the  $z$ ,  $y$ , and  $x$ -axes, respectively. You can check for yourself that these actually do solve Killing's equation. The overall signs don't matter, since minus a Killing vector is still a Killing vector.

This exercise leads directly to the Killing vectors for the two-sphere  $S^2$  with metric

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (3.187)$$

Since this sphere can be thought of as the locus of points at unit distance from the origin in  $\mathbf{R}^3$ , and the rotational Killing vectors all rotate such a sphere into itself, they also represent symmetries of  $S^2$ . To get explicit coordinate-basis representations for these vectors, we first transform the three-dimensional vectors (3.186) to polar coordinates  $x^{\mu'} = (r, \theta, \phi)$ . A straightforward calculation reveals

$$\begin{aligned} R &= \partial_\phi \\ S &= \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi \\ T &= -\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi. \end{aligned} \quad (3.188)$$

Notice that there are no components along  $\partial_r$ , which makes sense for a rotational isometry. Therefore the expressions (3.188) for the three rotational Killing vectors in  $\mathbf{R}^3$  are exactly the same as those of  $S^2$  in spherical polar coordinates.

In  $n \geq 2$  dimensions, there can be more Killing vectors than dimensions. This is because a set of Killing vector fields can be linearly independent, even though at any one point on the manifold the vectors at that point are linearly dependent. It is trivial to show (so you should do it yourself) that a linear combination of Killing vectors with *constant* coefficients is still a Killing vector (in which case the linear combination does not count as an independent Killing vector), but this is not generally true with coefficients that vary over the manifold. You can also show that the commutator of two Killing vector fields is a Killing vector field; this is very useful to know, but it may be the case that the commutator gives you a vector field that is not linearly independent (or it may simply vanish). The problem of finding all of the Killing vectors of a metric is therefore somewhat tricky, as it is not always clear when to stop looking.

### 3.9 ■ MAXIMALLY SYMMETRIC SPACES

How symmetric can a space possibly be? An example of a space with the highest possible degree of symmetry is  $\mathbf{R}^n$  with the flat Euclidean metric. Consider the isometries of this space, which we know to be translations and rotations in  $n$  dimensions, from the perspective of what they do in the neighborhood of some fixed point  $p$ . The translations are those transformations that move the point; there are  $n$  independent axes along which it can be moved, and hence  $n$  total translations. The rotations, centered at  $p$ , are those transformations that leave  $p$  invariant; they

can be thought of as moving one of the axes through  $p$  into one of the others. There are  $n$  axes, and for each axis there are  $n - 1$  other axes into which it can be rotated, but we shouldn't count a rotation of  $y$  into  $x$  as separate from a rotation of  $x$  into  $y$ , so the total number of independent rotations is  $\frac{1}{2}n(n - 1)$ . We therefore have

$$n + \frac{1}{2}n(n - 1) = \frac{1}{2}n(n + 1) \quad (3.189)$$

independent symmetries of  $\mathbf{R}^n$ . But our counting argument only referred to the behavior of the symmetry in a neighborhood of  $p$ , not globally all over the manifold; so even in the presence of curvature the counting should be the same. If the metric signature is not Euclidean, some of the rotations will actually be boosts, but again the counting will be the same. The number of isometries is, of course, the number of linearly independent Killing vector fields. We therefore refer to an  $n$ -dimensional manifold with  $\frac{1}{2}n(n + 1)$  Killing vectors as a **maximally symmetric space**. The most familiar examples of maximally symmetric spaces are  $n$ -dimensional Euclidean spaces  $\mathbf{R}^n$  and the  $n$ -dimensional spheres  $S^n$ . For an  $n$ -dimensional sphere we usually think of the isometries as consisting of  $\frac{1}{2}n(n + 1)$  independent rotations, rather than as some collection of both rotations and translations. However, if we consider the action of these rotations on some fixed point  $p$ , a moment's thought convinces us that the entire set can be decomposed into  $\frac{1}{2}n(n - 1)$  rotations around the point (keeping  $p$  fixed), and another  $n$  that move  $p$  along each direction, just as in  $\mathbf{R}^n$ .

If a manifold is maximally symmetric, the curvature is the same everywhere (as expressed by translation-like isometries) and the same in every direction (as expressed by rotation-like isometries). Hence, if we know the curvature of a maximally symmetric space at one point, we know it everywhere. Indeed, there are only a small number of possible maximally symmetric spaces; they are classified by the curvature scalar  $R$  (which will be constant everywhere), the dimensionality  $n$ , the metric signature, and perhaps some discrete pieces of information relating to the global topology (distinguishing, for example, an  $n$ -torus from  $\mathbf{R}^n$ , and tori of different sizes from each other). It follows that we should be able to reconstruct the entire Riemann tensor of such a space from the Ricci scalar  $R$ ; let's see how this works.

The basic idea is simply that, since the geometry looks the same in all directions, the curvature tensor should look the same in all directions. What might this mean? First choose locally inertial coordinates at some point  $p$ , so that  $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$ . Of course, locally inertial coordinates are not unique; for example, we can perform a Lorentz transformation at  $p$  and the metric components will remain those of  $\eta_{\hat{\mu}\hat{\nu}}$ . (By "doing a Lorentz transformation" we really are referring to a change of basis vectors in  $T_p$ ; in a curved spacetime, this only makes sense at a single point, not over a region.) Since the geometry is maximally symmetric, we want the same to be true of the Riemann tensor; that is, the components of  $R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}$  should not change under a Lorentz transformation either, since there is no preferred direction in spacetime. But there are unique tensors that do not change their components under Lorentz transformations—the metric, the Kronecker delta, and

the Levi-Civita tensor. This means that, in these coordinates and at this point, the components of  $R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}$  will be proportional to a tensor constructed from these invariant tensors. Attempting to match the symmetries of the Riemann tensor reveals that there is a unique possibility:

$$R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} \propto g_{\hat{\rho}\hat{\mu}}g_{\hat{\sigma}\hat{\nu}} - g_{\hat{\rho}\hat{\nu}}g_{\hat{\sigma}\hat{\mu}}. \quad (3.190)$$

But this is a completely tensorial relation, so it must be true in any coordinate system. We have argued in favor of this relation at a single point  $p$ , but in a maximally symmetric space all points are created equal, so it must also be true at any other point as well. The proportionality constant is easily fixed by contracting both sides twice [the left-hand side becomes  $R$ , and the right-hand side is  $n(n-1)$ ]. We end up with an equation true in any maximally symmetric space, at any point, in any coordinate system:

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}). \quad (3.191)$$

Likewise, if the Riemann tensor satisfies this condition (with  $R$  a constant over the manifold), the metric will be maximally symmetric. In two dimensions, finding that  $R$  is a constant is sufficient to prove that a space is maximally symmetric, since there is only one independent component of the curvature. In higher dimensions you have to work harder.

Locally, then (ignoring questions of global topology), a maximally symmetric space of given dimension and signature is fully specified by  $R$ . The basic classification of such spaces is simply whether  $R$  is positive, zero, or negative, since the magnitude of  $R$  represents an overall scaling of the size of the space. For Euclidean signatures, the flat maximally symmetric spaces are planes or appropriate higher-dimensional generalizations, while the positively curved ones are spheres. Maximally symmetric Euclidean spaces of negative curvature are hyperboloids, denoted  $H^n$ . These are less familiar because even a two-dimensional hyperboloid cannot be isometrically embedded in  $\mathbf{R}^3$ . Let's examine this two-dimensional hyperboloid briefly.

There are a number of ways of representing  $H^2$ , which has the same topology as  $\mathbf{R}^2$ . One simple way is as the **Poincaré half-plane**, which is the region  $y > 0$  of a two-dimensional region with coordinates  $(x, y)$  and metric

$$ds^2 = \frac{a^2}{y^2}(dx^2 + dy^2). \quad (3.192)$$

The geometry of the Poincaré half-plane is of course different from that of the upper half of  $\mathbf{R}^2$ , despite the use of similar coordinates. For example, we can compute the length of a line segment stretching vertically ( $x = \text{constant}$ ) from  $y_1$  to  $y_2$ :

$$\begin{aligned}
 \Delta s &= \int_{y_1}^{y_2} \sqrt{g_{\mu\nu} \frac{dx^\mu}{dy} \frac{dx^\nu}{dy}} dy \\
 &= a \int_{y_1}^{y_2} \frac{dy}{y} \\
 &= a \ln\left(\frac{y_2}{y_1}\right).
 \end{aligned} \tag{3.193}$$

This is not at all the result  $\Delta s = y_2 - y_1$  we would expect in Euclidean space. In particular, notice that the path length becomes infinite for paths that approach the boundary  $y = 0$ . In other words, it's not really a boundary at all; it's infinitely far away, as far as anyone living on the hyperboloid is concerned.

The nonvanishing Christoffel symbols for (3.192) are

$$\begin{aligned}
 \Gamma_{xy}^x &= \Gamma_{yx}^x = -y^{-1} \\
 \Gamma_{xx}^y &= y^{-1} \\
 \Gamma_{yy}^y &= -y^{-1}.
 \end{aligned} \tag{3.194}$$

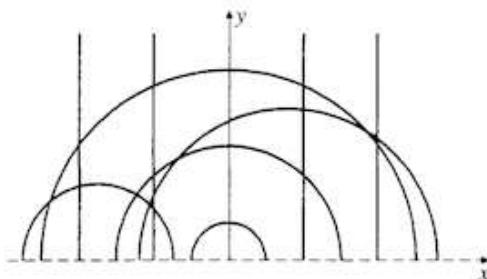
From these it is straightforward to show that geodesics satisfy

$$(x - x_0)^2 + y^2 = l^2, \tag{3.195}$$

for some constants  $x_0$  and  $l$ . Curves of this form are semicircles with centers located on the  $x$ -axis, as shown in Figure 3.8. In the limit as  $x_0 \rightarrow \infty$  and  $l \rightarrow \infty$  with  $l - x_0$  fixed, we get a straight vertical line. Following our discussion of  $S^2$  at the end of Section 3.7, we calculate a representative component of the Riemann tensor to be

$$R^x_{yxy} = -y^{-2}. \tag{3.196}$$

As with the two-sphere, all other components are either vanishing or related to this by symmetries. This is simply a reflection of the fact that we are in two di-



**FIGURE 3.8** The upper half plane with a negatively curved metric. Geodesics are semicircles and straight lines that intersect the  $x$ -axis vertically.

mensions, with only one independent component of curvature. Turning the crank yields the Ricci tensor,

$$\begin{aligned} R_{xx} &= -y^{-2} \\ R_{xy} &= 0 \\ R_{yy} &= -y^{-2}, \end{aligned} \tag{3.197}$$

and the curvature scalar,

$$R = -\frac{2}{a^2}. \tag{3.198}$$

We see that it matches that of  $S^2$  with the opposite sign, and in particular that it is a constant. Since we are in two dimensions, this is enough to ensure that our metric really is maximally symmetric. Of course there are coordinates in which  $H^2$  looks very different; one is introduced in the Exercises.

Locally, then, a maximally symmetric space of Euclidean signature is either a plane, a sphere, or a hyperboloid, depending on the sign of  $R$ . Globally, any maximally symmetric space (of Euclidean signature) can be constructed by taking a carefully chosen region of one of these three spaces and identifying different sides, as the flat torus can be constructed from  $\mathbf{R}^2$ . As an aside, let's briefly mention a connection between local geometry and global topology, encompassed by the Gauss–Bonnet theorem. For a two-dimensional compact boundaryless orientable manifold, this reads

$$\chi(M) = \frac{1}{4\pi} \int_M R \sqrt{|g|} d^n x, \tag{3.199}$$

where  $\chi(M)$  is a topological invariant of the space, known as the Euler characteristic. In general it can be calculated from the cohomology spaces mentioned in Chapter 2; in two dimensions, however, it is simply given by

$$\chi(M) = 2(1 - g), \tag{3.200}$$

where  $g$  is the genus of the surface (zero for a sphere, and equal to the number of handles of a torus or Riemann surface). The Gauss–Bonnet theorem holds whether or not the curvature  $R$  is a constant; when it is, however, we see that all Riemann surfaces of genus  $g \geq 2$  must have negative curvature, just as a sphere must be positively curved and a torus must be flat.

Continuing our aside, think for the moment about string theory, which claims that the fundamental objects comprising the universe are small one-dimensional loops of string. Such strings have two-dimensional “world-sheets” rather than one-dimensional worldlines. Doing perturbation theory in string theory (the equivalent of calculating Feynman diagrams in quantum field theory) involves summing over all world-sheet geometries (generally, for technical reasons, Eu-

clidean geometries). This sounds like a lot of geometries, but in two dimensions any metric can be written as some fiducial metric times a conformal factor. This should be plausible, since there is only one curvature component; you are asked to prove it in the Exercises. The fiducial metric can be chosen differently for each world-sheet topology, and we can make our lives easier by choosing it to be (locally) a metric of maximal symmetry—the round sphere for genus zero, the plane for genus one, and the hyperboloid for higher genera. Even more fortunately, the string theories of greatest physical interest are the so-called critical string theories, for which the conformal factor itself doesn't matter. This is one of the things that makes doing calculations in perturbative string theory possible; we only have to sum over a discrete set of topologies, with a finite number of modular parameters for each topology (such as the parameters telling us the sizes of the different directions in a torus).

We close this section with one last point. We have explored the maximally symmetric spaces of Euclidean signatures; there are, of course, corresponding spacetimes with Lorentzian signatures. We know that the maximally symmetric spacetime with  $R = 0$  is simply Minkowski space. The positively curved maximally symmetric spacetime is called de Sitter space, while that with negative curvature is imaginatively labeled anti-de Sitter space. These spacetimes will be more thoroughly discussed in Chapter 8.

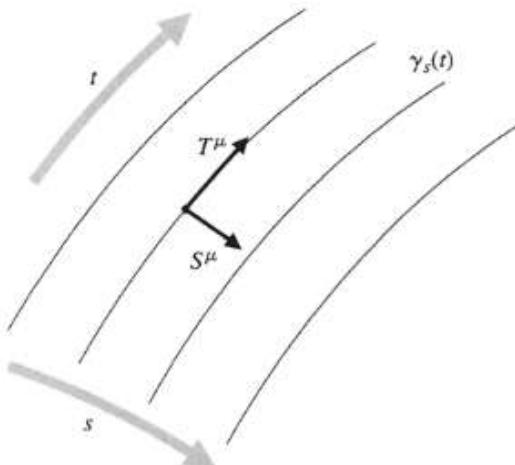
It should be clear by now that the Appendices flesh out these ideas in important ways. Impatient readers may skip over them, but it would be a shame to do so.

### 3.10 ■ GEODESIC DEVIATION

The Riemann tensor shows up as a consequence of curvature in one more way: geodesic deviation. You have undoubtedly heard that the defining property of Euclidean (flat) geometry is the parallel postulate: initially parallel lines remain parallel forever. Of course in a curved space this is not true; on a sphere, certainly, initially parallel geodesics will eventually cross. We would like to quantify this behavior for an arbitrary curved space.

The problem is that the notion of “parallel” does not extend naturally from flat to curved spaces. The best we can do is to consider geodesic curves that might be initially parallel, and see how they behave as we travel down the geodesics. To this end we consider a one-parameter family of geodesics,  $\gamma_s(t)$ . That is, for each  $s \in \mathbf{R}$ ,  $\gamma_s$  is a geodesic parameterized by the affine parameter  $t$ . The collection of these curves defines a smooth two-dimensional surface (embedded in a manifold  $M$  of arbitrary dimensionality). The coordinates on this surface may be chosen to be  $s$  and  $t$ , provided we have chosen a family of geodesics that do not cross. The entire surface is the set of points  $x^\mu(s, t) \in M$ . We have two natural vector fields: the tangent vectors to the geodesics,

$$T^\mu = \frac{\partial x^\mu}{\partial t}, \quad (3.201)$$



**FIGURE 3.9** A set of geodesics  $\gamma_s(t)$ , with tangent vectors  $T^\mu$ . The vector field  $S^\mu$  measures the deviation between nearby geodesics.

and the deviation vectors

$$S^\mu = \frac{\partial x^\mu}{\partial s}. \quad (3.202)$$

This name derives from the informal notion that  $S^\mu$  points from one geodesic toward the neighboring ones.

The idea that  $S^\mu$  points from one geodesic to the next inspires us to define the “relative velocity of geodesics,”

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu, \quad (3.203)$$

and the “relative acceleration of geodesics,”

$$A^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu. \quad (3.204)$$

You should take the names with a grain of salt, but these vectors are certainly well-defined. This notion of *relative* acceleration between geodesics should be distinguished from the acceleration of a path away from being a geodesic, which would be given (when  $t$  is the proper time) by  $a^\mu = T^\sigma \nabla_\sigma T^\mu$ .

Since  $S$  and  $T$  are basis vectors adapted to a coordinate system, their commutator vanishes:

$$[S, T] = 0. \quad (3.205)$$

From (3.37) we then have

$$S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu. \quad (3.206)$$

With this in mind, let's compute the acceleration:

$$\begin{aligned}
 A^\mu &= T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) \\
 &= T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu) \\
 &= (T^\rho \nabla_\rho S^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu \\
 &= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu{}_{\nu\rho\sigma} T^\nu) \\
 &= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + S^\sigma \nabla_\sigma (T^\rho \nabla_\rho T^\mu) - (S^\sigma \nabla_\sigma T^\rho) \nabla_\rho T^\mu \\
 &\quad + R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma \\
 &= R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma. \tag{3.207}
 \end{aligned}$$

Let's think about this line by line. The first line is the definition of  $A^\mu$ , and the second line comes directly from (3.206). The third line is simply the Leibniz rule. The fourth line replaces a double covariant derivative by the derivatives in the opposite order plus the Riemann tensor. In the fifth line we use Leibniz again (in the opposite order from usual), and then we cancel two identical terms and notice that the term involving  $T^\rho \nabla_\rho T^\mu$  vanishes because  $T^\mu$  is the tangent vector to a geodesic. The result,

$$A^\mu = \frac{D^2}{dt^2} S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma, \tag{3.208}$$

is the **geodesic deviation equation**. It expresses something that we might have expected: the relative acceleration between two neighboring geodesics is proportional to the curvature.

The geodesic deviation equation characterizes the behavior of a one-parameter family of neighboring geodesics. We will sometimes be interested in keeping track of the behavior of a multi-dimensional set of neighboring geodesics, perhaps representing a bundle of photons or a distribution of massive test particles. Such a set of geodesics forms a congruence; in Appendix F we derive equations that describe the evolution of such congruences.

Physically, of course, the acceleration of neighboring geodesics is interpreted as a manifestation of gravitational tidal forces. In the next chapter we explore in more detail how properties of curved spacetime are reflected by physics in a gravitational field.

### 3.11 ■ EXERCISES

- Verify these consequences of metric compatibility ( $\nabla_\sigma g_{\mu\nu} = 0$ ):

$$\begin{aligned}
 \nabla_\sigma g^{\mu\nu} &= 0 \\
 \nabla_\lambda \epsilon_{\mu\nu\rho\sigma} &= 0. \tag{3.209}
 \end{aligned}$$

2. You are familiar with the operations of gradient ( $\nabla\phi$ ), divergence ( $\nabla \cdot \mathbf{V}$ ) and curl ( $\nabla \times \mathbf{V}$ ) in ordinary vector analysis in three-dimensional Euclidean space. Using covariant derivatives, derive formulae for these operations in spherical polar coordinates  $\{r, \theta, \phi\}$  defined by

$$x = r \sin \theta \cos \phi \quad (3.210)$$

$$y = r \sin \theta \sin \phi \quad (3.211)$$

$$z = r \cos \theta. \quad (3.212)$$

Compare your results to those in Jackson (1999) or an equivalent text. Are they identical? Should they be?

3. Imagine we have a *diagonal* metric  $g_{\mu\nu}$ . Show that the Christoffel symbols are given by

$$\Gamma_{\mu\nu}^{\lambda} = 0 \quad (3.213)$$

$$\Gamma_{\mu\mu}^{\lambda} = -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_{\lambda}g_{\mu\mu} \quad (3.214)$$

$$\Gamma_{\mu\lambda}^{\lambda} = \partial_{\mu}(\ln \sqrt{|g_{\lambda\lambda}|}) \quad (3.215)$$

$$\Gamma_{\lambda\lambda}^{\lambda} = \partial_{\lambda}(\ln \sqrt{|g_{\lambda\lambda}|}) \quad (3.216)$$

In these expressions,  $\mu \neq \nu \neq \lambda$ , and repeated indices are *not* summed over.

4. In Euclidean three-space, we can define paraboloidal coordinates  $(u, v, \phi)$  via

$$x = uv \cos \phi \quad y = uv \sin \phi \quad z = \frac{1}{2}(u^2 - v^2).$$

- (a) Find the coordinate transformation matrix between paraboloidal and Cartesian coordinates  $\partial x^{\alpha} / \partial x^{\beta'}$  and the inverse transformation. Are there any singular points in the map?

- (b) Find the basis vectors and basis one-forms in terms of Cartesian basis vectors and forms.

- (c) Find the metric and inverse metric in paraboloidal coordinates.

- (d) Calculate the Christoffel symbols.

- (e) Calculate the divergence  $\nabla_{\mu} V^{\mu}$  and Laplacian  $\nabla_{\mu} \nabla^{\mu} f$ .

5. Consider a 2-sphere with coordinates  $(\theta, \phi)$  and metric

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (3.217)$$

- (a) Show that lines of constant longitude ( $\phi = \text{constant}$ ) are geodesics, and that the only line of constant latitude ( $\theta = \text{constant}$ ) that is a geodesic is the equator ( $\theta = \pi/2$ ).

- (b) Take a vector with components  $V^{\mu} = (1, 0)$  and parallel-transport it once around a circle of constant latitude. What are the components of the resulting vector, as a function of  $\theta$ ?

6. A good approximation to the metric outside the surface of the Earth is provided by

$$ds^2 = -(1 + 2\Phi)dr^2 + (1 - 2\Phi)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (3.218)$$

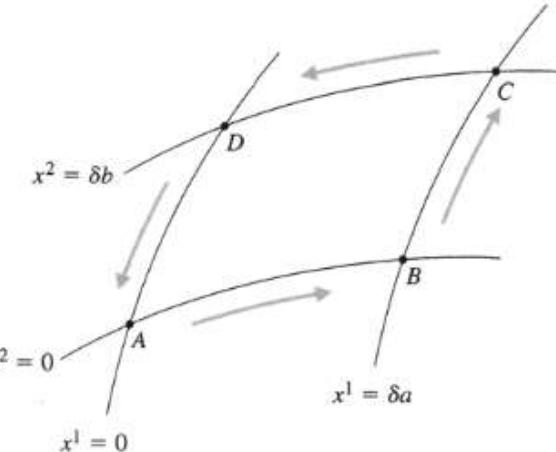
where

$$\Phi = -\frac{GM}{r} \quad (3.219)$$

may be thought of as the familiar Newtonian gravitational potential. Here  $G$  is Newton's constant and  $M$  is the mass of the earth. For this problem  $\Phi$  may be assumed to be small.

- (a) Imagine a clock on the surface of the Earth at distance  $R_1$  from the Earth's center, and another clock on a tall building at distance  $R_2$  from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time  $t$ . Which clock moves faster?
- (b) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ( $\theta = \pi/2$ ). What is  $d\phi/dt$ ?
- (c) How much proper time elapses while a satellite at radius  $R_1$  (skimming along the surface of the earth, neglecting air resistance) completes one orbit? You can work to first order in  $\Phi$  if you like. Plug in the actual numbers for the radius of the Earth and so on (don't forget to restore the speed of light) to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface?

7. For this problem you will use the parallel propagator introduced in Appendix I to see how the Riemann tensor arises from parallel transport around an infinitesimal loop. Consider the following loop:



Using the infinite series expression for the parallel propagator, compute to lowest nontrivial order in  $\delta a$  and  $\delta b$  the transformation induced on a vector that is parallel transported around this loop from  $A$  to  $B$  to  $C$  to  $D$  and back to  $A$ , and show it is proportional to the appropriate components of the Riemann tensor. To make things easy, you can use  $x^1$  and  $x^2$  as parameters on the appropriate legs of the journey.

8. The metric for the three-sphere in coordinates  $x^\mu = (\psi, \theta, \phi)$  can be written

$$ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2). \quad (3.220)$$

- (a) Calculate the Christoffel connection coefficients. Use whatever method you like, but it is good practice to get the connection coefficients by varying the integral (3.49).
- (b) Calculate the Riemann tensor, Ricci tensor, and Ricci scalar.
- (c) Show that (3.191) is obeyed by this metric, confirming that the three-sphere is a maximally symmetric space (as you would expect).
9. Show that the Weyl tensor  $C^\mu{}_{\nu\rho\sigma}$  is left invariant by a conformal transformation.
10. Show that, for  $n \geq 4$ , the Weyl tensor satisfies a version of the Bianchi identity,

$$\nabla_\rho C^\rho{}_{\sigma\mu\nu} = 2 \frac{(n-3)}{(n-2)} \left( \nabla_{[\mu} R_{\nu]\sigma} + \frac{1}{2(n-1)} g_{\sigma[\mu} \nabla_{\nu]} R \right). \quad (3.221)$$

11. Since the Poincaré half-plane with metric (3.192) is maximally symmetric, we might expect that it is rotationally symmetric around any point, although this certainly isn't evident in the  $\{x, y\}$  coordinates. If that is so, it should be possible to put the metric in a form where the rotational symmetry is manifest, such as

$$ds^2 = f^2(r)[dr^2 + r^2 d\theta^2]. \quad (3.222)$$

To show that this works, calculate the curvature scalar for this metric and solve for the function  $f(r)$  subject to the condition  $R = -2/a^2$  everywhere. What is the range of the coordinate  $r$ ?

12. Show that any Killing vector  $K^\mu$  satisfies the relations mentioned in the text:

$$\begin{aligned} \nabla_\mu \nabla_\sigma K^\rho &= R^\rho{}_{\sigma\mu\nu} K^\nu \\ K^\lambda \nabla_\lambda R &= 0. \end{aligned} \quad (3.223)$$

13. Find explicit expressions for a complete set of Killing vector fields for the following spaces:

- (a) Minkowski space, with metric  $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ .
- (b) A spacetime with coordinates  $\{u, v, x, y\}$  and metric

$$ds^2 = -(du dv + dv du) + a^2(u)dx^2 + b^2(u)dy^2, \quad (3.224)$$

where  $a$  and  $b$  are unspecified functions of  $u$ . This represents a gravitational wave spacetime. (*Hints*, which you need not show: there are five Killing vectors in all, and all of them have a vanishing  $u$  component  $K^u$ .)

Be careful, in all of these cases, about the distinction between upper and lower indices.

14. Consider the three Killing vectors of the two-sphere, (3.188). Show that their commutators satisfy the following algebra:

$$\begin{aligned} [R, S] &= T \\ [S, T] &= R \\ [T, R] &= S. \end{aligned} \quad (3.225)$$

15. Use Raychaudhuri's equation, discussed in Appendix F, to show that, if a fluid is flowing on geodesics through spacetime with zero shear and expansion, then spacetime must have a timelike Killing vector.

- 16.** Consider again the metric on a three-sphere,

$$ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2). \quad (3.226)$$

In this problem we make use of noncoordinate bases, discussed in Appendix J. In an orthonormal frame of one-forms  $\hat{\theta}^{(a)}$  the metric would become

$$ds^2 = \hat{\theta}^{(1)} \otimes \hat{\theta}^{(1)} + \hat{\theta}^{(2)} \otimes \hat{\theta}^{(2)} + \hat{\theta}^{(3)} \otimes \hat{\theta}^{(3)}. \quad (3.227)$$

- (a) Find such an orthonormal frame of one-forms, such that the matrix  $e_a^\mu$  is diagonal. Don't worry about covering the entire manifold.
- (b) Compute the components of the spin connection by solving  $de^a + \omega^a{}_b \wedge e^b = 0$ .
- (c) Compute the components of the Riemann tensor  $R^\rho{}_{\sigma\mu\nu}$  in the coordinate basis adapted to  $x^\mu$  by computing the components of the curvature two-form  $R^a{}_{b\mu\nu}$  and then converting.

## CHAPTER

## 4

## Gravitation

## 4.1 ■ PHYSICS IN CURVED SPACETIME

Having paid our mathematical dues, we are now prepared to examine the physics of gravitation as described by general relativity. This subject falls naturally into two pieces: how the gravitational field influences the behavior of matter, and how matter determines the gravitational field. In Newtonian gravity, these two elements consist of the expression for the acceleration of a body in a gravitational potential  $\Phi$ ,

$$\mathbf{a} = -\nabla \Phi, \quad (4.1)$$

and Poisson's differential equation for the potential in terms of the matter density  $\rho$  and Newton's gravitational constant  $G$ :

$$\nabla^2 \Phi = 4\pi G\rho. \quad (4.2)$$

In general relativity, the analogous statements will describe how the curvature of spacetime acts on matter to manifest itself as gravity, and how energy and momentum influence spacetime to create curvature. In either case it would be legitimate to start at the top, by stating outright the laws governing physics in curved spacetime and working out their consequences. Instead, we will try to be a little more motivational, starting with basic physical principles and attempting to argue that these lead naturally to an almost unique physical theory.

In Chapter 2 we motivated our discussion of manifolds by introducing the Einstein Equivalence Principle, or EEP: "In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments." The EEP arises from the idea that gravity is *universal*; it affects all particles (and indeed all forms of energy-momentum) in the same way. This feature of universality led Einstein to propose that what we experience as gravity is a manifestation of the curvature of spacetime. The idea is simply that something so universal as gravitation could be most easily described as a fundamental feature of the background on which matter fields propagate, as opposed to as a conventional force. At the same time, the identification of spacetime as a curved manifold is supported by the similarity between the undetectability of gravity in local regions and our ability to find locally inertial coordinates ( $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$ ,  $\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}} = 0$  at a point  $p$ ) on a manifold.

Best of all, this abstract philosophizing translates directly into a simple recipe for generalizing laws of physics to the curved-spacetime context, known as the **minimal-coupling principle**. In its baldest form, this recipe may be stated as follows:

1. Take a law of physics, valid in inertial coordinates in flat spacetime.
2. Write it in a coordinate-invariant (tensorial) form.
3. Assert that the resulting law remains true in curved spacetime.

It may seem somewhat melodramatic to take such a simple idea and spread it out into a three-part procedure. We hope only to make clear that there is nothing very complicated going on. Operationally, this recipe usually amounts to taking an agreed-upon law in flat space and replacing the Minkowski metric  $\eta_{\mu\nu}$  by the more general metric  $g_{\mu\nu}$ , and replacing partial derivatives  $\partial_\mu$  by covariant derivatives  $\nabla_\mu$ . For this reason, this recipe is sometimes known as the “Comma-Goes-to-Semicolon Rule,” by those who use commas and semicolons to denote partial and covariant derivatives.

As a straightforward example, we can consider the motion of freely-falling (unaccelerated) particles. In flat space such particles move in straight lines; in equations, this is expressed as the vanishing of the second derivative of the parameterized path  $x^\mu(\lambda)$ :

$$\frac{d^2x^\mu}{d\lambda^2} = 0. \quad (4.3)$$

This is not, in general coordinates, a tensorial equation; although  $dx^\mu/d\lambda$  are the components of a well-defined vector, the second derivative components  $d^2x^\mu/d\lambda^2$  are not. You might really think that this is a tensorial-looking equation; however, you can readily check that it's not even true in polar coordinates, unless you expect free particles to move in circles. We can use the chain rule to write

$$\frac{d^2x^\mu}{d\lambda^2} = \frac{dx^\nu}{d\lambda} \partial_\nu \frac{dx^\mu}{d\lambda}. \quad (4.4)$$

Now it is clear how to generalize this to curved space—simply replace the partial derivative by a covariant one,

$$\frac{dx^\nu}{d\lambda} \partial_\nu \frac{dx^\mu}{d\lambda} \rightarrow \frac{dx^\nu}{d\lambda} \nabla_\nu \frac{dx^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda}. \quad (4.5)$$

We recognize, then, that the appropriate general-relativistic version of the Newtonian relation (4.3) is simply the geodesic equation,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0. \quad (4.6)$$

In general relativity, therefore, free particles move along geodesics; we have mentioned this before, but now you have a slightly better idea why it is true.

As an even more straightforward example, and one that we have referred to already, we have the law of energy-momentum conservation in flat spacetime:

$$\partial_\mu T^{\mu\nu} = 0, \quad (4.7)$$

Plugging into our recipe reveals the appropriate generalization to curved spacetime:

$$\nabla_\mu T^{\mu\nu} = 0. \quad (4.8)$$

It really is just that simple—sufficiently so that we felt quite comfortable using this equation in Chapter 3, without any detailed justification. Of course, this simplicity should not detract from the profound consequences of the generalization to curved spacetime, as illustrated in the example of the expanding universe.

It is one thing to generalize an equation from flat to curved spacetime; it is something altogether different to argue that the result describes gravity. To do so, we can show how the usual results of Newtonian gravity fit into the picture. We define the Newtonian limit by three requirements: the particles are moving slowly (with respect to the speed of light), the gravitational field is weak (so that it can be considered as a perturbation of flat space), and the field is also static (unchanging with time). Let us see what these assumptions do to the geodesic equation, taking the proper time  $\tau$  as an affine parameter. “Moving slowly” means that

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}, \quad (4.9)$$

so the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma_{00}^\mu \left( \frac{dt}{d\tau} \right)^2 = 0. \quad (4.10)$$

Since the field is static ( $\partial_0 g_{\mu\nu} = 0$ ), the relevant Christoffel symbols  $\Gamma_{00}^\mu$  simplify:

$$\begin{aligned} \Gamma_{00}^\mu &= \frac{1}{2} g^{\mu\lambda} (\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \\ &= -\frac{1}{2} g^{\mu\lambda} \partial_\lambda g_{00}. \end{aligned} \quad (4.11)$$

Finally, the weakness of the gravitational field allows us to decompose the metric into the Minkowski form plus a small perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \quad (4.12)$$

We are working in inertial coordinates, so  $\eta_{\mu\nu}$  is the canonical form of the metric. The “smallness condition” on the metric perturbation  $h_{\mu\nu}$  doesn’t really make sense in arbitrary coordinates. From the definition of the inverse metric,  $g^{\mu\nu} g_{\nu\sigma} = \delta_\sigma^\mu$ , we find that to first order in  $h$ ,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad (4.13)$$

where  $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$ . In fact, we can use the Minkowski metric to raise and lower indices on an object of any definite order in  $h$ , since the corrections would only contribute at higher orders. If you like, think of  $h_{\mu\nu}$  as a symmetric  $(0, 2)$  tensor field propagating in Minkowski space and interacting with other fields.

Putting it all together, to first order in  $h_{\mu\nu}$  we find

$$\Gamma_{00}^\mu = -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}. \quad (4.14)$$

The geodesic equation (4.10) is therefore

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} \left(\frac{dt}{d\tau}\right)^2. \quad (4.15)$$

Using  $\partial_0 h_{00} = 0$ , the  $\mu = 0$  component of this is just

$$\frac{d^2t}{d\tau^2} = 0. \quad (4.16)$$

That is,  $dt/d\tau$  is constant. To examine the spacelike components of (4.15), recall that the spacelike components of  $\eta^{\mu\nu}$  are just those of a  $3 \times 3$  identity matrix. We therefore have

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2\partial_i h_{00}. \quad (4.17)$$

Dividing both sides by  $(dt/d\tau)^2$  has the effect of converting the derivative on the left-hand side from  $\tau$  to  $t$ , leaving us with

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}. \quad (4.18)$$

This begins to look a great deal like Newton's theory of gravitation. In fact, if we compare this equation to (4.1), we find that they are the same once we identify

$$h_{00} = -2\Phi, \quad (4.19)$$

or in other words

$$g_{00} = -(1 + 2\Phi). \quad (4.20)$$

Therefore, we have shown that the curvature of spacetime is indeed sufficient to describe gravity in the Newtonian limit, as long as the metric takes the form (4.20). It remains, of course, to find field equations for the metric that imply this is the form taken, and that for a single gravitating body we recover the Newtonian formula

$$\Phi = -\frac{GM}{r}, \quad (4.21)$$

but that will come soon enough.

The straightforward procedure we have outlined for generalizing laws of physics to curved spacetime does have some subtleties, which we address in Section 4.7. But it's more than good enough for our present purposes, so let's not delay our pursuit of the second half of our task, obtaining the field equation for the metric in general relativity.

## 4.2 ■ EINSTEIN'S EQUATION

Just as Maxwell's equations govern how the electric and magnetic fields respond to charges and currents, Einstein's field equation governs how the metric responds to energy and momentum. Ultimately the field equation must be postulated and tested against experiment, not derived from any bedrock principles; however, we can motivate it on the basis of plausibility arguments. We will actually do this in two ways: first by some informal reasoning by analogy, close to what Einstein himself was thinking, and then by starting with an action and deriving the corresponding equations of motion.

The informal argument begins with the realization that we would like to find an equation that supersedes the Poisson equation for the Newtonian potential:

$$\nabla^2 \Phi = 4\pi G\rho, \quad (4.22)$$

where  $\nabla^2 = \delta^{ij} \partial_i \partial_j$  is the Laplacian in space and  $\rho$  is the mass density. [The explicit form of  $\Phi$  given in (4.21) is one solution of (4.22), for the case of a pointlike mass distribution.] What characteristics should our sought-after equation possess? On the left-hand side of (4.22) we have a second-order differential operator acting on the gravitational potential, and on the right-hand side a measure of the mass distribution. A relativistic generalization should take the form of an equation between tensors. We know what the tensor generalization of the mass density is; it's the energy-momentum tensor  $T_{\mu\nu}$ . The gravitational potential, meanwhile, should get replaced by the metric tensor, because in (4.20) we had to relate a perturbation of the metric to the Newtonian potential to successfully reproduce gravity. We might therefore guess that our new equation will have  $T_{\mu\nu}$  set proportional to some tensor, which is second-order in derivatives of the metric; something along the lines of

$$[\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu}, \quad (4.23)$$

but of course we want it to be completely tensorial.

The left-hand side of (4.23) is not a sensible tensor; it's just a suggestive notation to indicate that we would like a symmetric  $(0, 2)$  tensor that is second-order in derivatives of the metric. The first choice might be to act the d'Alembertian  $\square = \nabla^\mu \nabla_\mu$  on the metric  $g_{\mu\nu}$ , but this is automatically zero by metric compatibility. Fortunately, there is an obvious quantity which is not zero and is constructed from second derivatives (and first derivatives) of the metric: the Riemann tensor  $R^\rho{}_{\sigma\mu\nu}$ . Recall that the Riemann tensor is constructed from the Christoffel sym-

bols and their first derivatives, and the Christoffel symbols are constructed from the metric and its first derivatives, so  $R^\rho{}_{\sigma\mu\nu}$  contains second derivatives of  $g_{\mu\nu}$ . It doesn't have the right number of indices, but we can contract it to form the Ricci tensor  $R_{\mu\nu}$ , which does (and is symmetric to boot). It is therefore tempting to guess that the gravitational field equations are

$$R_{\mu\nu} = \kappa T_{\mu\nu}, \quad (4.24)$$

for some constant  $\kappa$ . In fact, Einstein did suggest this equation at one point. There is a problem, unfortunately, with conservation of energy. If we want to preserve

$$\nabla^\mu T_{\mu\nu} = 0, \quad (4.25)$$

by (4.24) we would have

$$\nabla^\mu R_{\mu\nu} = 0. \quad (4.26)$$

This is certainly not true in an arbitrary geometry; we have seen from the Bianchi identity (3.150) that

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R. \quad (4.27)$$

But our proposed field equation implies that  $R = \kappa g^{\mu\nu} T_{\mu\nu} = \kappa T$ , so taking these together we have

$$\nabla_\mu T = 0. \quad (4.28)$$

The covariant derivative of a scalar is just the partial derivative, so (4.28) is telling us that  $T$  is constant throughout spacetime. This is highly implausible, since  $T = 0$  in vacuum while  $T \neq 0$  in matter. We have to try harder.

Of course we don't have to try much harder, since we already know of a symmetric  $(0, 2)$  tensor, constructed from the Ricci tensor, which is automatically conserved: the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \quad (4.29)$$

which always obeys  $\nabla^\mu G_{\mu\nu} = 0$ . We are therefore led to propose

$$G_{\mu\nu} = \kappa T_{\mu\nu} \quad (4.30)$$

as a field equation for the metric. (Actually it is probably more common to write out  $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ , rather than use the abbreviation  $G_{\mu\nu}$ .) This equation satisfies all of the obvious requirements: the right-hand side is a covariant expression of the energy and momentum density in the form of a symmetric and conserved  $(0, 2)$  tensor, while the left-hand side is a symmetric and conserved  $(0, 2)$  tensor constructed from the metric and its first and second derivatives. It only remains to fix the proportionality constant  $\kappa$ , and to see whether the result actually repro-

duces gravity as we know it. In other words, does this equation predict the Poisson equation for the gravitational potential in the Newtonian limit?

To answer this, note that contracting both sides of (4.30) yields (in four dimensions)

$$R = -\kappa T, \quad (4.31)$$

and using this we can rewrite (4.30) as

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}). \quad (4.32)$$

This is the same equation, just written slightly differently. We would like to see if it predicts Newtonian gravity in the weak-field, time-independent, slowly-moving-particles limit. We consider a perfect-fluid source of energy-momentum, for which

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}, \quad (4.33)$$

where  $U^\mu$  is the fluid four-velocity and  $\rho$  and  $p$  are the rest-frame energy and momentum densities. In fact for the Newtonian limit we may neglect the pressure; roughly speaking, the pressure of a body becomes important when its constituent particles are traveling at speeds close to that of light, which we exclude from the Newtonian limit by hypothesis. So we are actually considering the energy-momentum tensor of dust:

$$T_{\mu\nu} = \rho U_\mu U_\nu. \quad (4.34)$$

The “fluid” we are considering is some massive body, such as the Earth or the Sun. We will work in the fluid rest frame, in which

$$U^\mu = (U^0, 0, 0, 0). \quad (4.35)$$

The timelike component can be fixed by appealing to the normalization condition  $g_{\mu\nu}U^\mu U^\nu = -1$ . In the weak-field limit we write, in accordance with (4.12) and (4.13),

$$\begin{aligned} g_{00} &= -1 + h_{00}, \\ g^{00} &= -1 - h_{00}. \end{aligned} \quad (4.36)$$

Then to first order in  $h_{\mu\nu}$  we get

$$U^0 = 1 + \frac{1}{2}h_{00}. \quad (4.37)$$

In fact, however, this is needlessly careful, as we are going to plug the four-velocity into (4.34), and the energy density  $\rho$  is already considered small (space-time will be flat as  $\rho$  is taken to zero). So to our level of approximation, we can simply take  $U^0 = 1$ , and correspondingly  $U_0 = -1$ . Then

$$T_{00} = \rho, \quad (4.38)$$

and all other components vanish. In this limit the rest energy  $\rho = T_{00}$  will be much larger than the other terms in  $T_{\mu\nu}$ , so we want to focus on the  $\mu = 0, \nu = 0$  component of (4.32). The trace, to lowest nontrivial order, is

$$T = g^{00}T_{00} = -T_{00} = -\rho. \quad (4.39)$$

We plug this into the 00 component of our proposed gravitational field equation (4.32), to get

$$R_{00} = \frac{1}{2}\kappa\rho. \quad (4.40)$$

This is an equation relating derivatives of the metric to the energy density. To find the explicit expression in terms of the metric, we need to evaluate  $R_{00} = R^{\lambda}_{0;\lambda}$ . In fact we only need  $R^i_{0;j0}$ , since  $R^0_{000} = 0$ . We have

$$R^i_{0;j0} = \partial_j\Gamma^i_{00} - \partial_0\Gamma^i_{j0} + \Gamma^i_{j\lambda}\Gamma^{\lambda}_{00} - \Gamma^i_{0\lambda}\Gamma^{\lambda}_{j0}. \quad (4.41)$$

The second term here is a time derivative, which vanishes for static fields. The third and fourth terms are of the form  $(\Gamma)^2$ , and since  $\Gamma$  is first-order in the metric perturbation these contribute only at second order, and can be neglected. We are left with  $R^i_{0;j0} = \partial_j\Gamma^i_{00}$ . From this we get

$$\begin{aligned} R_{00} &= R^i_{0;i0} \\ &= \partial_i \left[ \frac{1}{2}g^{i\lambda}(\partial_0g_{\lambda 0} + \partial_0g_{0\lambda} - \partial_\lambda g_{00}) \right] \\ &= -\frac{1}{2}\delta^{ij}\partial_i\partial_j h_{00} \\ &= -\frac{1}{2}\nabla^2 h_{00}. \end{aligned} \quad (4.42)$$

Comparing to (4.40), we see that the 00 component of (4.30) in the Newtonian limit predicts

$$\nabla^2 h_{00} = -\kappa\rho. \quad (4.43)$$

Since (4.19) sets  $h_{00} = -2\Phi$ , this is precisely the Poisson equation (4.22), if we set  $\kappa = 8\pi G$ .

So our guess, (4.30), seems to have worked out. With the normalization chosen so as to correctly recover the Newtonian limit, we can present **Einstein's equation** for general relativity:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (4.44)$$

This tells us how the curvature of spacetime reacts to the presence of energy-momentum.  $G$  is of course Newton's constant of gravitation; it has nothing to do with the trace of  $G_{\mu\nu}$ . Einstein, you may have heard, thought that the left-hand side was nice and geometrical, while the right-hand side was somewhat less compelling.

It is sometimes useful to rewrite Einstein's equation in a slightly different form. Following (4.31) and (4.32), we can take the trace of (4.44) to find that  $R = -8\pi GT$ . Plugging this in and moving the trace term to the right-hand side, we obtain

$$R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right). \quad (4.45)$$

The difference between this and (4.44) is purely cosmetic; in substance they are precisely the same. We will often be interested in the Einstein's equation in vacuum, where  $T_{\mu\nu} = 0$  (for example, outside a star or planet). Then of course the right-hand side of (4.45) vanishes. Therefore the vacuum Einstein equation is simply

$$R_{\mu\nu} = 0. \quad (4.46)$$

This is both slightly less formidable, and of considerable physical usefulness.

### 4.3 ■ LAGRANGIAN FORMULATION

An alternative route to Einstein's equation is through the principle of least action, as we discussed for classical field theories in flat spacetime at the end of Chapter 1. Let's spend a moment to generalize those results to curved spacetime, and then see what kind of Lagrangian is appropriate for general relativity. We'll work in  $n$  dimensions, since our results will not depend on the dimensionality; we will, however, assume that our metric has Lorentzian signature.

Consider a field theory in which the dynamical variables are a set of fields  $\Phi^i(x)$ . The classical solutions to such a theory will be those that are critical points of an action  $S$ , generally expressed as an integral over space of a Lagrange density  $\mathcal{L}$ ,

$$S = \int \mathcal{L}(\Phi^i, \nabla_\mu \Phi^i) d^n x. \quad (4.47)$$

Note that we are now imagining that the Lagrangian is a function of the fields and their covariant (rather than partial) derivatives, as is appropriate in curved space. Note also that, since  $d^n x$  is a density rather than a tensor,  $\mathcal{L}$  is also a density (since their product must be a well-defined tensor); we typically write

$$\mathcal{L} = \sqrt{-g} \hat{\mathcal{L}}, \quad (4.48)$$

where  $\hat{\mathcal{L}}$  is indeed a scalar. You might think it would be sensible to forget about what we are calling  $\mathcal{L}$  and just focus on  $\hat{\mathcal{L}}$ , but in fact both quantities are useful in different circumstances; it is  $\mathcal{L}$  that will matter whenever we are varying with

respect to the metric itself. The associated Euler–Lagrange equations make use of the scalar  $\widehat{\mathcal{L}}$ , and are otherwise like those in flat space, but with covariant instead of partial derivatives:

$$\frac{\partial \widehat{\mathcal{L}}}{\partial \Phi} - \nabla_\mu \left( \frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi)} \right) = 0. \quad (4.49)$$

In deriving these equations, we make use of Stokes's theorem (3.35),

$$\int_{\Sigma} \nabla_\mu V^\mu \sqrt{|g|} d^n x = \int_{\partial \Sigma} n_\mu V^\mu \sqrt{|\gamma|} d^{n-1} x, \quad (4.50)$$

and set the variation equal to zero at infinity (the boundary). Integration by parts therefore takes the form

$$\int A^\mu (\nabla_\mu B) \sqrt{-g} d^n x = - \int (\nabla_\mu A^\mu) B \sqrt{-g} d^n x + \text{boundary terms}. \quad (4.51)$$

For example, the curved-spacetime generalization of the action for a single scalar field  $\phi$  considered in Chapter 1 would be

$$S_\phi = \int \left[ -\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - V(\phi) \right] \sqrt{-g} d^n x, \quad (4.52)$$

which would lead to an equation of motion

$$\square \phi - \frac{dV}{d\phi} = 0, \quad (4.53)$$

where the covariant d'Alembertian is

$$\square = \nabla^\mu \nabla_\mu = g^{\mu\nu} \nabla_\mu \nabla_\nu, \quad (4.54)$$

Just as in flat spacetime, the combination  $g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi)$  is often abbreviated as  $(\nabla \phi)^2$ . Of course, the covariant derivatives are equivalent to partial derivatives when acting on scalars, but it is wise to use the  $\nabla_\mu$  notation still; you never know when you might integrate by parts and suddenly be acting on a vector.

With that as a warm-up, we turn to the construction of an action for general relativity. Our dynamical variable is now the metric  $g_{\mu\nu}$ ; what scalars can we make out of the metric to serve as a Lagrangian? Since we know that the metric can be set equal to its canonical form and its first derivatives set to zero at any one point, any nontrivial scalar must involve at least second derivatives of the metric. The Riemann tensor is of course made from second derivatives of the metric, and we argued earlier that the only independent scalar we could construct from the Riemann tensor was the Ricci scalar  $R$ . What we did not show, but is nevertheless true, is that any nontrivial tensor made from products of the metric and its first and second derivatives can be expressed in terms of the metric and the Riemann tensor. Therefore, the *only* independent scalar constructed from the metric, which

is no higher than second order in its derivatives, is the Ricci scalar. Hilbert figured that this was therefore the simplest possible choice for a Lagrangian, and proposed

$$S_H = \int \sqrt{-g} R d^n x, \quad (4.55)$$

known as the **Hilbert action** (or sometimes the Einstein–Hilbert action). As we shall see, he was right.

The equation of motion should come from varying the action with respect to the metric. Unfortunately the action isn't quite in the form (4.47), since it can't be written in terms of covariant derivatives of  $g_{\mu\nu}$  (which would simply vanish). Therefore, instead of simply plugging into the Euler–Lagrange equations, we will consider directly the behavior of  $S_H$  under small variations of the metric. In fact it is more convenient to vary with respect to the inverse metric  $g^{\mu\nu}$ . Since  $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$ , and the Kronecker delta is unchanged under any variation, it is straightforward to express variations of the metric and inverse metric in terms of each other:

$$\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}, \quad (4.56)$$

so stationary points with respect to variations in  $g^{\mu\nu}$  are equivalent to those with respect to variations in  $g_{\mu\nu}$ . Using  $R = g^{\mu\nu} R_{\mu\nu}$ , we have

$$\delta S_H = (\delta S)_1 + (\delta S)_2 + (\delta S)_3, \quad (4.57)$$

where

$$\begin{aligned} (\delta S)_1 &= \int d^n x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \\ (\delta S)_2 &= \int d^n x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} \\ (\delta S)_3 &= \int d^n x R \delta \sqrt{-g}. \end{aligned} \quad (4.58)$$

The second term  $(\delta S)_2$  is already in the form of some expression multiplied by  $\delta g^{\mu\nu}$ ; let's examine the others more closely.

Recall that the Ricci tensor is the contraction of the Riemann tensor, which is given by

$$R^\rho{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\lambda\sigma} \Gamma^\sigma_{\nu\mu} - (\lambda \leftrightarrow \nu). \quad (4.59)$$

The variation of the Riemann tensor with respect to the metric can be found by first varying the connection with respect to the metric, and then substituting into this expression. However, let us consider arbitrary variations of the connection by replacing

$$\Gamma^\rho_{\nu\mu} \rightarrow \Gamma^\rho_{\nu\mu} + \delta \Gamma^\rho_{\nu\mu}. \quad (4.60)$$

The variation  $\delta\Gamma_{v\mu}^\rho$  is the difference of two connections, and therefore is itself a tensor. We can thus take its covariant derivative,

$$\nabla_\lambda(\delta\Gamma_{v\mu}^\rho) = \partial_\lambda(\delta\Gamma_{v\mu}^\rho) + \Gamma_{\lambda\sigma}^\rho \delta\Gamma_{v\mu}^\sigma - \Gamma_{\lambda\nu}^\sigma \delta\Gamma_{\sigma\mu}^\rho - \Gamma_{\lambda\mu}^\sigma \delta\Gamma_{v\sigma}^\rho. \quad (4.61)$$

Here and elsewhere, the covariant derivatives are taken with respect to  $g_{\mu\nu}$ , not  $g_{\mu\nu} + \delta g_{\mu\nu}$ . Given this expression and a small amount of labor, it is easy to show that, to first order in the variation,

$$\delta R^\rho_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma_{v\mu}^\rho) - \nabla_v(\delta\Gamma_{\lambda\mu}^\rho). \quad (4.62)$$

You are encouraged check this yourself. Therefore, the contribution of the first term in (4.58) to  $\delta S$  can be written

$$\begin{aligned} (\delta S)_1 &= \int d^n x \sqrt{-g} g^{\mu\nu} [\nabla_\lambda(\delta\Gamma_{v\mu}^\lambda) - \nabla_v(\delta\Gamma_{\lambda\mu}^\lambda)] \\ &= \int d^n x \sqrt{-g} \nabla_\sigma [g^{\mu\nu}(\delta\Gamma_{\mu\nu}^\sigma) - g^{\mu\sigma}(\delta\Gamma_{\lambda\mu}^\lambda)], \end{aligned} \quad (4.63)$$

where we have used metric compatibility and relabeled some dummy indices. We can now plug in the expression for  $\delta\Gamma_{\mu\nu}^\sigma$  in terms of  $\delta g^{\mu\nu}$ , which works out to be

$$\delta\Gamma_{\mu\nu}^\sigma = -\frac{1}{2} [g_{\lambda\mu}\nabla_v(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})], \quad (4.64)$$

leading to

$$(\delta S)_1 = \int d^n x \sqrt{-g} \nabla_\sigma [g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})], \quad (4.65)$$

as you are also welcome to check. But (4.63) [or (4.65)] is an integral with respect to the natural volume element of the covariant divergence of a vector; by Stokes's theorem, this is equal to a boundary contribution at infinity, which we can set to zero by making the variation vanish at infinity. Therefore this term contributes nothing to the total variation. Although to be honest, we have cheated. The boundary term will include not only the metric variation, but also its first derivative, which is not traditionally set to zero. For our present purposes it doesn't matter, but in principle we might care about what happens at the boundary, and would have to include an additional term in the action to take care of this subtlety.

To make sense of the  $(\delta S)_3$  term we need to use the following fact, true for any square matrix  $M$  with nonvanishing determinant:

$$\ln(\det M) = \text{Tr}(\ln M). \quad (4.66)$$

Here,  $\ln M$  is defined by  $\exp(\ln M) = M$ . For numbers this is obvious, for matrices it's a little less straightforward. The variation of this identity yields

$$\frac{1}{\det M} \delta(\det M) = \text{Tr}(M^{-1} \delta M). \quad (4.67)$$

We have used the cyclic property of the trace to allow us to ignore the fact that  $M^{-1}$  and  $\delta M$  may not commute. Taking the matrix  $M$  to be the metric  $g_{\mu\nu}$ , so that  $\det M = \det g_{\mu\nu} = g$ , we get

$$\begin{aligned}\delta g &= g(g^{\mu\nu}\delta g_{\mu\nu}) \\ &= -g(g_{\mu\nu}\delta g^{\mu\nu}).\end{aligned}\quad (4.68)$$

In the last step we converted from  $\delta g_{\mu\nu}$  to  $\delta g^{\mu\nu}$  using (4.56). Now we can just plug in to get

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g \\ &= \frac{1}{2}\frac{g}{\sqrt{-g}}g_{\mu\nu}\delta g^{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.\end{aligned}\quad (4.69)$$

Harkening back to (4.58), and remembering that  $(\delta S)_1$  does not contribute, we find

$$\delta S_H = \int d^n x \sqrt{-g} \left[ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \right] \delta g^{\mu\nu}. \quad (4.70)$$

Recall that the functional derivative of the action satisfies

$$\delta S = \int \sum_i \left( \frac{\delta S}{\delta \Phi^i} \delta \Phi^i \right) d^n x, \quad (4.71)$$

where  $\{\Phi^i\}$  is a complete set of fields being varied (in our case, it's just  $g^{\mu\nu}$ ). Stationary points are those for which each  $\delta S / \delta \Phi^i = 0$ , so we recover Einstein's equation in vacuum:

$$\frac{1}{\sqrt{-g}} \frac{\delta S_H}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \quad (4.72)$$

The advantage of the Lagrangian approach is manifested by the fact that our very first guess (which was practically unique) gave the right answer, in contrast with our previous trial-and-error method. This is a reflection of two elegant features of this technique: First, the Lagrangian is a scalar, rather than a tensor, and therefore more restricted; second, the symmetries of the theory are straightforwardly imposed (in this case, we automatically derived a tensor with vanishing divergence, which is related to diffeomorphism invariance, as discussed in Appendix B).

We derived Einstein's equation "in vacuum" because we only included the gravitational part of the action, not additional terms for matter fields. What we would really like, however, is to get the nonvacuum field equation as well. That

means we consider an action of the form

$$S = \frac{1}{16\pi G} S_H + S_M, \quad (4.73)$$

where  $S_M$  is the action for matter, and we have presciently normalized the gravitational action so that we get the right answer. Following through the same procedure as above leads to

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = 0. \quad (4.74)$$

We now boldly define the energy-momentum tensor to be

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \quad (4.75)$$

This allows us to recover the complete Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (4.76)$$

or equivalently,  $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ .

Why should we think that (4.75) is really the energy-momentum tensor? In some sense it is only because it is a symmetric, conserved,  $(0, 2)$  tensor with dimensions of energy density; if you prefer to call it by some other name, go ahead. But it also accords with our preconceived expectations. Consider again the action for a scalar field, (4.52). Now vary this action with respect, not to  $\phi$ , but to the inverse metric:

$$8S_\phi = \int d^n x \left[ \sqrt{-g} \left( -\frac{1}{2} \delta g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) + \delta \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right) \right] \quad (4.77)$$

$$= \int d^n x \sqrt{-g} \delta g^{\mu\nu} \left[ -\frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi + \left( -\frac{1}{2} g_{\mu\nu} \right) \left( -\frac{1}{2} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - V(\phi) \right) \right]. \quad (4.78)$$

We therefore have

$$\begin{aligned} T_{\mu\nu}^{(\phi)} &= -2 \frac{1}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} \\ &= \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - g_{\mu\nu} V(\phi). \end{aligned} \quad (4.79)$$

In flat spacetime this reduces to what we had asserted, in Chapter 1, was the correct energy-momentum tensor for a scalar field.

On the other hand, in Minkowski space there is an alternative definition for the energy-momentum tensor, which is sometimes given in books on electromagnetism or field theory. In this context energy-momentum conservation arises

as a consequence of symmetry of the Lagrangian under spacetime translations. **Noether's theorem** states that every symmetry of a Lagrangian implies the existence of a conservation law; invariance under the four spacetime translations leads to a tensor  $S^{\mu\nu}$ , which obeys  $\partial_\mu S^{\mu\nu} = 0$  (four relations, one for each value of  $\nu$ ). The details can be found in Wald (1984) or Peskin and Schroeder (1995). Applying Noether's procedure to a Lagrangian that depends on some fields  $\Phi^i$  and their first derivatives  $\partial_\mu \Phi^i$  (in flat spacetime), we obtain

$$S^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi^i)} \partial^\nu \Phi^i - \eta^{\mu\nu} \mathcal{L}, \quad (4.80)$$

where a sum over  $i$  is implied. You can check that this tensor is conserved by virtue of the equations of motion of the matter fields.  $S^{\mu\nu}$  often goes by the name “canonical energy-momentum tensor”; however, there are a number of reasons why it is more convenient for us to use (4.75). First, (4.75) is in fact what appears on the right hand side of Einstein's equation when it is derived from an action, and it is not always possible to generalize (4.80) to curved spacetime. But even in flat space (4.75) has its advantages; it is manifestly symmetric, and also guaranteed to be gauge invariant, neither of which is true for (4.80). We will therefore stick with (4.75) as the definition of the energy-momentum tensor.

Now that Einstein's equation has been derived, the rest of this chapter is devoted to exploring some of its properties. These discussions are fascinating but not strictly necessary; if you like, you can jump right to the applications discussed in subsequent chapters.

#### 4.4 ■ PROPERTIES OF EINSTEIN'S EQUATION

Einstein's equation may be thought of as a set of second-order differential equations for the metric tensor field  $g_{\mu\nu}$ . There are really ten independent equations (since both sides are symmetric two-index tensors), which seems to be exactly right for the ten unknown functions of the metric components. However, the Bianchi identity  $\nabla^\mu G_{\mu\nu} = 0$  represents four constraints on the functions  $R_{\mu\nu}(x)$ , so there are only six truly independent equations in (4.44). In fact this is appropriate, since if a metric is a solution to Einstein's equation in one coordinate system  $x^\mu$  it should also be a solution in any other coordinate system  $x^{\mu'}$ . This means that there are four unphysical degrees of freedom in  $g_{\mu\nu}$ , represented by the four functions  $x^{\mu'}(x^\mu)$ , and we should expect that Einstein's equation only constrains the six coordinate-independent degrees of freedom.

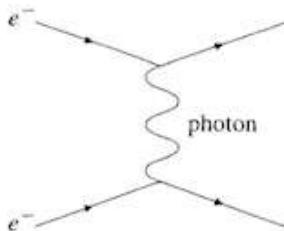
As differential equations, these are extremely complicated; the Ricci scalar and tensor are contractions of the Riemann tensor, which involves derivatives and products of the Christoffel symbols, which in turn involve the inverse metric and derivatives of the metric. Furthermore, the energy-momentum tensor  $T_{\mu\nu}$  will generally involve the metric as well. The equations are also nonlinear, so that two known solutions cannot be superposed to find a third. It is therefore very

difficult to solve Einstein's equation in any sort of generality, and it is usually necessary to make some simplifying assumptions. Even in vacuum, where we set the energy-momentum tensor to zero, the resulting equation (4.46) can be very difficult to solve. The most popular sort of simplifying assumption is that the metric has a significant degree of symmetry, and we will see later how isometries make life easier.

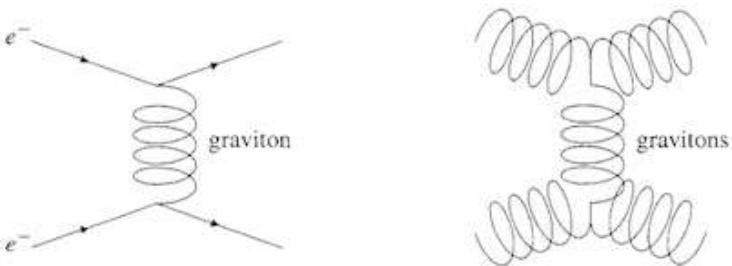
The nonlinearity of general relativity is worth a remark. In Newtonian gravity the potential due to two point masses is simply the sum of the potentials for each mass, but clearly this does not carry over to general relativity outside the weak-field limit. There is a physical reason for this, namely that in GR the gravitational field couples to itself. This can be thought of as a consequence of the equivalence principle—if gravitation did not couple to itself, a gravitational atom (two particles bound by their mutual gravitational attraction) would have a different inertial mass than gravitational mass (due to the negative binding energy). The nonlinearity of Einstein's equation is a reflection of the back-reaction of gravity on itself.

A nice way to think about this is provided by Feynman diagrams. These are used in quantum field theory to calculate the amplitudes for scattering processes, which can be obtained by summing the various contributions from different interactions, each represented by its own diagram. Even if we don't go so far as to quantize gravity and calculate scattering cross-sections (see the end of this section), we can still draw Feynman diagrams as a simple way of keeping track of which interactions exist and which do not. A simple example is provided by the electromagnetic interaction between two electrons; this can be thought of as due to exchange of a virtual photon, as shown in Figure 4.1.

In contrast, there is no diagram in which two photons exchange another photon between themselves, because electromagnetism is linear (there is no back-reaction). The gravitational interaction, meanwhile, can be thought of as deriving from the exchange of a virtual graviton (a quantized perturbation of the metric). The nonlinearity manifests itself as the fact that both electrons and gravi-



**FIGURE 4.1** A Feynman diagram for electromagnetism. In quantum field theory, such diagrams are used to calculate amplitudes for scattering processes; here, just think of it as a cartoon representing a certain interaction. The point of this particular diagram is that the coupling of photons to electrons is what causes the electromagnetic interaction between them. In contrast, there is no coupling of photons to other photons, and no analogous diagram in which photons interact.



**FIGURE 4.2** Feynman diagrams for gravity. Upon quantization, Einstein's equation predicts spin-two particles called gravitons. We don't know how to carry out such a quantization consistently, but the existence of gravitons is sufficiently robust that it is expected to be a feature of any well-defined scheme. Since gravity couples to energy-momentum, gravitons interact with every kind of particle, including other gravitons. This provides a way of thinking about the nonlinearity of Einstein's theory.

tons can exchange virtual gravitons, and therefore exert a gravitational force, as shown in Figure 4.2. There is nothing unique about this feature of gravity; it is shared by most gauge theories, such as quantum chromodynamics, the theory of the strong interactions. Electromagnetism is actually the exception; the linearity can be traced to the fact that the relevant gauge group,  $U(1)$ , is abelian. But non-linearity does represent a departure from the Newtonian theory. This difference is experimentally detectable; the reason why (as we shall see) the orbit of Mercury is different in GR versus Newtonian gravity is that the gravitational field influences itself, and the closer we get to the Sun, the more noticeable that influence is.

Beyond the fact that it is complicated and nonlinear, it is worth thinking a bit about what Einstein's equation is actually telling us. Clearly it relates the energy-momentum distribution to components of the curvature tensor; but from a physical point of view, precisely what kind of gravitational field is generated by a given kind of source? One way to answer this question is to consider the evolution of the *expansion*  $\theta$  of a family of neighboring timelike geodesics. We imagine a small ball of free test particles moving along geodesics with four-velocities  $U^\mu$ , and follow their evolution; the expansion  $\theta = \nabla_\mu U^\mu$  tells us how the volume of the ball is growing (or shrinking, if  $\theta < 0$ ) at any one moment of time. Clearly the value of the expansion will depend on the initial conditions for our test particles. The effects of gravity, on the other hand, are encoded in the *evolution* of the expansion, which is governed by Raychaudhuri's equation. This equation, discussed in Appendix F, tells us that the derivative of the expansion with respect to the proper time  $\tau$  along the geodesics is given by the following expression:

$$\frac{d\theta}{d\tau} = 2\omega^2 - 2\sigma^2 - \frac{1}{3}\theta^2 - R_{\mu\nu}U^\mu U^\nu. \quad (4.81)$$

The terms on the right-hand side are explained carefully in Appendix F;  $\omega$  encodes the rotation of the geodesics,  $\sigma$  encodes the shear, and  $R_{\mu\nu}$  is of course the Ricci tensor. Raychaudhuri's equation is a purely geometric relation, making no

reference to Einstein's equation. The combination of the two equations, however, can be used to describe how energy-momentum influences the motion of test particles, since Einstein's equation relates  $T_{\mu\nu}$  to  $R_{\mu\nu}$  and Raychaudhuri's equation relates  $R_{\mu\nu}$  to  $d\theta/d\tau$ .

Let us consider the simplest possible situation, where we start with all of the nearby particles at rest with respect to each other in a small region of spacetime. Then the expansion, rotation, and shear will all vanish at this initial moment. Let us further construct locally inertial coordinates  $x^{\hat{\mu}}$ , in which  $U^{\hat{\mu}}$  is in its rest frame, so that  $U^{\hat{\mu}} = (1, 0, 0, 0)$  and  $R_{\hat{\mu}\hat{\nu}} U^{\hat{\mu}} U^{\hat{\nu}} = R_{\hat{0}\hat{0}}$ . We therefore have (in these coordinates, at this point)

$$\frac{d\theta}{d\tau} = -R_{\hat{0}\hat{0}}. \quad (4.82)$$

Now we can turn to Einstein's equation, in the form

$$R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right). \quad (4.83)$$

Since we are in locally inertial coordinates, we have

$$g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} \quad (4.84)$$

$$T = g^{\hat{\mu}\hat{\nu}} T_{\hat{\mu}\hat{\nu}} = -\rho + p_x + p_y + p_z, \quad (4.85)$$

where  $\rho = T_{\hat{0}\hat{0}}$  is the rest-frame energy density and  $p_k = T_{\hat{k}\hat{k}}$  is the pressure in the  $x^{\hat{k}}$  direction. Thus, (4.82) becomes

$$\frac{d\theta}{d\tau} = -4\pi G(\rho + p_x + p_y + p_z). \quad (4.86)$$

This equation is telling us that energy and pressure create a gravitational field that works to decrease the volume of our initially stationary ball of test particles (if  $\rho$  and the  $p_i$ 's are all positive). In other words, gravity is attractive.

Of course, from (4.86) we see that gravity is not *necessarily* attractive; we could imagine sources for which  $\rho + p_x + p_y + p_z$  were a negative number. Clearly, the role of pressure bears noting. For one thing, it represents an unambiguous departure from Newtonian theory, in which the pressure does not influence gravity (it doesn't appear in Poisson's equation,  $\nabla^2 \Phi = 4\pi G\rho$ ). The difference is hard to notice in our Solar System, since the pressure in the Sun and planets is much less than the energy density, which is dominated by the rest masses of the constituent particles. For another thing, notice that the *gravitational* effect of the pressure is opposite to that of the *direct* effect with which we are more familiar, namely that positive pressure works to push things apart. In most circumstances the direct effect of pressure is much more noticeable. However, the pressure can only act directly when there is a pressure gradient (for example, a change in pressure between the interior and exterior of a piston), whereas the gravitational effect depends only on the value of the pressure locally. If there were a perfectly smooth

pressure, it would only be detectable through its gravitational effect; an example is provided by vacuum energy, discussed in Section 4.5.

As a final comment on (4.86), let's point out that it is completely equivalent to Einstein's equation—they convey identical information. This very specific relation will hold for any set of initially motionless test particles; the only way this can happen is if all of the components of Einstein's equation are true. If we like, then, we can state Einstein's equation in words<sup>1</sup> as follows: "The expansion of the volume of any set of particles initially at rest is proportional to (minus) the sum of the energy density and the three components of pressure."

So Einstein's equation tells us that energy density and pressure affect the Ricci tensor in such a way as to attract particles together when  $\rho$  and  $p$  are positive. What about the components of the Riemann tensor that are not included in the Ricci tensor? In Chapter 3 we found that these components were described by the Weyl tensor (expressed here in four dimensions),

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} + \frac{1}{3}g_{\rho[\mu}g_{\nu]\sigma}R - g_{\rho[\mu}R_{\nu]\sigma} + g_{\sigma[\mu}R_{\nu]\rho}. \quad (4.87)$$

The Ricci tensor is the trace of the Riemann tensor, while the Weyl tensor describes the trace-free part; together they provide a complete characterization of the curvature. Clearly, given some specified energy-momentum distribution, there is still some freedom in the choice of Weyl curvature, since there is no analogue of Einstein's equation to relate  $C^{\rho}_{\sigma\mu\nu}$  algebraically to  $T_{\mu\nu}$ . This is exactly as it should be. Imagine for example a spacetime that is vacuum everywhere,  $R_{\mu\nu} = 0$ . Flat Minkowski space is a possible solution in such a case, but so is a gravitational wave propagating through empty spacetime (as we will discuss in Chapter 7).

Since only  $R_{\mu\nu}$  enters Einstein's equation, it might appear that the components of  $C_{\rho\sigma\mu\nu}$  are completely unconstrained. But recall that we are not permitted to arbitrarily specify the components of the curvature tensor throughout a manifold; they are related by the Bianchi identity,

$$\nabla_{[\lambda}R_{\rho\sigma]\mu\nu} = 0. \quad (4.88)$$

As you showed in Exercise 10 of Chapter 3, this identity implies a differential relation for the Weyl tensor of the form

$$\nabla^{\rho}C_{\rho\sigma\mu\nu} = \nabla_{[\mu}R_{\nu]\sigma} + \frac{1}{6}g_{\sigma[\mu}\nabla_{\nu]}R. \quad (4.89)$$

On the right-hand side, the Riemann tensor only appears via its contractions—the Ricci scalar and tensor, which can be related to  $T_{\mu\nu}$  by Einstein's equation; we therefore have

$$\nabla^{\rho}C_{\rho\sigma\mu\nu} = 8\pi G \left( \nabla_{[\mu}T_{\nu]\sigma} + \frac{1}{3}g_{\sigma[\mu}\nabla_{\nu]}T \right). \quad (4.90)$$

So, while  $R_{\mu\nu}$  and  $T_{\mu\nu}$  are related algebraically through Einstein's equation,  $C_{\rho\sigma\mu\nu}$  and  $T_{\mu\nu}$  are related by this first-order differential equation. There will be

<sup>1</sup>J.C Baez, "The Meaning of Einstein's Equation," <http://arXiv.org/abs/gr-qc/0103044>.

a number of possible solutions for a given energy-momentum distribution, each specified by certain boundary conditions. This equation can be thought of as a propagation equation for gravitational waves, in close analogy with Maxwell's equations  $\nabla_\mu F^{\nu\mu} = J^\nu$ .

Having listed all of these lovely properties of Einstein's equation, it seems only fair that we should mention one distressing feature: the well-known difficulty of reconciling general relativity with quantum mechanics. GR is a classical field theory: the dynamical variable is a field (the metric) defined on spacetime, and coordinate-invariant quantities constructed from this field (such as the curvature scalar) can in principle be specified and measured to arbitrary accuracy. In the case of other field theories, such as electromagnetism, there are well-understood procedures for beginning with the classical theory and quantizing it, to obtain the dynamics of operators acting on wave functions living in a Hilbert space. For GR, the usual procedures run into both technical and conceptual difficulties, a description of which is beyond the scope of this book. One aspect of the technical difficulties is that GR is not "renormalizable" in the way that the Standard Model of particle physics is; when considering higher-order quantum effects, infinities appear that cannot be absorbed in any finite number of parameters. Nonrenormalizability does not mean that theory is fundamentally incorrect, but is a strong suggestion that it should only be taken seriously up to a certain energy scale.

Fortunately, the regime in which observable effects of quantum gravity are expected to become important is far from our everyday experience (or, for that matter, any conditions we can produce in the lab). Way back in 1899 Planck noticed that his constant  $\hbar$ , for which nowadays we more often substitute  $\hbar = \hbar/2\pi = 1.05 \times 10^{-27}$  cm<sup>2</sup> g/sec, could be combined with Newton's constant  $G = 6.67 \times 10^{-8}$  cm<sup>3</sup> g<sup>-1</sup> sec<sup>-2</sup> and the speed of light  $c = 3.00 \times 10^{10}$  cm sec<sup>-1</sup> to form a basic set of dimensionful quantities: the Planck mass,

$$m_P = \left( \frac{\hbar c}{G} \right)^{1/2} = 2.18 \times 10^{-5} \text{ g}, \quad (4.91)$$

the Planck length,

$$l_P = \left( \frac{\hbar G}{c^3} \right)^{1/2} = 1.62 \times 10^{-33} \text{ cm}, \quad (4.92)$$

the Planck time,

$$t_P = \left( \frac{\hbar G}{c^5} \right)^{1/2} = 5.39 \times 10^{-44} \text{ sec}, \quad (4.93)$$

and the Planck energy,

$$E_P = \left( \frac{\hbar c^5}{G} \right)^{1/2} = 1.95 \times 10^{16} \text{ erg} \quad (4.94)$$

$$= 1.22 \times 10^{19} \text{ GeV}. \quad (4.95)$$

A GeV is  $10^9$  electron volts, a common unit in particle physics, as it is approximately the mass of a proton. We usually set  $\hbar = c = 1$ , so that these quantities are all indistinguishable in the sense that  $m_P = l_P^{-1} = t_P^{-1} = E_P$ . You will hear people say things like “the Planck mass is  $10^{19}$  GeV”; or simply refer to “the Planck scale.” Another commonly used quantity is the reduced Planck scale,  $\bar{m}_P = m_P/\sqrt{8\pi} = 2.43 \times 10^{18}$  GeV, which is often more convenient in equations—note that the coefficient of the curvature scalar in (4.73) is  $\bar{m}_P^2/2$ . Most likely, quantum gravity does not become important until we consider particle masses greater than  $m_P$ , or times shorter than  $t_P$ , or lengths smaller than  $l_P$ , or energies higher than  $E_P$ ; at lower scales, classical GR should suffice. Since these are all far removed from observable phenomena, constructing a consistent theory of quantum gravity is more an issue of principle than of practice. On the other hand, quantum effects in curved spacetime might be important in the real world; as we will discuss in Chapter 8, they might lead to density fluctuations in the early universe, which grow into the galaxies and large-scale structure we observe today.

There is a leading contender for a fully quantum theory that would encompass GR in the appropriate limit: string theory. In string theory we imagine that the fundamental objects are not point particles like electrons or photons, but rather small one-dimensional objects called strings, which can be either closed loops or open segments. String theory was originally proposed as a model of the strong nuclear force, but it was soon realized that the theory inevitably predicted a massless spin-two particle: exactly what a quantum theory of gravity would require. String theory seems to be a consistent quantum theory, and it predicts gravity, but there is still a great deal about it that we don’t understand. In particular, the way in which a classical spacetime arises out of fundamental strings is somewhat mysterious, and the connection to direct experiments is tenuous at best. Nevertheless, string theory is remarkably rich and robust, and promises to be an important part of theoretical physics for the foreseeable future.

#### 4.5 ■ THE COSMOLOGICAL CONSTANT

A characteristic feature of general relativity is that the source for the gravitational field is the entire energy-momentum tensor. In nongravitational physics, only *changes* in energy from one state to another are measurable; the normalization of the energy is arbitrary. For example, the motion of a particle with potential energy  $V(x)$  is precisely the same as that with a potential energy  $V(x) + V_0$ , for any constant  $V_0$ . In gravitation, however, the actual value of the energy matters, not just the differences between states.

This behavior opens up the possibility of **vacuum energy**: an energy density characteristic of empty space. One feature that we might want the vacuum to exhibit is that it not pick out a preferred direction; it will still be possible to have a nonzero energy density if the associated energy-momentum tensor is Lorentz invariant in locally inertial coordinates. Lorentz invariance implies that the corre-

sponding energy-momentum tensor should be proportional to the metric,

$$T_{\hat{\mu}\hat{\nu}}^{(\text{vac})} = -\rho_{\text{vac}}\eta_{\hat{\mu}\hat{\nu}}, \quad (4.96)$$

since  $\eta_{\hat{\mu}\hat{\nu}}$  is the only Lorentz invariant  $(0, 2)$  tensor. This generalizes straightforwardly from inertial coordinates to arbitrary coordinates as

$$T_{\mu\nu}^{(\text{vac})} = -\rho_{\text{vac}}g_{\mu\nu}. \quad (4.97)$$

Comparing to the perfect-fluid energy-momentum tensor  $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$ , we find that the vacuum looks like a perfect fluid with an isotropic pressure opposite in sign to the energy density,

$$p_{\text{vac}} = -\rho_{\text{vac}}. \quad (4.98)$$

The energy density should be constant throughout spacetime, since a gradient would not be Lorentz invariant.

If we decompose the energy-momentum tensor into a matter piece  $T_{\mu\nu}^{(M)}$  and a vacuum piece  $T_{\mu\nu}^{(\text{vac})} = -\rho_{\text{vac}}g_{\mu\nu}$ , Einstein's equation is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G \left( T_{\mu\nu}^{(M)} - \rho_{\text{vac}}g_{\mu\nu} \right). \quad (4.99)$$

Soon after inventing GR, Einstein tried to find a static cosmological model, since that was what astronomical observations of the time seemed to imply. The result was the Einstein static universe, which will be discussed in Chapter 8. In order for this static cosmology to solve the field equation with an ordinary matter source, it was necessary to add a new term called the **cosmological constant**,  $\Lambda$ , which enters as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (4.100)$$

From comparison with (4.99), we see that the cosmological constant is precisely equivalent to introducing a vacuum energy density

$$\rho_{\text{vac}} = \frac{\Lambda}{8\pi G}. \quad (4.101)$$

The terms “cosmological constant” and “vacuum energy” are essentially interchangeable.

Is a nonzero vacuum energy something we should expect? We arrived at the Hilbert Lagrangian  $\widehat{\mathcal{L}}_H = R$  by looking for the simplest possible scalar we could construct from the metric. Of course there is an even simpler one, namely a constant. Using (4.69), it is straightforward to check that

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} (R - 2\Lambda) + \widehat{\mathcal{L}}_M \right] \quad (4.102)$$

leads to the modified equation (4.100); alternatively, the vacuum Lagrangian is simply

$$\hat{\mathcal{L}}_{\text{vac}} = -\rho_{\text{vac}}. \quad (4.103)$$

So it is certainly easy to introduce vacuum energy; however, we have no insight into its expected value, since it enters as an arbitrary constant.

The vacuum energy ultimately is a constant of nature in its own right. (An exception occurs in certain theories where a spacetime symmetry such as supersymmetry or conformal invariance governs the value of the vacuum energy; here we are considering a more generic field theory.) Nevertheless, there are various distinct contributions to the vacuum energy, and it would be strange if the total value were much smaller than the individual contributions. One such contribution comes from zero-point fluctuations—the energies of quantum fields in their vacuum state.

Consider a simple harmonic oscillator, a particle moving in a one-dimensional potential  $V(x) = \frac{1}{2}\omega^2x^2$ . Classically, the vacuum for this system is the state in which the particle is motionless and at the minimum of the potential ( $x = 0$ ), for which the energy in this case vanishes. Quantum-mechanically, however, the uncertainty principle forbids us from isolating the particle both in position and momentum, and we find that the lowest energy state has an energy  $E_0 = \frac{1}{2}\hbar\omega$  (where we have temporarily reintroduced explicit factors of  $\hbar$  for clarity). Of course, in the absence of gravity, either system actually has a vacuum energy that is completely arbitrary; we could add any constant to the potential without changing the theory. But quantum fluctuations have changed the zero-point energy from our classical expectation.

A precisely analogous situation holds in field theory. If we take the Fourier transform of a free quantum field (one where we ignore interactions for simplicity), we find that it becomes an infinite number of harmonic oscillators in momentum space, as we discuss in Chapter 9. The frequency  $\omega$  of each oscillator is  $\omega = \sqrt{m^2 + k^2}$ , where  $m$  is the mass of the field and  $k$  is the magnitude of the wave vector of the mode. If we set the classical vacuum energy to zero, each of these modes contributes a quantum zero-point energy of  $\hbar\omega/2$ . Formally, adding all of these contributions together yields an infinite result. If, however, we discard the very high-momentum modes on the grounds that we trust our theory only up to a certain ultraviolet momentum cutoff  $k_{\text{max}}$ , we find that the resulting energy density is of the form

$$\rho_{\text{vac}} \sim \hbar k_{\text{max}}^4. \quad (4.104)$$

This answer could have been guessed by dimensional analysis; the numerical constants that have been neglected will depend on the precise theory under consideration. If we are confident that we can use ordinary quantum field theory all the way up to the reduced Planck scale  $\bar{m}_P = (8\pi G)^{-1/2} \sim 10^{18}$  GeV, we expect a contribution of order

$$\rho_{\text{vac}} \sim (10^{18} \text{ GeV})^4 \sim 10^{112} \text{ erg/cm}^3. \quad (4.105)$$

Field theory may fail earlier, although quantum gravity is the best reason we have to believe it will fail at any specific scale.

As we will discuss in Chapter 8, cosmological observations imply

$$|\rho_{\Lambda}^{(\text{obs})}| \leq (10^{-12} \text{ GeV})^4 \sim 10^{-8} \text{ erg/cm}^3, \quad (4.106)$$

much smaller than the naive expectation just derived. The ratio of (4.105) to (4.106) is the origin of the famous discrepancy of 120 orders of magnitude between the theoretical and observational values of the cosmological constant. We are free to imagine that the bare vacuum energy is adjusted so that the net cosmological constant is consistent with the limit (4.106), except for one problem: we know of no special symmetry that could enforce a vanishing vacuum energy while remaining consistent with the known laws of physics; this conundrum is the “cosmological constant problem.” We will discuss the cosmological effects of vacuum energy more in Chapter 8.<sup>2</sup>

#### 4.6 ■ ENERGY CONDITIONS

Sometimes it is useful to think about Einstein’s equation without specifying the theory of matter from which  $T_{\mu\nu}$  is derived. This leaves us with a great deal of arbitrariness; consider for example the question, What metrics obey Einstein’s equation? In the absence of some constraints on  $T_{\mu\nu}$ , the answer is any metric at all; simply take the metric of your choice, compute the Einstein tensor  $G_{\mu\nu}$  for this metric, and then demand that  $T_{\mu\nu}$  be equal to  $G_{\mu\nu}$ . It will automatically be conserved, by the Bianchi identity. Our real concern is with the existence of solutions to Einstein’s equation in the presence of “realistic” sources of energy and momentum, whatever that means. One strategy is to consider specific kinds of sources, such as scalar fields, dust, or electromagnetic fields. However, we occasionally wish to understand properties of Einstein’s equations that hold for a variety of different sources. In this circumstance it is convenient to impose *energy conditions* that limit the arbitrariness of  $T_{\mu\nu}$ .

Energy conditions are coordinate-invariant restrictions on the energy-momentum tensor. We must therefore construct scalars from  $T_{\mu\nu}$ , which is typically accomplished by contracting with arbitrary timelike vectors  $t^\mu$  or null vectors  $\ell^\mu$ . For example, the weak energy condition (WEC) states that  $T_{\mu\nu}t^\mu t^\nu \geq 0$  for all timelike vectors  $t^\mu$ . For purposes of physical intuition, it is useful to consider the special case where the source is a perfect fluid, so that the energy-momentum tensor takes the form

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \quad (4.107)$$

where  $U^\mu$  is the fluid four-velocity. Let’s use this form to translate the WEC into physical terms. Because the pressure is isotropic,  $T_{\mu\nu}t^\mu t^\nu$  will be nonnegative

<sup>2</sup>For more on the physics and cosmology of vacuum energy, see S.M. Carroll, *Liv. Rev. Rel.* **4**, 1 (2001), <http://arxiv.org/astro-ph/0004075>.

for all timelike vectors  $t^\mu$  if both  $T_{\mu\nu}U^\mu U^\nu \geq 0$  and  $T_{\mu\nu}\ell^\mu \ell^\nu \geq 0$  for some null vector  $\ell^\mu$  (convince yourself of this; it's just adding vectors). We therefore evaluate

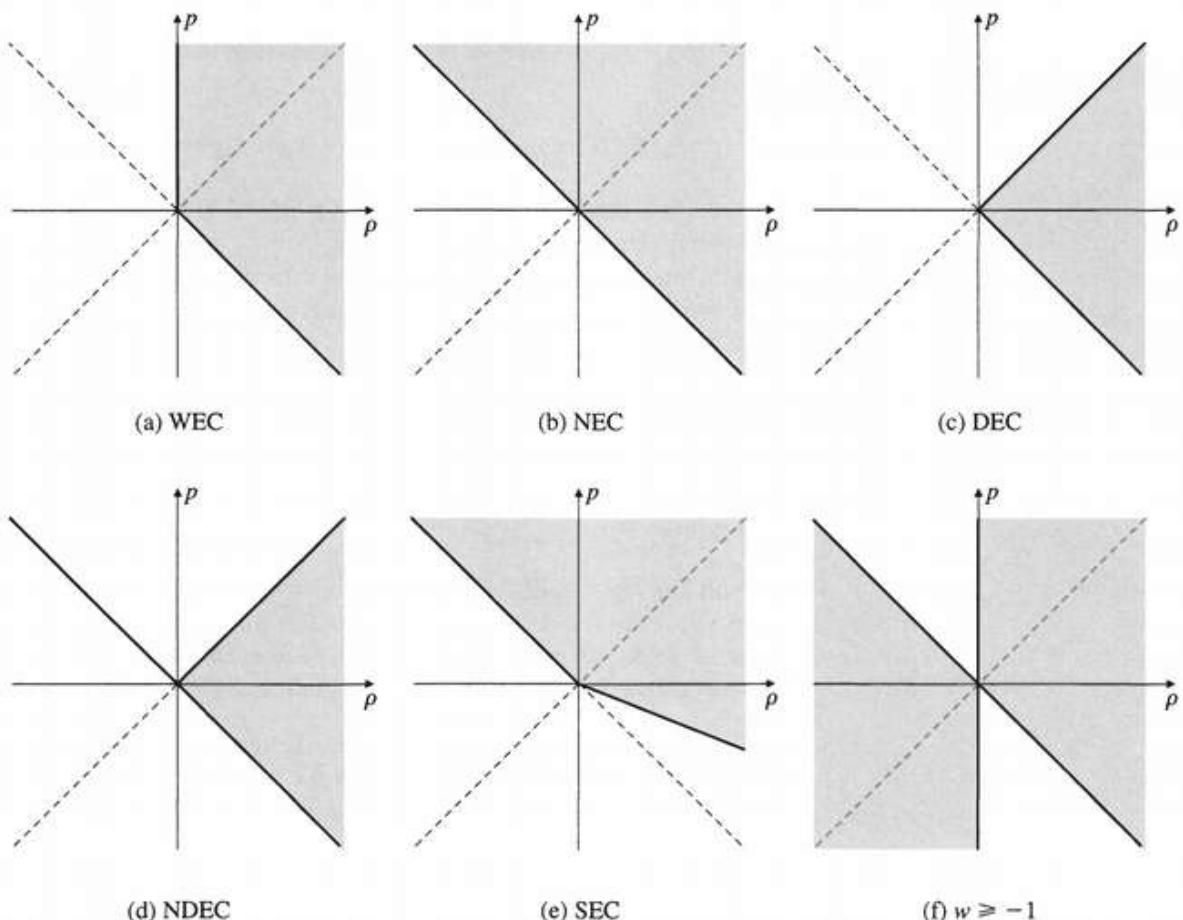
$$T_{\mu\nu}U^\mu U^\nu = \rho, \quad T_{\mu\nu}\ell^\mu \ell^\nu = (\rho + p)(U_\mu \ell^\mu)^2. \quad (4.108)$$

The WEC therefore implies  $\rho \geq 0$  and  $\rho + p \geq 0$ . These are simply the reasonable-sounding requirements that the energy density be nonnegative and the pressure not be too large compared to the energy density. Of course we need not restrict ourselves to perfect fluids, we merely use them to gain insight into the requirements the energy conditions impose.

There are a number of different energy conditions, appropriate to different circumstances. Some of the most popular are the following:

- The **Weak Energy Condition** or WEC, as just discussed, states that  $T_{\mu\nu}t^\mu t^\nu \geq 0$  for all timelike vectors  $t^\mu$ , or equivalently that  $\rho \geq 0$  and  $\rho + p \geq 0$ .
- The **Null Energy Condition** or NEC states that  $T_{\mu\nu}\ell^\mu \ell^\nu \geq 0$  for all null vectors  $\ell^\mu$ , or equivalently that  $\rho + p \geq 0$ . It is a special case of the WEC, with the timelike vector replaced by a null vector. The energy density may now be negative, so long as there is a compensating positive pressure.
- The **Dominant Energy Condition** or DEC includes the WEC ( $T_{\mu\nu}t^\mu t^\nu \geq 0$  for all timelike vectors  $t^\mu$ ), as well as the additional requirement that  $T^{\mu\nu}t_\mu$  is a nonspacelike vector (namely, that  $T_{\mu\nu}T^\nu{}_\lambda t^\mu t^\lambda \leq 0$ ). For a perfect fluid, these conditions together are equivalent to the simple requirement that  $\rho \geq |p|$ ; the energy density must be nonnegative, and greater than or equal the magnitude of the pressure.
- The **Null Dominant Energy Condition** or NDEC is the DEC condition for null vectors only: for any null vector  $\ell^\mu$ ,  $T_{\mu\nu}\ell^\mu \ell^\nu \geq 0$  and  $T^{\mu\nu}\ell_\mu$  is a nonspacelike vector. The allowed density and pressure are the same as for the DEC, except that negative densities are allowed so long as  $p = -\rho$ . In other words, the NDEC excludes all sources excluded by the DEC, except for a negative vacuum energy.
- The **Strong Energy Condition** or SEC states that  $T_{\mu\nu}t^\mu t^\nu \geq \frac{1}{2}T^\lambda{}_\lambda t^\sigma t_\sigma$  for all timelike vectors  $t^\mu$ , or equivalently that  $\rho + p \geq 0$  and  $\rho + 3p \geq 0$ . Note that the SEC does *not* imply the WEC. It implies the NEC, along with excluding excessively large negative pressures. From (4.86) we see that it is the SEC that implies gravitation is attractive.

These conditions are illustrated in Figure 4.3. In addition we have plotted the constraint  $w \geq -1$ , where  $w = p/\rho$  is called the **equation-of-state parameter**. This is a useful concept in cosmology, where sources often have equations of state  $p = w\rho$  with  $w$  being a constant (of course,  $w$  is defined whether it is constant



**FIGURE 4.3** Energy conditions as applied to perfect fluids, expressed as allowed regions of energy density  $\rho$  and pressure  $p$ . Illustrated are the Weak Energy Condition (WEC), Null Energy Condition (NEC), Dominant Energy Condition (DEC), Null Dominant Energy Condition (NDEC), and the Strong Energy Condition (SEC). For comparison, we also have illustrated the condition  $w \geq -1$ , where  $w = p/\rho$  is the equation-of-state parameter.

or not). If we restrict ourselves to sources with  $\rho \geq 0$ , then any of the energy conditions mentioned above will imply  $w \geq -1$ .

Most ordinary classical forms of matter, including scalar fields and electromagnetic fields, obey the DEC (see Exercises), and hence the less restrictive conditions (WEC, NEC, NDEC). The SEC is useful in the proof of some singularity theorems, but can be violated by certain forms of matter, such as a massive scalar field. It turns out that quantum fields can generically violate any of the energy conditions we have listed; there may, however, be inequalities involving integrals over regions of spacetime that are satisfied even by quantum fields. This is an area of current investigation.

The energy conditions are not, strictly speaking, related to energy conservation; the Bianchi identity guarantees that  $\nabla_\mu T^{\mu\nu} = 0$  regardless of whether we

impose any additional constraints on  $T^{\mu\nu}$ . Rather, they serve to prevent other properties that we think of as “unphysical,” such as energy propagating faster than the speed of light, or empty space spontaneously decaying into compensating regions of positive and negative energy. In particular, Hawking and Ellis (1973) prove a *conservation theorem*: Essentially, if the energy-momentum tensor obeys the DEC and vanishes in some spacelike region, then it will necessarily vanish everywhere in the future domain of dependence of that region (see Section 2.7 for the definition of the future domain of dependence). Thus, energy cannot spontaneously appear from nothing, nor can it sneak outside the light cone. The theorem does not include the converse statement (that sources violating the DEC are necessarily acausal), so it pays to be careful.

#### 4.7 ■ THE EQUIVALENCE PRINCIPLE REVISITED

In this section we will examine more carefully the underpinnings and consequences of the Principle of Equivalence, which we used in Section 4.1 to motivate the minimal-coupling procedure for generalizing physics to curved spacetime. We will see that the Principle of Equivalence is not a sacred physical law, nor is it even a mathematically rigorous statement; at a more fundamental level, it arises as a consequence of the nature of general relativity as an effective field theory valid at macroscopic distances, and our job is to determine which kinds of couplings between matter and the metric we would expect in such a theory.

In practice, it is common to invoke the Equivalence Principle to justify any of the following four ideas:

1. Laws of physics should be expressed (or at least be expressible) in generally covariant form.
2. There exists a metric on spacetime, the curvature of which we interpret as gravity.
3. There do not exist any other fields that resemble gravity.
4. The interactions of matter fields to curvature are minimal: they do not involve direct couplings to the Riemann tensor or its contractions.

These very different statements each have a very different status: the first is vacuous, the second is both profound and almost certainly true, the third is interesting and testable, and the fourth is just a useful approximation. Let’s examine each of them in turn.

The first statement is sometimes called the Principle of Covariance. It is more or less content-free. “Generally covariant” simply means that all of the terms in an equation transform in the same way under a change of coordinates, so that the form of the equation is coordinate-invariant. Due to the universal nature of the tensor transformation law, the most straightforward way of achieving this aim is to make the equation manifestly tensorial. Certainly there is nothing wrong if a law is expressed in a form that is not generally covariant, as long as we

know that it is possible to rewrite it in a coordinate-independent way. On the other hand, it is *always* possible to write laws in a coordinate-independent way, if the laws are well-defined to begin with. A physical system acting in a certain way doesn't know which coordinate system you are using to describe it; consequently, anything deserving of the name "law of physics" (as opposed to some particular statement of that law) must be independent of coordinates. An insistence on explicit coordinate-independence says nothing about the adaptation of laws to curved spacetime; as we have seen, manifestly tensorial equations take on the same form regardless of the geometry.

Consider Maxwell's equations in flat spacetime, as we wrote them in Chapter 1:

$$\partial_\mu F^{\nu\mu} = J^\nu. \quad (4.109)$$

The right-hand side is a well-defined tensor, while the left-hand side is not, due to the appearance of the partial derivative. That's okay, since we know that this equation is valid only in inertial coordinates in Minkowski space. A coordinate-invariant way of expressing the same law is

$$\nabla_\mu F^{\nu\mu} = J^\nu. \quad (4.110)$$

No physical principle needs to be invoked to conclude that this is the correct formulation in Minkowski space; it is the *unique* tensorial equation, which is equivalent to (4.109) in inertial coordinates. It is not the unique generalization to curved spacetime, since we could imagine new terms involving products of  $F_{\mu\nu}$  and  $R^\rho{}_{\sigma\mu\nu}$ ; the status of such additional terms is directly addressed by the minimal-coupling assumption, point four in the above list. By itself, however, making things "tensorial" or "generally covariant" is a simple matter of logical necessity, not a physical principle that one could imagine disproving by experiment. (Another spin on the same idea is "diffeomorphism invariance," discussed in the Appendix B.)

The second purported consequence of the Equivalence Principle from our list above is much deeper, and by no means obvious. Although he was inspired by the EP, this geometric insight was Einstein's great breakthrough. At the beginning of Chapter 2 we discussed why such an insight was warranted: the EP implies that gravity is universal, which implies in turn that gravitational fields become impossible to measure in small regions of spacetime, a feature which in turn is most directly implemented by identifying gravitation with the effects of spacetime geometry. These steps are well-motivated suggestions, not rigorously derived consequences; once we have the idea that there is a metric whose curvature gives rise to gravity, we can check its usefulness by comparing with experiment. As we've mentioned, it passes with flying colors. An accumulation of evidence (such as the gravitational redshift discussed in Chapter 2) is consistent with the idea that idealized rods and clocks behave as they should if the geometry of spacetime were curved. Still, one should not imagine proving that there really is a metric with the desired properties; we make the hypothesis, test it against ever-more precise ex-

periments, and deduce its range of usefulness. Indeed, the demands of eventually reconciling general relativity with quantum mechanics suggest to many that the metric will ultimately be revealed as a concept derived from a more fundamental collection of degrees of freedom. For our present purposes this ultimate resolution doesn't matter; the idea of a curved metric has proven its usefulness beyond a reasonable doubt, and we work to extend our understanding of its properties until they run up against insurmountable obstacles (either theoretical or empirical).

Given our conviction that the effects of gravitation are best ascribed to the curvature of a metric on spacetime, what would we conclude if experiments were to detect an apparent violation of the Equivalence Principle? For example, we might imagine an experiment that revealed that the acceleration of test bodies in the direction of the Earth or Sun actually did depend, ever so slightly, on the composition of the test body. (The best current limits on such anomalous accelerations constrain them to be less than  $10^{-12}$  times that due to gravity.)<sup>3</sup> In such a circumstance, nobody would really be tempted to declare that general relativity had been completely undermined and it was necessary to start over. Rather, we would return to the definition of "test body," which includes the proviso that the body be uncharged. An electron, for example, would not make a good test body, as it would be buffeted about by ambient electromagnetic fields as well as by gravity. Similarly, by far the most straightforward explanation of any hypothetical anomalous acceleration on purportedly neutral test bodies would be to imagine that we had discovered the existence of a new long-range field, under which our test bodies were actually charged. To have remain undetected thus far, such a field must be either very weakly coupled, or must couple almost universally, so as to mimic the effects of gravity. We could imagine, for example, scalar fields that couple to the trace of the energy-momentum tensor, or vector fields that couple to baryon number. The mass of ordinary test bodies is almost proportional to their baryon number, which counts the number of protons and neutrons in the body. It is therefore sometimes convenient to think of "tests of the Equivalence Principle" as tests of the third of our statements above—that there do not exist any other fields that resemble gravity (where a field resembles gravity if it is long-range and couples almost universally to mass). Again, detecting a violation of this hypothesis would be most directly interpreted as discovery of a new "fifth force" rather than as a repudiation of Einstein's ideas. As to whether we should expect to discover such a new field if we improve upon current experiments, it is hard to say; on the one hand, it is easy to concoct models with new long-range forces, but on the other hand, they would typically be strong enough to already have been detected. At this stage it is still worthwhile to keep an open mind.

Beyond the very existence of the metric, the heart of the Equivalence Principle lies in the fourth of our formulations, that the interactions of matter fields to curvature are minimal: they do not involve direct couplings to the Riemann tensor or its contractions. For example, we could consider the following possible alternative

<sup>3</sup>Y. Su et al., *Phys. Rev. D* **50**, 3614 (1994).

to the conventional geodesic equation:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = \alpha (\nabla_\sigma R) \frac{dx^\mu}{d\lambda} \frac{dx^\sigma}{d\lambda}, \quad (4.111)$$

where  $R$  is the Ricci scalar and  $\alpha$  is a coupling constant. This equation also reduces to straight-line motion in flat spacetime, but would allow for direct detection of spacetime curvature in small regions by measurement of the coupling to  $\nabla_\sigma R$ . Why, then, does nature choose the simple geodesic equation? As a first step toward an answer, consider the dimensions of the coupling  $\alpha$ . Since  $c = 1$  and space and time have the same units, we can use length as our basic dimension. The metric, the inverse metric, and  $dx^\mu/d\lambda$  are then dimensionless. The partial derivative operator has units of inverse length, as does the covariant derivative. The Christoffel symbols involve first derivatives of the metric, and thus have dimensions of inverse length; similarly, the Riemann tensor, Ricci tensor, and Ricci scalar have dimensions of inverse length squared:

$$\left[ \frac{dx^\mu}{d\lambda} \right] = [g_{\mu\nu}] = [g^{\mu\nu}] = L^0, \quad [\nabla_\mu] = [\Gamma_{\rho\sigma}^\mu] = L^{-1}, \quad [R] = L^{-2}. \quad (4.112)$$

To be consistent, the coupling  $\alpha$  must have dimensions of length squared:

$$[\alpha] = L^2. \quad (4.113)$$

The square root of  $\alpha$  therefore defines a length scale; what should the length scale be? We don't know for sure, but there is every reason to believe it should be extremely small. There are two arguments for this. One is that, since the coupling represented by  $\alpha$  is of gravitational origin, the only reasonable expectation for the relevant length scale is

$$\alpha \sim l_p^2, \quad (4.114)$$

where  $l_p$  is the Planck length. Another reason is simply a more sophisticated version of this "what else could it be?" rationale. Although general relativity is a classical theory, at a deeper level we expect that it is merely an effective field theory describing an underlying quantum-mechanical structure. Even without knowing what this structure may be, a generic expectation (derived from our experience with quantum field theories we do understand) is that the effective classical limit should contain all possible interactions, but with dimensionful length parameters representing scales at which new degrees of freedom become important (recall our discussion of effective field theory at the end of Chapter 1). Thus, the Fermi theory of the weak interactions contains a length scale, which we now know to correspond to the scale of electroweak symmetry breaking where  $W$  and  $Z$  bosons become relevant. Since we do not expect new gravitational physics to arise before the Planck scale, the higher-order interactions associated with gravity should be suppressed by appropriate powers of the Planck length.

How much suppression does this represent? One measure would be to compare  $l_p$  (and thus the likely value of the parameter  $\alpha$ ) to a typical gravitational length scale near the vicinity of the Earth. The strength of gravity on Earth is characterized by the acceleration due to gravity,  $a_g = 980 \text{ cm/sec}^2$ . To construct a quantity with dimensions of length, we define

$$l_{\oplus} = c^2/a_g \sim 10^{18} \text{ cm}, \quad (4.115)$$

where the symbol  $\oplus$  in this context stands for the Earth (not a direct sum). So the relative strength of higher-order gravitational effects is measured by

$$\frac{l_p}{l_{\oplus}} \sim 10^{-51}. \quad (4.116)$$

In fact, since we expect  $\alpha \sim l_p^2$ , the suppression will be of order  $10^{-102}$ . Consequently, there seems to be little need to worry about the possible role of such couplings. But dramatic departures should be kept in mind; recent ideas about large extra dimensions have opened up the possibility of observing direct gravitational interactions at particle accelerators. Ultimately, there is no way to resolve these problems by pure thought alone; only experiment can decide among the alternatives.

#### 4.8 ■ ALTERNATIVE THEORIES

General relativity has passed a wide variety of experimental tests. Nevertheless, it is always possible that the next experiment we do will reveal a deviation from Einstein's original formulation. Let us therefore briefly consider ways in which general relativity could be modified. There are an uncountable number of such ways, but we will consider four different possibilities:

- gravitational scalar fields
- extra spatial dimensions
- higher-order terms in the action
- nonChristoffel connections

A popular set of alternative models are known as **scalar-tensor theories** of gravity, since they involve both the metric tensor,  $g_{\mu\nu}$  and a scalar field,  $\lambda$ . In particular, the scalar field couples directly to the curvature scalar, not simply to the metric (as the Equivalence Principle would seem to imply). The action can be written as a sum of a gravitational piece, a pure-scalar piece, and a matter piece:

$$S = S_{fR} + S_{\lambda} + S_M. \quad (4.117)$$

where

$$S_{fR} = \int d^4x \sqrt{-g} f(\lambda) R, \quad (4.118)$$

$$S_\lambda = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} h(\lambda) g^{\mu\nu} (\partial_\mu \lambda)(\partial_\nu \lambda) - U(\lambda) \right], \quad (4.119)$$

and

$$S_M = \int d^4x \sqrt{-g} \hat{\mathcal{L}}_M(g_{\mu\nu}, \psi_i). \quad (4.120)$$

Here,  $f(\lambda)$ ,  $h(\lambda)$  and  $U(\lambda)$  are functions that define the theory, and the matter Lagrangian  $\hat{\mathcal{L}}_M$  depends on the metric and a set of matter fields  $\psi_i$ , but not on  $\lambda$ . By change of variables we can always set  $h(\lambda) = 1$ , but we leave it here to facilitate comparison with models found in the literature.

The equations of motion for this theory include the gravitational equation (from varying with respect to the metric), and the scalar equation (from varying with respect to  $\lambda$ ), as well as the appropriate matter equations. Let's start with the gravitational equation, which we can derive by following the same steps as for the ordinary Hilbert action (4.55). We consider perturbations of the metric,

$$g^{\mu\nu} \rightarrow g^{\mu\nu} + \delta g^{\mu\nu}. \quad (4.121)$$

Following the procedure from Section 4.3, the variation of the gravitational part of the action is

$$\begin{aligned} \delta S_{fR} = \int d^4x \sqrt{-g} f(\lambda) & \left[ \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} + \nabla_\sigma \nabla^\sigma (g_{\mu\nu} \delta g^{\mu\nu}) \right. \\ & \left. - \nabla_\mu \nabla_\nu (\delta g^{\mu\nu}) \right]. \end{aligned} \quad (4.122)$$

For the Hilbert action,  $f$  is a constant, so the last two terms are total derivatives, which can be converted to surface terms through integration by parts and therefore ignored. Now integration by parts (twice) picks up derivatives of  $f$ , and we obtain

$$\delta S_{fR} = \int d^4x \sqrt{-g} [f(\lambda) G_{\mu\nu} + g_{\mu\nu} \square f - \nabla_\mu \nabla_\nu f] \delta g^{\mu\nu}, \quad (4.123)$$

where  $G_{\mu\nu}$  is the Einstein tensor. We have discarded surface terms as usual, although there are subtleties concerning boundary contributions in this case; see Wald (1984) for a discussion. The gravitational equation of motion, including contributions from  $S_\lambda$  and  $S_M$ , is thus

$$G_{\mu\nu} = f^{-1}(\lambda) \left( \frac{1}{2} T_{\mu\nu}^{(M)} + \frac{1}{2} T_{\mu\nu}^{(\lambda)} + \nabla_\mu \nabla_\nu f - g_{\mu\nu} \square f \right), \quad (4.124)$$

where the energy-momentum tensors are  $T_{\mu\nu}^{(i)} = -2(-g)^{-1/2}\delta S_i/\delta g^{\mu\nu}$ ; in particular,

$$T_{\mu\nu}^{(\lambda)} = h(\lambda)\nabla_\mu\lambda\nabla_\nu\lambda - g_{\mu\nu}\left[\frac{1}{2}h(\lambda)g^{\rho\sigma}\nabla_\rho\lambda\nabla_\sigma\lambda + U(\lambda)\right]. \quad (4.125)$$

From looking at the coefficient of  $T_{\mu\nu}^{(\lambda)}$  in (4.124), we see that when the scalar field is *constant* (or practically so), we may identify  $f(\lambda) = 1/(16\pi G)$ , as makes sense from the original action (4.118). Meanwhile, if  $\lambda$  varies slightly from point to point in spacetime, it would be interpreted as a spacetime-dependent Newton's constant. The dynamics that control this variation are determined by the equation of motion for  $\lambda$ , which is straightforward to derive as

$$h\square\lambda + \frac{1}{2}h'g^{\mu\nu}\nabla_\mu\lambda\nabla_\nu\lambda - U' + f'R = 0, \quad (4.126)$$

where primes denote differentiation with respect to  $\lambda$ . Notice that if we set  $h(\lambda) = 1$  to get a conventional kinetic term for the scalar,  $\lambda$  obeys a conventional scalar-field equation of motion, with an additional coupling to the curvature scalar. In the real world, we don't want  $f(\lambda)$  to vary too much, as it would have observable consequences in the classic experimental tests of GR in the solar system, and also in cosmological tests such as primordial nucleosynthesis. This can be ensured either by choosing  $U(\lambda)$  so that there is a minimum to the potential and  $\lambda$  cannot deviate too far without a large input of energy—in other words,  $\lambda$  has a large mass—or by choosing  $f(\lambda)$  and  $h(\lambda)$  so that large changes in  $\lambda$  give rise to relatively small changes in the effective value of Newton's constant.

One of the earliest scalar-tensor models is known as Brans–Dicke theory, and corresponds in our notation to the choices

$$f(\lambda) = \frac{\lambda}{16\pi}, \quad h(\lambda) = \frac{\omega}{8\pi\lambda}, \quad U(\lambda) = 0. \quad (4.127)$$

where  $\omega$  is a coupling constant. The scalar-tensor action takes the form

$$S_{BD} = \int d^4x \sqrt{-g} \left[ \frac{\lambda}{16\pi} R - \frac{\omega}{16\pi} g^{\mu\nu} \frac{(\partial_\mu\lambda)(\partial_\nu\lambda)}{\lambda} \right]. \quad (4.128)$$

In the Brans–Dicke theory, the scalar field is massless, but in the  $\omega \rightarrow \infty$  limit the field becomes nondynamical and ordinary GR is recovered. Current bounds from Solar System tests imply  $\omega > 500$ , so if there is such a scalar field it must couple only weakly to the Ricci scalar.

A popular approach to dealing with scalar-tensor theories is to perform a conformal transformation to bring the theory in to a form that looks like conventional GR. We define a conformal metric

$$\tilde{g}_{\mu\nu} = 16\pi \tilde{G} f(\lambda) g_{\mu\nu}, \quad (4.129)$$

where  $\tilde{G}$  will become Newton's constant in the conformal frame. Using formulae for conformal transformations from the Appendix G, the action  $S_{fR}$  from (4.118)

becomes

$$\begin{aligned} S_{fR} &= \int d^4x \sqrt{-g} f(\lambda) R \\ &= \int d^4x \sqrt{-\tilde{g}} (16\pi \tilde{G})^{-1} \left[ \tilde{R} - \frac{3}{2} \tilde{g}^{\rho\sigma} f^{-2} \left( \frac{df}{d\lambda} \right)^2 (\tilde{\nabla}_\rho \lambda)(\tilde{\nabla}_\sigma \lambda) \right], \end{aligned} \quad (4.130)$$

where as usual we have integrated by parts and discarded surface terms. In the conformal frame, therefore, the curvature scalar appears by itself, not multiplied by any function of  $\lambda$ . This frame is sometimes called the **Einstein frame**, since Einstein's equations for the conformal metric  $\tilde{g}_{\mu\nu}$  take on their conventional form. The original frame with metric  $g_{\mu\nu}$  is called the **Jordan frame**, or sometimes the **string frame**. (String theory typically predicts a scalar-tensor theory rather than ordinary GR, and the string worldsheet responds to the metric  $g_{\mu\nu}$ .)

Before going on with our analysis of the conformally-transformed theory, consider what happens if we choose

$$f(\lambda) = e^{\lambda/\sqrt{3}}, \quad h(\lambda) = 0, \quad U(\lambda) = 0, \quad (4.131)$$

a specific choice for  $f(\lambda)$ , but turning off the pure scalar terms in  $S_\lambda$ . Then we notice that the Einstein frame action (4.130) actually includes a conventional kinetic term for the scalar, even though it wasn't present in the Jordan frame action (4.118). Even without an explicit kinetic term for  $\lambda$ , the degrees of freedom of this theory include a propagating scalar as well as the metric. This should hopefully become more clear after we examine the degrees of freedom of the gravitational field in Chapter 7. There we will find that the metric  $g_{\mu\nu}$  actually includes scalar (spin-0) and vector (spin-1) degrees of freedom as well as the expected tensor (spin-2) degrees of freedom; however, with the standard Hilbert action, these degrees of freedom are constrained rather than freely propagating. What we have just found is that multiplying  $R$  by a scalar in the action serves to bring the scalar degree of freedom to life, which is revealed explicitly in the Einstein frame.

If we do choose to include the pure-scalar action  $S_\lambda$ , we obtain

$$S_{fR} + S_\lambda = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{16\pi \tilde{G}} - \frac{1}{2} K(\lambda) \tilde{g}^{\rho\sigma} (\tilde{\nabla}_\rho \lambda)(\tilde{\nabla}_\sigma \lambda) - \frac{U(\lambda)}{(16\pi \tilde{G})^2 f^2(\lambda)} \right]. \quad (4.132)$$

where

$$K(\lambda) = \frac{1}{16\pi \tilde{G} f^2} [fh + 3(f')^2]. \quad (4.133)$$

We can make our action look utterly conventional by defining a new scalar field  $\phi$  via

$$\phi = \int K^{1/2} d\lambda, \quad (4.134)$$

in terms of which the action becomes

$$S_{fR} + S_\lambda = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{16\pi \tilde{G}} - \frac{1}{2} \tilde{g}^{\rho\sigma} (\tilde{\nabla}_\rho \phi)(\tilde{\nabla}_\sigma \phi) - V(\phi) \right], \quad (4.135)$$

where

$$V(\phi) = \frac{U(\lambda(\phi))}{(16\pi \tilde{G})^2 f^2(\lambda(\phi))}, \quad (4.136)$$

Amazingly, in the Einstein frame we have a completely ordinary theory of a scalar field in curved spacetime. So long as  $f(\lambda)$  is well-behaved, the variables  $(\tilde{g}_{\mu\nu}, \phi)$  can be used instead of  $(g_{\mu\nu}, \lambda)$ , in the sense that varying with respect to the new variables is equivalent to starting with the original equations of motion (4.124) and (4.126) and then doing the transformations (4.129) and (4.134).

Finally, we add in the matter action (4.120). Varying with respect to  $\tilde{g}_{\mu\nu}$  will yield an energy-momentum tensor in the Einstein frame. In the original variables  $(g_{\mu\nu}, \lambda)$ , we knew that  $S_M$  was independent of  $\lambda$ , but now it will depend on both of the new variables  $(\tilde{g}_{\mu\nu}, \phi)$ ; we can use the chain rule to characterize this dependence. Let us also assume that  $S_M$  depends on  $g_{\mu\nu}$  only algebraically, not through derivatives. This will hold for ordinary scalar-field or gauge-field matter; things become more complicated for fermions, which we won't discuss here. We obtain

$$\begin{aligned} \tilde{T}_{\mu\nu} &\equiv -2 \frac{1}{\sqrt{-\tilde{g}}} \frac{\delta S_M}{\delta \tilde{g}^{\mu\nu}} \\ &= -2 \frac{1}{\sqrt{-\tilde{g}}} \frac{\partial g^{\alpha\beta}}{\partial \tilde{g}^{\mu\nu}} \frac{\delta S_M}{\delta g^{\alpha\beta}} \\ &= -2(16\pi \tilde{G} f)^{-1} \frac{1}{\sqrt{-g}} \delta_\mu^\alpha \delta_\nu^\beta \frac{\delta S_M}{\delta g^{\alpha\beta}} \\ &= (16\pi \tilde{G} f)^{-1} T_{\mu\nu}. \end{aligned} \quad (4.137)$$

A similar trick works for the coupling of matter to  $\phi$ , which comes from varying  $S_M$  with respect to  $\phi$ , using  $g^{\alpha\beta} = 16\pi \tilde{G} f \tilde{g}^{\alpha\beta}$ :

$$\begin{aligned} \frac{\delta S_M}{\delta \phi} &= \frac{\partial g^{\alpha\beta}}{\partial \phi} \frac{\delta S_M}{\delta g^{\alpha\beta}} \\ &= \left( 16\pi \tilde{G} \frac{df}{d\phi} \tilde{g}^{\alpha\beta} \right) \left( -\frac{1}{2} \sqrt{-g} T_{\alpha\beta}^M \right) \\ &= -\frac{1}{2f} \frac{df}{d\phi} \sqrt{-\tilde{g}} \tilde{T}^M, \end{aligned} \quad (4.138)$$

where

$$\tilde{T}^{(M)} = \tilde{g}^{\alpha\beta} \tilde{T}_{\alpha\beta}^{(M)} = \frac{1}{(16\pi \tilde{G} f)^2} g^{\alpha\beta} T_{\alpha\beta}^{(M)} \quad (4.139)$$

is the trace of the energy-momentum tensor in the conformal frame.

Varying (4.135) with respect to  $\tilde{g}_{\mu\nu}$  and  $\phi$  returns equations of motion equivalent to Einstein's equations and an equation for  $\phi$ . The gravitational equation is

$$\tilde{G}_{\mu\nu} = 8\pi \tilde{G} \left( \tilde{T}_{\mu\nu}^{(M)} + \tilde{T}_{\mu\nu}^{(\phi)} \right), \quad (4.140)$$

where

$$\tilde{T}_{\mu\nu}^{(\phi)} = \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - \tilde{g}_{\mu\nu} \left[ \frac{1}{2} \tilde{g}^{\rho\sigma} \tilde{\nabla}_\rho \phi \tilde{\nabla}_\sigma \phi + V(\phi) \right], \quad (4.141)$$

and the scalar field equation is

$$\tilde{\square} \phi - \frac{dV}{d\phi} = \frac{1}{2f} \frac{df}{d\phi} \tilde{T}^{(M)}. \quad (4.142)$$

Given that (4.140) looks just like Einstein's equation with both matter and scalar-field sources, why should we even bother to call this scalar-tensor theory an alternative to GR? Isn't it the same theory, just in different variables? In fact it is not the same, because of the dependence of  $S_M$  on  $\phi$  in the Einstein frame. In particular, physical test particles will move along geodesics of  $g_{\mu\nu}$ , which will not generally coincide with those of  $\tilde{g}_{\mu\nu}$ . The original metric is the one that test particles "see." So either we work in the original variables  $(g_{\mu\nu}, \lambda)$ , where the gravitational field equation is altered, or we use the new variables  $(\tilde{g}_{\mu\nu}, \phi)$ , in which the equations of motion for matter are altered; either way, there will be unambiguously measurable departures (in principle) from ordinary GR.

Another way to modify general relativity is to allow for the existence of extra spatial dimensions; in fact the physical consequences of extra dimensions turn out to be closely related to those of scalar-tensor theories. By extra dimensions we don't simply mean considering GR in higher-dimensional spaces, but rather considering models in which the spacetime appears four-dimensional on large scales even though there are really  $4+d$  total dimensions. The simplest way for this to happen is if the extra  $d$  dimensions are "compactified" on some manifold; it is this possibility we consider here.<sup>4</sup> Models of this kind are known as Kaluza-Klein theories.

Let  $G_{ab}$  be the metric for a  $(4+d)$ -dimensional spacetime with coordinates  $X^a$ , where indices  $a, b$  run from 0 to  $d+3$ .

<sup>4</sup>We follow the analysis of S.M. Carroll, J. Geddes, M. Hoffman, and R.M. Wald, *Phys. Rev. D* **66**, 024036 (2002); <http://arxiv.org/hep-th/0110149>. The original papers on extra dimensions are those by Kaluza and Klein: T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **K1**, 966 (1921); O. Klein, *Z. Phys.* **37**, 895 (1926) [*Surveys High Energ. Phys.* **5**, 241 (1926)]; O. Klein, *Nature* **118**, 516 (1926).

$$ds^2 = G_{ab}dx^a dx^b = g_{\mu\nu}(x)dx^\mu dx^\nu + b^2(x)\gamma_{ij}(y)dy^i dy^j, \quad (4.143)$$

where the  $x^\mu$  are coordinates in the four-dimensional spacetime and the  $y^i$  are coordinates on the extra-dimensional manifold, taken to be a maximally symmetric space with metric  $\gamma_{ij}$ . Of course the geometry of the extra dimensions is actually something dynamical that should be determined by solving the full equations of motion, but we are going to take (4.143) as a simplifying ansatz. (In a more complete treatment, we would expand the dynamical modes of the compactified geometry as a Fourier series, and show that the modes we are presently neglecting have larger masses than the overall-size mode we are choosing to examine.) The action is the  $(4+d)$ -dimensional Hilbert action plus a matter term:

$$S = \int d^{4+d}X \sqrt{-G} \left( \frac{1}{16\pi G_{4+d}} R[G_{ab}] + \hat{\mathcal{L}}_M \right), \quad (4.144)$$

where  $\sqrt{-G}$  is the square root of minus the determinant of  $G_{ab}$ ,  $R[G_{ab}]$  is the Ricci scalar of  $G_{ab}$ , and  $\hat{\mathcal{L}}_M$  is the matter Lagrange density with the metric determinant factored out.

The first step is to dimensionally reduce the action (4.144). By this we mean to actually perform the integral over the extra dimensions, which is possible because we have assumed that the extra-dimensional scale factor  $b$  is independent of  $y^i$ . Therefore we can express everything in terms of  $g_{\mu\nu}$ ,  $\gamma_{IJ}$ , and  $b(x)$ , integrate over the extra dimensions, and arrive at an effective four-dimensional theory. From the metric (4.143) we have

$$\sqrt{-G} = b^d \sqrt{-g} \sqrt{\gamma}, \quad (4.145)$$

and we can evaluate the curvature scalar for this metric to obtain

$$\begin{aligned} R[G_{ab}] &= R[g_{\mu\nu}] + b^{-2}R[\gamma_{ij}] - 2db^{-1}g^{\mu\sigma}\nabla_\mu\nabla_\sigma b \\ &\quad - d(d-1)b^{-2}g^{\mu\sigma}(\nabla_\mu b)(\nabla_\sigma b), \end{aligned} \quad (4.146)$$

where  $\nabla_\mu$  is the covariant derivative associated with the four-dimensional metric  $g_{\mu\nu}$ . We denote by  $\mathcal{V}$  the volume of the extra dimensions when  $b = 1$ ; it is given by

$$\mathcal{V} = \int d^d y \sqrt{\gamma}. \quad (4.147)$$

The four-dimensional Newton's constant  $G_4$  is determined by evaluating the coefficient of the curvature scalar in the action; we find that  $G_4$  is related to its higher-dimensional analogue by

$$\frac{1}{16\pi G_4} = \frac{\mathcal{V}}{16\pi G_{4+d}}, \quad (4.148)$$

We are thus left with

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi G_4} \left[ b^d R[g_{\mu\nu}] + d(d-1)b^{d-2} g^{\mu\nu} (\nabla_\mu b)(\nabla_\nu b) + d(d-1)\kappa b^{d-2} \right] + \mathcal{V} b^d \hat{\mathcal{L}}_M \right\}, \quad (4.149)$$

where we have integrated by parts for convenience, and introduced the curvature parameter  $\kappa$  of  $\gamma_{ij}$ , given by

$$\kappa = \frac{R[\gamma_{ij}]}{d(d-1)}. \quad (4.150)$$

Comparing to (4.117)–(4.120), we see that the dimensionally-reduced action is precisely that of a scalar-tensor theory; the size of the extra dimensions plays the role of the scalar field. We can therefore make it look more conventional by performing a change of variables and a conformal transformation,

$$\begin{aligned} \beta(x) &= \ln b, \\ \tilde{g}_{\mu\nu} &= e^{d\beta} g_{\mu\nu}, \end{aligned} \quad (4.151)$$

which turns the reduced action into that of a scalar field coupled to gravity in the Einstein frame. Following the same procedure as outlined in our discussion of scalar-tensor theories yields

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{16\pi G_4} \left[ R[\tilde{g}_{\mu\nu}] - \frac{1}{2} d(d+2) \tilde{g}^{\mu\nu} (\tilde{\nabla}_\mu \beta)(\tilde{\nabla}_\nu \beta) + d(d-1)\kappa e^{(d+2)\beta} \right] + \mathcal{V} e^{-d\beta} \hat{\mathcal{L}}_M \right\}, \quad (4.152)$$

where we have dropped terms that are total derivatives.

To turn  $\beta$  into a canonically normalized scalar field, we make one final change of variables, to

$$\phi = \sqrt{\frac{d(d+2)}{2}} \bar{m}_P \beta, \quad (4.153)$$

where the reduced Planck mass is  $\bar{m}_P = (8\pi G_4)^{-1/2}$ . We are then left with

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{16\pi G_4} R[\tilde{g}_{\mu\nu}] - \frac{1}{2} \tilde{g}^{\mu\nu} (\tilde{\nabla}_\mu \phi)(\tilde{\nabla}_\nu \phi) + \frac{1}{2} \kappa d(d-1) \bar{m}_P^2 e^{-\sqrt{2(d+2)/d}\phi/\bar{m}_P} + \mathcal{V} e^{-\sqrt{2d/(d+2)}\phi/\bar{m}_P} \hat{\mathcal{L}}_M \right\}. \quad (4.154)$$

The scalar  $\phi$  is known as the **dilaton** or **radion**, and characterizes the size of the extra-dimensional manifold.

The last two terms in (4.154) represent (minus) the potential  $V(\phi)$ . If we ignore the matter term  $\hat{\mathcal{L}}_M$ , the behavior of the dilaton will depend only on the sign of  $\kappa$ . If the extra-dimensional manifold is flat ( $\kappa = 0$ ), the potential vanishes and we simply have a massless scalar field; this possibility runs afoul of the experimental constraints on scalar-tensor theories mentioned above. If there is curvature ( $\kappa \neq 0$ ), the potential has no minimum; for  $\kappa > 0$  the field will roll to  $-\infty$ , while for  $\kappa < 0$  the field will roll to  $+\infty$ . But  $\phi \propto \ln b$ , so this means the scale factor  $b(x)$  of the extra dimensions either shrinks to zero or becomes arbitrarily large, in either case ruining the hope for stable extra dimensions. Stability can be achieved, however, by choosing an appropriate matter Lagrangian, and an appropriate field configuration in the extra dimensions.

Let us now move on to a different kind of alternative theory, those that feature Lagrangians of more than second order in derivatives of the metric. We could imagine an action of the form

$$S = \int d^n x \sqrt{-g} (R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 g^{\mu\nu} \nabla_\mu R \nabla_\nu R + \dots), \quad (4.155)$$

where the  $\alpha$ 's are coupling constants and the dots represent every other scalar we can make from the curvature tensor, its contractions, and its derivatives. Traditionally, such terms have been neglected on the reasonable grounds that they merely complicate a theory that is already both aesthetically pleasing and empirically successful. There is also, classically speaking, a more substantive objection. In conventional form, Einstein's equation leads to a well-posed initial value problem for the metric, in which coordinates and momenta specified at an initial time can be used to predict future evolution. With higher-derivative terms, we would require not only those data, but also some number of derivatives of the momenta; the character of the theory is dramatically altered.

However, there are also good reasons to consider such additional terms. As mentioned in our brief discussion of quantum gravity, one of the technical obstacles to consistent quantization of general relativity is that the theory is non-renormalizable: Inclusion of higher-order quantum effects leads to infinite answers. With the appropriate combination of higher-order Lagrangian terms, it turns out that you can actually render the theory renormalizable, which gives some hope of constructing a consistent quantum theory.<sup>5</sup> Unfortunately, it turns out that renormalizability comes at too high a price; these models generally feature negative-energy field excitations (ghosts). Consequently, the purported vacuum state (empty space) would be unstable to decay into positive- and negative-energy modes, which is inconsistent with both empirical experience and theoretical prejudice.

Nevertheless, the prevailing current view is that GR is an effective theory valid at energies below the Planck scale, and we should actually include all of the pos-

<sup>5</sup>See, for example, K.S. Stelle, *Phys. Rev. D* **16**, 953 (1977).

sible higher-order terms; but they will be suppressed by appropriate powers of the Planck scale, just as we argued in our discussion of the Equivalence Principle in Section 4.7. They will therefore only become important when the length scale characteristic of the curvature approaches the Planck scale, which is far from any plausible experiment. Higher-order terms are therefore interesting in principle, but not in practice. On the other hand, similar reasoning would lead us to expect a huge vacuum energy term, since it is lower-order than the Hilbert action, which we know not to be true; so we should keep an open mind.

As a final alternative to general relativity, we should mention the possibility that the connection really is not derived from the metric, but in fact has an independent existence as a fundamental field. As one of the exercises you are asked to show that it is possible to consider the conventional action for general relativity but treat it as a function of both the metric  $g_{\mu\nu}$  and a torsion-free connection  $\Gamma_{\rho\sigma}^\lambda$ , and the equations of motion derived from varying such an action with respect to the connection imply that  $\Gamma_{\rho\sigma}^\lambda$  is actually the Christoffel connection associated with  $g_{\mu\nu}$ . We could drop the demand that the connection be torsion-free, in which case the torsion tensor could lead to additional propagating degrees of freedom. The basic reason why such theories do not receive much attention is simply because the torsion is itself a tensor; there is nothing to distinguish it from other, nongravitational tensor fields. Thus, we do not really lose any generality by considering theories of torsion-free connections (which lead to GR) plus any number of tensor fields, which we can name what we like. Similar considerations apply when we consider dropping the requirement of metric compatibility—any connection can be written as a metric-compatible connection plus a tensorial correction, so any such theory is equivalent to GR plus extra tensor fields, which wouldn't really deserve to be called an “alternative to general relativity”.

#### 4.9 ■ EXERCISES

1. The Lagrange density for electromagnetism in curved space is

$$\mathcal{L} = \sqrt{-g} \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu \right), \quad (4.156)$$

where  $J^\mu$  is the conserved current.

- (a) Derive the energy-momentum tensor by functional differentiation with respect to the metric. You can assume that the  $A_\mu J^\mu$  term does not contribute to the energy-momentum tensor.
- (b) Consider adding a new term to the Lagrangian,

$$\mathcal{L}' = \beta R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}.$$

How are Maxwell's equations altered in the presence of this term? Einstein's equation? Is the current still conserved?

2. We showed how to derive Einstein's equation by varying the Hilbert action with respect to the metric. They can also be derived by treating the metric and connection as independent degrees of freedom and varying separately with respect to them; this is known

as the **Palatini formalism**. That is, we consider the action

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma),$$

where the Ricci tensor is thought of as constructed purely from the connection, not using the metric. Variation with respect to the metric gives the usual Einstein's equations, but for a Ricci tensor constructed from a connection that has no a priori relationship to the metric. Imagining from the start that the connection is symmetric (torsion free), show that variation of this action with respect to the connection coefficients leads to the requirement that the connection be metric compatible, that is, the Christoffel connection. Remember that Stokes's theorem, relating the integral of the covariant divergence of a vector to an integral of the vector over the boundary, does not work for a general covariant derivative. The best strategy is to write the connection coefficients as a sum of the Christoffel symbols  $\tilde{\Gamma}_{\mu\nu}^\lambda$  and a tensor  $C_{\mu\nu}^\lambda$ ,

$$\Gamma_{\mu\nu}^\lambda = \tilde{\Gamma}_{\mu\nu}^\lambda + C_{\mu\nu}^\lambda,$$

and then show that  $C_{\mu\nu}^\lambda$  must vanish.

3. The four-dimensional  $\delta$ -function on a manifold  $M$  is defined by

$$\int_M F(x^\mu) \left[ \frac{\delta^{(4)}(x^\sigma - y^\sigma)}{\sqrt{-g}} \right] \sqrt{-g} d^4x = F(y^\sigma), \quad (4.157)$$

for an arbitrary function  $F(x^\mu)$ . Meanwhile, the energy-momentum tensor for a pressureless perfect fluid (dust) is

$$T^{\mu\nu} = \rho U^\mu U^\nu, \quad (4.158)$$

where  $\rho$  is the energy density and  $U^\mu$  is the four-velocity. Consider such a fluid that consists of a single particle traveling on a world line  $x^\mu(\tau)$ , with  $\tau$  the proper time. The energy-momentum tensor for this fluid is then given by

$$T^{\mu\nu}(y^\sigma) = m \int_M \left[ \frac{\delta^{(4)}(y^\sigma - x^\sigma(\tau))}{\sqrt{-g}} \right] \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau, \quad (4.159)$$

where  $m$  is the rest mass of the particle. Show that covariant conservation of the energy-momentum tensor,  $\nabla_\mu T^{\mu\nu} = 0$ , implies that  $x^\mu(\tau)$  satisfies the geodesic equation.

4. Show that the energy-momentum tensors for electromagnetism and for scalar field theory satisfy the dominant energy condition, and thus also the weak, null, and null dominant conditions. Show that they also satisfy  $w \geq -1$ .
5. A spacetime is static if there is a timelike Killing vector that is orthogonal to spacelike hypersurfaces. (See Appendices D and F for more discussion, including a definition of Raychaudhuri's equation.)
- (a) Generally speaking, if a vector field  $v^\mu$  is orthogonal to a set of hypersurfaces defined by  $f = \text{constant}$ , then we can write the vector as  $v_\mu = h \nabla_\mu f$  (here both  $f$  and  $h$  are functions). Show that this implies

$$v_{[\sigma} \nabla_\mu v_{\nu]} = 0.$$

- (b) Imagine we have a perfect fluid with zero pressure (dust), which generates a solution to Einstein's equations. Show that the metric can be static only if the fluid four-velocity is parallel to the timelike (and hypersurface-orthogonal) Killing vector.
- (c) Use Raychaudhuri's equation to prove that there is no static solution to Einstein's equations if the pressure is zero and the energy density is greater than zero.
6. Let  $K$  be a Killing vector field. Show that an electromagnetic field with potential  $A_\mu = K_\mu$  solves Maxwell's equations if the metric is a vacuum solution to Einstein's equations. This is a slight cheat, since you won't be in vacuum if there is a nonzero electromagnetic field strength, but we assume the field strength is small enough not to dramatically affect the geometry.

## 5

## The Schwarzschild Solution

## 5.1 ■ THE SCHWARZSCHILD METRIC

The most obvious application of a theory of gravity is to a spherically symmetric gravitational field. This would be the relevant situation to describe, for example, the field created by the Earth or the Sun (to a good approximation), in which apples fall or planets move. Furthermore, our first concern is with exterior solutions (empty space surrounding a gravitating body), since understanding the motion of test particles outside an object is both easier and more immediately useful than considering the relatively inaccessible interior. In addition to its practical usefulness, the answer to this problem in general relativity will lead us to remarkable solutions describing new phenomena of great interest to physicists and astronomers: black holes. In this chapter we examine the simple case of vacuum solutions with perfect spherical symmetry; in the next chapter we consider features of black holes in more general contexts.

In GR, the unique spherically symmetric vacuum solution is the **Schwarzschild metric**; it is second only to Minkowski space in the list of important space-times. In spherical coordinates  $\{t, r, \theta, \phi\}$ , the metric is given by

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (5.1)$$

where  $d\Omega^2$  is the metric on a unit two-sphere,

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (5.2)$$

The constant  $M$  is interpreted as the mass of the gravitating object (although some care is required in making this identification). In this section we will derive the Schwarzschild metric by trial and error; in the next section we will be more systematic in both the derivation of the solution and its consequences.

Since we are interested in the solution *outside* a spherical body, we care about Einstein's equation in vacuum,

$$R_{\mu\nu} = 0. \quad (5.3)$$

Our hypothesized source is static (unevolving) and spherically symmetric, so we will look for solutions that also have these properties. Rigorous definitions of both “static” and “spherically symmetric” require some care, due to subtleties of coordinate independence. For now we will interpret static to imply two conditions: that all metric components are independent of the time coordinate, and that there are no time-space cross terms ( $dr dx^i + dx^i dr$ ) in the metric. The latter condition makes sense if we imagine performing a time inversion  $t \rightarrow -t$ ; the  $dt^2$  term remains invariant, as do any  $dx^i dx^j$  terms, while cross terms would not. Since we hope to find a solution that is independent of time, it should be invariant under time reversal, and we therefore leave cross terms out. To impose spherical symmetry, we begin by writing the metric of Minkowski space (a spherically symmetric spacetime we know something about) in polar coordinates  $x^\mu = (t, r, \theta, \phi)$ :

$$ds_{\text{Minkowski}}^2 = -dt^2 + dr^2 + r^2 d\Omega^2. \quad (5.4)$$

One requirement to preserve spherical symmetry is that we maintain the form of  $d\Omega^2$ ; that is, if we want our spheres to be perfectly round, the coefficient of the  $d\phi^2$  term should be  $\sin^2 \theta$  times that of the  $d\theta^2$  term. But we are otherwise free to multiply all of the terms by separate coefficients, so long as they are only functions of the radial coordinate  $r$ :

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + e^{2\gamma(r)} r^2 d\Omega^2. \quad (5.5)$$

We've expressed our functions as exponentials so that the signature of the metric doesn't change. In a full treatment, we would allow for complete freedom and see what happens.

We can use our ability to change coordinates to make a slight simplification to the static, spherically-symmetric metric (5.5), even before imposing Einstein's equation. Unlike other theories of physics, in general relativity we simultaneously define coordinates and the metric as a function of those coordinates. In other words, we don't know ahead of time what, for example, the radial coordinate  $r$  really is; we can only interpret it once the solution is in our hands. Let us therefore imagine defining a new coordinate  $\tilde{r}$  via

$$\tilde{r} = e^{\gamma(r)} r, \quad (5.6)$$

with an associated basis one-form

$$d\tilde{r} = e^\gamma dr + e^\gamma r d\gamma = \left(1 + r \frac{d\gamma}{dr}\right) e^\gamma dr. \quad (5.7)$$

In terms of this new variable, the metric (5.5) becomes

$$ds^2 = -e^{2\alpha(r)} dt^2 + \left(1 + r \frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r)-2\gamma(r)} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2, \quad (5.8)$$

where each function of  $r$  is a function of  $\tilde{r}$  in the obvious way. But now let us make the following relabelings:

$$\tilde{r} \rightarrow r \quad (5.9)$$

$$\left(1 + r \frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r) - 2\gamma(r)} \rightarrow e^{2\beta}. \quad (5.10)$$

There is nothing to stop us from doing this, as they are simply labels, with no independent external definition. If you wish you can continue to use  $\tilde{r}$ , and set (5.10) equal to  $e^{2\tilde{\beta}}$ , but we won't bother. Our metric (5.8) becomes

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2. \quad (5.11)$$

This looks exactly like (5.5), except that the  $e^{2\gamma}$  factor has disappeared. We have not set  $e^{2\gamma}$  equal to one, which would be a statement about the geometry; we have simply chosen our radial coordinate such that this factor doesn't exist. Thus, (5.11) is precisely as general as (5.5).

Let's now take this metric and use Einstein's equation to solve for the functions  $\alpha(r)$  and  $\beta(r)$ . We begin by evaluating the Christoffel symbols. If we use labels  $(t, r, \theta, \phi)$  for  $(0, 1, 2, 3)$  in the usual way, the Christoffel symbols are given by

$$\begin{aligned} \Gamma_{tr}^t &= \partial_r \alpha & \Gamma_{tt}^r &= e^{2(\alpha-\beta)} \partial_r \alpha & \Gamma_{rr}^r &= \partial_r \beta \\ \Gamma_{r\theta}^\theta &= \frac{1}{r} & \Gamma_{\theta\theta}^r &= -re^{-2\beta} & \Gamma_{r\phi}^\phi &= \frac{1}{r} \\ \Gamma_{\phi\phi}^r &= -re^{-2\beta} \sin^2 \theta & \Gamma_{\phi\phi}^\theta &= -\sin \theta \cos \theta & \Gamma_{\theta\phi}^\phi &= \frac{\cos \theta}{\sin \theta}. \end{aligned} \quad (5.12)$$

Anything not written down explicitly is meant to be zero, or related to what is written by symmetries. From these we get the following nonvanishing components of the Riemann tensor:

$$\begin{aligned} R^t_{rrt} &= \partial_r \alpha \partial_r \beta - \partial_r^2 \alpha - (\partial_r \alpha)^2 \\ R^t_{\theta i \theta} &= -re^{-2\beta} \partial_r \alpha \\ R^t_{\phi t \phi} &= -re^{-2\beta} \sin^2 \theta \partial_r \alpha \\ R^r_{\theta r \theta} &= re^{-2\beta} \partial_r \beta \\ R^r_{\phi r \phi} &= re^{-2\beta} \sin^2 \theta \partial_r \beta \\ R^\theta_{\phi \theta \phi} &= (1 - e^{-2\beta}) \sin^2 \theta. \end{aligned} \quad (5.13)$$

Taking the contraction as usual yields the Ricci tensor:

$$\begin{aligned} R_{tt} &= e^{2(\alpha-\beta)} \left[ \partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \alpha \right] \\ R_{rr} &= -\partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \beta \\ R_{\theta\theta} &= e^{-2\beta} [r(\partial_r \beta - \partial_r \alpha) - 1] + 1 \\ R_{\phi\phi} &= \sin^2 \theta R_{\theta\theta}, \end{aligned} \quad (5.14)$$

and for future reference we calculate the curvature scalar,

$$R = -2e^{-2\beta} \left[ \partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} (\partial_r \alpha - \partial_r \beta) + \frac{1}{r^2} (1 - e^{2\beta}) \right]. \quad (5.15)$$

With the Ricci tensor calculated, we would like to set it equal to zero. Since  $R_{tt}$  and  $R_{rr}$  vanish independently, we can write

$$0 = e^{2(\beta-\alpha)} R_{tt} + R_{rr} = \frac{2}{r} (\partial_r \alpha + \partial_r \beta), \quad (5.16)$$

which implies  $\alpha = -\beta + c$ , where  $c$  is some constant. We can set this constant equal to zero by rescaling our time coordinate by  $t \rightarrow e^{-c}t$ , after which we have

$$\alpha = -\beta. \quad (5.17)$$

Next let us turn to  $R_{\theta\theta} = 0$ , which now reads

$$e^{2\alpha} (2r \partial_r \alpha + 1) = 1. \quad (5.18)$$

This is equivalent to

$$\partial_r (r e^{2\alpha}) = 1. \quad (5.19)$$

We can solve this to obtain

$$e^{2\alpha} = 1 - \frac{R_S}{r}, \quad (5.20)$$

where  $R_S$  is some undetermined constant. With (5.17) and (5.20), our metric becomes

$$ds^2 = - \left( 1 - \frac{R_S}{r} \right) dt^2 + \left( 1 - \frac{R_S}{r} \right)^{-1} dr^2 + r^2 d\Omega^2. \quad (5.21)$$

We now have no freedom left except for the single constant  $R_S$ , so this form had better solve the remaining equations  $R_{tt} = 0$  and  $R_{rr} = 0$ ; it is straightforward to check that it does, for any value of  $R_S$ .

The only thing left to do is to interpret the constant  $R_S$ , called the **Schwarzschild radius**, in terms of some physical parameter. Nothing could be simpler. In Chapter 4 we found that, in the weak-field limit, the  $tt$  component of the metric around a point mass satisfies

$$g_{tt} = - \left( 1 - \frac{2GM}{r} \right). \quad (5.22)$$

The Schwarzschild metric should reduce to the weak-field case when  $r \gg 2GM$ , but for the  $tt$  component the forms are already exactly the same; we need only identify

$$R_S = 2GM. \quad (5.23)$$

This can be thought of as the definition of the parameter  $M$ .

Our final result is the Schwarzschild metric, (5.1). We have shown that it is a static, spherically symmetric vacuum solution to Einstein's equation;  $M$  functions as a parameter, which we happen to know can be interpreted as the conventional Newtonian mass that we would measure by studying orbits at large distances from the gravitating source. It won't simply be the sum of the masses of the constituents of whatever body is curving spacetime, since there will be a contribution from what we might think of as the gravitational binding energy; however, in the weak field limit, the quantities will agree. Note that as  $M \rightarrow 0$  we recover Minkowski space, which is to be expected. Note also that the metric becomes progressively Minkowskian as  $r \rightarrow \infty$ ; this property is known as **asymptotic flatness**. A more technical definition involves matching regions at infinity in a conformal diagram, as discussed in the next chapter.

## 5.2 ■ BIRKHOFF'S THEOREM

**Birkhoff's theorem** is the statement that the Schwarzschild metric is the *unique* vacuum solution with spherical symmetry (and in particular, that there are no time-dependent solutions of this form); proving it is an instructive exercise, which consists of three major steps. First, we argue that a spherically symmetric spacetime can be foliated by two-spheres—in other words, that (almost) every point lies on a unique sphere that is left invariant by the generators of spherical symmetry. Second, we show on purely geometric grounds that the metric on such a space can always (at least in a local region) be put in the form

$$ds^2 = d\tau^2(a, b) + r^2(a, b) d\Omega^2(\theta, \phi), \quad (5.24)$$

where  $(a, b)$  are coordinates transverse to the spheres, and  $r$  is a function of these coordinates. Third, we plug this metric into Einstein's equation in vacuum to show that Schwarzschild is the unique solution. We will argue in favor of the first two points at a level of rigor that is likely to be convincing to most physicists, although mathematicians will be uneasy; the third point is straightforward calculation. For a more careful treatment see Hawking and Ellis (1973). We will use a few concepts from Appendix C, which may be useful to read at this point. Of course, if you are more interested in exploring properties of the Schwarzschild solution than in proving its uniqueness, you are welcome to skip right to the next section.

We begin with the concept of a four-dimensional spherically symmetric spacetime  $M$ . Spherically symmetric means having the same symmetries as a sphere. (In this chapter the word sphere refers specifically to  $S^2$ , not spheres of other dimension.) The symmetries of a sphere are precisely those of ordinary rotations in three-dimensional Euclidean space; in the language of group theory, they comprise the special orthogonal group  $SO(3)$ . (Recall the discussion of the Lorentz and rotation groups in Chapter 1.) In the case of a metric on a manifold, symmetries are characterized by the existence of Killing vectors. In Section 3.8 we found the three Killing vectors of  $S^2$ , labeled  $(R, S, T)$ ; in  $(\theta, \phi)$  coordinates they take

the form

$$\begin{aligned} R &= \partial_\phi \\ S &= \cos\phi \partial_\theta - \cot\theta \sin\phi \partial_\phi \\ T &= -\sin\phi \partial_\theta - \cot\theta \cos\phi \partial_\phi. \end{aligned} \quad (5.25)$$

A spherically symmetric manifold is one that has three Killing vector fields that are the same as those on  $S^2$ . But how do we know, in a coordinate-independent way, that a set of Killing vectors on one manifold is the same as that on some other manifold? The structure of a set of symmetry transformations is given by the commutation relations of the transformations, which express the difference between performing two infinitesimal transformations in one order versus the reversed order. In group theory these are expressed by the Lie algebra of the symmetry generators, while in differential geometry they are expressed by the commutators of the Killing vector fields. There is a deep connection here, which we don't have time to pursue; see Schutz (1980). In the Exercises for Chapter 3 you verified that the commutators of the rotational Killing vectors ( $R, S, T$ ) satisfied

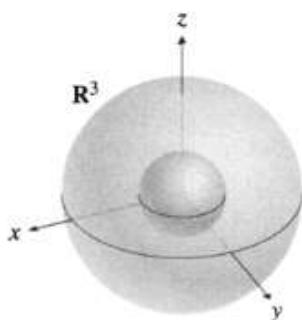
$$\begin{aligned} [R, S] &= T \\ [S, T] &= R \\ [T, R] &= S. \end{aligned} \quad (5.26)$$

This algebra of Killing vectors fully characterizes the kind of symmetry we have. A manifold will be said to possess **spherical symmetry** if and only if there are three Killing fields satisfying (5.26).

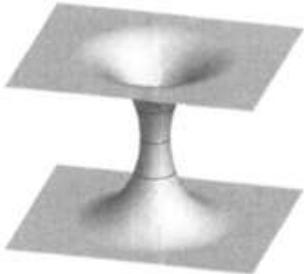
In Appendix C we discuss Frobenius's theorem, which states that if you have a set of vector fields whose commutator closes—the commutator of any two fields in the set is a linear combination of other fields in the set—then the integral curves of these vector fields fit together to describe submanifolds of the manifold on which they are all defined. The dimensionality of the submanifold may be smaller than the number of vectors, or it could be equal, but obviously not larger. Vector fields that obey (5.26) will of course form 2-spheres. Since the vector fields stretch throughout the space, every point will be on exactly one of these spheres. (Actually, it's almost every point—we will show below how it can fail to be absolutely every point.) Thus, we say that a spherically symmetric manifold can be foliated into spheres.

Let's consider some examples to bring this down to earth. The simplest example is flat three-dimensional Euclidean space. If we pick an origin, then  $\mathbf{R}^3$  is clearly spherically symmetric with respect to rotations around this origin. Under such rotations (that is, under the flow of the Killing vector fields), points move into each other, but each point stays on an  $S^2$  at a fixed distance from the origin.

These spheres foliate  $\mathbf{R}^3$ , as depicted in Figure 5.1. Of course, they don't really foliate all of the space, since the origin itself just stays put under rotations—it



**FIGURE 5.1** Foliating  $\mathbf{R}^3$  (minus the origin) by two-spheres.



**FIGURE 5.2** Foliation of a wormhole by two-spheres.

doesn't move around on some two-sphere. But it should be clear that almost all of the space is properly foliated, and this will turn out to be enough for us.

We can also have spherical symmetry without an origin to rotate things around. An example is provided by a wormhole, with topology  $\mathbf{R} \times S^2$ . If we suppress a dimension and draw our two-spheres as circles, such a space might look like Figure 5.2. In this case the entire manifold can be foliated by two-spheres.

Given that manifolds with  $SO(3)$  symmetry may be foliated by spheres, our second step is to show that the metric on  $M$  can be put into the form (5.24). The set of all the spheres forms a two-dimensional space (since a four-dimensional spacetime is being foliated with two-dimensional spheres). You might hope we could simply put coordinates  $(\theta, \phi)$  on each sphere, and coordinates  $(a, b)$  on the set of all spheres, for a complete set of coordinates  $(a, b, \theta, \phi)$  on  $M$ . Then each sphere is specified by  $a = \text{constant}$ ,  $b = \text{constant}$ . We know that the metric on a round sphere is  $d\Omega^2$ , so this strategy would be sufficient to guarantee that the metric restricted to any fixed values  $a = a_0$  and  $b = b_0$  (so that  $da = db = 0$ ) takes the form

$$ds^2(a_0, b_0, \theta, \phi) = f(a_0, b_0) d\Omega^2. \quad (5.27)$$

In particular, the function  $f$  must be independent of  $\theta$  and  $\phi$ , or the sphere would be lumpy rather than round. Furthermore, it's equally clear that the metric restricted to any fixed values  $\theta = \theta_0$  and  $\phi = \phi_0$  (so that  $d\theta = d\phi = 0$ ) takes the form

$$ds^2(a, b, \theta_0, \phi_0) = d\tau^2(a, b). \quad (5.28)$$

Again, any dependence on  $\theta$  or  $\phi$  would destroy the symmetry; it would mean that the geometry transverse to the spheres depended on where you were on the sphere.

However, we have been too reckless by slapping down these coordinates, since we cannot rule out cross terms of the form  $dad\theta + d\theta da$  and so on. In other words, we must be careful to line up our spheres appropriately, so that travel along a curve that is perpendicular to one of the spheres keeps us at constant  $\theta$  and  $\phi$ . To guarantee this we need to be more careful in setting up our coordinates. Begin by considering a single point  $q$  lying on a sphere  $S_q$  (note that  $q$  must not be a degenerate point at which all of the Killing vectors vanish). Put coordinates  $(\theta, \phi)$  on this particular sphere only, not yet through the manifold. At each point  $p$  on  $S_q$ , there will be a two-dimensional orthogonal subspace  $O_p$ , consisting of points along geodesics emanating from  $p$  whose tangent vectors at  $p$  are orthogonal to  $S_q$ . Note that there will be a one-dimensional subgroup  $R_p$  of rotations that leave  $p$  fixed; indeed, these rotations keep fixed any direction perpendicular to  $S_q$  at  $p$ , and hence the entire two-surface  $O_p$  is left invariant by  $R_p$ .

Consider a point  $r$  that is not on  $S_q$ , but on some other sphere  $S_r$  in the foliation, and that lies in the two-surface  $O_p$  orthogonal to  $S_q$  at  $p$ . Since  $p$  is arbitrary, this includes any possible point  $r$  in a neighborhood of  $S_q$ . Note that  $O_p$  will be orthogonal to  $S_r$  as well as to  $S_q$ . To see this, consider the two-dimensional plane

$V_r$  of vectors in the tangent space  $T_r M$  that are orthogonal to the two-surface  $O_p$ . Since  $O_p$  is left invariant by the rotations  $R_p$ , these rotations must take  $V_r$  into itself, because they are an isometry, and hence preserve orthogonality. But  $R_p$  also takes the set of vectors tangent to  $S_q$  into itself, since these rotations leave the spheres invariant. In four dimensions, two planes that are both orthogonal to a given plane at the same point must be the same plane; hence, the vectors tangent to  $S_r$  must be orthogonal to  $O_p$ .

There will be a unique geodesic that is orthogonal to  $S_q$  and connects  $p$  to  $r$ . Traveling down such geodesics provides a map  $f : S_q \rightarrow S_r$ , which is both one-to-one and onto (at least in a neighborhood of the original sphere). We use this map to define coordinates on  $S_r$  (and, similarly, on any other sphere) by assigning the same values of  $(\theta, \phi)$  to  $r \in S_r$  that were the coordinates at  $p \in S_q$ . We have therefore defined  $(\theta, \phi)$  throughout the manifold. Now to define coordinates  $(a, b)$ , choose two basis vectors  $S, T$  for the subspace of  $T_q M$  that generates the orthogonal space  $O_q$ . Any other sphere will be connected to  $q$  by a unique orthogonal geodesic, with tangent vector  $aS + bT \in T_q M$ . Assign those components  $(a, b)$  as coordinates everywhere on that sphere. This defines the full set of coordinates  $(a, b, \theta, \phi)$  throughout the manifold.

The metric in these coordinates satisfies (5.27) and (5.28); it remains to be shown that there are no cross terms between directions along the spheres and those transverse. This means, for example, that the vector field  $\partial_a$  should be orthogonal to  $\partial_\theta$ , and so on; it is straightforward to verify that this is so. First, consider  $\partial_\theta$  at some point  $r \in S_r$ ; this vector is the directional derivative along a curve of the form  $x^\mu(\theta) = (a_r, b_r, \theta, \phi_r)$ . Since  $a$  and  $b$  are constant along the curve, the entire curve remains in the sphere  $S_r$ , so that  $\partial_\theta$  is tangent to the sphere. Meanwhile,  $\partial_a$  is a derivative along  $x^\mu(a) = (a, b_r, \theta_r, \phi_r)$ . Since this curve remains in the orthogonal subspace  $O_r$ ,  $\partial_a$  will be orthogonal to  $S_r$ , and hence to  $\partial_\theta$ . Similar arguments guarantee that there will be no cross terms between  $(a, b)$  and  $(\theta, \phi)$ .

We have thus succeeded in putting the metric on a spherically symmetric spacetime in the form

$$ds^2 = g_{aa}(a, b) da^2 + g_{ab}(a, b)(dadb + dbda) + g_{bb}(a, b) db^2 + r^2(a, b) d\Omega^2. \quad (5.29)$$

Here  $r(a, b)$  is some as-yet-undetermined function, to which we have merely given a suggestive label. There is nothing to stop us, however, from changing coordinates from  $(a, b)$  to  $(a, r)$  by inverting  $r(a, b)$ , unless  $r$  were a function of  $a$  alone; in this case we could just as easily switch to  $(b, r)$ , so we will not consider this situation separately. The metric is then

$$ds^2 = g_{aa}(a, r) da^2 + g_{ar}(a, r)(da dr + dr da) + g_{rr}(a, r) dr^2 + r^2 d\Omega^2. \quad (5.30)$$

Our next step is to find a function  $t(a, r)$  such that, in the  $(t, r)$  coordinate system, there are no cross terms  $dt dr + dr dt$  in the metric. Notice that

$$dt = \frac{\partial t}{\partial a} da + \frac{\partial t}{\partial r} dr, \quad (5.31)$$

so

$$dt^2 = \left(\frac{\partial t}{\partial a}\right)^2 da^2 + \left(\frac{\partial t}{\partial a}\right) \left(\frac{\partial t}{\partial r}\right) (da dr + dr da) + \left(\frac{\partial t}{\partial r}\right)^2 dr^2. \quad (5.32)$$

We would like to replace the first three terms in the metric (5.30) by

$$mdt^2 + ndr^2, \quad (5.33)$$

for some functions  $m$  and  $n$ . This is equivalent to the requirements

$$m \left(\frac{\partial t}{\partial a}\right)^2 = g_{aa}, \quad (5.34)$$

$$n + m \left(\frac{\partial t}{\partial r}\right)^2 = g_{rr}, \quad (5.35)$$

and

$$m \left(\frac{\partial t}{\partial a}\right) \left(\frac{\partial t}{\partial r}\right) = g_{ar}. \quad (5.36)$$

We therefore have three equations for the three unknowns  $t(a, r)$ ,  $m(a, r)$ , and  $n(a, r)$ , just enough to determine them precisely, up to initial conditions for  $t$ . (Of course, they are “determined” in terms of the unknown functions  $g_{aa}$ ,  $g_{ar}$ , and  $g_{rr}$ , so in this sense they are still undetermined.) We can therefore put our metric in the form

$$ds^2 = m(t, r) dt^2 + n(t, r) dr^2 + r^2 d\Omega^2. \quad (5.37)$$

To this point the only difference between the two coordinates  $t$  and  $r$  is that we have chosen  $r$  to be the one that multiplies the metric for the two-sphere. This choice was motivated by what we know about the metric for flat Minkowski space, which can be written  $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$ . We know that the spacetime under consideration is Lorentzian, so either  $m$  or  $n$  will have to be negative. Let us choose  $m$ , the coefficient of  $dt^2$ , to be negative. This is not a choice we are simply allowed to make, and in fact we will see later that it can go wrong; but we will assume it for now. The assumption is not completely unreasonable, since we know that Minkowski space is itself spherically symmetric, and will therefore be described by (5.37). With this choice we can trade in the functions  $m$  and  $n$  for new functions  $\alpha$  and  $\beta$ , such that

$$ds^2 = -e^{2\alpha(t,r)} dt^2 + e^{2\beta(t,r)} dr^2 + r^2 d\Omega^2. \quad (5.38)$$

This is the best we can do using only geometry; spherical symmetry is certainly not enough to say anything substantive about the functions  $\alpha(t, r)$  and  $\beta(t, r)$ . Our next step is therefore to actually solve Einstein's equation; the steps follow closely

along those of Section 5.1, in which we considered a metric similar to (5.38) but with the additional assumption of time-independence. Here we will see that this assumption was unnecessary, as the solution will necessarily be static.

The nonvanishing Christoffel symbols for (5.38) are

$$\begin{aligned}\Gamma_{tt}^t &= \partial_t \alpha & \Gamma_{tr}^t &= \partial_r \alpha & \Gamma_{rr}^t &= e^{2(\beta-\alpha)} \partial_t \beta \\ \Gamma_{tt}^r &= e^{2(\alpha-\beta)} \partial_r \alpha & \Gamma_{tr}^r &= \partial_t \beta & \Gamma_{rr}^r &= \partial_r \beta \\ \Gamma_{r\theta}^\theta &= \frac{1}{r} & \Gamma_{\theta\theta}^r &= -re^{-2\beta} & \Gamma_{r\phi}^\phi &= \frac{1}{r} \\ \Gamma_{\phi\phi}^r &= -re^{-2\beta} \sin^2 \theta & \Gamma_{\phi\phi}^\theta &= -\sin \theta \cos \theta & \Gamma_{\theta\phi}^\phi &= \frac{\cos \theta}{\sin \theta},\end{aligned}\quad (5.39)$$

the nonvanishing components of the Riemann tensor are

$$\begin{aligned}R^t_{rrt} &= e^{2(\beta-\alpha)} [\partial_t^2 \beta + (\partial_t \beta)^2 - \partial_t \alpha \partial_t \beta] + [\partial_r \alpha \partial_r \beta - \partial_r^2 \alpha - (\partial_r \alpha)^2] \\ R^t_{\theta t\theta} &= -re^{-2\beta} \partial_r \alpha \\ R^t_{\phi t\phi} &= -re^{-2\beta} \sin^2 \theta \partial_r \alpha \\ R^t_{\theta r\theta} &= -re^{-2\alpha} \partial_t \beta \\ R^t_{\phi r\phi} &= -re^{-2\alpha} \sin^2 \theta \partial_t \beta \\ R^r_{\theta r\theta} &= re^{-2\beta} \partial_r \beta \\ R^r_{\phi r\phi} &= re^{-2\beta} \sin^2 \theta \partial_r \beta \\ R^\theta_{\phi\theta\phi} &= (1 - e^{-2\beta}) \sin^2 \theta,\end{aligned}\quad (5.40)$$

and the Ricci tensor is

$$\begin{aligned}R_{tt} &= \left[ \partial_t^2 \beta + (\partial_t \beta)^2 - \partial_t \alpha \partial_t \beta \right] + e^{2(\alpha-\beta)} \left[ \partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \alpha \right] \\ R_{rr} &= - \left[ \partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta - \frac{2}{r} \partial_r \beta \right] \\ &\quad + e^{2(\beta-\alpha)} \left[ \partial_t^2 \beta + (\partial_t \beta)^2 - \partial_t \alpha \partial_t \beta \right] \\ R_{tr} &= \frac{2}{r} \partial_t \beta \\ R_{\theta\theta} &= e^{-2\beta} [r(\partial_r \beta - \partial_r \alpha) - 1] + 1 \\ R_{\phi\phi} &= R_{\theta\theta} \sin^2 \theta.\end{aligned}\quad (5.41)$$

Our job is to solve Einstein's equation in vacuum,  $R_{\mu\nu} = 0$ . From  $R_{tr} = 0$  we get

$$\partial_t \beta = 0. \quad (5.42)$$

If we consider taking the time derivative of  $R_{\theta\theta} = 0$  and using  $\partial_t \beta = 0$ , we get

$$\partial_t \partial_r \alpha = 0. \quad (5.43)$$

We can therefore write

$$\begin{aligned}\beta &= \beta(r) \\ \alpha &= f(r) + g(t).\end{aligned}\quad (5.44)$$

The first term in the metric (5.38) is thus  $-e^{2f(r)} e^{2g(t)} dt^2$ . But we can always simply redefine our time coordinate by replacing  $dt \rightarrow e^{-g(t)} dt$ ; in other words, we are free to choose  $t$  such that  $g(t) = 0$ , whence  $\alpha(t, r) = f(r)$ . We therefore have

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2. \quad (5.45)$$

All of the metric components are independent of the coordinate  $t$ . We have therefore proven a crucial result: *any spherically symmetric vacuum metric possesses a timelike Killing vector.*

This property is so interesting that it gets its own name: a metric that possesses a Killing vector that is timelike near infinity is called **stationary**. (Often, including in Schwarzschild, the Killing vector that is timelike at infinity will become spacelike somewhere in the interior.) In a stationary metric we can choose coordinates  $(t, x^1, x^2, x^3)$  in which the Killing vector is  $\partial_t$  and the metric components are independent of  $t$ ; the general form of a stationary metric in these coordinates is thus

$$ds^2 = g_{00}(\vec{x}) dt^2 + g_{0i}(\vec{x})(dt dx^i + dx^i dt) + g_{ij}(\vec{x}) dx^i dx^j. \quad (5.46)$$

There is also a more restrictive property: a metric is called **static** if it possesses a timelike Killing vector that is orthogonal to a family of hypersurfaces. (For more details on hypersurfaces, see Appendix D.) In the Exercises for Chapter 4 you showed that a hypersurface-orthogonal vector field  $v^\mu$  obeys

$$v_{[\mu} \nabla_\nu v_{\sigma]} = 0. \quad (5.47)$$

But there is a simpler diagnostic: if we have adapted coordinates so that the components  $g_{\mu\nu}$  are all independent of  $t$ , the surfaces to which the Killing vector will be orthogonal are defined by the condition  $t = \text{constant}$ . Operationally, this means that the time-space cross terms in (5.46) will be absent; the general static metric can be written

$$ds^2 = g_{00}(\vec{x}) dt^2 + g_{ij}(\vec{x}) dx^i dx^j. \quad (5.48)$$

We notice that only even powers of the time coordinate  $t$  appear in this form; thus, an alternative definition of “static” is “stationary, and invariant under time reversal ( $t \rightarrow -t$ ).” The metric (5.45) is clearly static. You should think of stationary as meaning “doing exactly the same thing at every time,” while static means “not

doing anything at all." For example, the static spherically symmetric metric (5.45) will describe nonrotating stars or black holes, while rotating systems that keep rotating in the same way at all times will be described by metrics that are stationary but not static.

Notice that (5.45) is precisely the same as (5.11), the metric we originally used to derive the Schwarzschild solution in Section 5.1. We have therefore proven Birkhoff's theorem, that the unique spherically symmetric vacuum solution is the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (5.49)$$

as promised.

We did not say anything about the source of the Schwarzschild metric, except that it be spherically symmetric. Specifically, we did not demand that the source itself be static; it could be a collapsing star, as long as the collapse is symmetric. Therefore a process such as a supernova explosion would generate very little gravitational radiation (in comparison to the amount of energy released through other channels) if it were close to spherically symmetric, which a realistic supernova may or may not be depending on its origin. This is the same result we would have obtained in electromagnetism, where the electromagnetic fields around a spherical charge distribution do not depend on the radial distribution of the charges.

### 5.3 ■ SINGULARITIES

Before exploring the behavior of test particles in the Schwarzschild geometry, we should say something about singularities. From the form of (5.1), the metric coefficients become infinite at  $r = 0$  and  $r = 2GM$ —an apparent sign that something is going wrong. The metric coefficients, of course, are coordinate-dependent quantities, and as such we should not make too much of their values; it is certainly possible to have a coordinate singularity that results from a breakdown of a specific coordinate system rather than the underlying manifold. An example occurs at the origin of polar coordinates in the plane, where the metric  $ds^2 = dr^2 + r^2 d\theta^2$  becomes degenerate and the component  $g^{\theta\theta} = r^{-2}$  of the inverse metric blows up, even though that point of the manifold is no different from any other.

What kind of coordinate-independent signal should we look for as a warning that something about the geometry is out of control? This turns out to be a difficult question to answer, and entire books have been written about the nature of singularities in general relativity. We won't go into this issue in detail, but rather turn to one simple criterion for when something has gone wrong—when the curvature becomes infinite. The curvature is measured by the Riemann tensor, and it is hard to say when a tensor becomes infinite, since its components are coordinate-dependent. But from the curvature we can construct various scalar quantities, and since scalars are coordinate-independent it is meaningful to say that they become infinite. The simplest such scalar is the Ricci scalar

$R = g^{\mu\nu} R_{\mu\nu}$ , but we can also construct higher-order scalars such as  $R^{\mu\nu} R_{\mu\nu}$ ,  $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ ,  $R_{\mu\nu\rho\sigma} R^{\rho\sigma\lambda\tau} R_{\lambda\tau}^{\mu\nu}$ , and so on. If any of these scalars (but not necessarily all of them) goes to infinity as we approach some point, we regard that point as a singularity of the curvature. We should also check that the point is not infinitely far away; that is, that it can be reached by traveling a finite distance along a curve.

We therefore have a sufficient condition for a point to be considered a singularity. It is not a necessary condition, however, and it is generally harder to show that a given point is nonsingular; for our purposes we will simply test to see if geodesics are well-behaved at the point in question, and if so then we will consider the point nonsingular. In the case of the Schwarzschild metric (5.1), direct calculation reveals that

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48G^2 M^2}{r^6}. \quad (5.50)$$

This is enough to convince us that  $r = 0$  represents an honest singularity.

The other trouble spot is  $r = 2GM$ , the Schwarzschild radius. You could check that none of the curvature invariants blows up there. We therefore begin to think that it is actually not singular, and we have simply chosen a bad coordinate system. The best thing to do is to transform to more appropriate coordinates if possible. We will soon see that in this case it is in fact possible, and the surface  $r = 2GM$  is very well-behaved (although interesting) in the Schwarzschild metric—it demarcates the event horizon of a black hole.

Having worried a little about singularities, we should point out that the behavior of the Schwarzschild metric inside the Schwarzschild radius is of little day-to-day consequence. The solution we derived is valid only in vacuum, and we expect it to hold outside a spherical body such as a star. However, in the case of the Sun we are dealing with a body that extends to a radius of

$$R_\odot = 10^6 GM_\odot. \quad (5.51)$$

Thus,  $r = 2GM_\odot$  is far inside the solar interior, where we do not expect the Schwarzschild metric to apply. In fact, realistic stellar interior solutions consist of matching the exterior Schwarzschild metric to an interior metric that is perfectly smooth at the origin. Nevertheless, there are objects for which the full Schwarzschild metric is required—black holes—and therefore we will let our imaginations roam far outside the solar system in this chapter.

## 5.4 ■ GEODESICS OF SCHWARZSCHILD

The first step we will take to understand the Schwarzschild metric more fully is to consider the behavior of geodesics. We need the nonzero Christoffel symbols for Schwarzschild:

$$\begin{aligned}
 \Gamma_{tt}^r &= \frac{GM}{r^3}(r - 2GM) & \Gamma_{rr}^t &= \frac{-GM}{r(r - 2GM)} & \Gamma_{tr}^t &= \frac{GM}{r(r - 2GM)} \\
 \Gamma_{r\theta}^\theta &= \frac{1}{r} & \Gamma_{\theta\theta}^r &= -(r - 2GM) & \Gamma_{r\phi}^\phi &= \frac{1}{r} \\
 \Gamma_{\phi\phi}^r &= -(r - 2GM)\sin^2\theta & \Gamma_{\phi\phi}^\theta &= -\sin\theta\cos\theta & \Gamma_{\theta\phi}^\phi &= \frac{\cos\theta}{\sin\theta}.
 \end{aligned} \tag{5.52}$$

The geodesic equation therefore turns into the following four equations, where  $\lambda$  is an affine parameter:

$$\begin{aligned}
 \frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r - 2GM)} \frac{dr}{d\lambda} \frac{dt}{d\lambda} &= 0, \\
 \frac{d^2r}{d\lambda^2} + \frac{GM}{r^3}(r - 2GM) \left( \frac{dt}{d\lambda} \right)^2 - \frac{GM}{r(r - 2GM)} \left( \frac{dr}{d\lambda} \right)^2 \\
 -(r - 2GM) \left[ \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2\theta \left( \frac{d\phi}{d\lambda} \right)^2 \right] &= 0, \\
 \frac{d^2\theta}{d\lambda^2} + \frac{2}{r} \frac{d\theta}{d\lambda} \frac{dr}{d\lambda} - \sin\theta\cos\theta \left( \frac{d\phi}{d\lambda} \right)^2 &= 0, \\
 \frac{d^2\phi}{d\lambda^2} + \frac{2}{r} \frac{d\phi}{d\lambda} \frac{dr}{d\lambda} + 2 \frac{\cos\theta}{\sin\theta} \frac{d\theta}{d\lambda} \frac{d\phi}{d\lambda} &= 0. \tag{5.53}
 \end{aligned}$$

There does not seem to be much hope for simply solving this set of coupled equations by inspection. Fortunately our task is greatly simplified by the high degree of symmetry of the Schwarzschild metric. We know that there are four Killing vectors: three for the spherical symmetry, and one for time translations. Each of these will lead to a constant of the motion for a free particle. If  $K^\mu$  is a Killing vector, we know that

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{constant}. \tag{5.54}$$

In addition, we always have another constant of the motion for geodesics: the geodesic equation (together with metric compatibility) implies that the quantity

$$\epsilon = -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{5.55}$$

is constant along the path. (For any trajectory we can choose the parameter  $\lambda$  such that  $\epsilon$  is a constant; we are simply noting that this is compatible with affine parameterization along a geodesic.) Of course, for a massive particle we typically choose  $\lambda = \tau$ , and this relation simply becomes  $\epsilon = -g_{\mu\nu}U^\mu U^\nu = +1$ . For massless particles, which move along null trajectories, we always have  $\epsilon = 0$ ,

and this equation does not fix the parameter  $\lambda$ . As discussed in Section 3.4, it is convenient to normalize  $\lambda$  along null geodesics such that the four-momentum and four-velocity are equal,  $p^\mu = dx^\mu/d\lambda$ . We might also be concerned with spacelike geodesics (even though they do not correspond to paths of particles), for which we will choose  $\epsilon = -1$ .

Rather than immediately writing out explicit expressions for the four conserved quantities associated with Killing vectors, let's think about what they are telling us. Notice that the symmetries they represent are also present in flat spacetime, where the conserved quantities they lead to are very familiar. Invariance under time translations leads to conservation of energy, while invariance under spatial rotations leads to conservation of the three components of angular momentum. Essentially the same applies to the Schwarzschild metric. We can think of the angular momentum as a three-vector with a magnitude (one component) and direction (two components). Conservation of the direction of angular momentum means that the particle will move in a plane. We can choose this to be the equatorial plane of our coordinate system; if the particle is not in this plane, we can rotate coordinates until it is. Thus, the two Killing vectors that lead to conservation of the direction of angular momentum imply that, for a single particle, we can choose

$$\theta = \frac{\pi}{2}. \quad (5.56)$$

The two remaining Killing vectors correspond to energy and the magnitude of angular momentum. The energy arises from the timelike Killing vector

$$K^\mu = (\partial_t)^\mu = (1, 0, 0, 0). \quad (5.57)$$

The Killing vector whose conserved quantity is the magnitude of the angular momentum is

$$R^\mu = (\partial_\phi)^\mu = (0, 0, 0, 1). \quad (5.58)$$

In both cases it is convenient to lower the index to obtain

$$K_\mu = \left( -\left( 1 - \frac{2GM}{r} \right), 0, 0, 0 \right) \quad (5.59)$$

and

$$R_\mu = \left( 0, 0, 0, r^2 \sin^2 \theta \right). \quad (5.60)$$

Since (5.56) implies that  $\sin \theta = 1$  along the geodesics of interest to us, the two conserved quantities are

$$E = -K_\mu \frac{dx^\mu}{d\lambda} = \left( 1 - \frac{2GM}{r} \right) \frac{dt}{d\lambda} \quad (5.61)$$

and

$$L = R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda}. \quad (5.62)$$

For massless particles, these can be thought of as the conserved energy and angular momentum, while for massive particles they are the conserved energy and angular momentum per unit mass of the particle. In the discussion of rotating black holes in the next chapter, we will use  $E$  and  $L$  to refer to the actual energy and angular momentum, not “per unit mass”; the meaning should be clear from context. Note that the constancy of (5.62) is the GR equivalent of Kepler’s second law—equal areas are swept out in equal times.

Recall that in Section 3.4 we claimed that the energy of a particle with four-momentum  $p^\mu$ , as measured by an observer with four-velocity  $U^\mu$ , would be  $-p_\mu U^\mu$ . This is *not* equal, or even proportional, to (5.61), even if the observer is taken to be stationary ( $U^i = 0$ ). Mathematically, this is because the four-velocity is normalized to  $U_\mu U^\mu = -1$ , which the Killing vector  $K^\mu$  is not: If we tried to normalize it in that way, it would no longer solve Killing’s equation. At a slightly deeper level,  $-p_\mu U^\mu$  may be thought of as the inertial/kinetic energy of the particle, while  $-p_\mu K^\mu$  is the total conserved energy, including the potential energy due to the gravitational field. The notion of gravitational potential energy is not always well-defined, but the total energy is well-defined in the presence of a time-like Killing vector. We will presently use  $E$  to help characterize geodesics of Schwarzschild; later we will also use  $-p_\mu U^\mu$  for massless particles, where it can be thought of as the observed frequency of a photon, to describe gravitational redshift.

Together the conserved quantities  $E$  and  $L$  provide a convenient way to understand the orbits of particles in the Schwarzschild geometry. Let us expand the expression (5.55) for  $\epsilon$  to obtain

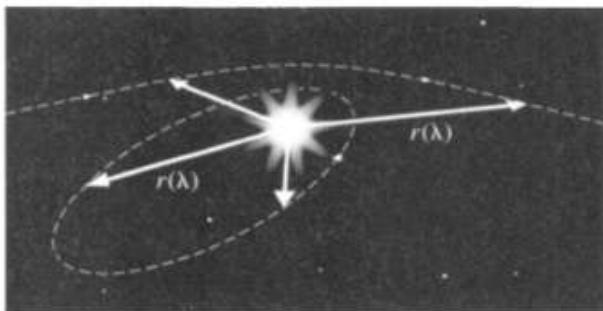
$$-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = -\epsilon. \quad (5.63)$$

If we multiply this by  $(1 - 2GM/r)$  and use our expressions for  $E$  and  $L$ , we obtain

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = 0. \quad (5.64)$$

This is certainly progress, since we have taken a messy system of coupled equations and obtained a single equation for  $r(\lambda)$ . It looks even nicer if we rewrite it as

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E}, \quad (5.65)$$



**FIGURE 5.3** Orbits around a star are characterized by giving the radius  $r$  as a function of a parameter  $\lambda$ .

where

$$V(r) = \frac{1}{2}\epsilon - \epsilon \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \quad (5.66)$$

and

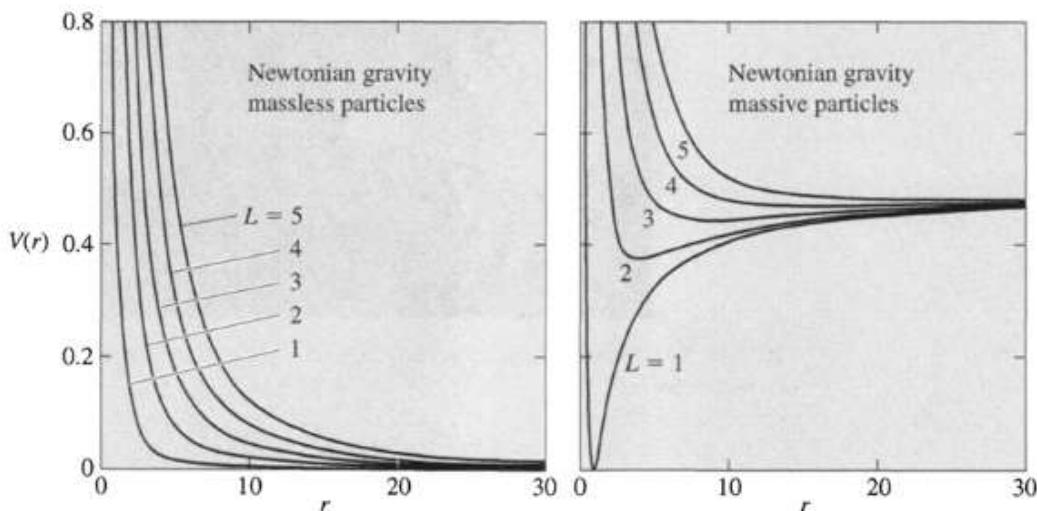
$$\mathcal{E} = \frac{1}{2}E^2. \quad (5.67)$$

In (5.65) we have precisely the equation for a classical particle of unit mass and “energy”  $\mathcal{E}$  moving in a one-dimensional potential given by  $V(r)$ . It’s a little confusing, but not too bad: the conserved energy per unit mass is  $E$ , but the effective potential for the coordinate  $r$  responds to  $\mathcal{E} = E^2/2$ .

Of course, our physical situation is quite different from a classical particle moving in one dimension; the trajectories under consideration are orbits around a star or other object, as shown in Figure 5.3. The quantities of interest to us are not only  $r(\lambda)$ , but also  $t(\lambda)$  and  $\phi(\lambda)$ . Nevertheless, we can go a long way toward understanding all of the orbits by understanding their radial behavior, and it is a great help to reduce this behavior to a problem we know how to solve.

A similar analysis of orbits in Newtonian gravity would have produced a similar result; the general equation (5.65) would have been the same, but the effective potential (5.66) would not have had the last term. (Note that this equation is not a power series in  $1/r$ , it is exact.) In the potential (5.66) the first term is just a constant, the second term corresponds exactly to the Newtonian gravitational potential, and the third term is a contribution from angular momentum that takes the same form in Newtonian gravity and general relativity. The last term, the GR contribution, will turn out to make a great deal of difference, especially at small  $r$ .

Let us examine the effective potentials for different kinds of possible orbits, as illustrated in Figures 5.4 and 5.5. There are different curves  $V(r)$  for different values of  $L$ ; for any one of these curves, the behavior of the orbit can be judged by comparing  $\mathcal{E}$  to  $V(r)$ . The general behavior of the particle will be to move in the potential until it reaches a “turning point” where  $V(r) = \mathcal{E}$ , when it will begin



**FIGURE 5.4** Effective potentials in Newtonian gravity. Five curves are shown, corresponding to the listed values of the angular momentum (per unit mass)  $L$ , and we have chosen  $GM = 1$ . Note that, for large enough energy, every orbit reaches a turning point and returns to infinity.

moving in the other direction. Sometimes there may be no turning point to hit, in which case the particle just keeps going. In other cases the particle may simply move in a circular orbit at radius  $r_c = \text{constant}$ ; this can happen at points where the potential is flat,  $dV/dr = 0$ . Differentiating (5.66), we find that the circular orbits occur when

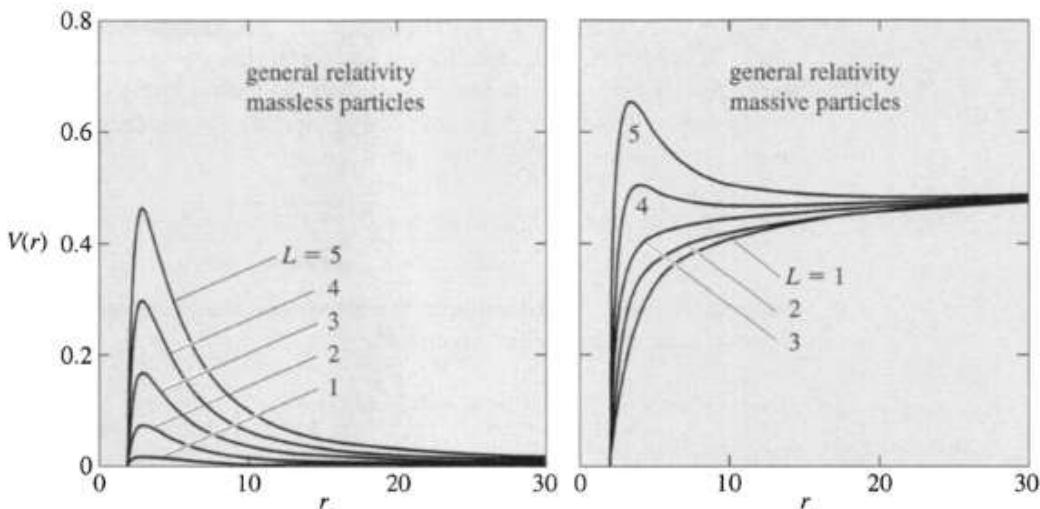
$$\epsilon GM r_c^2 - L^2 r_c + 3GML^2 \gamma = 0, \quad (5.68)$$

where  $\gamma = 0$  in Newtonian gravity and  $\gamma = 1$  in general relativity. Circular orbits will be stable if they correspond to a minimum of the potential, and unstable if they correspond to a maximum. Bound orbits that are not circular will oscillate around the radius of the stable circular orbit.

Turning to Newtonian gravity, we find that circular orbits appear at

$$r_c = \frac{L^2}{\epsilon GM}. \quad (5.69)$$

For massless particles,  $\epsilon = 0$ , and there are no circular orbits; this is consistent with the first plot in Figure 5.4, which illustrates that there are no bound orbits of any sort. Although it is somewhat obscured in polar coordinates, massless particles actually move in a straight line, since the Newtonian gravitational force on a massless particle is zero. Of course the standing of massless particles in Newtonian theory is somewhat problematic, so you can get different answers depending on what assumptions you make. In terms of the effective potential, a photon with a given energy  $E$  will come in from  $r = \infty$  and gradually slow down (actually  $dr/d\lambda$  will decrease, but the speed of light isn't changing) until it reaches the



**FIGURE 5.5** Effective potentials in general relativity. Again, five curves are shown, corresponding to the listed values of the angular momentum (per unit mass)  $L$ , and we have chosen  $GM = 1$ . In GR there is an innermost circular orbit greater than or equal to  $3GM$ , and any orbit that falls inside this radius continues to  $r = 0$  (for particles on geodesics).

turning point, when it will start moving away back to  $r = \infty$ . The lower values of  $L$ , for which the photon will come closer before it starts moving away, are simply those trajectories that are initially aimed closer to the gravitating body. For massive particles there will be stable circular orbits at the radius (5.69), as well as bound orbits that oscillate around this radius. If the energy is greater than the asymptotic value  $E = 1$ , the orbits will be unbound, describing a particle that approaches the star and then recedes. We know that the orbits in Newton's theory are conic sections—bound orbits are either circles or ellipses, while unbound ones are either parabolas or hyperbolas—although we won't show that here.

In general relativity the situation is different, but only for  $r$  sufficiently small. Since the difference resides in the term  $-GML^2/r^3$ , as  $r \rightarrow \infty$  the behaviors are identical in the two theories. But as  $r \rightarrow 0$  the potential goes to  $-\infty$  rather than  $+\infty$  as in the Newtonian case. At  $r = 2GM$  the potential is always zero; inside this radius is the black hole, which we will discuss more thoroughly later. For massless particles there is always a barrier (except for  $L = 0$ , for which the potential vanishes identically), but a sufficiently energetic photon will nevertheless go over the barrier and be dragged inexorably down to the center. Note that “sufficiently energetic” means “in comparison to its angular momentum”—in fact the frequency of the photon is immaterial, only the direction in which it is pointing. At the top of the barrier are unstable circular orbits. For  $\epsilon = 0, \gamma = 1$ , we can easily solve (5.68) to obtain

$$r_c = 3GM. \quad (5.70)$$

This is borne out by the first part of Figure 5.5, which shows a maximum of  $V(r)$  at  $r = 3GM$  for every  $L$ . This means that a photon can orbit forever in a circle at this radius, but any perturbation will cause it to fly away either to  $r = 0$  or  $r = \infty$ .

For massive particles there are once again different regimes depending on the angular momentum. The circular orbits are at

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM}. \quad (5.71)$$

For large  $L$  there will be two circular orbits, one stable and one unstable. In the  $L \rightarrow \infty$  limit their radii are given by

$$r_c = \frac{L^2 \pm L^2(1 - 6G^2M^2/L^2)}{2GM} = \left( \frac{L^2}{GM}, 3GM \right). \quad (5.72)$$

In this limit the stable circular orbit becomes farther away, while the unstable one approaches  $3GM$ , behavior that parallels the massless case. As we decrease  $L$ , the two circular orbits come closer together; they coincide when the discriminant in (5.71) vanishes, which is at

$$L = \sqrt{12GM}, \quad (5.73)$$

for which

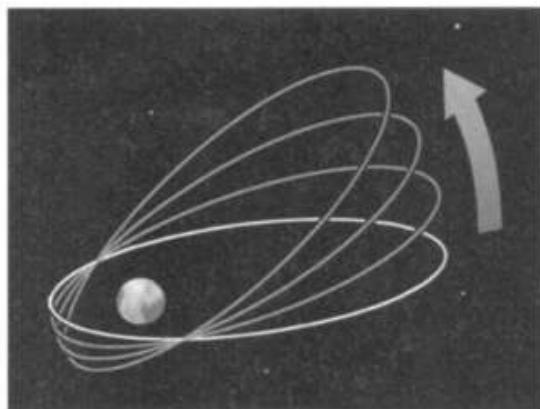
$$r_c = 6GM, \quad (5.74)$$

and they disappear entirely for smaller  $L$ . Thus  $6GM$  is the smallest possible radius of a stable circular orbit in the Schwarzschild metric. There are also unbound orbits, which come in from infinity and turn around, and bound but noncircular orbits, which oscillate around the stable circular radius. Note that such orbits, which would describe exact conic sections in Newtonian gravity, will not do so in GR, although we would have to solve the equation for  $d\phi/d\lambda$  to demonstrate it. Finally, there are orbits that come in from infinity and continue all the way in to  $r = 0$ ; this can happen either if the energy is higher than the barrier, or for  $L < \sqrt{12GM}$ , when the barrier goes away entirely.

We have therefore found that the Schwarzschild solution possesses stable circular orbits for  $r > 6GM$  and unstable circular orbits for  $3GM < r < 6GM$ . It's important to remember that these are only the geodesics; there is nothing to stop an accelerating particle from dipping below  $r = 3GM$  and emerging, as long as it stays beyond  $r = 2GM$ .

## 5.5 ■ EXPERIMENTAL TESTS

Most experimental tests of general relativity involve the motion of test particles in the solar system, and hence geodesics of the Schwarzschild metric. Einstein



**FIGURE 5.6** Orbits in general relativity describe precessing ellipses.

suggested three tests: the deflection of light, the precession of perihelia, and gravitational redshift. The deflection of light is observable in the weak-field limit, and is therefore discussed in Chapter 7. In this section we will discuss the precession of perihelia and the gravitational redshift. (The perihelion of an elliptical orbit is its point of closest approach to the Sun; orbits around the Earth or a star would have perigee or periastron, respectively.)

The precession of perihelia reflects the fact that noncircular orbits in GR are not perfect closed ellipses; to a good approximation they are ellipses that precess, describing a flower pattern as shown in Figure 5.6. Despite its conceptual simplicity, the rate of perihelion precession is somewhat cumbersome to calculate; here we follow d’Inverno (1992). The strategy is to describe the evolution of the radial coordinate  $r$  as a function of the angular coordinate  $\phi$ ; for a perfect ellipse,  $r(\phi)$  would be periodic with period  $2\pi$ , reflecting the fact that perihelion occurred at the same angular position each orbit. Using perturbation theory we can show how GR introduces a slight alteration of the period, giving rise to precession.

We start with our radial equation of motion of a massive particle in a Schwarzschild metric (5.65). To get an equation for  $dr/d\phi$  we multiply by

$$\left(\frac{d\phi}{d\lambda}\right)^{-2} = \frac{r^4}{L^2}, \quad (5.75)$$

which yields

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{L^2}r^4 - \frac{2GM}{L^2}r^3 + r^2 - 2GMr = \frac{2\mathcal{E}}{L^2}r^4. \quad (5.76)$$

Two tricks are useful in solving this equation. The first trick is to define a new variable

$$x = \frac{L^2}{GMr}. \quad (5.77)$$

From (5.69) we see that  $x = 1$  at a Newtonian circular orbit. Our equation of motion (5.76) becomes

$$\left(\frac{dx}{d\phi}\right)^2 + \frac{L^2}{G^2 M^2} - 2x + x^2 - \frac{2G^2 M^2}{L^2} x^3 = \frac{2\mathcal{E} L^2}{G^2 M^2}. \quad (5.78)$$

The second trick is to differentiate this with respect to  $\phi$ , obtaining a second-order equation for  $x(\phi)$ :

$$\frac{d^2 x}{d\phi^2} - 1 + x = \frac{3G^2 M^2}{L^2} x^2. \quad (5.79)$$

In a Newtonian calculation, the last term would be absent, and we could solve for  $x$  exactly; here, we can treat it as a perturbation.

We expand  $x$  into a Newtonian solution plus a small deviation,

$$x = x_0 + x_1. \quad (5.80)$$

The zeroth-order part of (5.79) is then

$$\frac{d^2 x_0}{d\phi^2} - 1 + x_0 = 0 \quad (5.81)$$

and the first-order part is

$$\frac{d^2 x_1}{d\phi^2} + x_1 = \frac{3G^2 M^2}{L^2} x_0^2. \quad (5.82)$$

The solution for the zeroth-order equation can be written

$$x_0 = 1 + e \cos \phi. \quad (5.83)$$

This is the standard result of Newton or Kepler; it describes a perfect ellipse, with  $e$  the eccentricity. An ellipse is specified by the semi-major axis  $a$ , the distance from the center to the farthest point on the ellipse, and the semi-minor axis  $b$ , the distance from the center to the closest point. The eccentricity satisfies  $e^2 = 1 - b^2/a^2$ .

Plugging the Newtonian solution into the first-order equation (5.82), we obtain

$$\begin{aligned} \frac{d^2 x_1}{d\phi^2} + x_1 &= \frac{3G^2 M^2}{L^2} (1 + e \cos \phi)^2 \\ &= \frac{3G^2 M^2}{L^2} \left[ \left(1 + \frac{1}{2} e^2\right) + 2e \cos \phi + \frac{1}{2} e^2 \cos 2\phi \right]. \end{aligned} \quad (5.84)$$

To solve this equation, notice that

$$\frac{d^2}{d\phi^2}(\phi \sin \phi) + \phi \sin \phi = 2 \cos \phi \quad (5.85)$$

and

$$\frac{d^2}{d\phi^2}(\cos 2\phi) + \cos 2\phi = -3 \cos 2\phi. \quad (5.86)$$

Comparing these to (5.84), we see that a solution is provided by

$$x_1 = \frac{3G^2 M^2}{L^2} \left[ \left( 1 + \frac{1}{2} e^2 \right) + e \phi \sin \phi - \frac{1}{6} e^2 \cos 2\phi \right], \quad (5.87)$$

as you are welcome to check. The three terms here have different characters. The first is simply a constant displacement, while the third oscillates around zero. The important effect is thus contained in the second term, which accumulates over successive orbits. We therefore combine this term with the zeroth-order solution to write

$$x = 1 + e \cos \phi + \frac{3G^2 M^2 e}{L^2} \phi \sin \phi. \quad (5.88)$$

This is not a full solution, even to the perturbed equation, but it encapsulates the part that we care about. In particular, this expression for  $x$  can be conveniently rewritten as the equation for an ellipse with an angular period that is not quite  $2\pi$ :

$$x = 1 + e \cos [(1 - \alpha)\phi], \quad (5.89)$$

where we have introduced

$$\alpha = \frac{3G^2 M^2}{L^2}. \quad (5.90)$$

The equivalence of (5.88) and (5.89) can be seen by expanding  $\cos[(1 - \alpha)\phi]$  as a power series in the small parameter  $\alpha$ :

$$\begin{aligned} \cos[(1 - \alpha)\phi] &= \cos \phi + \alpha \frac{d}{d\alpha} \cos[(1 - \alpha)\phi]_{\alpha=0} \\ &= \cos \phi + \alpha \phi \sin \phi. \end{aligned} \quad (5.91)$$

We have therefore found that, during each orbit of the planet, perihelion advances by an angle

$$\Delta\phi = 2\pi\alpha = \frac{6\pi G^2 M^2}{L^2}. \quad (5.92)$$

To convert from the angular momentum  $L$  to more conventional quantities, we may use expressions valid for Newtonian orbits, since the quantity we're looking

at is already a small perturbation. An ordinary ellipse satisfies

$$r = \frac{(1 - e^2)a}{1 + e \cos \phi}, \quad (5.93)$$

where  $a$  is the semi-major axis. Comparing to our zeroth-order solution (5.83) and the definition (5.77) of  $x$ , we see that

$$L^2 \approx GM(1 - e^2)a. \quad (5.94)$$

This is an approximation, valid if the orbit were a perfect closed ellipse. Plugging this into (5.92) and restoring explicit factors of the speed of light, we obtain

$$\Delta\phi = \frac{6\pi GM}{c^2(1 - e^2)a}. \quad (5.95)$$

Historically, the precession of Mercury was the first test of GR. In fact it was known before Einstein invented GR that there was an apparent discrepancy in Mercury's orbit, and a number of solutions had been proposed (including "dark matter" in the inner Solar System). Einstein knew of the discrepancy, and one of his first tasks after formulating GR was to show that it correctly accounted for Mercury's perihelion precession. For the motion of Mercury around the Sun, the relevant orbital parameters are

$$\begin{aligned} \frac{GM_{\odot}}{c^2} &= 1.48 \times 10^5 \text{ cm,} \\ a &= 5.79 \times 10^{12} \text{ cm} \\ e &= 0.2056, \end{aligned} \quad (5.96)$$

and of course  $c = 3.00 \times 10^{10}$  cm/sec. This gives

$$\Delta\phi_{\text{Mercury}} = 5.01 \times 10^{-7} \text{ radians/orbit} = 0.103''/\text{orbit}, \quad (5.97)$$

where " stands for arcseconds. It is more conventional to express this in terms of precession per century; Mercury orbits once every 88 days, yielding

$$\Delta\phi_{\text{Mercury}} = 43.0''/\text{century}. \quad (5.98)$$

So the major axis of Mercury's orbit precesses at a rate of 43.0 arcsecs every 100 years. The observed value is 5601 arcsecs/100 years. However, much of that is due to the precession of equinoxes in our geocentric coordinate system; 5025 arcsecs/100 years, to be precise. The gravitational perturbations of the other planets contribute an additional 532 arcsecs/100 years, leaving 43 arcsecs/100 years to be explained by GR, which it does quite well. You can imagine that Einstein must have been very pleased when he first figured this out.

In Chapter 2 we discussed the gravitational redshift of photons as a consequence of the Principle of Equivalence. The Schwarzschild metric is an exact

solution of GR, and should therefore predict a redshift that reduces to the EP prediction in small regions of spacetime. Let's see how that works.

Consider an observer with four-velocity  $U^\mu$ , who is stationary in the Schwarzschild coordinates ( $U^t = 0$ ). We could allow the observer to be moving, but that would merely superimpose a conventional Doppler shift over the gravitational effect. The four-velocity satisfies  $U_\mu U^\mu = -1$ , which for a stationary observer in Schwarzschild implies

$$U^0 = \left(1 - \frac{2GM}{r}\right)^{-1/2}. \quad (5.99)$$

Any such observer measures the frequency of a photon following along a null geodesic  $x^\mu(\lambda)$  to be

$$\omega = -g_{\mu\nu} U^\mu \frac{dx^\nu}{d\lambda}. \quad (5.100)$$

Indeed, this relation defines the normalization of  $\lambda$ . We therefore have

$$\omega = \left(1 - \frac{2GM}{r}\right)^{1/2} \frac{dt}{d\lambda} \quad (5.101)$$

$$= \left(1 - \frac{2GM}{r}\right)^{-1/2} E, \quad (5.102)$$

where  $E$  is defined by (5.61), applied to the photon trajectory.  $E$  is conserved, so  $\omega$  will clearly take on different values when measured at different radial distances. For a photon emitted at  $r_1$  and observed at  $r_2$ , the observed frequencies will be related by

$$\frac{\omega_2}{\omega_1} = \left(\frac{1-2GM/r_1}{1-2GM/r_2}\right)^{1/2}. \quad (5.103)$$

This is an exact result for the frequency shift; in the limit  $r \gg 2GM$  we have

$$\begin{aligned} \frac{\omega_2}{\omega_1} &= 1 - \frac{GM}{r_1} + \frac{GM}{r_2} \\ &= 1 + \Phi_1 - \Phi_2, \end{aligned} \quad (5.104)$$

where  $\Phi = -GM/r$  is the Newtonian potential. This tells us that the frequency goes down as  $\Phi$  increases, which happens as we climb out of a gravitational field; thus, a redshift. (Photons that fall toward a gravitating body are blueshifted.) We see that the  $r \gg 2GM$  result agrees with the calculation based on the Equivalence Principle.

The gravitational redshift was first detected in 1960 by Pound and Rebka, using gamma rays traveling upward a distance of only 72 feet (the height of the physics building at Harvard). Subsequent tests have become increasingly precise, often

making use of artificial spacecraft or atomic clocks carried aboard airplanes. The agreement with Einstein's predictions has been excellent in all cases.

Since Einstein's proposal of the three classic tests, further tests of GR have been proposed. The most famous is of course the binary pulsar, to be discussed in Chapter 7. Another is the gravitational time delay, discovered and observed by Shapiro, also discussed in Chapter 7. In a very different context, Big-Bang nucleosynthesis provides a cosmological test of GR at an epoch when the universe was only seconds old, as discussed in Chapter 8. Modern advances have also introduced a host of new tests; for a comprehensive introduction see Will (1981).

## 5.6 ■ SCHWARZSCHILD BLACK HOLES

We now know something about the behavior of geodesics outside the troublesome radius  $r = 2GM$ , which is the regime of interest for the solar system and most other astrophysical situations. We next turn to the study of objects that are described by the Schwarzschild solution even at radii smaller than  $2GM$ —black holes. (We'll use the term “black hole” for the moment, even though we haven't introduced a precise meaning for such an object.)

One way to understand the geometry of a spacetime is to explore its causal structure, as defined by the light cones. We therefore consider radial null curves, those for which  $\theta$  and  $\phi$  are constant and  $ds^2 = 0$ :

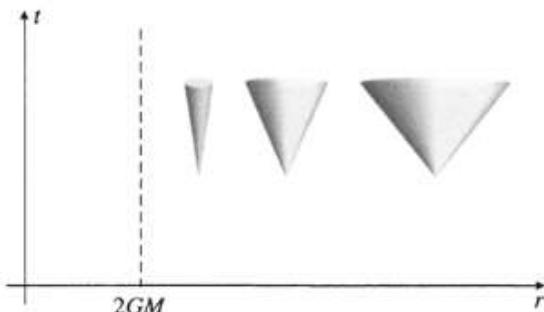
$$ds^2 = 0 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2, \quad (5.105)$$

from which we see that

$$\frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1}. \quad (5.106)$$

This of course measures the slope of the light cones on a spacetime diagram of the  $t$ - $r$  plane. For large  $r$  the slope is  $\pm 1$ , as it would be in flat space, while as we approach  $r = 2GM$  we get  $dt/dr \rightarrow \pm\infty$ , and the light cones “close up,” as shown in Figure 5.7. Thus a light ray that approaches  $r = 2GM$  never seems to get there, at least in this coordinate system; instead it seems to asymptote to this radius.

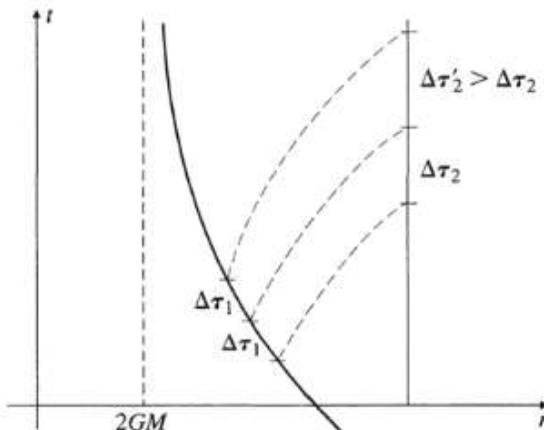
As we will see, the apparent inability to get to  $r = 2GM$  is an illusion, and the light ray (or a massive particle) actually has no trouble reaching this radius. But an observer far away would never be able to tell. If we stayed outside while an intrepid observational general relativist dove into the black hole, sending back signals all the time, we would simply see the signals reach us more and more slowly, as portrayed in Figure 5.8. In the Exercises you are asked to look at this phenomenon more carefully. As an infalling observer approaches  $r = 2GM$ , any fixed interval  $\Delta\tau_1$  of their proper time corresponds to a longer and longer interval  $\Delta\tau_2$  from our point of view. This continues forever; we would never see



**FIGURE 5.7** In Schwarzschild coordinates, light cones appear to close up as we approach  $r = 2GM$ .

the observer crosses  $r = 2GM$ , we would just see them move more and more slowly (and become redder and redder, as if embarrassed to have done something as stupid as diving into a black hole).

The fact that we never see the infalling observer reach  $r = 2GM$  is a meaningful statement, but the fact that their trajectory in the  $t$ - $r$  plane never reaches there is not. It is highly dependent on our coordinate system, and we would like to ask a more coordinate-independent question (such as, “Does the observer reach this radius in a finite amount of their proper time?”). The best way to do this is to change coordinates to a system that is better behaved at  $r = 2GM$ . We now set out to find an appropriate set of such coordinates. There is no way to “derive” a coordinate transformation, of course, we just say what the new coordinates are and plug in the formulas. But we will develop these coordinates in several steps, in hopes of making the choices seem somewhat motivated.



**FIGURE 5.8** A beacon falling freely into a black hole emits signals at intervals of constant proper time  $\Delta\tau_1$ . An observer at fixed  $r$  receives the signals at successively longer time intervals  $\Delta\tau_2$ .

The problem with our current coordinates is that  $dt/dr \rightarrow \infty$  along radial null geodesics that approach  $r = 2GM$ ; progress in the  $r$  direction becomes slower and slower with respect to the coordinate time  $t$ . We can try to fix this problem by replacing  $t$  with a coordinate that moves more slowly along null geodesics. First notice that we can explicitly solve the condition (5.106) characterizing radial null curves to obtain

$$t = \pm r^* + \text{constant}, \quad (5.107)$$

where the **tortoise coordinate**  $r^*$  is defined by

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right). \quad (5.108)$$

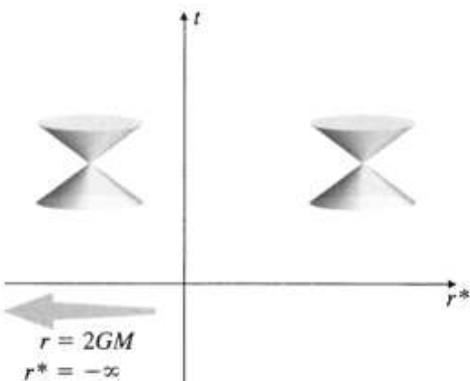
(The tortoise coordinate is only sensibly related to  $r$  when  $r \geq 2GM$ , but beyond there our coordinates aren't very good anyway.) In terms of the tortoise coordinate the Schwarzschild metric becomes

$$ds^2 = \left(1 - \frac{2GM}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 d\Omega^2, \quad (5.109)$$

where  $r$  is thought of as a function of  $r^*$ . This represents some progress, since the light cones now don't seem to close up, as shown in Figure 5.9; furthermore, none of the metric coefficients becomes infinite at  $r = 2GM$  (although both  $g_{tt}$  and  $g_{rr^*r^*}$  become zero). The price we pay, however, is that the surface of interest at  $r = 2GM$  has just been pushed to infinity.

Our next move is to define coordinates that are naturally adapted to the null geodesics. If we let

$$\begin{aligned} v &= t + r^* \\ u &= t - r^*, \end{aligned} \quad (5.110)$$



**FIGURE 5.9** Schwarzschild light cones in tortoise coordinates, equation (5.109). Light cones remain nondegenerate, but the surface  $r = 2GM$  has been pushed to infinity.

then infalling radial null geodesics are characterized by  $v = \text{constant}$ , while the outgoing ones satisfy  $u = \text{constant}$ . Now consider going back to the original radial coordinate  $r$ , but replacing the timelike coordinate  $t$  with the new coordinate  $v$ . These are known as **Eddington–Finkelstein coordinates**. In terms of these coordinates the metric is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + (dv dr + dr dv) + r^2 d\Omega^2. \quad (5.111)$$

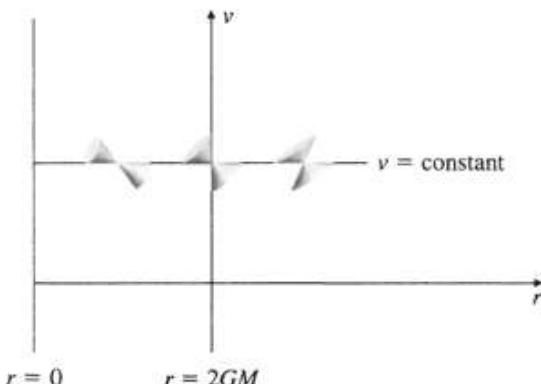
Here we see our first sign of real progress. Even though the metric coefficient  $g_{vv}$  vanishes at  $r = 2GM$ , there is no real degeneracy; the determinant of the metric is

$$g = -r^4 \sin^2 \theta, \quad (5.112)$$

which is perfectly regular at  $r = 2GM$ . Therefore the metric is invertible, and we see once and for all that  $r = 2GM$  is simply a coordinate singularity in our original  $(t, r, \theta, \phi)$  system. In the Eddington–Finkelstein coordinates the condition for radial null curves is solved by

$$\frac{dv}{dr} = \begin{cases} 0, & (\text{infalling}) \\ 2\left(1 - \frac{2GM}{r}\right)^{-1}, & (\text{outgoing}) \end{cases} \quad (5.113)$$

We can therefore see what has happened: In this coordinate system the light cones remain well-behaved at  $r = 2GM$ , and this surface is at a finite coordinate value. There is no problem in tracing the paths of null or timelike particles past the surface. On the other hand, something interesting is certainly going on. Although the light cones don't close up, they do tilt over, such that for  $r < 2GM$  all future-directed paths are in the direction of decreasing  $r$ , as shown in Figure 5.10.



**FIGURE 5.10** Schwarzschild light cones in the  $(v, r)$  coordinates of (5.111). In these coordinates we can follow future-directed timelike paths past  $r = 2GM$ .

The surface  $r = 2GM$ , while being locally perfectly regular, globally functions as a point of no return—once a test particle dips below it, it can never come back. We define an **event horizon** to be a surface past which particles can never escape to infinity; in Schwarzschild the event horizon is located at  $r = 2GM$ . (This is a rough definition; we will be somewhat more precise in the next chapter.) Despite being located at fixed radial coordinate, the event horizon is a null surface rather than a timelike one, so it is really the causal structure of spacetime itself that makes it impossible to cross the horizon in an outward-going direction. Since nothing can escape the event horizon, it is impossible for us to see inside—thus the name **black hole**. A black hole is simply a region of spacetime separated from infinity by an event horizon. The notion of an event horizon is a global one; the location of the horizon is a statement about the spacetime as a whole, not something you could determine just by knowing the geometry at that location. This will continue to be true in more general spacetimes.

We should mention a couple of features of black holes that sometimes get confused in the popular imagination. First, the external geometry of a black hole is the same Schwarzschild solution that we would have outside a star or planet. In particular, a black hole does not suck in everything around it any more than the Sun does; a particle well outside  $r = 2GM$  behaves in exactly the same way regardless of whether the gravitating source is a black hole or not. Second, there is a misleading Newtonian analogy for black holes. The Newtonian escape velocity of a particle at distance  $r$  from a gravitating body of mass  $M$  is

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}. \quad (5.114)$$

If we naively ask where the Newtonian escape velocity equals the velocity of light, we find exactly  $r = 2GM$ . Despite the fact that the speed of light plays no fundamental role in Newtonian theory, it might seem provocative that light, thought of as inertial particles moving at a velocity  $c$ , is seemingly not able to escape from a body with mass  $M$  and radius less than  $2GM$ . But there is a profound difference between this case and what we see in GR. The escape velocity is the velocity that a particle would initially need to have in order to escape from a gravitating source on a free trajectory. But nothing stops us from considering accelerated trajectories; for example, one could imagine an acceleration chosen such that the particle moved steadily away from the massive body at some constant velocity. Therefore, a purported Newtonian black hole would not have the crucial property that *nothing* can escape; whereas in GR, arbitrary timelike paths must stay inside their light cones, and hence never escape the event horizon.

## 5.7 ■ THE MAXIMALLY EXTENDED SCHWARZSCHILD SOLUTION

Let's review what we have done. Acting under the suspicion that our coordinates may not have been good for the entire manifold, we have changed from our original coordinate  $t$  to the new one  $v$ , which has the nice property that if we decrease

$r$  along a radial null curve  $v = \text{constant}$ , we go right through the event horizon without any problems. Indeed, a local observer actually making the trip would not necessarily know when the event horizon had been crossed—the local geometry is no different from anywhere else. We therefore conclude that our suspicion was correct and our initial coordinate system didn't do a good job of covering the entire manifold. The region  $r \leq 2GM$  should certainly be included in our spacetime, since physical particles can easily reach there and pass through. However, there is no guarantee that we are finished; perhaps we can extend our manifold in other directions.

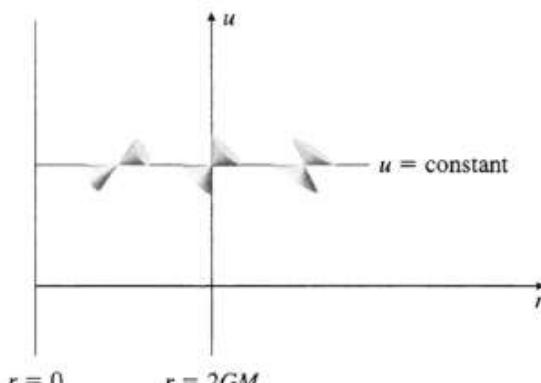
In fact there are other directions. In the  $(v, r)$  coordinate system we can cross the event horizon on future-directed paths, but not on past-directed ones. This seems unreasonable, since we started with a time-independent solution. But we could have chosen  $u$  instead of  $v$ , in which case the metric would have been

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du^2 - (du dr + dr du) + r^2 d\Omega^2. \quad (5.115)$$

Now we can once again pass through the event horizon, but this time only along past-directed curves, as shown in Figure 5.11.

This is perhaps a surprise: we can consistently follow either future-directed or past-directed curves through  $r = 2GM$ , but we arrive at different places. It was actually to be expected, since from the definitions (5.110), if we keep  $v$  constant and decrease  $r$  we must have  $t \rightarrow +\infty$ , while if we keep  $u$  constant and decrease  $r$  we must have  $t \rightarrow -\infty$ . (The tortoise coordinate  $r^*$  goes to  $-\infty$  as  $r \rightarrow 2GM$ .) So we have extended spacetime in two different directions, one to the future and one to the past.

The next step would be to follow spacelike geodesics to see if we would uncover still more regions. The answer is yes, we would reach yet another piece of the spacetime, but let's shortcut the process by defining coordinates that are good all over. A first guess might be to use both  $u$  and  $v$  at once (in place of  $t$  and  $r$ ),



**FIGURE 5.11** Schwarzschild light cones in the  $(u, r)$  coordinates of (5.115). In these coordinates we can follow past-directed timelike paths past  $r = 2GM$ .

which leads to

$$ds^2 = -\frac{1}{2} \left(1 - \frac{2GM}{r}\right) (dv du + du dv) + r^2 d\Omega^2, \quad (5.116)$$

with  $r$  defined implicitly in terms of  $v$  and  $u$  by

$$\frac{1}{2}(v - u) = r + 2GM \ln \left( \frac{r}{2GM} - 1 \right). \quad (5.117)$$

We have actually reintroduced the degeneracy with which we started out; in these coordinates  $r = 2GM$  is “infinitely far away” (at either  $v = -\infty$  or  $u = +\infty$ ). The thing to do is to change to coordinates that pull these points into finite coordinate values; a good choice is

$$\begin{aligned} v' &= e^{v/4GM} \\ u' &= -e^{-u/4GM}, \end{aligned} \quad (5.118)$$

which in terms of our original  $(t, r)$  system is

$$\begin{aligned} v' &= \left( \frac{r}{2GM} - 1 \right)^{1/2} e^{(r+t)/4GM} \\ u' &= -\left( \frac{r}{2GM} - 1 \right)^{1/2} e^{(r-t)/4GM}. \end{aligned} \quad (5.119)$$

In the  $(v', u', \theta, \phi)$  system the Schwarzschild metric is

$$ds^2 = -\frac{16G^3M^3}{r} e^{-r/2GM} (dv' du' + du' dv') + r^2 d\Omega^2. \quad (5.120)$$

Finally the nonsingular nature of  $r = 2GM$  becomes completely manifest; in this form none of the metric coefficients behaves in any special way at the event horizon.

Both  $v'$  and  $u'$  are null coordinates, in the sense that their partial derivatives  $\partial/\partial v'$  and  $\partial/\partial u'$  are null vectors. There is nothing wrong with this, since the collection of four partial derivative vectors (two null and two spacelike) in this system serve as a perfectly good basis for the tangent space. Nevertheless, we are somewhat more comfortable working in a system where one coordinate is timelike and the rest are spacelike. We therefore define

$$T = \frac{1}{2}(v' + u') = \left( \frac{r}{2GM} - 1 \right)^{1/2} e^{r/4GM} \sinh \left( \frac{t}{4GM} \right) \quad (5.121)$$

and

$$R = \frac{1}{2}(v' - u') = \left( \frac{r}{2GM} - 1 \right)^{1/2} e^{r/4GM} \cosh \left( \frac{t}{4GM} \right), \quad (5.122)$$

in terms of which the metric becomes

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM} (-dT^2 + dR^2) + r^2 d\Omega^2, \quad (5.123)$$

where  $r$  is defined implicitly from

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}. \quad (5.124)$$

The coordinates  $(T, R, \theta, \phi)$  are known as **Kruskal coordinates**, or sometimes Kruskal–Szekeres coordinates.

The Kruskal coordinates have a number of miraculous properties. Like the  $(t, r^*)$  coordinates, the radial null curves look like they do in flat space:

$$T = \pm R + \text{constant}. \quad (5.125)$$

Unlike the  $(t, r^*)$  coordinates, however, the event horizon  $r = 2GM$  is not infinitely far away; in fact it is defined by

$$T = \pm R, \quad (5.126)$$

consistent with it being a null surface. More generally, we can consider the surfaces  $r = \text{constant}$ . From (5.124) these satisfy

$$T^2 - R^2 = \text{constant}. \quad (5.127)$$

Thus, they appear as hyperbolae in the  $R$ - $T$  plane. Furthermore, the surfaces of constant  $t$  are given by

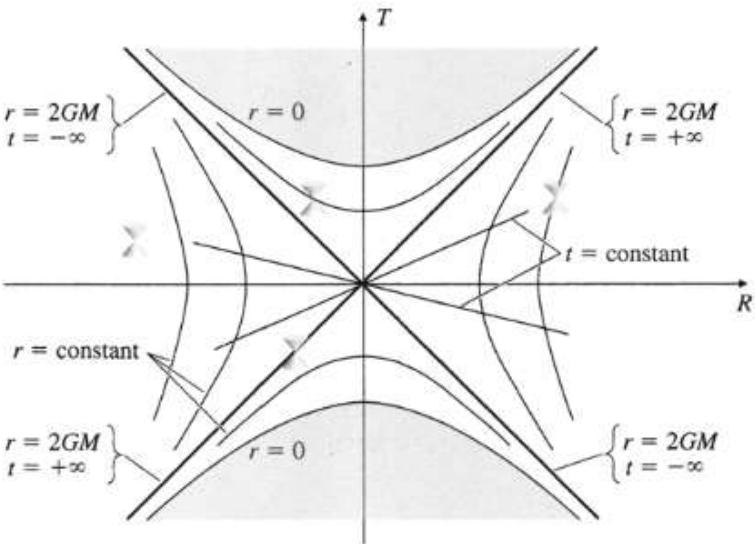
$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right), \quad (5.128)$$

which defines straight lines through the origin with slope  $\tanh(t/4GM)$ . Note that as  $t \rightarrow \pm\infty$  (5.128) becomes the same as (5.126); therefore  $t = \pm\infty$  represents the same surface as  $r = 2GM$ .

Our coordinates  $(T, R)$  should be allowed to range over every value they can take without hitting the real singularity at  $r = 0$ ; the allowed region is therefore

$$\begin{aligned} -\infty &\leq R \leq \infty \\ T^2 &< R^2 + 1. \end{aligned} \quad (5.129)$$

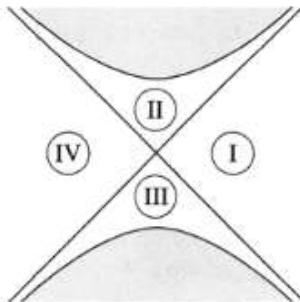
From (5.121) and (5.122),  $T$  and  $R$  seem to become imaginary for  $r < 2GM$ , but this is an illusion; in that region the  $(r, t)$  coordinates are no good (specifically,  $|t| > \infty$ ). We can now draw a spacetime diagram in the  $T$ - $R$  plane (with  $\theta$  and  $\phi$  suppressed), known as a **Kruskal diagram**, shown in Figure 5.12. Each point on the diagram is a two-sphere. This diagram represents the maximal extension



**FIGURE 5.12** The Kruskal diagram—the Schwarzschild solution in Kruskal coordinates, where all light cones are at  $\pm 45^\circ$ .

of the Schwarzschild geometry; the coordinates cover what we should think of as the entire manifold described by this solution.

The original Schwarzschild coordinates  $(t, r)$  were good for  $r > 2GM$ , which is only a part of the manifold portrayed on the Kruskal diagram. It is convenient to divide the diagram into four regions, as shown in Figure 5.13. Region I corresponds to  $r > 2GM$ , the patch in which our original coordinates were well-defined. By following future-directed null rays we reach region II, and by following past-directed null rays we reach region III. If we had explored space-like geodesics, we would have been led to region IV. The definitions (5.121) and (5.122), which relate  $(T, R)$  to  $(t, r)$ , are really only good in region I; in the other regions it is necessary to introduce appropriate minus signs to prevent the coordinates from becoming imaginary.



**FIGURE 5.13** Regions of the Kruskal diagram.

Having extended the Schwarzschild geometry as far as it will go, we have described a remarkable spacetime. Region II, of course, is what we think of as the black hole. Once anything travels from region I into II, it can never return. In fact, every future-directed path in region II ends up hitting the singularity at  $r = 0$ ; once you enter the event horizon, you are utterly doomed. This is worth stressing; not only can you not escape back to region I, you cannot even stop yourself from moving in the direction of decreasing  $r$ , since this is simply the timelike direction. This could have been seen in our original coordinate system; for  $r < 2GM$ ,  $t$  becomes spacelike and  $r$  becomes timelike. Thus you can no more stop moving toward the singularity than you can stop getting older. Since proper time is maximized along a geodesic, you will live the longest if you don't struggle, but just relax as you approach the singularity. Not that you will have long to relax, nor will the voyage be very relaxing; as you approach the singularity the tidal forces become infinite. As you fall toward the singularity your feet and head will be pulled apart from each other, while your torso is squeezed to infinitesimal thinness. The grisly demise of an astrophysicist falling into a black hole is detailed in Misner, Thorne, and Wheeler (1973), Section 32.6. Note that they use orthonormal frames, as we discuss in Appendix J (not that it makes the trip any more enjoyable).

Regions III and IV might be somewhat unexpected. Region III is simply the time-reverse of region II, a part of spacetime from which things can escape to us, while we can never get there. It can be thought of as a **white hole**. There is a singularity in the past, out of which the universe appears to spring. The boundary of region III is the past event horizon, while the boundary of region II is the future event horizon. Region IV, meanwhile, cannot be reached from our region I either forward or backward in time, nor can anybody from over there reach us. It is another asymptotically flat region of spacetime, a mirror image of ours. It can be thought of as being connected to region I by a wormhole (or Einstein-Rosen bridge), a neck-like configuration joining two distinct regions. Consider slicing up the Kruskal diagram into spacelike surfaces of constant  $T$ , as shown in Figure 5.14. Now we can draw pictures of each slice, restoring one of the angular coordinates for clarity, as in Figure 5.15. In this way of slicing, the Schwarzschild geometry describes two asymptotically flat regions that reach toward each other, join together via a wormhole for a while, and then disconnect. But the wormhole closes up too quickly for any timelike observer to cross it from one region into the next.

As pleasing as the Kruskal diagram is, it is often even more useful to collapse the Schwarzschild solution into a finite region by constructing its conformal diagram. The idea of a conformal diagram is discussed in Appendix H; it is a crucial tool for analyzing spacetimes in general relativity, and you are encouraged to review that discussion now. We will not go through the manipulations necessary to construct the conformal diagram of Schwarzschild in full detail, since they parallel the Minkowski case with considerable additional algebraic complexity. We would start with the null version of the Kruskal coordinates, in which the metric

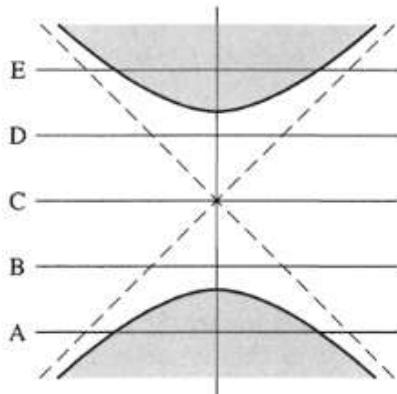


FIGURE 5.14 Spacelike slices in Kruskal coordinates.

takes the form

$$ds^2 = -\frac{16G^3M^3}{r}e^{-r/2GM}(dv'du' + du'dv') + r^2 d\Omega^2, \quad (5.130)$$

where  $r$  is defined implicitly via

$$v'u' = -\left(\frac{r}{2GM} - 1\right)e^{r/2GM}. \quad (5.131)$$

Then essentially the same transformation used in the flat spacetime case suffices to bring infinity into finite coordinate values:

$$\begin{aligned} v'' &= \arctan\left(\frac{v'}{\sqrt{2GM}}\right) \\ u'' &= \arctan\left(\frac{u'}{\sqrt{2GM}}\right). \end{aligned} \quad (5.132)$$

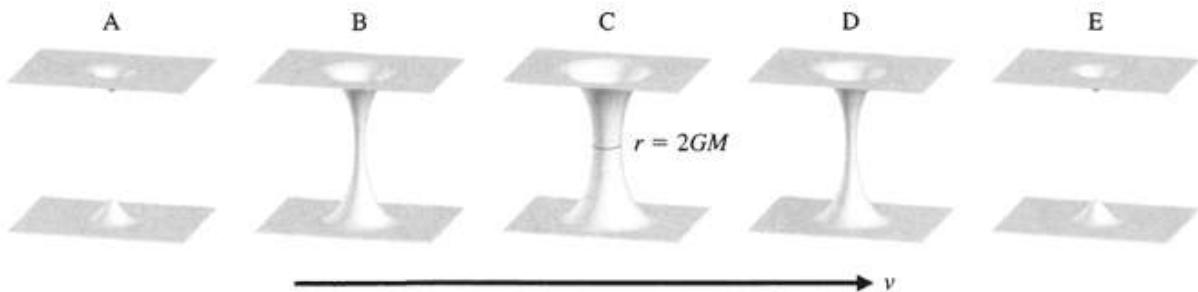
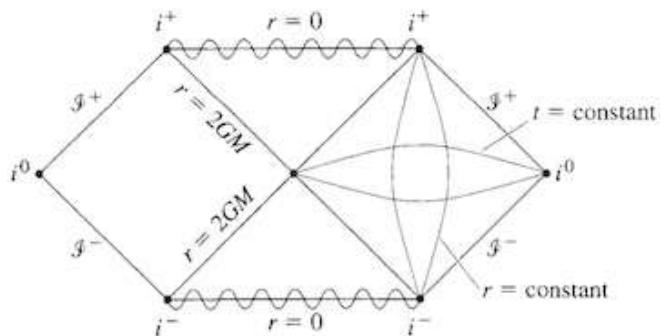


FIGURE 5.15 Geometry of the spacelike slices in Figure 5.14.



**FIGURE 5.16** Conformal diagram for Schwarzschild spacetime.

with ranges

$$\begin{aligned} -\frac{\pi}{2} < v'' &< +\frac{\pi}{2} \\ -\frac{\pi}{2} < u'' &< +\frac{\pi}{2} \\ -\frac{\pi}{2} < v'' + u'' &< \frac{\pi}{2}. \end{aligned}$$

The  $(v'', u'')$  part of the metric (that is, at constant angular coordinates) is now conformally related to Minkowski space. In the new coordinates the singularities at  $r = 0$  are straight lines that stretch from timelike infinity in one asymptotic region to timelike infinity in the other.

The conformal diagram for the maximally extended Schwarzschild solution thus looks like Figure 5.16. The only real subtlety about this diagram is the necessity to understand that  $i^+$  and  $i^-$  (future and past infinity) are distinct from  $r = 0$ —there are plenty of timelike paths that do not hit the singularity. As in the Kruskal diagram, light cones in the conformal diagram are at  $45^\circ$ ; the major difference is that the entire spacetime is represented in a finite region. Notice also that the structure of conformal infinity is just like that of Minkowski space, consistent with the claim that Schwarzschild is asymptotically flat.

## 5.8 ■ STARS AND BLACK HOLES

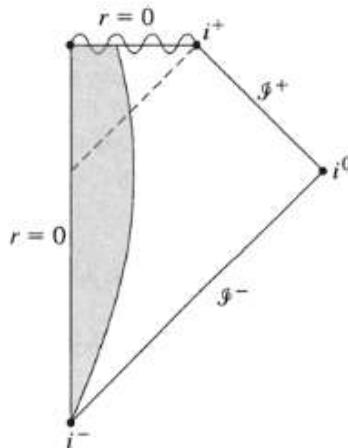
The maximally extended Schwarzschild solution we have just constructed tells a remarkable story, including not only the sought-after black hole, but also a white hole and an additional asymptotically flat region, connected to our universe by a wormhole. It would be premature, however, to imagine that such features are common in the real world. The Schwarzschild solution represents a highly idealized situation: not only spherically symmetric, but completely free of energy-momentum throughout spacetime. Birkhoff's theorem implies that any vacuum

region of a spherically symmetric spacetime will be described by *part of* the Schwarzschild metric, but the existence of matter somewhere in the universe may dramatically alter the global picture.

A static spherical object—let's call it a star for definiteness—with radius larger than  $2GM$  will be Schwarzschild in the exterior, but there won't be any singularities or horizons, and the global structure will actually be very similar to Minkowski spacetime. Of course, real stars evolve, and it may happen that a star eventually collapses under its own gravitational pull, shrinking down to below  $r = 2GM$  and further into a singularity, resulting in a black hole. There is no need for a white hole, however, because the past of such a spacetime looks nothing like that of the full Schwarzschild solution. A conformal diagram describing stellar collapse would look like Figure 5.17. The interior shaded region is nonvacuum, so is not described by Schwarzschild; in particular, there is no wormhole connecting to another universe. It is asymptotically Minkowskian, except for a future region giving rise to an event horizon. We see that a realistic black hole may share the singularity and future horizon with the maximally extended Schwarzschild solution, without any white hole, past horizon, or separate asymptotic region.

We believe that gravitational collapse of this kind is by no means a necessary endpoint of stellar evolution, but will occur under certain conditions. General relativity places rigorous limits on the kind of stars that can resist gravitational collapse; for any given sort of matter, enough mass will always lead to the collapse to a black hole. Furthermore, from astrophysical observations we have excellent evidence that black holes exist in our universe.

To understand gravitational collapse to a black hole, we should first understand static configurations describing the interiors of spherically symmetric stars. We won't delve into this subject in detail, only enough to get a feeling for the basic features of interior solutions. Consider the general static, spherically symmetric



**FIGURE 5.17** Conformal diagram for a black hole formed from a collapsing star. The shaded region contains matter, and will be described by an appropriate dynamical interior solution; the exterior region is Schwarzschild.

metric from (5.11):

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2. \quad (5.133)$$

We are now looking for nonvacuum solutions, so we turn to the full Einstein equation,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (5.134)$$

The Einstein tensor follows from the Ricci tensor (5.14) and curvature scalar (5.15),

$$\begin{aligned} G_{tt} &= \frac{1}{r^2} e^{2(\alpha-\beta)} \left( 2r\partial_r\beta - 1 + e^{2\beta} \right) \\ G_{rr} &= \frac{1}{r^2} \left( 2r\partial_r\alpha + 1 - e^{2\beta} \right) \\ G_{\theta\theta} &= r^2 e^{-2\beta} \left[ \partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{1}{r}(\partial_r\alpha - \partial_r\beta) \right] \\ G_{\phi\phi} &= \sin^2\theta G_{\theta\theta}, \end{aligned} \quad (5.135)$$

We model the star itself as a perfect fluid, with energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}. \quad (5.136)$$

The energy density  $\rho$  and pressure  $p$  will be functions of  $r$  alone. Since we seek static solutions, we can take the four-velocity to be pointing in the timelike direction. Normalized to  $U^\mu U_\mu = -1$ , it becomes

$$U_\mu = (e^\alpha, 0, 0, 0), \quad (5.137)$$

so that the components of the energy-momentum tensor are

$$T_{\mu\nu} = \begin{pmatrix} e^{2\alpha}\rho & & & \\ & e^{2\beta}p & & \\ & & r^2p & \\ & & & r^2(\sin^2\theta)p \end{pmatrix}. \quad (5.138)$$

We therefore have three independent components of Einstein's equation: the  $tt$  component,

$$\frac{1}{r^2} e^{-2\beta} \left( 2r\partial_r\beta - 1 + e^{2\beta} \right) = 8\pi G\rho, \quad (5.139)$$

the  $rr$  component,

$$\frac{1}{r^2} e^{-2\beta} \left( 2r\partial_r\alpha + 1 - e^{2\beta} \right) = 8\pi Gp, \quad (5.140)$$

and the  $\theta\theta$  component,

$$e^{-2\beta} \left[ \partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{1}{r} (\partial_r \alpha - \partial_r \beta) \right] = 8\pi G\rho. \quad (5.141)$$

The  $\phi\phi$  equation is proportional to the  $\theta\theta$  equation, so there is no need to consider it separately.

We notice that the  $tt$  equation (5.139) involves only  $\beta$  and  $\rho$ . It is convenient to replace  $\beta(r)$  with a new function  $m(r)$ , given by

$$m(r) = \frac{1}{2G}(r - re^{-2\beta}), \quad (5.142)$$

or equivalently

$$e^{2\beta} = \left[ 1 - \frac{2Gm(r)}{r} \right]^{-1}, \quad (5.143)$$

so that

$$ds^2 = -e^{2\alpha(r)} dt^2 + \left[ 1 - \frac{2Gm(r)}{r} \right]^{-1} dr^2 + r^2 d\Omega^2. \quad (5.144)$$

The metric component  $g_{rr}$  is an obvious generalization of the Schwarzschild case, but this will not be true for  $g_{tt}$ . The  $tt$  equation (5.139) becomes

$$\frac{dm}{dr} = 4\pi r^2 \rho, \quad (5.145)$$

which can be integrated to obtain

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'. \quad (5.146)$$

Let's imagine that our star extends to a radius  $R$ , after which we are in vacuum and described by Schwarzschild. In order that the metrics match at this radius, the Schwarzschild mass  $M$  must be given by

$$M = m(R) = 4\pi \int_0^R \rho(r) r^2 dr. \quad (5.147)$$

It looks like  $m(r)$  is simply the integral of the energy density over the stellar interior, and can be interpreted as the mass within a radius  $r$ .

There is one subtlety with interpreting  $m(r)$  as the integrated energy density; in a proper spatial integral, the volume element should be

$$\sqrt{\gamma} d^3x = e^\beta r^2 \sin\theta dr d\theta d\phi, \quad (5.148)$$

where

$$\gamma_{ij} dx^i dx^j = e^{2\beta} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad (5.149)$$

is the spatial metric. The true integrated energy density is therefore

$$\begin{aligned}\hat{M} &= 4\pi \int_0^R \rho(r) r^2 e^{\beta(r)} dr \\ &= 4\pi \int_0^R \frac{\rho(r) r^2}{\left[1 - \frac{2Gm(r)}{r}\right]^{1/2}} dr.\end{aligned}\quad (5.150)$$

The difference, of course, arises because there is a binding energy due to the mutual gravitational attraction of the fluid elements in the star, which is given by

$$E_B = \hat{M} - M > 0. \quad (5.151)$$

The binding energy is the amount of energy that would be required to disperse the matter in the star to infinity. It is not always a well-defined notion in general relativity, but makes sense for spherical stars.

In terms of  $m(r)$ , the  $rr$  equation (5.140) can be written

$$\frac{d\alpha}{dr} = \frac{Gm(r) + 4\pi Gr^3 p}{r[r - 2Gm(r)]}. \quad (5.152)$$

It is convenient not to use the  $\theta\theta$  equation directly, but instead appeal to energy-momentum conservation,  $\nabla_\mu T^{\mu\nu} = 0$ . For our metric (5.144), it is straightforward to derive that  $v = r$  is the only nontrivial component, and it gives

$$(\rho + p) \frac{d\alpha}{dr} = -\frac{dp}{dr}. \quad (5.153)$$

Combining this with (5.152) allows us to eliminate  $\alpha(r)$  to obtain

$$\frac{dp}{dr} = -\frac{(\rho + p)[Gm(r) + 4\pi Gr^3 p]}{r[r - 2Gm(r)]}. \quad (5.154)$$

This is the **Tolman–Oppenheimer–Volkoff equation**, or simply the equation of hydrostatic equilibrium. Since  $m(r)$  is related to  $\rho(r)$  via (5.146), this equation relates  $p(r)$  to  $\rho(r)$ . To get a closed system of equations, we need one more relation: the equation of state. In general this will give the pressure in terms of the energy density and specific entropy,  $p = p(\rho, S)$ . Often we care about situations in which the entropy is very small, and can be neglected; the equation of state then takes the form

$$p = p(\rho). \quad (5.155)$$

Astrophysical systems often obey a polytropic equation of state,  $p = K\rho^\gamma$  for some constants  $K$  and  $\gamma$ .

A simple and semi-realistic model of a star comes from assuming that the fluid is incompressible: the density is a constant  $\rho_*$  out to the surface of the star, after

which it vanishes,

$$\rho(r) = \begin{cases} \rho_*, & r < R \\ 0, & r > R. \end{cases} \quad (5.156)$$

Specifying  $\rho(r)$  explicitly takes the place of an equation of state, since  $p(r)$  can be determined from hydrostatic equilibrium. It is then straightforward to integrate (5.146) to get

$$m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_*, & r < R \\ \frac{4}{3}\pi R^3 \rho_* = M, & r > R. \end{cases} \quad (5.157)$$

Integrating the equation of hydrostatic equilibrium yields

$$p(r) = \rho_* \left[ \frac{R\sqrt{R - 2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3R\sqrt{R - 2GM}} \right]. \quad (5.158)$$

Finally we can get the metric component  $g_{tt} = -e^{2\alpha(r)}$  from (5.152); we find that

$$e^{\alpha(r)} = \frac{3}{2} \left( 1 - \frac{2GM}{R} \right)^{1/2} - \frac{1}{2} \left( 1 - \frac{2GMr^2}{R^3} \right)^{1/2}, \quad r < R. \quad (5.159)$$

The pressure increases near the core of the star, as one would expect. Indeed, for a star of fixed radius  $R$ , the central pressure  $p(0)$  will need to be greater than infinity if the mass exceeds

$$M_{\max} = \frac{4}{9G} R. \quad (5.160)$$

Thus, if we try to squeeze a greater mass than this inside a radius  $R$ , general relativity admits no static solutions; a star that shrinks to such a size must inevitably keep shrinking, eventually forming a black hole. We derived this result from the rather strong assumption that the density is constant, but it continues to hold when that assumption considerably weakened: **Buchdahl's theorem** states that any reasonable static, spherically symmetric interior solution has  $M < 4R/9G$ . Although a careful proof requires more work, this result makes sense; if we imagine that there is some maximum sustainable density in nature, the most massive object we could in principle make would have that density everywhere, which is the specific case we considered.

Of course, this still doesn't mean that realistic astrophysical objects will always ultimately collapse to black holes. An ordinary planet, supported by material pressures, will persist essentially forever (apart from some fantastically unlikely quantum tunneling from a planet to something very different, or the possibility of eventual proton decay). But massive stars are a different story. The pressure supporting a star comes from the heat produced by fusion of light nuclei into heavier ones. When the nuclear fuel is used up, the temperature declines and the

star begins to shrink under the influence of gravity. The collapse may eventually be halted by Fermi degeneracy pressure: Electrons are pushed so close together that they resist further compression simply on the basis of the Pauli exclusion principle (no two fermions can be in the same state). A stellar remnant supported by electron degeneracy pressure is called a **white dwarf**; a typical white dwarf is comparable in size to the Earth. Lower-mass particles become degenerate at lower number densities than high-mass particles, so nucleons do not contribute appreciably to the pressure in a white dwarf. White dwarfs are the end state for most stars, and are extremely common throughout the universe.

If the total mass is sufficiently high, however, the star will reach the **Chandrasekhar limit**, where even the electron degeneracy pressure is not enough to resist the pull of gravity. Calculations put the Chandrasekhar limit at about  $1.4 M_{\odot}$ , where  $M_{\odot} = 2 \times 10^{33}$  g is the mass of the Sun. When it is reached, the star is forced to collapse to an even smaller radius. At this point electrons combine with protons to make neutrons and neutrinos (inverse beta decay), and the neutrinos simply fly away. The result is a **neutron star**, with a typical radius of about 10 km. Neutron stars have a low total luminosity, but often are rapidly spinning and possess strong magnetic fields. This combination gives rise to **pulsars**, which accelerate particles in jets emanating from the magnetic poles, appearing to rapidly flash as the neutron star spins. Pulsars were discovered by Bell in 1967; after a brief speculation that they might represent signals from an extraterrestrial civilization, the more prosaic astrophysical explanation was settled on.

Since the conditions at the center of a neutron star are very different from those on Earth, we do not have a perfect understanding of the equation of state. Nevertheless, we believe that a sufficiently massive neutron star will itself be unable to resist the pull of gravity, and will continue to collapse; current estimates of the maximum possible neutron-star mass are around  $3\text{--}4 M_{\odot}$ , the **Oppenheimer-Volkoff limit**. Since a fluid of neutrons is the densest material we know about (apart from some very speculative suggestions), it is believed that the outcome of such a collapse is a black hole.

How would we know if there were a black hole? The fundamental obstacle to direct detection is, of course, blackness: a black hole will not itself give off any radiation (neglecting Hawking radiation, which is a very small effect to be discussed in Chapter 9). But black holes will feature extremely strong gravitational fields, so we can hope to detect them indirectly by observing matter being influenced by these fields. As matter falls into a black hole, it will heat up and emit X-rays, which we can detect with satellite observatories. A large number of black-hole candidates have been detected by this method, and the case for real black holes in our universe is extremely strong.<sup>1</sup> The large majority of candidates fall into one of two classes. There are black holes with masses of order a solar mass or somewhat higher; these are thought to be the endpoints of evolution for very massive stars. The other category describes supermassive black holes, be-

<sup>1</sup>For a review on astrophysical evidence for black holes, see A. Celotti, J.C. Miller, and D.W. Sciama (1999), *Class. Quant. Grav.* **16**, A3; <http://arxiv.org/abs/astro-ph/9912186>.

tween  $10^6$  and  $10^9$  solar masses. These are found at the centers of galaxies, and are thought to be the engines that powered quasars in the early era of galaxy formation. Our own Milky Way galaxy contains an object (Sgr A\*) that is believed to be a black hole of at least  $2 \times 10^6 M_\odot$ . The precise history of the formation of these supermassive holes is not well understood. Other possibilities include very small primordial black holes produced in the very early universe, and so-called “middleweight” black holes of order a thousand solar masses.

As matter falls into a black hole, it tends to settle into a rotating accretion disk, and both energy and angular momentum are gradually fed into the hole. As a result, the black holes we expect to see in astrophysical situations should be spinning, and indeed observations are consistent with very high spin rates for observed black holes. In this chapter we have excluded the possibility of black hole spin by focusing on the spherically symmetric Schwarzschild solution; in the next chapter we turn to more general types of black holes.

### 5.9 ■ EXERCISES

1. A space monkey is happily orbiting a Schwarzschild black hole in a circular geodesic orbit. An evil baboon, far from the black hole, tries to send the monkey to its death inside the black hole by dropping a carefully timed coconut radially toward the black hole, knowing that the monkey can't resist catching the falling coconut. Given the monkey's mass and initial orbital radius and the mass of the coconut, explain how you would go about solving the problem (but do not do the calculation). What are the possible fates for our intrepid space monkey?
2. Consider a perfect fluid in a static, circularly symmetric  $(2+1)$ -dimensional spacetime, equivalently, a cylindrical configuration in  $(3+1)$  dimensions with perfect rotational symmetry.
  - (a) Derive the analogue of the Tolman–Oppenheimer–Volkov (TOV) equation for  $(2+1)$  dimensions.
  - (b) Show that the vacuum solution can be written as

$$ds^2 = -dt^2 + \frac{1}{1-8GM} dr^2 + r^2 d\theta^2$$

Here  $M$  is a constant.

- (c) Show that another way to write the same solution is

$$ds^2 = -d\tau^2 + d\xi^2 + \xi^2 d\phi^2$$

where  $\phi \in [0, 2\pi(1-8GM)^{1/2}]$ .

- (d) Solve the  $(2+1)$  TOV equation for a constant density star. Find  $p(r)$  and solve for the metric.
- (e) Solve the  $(2+1)$  TOV equation for a star with equation of state  $p = \kappa\rho^{3/2}$ . Find  $p(r)$  and solve for the metric.
- (f) Find the mass  $M(R) = \int_0^{2\pi} \int_0^R \rho dr d\theta$  and the proper mass  $\tilde{M}(R) = \int_0^{2\pi} \int_0^R \rho \sqrt{-g} dr d\theta$  for the solutions in parts (d) and (e).

3. Consider a particle (not necessarily on a geodesic) that has fallen inside the event horizon,  $r < 2GM$ . Use the ordinary Schwarzschild coordinates  $(t, r, \theta, \phi)$ . Show that the radial coordinate must decrease at a minimum rate given by

$$\left| \frac{dr}{d\tau} \right| \geq \sqrt{\frac{2GM}{r} - 1}.$$

Calculate the maximum lifetime for a particle along a trajectory from  $r = 2GM$  to  $r = 0$ . Express this in seconds for a black hole with mass measured in solar masses. Show that this maximum proper time is achieved by falling freely with  $E \rightarrow 0$ .

4. Consider Einstein's equations in vacuum, but with a cosmological constant,  $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ .
- (a) Solve for the most general spherically symmetric metric, in coordinates  $(t, r)$  that reduce to the ordinary Schwarzschild coordinates when  $\Lambda = 0$ .
  - (b) Write down the equation of motion for radial geodesics in terms of an effective potential, as in (5.66). Sketch the effective potential for massive particles.
5. Consider a comoving observer sitting at constant spatial coordinates  $(r_*, \theta_*, \phi_*)$ , around a Schwarzschild black hole of mass  $M$ . The observer drops a beacon into the black hole (straight down, along a radial trajectory). The beacon emits radiation at a constant wavelength  $\lambda_{\text{em}}$  (in the beacon rest frame).
- (a) Calculate the coordinate speed  $dr/dt$  of the beacon, as a function of  $r$ .
  - (b) Calculate the proper speed of the beacon. That is, imagine there is a comoving observer at fixed  $r$ , with a locally inertial coordinate system set up as the beacon passes by, and calculate the speed as measured by the comoving observer. What is it at  $r = 2GM$ ?
  - (c) Calculate the wavelength  $\lambda_{\text{obs}}$ , measured by the observer at  $r_*$ , as a function of the radius  $r_{\text{em}}$  at which the radiation was emitted.
  - (d) Calculate the time  $t_{\text{obs}}$  at which a beam emitted by the beacon at radius  $r_{\text{em}}$  will be observed at  $r_*$ .
  - (e) Show that at late times, the redshift grows exponentially:  $\lambda_{\text{obs}}/\lambda_{\text{em}} \propto e^{t_{\text{obs}}/T}$ . Give an expression for the time constant  $T$  in terms of the black hole mass  $M$ .

## 6

## More General Black Holes

## 6.1 ■ THE BLACK HOLE ZOO

Birkhoff's theorem ensures that the Schwarzschild metric is the only spherically symmetric vacuum solution to general relativity. This shouldn't be too surprising, as it is reminiscent of the situation in electromagnetism, where the only spherically symmetric field configuration in a region free of charges will be a Coulomb field. Moving beyond spherical symmetry, there is an unlimited variety of possible gravitational fields. For a planet like the Earth, for example, the external field will depend on the density and profile of all the various mountain ranges and valleys on the surface. We could imagine decomposing the metric into multipole moments, and an infinite number of coefficients would have to be specified to describe the field exactly.

It might therefore come as something of a surprise that black holes do not share this property. Only a small number of stationary black-hole solutions exist, described by a small number of parameters. The specific set of parameters will depend on what matter fields we include in our theory; if electromagnetism is the only long-range nongravitational field, we have a **no-hair theorem**:

Stationary, asymptotically flat black hole solutions to general relativity coupled to electromagnetism that are nonsingular outside the event horizon are fully characterized by the parameters of mass, electric and magnetic charge, and angular momentum.

Stationary solutions are of special interest because we expect them to be the end states of gravitational collapse. The alternative might be some sort of oscillating configuration, but oscillations will ultimately be damped as energy is lost through the emission of gravitational radiation; in fact, typical evolutions will evolve quite rapidly to a stationary configuration.

We speak of "a" no-hair theorem, rather than "the" no-hair theorem, because the result depends not only on general relativity, but also on the matter content of our theory. In the Standard Model of particle physics, electromagnetism is the only long-range field, and the above theorem applies; but for different kinds of fields there might be other sorts of hair.<sup>1</sup> Examples have even been found of static (nonrotating) black holes that are axisymmetric but not completely spherically

<sup>1</sup>For a discussion see M. Heusler, "Stationary Black Holes: Uniqueness and Beyond," *Living Rev. Relativity* 1, (1998), 6; <http://www.livingreviews.org/Articles/Volume1/1998-6heusler/>.

symmetric. The central point, however, remains unaltered: black hole solutions are characterized by a very small number of parameters, rather than the potentially infinite set of parameters characterizing, say, a planet.

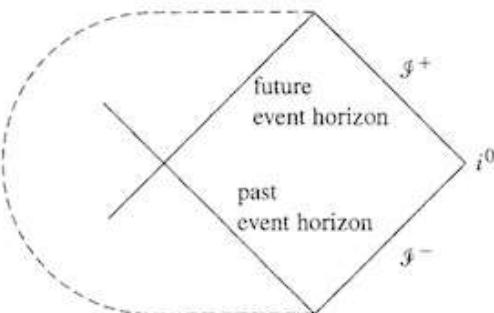
As we will discuss at the end of this chapter and again in Chapter 9, the no-hair property leads to a puzzling situation. In most physical theories, we hope to have a well-defined initial value problem, so that information about a state at any one moment of time can be used to predict (or retrodict) the state at any other moment of time. As a consequence, any two states that are connected by a solution to the equations of motion should require the same amount of information to be specified. But in GR, it seems, we can take a very complicated collection of matter, collapse it into a black hole, and end up with a configuration described completely by mass, charge, and spin. In classical GR this might not bother us so much, since the information can be thought of as hidden behind the event horizon rather than truly being lost. But when quantum field theory is taken into account, we find that black holes evaporate and eventually disappear, and the information seems to be truly lost. Conceivably, the outgoing Hawking radiation responsible for the evaporation somehow encodes information about what state was originally used to make a black hole, but how that could happen is completely unclear. Understanding this “information loss paradox” is considered by many to be a crucial step in building a sensible theory of quantum gravity.

In this chapter, however, we will stick to considerations of classical GR. We begin with some general discussion of black hole properties, especially those of event horizons and Killing horizons. This subject can be subtle and technical, and our philosophy here will be to try to convey the main ideas without being rigorous about definitions or proofs of theorems. We then discuss the specific solutions corresponding to charged (Reissner–Nordström) and spinning (Kerr) black holes; consistent with our approach, we will not carefully go through the coordinate redefinitions necessary to construct the maximally extended spacetimes, but instead simply draw the associated conformal diagrams. The reader interested in further details should consult the review article by Townsend,<sup>2</sup> or the books by Hawking and Ellis (1973) and Wald (1984), all of which we draw on heavily in this chapter.

## 6.2 ■ EVENT HORIZONS

Black holes are characterized by the fact that you can enter them, but never exit. Thus, their most important feature is actually not the singularity at the center, but the event horizon at the boundary. An event horizon is a hypersurface separating those spacetime points that are connected to infinity by a timelike path from those that are not. To understand what this means in practice, we should think a little more carefully about what we mean by “infinity.” In general relativity, the global structure of spacetime can take many different forms, with correspondingly different notions of infinity. But to think about black holes in the real universe, we

<sup>2</sup>P.K. Townsend, “Black Holes: Lecture Notes,” <http://arxiv.org/abs/gr-qc/9707012>.



**FIGURE 6.1** An asymptotically flat spacetime is one for which infinity in a conformal diagram matches that of Minkowski spacetime, with future null infinity  $\mathcal{J}^+$ , spacelike infinity  $i^0$ , and past null infinity  $\mathcal{J}^-$ . The future event horizon is the boundary of the past of  $\mathcal{J}^+$ . The dashed region represents the rest of the spacetime, which may take a number of different forms in different examples.

aren't actually concerned with what happens infinitely far away; we use infinity as a proxy for "well outside the black hole," and imagine that spacetime sufficiently far away from the hole can be approximated by Minkowski space.

As mentioned in the Chapter 5, a spacetime that looks Minkowskian at infinity is referred to as asymptotically flat. The meaning of this concept is made clear in a conformal diagram such as in Figure 6.1. From our discussion in Appendix H of the conformal diagram for Minkowski, we know that conformal infinity comes in five pieces: future and past timelike infinity  $i^\pm$ , future and past null infinity  $\mathcal{J}^\pm$ , and spatial infinity  $i^0$ . An asymptotically flat spacetime (or region of spacetime) is one for which  $\mathcal{J}^\pm$  and  $i^0$  have the same structure as for Minkowski; timelike infinity is not necessary. Such spacetimes will have the general form shown in Figure 6.1.

With this picture, it is clear how we should think of the future event horizon: it is the surface beyond which timelike curves cannot escape to infinity. Recalling that the causal past  $J^-$  of a region is the set of all points we can reach from that region by moving along past-directed timelike paths, the event horizon can be equivalently defined as the boundary of  $J^-(\mathcal{J}^+)$ , the causal past of future null infinity. (The event horizon is really the boundary of the *closure* of this set, but we're not being rigorous.) Analogous definitions hold for the past horizon. As we have seen in the case of maximally extended Schwarzschild, there may be more than one asymptotically flat region in a spacetime, and correspondingly more than one event horizon.

From the definition, it is clear that the event horizon is a null hypersurface. Properties of null hypersurfaces are discussed in Appendix D; here we can recall the major features. A hypersurface  $\Sigma$  can be defined by  $f(x) = \text{constant}$  for some function  $f(x)$ . The gradient  $\partial_\mu f$  is normal to  $\Sigma$ ; if the normal vector is null, the hypersurface is said to be null, and the normal vector is also tangent to  $\Sigma$ . Null hypersurfaces can be thought of as a collection of null geodesics  $x^\mu(\lambda)$ , called the generators of the hypersurface. The tangent vectors  $\xi^\mu$  to these geodesics are

proportional to the normal vectors,

$$\xi^\mu = \frac{dx^\mu}{d\lambda} = h(x)g^{\mu\nu}\partial_\nu f, \quad (6.1)$$

and therefore also serve as normal vectors to the hypersurface. We may choose the function  $h(x)$  so that the geodesics are affinely parameterized, so the tangent vectors will obey

$$\xi_\mu \xi^\mu = 0, \quad \xi^\mu \nabla_\mu \xi^\nu = 0. \quad (6.2)$$

For future event horizons, the generators may end in the past (for example, when a black hole is formed by stellar collapse) but will always continue indefinitely into the future (and similarly with future and past interchanged).

Because the event horizon is a global concept, it might be difficult to actually locate one when you are handed a metric in an arbitrary set of coordinates. Fortunately, in this chapter we will be concerned with quite special metrics—stationary, asymptotically flat, and containing event horizons with spherical topology. In such spacetimes, there are convenient coordinate systems in which there is a simple way to identify the event horizon. For the Schwarzschild solution, the event horizon is a place where the light cones “tilt over” so that  $r = 2GM$  is a null surface rather than a timelike surface, as  $r = \text{constant}$  would be for large  $r$ . Light-cone tilting is clearly a coordinate-dependent notion (it doesn’t happen, for example, in Kruskal coordinates), but the metrics of concern to us will allow for analogous constructions. A stationary metric has a Killing vector  $\partial_t$  that is asymptotically timelike, and we can adapt the metric components to be time-independent ( $\partial_t g_{\mu\nu} = 0$ ). On hypersurfaces  $t = \text{constant}$ , we can choose coordinates  $(r, \theta, \phi)$  in which the metric at infinity looks like Minkowski space in spherical polar coordinates. Hypersurfaces  $r = \text{constant}$  will be timelike cylinders with topology  $S^2 \times \mathbf{R}$  at  $r \rightarrow \infty$ . Now imagine we have chosen our coordinates cleverly, so that as we decrease  $r$  from infinity the  $r = \text{constant}$  hypersurfaces remain timelike until some fixed  $r = r_H$ , for which the surface is everywhere null. (In nonclever coordinates,  $r = \text{constant}$  hypersurfaces will become null or spacelike for some values of  $\theta$  and  $\phi$  but remain timelike for others.) This will clearly represent an event horizon, since timelike paths crossing into the region  $r < r_H$  will never be able to escape back to infinity. Determining the point at which  $r = \text{constant}$  hypersurfaces become null is easy;  $\partial_\mu r$  is a one-form normal to such hypersurfaces, with norm

$$g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = g^{rr}. \quad (6.3)$$

We are looking for the place where the norm of our one-form vanishes; hence, in the coordinates we have described, the event horizon  $r = r_H$  will simply be the hypersurface at which  $g^{rr}$  switches from being positive to negative,

$$g^{rr}(r_H) = 0. \quad (6.4)$$

This criterion clearly works for Schwarzschild, for which  $g^{rr} = 1 - 2GM/r$ . We will present the Reissner–Nordström and Kerr solutions in coordinates that are similarly adapted to the horizons.

The reason why we make such a big deal about event horizons is that they are nearly inevitable in general relativity. This conclusion is reached by concatenating two interesting results: Singularities are nearly inevitable, and singularities are hidden behind event horizons. Of course both results hold under appropriate sets of assumptions; it is not that hard to come up with spacetimes that have no singularities (Minkowski would be an example), nor is it even that hard to find singularities without horizons (as we will see below in our discussion of charged black holes). But we believe that “generic” solutions will have singularities hidden behind horizons.

The ubiquity of singularities is guaranteed by the **singularity theorems** of Hawking and Penrose. Before these theorems were proven, it was possible to hope that collapse to a Schwarzschild singularity was an artifact of spherical symmetry, and typical geometries would remain nonsingular (as happens, for example, in Newtonian gravity). But the Hawking–Penrose theorems demonstrate that once collapse reaches a certain point, evolution to a singularity is inevitable. The way we know there is a singularity is through geodesic incompleteness—there exists some geodesic that cannot be extended within the manifold, but nevertheless ends at a finite value of the affine parameter. The way we know collapse has reached a point of no return is the appearance of a **trapped surface**. To understand what a trapped surface is, first picture a two-sphere in Minkowski space, taken as a set of points some fixed radial distance from the origin, embedded in a constant-time slice. If we follow null rays emanating into spacetime from this spatial sphere, one set (pointed inward) will describe a shrinking set of spheres, while the other (pointed outward) will describe a growing set of spheres. But this would not be the case for a sphere of fixed radius  $r < 2GM$  in the Schwarzschild geometry; inside the event horizon, both sets of null rays emanating from such a sphere would evolve to smaller values of  $r$  (since  $r$  is a timelike coordinate), and thus to smaller areas  $4\pi r^2$ . This is what is meant by a trapped surface: a compact, spacelike, two-dimensional submanifold with the property that outgoing future-directed light rays *converge* in both directions everywhere on the submanifold. (The formal definition of “converge” is that the expansion  $\theta$ , as described in the discussion of geodesic congruences in Appendix F, is negative.)

With these definitions in hand, we can present an example of a singularity theorem.

Let  $M$  be a manifold with a generic metric  $g_{\mu\nu}$ , satisfying Einstein’s equation with the strong energy condition imposed. If there is a trapped surface in  $M$ , there must be either a closed timelike curve or a singularity (as manifested by an incomplete timelike or null geodesic).

In this case, by “a generic metric” we mean that the **generic condition** is satisfied for both timelike and null geodesics. For timelike geodesics, the generic condition

states that every geodesic with tangent vector  $U^\mu$  must have at least one point on which  $R_{\alpha\beta\gamma\delta}U^\alpha U^\delta \neq 0$ ; for null geodesics, the generic condition states that every geodesic with tangent vector  $k^\mu$  must have at least one point on which  $k_{[\alpha}R_{\beta]\gamma\delta}k_\gamma k^\delta \neq 0$ . These fancy conditions simply serve to exclude very special metrics for which the curvature consistently vanishes in some directions.

Singularity theorems exist in many forms, proceeding from various different sets of assumptions. The moral of the story seems to be that typical time-dependent solutions in general relativity usually end in singularities. (Or begin in them; some theorems imply the existence of cosmological singularities, such as the Big Bang.) This represents something of a problem for GR, in the sense that the theory doesn't really apply to the singularities themselves, whose existence therefore represents an incompleteness of description. The traditional attitude toward this issue is to hope that a sought-after quantum theory of gravity will somehow resolve the singularities of classical GR.

In the meantime, we can take solace in the idea that singularities are hidden behind event horizons. This belief is encompassed in the **cosmic censorship conjecture**:

Naked singularities cannot form in gravitational collapse from generic, initially nonsingular states in an asymptotically flat spacetime obeying the dominant energy condition.

A **naked singularity** is one from which signals can reach  $\mathcal{I}^+$ ; that is, one that is not hidden behind an event horizon. Notice that the conjecture refers to the formation of naked singularities, not their existence; there are certainly solutions in which spacelike naked singularities exist in the past (such as the Schwarzschild white hole) or timelike singularities exist for all times (such as in super-extremal charged black holes, discussed below). The cosmic censorship conjecture has not been proven, although a great deal of effort has gone into finding convincing counterexamples, without success. The requirement that the initial data be in some sense “generic” is important, as numerical experiments have shown that finely-tuned initial conditions are able to give rise to naked singularities. A precise proof of some form of the cosmic censorship conjecture remains one of the outstanding problems of classical general relativity.<sup>3</sup>

A consequence of cosmic censorship (or of certain equivalent assumptions) is that classical black holes never shrink, they only grow bigger. The size of a black hole is measured by the area of the event horizon, by which we mean the spatial area of the intersection of the event horizon with a spacelike slice. We then have Hawking's **area theorem**:

Assuming the weak energy condition and cosmic censorship, the area of a future event horizon in an asymptotically flat spacetime is non-decreasing.

<sup>3</sup>For a review of cosmic censorship see R.M. Wald, “Gravitational Collapse and Cosmic Censorship,” <http://arxiv.org/abs/gr-qc/9710068>.

For a Schwarzschild black hole, the area depends monotonically on the mass, so this theorem implies that Schwarzschild black holes can only increase in mass. But for spinning black holes this is no longer the case; the area depends on a combination of mass and angular momentum, and we will see below that we can actually extract energy from a black hole by decreasing its spin. We can also decrease the mass of a black hole through quantum-mechanical Hawking radiation; this can be traced to the fact that quantum field theory in curved spacetime can violate the weak energy condition.

### 6.3 ■ KILLING HORIZONS

In the Schwarzschild metric, the Killing vector  $K = \partial_t$  goes from being timelike to spacelike at the event horizon. In general, if a Killing vector field  $\chi^\mu$  is null along some null hypersurface  $\Sigma$ , we say that  $\Sigma$  is a **Killing horizon** of  $\chi^\mu$ . Note that the vector field  $\chi^\mu$  will be normal to  $\Sigma$ , since a null surface cannot have two linearly independent null tangent vectors.

The notion of a Killing horizon is logically independent from that of an event horizon, but in spacetimes with time-translation symmetry the two are closely related. Under certain reasonable conditions (made explicit below), we have the following classification:

Every event horizon  $\Sigma$  in a stationary, asymptotically flat spacetime is a Killing horizon for some Killing vector field  $\chi^\mu$ .

If the spacetime is static,  $\chi^\mu$  will be the Killing vector field  $K^\mu = (\partial_t)^\mu$  representing time translations at infinity.

If the spacetime is stationary but not static, it will be axisymmetric with a rotational Killing vector field  $R^\mu = (\partial_\phi)^\mu$ , and  $\chi^\mu$  will be a linear combination  $K^\mu + \Omega_H R^\mu$  for some constant  $\Omega_H$ .

For example, below we will examine the Kerr metric for spinning black holes, in which the event horizon is a Killing horizon for a linear combination of the Killing vectors for rotations and time translations. In Kerr, the hypersurface on which  $\partial_t$  becomes null is actually timelike, so is not a Killing horizon.

Let's be precise about the conditions under which this classification scheme actually holds.<sup>4</sup> Carter has shown that, for static black holes, the event horizon is a Killing horizon for  $K^\mu$ ; this is a purely geometric fact, which holds even without invoking Einstein's equation. In the stationary case, if we assume the existence of a rotational Killing field  $R^\mu$  with the property that 2-planes spanned by  $K^\mu$  and  $R^\mu$  are orthogonal to a family of two-dimensional surfaces, then the event horizon will be a Killing horizon for a linear combination of the two Killing fields, again from purely geometric considerations. If on the other hand we only assume that the black hole is stationary, we cannot prove in general that the event horizon

<sup>4</sup>For a discussion see R. M. Wald, "The thermodynamics of black holes," *Living Rev. Rel.* **4**, 6 (2001). <http://arxiv.org/gr-qc/9912119>.

is axisymmetric. Given Einstein's equation and some conditions on the matter fields, Hawking was able to show that the event horizon of any stationary black hole must be a Killing horizon for some vector field, and furthermore that such horizons must either be stationary or axisymmetric. For the rest of this chapter we will speak as if the above classification holds; however, making assumptions about matter fields is notoriously tricky, and we should keep in mind the possibility in principle of finding black holes that are not static or axisymmetric, for which the event horizon might not be a Killing horizon.

It's important to point out that, while event horizons for stationary asymptotically flat spacetimes will typically be Killing horizons, it's easy to have Killing horizons that have nothing to do with event horizons. Consider Minkowski space in inertial coordinates,  $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ ; clearly there are no event horizons in this spacetime. The Killing vector that generates boosts in the  $x$ -direction is

$$\chi = x\partial_t + t\partial_x, \quad (6.5)$$

with norm

$$\chi_\mu \chi^\mu = -x^2 + t^2, \quad (6.6)$$

This goes null at the null surfaces

$$x = \pm t, \quad (6.7)$$

which are therefore Killing horizons. By combining the boost Killing vector with translational and rotational Killing vectors, we can move these horizons through the manifold; there are Killing horizons all over. In more interesting spacetimes, of course, there will be fewer Killing vector fields, and the associated horizons (if any) will have greater physical significance.

To every Killing horizon we can associate a quantity called the **surface gravity**. Consider a Killing vector  $\chi^\mu$  with Killing horizon  $\Sigma$ . Because  $\chi^\mu$  is a normal vector to  $\Sigma$ , along the Killing horizon it obeys the geodesic equation,

$$\chi^\mu \nabla_\mu \chi^\nu = -\kappa \chi^\nu, \quad (6.8)$$

where the right-hand side arises because the integral curves of  $\chi^\mu$  may not be affinely parameterized. The parameter  $\kappa$  is the surface gravity; it will be constant over the horizon, except for a “bifurcation two-sphere” where the Killing vector vanishes and  $\kappa$  can change sign. (This happens, for example, at the center of the Kruskal diagram in the Schwarzschild solution.) Using Killing's equation  $\nabla_{(\mu} \chi_{\nu)} = 0$  and the fact that  $\chi_{[\mu} \nabla_{\nu]} \chi_{\sigma]} = 0$  (since  $\chi^\mu$  is normal to  $\Sigma$ ), it is straightforward to derive a nice formula for the surface gravity:

$$\kappa^2 = -\tfrac{1}{2} (\nabla_\mu \chi_\nu) (\nabla^\mu \chi^\nu). \quad (6.9)$$

The expression on the right-hand side is to be evaluated at the horizon  $\Sigma$ . You are encouraged to check this formula yourself.

The surface gravity associated with a Killing horizon is in principle arbitrary, since we can always scale a Killing field by a real constant and obtain another Killing field. In a static, asymptotically flat spacetime, the time-translation Killing vector  $K = \partial_t$  can be normalized by setting

$$K_\mu K^\mu(r \rightarrow \infty) = -1. \quad (6.10)$$

This in turn fixes the surface gravity of any associated Killing horizon. If we are in a stationary spacetime, where the Killing horizon is associated with a linear combination of time translations and rotations, fixing the normalization of  $K = \partial_t$  also fixes this linear combination, so the surface gravity remains unique.

The reason why  $\kappa$  is called the “surface gravity” becomes clear only when the spacetime is static. In that case we have the following interpretation:

In a static, asymptotically flat spacetime, the surface gravity is the acceleration of a static observer near the horizon, as measured by a static observer at infinity.

To make sense of such a statement, let’s first consider static observers. By a static observer we mean one whose four-velocity  $U^\mu$  is proportional to the time-translation Killing field  $K^\mu$ :

$$K^\mu = V(x)U^\mu. \quad (6.11)$$

Since the four-velocity is normalized to  $U_\mu U^\mu = -1$ , the function  $V$  is simply the magnitude of the Killing field,

$$V = \sqrt{-K_\mu K^\mu}, \quad (6.12)$$

and hence ranges from zero at the Killing horizon to unity at infinity.  $V$  is sometimes called the “redshift factor,” since it relates the emitted and observed frequencies of a photon as measured by static observers. Recall that the conserved energy of a photon with four-momentum  $p^\mu$  is  $E = -p_\mu K^\mu$ , while the frequency measured by an observer with four-velocity  $U^\mu$  will be  $\omega = -p_\mu U^\mu$ . Therefore

$$\omega = \frac{E}{V}, \quad (6.13)$$

and a photon emitted by static observer 1 will be observed by static observer 2 to have wavelength  $\lambda = 2\pi/\omega$  given by

$$\lambda_2 = \frac{V_2}{V_1} \lambda_1. \quad (6.14)$$

In particular, at infinity where  $V = 1$ , we will observe a wavelength  $\lambda_\infty = \lambda_1/V_1$ .

Now we turn to the idea of “acceleration as viewed from infinity.” A static observer will not typically be moving on a geodesic; for example, particles tend to fall into black holes rather than hovering next to them at fixed spatial coordinates.

We can express the four-acceleration  $a^\mu = U^\sigma \nabla_\sigma U^\mu$  in terms of the redshift factor as

$$a_\mu = \nabla_\mu \ln V, \quad (6.15)$$

as you can easily check. The magnitude of the acceleration,

$$a = \sqrt{a_\mu a^\mu} = V^{-1} \sqrt{\nabla_\mu V \nabla^\mu V}, \quad (6.16)$$

will go to infinity at the Killing horizon—it will take an infinite acceleration to keep an object on a static trajectory. But an observer at infinity will detect the acceleration to be ‘redshifted’ by a factor  $V$ ; this turns out to be the surface gravity. Thus, we claim that

$$\kappa = Va = \sqrt{\nabla_\mu V \nabla^\mu V}, \quad (6.17)$$

evaluated at the horizon  $\Sigma$ . You can check that this expression agrees with (6.9). The surface gravity is the product of zero ( $V$ ) and infinity ( $a$ ), but will typically be finite. When we say that the observed acceleration is redshifted, we have in mind stretching a test string from a static object at the horizon to an observer at infinity, and measuring the acceleration on the end of the string at infinity. (It is worth taking the time to see if you can promote this hand-waving argument to something more rigorous.)

What goes wrong with the above considerations if the spacetime is stationary but not static? We still have an asymptotically time-translation Killing vector  $K = \partial_t$ , and we can define stationary observers as ones whose four-velocities are parallel to  $K^\mu$ , as in (6.11); the redshift will continue to be given by (6.14). The problem is that  $K^\mu$  won’t become null at a Killing horizon, but generally at some timelike surface outside the horizon. This place where  $K^\mu K_\mu = 0$  is called the **stationary limit surface** (or sometimes ‘ergosurface’), since inside this surface  $K^\mu$  is spacelike, and consequently no observer can remain stationary, even if it is still outside the event horizon. Such an observer has to move with respect to the Killing field, but need not move in the direction of the black hole. From (6.12) and (6.14), the redshift of a stationary observer diverges as we approach the stationary limit surface, which is therefore also called the **infinite redshift surface**. As we will see in our discussion of the Kerr metric, the region between the stationary limit surface and the event horizon, the ergosphere, is a place where timelike paths are inevitably dragged along with the rotation of the black hole. We will continue to use ‘surface gravity’ as a label in stationary spacetimes, which we will calculate using the Killing vector  $\chi^\mu$ , which actually does go null on the event horizon, even if the resulting quantity cannot be interpreted as the gravitational acceleration of a stationary observer as seen at infinity.

Let’s apply these notions to Schwarzschild to see how they work. For the metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (6.18)$$

the Killing vector and static four-velocity are

$$K^\mu = (1, 0, 0, 0), \quad U^\mu = \left[ \left( 1 - \frac{2GM}{r} \right)^{-1/2}, 0, 0, 0 \right], \quad (6.19)$$

so the redshift factor is

$$V = \sqrt{1 - \frac{2GM}{r}}. \quad (6.20)$$

(Note the agreement with our calculation of the redshift in the previous chapter.) From (6.15), the acceleration is

$$a_\mu = \frac{GM}{r^2 \left( 1 - \frac{2GM}{r} \right)^{1/2}} \nabla_\mu r, \quad (6.21)$$

where of course  $\nabla_\mu r = \delta'_\mu$ . The magnitude of the acceleration is thus

$$a = \frac{GM}{r^2 \left( 1 - \frac{2GM}{r} \right)^{1/2}}, \quad (6.22)$$

The surface gravity is  $\kappa = Va$  evaluated at the event horizon  $r = 2GM$ , and

$$Va = \frac{GM}{r^2}, \quad (6.23)$$

so the surface gravity of a Schwarzschild black hole is

$$\kappa = \frac{1}{4GM}. \quad (6.24)$$

It might seem surprising that the surface gravity decreases as the mass increases, but a glance at (6.23) reveals what is going on; at fixed radius increasing  $M$  acts to increase the combination  $Va$ , but increasing the mass also increases the Schwarzschild radius, and that effect wins out. Thus, the surface gravity of a big black hole is actually weaker than that of a small black hole; this is consistent with an examination of the tidal forces, which are also smaller for bigger black holes.

## 6.4 ■ MASS, CHARGE, AND SPIN

Since we have claimed above that the most general stationary black-hole solution to general relativity is characterized by mass, charge, and spin, we should consider how these quantities might be defined in GR. Charge is the easiest to consider, so we start there; more details are found in our discussion of Stokes's theorem in Appendix E. We'll look specifically at electric charge, although magnetic charge could be examined in the same way.

Maxwell's equations relate the electromagnetic field strength tensor  $F_{\mu\nu}$  to the electric current four-vector  $J_e^\mu$ ,

$$\nabla_\nu F^{\mu\nu} = J_e^\mu. \quad (6.25)$$

The charge passing through a spacelike hypersurface  $\Sigma$  is given by an integral over coordinates  $x^i$  on the hypersurface,

$$\begin{aligned} Q &= - \int_{\Sigma} d^3x \sqrt{\gamma} n_\mu J_e^\mu \\ &= - \int_{\Sigma} d^3x \sqrt{\gamma} n_\mu \nabla_\nu F^{\mu\nu}, \end{aligned} \quad (6.26)$$

where  $\gamma_{ij}$  is the induced metric, and  $n^\mu$  is the unit normal vector, associated with  $\Sigma$ . The minus sign ensures that a positive charge density and a future-pointing normal vector will give a positive total charge. Stokes's theorem can then be used to express the charge as a boundary integral,

$$Q = - \int_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}} n_\mu \sigma_\nu F^{\mu\nu}, \quad (6.27)$$

where the boundary  $\partial\Sigma$ , typically a two-sphere at spatial infinity, has metric  $\gamma_{ij}^{(2)}$  and outward-pointing normal vector  $\sigma^\mu$ . The magnetic charge could be determined by replacing  $F^{\mu\nu}$  with the dual tensor  $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ . Thus, to calculate the total charge, we need know only the behavior of the electromagnetic field at spatial infinity. In Appendix E we do an explicit calculation for a point charge in Minkowski space, which yields a predictable result but serves as a good check that our conventions work out correctly.

We turn now to the concept of the total energy (or mass) of an asymptotically flat spacetime. This is a much trickier notion than that of the charge; for one thing, energy-momentum is a tensor rather than a vector in general relativity, and for another, the energy-momentum tensor  $T_{\mu\nu}$  only describes the properties of matter, not of the gravitational field. But recall that in Chapter 3 we discussed how we could nevertheless define a conserved total energy if spacetime were stationary, with a timelike Killing vector field  $K^\mu$ . We first construct a current

$$J_T^\mu = K_\nu T^{\mu\nu}, \quad (6.28)$$

where  $T^{\mu\nu}$  is the energy-momentum tensor. Because this current is divergenceless (from Killing's equation and conservation of  $T^{\mu\nu}$ ), we can find a conserved energy by integrating over a spacelike surface  $\Sigma$ ,

$$E_T = \int_{\Sigma} d^3x \sqrt{\gamma} n_\mu J_T^\mu, \quad (6.29)$$

just as for the charge. As interesting as this expression is, there are clearly some inadequacies with it. For example, consider the Schwarzschild metric. It has a

Killing vector, but  $T^{\mu\nu}$  vanishes everywhere. Is the energy of a Schwarzschild black hole therefore zero? On both physical and mathematical grounds, there is reason to suspect not; there is a singularity, after all, which renders the integral difficult to evaluate. Furthermore, a Schwarzschild black hole can evolve from a massive star with a definite nonzero energy, and we might like that energy to be conserved. It is worth searching for an alternative definition of energy that better captures our intuitive picture for black hole spacetimes.

Sticking for the moment to spacetimes with a timelike Killing vector  $K^\mu$ , consider a new current

$$J_R^\mu = K_\nu R^{\mu\nu}. \quad (6.30)$$

Using Einstein's equation, we can equivalently write this as

$$J_R^\mu = 8\pi G K_\nu \left( T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu} \right). \quad (6.31)$$

The Ricci tensor is not divergenceless; instead we have the contracted Bianchi identity,

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2} \nabla^\nu R. \quad (6.32)$$

But this and Killing's equation suffice to guarantee that our new current is conserved. To see this, we simply compute

$$\nabla_\mu J_R^\mu = (\nabla_\mu K_\nu) R^{\mu\nu} + K_\nu (\nabla_\mu R^{\mu\nu}). \quad (6.33)$$

The first term vanishes because  $R^{\mu\nu}$  is symmetric and  $\nabla_\mu K_\nu$  is antisymmetric (from Killing's equation). Using (6.32) we therefore have

$$\nabla_\mu J_R^\mu = \frac{1}{2} K_\nu \nabla^\nu R = 0, \quad (6.34)$$

which we know vanishes because the directional derivative of  $R$  vanishes along a Killing vector, (3.178).

As before, we can define a conserved energy associated with this current,

$$E_R = \frac{1}{4\pi G} \int_{\Sigma} d^3x \sqrt{\gamma} n_\mu J_R^\mu, \quad (6.35)$$

where the normalization is chosen for future convenience. The energy  $E_R$  will be independent of the spacelike hypersurface  $\Sigma$ , and hence conserved. This notion of energy has a significant advantage over  $E_T$ , arising from the fact that  $E_R$  can be rewritten as a surface integral over a two-sphere at spatial infinity. To see this, recall from (3.177) that any Killing vector satisfies  $\nabla_\mu \nabla_\nu K^\mu = K^\mu R_{\mu\nu}$ ; the current itself can thus be written as a total derivative,

$$J_R^\mu = \nabla_\nu (\nabla^\mu K^\nu), \quad (6.36)$$

so that

$$E_R = \frac{1}{4\pi G} \int_{\Sigma} d^3x \sqrt{\gamma} n_\mu \nabla_\nu (\nabla^\mu K^\nu). \quad (6.37)$$

Note that, from raising indices on Killing's equation,  $\nabla^\mu K^\nu = -\nabla^\nu K^\mu$ . We can therefore again use Stokes's theorem just as we did for electric charge, to write  $E_R$  as an integral at spatial infinity,

$$E_R = \frac{1}{4\pi G} \int_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}} n_\mu \sigma_\nu \nabla^\mu K^\nu. \quad (6.38)$$

This expression is the **Komar integral** associated with the timelike Killing vector  $K^\mu$ ; it can be interpreted as the total energy of a stationary spacetime.

To convince ourselves that we're on the right track, let's calculate the Komar integral for Schwarzschild, with metric (6.18). The normal vectors, normalized to  $n_\mu n^\mu = -1$  and  $\sigma_\mu \sigma^\mu = +1$ , have nonzero components

$$n_0 = -\left(1 - \frac{2GM}{r}\right)^{1/2}, \quad \sigma_1 = \left(1 - \frac{2GM}{r}\right)^{-1/2}, \quad (6.39)$$

with other components vanishing. We therefore have

$$n_\mu \sigma_\nu \nabla^\mu K^\nu = -\nabla^0 K^1. \quad (6.40)$$

The Killing vector is  $K^\mu = (1, 0, 0, 0)$ , so we can readily calculate

$$\begin{aligned} \nabla^0 K^1 &= g^{00} \nabla_0 K^1 \\ &= g^{00} \left( \partial_0 K^1 + \Gamma_{0\lambda}^1 K^\lambda \right) \\ &= g^{00} \Gamma_{00}^1 K^0 \\ &= -\left(1 - \frac{2GM}{r}\right)^{-1} \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right) \\ &= -\frac{GM}{r^2}. \end{aligned} \quad (6.41)$$

The metric on the two-sphere at infinity is

$$\gamma_{ij}^{(2)} dx^i dx^j = r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (6.42)$$

so that

$$\sqrt{\gamma^{(2)}} = r^2 \sin \theta. \quad (6.43)$$

Putting it all together, the energy of a Schwarzschild black hole is

$$\begin{aligned} E_R &= \frac{1}{4\pi G} \int d\theta d\phi r^2 \sin\theta \left( \frac{GM}{r^2} \right) \\ &= M. \end{aligned} \quad (6.44)$$

This is of course the desired result, explaining the normalization chosen in (6.35).

Despite getting the right answer, we should think about what just happened. In particular, we obtained this energy by integrating the current  $J_R^\mu = K_\nu R^{\mu\nu}$  over a spacelike slice, finding that the result could be written as an integral at spatial infinity. But for Schwarzschild, the metric solves the vacuum Einstein equation,  $R_{\mu\nu} = 0$ ; it therefore seems difficult to get a nonzero answer from integrating  $J_R^\mu$ , just as it did for (6.29). If we think about the structure of the maximally extended Schwarzschild solution, we realize that we could draw two kinds of spacelike slices: those that extend through the wormhole to the second asymptotic region, and those that end on the singularity. If the slice extends through the wormhole, the other asymptotic region provides another component to  $\partial\Sigma$ , and thus another contribution to (6.38); this contribution would exactly cancel, so the total energy would indeed be zero. If the slice intersected the singularity, we wouldn't know quite how to deal with it. Nevertheless, in either case it is sensible to treat our result (6.44) as the correct answer. The point is that, since (6.38) involves contributions only at spatial infinity, it should be a valid expression for the energy no matter what happens in the interior. We could even imagine time-dependent behavior in the interior; so long as  $K^\mu$  was *asymptotically* a timelike Killing vector, the Komar energy will be well-defined. We could, for example, consider spherically symmetric gravitational collapse from an initially static star. Evaluating the integral (6.35) directly over  $\Sigma$  would give a sensible answer for the total mass, which should not change as the star collapsed to a black hole (we are imagining spherical symmetry, so that gravitational radiation cannot carry away energy to infinity). So the Komar integral (6.38), which would be valid before the collapse, may be safely interpreted as the energy even after collapse to a black hole. Of course for some purposes we might want to allow for energy loss through gravitational radiation, in which case we need to be careful about how we extend our slice to infinity; one can define a “Bondi mass” at future null infinity which allows us to keep track of energy loss through radiation.

Another worry about the Komar formula is whether it is really what we should think of as the “energy,” which is typically the conserved quantity associated with time translation invariance. The best argument in favor of this interpretation is simply that  $E_R$  is certainly a conserved quantity of some sort, and it agrees with what we think should be the energy of Schwarzschild (and of a collection of masses in the Newtonian limit, as you could check), so what else could it be? Alternatively, one could think about a Hamiltonian formulation of general relativity, and carefully define the generator of time translations in an asymptotically flat spacetime, and then identify that with the total energy. This was first done by Arnowitt, Deser, and Misner, and their result is known as the **ADM energy**. In an

asymptotically flat spacetime, we can write the metric just as we do in perturbation theory,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (6.45)$$

except that here we only ask that the components  $h_{\mu\nu}$  be small at spatial infinity, not necessarily everywhere. The ADM energy can then be written as an integral over a two-sphere at spatial infinity, as

$$E_{\text{ADM}} = \frac{1}{16\pi G} \int_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}} \sigma^i \left( \partial_j h^j{}_i - \partial_i h^j{}_j \right), \quad (6.46)$$

where spatial indices are raised with  $\delta^{ij}$  (the spatial metric at infinity). This formula looks coordinate-dependent, but is actually well-defined under our assumptions. If  $h_{\mu\nu}$  is time-independent at infinity, it can be verified that the ADM energy and the Komar energy actually agree. This gives us even more confidence that the Komar integral really represents the energy. However, there is a sense in which the ADM energy is more respectable; for example, the Komar integral can run into trouble if we have long-range scalar fields nonminimally coupled to gravity. But for our immediate purposes the Komar energy is quite acceptable.

One quality that we would like something called “energy” to have is that it be positive for any physical configuration; otherwise a zero-energy state could decay into pieces of positive energy and negative energy. The energy conditions discussed in Chapter 4 give a notion of positive energy for matter fields, but we might worry about a negative gravitational contribution leading to problems. Happily, in GR we have the **positive energy theorem**, first proven by Shoen and Yau:

The ADM energy of a nonsingular, asymptotically flat spacetime obeying Einstein’s equation and the dominant energy condition is nonnegative. Furthermore, Minkowski is the only such spacetime with vanishing ADM energy.

If we allow for singularities, there are clearly counterexamples, such as Schwarzschild with  $M < 0$ . However, if a spacetime with a singularity (such as Schwarzschild with  $M > 0$ ) is reached as the evolution of nonsingular initial data, the theorem will apply. Thus we seem to be safe from negative-energy isolated systems in general relativity.

Finally, we may turn to spin (angular momentum), which is perfectly straightforward after our discussion of energy. Imagine that we have a rotational Killing vector  $R = \partial_\phi$ . In exact analogy with the time-translation case, we can define a conserved current

$$J_\phi^\mu = R_\nu R^{\mu\nu}, \quad (6.47)$$

which will lead to an expression for the conserved angular momentum  $J$  as an integral over spatial infinity,

$$J = -\frac{1}{8\pi G} \int_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}} n_\mu \sigma_\nu \nabla^\mu R^\nu, \quad (6.48)$$

(It is too bad that “ $J$ ” is used for both the current and the angular momentum, just as it is too bad that “ $R$ ” is both the rotational Killing vector and the Ricci tensor. But there are only so many letters to go around.) Just as with the energy, this expression will still be valid even if  $R^\mu$  is only asymptotically a Killing vector. Note that the normalization is different than in the energy integral; it could be justified, for example, by evaluating the expression for slowly-moving masses with weak gravitational fields.

## 6.5 ■ CHARGED (REISSNER–NORDSTRÖM) BLACK HOLES

We turn now to the exact solutions representing electrically charged black holes. Such solutions are not extremely relevant to realistic astrophysical situations; in the real world, a highly-charged black hole would be quickly neutralized by interactions with matter in the vicinity of the hole. But charged holes nevertheless illustrate a number of important features of more general situations. In this case the full spherical symmetry of the problem is still present; we know therefore that we can write the metric as

$$ds^2 = -e^{2\alpha(r,t)} dt^2 + e^{2\beta(r,t)} dr^2 + r^2 d\Omega^2. \quad (6.49)$$

Now, however, we are no longer in vacuum, since the hole will have a nonzero electromagnetic field, which in turn acts as a source of energy-momentum. The energy-momentum tensor for electromagnetism is given by

$$T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}, \quad (6.50)$$

where  $F_{\mu\nu}$  is the electromagnetic field strength tensor. Since we have spherical symmetry, the most general field strength tensor will have components

$$\begin{aligned} F_{tr} &= f(r, t) = -F_{rt} \\ F_{\theta\phi} &= g(r, t) \sin\theta = -F_{\phi\theta}, \end{aligned} \quad (6.51)$$

where  $f(r, t)$  and  $g(r, t)$  are some functions to be determined by the field equations, and components not written are zero.  $F_{tr}$  corresponds to a radial electric field, while  $F_{\theta\phi}$  corresponds to a radial magnetic field. For those of you wondering about the  $\sin\theta$ , recall that the thing that should be independent of  $\theta$  and  $\phi$  is the radial component of the magnetic field,  $B^r = \epsilon^{01\mu\nu} F_{\mu\nu}$ . For a spherically symmetric metric,

$$\epsilon^{\rho\sigma\mu\nu} = \frac{1}{\sqrt{-g}} \bar{\epsilon}^{\rho\sigma\mu\nu}$$

is proportional to  $(\sin\theta)^{-1}$ , so we want a factor of  $\sin\theta$  in  $F_{\theta\phi}$ .

The field equations in this case are both Einstein's equation and Maxwell's equations:

$$\begin{aligned} g^{\mu\nu}\nabla_\mu F_{\nu\sigma} &= 0 \\ \nabla_{[\mu} F_{\nu\rho]} &= 0. \end{aligned} \quad (6.52)$$

The two sets are coupled together, since the electromagnetic field strength tensor enters Einstein's equation through the energy-momentum tensor, while the metric enters explicitly into Maxwell's equations.

The difficulties are not insurmountable, however, and a procedure similar to the one we followed for the vacuum case leads to a solution for the charged case as well. We will not go through the steps explicitly, but merely quote the final answer. The solution is known as the **Reissner–Nordström (RN) metric**, and is given by

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2, \quad (6.53)$$

where

$$\Delta = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2}. \quad (6.54)$$

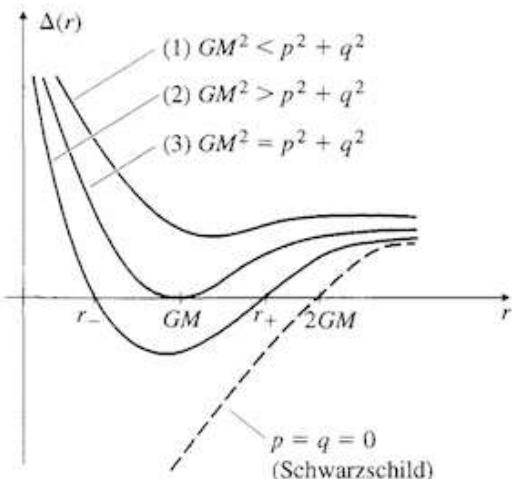
In this expression,  $M$  is once again interpreted as the mass of the hole;  $Q$  is the total electric charge, and  $P$  is the total magnetic charge. Isolated magnetic charges (monopoles) have never been observed in nature, but that doesn't stop us from writing down the metric that they would produce if they did exist.<sup>5</sup> There are good theoretical reasons to think that monopoles may exist if forces are "grand unified" at very high energies, but they must be very heavy and extremely rare. Of course, a black hole could possibly have magnetic charge even if there aren't any monopoles. In fact, the electric and magnetic charges enter the metric in the same way, so we are not introducing any additional complications by keeping  $P$  in our expressions. Conservatives are welcome to set  $P = 0$  if they like. The electromagnetic fields associated with this solution are given by

$$\begin{aligned} E_r &= F_{rt} = \frac{Q}{r^2} \\ B_r &= \frac{F_{\theta\phi}}{r^2 \sin \theta} = \frac{P}{r^2}. \end{aligned} \quad (6.55)$$

The  $1/r^2$  dependence of these fields is just what we are used to in flat space; of course, here we know that this depends on our precise choice of radial coordinate.

The RN metric has a true curvature singularity at  $r = 0$ , as could be checked by computing the curvature invariant scalar  $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ . The horizon structure,

<sup>5</sup>In this chapter we are using units in which there is no factor of  $4\pi$  in Coulomb's law. To compare with other chapters, divide each appearance of  $Q$  or  $P$  by  $4\pi$ .



**FIGURE 6.2** The function  $\Delta(r) = 1 - 2GM/r + G(Q^2 + P^2)/r^2$  for the Reissner–Nordström solutions; zeroes indicate the location of an event horizon.

however, is more complicated than in Schwarzschild. In the discussion of event horizons above, we suggested that  $g^{rr} = 0$  would be a useful diagnostic for locating event horizons, if we had cleverly chosen coordinates so that this condition is satisfied at some fixed value of  $r$ . Fortunately the coordinates of (6.53) have this property, and the event horizon will be located at

$$g^{rr}(r) = \Delta(r) = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2} = 0. \quad (6.56)$$

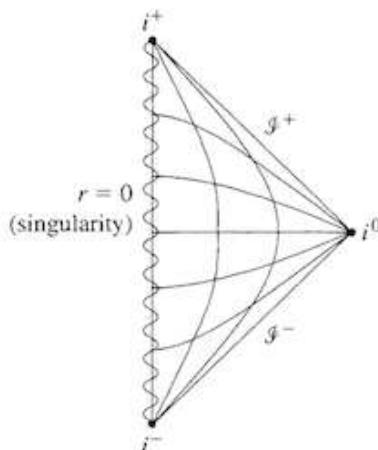
This will occur at

$$r_{\pm} = GM \pm \sqrt{G^2 M^2 - G(Q^2 + P^2)}. \quad (6.57)$$

As shown in Figure 6.2, this might constitute two, one, or zero solutions, depending on the relative values of  $GM^2$  and  $Q^2 + P^2$ . We therefore consider each case separately.

#### Case One: $GM^2 < Q^2 + P^2$

In this case the coefficient  $\Delta$  is always positive (never zero), and the metric is completely regular in the  $(t, r, \theta, \phi)$  coordinates all the way down to  $r = 0$ . The coordinate  $t$  is always timelike, and  $r$  is always spacelike. But still there is the singularity at  $r = 0$ , which is now a timelike line. Since there is no event horizon, there is no obstruction to an observer traveling to the singularity and returning to report on what was observed. This is a naked singularity, as discussed earlier. A careful analysis of the geodesics reveals that the singularity is repulsive—timelike geodesics never intersect  $r = 0$ ; instead they approach and then reverse course and move away. (Null geodesics can reach the singularity, as can nongeodesic timelike curves.)



**FIGURE 6.3** Conformal diagram for Reissner–Nordström solution with  $Q^2 + P^2 > GM^2$ . There is a naked singularity at the origin.

As  $r \rightarrow \infty$  the solution approaches flat spacetime, and as we have just seen the causal structure seems normal everywhere. The conformal diagram will therefore be just like that of Minkowski space, except that now  $r = 0$  is a singularity, as shown in Figure 6.3.

The nakedness of the singularity offends our sense of decency, as well as the cosmic censorship conjecture. In fact, we should never expect to find a black hole with  $GM^2 < Q^2 + P^2$  as the result of gravitational collapse. Roughly speaking, this condition states that the total energy of the hole is less than the contribution to the energy from the electromagnetic fields alone—that is, the mass of the matter that carried the charge would have had to be negative. This solution is therefore generally considered to be unphysical. Notice also that there are no Cauchy surfaces in this spacetime, since timelike lines can begin and end at the singularity.

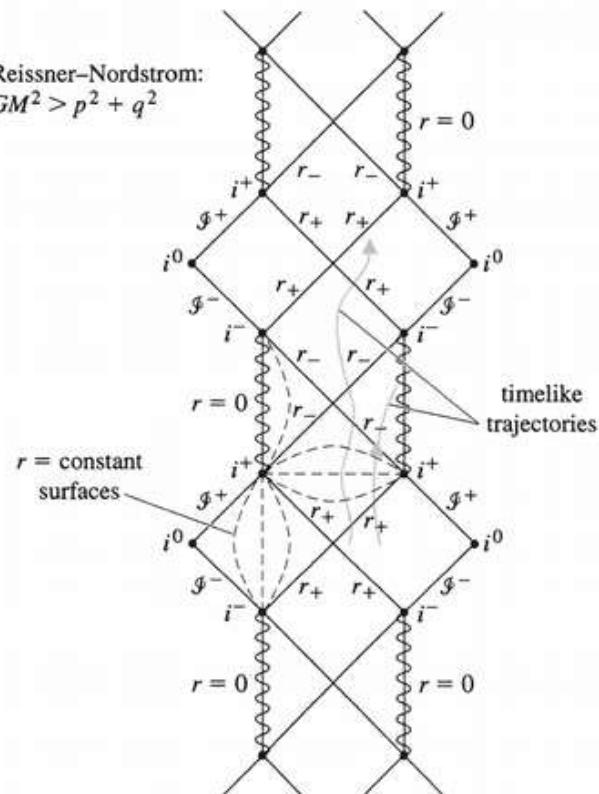
#### Case Two: $GM^2 > Q^2 + P^2$

We expect this situation to apply in realistic gravitational collapse; the energy in the electromagnetic field is less than the total energy. In this case the metric coefficient  $\Delta(r)$  is positive at large  $r$  and small  $r$ , and negative inside the two vanishing points  $r_{\pm} = GM \pm \sqrt{G^2M^2 - G(Q^2 + P^2)}$ . The metric has coordinate singularities at both  $r_+$  and  $r_-$ ; in both cases these could be removed by a change of coordinates as we did with Schwarzschild.

The surfaces defined by  $r = r_{\pm}$  are both null, and they are both event horizons. The singularity at  $r = 0$  is a timelike line, not a spacelike surface as in Schwarzschild. If you are an observer falling into the black hole from far away,  $r_+$  is just like  $2GM$  in the Schwarzschild metric: at this radius  $r$  switches from being a spacelike coordinate to a timelike coordinate, and you necessarily move in the direction of decreasing  $r$ . Witnesses outside the black hole also see the same

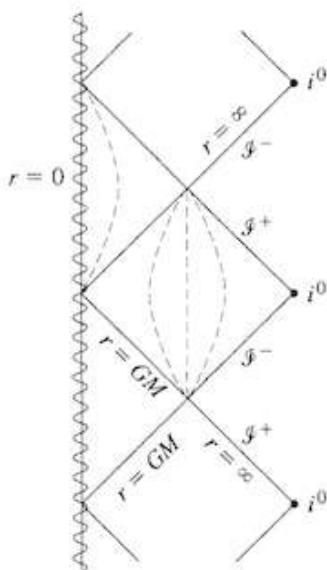
phenomena that they would outside an uncharged hole—the infalling observer is seen to move more and more slowly, and is increasingly redshifted.

But the inevitable fall from  $r_+$  to ever-decreasing radii only lasts until you reach the null surface  $r = r_-$ , where  $r$  switches back to being a spacelike coordinate and the motion in the direction of decreasing  $r$  can be arrested. Therefore you do not have to hit the singularity at  $r = 0$ ; this is to be expected, since  $r = 0$  is a timelike line (and therefore not necessarily in your future). In fact you can choose either to continue on to  $r = 0$ , or begin to move in the direction of increasing  $r$  back through the null surface at  $r = r_-$ . Then  $r$  will once again be a timelike coordinate, but with reversed orientation; you are forced to move in the direction of *increasing*  $r$ . You will eventually be spit out past  $r = r_+$  once more, which is like emerging from a white hole into the rest of the universe. From here you can choose to go back into the black hole—this time, a different hole than the one you entered in the first place—and repeat the voyage as many times as you like. This little story corresponds to the conformal diagram in Figure 6.4, which of course can be derived more rigorously by choosing appropriate coordinates and analytically extending the Reissner–Nordström metric as far as it will go.



**FIGURE 6.4** Conformal diagram for Reissner–Nordström solution with  $GM^2 > Q^2 + P^2$ . There are an infinite number of copies of the region outside the black hole.

How much of this is science, as opposed to science fiction? Probably not much. If you think about the world as seen from an observer inside the black hole who is about to cross the event horizon at  $r_-$ , you notice that the observer can look back in time to see the entire history of the external (asymptotically flat) universe, at least as seen from the black hole. But they see this (infinitely long) history in a finite amount of their proper time—thus, any signal that gets to them as they approach  $r_-$  is infinitely blueshifted. Therefore it is likely that any nonspherically-symmetric perturbation that comes into an RN black hole will violently disturb the geometry we have described. It's hard to say what the actual geometry will look like, but there is no very good reason to believe that it must contain an infinite number of asymptotically flat regions connecting to each other via various wormholes.<sup>5</sup>



**FIGURE 6.5** Conformal diagram for the extremal Reissner–Nordström solution,  $GM^2 = Q^2 + P^2$ . There is a naked singularity at the origin, and an infinite number of external regions.

### Case Three: $GM^2 = Q^2 + P^2$

This case is known as the **extreme** Reissner–Nordström solution. On the one hand the extremal hole is an amusing theoretical toy; this solution is often examined in studies of the role of black holes in quantum gravity. In supersymmetric theories, extremal black holes can leave certain symmetries unbroken, which is a considerable aid in calculations. On the other hand it appears unstable, since adding just a little bit of matter will bring it to Case Two.

The extremal black holes have  $\Delta(r) = 0$  at a single radius,  $r = GM$ . This represents an event horizon, but the  $r$  coordinate is never timelike; it becomes null at  $r = GM$ , but is spacelike on either side. The singularity at  $r = 0$  is a timelike line, as in the other cases. So for this black hole you can again avoid the singularity and continue to move to the future to extra copies of the asymptotically flat region, but the singularity is always “to the left.” The conformal diagram is shown in Figure 6.5.

A fascinating property of extremal black holes is that the mass is in some sense balanced by the charge. More specifically, two extremal holes with same-sign charges will attract each other gravitationally, but repel each other electromagnetically, and it turns out that these effects precisely cancel. Indeed, we can find *exact* solutions to the coupled Einstein–Maxwell equations representing any number of such black holes in a stationary configuration. To see this, turn first to the Reissner–Nordström metric itself, and let's stick with electric charges rather than magnetic charges, just for simplicity. At extremality,  $GM^2 = Q^2$ , and the metric takes the form

$$ds^2 = - \left(1 - \frac{GM}{r}\right)^2 dt^2 + \left(1 - \frac{GM}{r}\right)^{-2} dr^2 + r^2 d\Omega^2. \quad (6.58)$$

By defining a shifted radial coordinate

$$\rho = r - GM, \quad (6.59)$$

<sup>5</sup>For some work on this issue, see E. Poisson and W. Israel, *Phys. Rev. D* **41**, 1796 (1990).

the metric takes the isotropic form

$$ds^2 = -H^{-2}(\rho)dt^2 + H^2(\rho)[d\rho^2 + \rho^2d\Omega^2], \quad (6.60)$$

where

$$H(\rho) = 1 + \frac{GM}{\rho}. \quad (6.61)$$

Because  $d\rho^2 + \rho^2d\Omega^2$  is just the flat metric in three spatial dimensions, we can write (6.60) equally well as

$$ds^2 = -H^{-2}(\vec{x})dt^2 + H^2(\vec{x})[dx^2 + dy^2 + dz^2], \quad (6.62)$$

where  $H$  can be written

$$H = 1 + \frac{GM}{|\vec{x}|}. \quad (6.63)$$

In the original  $r$  coordinate, the electric field of the extremal solution can be expressed in terms of a vector potential  $A_\mu$  as

$$E_r = F_{rt} = \frac{Q}{r^2} = \partial_r A_0, \quad (6.64)$$

where the timelike component of the vector potential is

$$A_0 = -\frac{Q}{r}, \quad (6.65)$$

and we imagine the spatial components vanish (having set the magnetic field to zero). In our new  $\rho$  coordinate, and with the extremality condition  $Q^2 = GM^2$ , this becomes

$$A_0 = -\frac{\sqrt{GM}}{\rho + GM}, \quad (6.66)$$

or equivalently

$$\sqrt{G}A_0 = H^{-1} - 1. \quad (6.67)$$

But now let's forget that we know that  $H$  obeys (6.61), and simply plug the metric (6.62) and the electrostatic potential (6.67) into Einstein's equation and Maxwell's equations, imagining that  $H$  is time-independent ( $\partial_0 H = 0$ ) but otherwise unconstrained. We can straightforwardly show (see the Exercises) that they can be simultaneously satisfied by any time-independent function  $H(\vec{x})$  that obeys

$$\nabla^2 H = 0, \quad (6.68)$$

where  $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ . This is simply Laplace's equation, and it is straightforward to write down all of the solutions that are well-behaved at infinity; they take the form

$$H = 1 + \sum_{a=1}^N \frac{GM_a}{|\vec{x} - \vec{x}_a|}, \quad (6.69)$$

for some set of  $N$  spatial points defined by  $\vec{x}_a$ . These points describe the locations of  $N$  extremal RN black holes with masses  $M_a$  and charges  $Q_a = \sqrt{GM_a}$ . This multi-extremal-black hole metric is undoubtedly one of the most remarkable exact solutions to Einstein's equation.

## 6.6 ■ ROTATING (KERR) BLACK HOLES

We could go into a good deal more detail about the charged solutions, but let's instead move on to rotating black holes. To find the exact solution for the metric in this case is much more difficult, since we have given up on spherical symmetry. Instead we look for solutions with axial symmetry around the axis of rotation that are also stationary (a timelike Killing vector). Although the Schwarzschild and Reissner–Nordström solutions were discovered soon after general relativity was invented, the solution for a rotating black hole was found by Kerr only in 1963. His result, the **Kerr metric**, is given by the following mess:

$$\boxed{ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMar \sin^2 \theta}{\rho^2}(dt d\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2}[(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta]d\phi^2,} \quad (6.70)$$

where

$$\boxed{\Delta(r) = r^2 - 2GMr + a^2} \quad (6.71)$$

and

$$\boxed{\rho^2(r, \theta) = r^2 + a^2 \cos^2 \theta.} \quad (6.72)$$

The two constants  $M$  and  $a$  parameterize the possible solutions. To verify that the mass  $M$  is equal to the Komar energy (6.38) is straightforward but tedious, while  $a$  is the angular momentum per unit mass,

$$a = J/M, \quad (6.73)$$

where  $J$  is the Komar angular momentum (6.48). It is easy to include electric and magnetic charges  $Q$  and  $P$ , simply by replacing  $2GMr$  with  $2GMr - G(Q^2 +$

$P^2$ ); the result is the **Kerr–Newman metric**. The associated one-form potential has nonvanishing components

$$A_t = \frac{Qr - Pa \cos \theta}{\rho^2}, \quad A_\phi = \frac{-Qar \sin^2 \theta + P(r^2 + a^2) \cos \theta}{\rho^2}. \quad (6.74)$$

All of the essential phenomena persist in the absence of charges, so we will set  $Q = P = 0$  from now on.

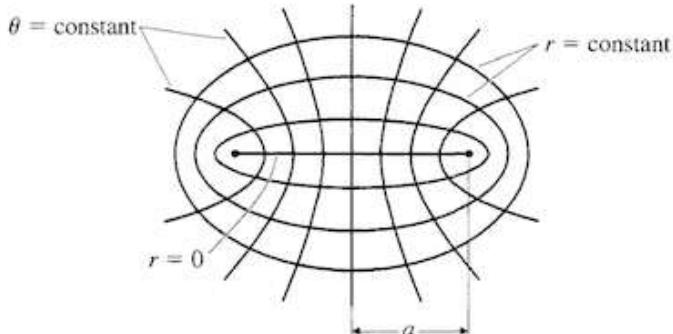
The coordinates  $(t, r, \theta, \phi)$  are known as **Boyer–Lindquist coordinates**. It is straightforward to check that as  $a \rightarrow 0$  they reduce to Schwarzschild coordinates. If we keep  $a$  fixed and let  $M \rightarrow 0$ , however, we recover flat spacetime but not in ordinary polar coordinates. The metric becomes

$$ds^2 = -dt^2 + \frac{(r^2 + a^2 \cos^2 \theta)}{(r^2 + a^2)} dr^2 + (r^2 + a^2 \cos^2 \theta)^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2, \quad (6.75)$$

and we recognize the spatial part of this as flat space in ellipsoidal coordinates, as shown in Figure 6.6. They are related to Cartesian coordinates in Euclidean 3-space by

$$\begin{aligned} x &= (r^2 + a^2)^{1/2} \sin \theta \cos \phi \\ y &= (r^2 + a^2)^{1/2} \sin \theta \sin \phi \\ z &= r \cos \theta. \end{aligned} \quad (6.76)$$

There are two Killing vectors of the metric (6.70), both of which are manifest; since the metric coefficients are independent of  $t$  and  $\phi$ , both  $K = \partial_t$  and  $R = \partial_\phi$  are Killing vectors. Of course  $R^\mu$  expresses the axial symmetry of the solution. The vector  $K^\mu$  is not orthogonal to  $t = \text{constant}$  hypersurfaces, and in fact is not orthogonal to any hypersurfaces at all; hence this metric is stationary, but not



**FIGURE 6.6** Ellipsoidal coordinates  $(r, \theta)$ , used in the Kerr metric.  $r = 0$  is a two-dimensional disk; the intersection of  $r = 0$  with  $\theta = \pi/2$  is the ring at the boundary of this disk.

static. This makes sense; the black hole is spinning, so it's not static, but it is spinning in exactly the same way at all times, so it's stationary. Alternatively, the metric can't be static because it's not time-reversal invariant, since that would reverse the angular momentum of the hole.

The Kerr metric also possesses a Killing tensor. These were defined in Chapter 3 as any symmetric  $(0, n)$  tensor  $\sigma_{\mu_1 \dots \mu_n}$  satisfying

$$\nabla_{(\lambda} \sigma_{\mu_1 \dots \mu_n)} = 0. \quad (6.77)$$

In the Kerr geometry we can define the  $(0, 2)$  tensor

$$\sigma_{\mu\nu} = 2\rho^2 l_{(\mu} n_{\nu)} + r^2 g_{\mu\nu}. \quad (6.78)$$

In this expression the two vectors  $l$  and  $n$  are given (with indices raised) by

$$\begin{aligned} l^\mu &= \frac{1}{\Delta} (r^2 + a^2, \Delta, 0, a) \\ n^\mu &= \frac{1}{2\rho^2} (r^2 + a^2, -\Delta, 0, a). \end{aligned} \quad (6.79)$$

Both vectors are null and satisfy

$$l^\mu l_\mu = 0, \quad n^\mu n_\mu = 0, \quad l^\mu n_\mu = -1. \quad (6.80)$$

With these definitions, you can check for yourself that  $\sigma_{\mu\nu}$  is a Killing tensor.

We have chosen coordinates for Kerr such that the event horizons occur at those fixed values of  $r$  for which  $g^{rr} = 0$ . Since  $g^{rr} = \Delta/\rho^2$ , and  $\rho^2 \geq 0$ , this occurs when

$$\Delta(r) = r^2 - 2GMr + a^2 = 0. \quad (6.81)$$

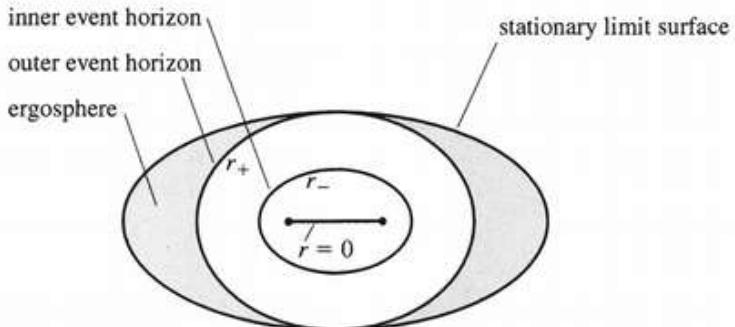
As in the Reissner–Nordström solution, there are three possibilities:  $GM > a$ ,  $GM = a$ , and  $GM < a$ . The last case features a naked singularity, and the extremal case  $GM = a$  is unstable, just as in Reissner–Nordström. Since these cases are of less physical interest, we will concentrate on  $GM > a$ . Then there are two radii at which  $\Delta$  vanishes, given by

$$r_\pm = GM \pm \sqrt{G^2 M^2 - a^2}. \quad (6.82)$$

Both radii are null surfaces that will turn out to be event horizons; a side view of a Kerr black hole is portrayed in Figure 6.7. The analysis of these surfaces proceeds in close analogy with the Reissner–Nordström case; it is straightforward to find coordinates that extend through the horizons.

Because Kerr is stationary but not static, the event horizons at  $r_\pm$  are not Killing horizons for the asymptotic time-translation Killing vector  $K = \partial_t$ . The norm of  $K^\mu$  is given by

$$K^\mu K_\mu = -\frac{1}{\rho^2} (\Delta - a^2 \sin^2 \theta). \quad (6.83)$$



**FIGURE 6.7** Horizon structure around the Kerr solution (side view). The event horizons are null surfaces that demarcate points past which it becomes impossible to return to a certain region of space. The stationary limit surface, in contrast, is timelike except where it is tangent to the event horizon (at the poles); it represents the place past which it is impossible to be a stationary observer. The ergosphere between the stationary limit surface and the outer event horizon is a region in which it is possible to enter and leave again, but not to remain stationary.

This does not vanish at the outer event horizon; in fact, at  $r = r_+$  (where  $\Delta = 0$ ), we have

$$K^\mu K_\mu = \frac{a^2}{\rho^2} \sin^2 \theta \geq 0. \quad (6.84)$$

So the Killing vector is already spacelike at the outer horizon, except at the north and south poles ( $\theta = 0, \pi$ ) where it is null. The locus of points where  $K^\mu K_\mu = 0$  is of course the stationary limit surface, and is given by

$$(r - GM)^2 = G^2 M^2 - a^2 \cos^2 \theta, \quad (6.85)$$

while the outer event horizon is given by

$$(r_+ - GM)^2 = G^2 M^2 - a^2. \quad (6.86)$$

There is thus a region between these two surfaces, known as the **ergosphere**. Inside the ergosphere, you must move in the direction of the rotation of the black hole (the  $\phi$  direction); however, you can still move toward or away from the event horizon (and have no trouble exiting the ergosphere). The ergosphere is evidently a place where interesting things can happen even before you cross the horizon; more details on this later.

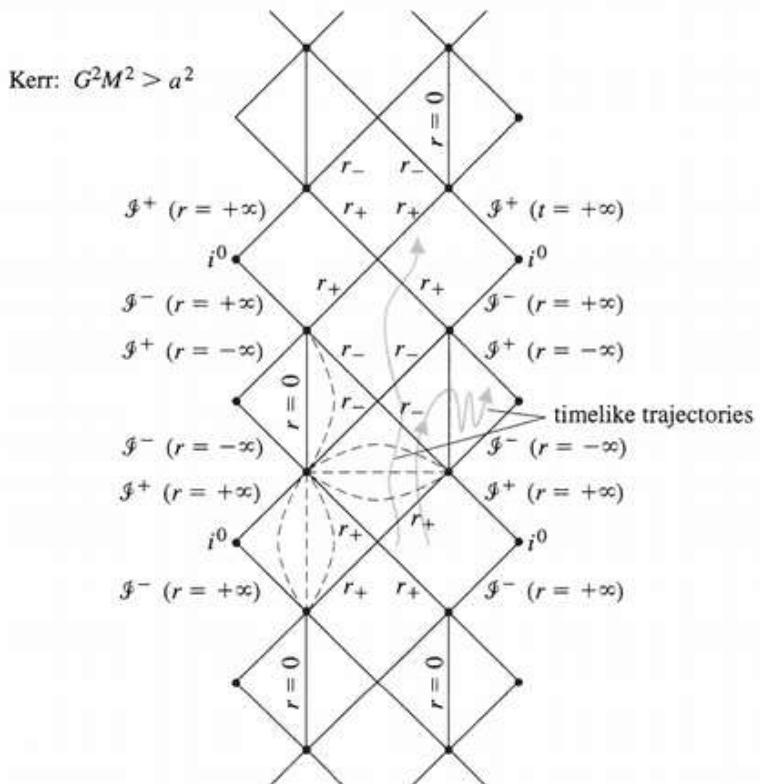
Before rushing to draw conformal diagrams, we need to understand the nature of the true curvature singularity; this does not occur at  $r = 0$  in this spacetime, but rather at  $\rho = 0$  (where the curvature invariant  $R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu}$  diverges). Since  $\rho^2 = r^2 + a^2 \cos^2 \theta$  is the sum of two manifestly nonnegative quantities, it can

only vanish when both quantities are zero, or

$$r = 0, \quad \theta = \frac{\pi}{2}. \quad (6.87)$$

This seems like a funny result, but remember that  $r = 0$  is not a point in space, but a disk; the set of points  $r = 0, \theta = \pi/2$  is actually the *ring* at the edge of this disk. The rotation has “softened” the Schwarzschild singularity, spreading it out over a ring.

What happens if you go inside the ring? A careful analytic continuation (which we will not perform) would reveal that you exit to another asymptotically flat spacetime, but not an identical copy of the one you came from. The new spacetime is described by the Kerr metric with  $r < 0$ . As a result,  $\Delta$  never vanishes and there are no horizons. The conformal diagram, Figure 6.8, is much like that for Reissner–Nordström, except now you can pass through the singularity. Because the Kerr metric is not spherically symmetric, the conformal diagram is not quite



**FIGURE 6.8** Conformal diagram for the Kerr solution with  $G^2M^2 > a^2$ . As with the analogous charged solution, there are an infinite number of copies of the region outside the black hole.

as faithful as in the previous cases; a single point on the diagram represents fixed values of  $t$  and  $r$ , and will have a different geometry for different values of  $\theta$ .

Not only do we have the usual strangeness of these distinct asymptotically flat regions connected to ours through the black hole, but the region near the ring singularity has additional pathologies: closed timelike curves. If you consider trajectories that wind around in  $\phi$  while keeping  $\theta$  and  $t$  constant and  $r$  a small negative value, the line element along such a path is

$$ds^2 \approx a^2 \left(1 + \frac{2GM}{r}\right) d\phi^2, \quad (6.88)$$

which is negative for small negative  $r$ . Since these paths are closed, they are obviously CTCs. You can therefore meet yourself in the past, with all that entails.

Of course, everything we say about the analytic extension of Kerr is subject to the same caveats we mentioned for Schwarzschild and Reissner–Nordström; it is unlikely that realistic gravitational collapse leads to these bizarre spacetimes. It is nevertheless always useful to have exact solutions. Furthermore, for the Kerr metric strange things are happening even if we stay outside the event horizon, to which we now turn.

We begin by considering more carefully the angular velocity of the hole. Obviously the conventional definition of angular velocity will have to be modified somewhat before we can apply it to something as abstract as the metric of space-time. Let us consider the fate of a photon that is emitted in the  $\phi$  direction at some radius  $r$  in the equatorial plane ( $\theta = \pi/2$ ) of a Kerr black hole. The instant it is emitted its momentum has no components in the  $r$  or  $\theta$  direction, and therefore the condition that the trajectory be null is

$$ds^2 = 0 = g_{tt} dt^2 + g_{t\phi} (dt d\phi + d\phi dt) + g_{\phi\phi} d\phi^2. \quad (6.89)$$

This can be immediately solved to obtain

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}. \quad (6.90)$$

If we evaluate this quantity on the stationary limit surface of the Kerr metric, we have  $g_{tt} = 0$ , and the two solutions are

$$\frac{d\phi}{dt} = 0, \quad \frac{d\phi}{dt} = \frac{a}{2G^2M^2 + a^2}. \quad (6.91)$$

The nonzero solution has the same sign as  $a$ ; we interpret this as the photon moving around the hole in the same direction as the hole's rotation. The zero solution means that the photon directed against the hole's rotation doesn't move at all in this coordinate system. Note that we haven't given a full solution to the photon's trajectory, only shown that its instantaneous velocity is zero. This is an example of a phenomenon known as the "dragging of inertial frames"; it is ex-

plored more in one of the exercises to Chapter 7. Massive particles, which must move more slowly than photons, are necessarily dragged along with the hole's rotation once they are inside the stationary limit surface. This dragging continues as we approach the outer event horizon at  $r_+$ ; we can define the angular velocity of the event horizon itself,  $\Omega_H$ , to be the minimum angular velocity of a particle at the horizon. Directly from (6.90) we find that

$$\Omega_H = \left( \frac{d\phi}{dt} \right)_- (r_+) = \frac{a}{r_+^2 + a^2}. \quad (6.92)$$

## 6.7 ■ THE PENROSE PROCESS AND BLACK-HOLE THERMODYNAMICS

Black hole thermodynamics is one of the most fascinating and mysterious subjects in general relativity. To get there, however, let us begin with something apparently very straightforward: motion along geodesics in the Kerr metric. We know that such a discussion will be simplified by considering the conserved quantities associated with the Killing vectors  $K = \partial_t$  and  $R = \partial_\phi$ . For the purposes at hand we can restrict our attention to massive particles, for which we can work with the four-momentum

$$p^\mu = m \frac{dx^\mu}{d\tau}, \quad (6.93)$$

where  $m$  is the rest mass of the particle. Then we can take as our two conserved quantities the actual energy and angular momentum of the particle,

$$E = -K_\mu p^\mu = m \left( 1 - \frac{2GMr}{\rho^2} \right) \frac{dt}{d\tau} + \frac{2mGMar}{\rho^2} \sin^2 \theta \frac{d\phi}{d\tau} \quad (6.94)$$

and

$$L = R_\mu p^\mu = -\frac{2mGMar}{\rho^2} \sin^2 \theta \frac{dt}{d\tau} + \frac{m(r^2 + a^2)^2 - m\Delta a^2 \sin^2 \theta}{\rho^2} \sin^2 \theta \frac{d\phi}{d\tau}. \quad (6.95)$$

These differ from the definitions for the conserved quantities used in the last chapter, where  $E$  and  $L$  were taken to be the energy and angular momentum *per unit mass*. They are conserved either way, of course.

The minus sign in the definition of  $E$  is there because at infinity both  $K^\mu$  and  $p^\mu$  are timelike, so their inner product is negative, but we want the energy to be positive. Inside the ergosphere, however,  $K^\mu$  becomes spacelike; we can therefore imagine particles for which

$$E = -K_\mu p^\mu < 0. \quad (6.96)$$

The extent to which this bothers us is ameliorated somewhat by the realization that *all* particles must have positive energies if they are outside the stationary limit surface; therefore a particle inside the ergosphere with negative energy must either remain in the ergosphere, or be accelerated until its energy is positive if it is to escape.

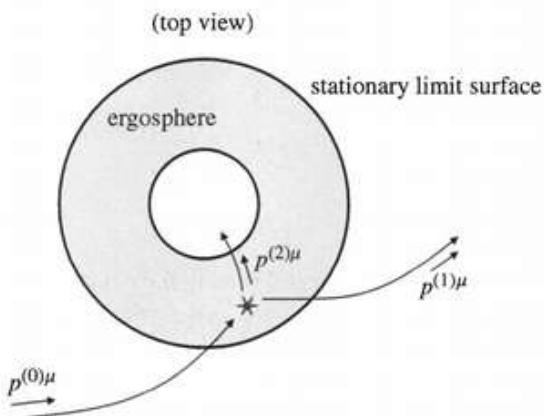
Still, this realization leads to a way to extract energy from a rotating black hole; the method is known as the **Penrose process**. The idea is simple; starting from outside the ergosphere, you arm yourself with a large rock and leap toward the black hole. If we call the four-momentum of the (you + rock) system  $p^{(0)\mu}$ , then the energy  $E^{(0)} = -K_\mu p^{(0)\mu}$  is certainly positive, and conserved as you move along your geodesic. Once you enter the ergosphere, you hurl the rock with all your might, in a very specific way. If we call your momentum  $p^{(1)\mu}$  and that of the rock  $p^{(2)\mu}$ , then at the instant you throw it we have conservation of momentum just as in special relativity:

$$p^{(0)\mu} = p^{(1)\mu} + p^{(2)\mu}. \quad (6.97)$$

Contracting with the Killing vector  $K_\mu$  gives

$$E^{(0)} = E^{(1)} + E^{(2)}. \quad (6.98)$$

But, if we imagine that you are arbitrarily strong (and accurate), you can arrange your throw such that  $E^{(2)} < 0$ , as per (6.96). Furthermore, Penrose was able to show that you can arrange the initial trajectory and the throw as shown in Figure 6.9, such that afterward you follow a geodesic trajectory back outside the stationary limit surface into the external universe. Since your energy is conserved



**FIGURE 6.9** The Penrose process (top view). An object falls toward a Kerr black hole and splits in two while in the ergosphere (within the stationary limit surface, but outside the outer event horizon). One piece falls into the horizon with a negative energy  $E^{(2)}$ , while the other escapes to infinity with a larger energy than that of the original infalling object.

along the way, at the end we will have

$$E^{(1)} > E^{(0)}. \quad (6.99)$$

Thus, you have emerged with *more* energy than you entered with.

There is no such thing as a free lunch; the energy you gained came from somewhere, and that somewhere is the black hole. In fact, the Penrose process extracts energy from the rotating black hole by decreasing its angular momentum; you have to throw the rock against the hole's rotation to get the trick to work. To see this more precisely, recall that we claimed earlier in this chapter that any event horizon in a stationary spacetime would be a Killing horizon for some Killing vector. For Kerr this is a linear combination of the time-translation and rotational Killing vectors,

$$\chi^\mu = K^\mu + \Omega_H R^\mu, \quad (6.100)$$

where  $\Omega_H$  is precisely the angular velocity of the horizon as defined in (6.92). Using  $K = \partial_t$  and  $R = \partial_\phi$ , it is straightforward to verify that  $\chi^\mu$  becomes null at the outer event horizon. The statement that the particle with momentum  $p^{(2)\mu}$  crosses the event horizon “moving forward in time” is simply

$$p^{(2)\mu} \chi_\mu < 0. \quad (6.101)$$

Plugging in the definitions of  $E$  and  $L$ , we see that this condition is equivalent to

$$L^{(2)} < \frac{E^{(2)}}{\Omega_H}. \quad (6.102)$$

Since we have arranged  $E^{(2)}$  to be negative, and  $\Omega_H$  positive, we see that the particle must have a negative angular momentum—it is moving against the hole's rotation. Once you have escaped the ergosphere and the rock has fallen inside the event horizon, the mass and angular momentum of the hole are what they used to be plus the negative contributions of the rock:

$$\begin{aligned} \delta M &= E^{(2)} \\ \delta J &= L^{(2)}, \end{aligned} \quad (6.103)$$

where  $J = Ma$  is the angular momentum of the black hole. Then (6.102) becomes a limit on how much you can decrease the angular momentum:

$$\delta J < \frac{\delta M}{\Omega_H}. \quad (6.104)$$

If we exactly reach this limit, as the rock we throw in becomes more and more null, we have the “ideal” process, in which  $\delta J = \delta M / \Omega_H$ .

We will now use these ideas to verify that, although you can use the Penrose process to extract energy from the black hole (thereby decreasing  $M$ ), you cannot

violate the area theorem: The area of the event horizon is nondecreasing. Although the mass decreases, the angular momentum must also decrease, in a combination which only allows the area to increase. To see this, let's calculate the area of the outer event horizon, which is located at

$$r_+ = GM + \sqrt{G^2M^2 - a^2}. \quad (6.105)$$

The induced metric  $\gamma_{ij}$  on the horizon (where  $i$  and  $j$  run over  $\{\theta, \phi\}$ ) can be found straightforwardly by setting  $r = r_+$  (so  $\Delta = 0$ ),  $dt = 0$  and  $dr = 0$  in (6.70):

$$\begin{aligned} \gamma_{ij} dx^i dx^j &= ds^2(dt = 0, dr = 0, r = r_+) \\ &= (r_+^2 + a^2 \cos^2 \theta) d\theta^2 + \left[ \frac{(r_+^2 + a^2)^2 \sin^2 \theta}{r_+^2 + a^2 \cos^2 \theta} \right] d\phi^2. \end{aligned} \quad (6.106)$$

The horizon area is then the integral of the induced volume element,

$$A = \int \sqrt{|\gamma|} d\theta d\phi. \quad (6.107)$$

The determinant is

$$|\gamma| = (r_+^2 + a^2)^2 \sin^2 \theta, \quad (6.108)$$

so the horizon area is simply

$$A = 4\pi(r_+^2 + a^2). \quad (6.109)$$

To show that the area doesn't decrease, it is convenient to work instead in terms of the **irreducible mass** of the black hole, defined by

$$\begin{aligned} M_{\text{irr}}^2 &= \frac{A}{16\pi G^2} \\ &= \frac{1}{4G^2}(r_+^2 + a^2) \\ &= \frac{1}{2} \left( M^2 + \sqrt{M^4 - (Ma/G)^2} \right) \\ &= \frac{1}{2} \left( M^2 + \sqrt{M^4 - (J/G)^2} \right). \end{aligned} \quad (6.110)$$

We can differentiate to obtain, after a bit of work, how  $M_{\text{irr}}$  is affected by changes in the mass or angular momentum,

$$\delta M_{\text{irr}} = \frac{a}{4GM_{\text{irr}}\sqrt{G^2M^2 - a^2}} (\Omega_H^{-1}\delta M - \delta J). \quad (6.111)$$

Then our limit (6.104) becomes

$$\delta M_{\text{irr}} > 0. \quad (6.112)$$

The irreducible mass can never be reduced; hence the name. It follows that the maximum amount of energy we can extract from a black hole before we slow its rotation to zero is

$$M - M_{\text{irr}} = M - \frac{1}{\sqrt{2}} \left( M^2 + \sqrt{M^4 - (J/G)^2} \right)^{1/2}. \quad (6.113)$$

The result of this complete extraction is a Schwarzschild black hole of mass  $M_{\text{irr}}$ . It turns out that the best we can do is to start with an extreme Kerr black hole; then we can get out approximately 29% of its total energy.

The irreducibility of  $M_{\text{irr}}$  leads immediately to the fact that the area  $A$  can never decrease. From (6.110) and (6.111) we have

$$\delta A = 8\pi G \frac{a}{\Omega_H \sqrt{G^2 M^2 - a^2}} (\delta M - \Omega_H \delta J), \quad (6.114)$$

which can be recast as

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega_H \delta J, \quad (6.115)$$

where we have introduced

$$\kappa = \frac{\sqrt{G^2 M^2 - a^2}}{2GM(GM + \sqrt{G^2 M^2 - a^2})}. \quad (6.116)$$

The quantity  $\kappa$  is of course just the surface gravity of the Kerr solution, as you could verify by plugging (6.100) into (6.9).

Equations like (6.115) first started people thinking about a correspondence between black holes and thermodynamics. Consider the first law of thermodynamics,

$$dE = T dS - p dV, \quad (6.117)$$

where  $T$  is the temperature,  $S$  is the entropy,  $p$  is the pressure, and  $V$  is the volume, so the  $p dV$  term represents work we do to the system. It is natural to think of the term  $\Omega_H \delta J$  in (6.115) as work that we do on the black hole by throwing rocks into it. Then the correspondence begins to take shape if we think of identifying the thermodynamic quantities energy, entropy, and temperature with the black-hole mass, area, and surface gravity:

$$\begin{aligned} E &\leftrightarrow M \\ S &\leftrightarrow A/4G \\ T &\leftrightarrow \kappa/2\pi. \end{aligned} \quad (6.118)$$

(Remember we are using units in which  $\hbar = c = k = 1$ .) In the context of classical general relativity the analogy is essentially perfect, with each law of thermodynamics corresponding to a law of black hole mechanics. A system in thermal equilibrium will have settled to a stationary state, corresponding to a stationary black hole. The zeroth law of thermodynamics states that in thermal equilibrium the temperature is constant throughout the system; the analogous statement for black holes is that stationary black holes have constant surface gravity on the entire horizon. This will be true, at least under the same reasonable assumptions under which the event horizon is a Killing horizon. As we have seen, the first law (6.117) is equivalent to (6.115). The second law, that entropy never decreases, is simply the statement that the area of the horizon never decreases. Finally, the usual statement of the third law is that it is impossible to achieve  $T = 0$  in any physical process, or that the entropy must go to zero as the temperature goes to zero. For black holes this doesn't quite work; it turns out that  $\kappa = 0$  corresponds to extremal black holes, which don't necessarily have a vanishing area. But the thermodynamic third law doesn't really work either, in the sense that there are ordinary physical systems that violate it; the third law applies to some situations but is not truly fundamental.

We have cheated a little in proposing the correspondence (6.118); you will notice that by equating  $TdS$  with  $\kappa dA/8\pi G$  we do not know how to separately normalize  $S/A$  or  $T/\kappa$ , only their combination. As we will discuss in Chapter 9, however, Hawking showed that quantum fields in a black-hole background allow the hole to radiate at a temperature  $T = \kappa/2\pi$ . Once this is known, we can interpret  $A/4G$  as an actual entropy of the black hole. Bekenstein has proposed a **generalized second law**, that the combined entropy of matter and black holes never decreases:

$$\delta \left( S + \frac{A}{4G} \right) \geq 0. \quad (6.119)$$

The generalized second law can actually be proven under a variety of assumptions. Usually, however, we like to associate the entropy of a system with the logarithm of the number of accessible quantum states. There is therefore some tension between this concept and the no-hair theorem, which indicates that there are very few possible states for a black hole of fixed charge, mass, and spin (only one, in fact). It seems likely that this behavior is an indication of a profound feature of the interaction between quantum mechanics and gravitation.

## 6.8 ■ EXERCISES

1. Show that the coupled Einstein–Maxwell equations can be simultaneously solved by the metric (6.62) and the electrostatic potential (6.67) if  $H(\vec{x})$  obeys Laplace's equation,

$$\nabla^2 H = 0. \quad (6.120)$$

2. Consider the orbits of massless particles, with affine parameter  $\lambda$ , in the equatorial plane of a Kerr black hole.

- (a) Show that

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{\Sigma^2}{\rho^4} (E - LW_+(r))(E - LW_-(r)), \quad (6.121)$$

where  $\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta(r) \sin^2 \theta$ ,  $E$  and  $L$  are the conserved energy and angular momentum, and you have to find expressions for  $W_{\pm}(r)$ .

- (b) Using this result, and assuming that  $\Sigma^2 > 0$  everywhere, show that the orbit of a photon in the equatorial plane cannot have a turning point inside the outer event horizon  $r_+$ . This means that ingoing light rays cannot escape once they cross  $r_+$ , so it really is an event horizon.

3. In the presence of an electromagnetic field, a particle of charge  $e$  and mass  $m$  obeys

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = \frac{e}{m} F^\mu_\nu \frac{dx^\nu}{d\tau}. \quad (6.122)$$

Imagine that such a particle is moving in the field of a Reissner–Nordström black hole with charge  $Q$  and mass  $M$ .

- (a) Show that the energy

$$E = m \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right) \frac{dt}{d\tau} + \frac{eQ}{r} \quad (6.123)$$

is conserved.

- (b) Will a Penrose-type process work for a charged black hole? What is the change in the black hole mass,  $\delta M$ , for the maximum physical process?

4. Consider de Sitter space in static coordinates:

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right) dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3}r^2} + r^2 d\Omega^2.$$

This space has a Killing vector  $\partial_t$  that is timelike near  $r = 0$  and null on a Killing horizon. Locate the radial position of the Killing horizon,  $r_K$ . What is the surface gravity,  $\kappa$ , of the horizon? Consider the Euclidean signature version of de Sitter space obtained by making the replacement  $t \rightarrow i\tau$ . Show that a coordinate transformation can be made to make the Euclidean metric regular at the horizon, so long as  $\tau$  is made periodic.

5. What is the magnetic field seen by an observer orbiting a Reissner–Nordström black hole of electric charge  $Q$  and mass  $M$  in a circular orbit with circumference  $2\pi R$ ?
6. Consider a Kerr black hole with an accretion disk of negligible mass in the equatorial plane. Assume that particles in the disk follow geodesics (that is, ignore any pressure support). Now suppose the disk contains some iron atoms that are being excited by a source of radiation. When the iron atoms de-excite they emit radiation with a known frequency  $v_0$ , as measured in their rest frame. Suppose we detect this radiation far from the black hole (we also lie in the equatorial plane). What is the observed frequency of photons emitted from either edge of the disk, and from the center of the disk? Consider cases where the disk and the black hole are rotating in the same and opposite directions. Can we use these measurements to determine the mass and angular momentum of the black hole?

# Perturbation Theory and Gravitational Radiation

## 7.1 ■ LINEARIZED GRAVITY AND GAUGE TRANSFORMATIONS

When we first derived Einstein's equation, we checked that we were on the right track by considering the Newtonian limit. We took this to mean not only that the gravitational field was weak, but also that it was static (no time derivatives), and that test particles were moving slowly. The weak-field limit described in this chapter is less restrictive, assuming that the field is still weak but it can vary with time, and without any restrictions on the motion of test particles. This will allow us to discuss phenomena that are absent or ambiguous in the Newtonian theory, such as gravitational radiation (where the field varies with time) and the deflection of light (which involves fast-moving particles).

The weakness of the gravitational field is once again expressed as our ability to decompose the metric into the flat Minkowski metric plus a small perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \quad (7.1)$$

We will restrict ourselves to coordinates in which  $\eta_{\mu\nu}$  takes its canonical form,  $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ . The assumption that  $h_{\mu\nu}$  is small allows us to ignore anything that is higher than first order in this quantity, from which we immediately obtain

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad (7.2)$$

where  $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$ . As before, we can raise and lower indices using  $\eta^{\mu\nu}$  and  $\eta_{\mu\nu}$ , since the corrections would be of higher order in the perturbation. In fact, we can think of the linearized version of general relativity (where effects of higher than first order in  $h_{\mu\nu}$  are neglected) as describing a theory of a symmetric tensor field  $h_{\mu\nu}$  propagating on a flat background spacetime. This theory is Lorentz invariant in the sense of special relativity; under a Lorentz transformation  $x^{\mu'} = \Lambda^{\mu'}_{\mu}x^{\mu}$ , the flat metric  $\eta_{\mu\nu}$  is invariant, while the perturbation transforms as

$$h_{\mu'\nu'} = \Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}h_{\mu\nu}. \quad (7.3)$$

Note that we could have considered small perturbations about some other background spacetime besides Minkowski space. In that case the metric would have been written  $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$ , and we would have derived a theory of a symmetric tensor propagating on the curved space with metric  $g_{\mu\nu}^{(0)}$ . Such an approach is necessary, for example, in cosmology.

We want to find the equations of motion obeyed by the perturbations  $h_{\mu\nu}$ , which come by examining Einstein's equation to first order. We begin with the Christoffel symbols, which are given by

$$\begin{aligned}\Gamma_{\mu\nu}^{\rho} &= \frac{1}{2}g^{\rho\lambda}(\partial_{\mu}g_{\nu\lambda} + \partial_{\nu}g_{\lambda\mu} - \partial_{\lambda}g_{\mu\nu}) \\ &= \frac{1}{2}\eta^{\rho\lambda}(\partial_{\mu}h_{\nu\lambda} + \partial_{\nu}h_{\lambda\mu} - \partial_{\lambda}h_{\mu\nu}).\end{aligned}\quad (7.4)$$

Since the connection coefficients are first-order quantities, the only contribution to the Riemann tensor will come from the derivatives of the  $\Gamma$ 's, not the  $\Gamma^2$  terms. Lowering an index for convenience, we obtain

$$\begin{aligned}R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_{\rho}\Gamma_{\nu\sigma}^{\lambda} - \eta_{\mu\lambda}\partial_{\sigma}\Gamma_{\nu\rho}^{\lambda} \\ &= \frac{1}{2}(\partial_{\rho}\partial_{\nu}h_{\mu\sigma} + \partial_{\sigma}\partial_{\mu}h_{\nu\rho} - \partial_{\sigma}\partial_{\nu}h_{\mu\rho} - \partial_{\rho}\partial_{\mu}h_{\nu\sigma}).\end{aligned}\quad (7.5)$$

The Ricci tensor comes from contracting over  $\mu$  and  $\rho$ , giving

$$R_{\mu\nu} = \frac{1}{2}(\partial_{\sigma}\partial_{\nu}h^{\sigma}_{\mu} + \partial_{\sigma}\partial_{\mu}h^{\sigma}_{\nu} - \partial_{\mu}\partial_{\nu}h - \square h_{\mu\nu}), \quad (7.6)$$

which is manifestly symmetric in  $\mu$  and  $\nu$ . In this expression we have defined the trace of the perturbation as  $h = \eta^{\mu\nu}h_{\mu\nu} = h^{\mu}_{\mu}$ , and the d'Alembertian is simply the one from flat space,  $\square = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$ . Contracting again to obtain the Ricci scalar yields

$$R = \partial_{\mu}\partial_{\nu}h^{\mu\nu} - \square h. \quad (7.7)$$

Putting it all together we obtain the Einstein tensor:

$$\begin{aligned}G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\ &= \frac{1}{2}(\partial_{\sigma}\partial_{\nu}h^{\sigma}_{\mu} + \partial_{\sigma}\partial_{\mu}h^{\sigma}_{\nu} - \partial_{\mu}\partial_{\nu}h - \square h_{\mu\nu} - \eta_{\mu\nu}\partial_{\rho}\partial_{\lambda}h^{\rho\lambda} + \eta_{\mu\nu}\square h).\end{aligned}\quad (7.8)$$

Consistent with our interpretation of the linearized theory as one describing a symmetric tensor on a flat background, the linearized Einstein tensor (7.8) can be derived by varying the following Lagrangian with respect to  $h_{\mu\nu}$ :

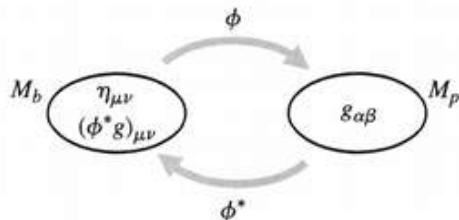
$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left[(\partial_{\mu}h^{\mu\nu})(\partial_{\nu}h) - (\partial_{\mu}h^{\rho\sigma})(\partial_{\rho}h^{\mu}_{\sigma}) + \frac{1}{2}\eta^{\mu\nu}(\partial_{\mu}h^{\rho\sigma})(\partial_{\nu}h_{\rho\sigma})\right. \\ &\quad \left.- \frac{1}{2}\eta^{\mu\nu}(\partial_{\mu}h)(\partial_{\nu}h)\right].\end{aligned}\quad (7.9)$$

You are asked to verify the appropriateness of the Lagrangian in the exercises.

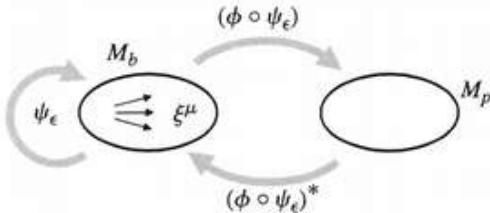
The linearized field equation is of course  $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ , where  $G_{\mu\nu}$  is given by (7.8) and  $T_{\mu\nu}$  is the energy-momentum tensor, calculated to zeroth order in  $h_{\mu\nu}$ . We do not include higher-order corrections to the energy-momentum tensor because the amount of energy and momentum must itself be small for the weak-field limit to apply. In other words, the lowest nonvanishing order in  $T_{\mu\nu}$  is automatically of the same order of magnitude as the perturbation. Notice that the conservation law to lowest order is simply  $\partial_\mu T^{\mu\nu} = 0$ . We will often be concerned with the vacuum equation, which as usual is just  $R_{\mu\nu} = 0$ , where  $R_{\mu\nu}$  is given by (7.6).

With the linearized field equation in hand, we are almost prepared to set about solving it. First, however, we should deal with the thorny issue of gauge invariance. This issue arises because the demand that  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  does not completely specify the coordinate system on spacetime; there may be other coordinate systems in which the metric can still be written as the Minkowski metric plus a small perturbation, but the perturbation will be different. Thus, the decomposition of the metric into a flat background plus a perturbation is not unique. To examine this issue, we will draw upon ideas about diffeomorphisms discussed in Appendices A and B; readers who have not yet read those sections can skip to equation (7.14) and the two paragraphs following, which contain the essential ideas.

Let's think about gauge invariance from a highbrow point of view. The notion that the linearized theory can be thought of as one governing the behavior of tensor fields on a flat background can be formalized in terms of a *background spacetime*  $M_b$ , a *physical spacetime*  $M_p$ , and a diffeomorphism  $\phi : M_b \rightarrow M_p$ . As manifolds  $M_b$  and  $M_p$  are the same (since they are diffeomorphic), but we imagine that they possess some different tensor fields; on  $M_b$  we have defined the flat Minkowski metric  $\eta_{\mu\nu}$ , while on  $M_p$  we have some metric  $g_{\alpha\beta}$  that obeys Einstein's equation. (We imagine that  $M_b$  is equipped with coordinates  $x^\mu$  and  $M_p$  is equipped with coordinates  $y^\alpha$ , although these will not play a prominent role.) The diffeomorphism  $\phi$  allows us to move tensors back and forth between the background and physical spacetimes, as in Figure 7.1. Since we would like to construct our linearized theory as one taking place on the flat background spacetime, we are interested in the pullback  $(\phi^* g)_{\mu\nu}$  of the physical metric. We can define the perturbation as the difference between the pulled-back physical metric



**FIGURE 7.1** A diffeomorphism relating the background spacetime  $M_b$  (with flat metric  $\eta_{\mu\nu}$ ) to the physical spacetime  $M_p$ .



**FIGURE 7.2** A one-parameter family of diffeomorphisms  $\psi_\epsilon$ , generated by the vector field  $\xi^\mu$  on the background spacetime  $M_b$ .

and the flat one:

$$h_{\mu\nu} = (\phi^* g)_{\mu\nu} - \eta_{\mu\nu}. \quad (7.10)$$

From this definition, there is no reason for the components of  $h_{\mu\nu}$  to be small; however, if the gravitational fields on  $M_p$  are weak, then for *some* diffeomorphisms  $\phi$  we will have  $|h_{\mu\nu}| \ll 1$ . We therefore limit our attention only to those diffeomorphisms for which this is true. Then the fact that  $g_{\alpha\beta}$  obeys Einstein's equation on the physical spacetime means that  $h_{\mu\nu}$  will obey the linearized equation on the background spacetime (since  $\phi$ , as a diffeomorphism, can be used to pull back Einstein's equation themselves).

In this language, the issue of gauge invariance is simply that there are a large number of permissible diffeomorphisms between  $M_b$  and  $M_p$  (where “permissible” means that the perturbation is small). Consider a vector field  $\xi^\mu(x)$  on the background spacetime. This vector field generates a one-parameter family of diffeomorphisms  $\psi_\epsilon : M_b \rightarrow M_b$ , as shown in Figure 7.2. For  $\epsilon$  sufficiently small, if  $\phi$  is a diffeomorphism for which the perturbation defined by (7.10) is small, then so will  $(\phi \circ \psi_\epsilon)$  be, although the perturbation will have a different value. Specifically, we can define a family of perturbations parameterized by  $\epsilon$ :

$$\begin{aligned} h_{\mu\nu}^{(\epsilon)} &= [(\phi \circ \psi_\epsilon)^* g]_{\mu\nu} - \eta_{\mu\nu} \\ &= [\psi_\epsilon^* (\phi^* g)]_{\mu\nu} - \eta_{\mu\nu}. \end{aligned} \quad (7.11)$$

The second equality is based on the fact that the pullback under a composition is given by the composition of the pullbacks in the opposite order, which follows from the fact that the pullback itself moves things in the opposite direction from the original map. Plugging in the relation (7.10), we find

$$\begin{aligned} h_{\mu\nu}^{(\epsilon)} &= \psi_\epsilon^* (h + \eta)_{\mu\nu} - \eta_{\mu\nu} \\ &= \psi_\epsilon^* (h_{\mu\nu}) + \psi_\epsilon^* (\eta_{\mu\nu}) - \eta_{\mu\nu}, \end{aligned} \quad (7.12)$$

since the pullback of the sum of two tensors is the sum of the pullbacks. Now we use our assumption that  $\epsilon$  is small; in this case  $\psi_\epsilon^*(h_{\mu\nu})$  will be equal to  $h_{\mu\nu}$  to lowest order, while the other two terms give us a Lie derivative:

$$\begin{aligned} h_{\mu\nu}^{(\epsilon)} &= \psi_\epsilon^*(h_{\mu\nu}) + \epsilon \left[ \frac{\psi_\epsilon^*(\eta_{\mu\nu}) - \eta_{\mu\nu}}{\epsilon} \right] \\ &= h_{\mu\nu} + \epsilon \mathcal{L}_\xi \eta_{\mu\nu}. \end{aligned} \quad (7.13)$$

In Appendix B we show that the Lie derivative of the metric along a vector field  $\xi_\mu$  is  $\mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$ . In the current context the background metric is flat, and covariant derivatives become partial derivatives; we therefore have

$$h_{\mu\nu}^{(\epsilon)} = h_{\mu\nu} + 2\epsilon \partial_{(\mu} \xi_{\nu)}. \quad (7.14)$$

This formula represents the change of the metric perturbation under an infinitesimal diffeomorphism along the vector field  $\epsilon\xi^\mu$ : we will call this a **gauge transformation** in linearized theory.

The diffeomorphisms  $\psi_\epsilon$  provide a different representation of the same physical situation, while maintaining our requirement that the perturbation be small. Therefore, the result (7.12) tells us what kind of metric perturbations denote physically equivalent spacetimes—those related to each other by  $2\epsilon \partial_{(\mu} \xi_{\nu)}$ , for some vector  $\xi^\mu$ . The invariance of our theory under such transformations is analogous to traditional gauge invariance of electromagnetism under  $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$ . (The analogy is different from another analogy we draw with electromagnetism in Appendix J, relating local Lorentz transformations in the orthonormal-frame formalism to changes of basis in an internal vector bundle.) In electromagnetism the invariance comes about because the field strength  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is left unchanged by gauge transformations; similarly, we find that the transformation (7.14) changes the linearized Riemann tensor by

$$\begin{aligned} \delta R_{\mu\nu\rho\sigma} &= \frac{1}{2} (\partial_\rho \partial_\nu \partial_\mu \xi_\sigma + \partial_\rho \partial_\nu \partial_\sigma \xi_\mu + \partial_\sigma \partial_\mu \partial_\nu \xi_\rho + \partial_\sigma \partial_\mu \partial_\rho \xi_\nu \\ &\quad - \partial_\sigma \partial_\nu \partial_\mu \xi_\rho - \partial_\sigma \partial_\nu \partial_\rho \xi_\mu - \partial_\rho \partial_\mu \partial_\nu \xi_\sigma - \partial_\rho \partial_\mu \partial_\sigma \xi_\nu) \\ &= 0. \end{aligned} \quad (7.15)$$

Our abstract derivation of the appropriate gauge transformation for the metric perturbation is verified by the fact that it leaves the curvature (and hence the physical spacetime) unchanged.

Gauge invariance can also be understood from the slightly more lowbrow but considerably more direct route of infinitesimal coordinate transformations. Our diffeomorphism  $\psi_\epsilon$  can be thought of as changing coordinates from  $x^\mu$  to  $x^\mu - \epsilon \xi^\mu$ . (The minus sign, which is unconventional, comes from the fact that the “new” metric is pulled back from a small distance forward along the integral curves, which is equivalent to replacing the coordinates by those a small distance backward along the curves.) Following through the usual rules for transforming tensors under coordinate transformations, you can derive precisely (7.14)—although you have to cheat somewhat by equating components of tensors in two different coordinate systems.

## 7.2 ■ DEGREES OF FREEDOM

With the expression (7.8) for the linearized Einstein tensor, and the expression (7.14) for the effect of gauge transformations, we could immediately set about choosing a gauge and solving Einstein's equation. However, we can accumulate some additional physical insight by first choosing a fixed inertial coordinate system in the Minkowski background spacetime, and decomposing the components of the metric perturbation according to their transformation properties under spatial rotations. You might worry that such a decomposition is contrary to the coordinate-independent spirit of general relativity, but it is really no different than decomposing the electromagnetic field strength tensor into electric and magnetic fields. Even though both  $\mathbf{E}$  and  $\mathbf{B}$  are components of a  $(0, 2)$  tensor, it is nevertheless sometimes convenient to assume the role of some fixed observer and think of them as three-vectors.<sup>1</sup>

The metric perturbation is a  $(0, 2)$  tensor, but symmetric rather than antisymmetric. Under spatial rotations, the  $00$  component is a scalar, the  $0i$  components (equal to the  $i0$  components) form a three-vector, and the  $ij$  components form a two-index symmetric spatial tensor. This spatial tensor can be further decomposed into a trace and a trace-free part. (In group theory language, we are looking for “irreducible representations” of the rotation group. In other words, we decompose the tensor into individual pieces, which transform only into themselves under spatial rotations.) We therefore write  $h_{\mu\nu}$  as

$$\begin{aligned} h_{00} &= -2\Phi \\ h_{0i} &= w_i \\ h_{ij} &= 2s_{ij} - 2\Psi\delta_{ij}, \end{aligned} \tag{7.16}$$

where  $\Psi$  encodes the trace of  $h_{ij}$ , and  $s_{ij}$  is traceless:

$$\begin{aligned} \Psi &= -\frac{1}{6}\delta^{ij}h_{ij} \\ s_{ij} &= \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right). \end{aligned} \tag{7.17}$$

The entire metric is thus written as

$$ds^2 = -(1 + 2\Phi)dt^2 + w_i(dt dx^i + dx^i dt) + [(1 - 2\Psi)\delta_{ij} + 2s_{ij}]dx^i dx^j.$$

(7.18)

<sup>1</sup>The discussion here follows that in E. Bertschinger, “Cosmological Dynamics,” a talk given at Summer School on Cosmology and Large Scale Structure (Session 60), Les Houches, France, 1–28 Aug 1993; <http://arXiv.org/abs/astro-ph/9503125>. Bertschinger focuses on cosmological perturbation theory, in which spacelike hypersurfaces are expanding with time, but it is simple enough to specialize to the case of a nonexpanding universe.

We have not chosen a gauge or solved any equations, just defined some convenient notation. The traceless tensor  $s_{ij}$  is known as the *strain*, and will turn out to contain gravitational radiation. Sometimes the decomposition of the spatial components into trace and trace-free parts is not helpful, and we can just stick with  $h_{ij}$ ; we will use whichever notation is appropriate in individual cases. Note that, just as in Chapter 1, the spatial metric is now simply  $\delta_{ij}$ , and we can freely raise and lower spatial indices without changing the components.

To get a feeling for the physical interpretation of the different fields in the metric perturbation, we consider the motion of test particles as described by the geodesic equation. The Christoffel symbols for (7.18) are

$$\begin{aligned}\Gamma_{00}^0 &= \partial_0 \Phi \\ \Gamma_{00}^i &= \partial_i \Phi + \partial_0 w_i \\ \Gamma_{j0}^0 &= \partial_j \Phi \\ \Gamma_{j0}^i &= \partial_{[j} w_{i]} + \frac{1}{2} \partial_0 h_{ij} \\ \Gamma_{jk}^0 &= -\partial_{(j} w_{k)} + \frac{1}{2} \partial_0 h_{jk} \\ \Gamma_{jk}^i &= \partial_{(j} h_{k)i} - \frac{1}{2} \partial_i h_{jk}.\end{aligned}\quad (7.19)$$

In these expressions we have stuck with  $h_{ij}$  rather than  $s_{ij}$  and  $\Psi$ , since they enter only in the combination  $h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$ . The distinction will become important once we start taking traces to get to the Ricci tensor and Einstein's equation. Since we have fixed an inertial frame, it is convenient to express the four-momentum  $p^\mu = dx^\mu/d\lambda$  (where  $\lambda = \tau/m$  if the particle is massive) in terms of the energy  $E$  and three-velocity  $v^i = dx^i/dt$ , as

$$p^0 = \frac{dt}{d\lambda} = E, \quad p^i = Ev^i. \quad (7.20)$$

Then we can take the geodesic equation

$$\frac{dp^\mu}{d\lambda} + \Gamma_{\rho\sigma}^\mu p^\rho p^\sigma = 0, \quad (7.21)$$

move the second term to the right-hand side so that it takes on the appearance of a force term, and divide both sides by  $E$  to obtain

$$\frac{dp^\mu}{dt} = -\Gamma_{\rho\sigma}^\mu \frac{p^\rho p^\sigma}{E}. \quad (7.22)$$

The  $\mu = 0$  component describes the evolution of the energy,

$$\frac{dE}{dt} = -E \left[ \partial_0 \Phi + 2(\partial_k \Phi)v^k - \left( \partial_{(j} w_{k)} - \frac{1}{2} \partial_0 h_{jk} \right) v^j v^k \right]. \quad (7.23)$$

You might think that the energy should be conserved, but  $E = p^0 = m\gamma$  only includes the “inertial” energy of the particle—in the slowly-moving limit, the rest energy and the kinetic energy—and not the energy from interactions with the gravitational field.

The spatial components  $\mu = i$  of the geodesic equation become

$$\frac{dp^i}{dt} = -E \left[ \partial_i \Phi + \partial_0 w_i + 2(\partial_{[i} w_{j]} + \partial_0 h_{ij}) v^j + \left( \partial_{(j} h_{k)i} - \frac{1}{2} \partial_i h_{jk} \right) v^j v^k \right]. \quad (7.24)$$

To interpret this physically, it is convenient to define the “gravito-electric” and “gravito-magnetic” three-vector fields,

$$\begin{aligned} G^i &\equiv -\partial_i \Phi - \partial_0 w_i \\ H^i &\equiv (\nabla \times \vec{w})^i = \epsilon^{ijk} \partial_j w_k, \end{aligned} \quad (7.25)$$

which bear an obvious resemblance to the definitions of the ordinary electric and magnetic field in terms of a scalar and vector potential. Then (7.24) becomes

$$\frac{dp^i}{dt} = E \left[ G^i + (\vec{v} \times H)^i - 2(\partial_0 h_{ij}) v^j - \left( \partial_{(j} h_{k)i} - \frac{1}{2} \partial_i h_{jk} \right) v^j v^k \right]. \quad (7.26)$$

The first two terms on the right-hand side describe how the test particle, moving along a geodesic, responds to the scalar and vector perturbations  $\Phi$  and  $w_i$  in a way reminiscent of the Lorentz force law in electromagnetism. We also find couplings to the spatial perturbations  $h_{ij}$ , of linear and quadratic order in the three-velocity. The relative importance of the different perturbations will of course depend on the physical situation under consideration, as we will soon demonstrate.

In addition to the motion of test particles, we should examine the field equations for the metric perturbations, which are of course the linearized Einstein equations. The Riemann tensor in our variables is

$$\begin{aligned} R_{0j0l} &= \partial_j \partial_l \Phi + \partial_0 \partial_{(j} w_{l)} - \frac{1}{2} \partial_0 \partial_0 h_{jl} \\ R_{0jkl} &= \partial_j \partial_{[k} w_{l]} - \partial_0 \partial_{[k} h_{l]j} \\ R_{ijkl} &= \partial_j \partial_{[k} h_{l]i} - \partial_i \partial_{[k} h_{l]j}, \end{aligned} \quad (7.27)$$

with other components related by symmetries. We contract using  $\eta^{\mu\nu}$  to obtain the Ricci tensor,

$$\begin{aligned} R_{00} &= \nabla^2 \Phi + \partial_0 \partial_k w^k + 3 \partial_0^2 \Psi \\ R_{0j} &= -\frac{1}{2} \nabla^2 w_j + \frac{1}{2} \partial_j \partial_k w^k + 2 \partial_0 \partial_j \Psi + \partial_0 \partial_k s_j^k \\ R_{ij} &= -\partial_i \partial_j (\Phi - \Psi) - \partial_0 \partial_{(i} w_{j)} + \square \Psi \delta_{ij} - \square s_{ij} + 2 \partial_k \partial_{(i} s_{j)}^k, \end{aligned} \quad (7.28)$$

where  $\nabla^2 = \delta^{ij} \partial_i \partial_j$  is the three-dimensional flat Laplacian. Since the Ricci tensor involves contractions, the trace-free and trace parts of the spatial perturbations now enter in different ways. Finally, we can calculate the Einstein tensor,

$$\begin{aligned} G_{00} &= 2\nabla^2\Psi + \partial_k \partial_l s^{kl} \\ G_{0j} &= -\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}\partial_j \partial_k w^k + 2\partial_0 \partial_j \Psi + \partial_0 \partial_k s_j{}^k \\ G_{ij} &= (\delta_{ij} \nabla^2 - \partial_i \partial_j)(\Phi - \Psi) + \delta_{ij} \partial_0 \partial_k w^k - \partial_0 \partial_{(i} w_{j)} \\ &\quad + 2\delta_{ij} \partial_0^2 \Psi - \square s_{ij} + 2\partial_k \partial_{(i} s_{j)}{}^k - \delta_{ij} \partial_k \partial_l s^{kl}. \end{aligned} \quad (7.29)$$

Using this expression in Einstein's equation  $G_{\mu\nu} = 8\pi G T_{\mu\nu}$  reveals that only a small fraction of the metric components are true degrees of freedom of the gravitational field; the rest obey constraints that determine them in terms of the other fields. To see this, start with  $G_{00} = 8\pi G T_{00}$ , which we write using (7.29) as

$$\nabla^2 \Psi = 4\pi G T_{00} - \tfrac{1}{2} \partial_k \partial_l s^{kl}. \quad (7.30)$$

This is an equation for  $\Psi$  with no time derivatives; if we know what  $T_{00}$  and  $s_{ij}$  are doing at any time, we can determine what  $\Psi$  must be (up to boundary conditions at spatial infinity). Thus,  $\Psi$  is not by itself a propagating degree of freedom; it is determined by the energy-momentum tensor and the gravitational strain  $s_{ij}$ . Next turn to the  $0j$  equation, which we write as

$$(\delta_{jk} \nabla^2 - \partial_j \partial_k) w^k = -16\pi G T_{0j} + 4\partial_0 \partial_j \Psi + 2\partial_0 \partial_k s_j{}^k. \quad (7.31)$$

This is an equation for  $w^i$  with no time derivatives; once again, if we know the energy-momentum tensor and the strain (from which we can find  $\Psi$ ), the vector  $w^i$  will be determined. Finally, the  $ij$  equation is

$$\begin{aligned} (\delta_{ij} \nabla^2 - \partial_i \partial_j) \Phi &= 8\pi G T_{ij} + (\delta_{ij} \nabla^2 - \partial_i \partial_j - 2\delta_{ij} \partial_0^2) \Psi \\ &\quad - \delta_{ij} \partial_0 \partial_k w^k + \partial_0 \partial_{(i} w_{j)} + \square s_{ij} - 2\partial_k \partial_{(i} s_{j)}{}^k - \delta_{ij} \partial_k \partial_l s^{kl}, \end{aligned} \quad (7.32)$$

Once again, we see that there are no time derivatives acting on  $\Phi$ , which is therefore determined as a function of the other fields.

Thus, the only propagating degrees of freedom in Einstein's equations are those in the strain tensor  $s_{ij}$ ; as we will see, these are used to describe gravitational waves. The other components of  $h_{\mu\nu}$  are determined in terms of  $s_{ij}$  and the matter fields—they do not require separate initial data. In alternative theories, such as those discussed in Section 4.8 with either additional fields or higher-order terms in the action, the other components of the metric may become dynamical variables. As we discuss briefly at the end of Section 7.4, propagating tensor fields give rise upon quantization to particles of different spins, depending on the behav-

ior of the field under spatial rotations. Thus, the scalars  $\Phi$  and  $\Psi$  would be spin-0, the vector  $w_i$  would be spin-1, and the tensor  $s_{ij}$  is spin-2. Only the spin-2 piece is a true particle excitation in ordinary GR.

In the previous section we showed how gauge transformations  $h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$  are generated by a vector field  $\xi^\mu$ . Henceforth we set the parameter  $\epsilon$  of (7.14) equal to unity, and think of the vector field  $\xi^\mu$  itself as being small. Under such a transformation, the different metric perturbation fields change by

$$\begin{aligned}\Phi &\rightarrow \Phi + \partial_0 \xi^0 \\ w_i &\rightarrow w_i + \partial_0 \xi^i - \partial_i \xi^0 \\ \Psi &\rightarrow \Psi - \frac{1}{3} \partial_i \xi^i \\ s_{ij} &\rightarrow s_{ij} + \partial_{(i} \xi_{j)} - \frac{1}{3} \partial_k \xi^k \delta_{ij},\end{aligned}\tag{7.33}$$

as you can easily check. Just as in electromagnetism and other gauge theories, different gauges can be appropriate to different circumstances; here we list some popular choices.

Consider first the **transverse gauge** (a generalization of the conformal Newtonian or Poisson gauge sometimes used in cosmology.) The transverse gauge is closely related to the Coulomb gauge of electromagnetism,  $\partial_i A^i = 0$ . We begin by fixing the strain to be spatially transverse,

$$\partial_i s^{ij} = 0,\tag{7.34}$$

by choosing  $\xi^j$  to satisfy

$$\nabla^2 \xi^j + \frac{1}{3} \partial_j \partial_i \xi^i = -2 \partial_i s^{ij}.\tag{7.35}$$

The value of  $\xi^0$  is still undetermined, so we can use this remaining freedom to render the vector perturbation transverse,

$$\partial_i w^i = 0,\tag{7.36}$$

by choosing  $\xi^0$  to satisfy

$$\nabla^2 \xi^0 = \partial_i w^i + \partial_0 \partial_i \xi^i.\tag{7.37}$$

The meaning of transverse becomes clear upon taking the Fourier transform, after which a vanishing divergence implies that a tensor is orthogonal to the wave vector. Neither (7.35) nor (7.37) completely fixes the value of  $\xi^\mu$ ; they are both second-order differential equations in spatial derivatives, which require boundary conditions to specify a solution. For our present purposes, it suffices that solutions will always exist. The conditions (7.34) and (7.36) together define the transverse gauge. In this gauge, Einstein's equation becomes

$$G_{00} = 2 \nabla^2 \Psi = 8\pi G T_{00},\tag{7.38}$$

$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + 2\partial_0\partial_j\Psi = 8\pi GT_{0j}, \quad (7.39)$$

and

$$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + 2\delta_{ij}\partial_0^2\Psi - \square s_{ij} = 8\pi GT_{ij}. \quad (7.40)$$

In the remainder of this chapter, we will use these equations to find weak-field solutions in different situations.

Another popular gauge is known as the **synchronous gauge**. It is equivalent to the choice of Gaussian normal coordinates, discussed in Appendix D. It may be thought of as the gravitational analogue of the temporal gauge of electromagnetism,  $A^0 = 0$ , since it kills off the nonspatial components of the perturbation. We begin by setting the scalar potential  $\Phi$  to vanish,

$$\Phi = 0, \quad (7.41)$$

by choosing  $\xi^0$  to satisfy

$$\partial_0\xi^0 = -\Phi. \quad (7.42)$$

This leaves us the ability to choose  $\xi^i$ . We can set the vector components to zero,

$$w^i = 0, \quad (7.43)$$

by choosing  $\xi^i$  to satisfy

$$\partial_0\xi^i = -w^i + \partial_i\xi^0. \quad (7.44)$$

The metric in synchronous gauge therefore takes on the attractive form

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j. \quad (7.45)$$

This is just a matter of gauge choice, and is applicable to any spacetime slightly perturbed away from Minkowski. It is straightforward to write down Einstein's equation in synchronous gauge, but we won't bother as we won't actually be using it in the rest of this chapter.

In addition to transverse and synchronous gauges, in calculating the production of gravitational waves it is convenient to use yet a third choice, the Lorenz/harmonic gauge. As we will discuss below, it is equivalent to setting

$$\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h = 0, \quad (7.46)$$

where  $h = \eta^{\mu\nu}h_{\mu\nu}$ . This gauge does not have any especially simple expression in terms of our decomposed perturbation fields, but it does make the linearized Einstein equation take on a particularly simple form.

Before moving on to applications of the weak-field limit, we conclude our discussion of degrees of freedom by drawing attention to the distinction between our *algebraic* decomposition of the metric perturbation components in (7.16), and an additional decomposition that becomes possible if we consider tensor *fields* rather than tensors defined at a point. This additional decomposition helps to bring out the physical degrees of freedom more directly, and is crucial in cosmological perturbation theory. Its basis is the standard observation that a vector field can be decomposed into a transverse part  $w_\perp^i$  and a longitudinal part  $w_\parallel^i$ :

$$w^i = w_\perp^i + w_\parallel^i, \quad (7.47)$$

where a transverse vector is divergenceless and a longitudinal vector is curl-free,

$$\partial_i w_\perp^i = 0, \quad \epsilon^{ijk}\partial_j w_\parallel^k = 0. \quad (7.48)$$

Notice that these are differential equations, so clearly they only make sense when applied to tensor fields. A transverse vector can be represented as the curl of some other vector  $\xi^i$ , although the choice of  $\xi^i$  is not unique unless we impose a subsidiary condition such as  $\partial_i \xi^i = 0$ . A longitudinal vector is the divergence of a scalar  $\lambda$ ,

$$w_\perp^i = \epsilon^{ijk}\partial_j \xi_k, \quad w_\parallel^i = \partial_i \lambda. \quad (7.49)$$

Just like our original decomposition of the metric perturbation into scalar, vector and tensor pieces, this decomposition of a vector field into parts depending on a scalar and a transverse vector is invariant under spatial rotations. The scalar  $\lambda$  clearly represents one degree of freedom; the vector  $\xi^i$  looks like three degrees of freedom, but one of these is illusory due to the nonuniqueness of the choice of  $\xi^i$  (which you will notice is equivalent to the freedom to make gauge transformations  $\xi_i \rightarrow \xi_i + \partial_i \omega$ ). There are thus three degrees of freedom in total, as there should be to describe the original vector field  $w^i$ .

A similar procedure applies to the traceless symmetric tensor  $s^{ij}$ , which can be decomposed into a transverse part  $s_\perp^{ij}$ , a solenoidal part  $s_S^{ij}$ , and a longitudinal part  $s_\parallel^{ij}$ .

$$s^{ij} = s_\perp^{ij} + s_S^{ij} + s_\parallel^{ij}. \quad (7.50)$$

The transverse part is divergenceless, while the divergence of the solenoidal part is a transverse (divergenceless) vector, and the divergence of the longitudinal part is a longitudinal (curl-free) vector:

$$\partial_i s_\perp^{ij} = 0$$

$$\begin{aligned}\partial_i \partial_j s_S^{ij} &= 0 \\ \epsilon^{jkl} \partial_k \partial_l s_{\parallel}^{\phantom{\parallel}i}{}_j &= 0.\end{aligned}\quad (7.51)$$

This means that the longitudinal part can be derived from a scalar field  $\theta$ , and the solenoidal part can be derived from a transverse vector  $\zeta^i$ ,

$$\begin{aligned}s_{\parallel ij} &= \left( \partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \theta \\ s_{Sij} &= \partial_{(i} \zeta_{j)},\end{aligned}\quad (7.52)$$

where

$$\partial_i \zeta^i = 0. \quad (7.53)$$

Thus, the longitudinal part describes a single degree of freedom, while the solenoidal part describes two degrees of freedom. The transverse part cannot be further decomposed; it describes the remaining two degrees of freedom of the symmetric traceless  $3 \times 3$  tensor  $s_{ij}$ . Later in this chapter we will introduce the transverse-traceless gauge for describing gravitational waves propagating in vacuum; in this gauge, the only nonvanishing metric perturbation is the transverse tensor perturbation  $s_{\perp}^{ij}$ .

With this decomposition of tensor fields, we have succeeded in writing the original ten-component metric perturbation  $h_{\mu\nu}$  in terms of four scalars ( $\Phi, \Psi, \lambda$ , and  $\theta$ ) with one degree of freedom each, two transverse vectors ( $\xi^i$  and  $\zeta^i$ ) with two degrees of freedom each, and one transverse-traceless tensor ( $s_{\perp}^{ij}$ ) with two degrees of freedom. People refer to this set of fields when they speak of “scalar,” “vector,” and “tensor” modes. We can then decompose the energy-momentum tensor in a similar way, write Einstein’s equation in terms of these variables, and isolate the physical (gauge-invariant) degrees of freedom. We won’t use this decomposition in this book, but you should be aware of its existence when referring to the literature.

### 7.3 ■ NEWTONIAN FIELDS AND PHOTON TRAJECTORIES

We previously defined the “Newtonian limit” as describing weak fields for which sources were static and test particles were slowly moving. In this section we will extend this definition somewhat, still restricting ourselves to static sources but allowing the test particles to move at any velocity. There is clearly an important difference, as we previously only needed to consider effects of the  $g_{00}$  component of the metric, but we will find that relativistic particles respond to spatial components of the metric as well.

We can model our static gravitating sources by dust, a perfect fluid for which the pressure vanishes. (Most of the matter in the universe is well approximated by dust, including stars, planets, galaxies, and even dark matter.) We work in the rest

frame of the dust, where the energy-momentum tensor takes the form

$$T_{\mu\nu} = \rho U_\mu U_\nu = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (7.54)$$

Since our background is flat Minkowski space, it is straightforward to accommodate moving sources by simply Lorentz-transforming into their rest frame; what we are unable to deal with in this limit is multiple sources with large relative velocities.

Turn to Einstein's equation in the transverse gauge, (7.38)–(7.40). For static sources we drop all time-derivative terms, and simultaneously plug in the energy-momentum tensor (7.54), to obtain

$$\begin{aligned} \nabla^2 \Psi &= 4\pi G\rho \\ \nabla^2 w_j &= 0 \\ (\delta_{ij} \nabla^2 - \partial_i \partial_j)(\Phi - \Psi) - \nabla^2 s_{ij} &= 0. \end{aligned} \quad (7.55)$$

We will look for solutions that are both nonsingular and well-behaved at infinity; consequently, only those fields that are sourced by the right-hand side will be nonvanishing. For example, the second equation in (7.55) immediately implies  $w^i = 0$ . We next take the trace of the third equation (summing over  $\delta^{ij}$ ):

$$2\nabla^2(\Phi - \Psi) = 0. \quad (7.56)$$

This enforces equality of the two scalar potentials,

$$\Phi = \Psi. \quad (7.57)$$

Recall that in our initial discussion of the Newtonian limit in Chapter 4, we argued that the 00 component  $\Phi$  of the perturbation (which is responsible for the motion of nonrelativistic particles) obeyed the Poisson equation; from (7.55) it appears as if it is actually the scalar perturbation  $\Psi$  to the spatial components that obeys this equation. The implicit connection is provided by (7.56), which sets the two potentials equal when the trace of  $T_{ij}$  (the sum of the three principle pressures) vanishes. Finally we can plug  $\Phi = \Psi$  into the last equation of (7.55) to get

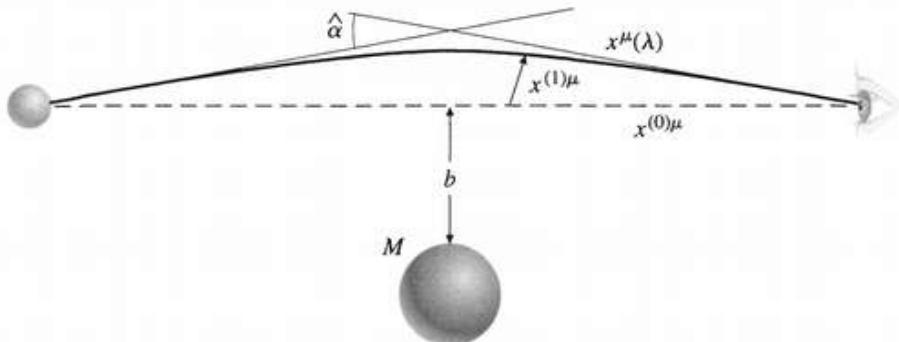
$$\nabla^2 s_{ij} = 0, \quad (7.58)$$

which implies  $s_{ij} = 0$  for a well-behaved solution.

The perturbed metric for static Newtonian sources is therefore

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2),$$

(7.59)



**FIGURE 7.3** A deflected geodesic  $x^\mu(\lambda)$ , decomposed into a background geodesic  $x^{(0)\mu}$  and a perturbation  $x^{(1)\mu}$ . The deflection angle  $\hat{\alpha}$  represents (minus) the amount by which the wave vector rotates along the path. A single mass  $M$  with impact parameter  $b$  is depicted, although the setup is more general.

or equivalently

$$h_{\mu\nu} = \begin{pmatrix} -2\Phi & & & \\ & -2\Phi & & \\ & & -2\Phi & \\ & & & -2\Phi \end{pmatrix}, \quad (7.60)$$

where the potential obeys the conventional Poisson equation,

$$\nabla^2 \Phi = 4\pi G\rho. \quad (7.61)$$

This is an important extension of our result from Chapter 4, since we now know the perturbation of the spatial metric as well as  $h_{00}$ .

Now let us consider the path of a photon (or other massless particle) through this geometry; in other words, solve the perturbed geodesic equation for a null trajectory  $x^\mu(\lambda)$ .<sup>2</sup> The geometry we consider is portrayed in Figure 7.3. Recall that our philosophy is to consider the metric perturbation as a field defined on a flat background spacetime. Similarly, we can decompose the geodesic into a background path plus a perturbation,

$$x^\mu(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda), \quad (7.62)$$

where  $x^{(0)\mu}$  solves the geodesic equation in the background (in other words, is just a straight null path). We then evaluate all quantities along the background path, to solve for  $x^{(1)\mu}(\lambda)$ . For this procedure to make sense, we need to assume that the potential  $\Phi$  is not appreciably different along the background and true geodesics; this condition amounts to requiring that  $x^{(1)i}\partial_i\Phi \ll \Phi$ . If this condition is not true, however, all is not lost. If we consider only very short paths, the deviation

<sup>2</sup>The approach we use is outlined in T. Pyne and M. Birkinshaw, *Astrophys. Journ.* **458**, 46 (1996), <http://arxiv.org/abs/astro-ph/9504060>.

$x^{(1)\mu}$  will necessarily be small, and our approximation will be valid. But then we can assemble larger paths out of such short segments. As a result, we will derive true equations, but the paths over which we integrate will be the *actual* path  $x^\mu(\lambda)$ , rather than the background path  $x^{(0)\mu}(\lambda)$ . As long as this is understood, our results will be valid for any trajectories in the perturbed spacetime.

For convenience we denote the wave vector of the background path as  $k^\mu$ , and the derivative of the deviation vector as  $\ell^\mu$ :

$$k^\mu \equiv \frac{dx^{(0)\mu}}{d\lambda}, \quad \ell^\mu \equiv \frac{dx^{(1)\mu}}{d\lambda}. \quad (7.63)$$

The condition that a path be null is of course

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0, \quad (7.64)$$

which we must solve order-by-order. At zeroth order we simply have  $\eta_{\mu\nu} k^\mu k^\nu = 0$ , or

$$(k^0)^2 = (\vec{k})^2 \equiv k^2, \quad (7.65)$$

where  $\vec{k}$  is the three-vector with components  $k^i$ . This equation serves as the definition of the constant  $k$ . Then at first order we obtain

$$2\eta_{\mu\nu} k^\mu \ell^\nu + h_{\mu\nu} k^\mu k^\nu = 0, \quad (7.66)$$

or

$$-k\ell^0 + \vec{\ell} \cdot \vec{k} = 2k^2 \Phi. \quad (7.67)$$

We now turn to the perturbed geodesic equation,

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0. \quad (7.68)$$

The Christoffel symbols can be found by setting  $w^i = 0$  and  $h_{ij} = -2\Phi\delta_{ij}$  in (7.19):

$$\begin{aligned} \Gamma_{0i}^0 &= \Gamma_{00}^i = \partial_i \Phi, \\ \Gamma_{jk}^i &= \delta_{jk} \partial_i \Phi - \delta_{ik} \partial_j \Phi - \delta_{ij} \partial_k \Phi. \end{aligned} \quad (7.69)$$

The zeroth-order geodesic equation simply tells us that  $x^{(0)\mu}$  is a straight trajectory, while at first order we have

$$\frac{d\ell^\mu}{d\lambda} = -\Gamma_{\rho\sigma}^\mu k^\rho k^\sigma. \quad (7.70)$$

There are no factors of  $\ell^\mu$  on the right-hand side, since the Christoffel symbols are already first-order in the perturbation. The  $\mu = 0$  component of (7.70) is

$$\frac{d\ell^0}{d\lambda} = -2k(\vec{k} \cdot \vec{\nabla}\Phi), \quad (7.71)$$

while the spatial components are

$$\frac{d\vec{\ell}}{d\lambda} = -2k^2 \vec{\nabla}_\perp \Phi. \quad (7.72)$$

Here we have introduced the gradient transverse to the path, defined as the total gradient minus the gradient along the path,

$$\begin{aligned} \vec{\nabla}_\perp \Phi &\equiv \vec{\nabla}\Phi - \vec{\nabla}_\parallel \Phi \\ &= \vec{\nabla}\Phi - k^{-2}(\vec{k} \cdot \vec{\nabla}\Phi)\vec{k}. \end{aligned} \quad (7.73)$$

In all of these expressions, the path means the background path.

Note that, to first order in  $\Phi$ , the spatial wave vector perturbation  $\vec{\ell}$  is orthogonal to the original spatial wave vector  $\vec{k}$ . To see this, we can get an expression for  $\ell^0$  by integrating (7.71) to get

$$\begin{aligned} \ell^0 &= \int \frac{d\ell^0}{d\lambda} d\lambda \\ &= -2k \int (\vec{k} \cdot \vec{\nabla}\Phi) d\lambda \\ &= -2k \int \left( \frac{d\vec{x}}{d\lambda} \cdot \vec{\nabla}\Phi \right) d\lambda \\ &= -2k \int \vec{\nabla}\Phi \cdot d\vec{x} \\ &= -2k\Phi. \end{aligned} \quad (7.74)$$

The constant of integration is fixed by demanding that  $\ell^0 = 0$  when  $\Phi = 0$ . Plugging this into (7.67) reveals

$$\vec{\ell} \cdot \vec{k} = k\ell^0 + 2k^2\Phi = 0, \quad (7.75)$$

verifying that  $\vec{\ell}$  and  $\vec{k}$  are orthogonal to first order.

The **deflection angle**  $\hat{\alpha}$  is the amount by which the original spatial wave vector is deflected as it travels from a source to the observer; it is a two-dimensional vector in the plane perpendicular to  $\vec{k}$ . (We use the notation  $\hat{\alpha}$  rather than  $\vec{\alpha}$ , as the latter is used for the reduced deflection angle introduced in Chapter 8.) From the geometry portrayed in Figure 7.3, the deflection angle can be expressed as

$$\hat{\alpha} = -\frac{\Delta \vec{\ell}}{k}, \quad (7.76)$$

where the minus sign simply accounts for the fact that the deflection angle is measured by an observer looking backward along the photon path. The rotation of the wave vector can be calculated from (7.72) as

$$\begin{aligned} \Delta \vec{\ell} &= \int \frac{d\vec{\ell}}{d\lambda} d\lambda \\ &= -2k^2 \int \vec{\nabla}_\perp \Phi d\lambda. \end{aligned} \quad (7.77)$$

The deflection angle can therefore be expressed as an integral over the physical spatial distance traversed,  $s = k\lambda$ , as

$$\boxed{\hat{\alpha} = 2 \int \vec{\nabla}_\perp \Phi ds.} \quad (7.78)$$

We can evaluate the deflection angle in the case of a point mass, where we imagine the background path to be along the  $x$ -direction with an impact parameter defined by a transverse vector  $\vec{b}$  pointing from the path to the mass at the point of closest approach. Setting  $b = |\vec{b}|$ , the potential is

$$\Phi = -\frac{GM}{r} = -\frac{GM}{(b^2 + x^2)^{1/2}}, \quad (7.79)$$

and its transverse gradient is therefore

$$\vec{\nabla}_\perp \Phi = \frac{GM}{(b^2 + x^2)^{3/2}} \vec{b}. \quad (7.80)$$

The deflection angle is thus

$$\begin{aligned} \hat{\alpha} &= 2GMb \int \frac{dx}{(b^2 + x^2)^{3/2}} \\ &= \frac{4GM}{b}, \end{aligned} \quad (7.81)$$

where the integral has been taken from  $-\infty$  to  $\infty$ , presuming that both source and observer are very far from the deflecting mass. Note that  $c = 1$  in our units; a factor of  $c^2$  should be inserted in the denominator of (7.81) in other systems.

Deflection of light by the Sun was historically a crucial test of general relativity. Einstein proposed three such tests: precession of the perihelion of Mercury, gravitational redshift, and deflection of light. The precession of Mercury's perihelion was successfully explained by GR, but this explained a discrepancy that had already been observed; gravitational redshift was not observed until much later,

so deflection of light was the first time that Einstein's theory correctly predicted a phenomenon that had not yet been detected. A famous expedition led by Eddington observed the positions of stars near the Sun during a 1919 total eclipse; the observations were in agreement with the GR prediction, leading to front-page stories in newspapers around the world. The predicted effect is quite small: for the Sun we have  $G M_{\odot}/c^2 = 1.48 \times 10^5$  cm, and the solar radius is  $R_{\odot} = 6.96 \times 10^{10}$  cm, leading to a maximum deflection angle of  $\hat{\alpha} = 1.75$  arcsecs. Later re-evaluation of Eddington's results has cast doubt upon whether he actually obtained the precision that was originally claimed; contemporary measurements use high-precision interferometric observations of quasars passing behind the Sun to obtain very accurate tests of GR (which it has so far passed). Meanwhile, observation of light deflection by astrophysical sources such as galaxies and stars has become a vibrant area of research, under the name of "gravitational lensing." Of course in these circumstances we rarely know the mass of the lens well enough to provide precision tests of GR; instead, it is more common to use the observed deflection angle as a way to measure the mass. We will discuss lensing more in Chapter 8.

In addition to the deflection of light, in 1964 Shapiro pointed out another observable consequence of weak-field general relativity on photon trajectories: gravitational time delay. The total coordinate time elapsed along a null path is

$$t = \int \frac{dx^0}{d\lambda} d\lambda. \quad (7.82)$$

We are putting ourselves in the position of an observer far from any sources, at rest in the background inertial frame, so coordinate time is our proper time. In the presence of a Newtonian potential, the photons appear to "slow down" with respect to the background light cones, leading to an additional time delay of

$$\begin{aligned} \Delta t &= \int \frac{dx^{(1)0}}{d\lambda} d\lambda \\ &= \int \ell^0 d\lambda \\ &= -2k \int \Phi d\lambda, \end{aligned} \quad (7.83)$$

or

$$\Delta t = -2 \int \Phi ds. \quad (7.84)$$

According to our rules, the integral is performed over the background path. In addition to this Shapiro delay, there can be an additional "geometric" time delay because the spatial distance traversed by the real path is longer than that of the background path. For deflection of light by the Sun the geometric delay effect is negligible, but in cosmological applications it can be comparable to the Shapiro

effect. The time delay has been observed, most precisely by making use of space-craft rather than naturally-occurring objects; for details see Will (1981).

The motion of photons through a Newtonian potential, leading to both the deflection of light and the gravitational time delay, could equivalently be derived by imagining that the photons are propagating in a medium with refractive index

$$n = 1 - 2\Phi, \quad (7.85)$$

to first order. Indeed, we could have found the equations of motion for the photon by using Fermat's principle of least time; you are asked to demonstrate this in the exercises.

#### 7.4 ■ GRAVITATIONAL WAVE SOLUTIONS

An even more exciting application of the weak-field limit is to gravitational radiation. Here we are studying the freely-propagating degrees of freedom of the gravitational field, requiring no local sources for their existence (although they can of course be generated by such sources). We therefore turn once again to the weak-field equations in transverse gauge, (7.38)–(7.40), this time keeping time derivatives but completely turning off the energy-momentum tensor,  $T_{\mu\nu} = 0$ . The 00 equation is then

$$\nabla^2 \Psi = 0, \quad (7.86)$$

which with well-behaved boundary conditions implies  $\Psi = 0$ . Then the 0j equation is

$$\nabla^2 w_j = 0, \quad (7.87)$$

which again implies  $w_j = 0$ .

We turn next to the trace of the ij equation, which (plugging in the above results) yields

$$\nabla^2 \Phi = 0, \quad (7.88)$$

which implies  $\Phi = 0$ .

We are therefore left with the trace-free part of the ij equation, which becomes a wave equation for the traceless strain tensor:

$$\square s_{ij} = 0. \quad (7.89)$$

Although it has been convenient thus far to work with  $s_{ij}$ , it is far more common in the literature to find expressions written in terms of the entire metric perturbation  $h_{\mu\nu}$ , but in an ansatz where all of the other degrees of freedom ( $\Phi$ ,  $\Psi$ ,  $w_i$ ) are set to zero (and  $s_{ij}$  is transverse). This is commonly known as the **transverse**

**traceless gauge**, in which we have

$$h_{\mu\nu}^{\text{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2s_{ij} & \\ 0 & & & \end{pmatrix}. \quad (7.90)$$

The equation of motion is then

$$\square h_{\mu\nu}^{\text{TT}} = 0. \quad (7.91)$$

To make it easier to compare with other resources, in our discussion of gravitational waves we will use  $h_{\mu\nu}^{\text{TT}}$  rather than  $s_{ij}$ , keeping in mind that  $h_{\mu\nu}^{\text{TT}}$  is purely spatial, traceless and transverse:

$$\begin{aligned} h_{0v}^{\text{TT}} &= 0 \\ \eta^{\mu\nu} h_{\mu\nu}^{\text{TT}} &= 0 \\ \partial_\mu h_{\nu\nu}^{\text{TT}} &= 0. \end{aligned} \quad (7.92)$$

From the wave equation (7.91) we begin finding solutions. Those familiar with the analogous problem in electromagnetism will notice that the procedure is almost precisely the same. A particularly useful set of solutions to this wave equation are the plane waves, given by

$$h_{\mu\nu}^{\text{TT}} = C_{\mu\nu} e^{ik_\sigma x^\sigma}, \quad (7.93)$$

where  $C_{\mu\nu}$  is a constant, symmetric,  $(0, 2)$  tensor, which is obviously traceless and purely spatial:

$$\begin{aligned} C_{0v} &= 0 \\ \eta^{\mu\nu} C_{\mu\nu} &= 0. \end{aligned} \quad (7.94)$$

Of course  $e^{ik_\sigma x^\sigma}$  is complex, while  $h_{\mu\nu}^{\text{TT}}$  is real; we carry both real and imaginary parts through the calculation, and take the real part at the end. The constant vector  $k^\sigma$  is the wave vector. To check that we have a solution, we plug in:

$$\begin{aligned} 0 &= \square h_{\mu\nu}^{\text{TT}} \\ &= \eta^{\rho\sigma} \partial_\rho \partial_\sigma h_{\mu\nu}^{\text{TT}} \\ &= \eta^{\rho\sigma} \partial_\rho (ik_\sigma h_{\mu\nu}^{\text{TT}}) \\ &= -\eta^{\rho\sigma} k_\rho k_\sigma h_{\mu\nu}^{\text{TT}} \\ &= -k_\sigma k^\sigma h_{\mu\nu}^{\text{TT}}. \end{aligned} \quad (7.95)$$

Since, for an interesting solution, not all of the components of  $h_{\mu\nu}^{\text{TT}}$  will be zero everywhere, we must have

$$k_\sigma k^\sigma = 0. \quad (7.96)$$

The plane wave (7.93) is therefore a solution to the linearized equation if the wave vector is null; this is loosely translated into the statement that gravitational waves propagate at the speed of light. The timelike component of the wave vector is the frequency of the wave, and we write  $k^\sigma = (\omega, k^1, k^2, k^3)$ . (More generally, an observer moving with four-velocity  $U^\mu$  would observe the wave to have a frequency  $\omega = -k_\mu U^\mu$ .) Then the condition that the wave vector be null becomes

$$\omega^2 = \delta_{ij} k^i k^j. \quad (7.97)$$

Of course our wave is far from the most general solution; any (possibly infinite) number of distinct plane waves can be added together and will still solve the linear equation (7.91). Indeed, any solution can be written as such a superposition.

We still need to ensure that the perturbation is transverse. This means that

$$\begin{aligned} 0 &= \partial_\mu h_{\text{TT}}^{\mu\nu} \\ &= i C^{\mu\nu} k_\mu e^{ik_\sigma x^\sigma}. \end{aligned} \quad (7.98)$$

which is only true if

$$k_\mu C^{\mu\nu} = 0. \quad (7.99)$$

We say that the wave vector is orthogonal to  $C^{\mu\nu}$ .

Our solution can be made more explicit by choosing spatial coordinates such that the wave is traveling in the  $x^3$  direction; that is,

$$k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega), \quad (7.100)$$

where we know that  $k^3 = \omega$  because the wave vector is null. In this case,  $k^\mu C_{\mu\nu} = 0$  and  $C_{0\nu} = 0$  together imply

$$C_{3\nu} = 0. \quad (7.101)$$

The only nonzero components of  $C_{\mu\nu}$  are therefore  $C_{11}$ ,  $C_{12}$ ,  $C_{21}$ , and  $C_{22}$ . But  $C_{\mu\nu}$  is traceless and symmetric, so in general we can write

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (7.102)$$

Thus, for a plane wave in this gauge traveling in the  $x^3$  direction, the two components  $C_{11}$  and  $C_{12}$  (along with the frequency  $\omega$ ) completely characterize the wave.

To get a feeling for the physical effect of a passing gravitational wave, consider the motion of test particles in the presence of a wave. It is certainly insufficient to solve for the trajectory of a single particle, since that would only tell us about the values of the coordinates along the world line. In fact, for any single particle we can find transverse traceless coordinates in which the particle appears stationary to first order in  $h_{\mu\nu}^{\text{TT}}$ . To obtain a coordinate-independent measure of the wave's effects, we consider the relative motion of nearby particles, as described by the geodesic deviation equation. If we consider some nearby particles with four-velocities described by a single vector field  $U^\mu(x)$  and separation vector  $S^\mu$ , we have

$$\frac{D^2}{d\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} \dot{U}^\nu U^\rho S^\sigma. \quad (7.103)$$

We would like to compute the right-hand side to first order in  $h_{\mu\nu}^{\text{TT}}$ . If we take our test particles to be moving slowly, we can express the four-velocity as a unit vector in the time direction plus corrections of order  $h_{\mu\nu}^{\text{TT}}$  and higher; but we know that the Riemann tensor is already first order, so the corrections to  $U^\nu$  may be ignored, and we write

$$U^\nu = (1, 0, 0, 0). \quad (7.104)$$

Therefore we only need to compute  $R^\mu{}_{00\sigma}$ , or equivalently  $R_{\mu 00\sigma}$ . From (7.5) we have

$$R_{\mu 00\sigma} = \frac{1}{2} (\partial_0 \partial_0 h_{\mu\sigma}^{\text{TT}} + \partial_\sigma \partial_\mu h_{00}^{\text{TT}} - \partial_\sigma \partial_0 h_{\mu 0}^{\text{TT}} - \partial_\mu \partial_0 h_{\sigma 0}^{\text{TT}}). \quad (7.105)$$

But  $h_{\mu 0}^{\text{TT}} = 0$ , so

$$R_{\mu 00\sigma} = \frac{1}{2} \partial_0 \partial_0 h_{\mu\sigma}^{\text{TT}}. \quad (7.106)$$

Meanwhile, for our slowly-moving particles we have  $\tau = x^0 = t$  to lowest order, so the geodesic deviation equation becomes

$$\frac{\partial^2}{\partial t^2} S^\mu = \frac{1}{2} S^\sigma \frac{\partial^2}{\partial t^2} h^{\text{TT}\mu}{}_\sigma. \quad (7.107)$$

For our wave traveling in the  $x^3$  direction, this implies that only  $S^1$  and  $S^2$  will be affected—the test particles are only disturbed in directions perpendicular to the wave vector. This is of course familiar from electromagnetism, where the electric and magnetic fields in a plane wave are perpendicular to the wave vector.

Our wave is characterized by the two numbers, which for future convenience we will rename as follows:

$$\begin{aligned} h_+ &= C_{11} \\ h_\times &= C_{12}, \end{aligned} \quad (7.108)$$

so that

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_x & 0 \\ 0 & h_x & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (7.109)$$

Let's consider their effects separately, beginning with the case  $h_x = 0$ . Then we have

$$\frac{\partial^2}{\partial t^2} S^1 = \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2} (h_+ e^{ik_\sigma x^\sigma}) \quad (7.110)$$

and

$$\frac{\partial^2}{\partial t^2} S^2 = -\frac{1}{2} S^2 \frac{\partial^2}{\partial t^2} (h_+ e^{ik_\sigma x^\sigma}). \quad (7.111)$$

These can be immediately solved to yield, to lowest order,

$$S^1 = \left(1 + \frac{1}{2} h_+ e^{ik_\sigma x^\sigma}\right) S^1(0) \quad (7.112)$$

and

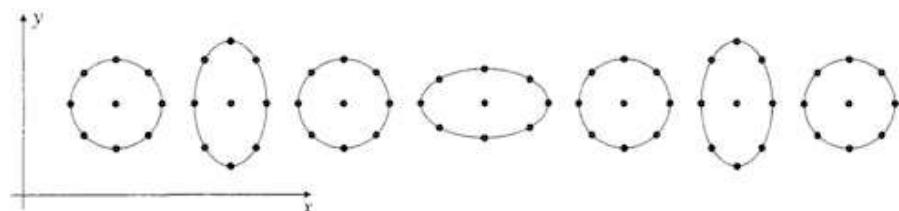
$$S^2 = \left(1 - \frac{1}{2} h_+ e^{ik_\sigma x^\sigma}\right) S^2(0). \quad (7.113)$$

Thus, particles initially separated in the  $x^1$  direction will oscillate in the  $x^1$  direction, and likewise for those with an initial  $x^2$  separation. That is, if we start with a ring of stationary particles in the  $x$ - $y$  plane, as the wave passes they will bounce back and forth in the shape of a "+," as shown in Figure 7.4. On the other hand, the equivalent analysis for the case where  $h_+ = 0$  but  $h_x \neq 0$  would yield the solution

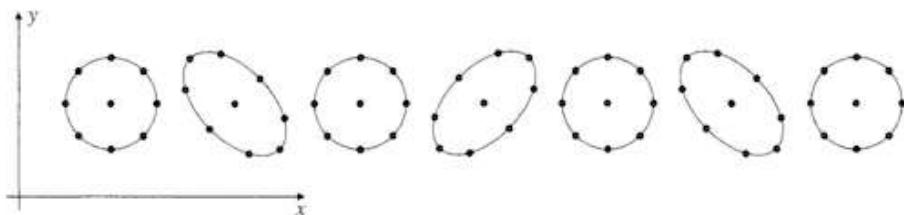
$$S^1 = S^1(0) + \frac{1}{2} h_x e^{ik_\sigma x^\sigma} S^2(0) \quad (7.114)$$

and

$$S^2 = S^2(0) + \frac{1}{2} h_x e^{ik_\sigma x^\sigma} S^1(0). \quad (7.115)$$



**FIGURE 7.4** The effect of a gravitational wave with + polarization is to distort a circle of test particles into ellipses oscillating in a + pattern.



**FIGURE 7.5** The effect of a gravitational wave with  $\times$  polarization is to distort a circle of test particles into ellipses oscillating in a  $\times$  pattern.

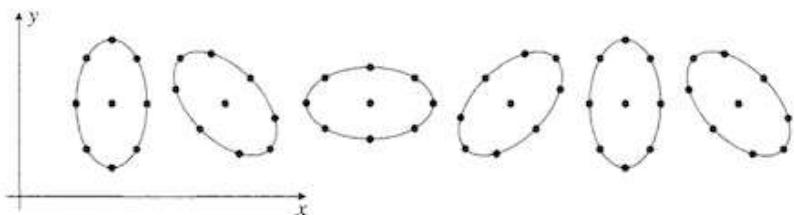
In this case the circle of particles would bounce back and forth in the shape of a “ $\times$ ,” as shown in Figure 7.5. The notation  $h_+$  and  $h_\times$  should therefore be clear. These two quantities measure the two independent modes of linear polarization of the gravitational wave, known as the “plus” and “cross” polarizations. If we liked we could consider right- and left-handed circularly polarized modes by defining

$$h_R = \frac{1}{\sqrt{2}}(h_+ + i h_\times),$$

$$h_L = \frac{1}{\sqrt{2}}(h_+ - i h_\times). \quad (7.116)$$

The effect of a pure  $h_R$  wave would be to rotate the particles in a right-handed sense, as shown in Figure 7.6, and similarly for the left-handed mode  $h_L$ . Note that the individual particles do not travel around the ring; they just move in little epicycles.

We can relate the polarization states of classical gravitational waves to the kinds of particles we would expect to find upon quantization. The spin of a quantized field is directly related to the transformation properties of that field under spatial rotations. The electromagnetic field has two independent polarization states described by vectors in the  $x$ - $y$  plane; equivalently, a single polarization mode is invariant under a rotation by  $360^\circ$  in this plane. Upon quantization this theory yields the photon, a massless spin-1 particle. The neutrino, on the other hand, is also a massless particle, described by a field that picks up a minus sign under rotations by  $360^\circ$ ; it is invariant under rotations of  $720^\circ$ , and we say it has

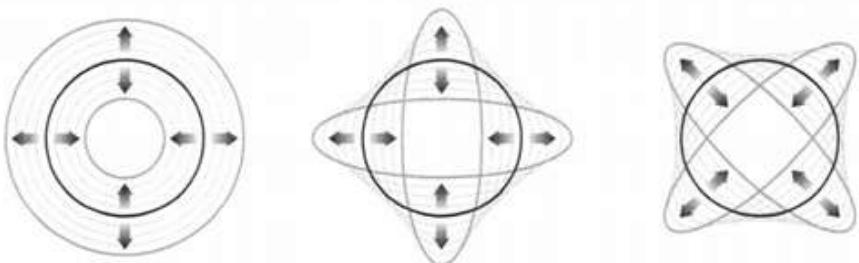


**FIGURE 7.6** The effect of a gravitational wave with  $R$  polarization is to distort a circle of test particles into an ellipse that rotates in a right-handed sense.

spin- $\frac{1}{2}$ . The general rule is that the spin  $S$  is related to the angle  $\theta$  under which the polarization modes are invariant by  $S = 360^\circ/\theta$ . The gravitational field, whose waves propagate at the speed of light, should lead to massless particles in the quantum theory. Noticing that the polarization modes we have described are invariant under rotations of  $180^\circ$  in the  $x$ - $y$  plane, we expect the associated particles—gravitons—to be spin-2. We are a long way from detecting such particles (and it would not be a surprise if we never detected them directly), but any respectable quantum theory of gravity should predict their existence.

In fact, starting with a theory of spin-2 gravitons and requiring some simple properties provides a nice way to *derive* the full Einstein's equation of general relativity. Imagine starting with the Lagrangian (7.9) for the symmetric tensor  $h_{\mu\nu}$ , but now imagining that this “really is” a physical field propagating in Minkowski spacetime rather than a perturbation to a dynamical metric. (This Lagrangian doesn't include couplings to matter, but it is straightforward to do so.) Now make the additional demand that  $h_{\mu\nu}$  couple to its own energy-momentum tensor (discussed below), as well as to the matter energy-momentum tensor. This induces higher-order nonlinear terms in the action, and consequently induces additional “energy-momentum” terms of even higher order. By repeating this process, an infinite series of terms is introduced, but the series can be summed to a simple expression, perhaps because you already know the answer—the Einstein–Hilbert action (possibly with some higher-order terms). In the process, we find that matter couples to the unique combination  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ . In other words, by asking for a theory of a spin-2 field coupling to the energy-momentum tensor, we end up with the fully nonlinear glory of general relativity. The background metric  $\eta_{\mu\nu}$  becomes completely unobservable. Of course, some of the global geometric aspects of GR are obscured by this procedure, which ultimately is just another way of justifying Einstein's equation.

While we are noting amusing things, let's point out that the behavior of gravitational waves yields a clue as to why string theory gives rise to a quantum theory of gravity. Consider the fundamental vibrational modes of a loop of string, as shown in Figure 7.7. There are three lowest-energy modes for a loop of string: an over-



**FIGURE 7.7** The three fundamental vibrational modes of a loop of string. The overall “breathing” mode (far left) is invariant under rotations, and gives rise to a spin-0 particle. The other two modes match the two polarizations of a gravitational wave, and represent the two states of a massless spin-2 particle.

all oscillation of its size, plus two independent ways it can oscillate into ellipses. These give rise to three massless degrees of freedom: a spin-0 particle (the dilaton) and a massless spin-2 particle (the graviton). Notice the obvious similarity between the string oscillations and the motion of test particles under the influence of a gravitational wave; this is no accident, and is the reason why quantized strings inevitably give rise to gravity. (String theory was originally investigated as a theory of the strong interactions, but different models would inevitably predict an unnecessary massless spin-2 particle; eventually it was realized that this flaw could be a virtue, if the theory came to be thought of as a quantum theory of gravity.) The extra unwanted spin-0 (scalar) mode reflects the fact that string theory actually predicts a scalar-tensor theory of gravity (as discussed in Section 4.8) rather than ordinary GR. Since a massless scalar of this sort is not observed in nature, some mechanism must work to give a mass to the scalar at low energies.

## 7.5 ■ PRODUCTION OF GRAVITATIONAL WAVES

With plane-wave solutions to the linearized vacuum equation in our possession, it remains to discuss the generation of gravitational radiation by sources. For this purpose it is necessary to consider Einstein's equation coupled to matter,  $G_{\mu\nu} = 8\pi GT_{\mu\nu}$ . Because  $T_{\mu\nu}$  doesn't vanish, the metric perturbation will include nonzero scalar and vector components as well as the strain tensor representing gravitational waves; we cannot assume that our solution takes the transverse-traceless form (7.90). Instead, we will keep the entire perturbation  $h_{\mu\nu}$  and solve for the produced gravitational wave far from the source, where we can then impose transverse-traceless gauge.

There are still some convenient simplifications we can introduce, even in the presence of sources. We first define the trace-reversed perturbation,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}. \quad (7.117)$$

The name of the trace-reversed perturbation makes sense, since

$$\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h. \quad (7.118)$$

Obviously we can reconstruct the original perturbation from the trace-reversed form, so no information has been lost. Note also that, if we are in vacuum far away from any sources and can go to transverse-traceless gauge, the trace-reversed perturbation will be equal to the original perturbation,

$$\bar{h}_{\mu\nu}^{\text{TT}} = h_{\mu\nu}^{\text{TT}}. \quad (7.119)$$

Meanwhile, we are still free to choose some sort of gauge. Under a gauge transformation (7.14), the trace-reversed perturbation transforms as

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} - \partial_\lambda\xi^\lambda\eta_{\mu\nu}. \quad (7.120)$$

By choosing a gauge parameter  $\xi_\mu$  satisfying

$$\square \xi_\mu = -\partial_\lambda \bar{h}^\lambda{}_\mu, \quad (7.121)$$

we can therefore set

$$\partial_\mu \bar{h}^{\mu\nu} = 0. \quad (7.122)$$

This condition is known as the **Lorenz gauge**, analogous with the similar condition  $\partial_\mu A^\mu = 0$  often used in electromagnetism.<sup>3</sup> Note that the original perturbation is not transverse in this gauge; rather, we have

$$\partial_\mu h^{\mu\nu} = \frac{1}{2} \partial^\nu h. \quad (7.123)$$

Plugging the definition of the trace-reversed perturbation into our expression for the Einstein tensor (7.8), and using the Lorenz gauge condition, yields the very concise expression

$$G_{\mu\nu} = -\frac{1}{2} \square \bar{h}_{\mu\nu}. \quad (7.124)$$

The analogous expression in terms of the original perturbation  $h_{\mu\nu}$  is slightly messier; this is the reason for introducing  $\bar{h}_{\mu\nu}$ . The linearized Einstein equation in this gauge is therefore simply a wave equation for each component,

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \quad (7.125)$$

The solution to such an equation can be obtained using a Green function, in precisely the same way as the analogous problem in electromagnetism. Here we will review the outline of the method, following Wald (1984).

The Green function  $G(x^\sigma - y^\sigma)$  for the d'Alembertian operator  $\square$  is the solution of the wave equation in the presence of a delta-function source:

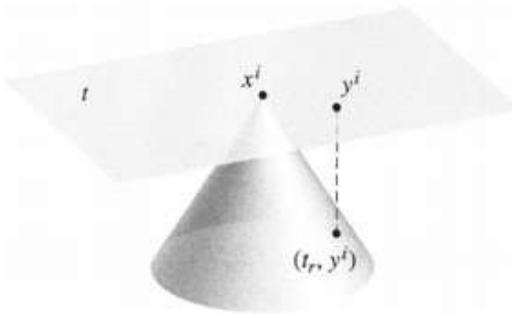
$$\square_x G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma), \quad (7.126)$$

where  $\square_x$  denotes the d'Alembertian with respect to the coordinates  $x^\sigma$ . The usefulness of such a function resides in the fact that the general solution to an equation such as (7.125) can be written

$$\bar{h}_{\mu\nu}(x^\sigma) = -16\pi G \int G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma) d^4y, \quad (7.127)$$

as can be verified immediately. (Notice that no factors of  $\sqrt{-g}$  are necessary, since our background is simply flat spacetime.) The solutions to (7.126) have of course been worked out long ago, and they can be thought of as either “retarded” or “advanced,” depending on whether they represent waves traveling forward or

<sup>3</sup>Note the spelling. The “gauge” was originated by Ludwig Lorenz (1829–1891), while the more famous “transformation” was invented by Hendrick Antoon Lorentz (1853–1928). See J.D. Jackson and L.B. Okun, *Rev. Mod. Phys.* **73**, 663 (2001).



**FIGURE 7.8** Disturbances in the gravitational field at  $(t, x^i)$  are calculated in terms of events on the past light cone.

backward in time. Our interest is in the retarded Green function, which represents the accumulated effect of signals to the past of the points under consideration. It is given by

$$G(x^\sigma - y^\sigma) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \delta[|\mathbf{x} - \mathbf{y}| - (x^0 - y^0)] \theta(x^0 - y^0). \quad (7.128)$$

Here we have used boldface to denote the spatial vectors  $\mathbf{x} = (x^1, x^2, x^3)$  and  $\mathbf{y} = (y^1, y^2, y^3)$ , with norm  $|\mathbf{x} - \mathbf{y}| = [\delta_{ij}(x^i - y^i)(x^j - y^j)]^{1/2}$ . The theta function  $\theta(x^0 - y^0)$  equals 1 when  $x^0 > y^0$ , and zero otherwise. The derivation of (7.128) would take us too far afield, but it can be found in any standard text on electrodynamics or partial differential equations in physics.

Upon plugging (7.128) into (7.127), we can use the delta function to perform the integral over  $y^0$ , leaving us with

$$\tilde{h}_{\mu\nu}(t, \mathbf{x}) = 4G \int \frac{1}{|\mathbf{x} - \mathbf{y}|} T_{\mu\nu}(t - |\mathbf{x} - \mathbf{y}|, \mathbf{y}) d^3y, \quad (7.129)$$

where  $t = x^0$ . The term “retarded time” is used to refer to the quantity

$$t_r = t - |\mathbf{x} - \mathbf{y}|. \quad (7.130)$$

The interpretation of (7.129) should be clear: the disturbance in the gravitational field at  $(t, \mathbf{x})$  is a sum of the influences from the energy and momentum sources at the point  $(t_r, \mathbf{x} - \mathbf{y})$  on the past light cone, as depicted in Figure 7.8.

Let us take this general solution and consider the case where the gravitational radiation is emitted by an isolated source, fairly far away, comprised of nonrelativistic matter; these approximations will be made more precise as we go on. First we need to set up some conventions for Fourier transforms, which always make life easier when dealing with oscillatory phenomena. Given a function of space-time  $\phi(t, \mathbf{x})$ , we are interested in its Fourier transform (and inverse) with respect to time alone,

$$\begin{aligned}\tilde{\phi}(\omega, \mathbf{x}) &= \frac{1}{\sqrt{2\pi}} \int dt e^{i\omega t} \phi(t, \mathbf{x}), \\ \phi(t, \mathbf{x}) &= \frac{1}{\sqrt{2\pi}} \int d\omega e^{-i\omega t} \tilde{\phi}(\omega, \mathbf{x}).\end{aligned}\quad (7.131)$$

Taking the transform of the metric perturbation, we obtain

$$\begin{aligned}\tilde{h}_{\mu\nu}(\omega, \mathbf{x}) &= \frac{1}{\sqrt{2\pi}} \int dt e^{i\omega t} \tilde{h}_{\mu\nu}(t, \mathbf{x}) \\ &= \frac{4G}{\sqrt{2\pi}} \int dt d^3y e^{i\omega t} \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{y}|, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \\ &= \frac{4G}{\sqrt{2\pi}} \int dt_r d^3y e^{i\omega t_r} e^{i\omega|\mathbf{x}-\mathbf{y}|} \frac{T_{\mu\nu}(t_r, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \\ &= 4G \int d^3y e^{i\omega|\mathbf{x}-\mathbf{y}|} \frac{\tilde{T}_{\mu\nu}(\omega, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|}.\end{aligned}\quad (7.132)$$

In this sequence, the first equation is simply the definition of the Fourier transform, the second line comes from the solution (7.129), the third line is a change of variables from  $t$  to  $t_r$ , and the fourth line is once again the definition of the Fourier transform.

We now make the approximations that our source is isolated, far away, and slowly moving. This means that we can consider the source to be centered at a (spatial) distance  $r$ , with the different parts of the source at distances  $r + \delta r$  such that  $\delta r \ll r$ , as shown in Figure 7.9. Since it is slowly moving, most of the radiation emitted will be at frequencies  $\omega$  sufficiently low that  $\delta r \ll \omega^{-1}$ . (Essentially, light traverses the source much faster than the components of the source itself do.) Under these approximations, the term  $e^{i\omega|\mathbf{x}-\mathbf{y}|}/|\mathbf{x} - \mathbf{y}|$  can be replaced by  $e^{i\omega r}/r$  and brought outside the integral. This leaves us with

$$\tilde{h}_{\mu\nu}(\omega, \mathbf{x}) = 4G \frac{e^{i\omega r}}{r} \int d^3y \tilde{T}_{\mu\nu}(\omega, \mathbf{y}).\quad (7.133)$$



FIGURE 7.9 A source of size  $\delta r$ , at a distance  $r$  from the observer.

In fact there is no need to compute all of the components of  $\tilde{h}_{\mu\nu}(\omega, \mathbf{x})$ , since the Lorenz gauge condition  $\partial_\mu \tilde{h}^{\mu\nu}(t, \mathbf{x}) = 0$  in Fourier space implies

$$\tilde{h}^{0\nu} = \frac{i}{\omega} \partial_i \tilde{h}^{i\nu}. \quad (7.134)$$

We therefore only need to concern ourselves with the spacelike components of  $\tilde{h}_{\mu\nu}(\omega, \mathbf{x})$ , and recover  $\tilde{h}^{0\nu}$  from (7.134). The first thing to do is to set  $\nu = j$  to find  $\tilde{h}^{0j}$  from  $\tilde{h}^{ij}$ , which we would then use to find  $\tilde{h}^{00}$  from  $\tilde{h}^{i0}$ . From (7.133) we therefore want to take the integral of the spacelike components of  $\tilde{T}_{\mu\nu}(\omega, \mathbf{y})$ . We begin by integrating by parts in reverse:

$$\int d^3y \tilde{T}^{ij}(\omega, \mathbf{y}) = \int \partial_k(y^i \tilde{T}^{kj}) d^3y - \int y^i (\partial_k \tilde{T}^{kj}) d^3y. \quad (7.135)$$

The first term is a surface integral which will vanish since the source is isolated, while the second can be related to  $\tilde{T}^{0j}$  by the Fourier-space version of  $\partial_\mu T^{\mu\nu} = 0$ :

$$-\partial_k \tilde{T}^{k\mu} = i\omega \tilde{T}^{0\mu}. \quad (7.136)$$

Thus,

$$\begin{aligned} \int d^3y \tilde{T}^{ij}(\omega, \mathbf{y}) &= i\omega \int y^i \tilde{T}^{0j} d^3y \\ &= \frac{i\omega}{2} \int (y^i \tilde{T}^{0j} + y^j \tilde{T}^{0i}) d^3y \\ &= \frac{i\omega}{2} \int [\partial_l(y^i y^j \tilde{T}^{0l}) - y^i y^j (\partial_l \tilde{T}^{0l})] d^3y \\ &= -\frac{\omega^2}{2} \int y^i y^j \tilde{T}^{00} d^3y. \end{aligned} \quad (7.137)$$

The second line is justified since we know that the left-hand side is symmetric in  $i$  and  $j$ , while the third and fourth lines are simply repetitions of reverse integration by parts and conservation of  $T^{\mu\nu}$ . It is conventional to define the **quadrupole moment tensor** of the energy density of the source,

$$I_{ij}(t) = \int y^i y^j T^{00}(t, \mathbf{y}) d^3y. \quad (7.138)$$

a constant tensor on each surface of constant time. The overall normalization of the quadrupole tensor is a matter of convention, and by no means universal, so be careful in comparing different references. In terms of the Fourier transform of the quadrupole moment, our solution takes on the compact form

$$\tilde{h}_{ij}(\omega, \mathbf{x}) = -2G\omega^2 \frac{e^{i\omega r}}{r} I_{ij}(\omega). \quad (7.139)$$

We can transform this back to  $t$  to obtain the **quadrupole formula**,

$$\tilde{h}_{ij}(t, \mathbf{x}) = \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2}(t_r), \quad (7.140)$$

where as before  $t_r = t - r$ .

The gravitational wave produced by an isolated nonrelativistic object is therefore proportional to the second derivative of the quadrupole moment of the energy density at the point where the past light cone of the observer intersects the source. In contrast, the leading contribution to electromagnetic radiation comes from the changing *dipole* moment of the charge density. The difference can be traced back to the universal nature of gravitation. A changing dipole moment corresponds to motion of the center of density—charge density in the case of electromagnetism, energy density in the case of gravitation. While there is nothing to stop the center of charge of an object from oscillating, oscillation of the center of mass of an isolated system violates conservation of momentum. (You can shake a body up and down, but you and the earth shake ever so slightly in the opposite direction to compensate.) The quadrupole moment, which measures the shape of the system, is generally smaller than the dipole moment, and for this reason, as well as the weak coupling of matter to gravity, gravitational radiation is typically much weaker than electromagnetic radiation.

One case of special interest is the gravitational radiation emitted by a binary star (two stars in orbit around each other). For simplicity let us consider two stars of mass  $M$  in a circular orbit in the  $x^1$ - $x^2$  plane, at distance  $R$  from their common center of mass, as shown in Figure 7.10. We will treat the motion of the stars in the Newtonian approximation, where we can discuss their orbit just as Kepler would have. Circular orbits are most easily characterized by equating the force due to gravity to the outward “centrifugal” force:

$$\frac{GM^2}{(2R)^2} = \frac{Mv^2}{R}, \quad (7.141)$$

which gives us

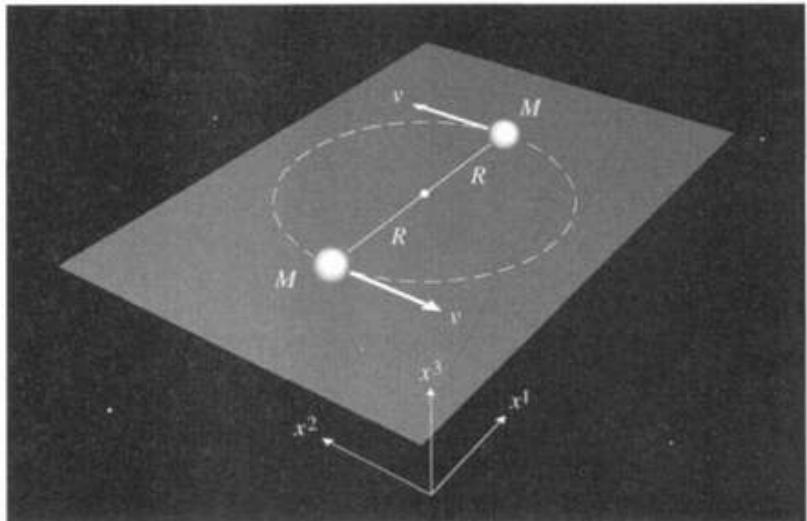
$$v = \left( \frac{GM}{4R} \right)^{1/2}. \quad (7.142)$$

The time it takes to complete a single orbit is simply

$$T = \frac{2\pi R}{v}. \quad (7.143)$$

but more useful to us is the angular frequency of the orbit,

$$\Omega = \frac{2\pi}{T} = \left( \frac{GM}{4R^3} \right)^{1/2}. \quad (7.144)$$



**FIGURE 7.10** A binary star system. Two stars of mass  $M$  orbit in the  $x^1$ - $x^2$  plane with an orbital radius  $R$ .

In terms of  $\Omega$  we can write down the explicit path of star  $a$ ,

$$x_a^1 = R \cos \Omega t, \quad x_a^2 = R \sin \Omega t, \quad (7.145)$$

and star  $b$ ,

$$x_b^1 = -R \cos \Omega t, \quad x_b^2 = -R \sin \Omega t. \quad (7.146)$$

The corresponding energy density is

$$\begin{aligned} T^{00}(t, \mathbf{x}) = M\delta(x^3)[ & \delta(x^1 - R \cos \Omega t)\delta(x^2 - R \sin \Omega t) \\ & + \delta(x^1 + R \cos \Omega t)\delta(x^2 + R \sin \Omega t)]. \end{aligned} \quad (7.147)$$

The profusion of delta functions allows us to integrate this straightforwardly to obtain the quadrupole moment from (7.138):

$$\begin{aligned} I_{11} &= 2MR^2 \cos^2 \Omega t = MR^2(1 + \cos 2\Omega t) \\ I_{22} &= 2MR^2 \sin^2 \Omega t = MR^2(1 - \cos 2\Omega t) \\ I_{12} &= I_{21} = 2MR^2(\cos \Omega t)(\sin \Omega t) = MR^2 \sin 2\Omega t \\ I_{33} &= 0. \end{aligned} \quad (7.148)$$

From this in turn it is easy to get the components of the metric perturbation from (7.140):

$$\tilde{h}_{ij}(t, \mathbf{x}) = \frac{8GM}{r} \Omega^2 R^2 \begin{pmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (7.149)$$

The remaining components of  $\tilde{h}_{\mu\nu}$  could be derived from demanding that the Lorenz gauge condition be satisfied.

## 7.6 ■ ENERGY LOSS DUE TO GRAVITATIONAL RADIATION

It is natural at this point to talk about the energy emitted via gravitational radiation. Such a discussion, however, is immediately beset by problems, both technical and philosophical. As we have mentioned before, there is no true local measure of the energy in the gravitational field. Of course, in the weak field limit, where we think of gravitation as being described by a symmetric tensor propagating on a fixed background metric, we might hope to derive an energy-momentum tensor for the fluctuations  $h_{\mu\nu}$ , just as we would for electromagnetism or any other field theory. To some extent this is possible, but still difficult. As a result of these difficulties there are a number of different proposals in the literature for what we should use as the energy-momentum tensor for gravitation in the weak field limit; all of them are different, but for the most part they give the same answers for physically well-posed questions such as the rate of energy emitted by a binary system.

At a technical level, the difficulties begin to arise when we consider what form the energy-momentum tensor should take. We have previously mentioned the energy-momentum tensors for electromagnetism and scalar field theory, both of which share an important feature—they are quadratic in the relevant fields. By hypothesis, our approach to the weak field limit has been to keep only terms that are linear in the metric perturbation. Hence, in order to keep track of the energy carried by the gravitational waves, we will have to extend our calculations to at least second order in  $h_{\mu\nu}$ . In fact we have been cheating slightly all along. In discussing the effects of gravitational waves on test particles, and the generation of waves by a binary system, we have been using the fact that test particles move along geodesics. But as we know, this is derived from the covariant conservation of energy-momentum,  $\nabla_\mu T^{\mu\nu} = 0$ . In the order to which we have been working, however, we actually have  $\partial_\mu T^{\mu\nu} = 0$ , which would imply that test particles move on straight lines in the flat background metric. This is a symptom of the inability of the weak field limit to describe self-gravitating systems. In practice, the best that can be done is to solve the weak field equation to some appropriate order, and then justify after the fact the validity of the solution. We will follow the procedure outlined in Chapters 35 and 36 of Misner, Thorne, and Wheeler (1973), where additional discussion of subtleties may be found. See also Wald (1984) and Schutz (1985).

Let us now examine Einstein's vacuum equation  $R_{\mu\nu} = 0$  to second order, and see how the result can be interpreted in terms of an energy-momentum tensor for

the gravitational field. We expand both the metric and the Ricci tensor,

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} \\ R_{\mu\nu} &= R_{\mu\nu}^{(0)} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)}, \end{aligned} \quad (7.150)$$

where  $R_{\mu\nu}^{(1)}$  is taken to be of the same order as  $h_{\mu\nu}^{(1)}$ , while  $R_{\mu\nu}^{(2)}$  and  $h_{\mu\nu}^{(2)}$  are of order  $(h_{\mu\nu}^{(1)})^2$ . Because we work in a flat background, the zeroth-order equation  $R_{\mu\nu}^{(0)} = 0$  is automatically solved. The first-order vacuum equation is simply

$$R_{\mu\nu}^{(1)}[h^{(1)}] = 0, \quad (7.151)$$

which determines the first-order perturbation  $h_{\mu\nu}^{(1)}$  (up to gauge transformations). The second-order perturbation  $h_{\mu\nu}^{(2)}$  will be determined by the second-order equation

$$R_{\mu\nu}^{(1)}[h^{(2)}] + R_{\mu\nu}^{(2)}[h^{(1)}] = 0. \quad (7.152)$$

The notation  $R_{\mu\nu}^{(1)}[h^{(2)}]$  indicates the parts of the expanded Ricci tensor that are linear in the metric perturbation, as given by (7.6), applied to the second-order perturbation  $h_{\mu\nu}^{(2)}$ ; meanwhile,  $R_{\mu\nu}^{(2)}[h^{(1)}]$  stands for the quadratic part of the expanded Ricci tensor,

$$\begin{aligned} R_{\mu\nu}^{(2)} &= \frac{1}{2} h^{\rho\sigma} \partial_\mu \partial_\nu h_{\rho\sigma} + \frac{1}{4} (\partial_\mu h_{\rho\sigma}) \partial_\nu h^{\rho\sigma} + (\partial^\sigma h^\rho{}_\nu) \partial_{[\sigma} h_{\rho]\mu} - h^{\rho\sigma} \partial_\rho \partial_{(\mu} h_{\nu)\sigma} \\ &\quad + \frac{1}{2} \partial_\sigma (h^{\rho\sigma} \partial_\rho h_{\mu\nu}) - \frac{1}{4} (\partial_\rho h_{\mu\nu}) \partial^\rho h - (\partial_\sigma h^{\rho\sigma} - \frac{1}{2} \partial^\rho h) \partial_{(\mu} h_{\nu)\rho}, \end{aligned} \quad (7.153)$$

applied to the first-order perturbation  $h_{\mu\nu}^{(1)}$ . There are no cross terms, as they would necessarily be higher order.

Now let's write the vacuum equation as  $G_{\mu\nu} = 0$ ; this is of course equivalent to  $R_{\mu\nu} = 0$ , but will enable us to express the result in a suggestive form. At second order we have

$$R_{\mu\nu}^{(1)}[h^{(2)}] - \frac{1}{2} \eta^{\rho\sigma} R_{\rho\sigma}^{(1)}[h^{(2)}] \eta_{\mu\nu} = 8\pi G t_{\mu\nu}, \quad (7.154)$$

where we have defined

$$t_{\mu\nu} \equiv -\frac{1}{8\pi G} \left\{ R_{\mu\nu}^{(2)}[h^{(1)}] - \frac{1}{2} \eta^{\rho\sigma} R_{\rho\sigma}^{(2)}[h^{(1)}] \eta_{\mu\nu} \right\}. \quad (7.155)$$

Notice a couple of things about this expression. First, we have not included terms of the form  $h^{(1)\rho\sigma} R_{\rho\sigma}^{(1)}[h^{(1)}]$ , since  $R_{\mu\nu}^{(1)}[h^{(1)}] = 0$ . Second, the left-hand side of (7.154) is not the full second-order Einstein tensor, as we have moved terms involving  $R_{\mu\nu}^{(2)}[h^{(1)}]$  to the right-hand side and provocatively relabeled them as an energy-momentum tensor for the first-order perturbations,  $t_{\mu\nu}$ . Such an identifi-

cation seems eminently reasonable;  $t_{\mu\nu}$  is a symmetric tensor, quadratic in  $h_{\mu\nu}$ , which represents how the perturbations affect the spacetime metric in just the way that a matter energy-momentum tensor would. (Linear terms in  $h_{\mu\nu}$  have no effect, since  $G_{\mu\nu}^{(1)}[h^{(1)}]$  is simply set to zero by the first-order equation.) Notice that  $t_{\mu\nu}$  is also conserved, in the background flat-space sense,

$$\partial_\mu t^{\mu\nu} = 0, \quad (7.156)$$

which we know from the Bianchi identity  $\partial_\mu G^{\mu\nu} = 0$ .

Unfortunately there are some limitations on our interpretation of  $t_{\mu\nu}$  as an energy-momentum tensor. Of course it is not a tensor at all in the full theory, but we are leaving that aside by hypothesis. More importantly, it is not invariant under gauge transformations (infinitesimal diffeomorphisms), as you could check by direct calculation. One way of circumventing this difficulty is to average the energy-momentum tensor over several wavelengths, an operation we denote by angle brackets  $\langle \dots \rangle$ . This procedure has both philosophical and practical advantages. From a philosophical viewpoint, we know that our ability to choose Riemann normal coordinates at any one point makes it impossible to define a reliable measure of the gravitational energy-momentum that is purely local (defined at each point in terms of the metric and its first derivatives at precisely that point). If we average over several wavelengths, however, we may hope to capture enough of the physical curvature in a small region to describe a gauge-invariant measure. From a practical standpoint, any terms that are derivatives (as opposed to products of derivatives) will average to zero,

$$\langle \partial_\mu (X) \rangle = 0. \quad (7.157)$$

We are therefore empowered to integrate by parts under the averaging brackets,

$$\langle A(\partial_\mu B) \rangle = -\langle (\partial_\mu A)B \rangle, \quad (7.158)$$

which will greatly simplify our expressions.

With this in mind, let us calculate  $t_{\mu\nu}$  as defined in (7.155), using the expression (7.153) for the second-order Ricci tensor. (Henceforth we will no longer use superscripts on the metric perturbation, as we will only be interested in the first-order perturbation.) Although part of the motivation for averaging is to obtain a gauge-invariant answer, the actual calculation is a mess, so for illustrative purposes we will carry it out in transverse-traceless gauge,

$$\partial^\mu h_{\mu\nu}^{TT} = 0, \quad h^{TT} = 0. \quad (7.159)$$

Don't forget that we are only allowed to choose this gauge in vacuum. The non-vanishing parts of  $R_{\mu\nu}^{(2)TT}$  in this gauge can be written as

$$\begin{aligned} R_{\mu\nu}^{(2)TT} = & \frac{1}{2} h_{TT}^{\rho\sigma} \partial_\mu \partial_\nu h_{\rho\sigma}^{TT} + \frac{1}{4} (\partial_\mu h_{\rho\sigma}^{TT}) \partial_\nu h_{\rho\sigma}^{\rho\sigma} + \frac{1}{2} \eta^{\rho\lambda} (\partial^\sigma h_{\rho\lambda}^{TT}) \partial_\sigma h_{\mu\nu}^{TT} \\ & - \frac{1}{2} (\partial^\sigma h_{\rho\lambda}^{TT}) \partial^\rho h_{\sigma\mu}^{TT} - h_{TT}^{\rho\sigma} \partial_\rho \partial_{(\mu} h_{\nu)\sigma}^{TT} + \frac{1}{2} h_{TT}^{\rho\sigma} \partial_\sigma \partial_\rho h_{\mu\nu}^{TT}. \end{aligned} \quad (7.160)$$

Now let's apply the averaging brackets, and integrate by parts where convenient. The last three terms in (7.160) all go away, as integration by parts leads to divergences that vanish. We are left with

$$\left\langle R_{\mu\nu}^{(2)\text{TT}} \right\rangle = -\frac{1}{4} \left\langle (\partial_\mu h_{\rho\sigma}^{\text{TT}})(\partial_\nu h_{\lambda\mu}^{\rho\sigma}) + 2\eta^{\rho\lambda} (\square h_{\rho\nu}^{\text{TT}}) h_{\lambda\mu}^{\text{TT}} \right\rangle. \quad (7.161)$$

But the perturbation obeys the first-order equation of motion, which sets  $\square h_{\mu\nu}^{\text{TT}} = 0$ . So we are finally left with

$$\left\langle R_{\mu\nu}^{(2)\text{TT}} \right\rangle = -\frac{1}{4} \left\langle (\partial_\mu h_{\rho\sigma}^{\text{TT}})(\partial_\nu h_{\lambda\mu}^{\rho\sigma}) \right\rangle. \quad (7.162)$$

We can take the trace to get the curvature scalar; after integration by parts we again find a  $\square h_{\mu\nu}^{\text{TT}}$  term which we set to zero, so

$$\left\langle \eta^{\mu\nu} R_{\mu\nu}^{(2)\text{TT}} \right\rangle = 0. \quad (7.163)$$

These expressions can be inserted into (7.155) to obtain a simple expression for the gravitational-wave energy-momentum tensor in transverse-traceless gauge:

$$t_{\mu\nu} = \frac{1}{32\pi G} \left\langle (\partial_\mu h_{\rho\sigma}^{\text{TT}})(\partial_\nu h_{\lambda\mu}^{\rho\sigma}) \right\rangle. \quad (7.164)$$

Remember that, in this gauge, nonspatial components vanish,  $h_{0\nu}^{\text{TT}} = 0$ . You will therefore sometimes see the above expression written with spatial indices  $ij$  instead of spacetime indices  $\rho\sigma$ ; the two versions are clearly equivalent. If we had been strong enough to do the corresponding calculation without first choosing a gauge, we would have found

$$\begin{aligned} t_{\mu\nu} = \frac{1}{32\pi G} & \left\langle (\partial_\mu h_{\rho\sigma})(\partial_\nu h^{\rho\sigma}) - \frac{1}{2}(\partial_\mu h)(\partial_\nu h) \right. \\ & \left. - (\partial_\rho h^{\rho\sigma})(\partial_\mu h_{\nu\sigma}) - (\partial_\rho h^{\rho\sigma})(\partial_\nu h_{\mu\sigma}) \right\rangle. \end{aligned} \quad (7.165)$$

A bit of straightforward manipulation suffices to check that this expression is actually gauge invariant, as you are asked to show in the exercises.

Let's calculate the transverse-traceless expression (7.164) for a single plane wave,

$$h_{\mu\nu}^{\text{TT}} = C_{\mu\nu} \sin(k_\lambda x^\lambda). \quad (7.166)$$

We have taken the real part and set the phase arbitrarily so that the wave is a sine rather than cosine. The energy-momentum tensor is then

$$t_{\mu\nu} = \frac{1}{32\pi G} k_\mu k_\nu C_{\rho\sigma} C^{\rho\sigma} \left\langle \cos^2(k_\lambda x^\lambda) \right\rangle. \quad (7.167)$$

Averaging the  $\cos^2$  term over several wavelengths yields

$$\langle \cos^2(k_\lambda x^\lambda) \rangle = \frac{1}{2}. \quad (7.168)$$

For simplicity we can take the wave to be moving along the  $z$ -axis, so that

$$k_\lambda = (-\omega, 0, 0, \omega) \quad (7.169)$$

the minus sign coming from lowering an index on  $k^\lambda$ , and from (7.109),

$$C_{\rho\sigma} C^{\rho\sigma} = 2(h_+^2 + h_x^2). \quad (7.170)$$

It is more common in the gravitational-wave literature to express observables in terms of the ordinary frequency  $f = \omega/2\pi$ , rather than the angular frequency  $\omega$ . Putting it all together reveals

$$t_{\mu\nu} = \frac{\pi}{8G} f^2 (h_+^2 + h_x^2) \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \quad (7.171)$$

As we will discuss in the next section, typical gravitational-wave sources we might expect to observe at Earth will have frequencies between  $10^{-4}$  and  $10^4$  Hz, and amplitudes  $h \sim 10^{-22}$ . It is therefore useful to express the energy flux in the  $z$  direction,  $-T_{0z}$ , at an order-of-magnitude level as

$$-T_{0z} \sim 10^{-4} \left( \frac{f}{\text{Hz}} \right)^2 \frac{(h_+^2 + h_x^2)}{(10^{-21})^2} \frac{\text{erg}}{\text{cm}^2 \cdot \text{s}}. \quad (7.172)$$

This is the amount of energy that could in principle be deposited in each square centimeter of a detector every second. As pointed out by Thorne,<sup>4</sup> this is actually a substantial energy flux, especially at the upper end of the frequency range. For comparison purposes, a supernova at cosmological distances is characterized by a peak electromagnetic flux of approximately  $10^{-9}$  erg/cm<sup>2</sup>/s; the gravitational-wave signal, however, only lasts for milliseconds, while the visible electromagnetic signal extends for months.

Now let's use our formula for the gravitational-wave energy-momentum tensor to calculate the rate of energy loss from a system emitting gravitational radiation according to the quadrupole formula (7.140). The total energy contained in gravitational radiation on a surface  $\Sigma$  of constant time is defined as

$$E = \int_{\Sigma} t_{00} d^3x, \quad (7.173)$$

while the total energy radiated through to infinity may be expressed as

<sup>4</sup>K.S. Thorne, in *Three Hundred Years of Gravitation*, Cambridge: Cambridge University Press, 1987.

$$\Delta E = \int P dt, \quad (7.174)$$

where the power  $P$  is

$$P = \int_{S_\infty^2} t_{0\mu} n^\mu r^2 d\Omega. \quad (7.175)$$

Here, the integral is taken over a two-sphere at spatial infinity  $S_\infty^2$ , and  $n^\mu$  is a unit spacelike vector normal to  $S_\infty^2$ . In polar coordinates  $\{t, r, \theta, \phi\}$ , the components of the normal vector are

$$n^\mu = (0, 1, 0, 0). \quad (7.176)$$

We would like to calculate the power  $P$  using our expression for  $t_{\mu\nu}$ , (7.164). The first issue we face is that this expression is written in terms of the transverse-traceless perturbation, while the quadrupole formula (7.140) is written in terms of the spatial components  $\bar{h}_{ij}$  of the Lorenz-gauge trace-reversed perturbation. The simplest procedure (although it's not that simple) is to first convert  $\bar{h}_{ij}$  into transverse-traceless gauge, which is permissible because we are interested in the behavior of the waves in vacuum, far from the source from which they are emitted, plug into the formula for  $t_{\mu\nu}$ , then convert back into nontransverse-traceless form. Let's see how this works.

We begin by introducing the (spatial) projection tensor

$$P_{ij} = \delta_{ij} - n_i n_j, \quad (7.177)$$

which projects tensor components into a surface orthogonal to the unit vector  $n^i$ . (See Appendix D for more discussion.) In our case, we choose  $n^i$  to point along the direction of propagation of the wave, so that  $P_{ij}$  will project onto the two-sphere at spatial infinity. We can use the projection tensor to construct the transverse-traceless version of a symmetric spatial tensor  $X_{ij}$  via

$$X_{ij}^{TT} = \left( P_i^k P_j^l - \frac{1}{2} P_{ij} P^{kl} \right) X_{kl}. \quad (7.178)$$

You can check for yourself that  $X_{ij}^{TT}$  is indeed transverse and traceless. Because it is traceless,  $\bar{h}_{ij}^{TT}$  is equal to the original perturbation  $h_{ij}^{TT}$ ; plugging into the quadrupole formula (7.140), we get

$$h_{ij}^{TT} = \bar{h}_{ij}^{TT} = \frac{2G}{r} \frac{d^2 I_{ij}^{TT}}{dt^2} (t - r), \quad (7.179)$$

where the transverse-traceless part of the quadrupole moment is also constructed via (7.178). In fact the quadrupole moment defined by (7.138) is not the most convenient quantity to use in expressing the generated wave, as it involves an

integral over the energy density that might be difficult to determine. Instead we can use the **reduced quadrupole moment**,

$$J_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} \delta^{kl} I_{kl}, \quad (7.180)$$

which is just the traceless part of  $I_{ij}$ . The reduced quadrupole moment has the nice property of being the coefficient of the  $r^{-3}$  term in the multipole expansion of the Newtonian potential,

$$\Phi = -\frac{GM}{r} - \frac{G}{r^3} D_i x^i - \frac{3G}{2r^5} J_{ij} x^i x^j + \dots, \quad (7.181)$$

and is therefore more readily approximated for realistic sources. (Here  $D_i$  is the dipole moment,  $D_i = \int T^{00} x^i d^3x$ .) Of course, the transverse-traceless part of the quadrupole moment is the same as the transverse-traceless part of the reduced (that is, traceless) quadrupole moment, so (7.179) becomes

$$h_{ij}^{\text{TT}} = \frac{2G}{r} \frac{d^2 J_{ij}^{\text{TT}}}{dt^2} (t - r). \quad (7.182)$$

To calculate the power, we are interested in  $t_{0\mu} n^\mu = t_{0r}$ . Because the quadrupole moment depends only on the retarded time  $t_r = t - r$ , we have

$$\begin{aligned} \partial_0 h_{ij}^{\text{TT}} &= \frac{2G}{r} \frac{d^3 J_{ij}^{\text{TT}}}{dt^3}, \\ \partial_r h_{ij}^{\text{TT}} &= -\frac{2G}{r} \frac{d^3 J_{ij}^{\text{TT}}}{dt^3} - \frac{2G}{r^2} \frac{d^2 J_{ij}^{\text{TT}}}{dt^2} \\ &\approx -\frac{2G}{r} \frac{d^3 J_{ij}^{\text{TT}}}{dt^3}, \end{aligned} \quad (7.183)$$

where we have dropped the  $r^{-2}$  term because we are interested in the  $r \rightarrow \infty$  limit. The important component of the energy-momentum tensor is therefore

$$t_{0r} = -\frac{G}{8\pi r^2} \left( \left( \frac{d^3 J_{ij}^{\text{TT}}}{dt^3} \right) \left( \frac{d^3 J_{ij}^{\text{TT}}}{dt^3} \right) \right). \quad (7.184)$$

The next step is to convert back to  $J_{ij}$  from the transverse-traceless part. Applying (7.178) and some messy algebra, it is straightforward to show that

$$X_{ij}^{\text{TT}} X_{\text{TT}}^{ij} = X_{ij} X^{ij} - 2X_i^j X^{ik} n_j n_k + \frac{1}{2} X^{ij} X^{kl} n_i n_j n_k n_l - \frac{1}{2} X^2 + X X^{ij} n_i n_j, \quad (7.185)$$

where  $X = \delta^{ij} X_{ij}$ . Because  $J_{ij}$  is traceless, we have

$$J_{ij}^{\text{TT}} J_{\text{TT}}^{ij} = J_{ij} J^{ij} - 2J_i^j J^{ik} n_j n_k + \frac{1}{2} J^{ij} J^{kl} n_i n_j n_k n_l, \quad (7.186)$$

and the power is

$$P = -\frac{G}{8\pi} \int_{S_\infty^2} \left\{ \frac{d^3 J_{ij}}{dt^3} \frac{d^3 J^{ij}}{dt^3} - 2 \frac{d^3 J_i^j}{dt^3} \frac{d^3 J^{ik}}{dt^3} n_j n_k + \frac{1}{2} \frac{d^3 J^{ij}}{dt^3} \frac{d^3 J^{kl}}{dt^3} n_i n_j n_k n_l \right\} d\Omega. \quad (7.187)$$

To evaluate this expression, it is best to switch back to Cartesian coordinates in space, where  $n^i = x^i/r$ . The quadrupole tensors are independent of the angular coordinates, since they are defined by integrals over all of space. We may therefore pull them outside the integral, and use the identities

$$\begin{aligned} \int d\Omega &= 4\pi \\ \int n_i n_j d\Omega &= \frac{4\pi}{3} \delta_{ij} \\ \int n_i n_j n_k n_l d\Omega &= \frac{4\pi}{15} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \end{aligned} \quad (7.188)$$

When all is said and done, the expression for the power collapses to

$$P = -\frac{G}{5} \left\langle \frac{d^3 J_{ij}}{dt^3} \frac{d^3 J^{ij}}{dt^3} \right\rangle, \quad (7.189)$$

where we should remember that the quadrupole moment is evaluated at the retarded time  $t_r = t - r$ . Our formula has a minus sign because it represents the rate at which the energy is changing, and radiating sources will be losing energy.

For the binary system represented by (7.148), the reduced quadrupole moment is

$$J_{ij} = \frac{MR^2}{3} \begin{pmatrix} (1+3\cos 2\Omega t) & 3\sin 2\Omega t & 0 \\ 3\sin 2\Omega t & (1-3\cos 2\Omega t) & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad (7.190)$$

and its third time derivative is therefore

$$\frac{d^3 J_{ij}}{dt^3} = 8MR^2\Omega^3 \begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (7.191)$$

The power radiated by the binary is thus

$$P = -\frac{128}{5} GM^2 R^4 \Omega^6, \quad (7.192)$$

or, using expression (7.144) for the frequency,

$$P = -\frac{2}{5} \frac{G^4 M^5}{R^5}. \quad (7.193)$$

Of course, energy loss through the emission of gravitational radiation has been observed. In 1974 Hulse and Taylor discovered a binary system, PSR1913+16, in which both stars are very small, so classical effects are negligible, or at least under control, and one is a pulsar. The period of the orbit is eight hours, extremely small by astrophysical standards. The fact that one of the stars is a pulsar provides a very accurate clock, with respect to which the change in the period as the system loses energy can be measured. The result is consistent with the prediction of general relativity for energy loss through gravitational radiation.

## 7.7 ■ DETECTION OF GRAVITATIONAL WAVES

One of the highest-priority goals of contemporary gravitational physics and astrophysics is to detect gravitational radiation directly. (By direct we mean “by observing the influence of the gravitational wave on test bodies,” in contrast to observing the indirect effect of energy loss, as in the binary pulsar.) There is every reason to believe that such a detection will happen soon, either in already-existing gravitational-wave observatories or those being planned for the near future. Once we detect gravitational radiation, of course, the goal will immediately become to extract useful astrophysical information from the observations. Our current understanding of the universe outside the Solar System comes almost exclusively from observations of electromagnetic radiation, with some additional input from neutrinos and cosmic rays; the advent of gravitational-wave astrophysics will open an entirely new window onto energetic phenomena in the distant universe.<sup>5</sup>

Before discussing how we might go about detecting astrophysical gravitational waves, we should think about what sources are likely to be most readily observable. The first important realization is that the necessary conditions for the generation of appreciable gravitational radiation are very different from those for electromagnetic radiation. The difference can be traced to the fact that gravitational waves are produced by the bulk motion of large masses, while electromagnetic waves are produced (typically) by incoherent excitations of individual particles. Electromagnetic radiation can therefore be produced by a source that is static in bulk, such as a star, which is a substantial advantage to the astronomer. However, gravitational waves are produced coherently by large moving masses (every particle in the mass contributes in the same sense to the wave), which can partially compensate for the impossibility of emission from static sources.

We therefore need massive sources with substantial bulk motions. As a simple example, consider the binary system of Section 7.5, in which both stars have mass  $M$  and the orbital radius is  $R$ . We will cheat somewhat by applying the Newtonian formulae for the orbital parameters in a regime where GR has begun to become important, but this will suffice for an order-of-magnitude estimate. The relevant parameters can be distilled down to the Schwarzschild radius  $R_S = 2GM/c^2$ ,

<sup>5</sup>For an overview of gravitational-wave astrophysics, see S.A. Hughes, S. Márka, P.L. Bender, and C.J. Hogan, “New physics and astronomy with the new gravitational-wave observatories,” <http://arxiv.org/astro-ph/0110349>.

the orbital radius  $R$ , and the distance  $r$  between us and the binary. (We will now restore explicit factors of  $c$ , to facilitate comparison with experiment.) In terms of these, the frequency of the orbit and thus of the produced gravitational waves is approximately

$$f = \frac{\Omega}{2\pi} \sim \frac{cR_S^{1/2}}{10R^{3/2}}. \quad (7.194)$$

From the formula (7.149) for the resulting perturbation, we can estimate the gravitational-wave amplitude received as

$$h \sim \frac{R_S^2}{rR}. \quad (7.195)$$

Let's see what this implies for the kind of source we might hope to observe. A paradigmatic example is the coalescence of a black-hole/black-hole binary. For typical parameters we can take both black holes to be 10 solar masses, the binary to be at cosmological distances  $\sim 100$  Mpc, and the components to be separated by ten times their Schwarzschild radii:

$$\begin{aligned} R_S &\sim 10^6 \text{ cm} \\ R &\sim 10^7 \text{ cm} \\ r &\sim 10^{26} \text{ cm}. \end{aligned} \quad (7.196)$$

Such a source is thus characterized by

$$f \sim 10^2 \text{ s}^{-1}, \quad h \sim 10^{-21}. \quad (7.197)$$

If we are to have any hope of detecting the coalescence of a binary with these parameters, we need to be sensitive to frequencies near 100 Hz and strains of order  $10^{-21}$  or less.

Fortunately, these parameters are within the reach of our experimental capabilities (with the heroic efforts of many scientists). The most promising technique for gravitational-wave detection currently under consideration is interferometry, and here we will stick exclusively to a discussion of interferometers, although it is certainly conceivable that a new technology could be invented that would have better sensitivity.

Recall that the physical effect of a passing gravitational wave is to slightly perturb the relative positions of freely-falling masses. If two test masses are separated by a distance  $L$ , the change in their distance will be roughly

$$\frac{\delta L}{L} \sim h. \quad (7.198)$$

Imagine that we contemplate building an observatory with test bodies separated by some distance of order kilometers. Then to detect a wave with amplitude of

order  $h \sim 10^{-21}$  would require a sensitivity to changes of

$$\delta L \sim 10^{-16} \left( \frac{h}{10^{-21}} \right) \left( \frac{L}{\text{km}} \right) \text{ cm.} \quad (7.199)$$

Compare this to the size of a typical atom, set by the Bohr radius,

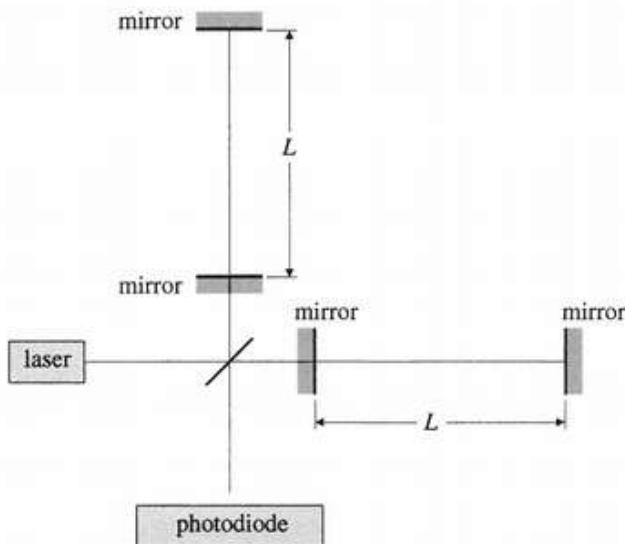
$$a_0 \sim 5 \times 10^{-9} \text{ cm}, \quad (7.200)$$

or for that matter the size of a typical nucleus, of approximately a Fermi,

$$1 \text{ fm} = 10^{-13} \text{ cm}. \quad (7.201)$$

The point we are belaboring here is that a feasible terrestrial gravitational-wave observatory will have to be sensitive to changes in distance much smaller than the size of the constituent atoms out of which any conceivable test masses would have to be made.

Laser interferometers provide a way to overcome the difficulty of measuring such minuscule perturbations. Consider the schematic set-up portrayed in Figure 7.11. A laser (typically with characteristic wavelength  $\lambda \sim 10^{-4}$  cm) is directed at a beamsplitter, which sends the photons down two evacuated tubes of length  $L$ . At the ends of the cavities are test masses, represented by mirrors suspended from pendulums. The light actually bounces off partially-reflective mirrors near the beamsplitter, so that a typical photon travels up and down the cavity



**FIGURE 7.11** A schematic design for a gravitational-wave interferometer.

of order 100 times before returning to the beamsplitter and being directed into a photodiode. The system is arranged such that, if the test masses are perfectly stationary, the returning beams destructively interfere, sending no signal to the photodiode. As we have seen, the effect of a passing gravitational wave will be to perturb orthogonal lengths in opposite senses, leading to a phase shift in the laser pulse that will disturb the destructive interference. During 100 round trips through the cavity arms, the accumulated phase shift will be

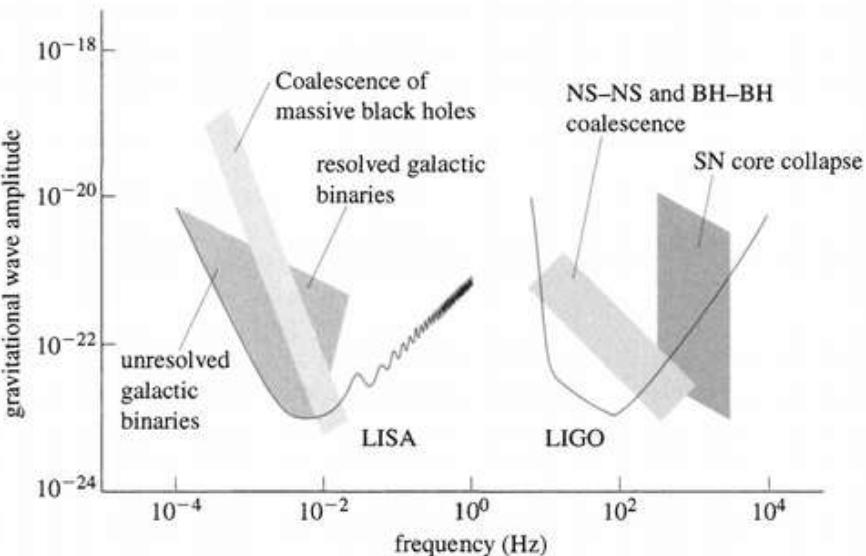
$$\delta\phi \sim 200 \left( \frac{2\pi}{\lambda} \right) \delta L \sim 10^{-9}. \quad (7.202)$$

where 200 rather than 100 represents the fact that the shifts in the two arms add together. Such a tiny shift can be measured if the number of photons  $N$  is sufficiently large to overcome the “shot noise”; in particular, if  $\sqrt{N} > \delta\phi$ .

The technological challenges associated with building sufficiently quiet and sensitive gravitational-wave observatories are being tackled in a number of different locations, including the United States (LIGO), Italy (Virgo), Germany (GEO), Japan (TAMA), and Australia (ACIGA). LIGO (Laser Interferometric Gravitational-Wave Observatory) is presently the most advanced detector; it consists of two facilities (one in Washington state and one in Louisiana), each with four-kilometer arms. A single gravitational-wave observatory will be unable to localize a source’s position on the sky; multiple detectors will be crucial for this task (as well as for verifying that an apparent signal is actually real).

Fundamental noise sources limit the ability of terrestrial observatories to detect low-frequency gravitational waves. Figure 7.12 shows the sensitivity regions, as a function of frequency, for two dramatically different designs: a terrestrial observatory such as LIGO, and a space-based mission such as LISA (Laser Interferometer Space Antenna). The general principle behind LISA is the same as any other interferometer, but the implementation is (or will be, if it is actually built) dramatically different. Current designs envision three spacecraft orbiting the Sun at approximately 30 million kilometers behind the Earth, separated from each other by 5 million kilometers. Due to the much larger separations, LISA is sensitive to frequencies in the vicinity of  $10^{-2}$  Hz. The sensitivities portrayed in this plot should be taken as suggestive, as they depend on integration times and other factors.

Many potential noise sources confront the gravitational-wave astronomer. For ground-based observatories, the dominant effect at low frequencies is typically seismic noise, while at high frequencies it comes from photon shot noise and at intermediate frequencies from thermal noise. Advanced versions of ground-based detectors may be able to compensate for seismic noise at low frequencies, but will encounter irreducible noise from gravity gradients due to atmospheric phenomena or objects (such as cars) passing nearby. Satellite observatories, of course, are immune from such effects. Instead, the fundamental limitations are expected to come from errors in measuring changes in the distances between the spacecraft (or more properly, between the shielded proof masses within the spacecraft) and from nongravitational accelerations of the spacecraft.



**FIGURE 7.12** Sensitivities as a function of frequency for representative ground-based (LIGO) and space-based (LISA) gravitational-wave observatories, along with the expected signals from possible sources. Figure from the LISA collaboration home page (<http://lisa.jpl.nasa.gov/>).

We can conclude with a very brief overview of possible sources for gravitational-wave observatories. We have already mentioned the possibility of compact binaries of various sorts. For ground-based observatories, such sources will not become visible until they are very close to coalescence, and then only if the components are sufficiently massive (neutron stars or black holes). Extrapolating from what we know about such systems suggests that there may be several coalescences per year within a distance of a few hundred Mpc. Another promising possibility is core collapse in massive stars, giving rise to supernovae. Although a perfectly spherically-symmetric collapse would not generate any gravitational waves, realistic events are expected to be subject to instabilities that would break this symmetry. An exciting prospect is the coordinated observation of supernovae by ordinary telescopes and gravitational-wave observatories. Lastly, among possible sources for ground-based observatories are periodic sources such as (not-completely-axially-symmetric) rotating neutron stars. The amplitudes from such sources are expected to be small, but not necessarily completely out of reach of advanced detectors.

The interesting sources for space-based detectors are somewhat different. Most importantly, the known population of binaries in our galaxy will certainly provide a gravitational-wave signal of detectable magnitude. Indeed, unresolved binaries represent a source of confusion noise for the detector, as it will be impossible to pick out individual low-intensity sources from the background. Nevertheless, numerous higher-intensity sources should be easily observable. In addition, various

processes in the evolution of supermassive black holes (greater than  $1000 M_\odot$ , such as those found in the centers of galaxies) lead to interesting sources; the formation of such objects, their subsequent growth via accretion of smaller objects, and possible coalescence of multiple supermassive holes. Tracking the evolution of the gravitational-wave signal from a solar-mass black hole orbiting and eventually falling into a supermassive hole will allow for precision mapping of the spacetime metric, providing a novel test of GR.

In addition to waves produced by localized sources, we also face the possibility of stochastic gravitational-wave backgrounds. By this we mean an isotropic set of gravitational waves, perhaps generated in the early universe, characterized by a smoothly-varying power spectrum as a function of frequency. One possibility is a nearly scale-free spectrum of gravitational waves produced by inflation, as discussed in Chapter 8. Such waves will be essentially impossible to detect directly on the ground (falling perhaps five orders of magnitude below the capabilities of advanced detectors), or even by LISA, but could conceivably be observable by a next-generation space-based mission. More likely, any such waves will first become manifest in the polarization of the cosmic microwave background. Another possibility, however, is generation of primordial gravitational waves from a violent (first-order) phase transition. Such waves will have a spectrum with a well-defined peak frequency, related to the temperature  $T$  of the phase transition by

$$f_{\text{peak}} \sim 10^{-3} \left( \frac{T}{1000 \text{ GeV}} \right) \text{ Hz}. \quad (7.203)$$

Thus, a first-order electroweak phase transition ( $T \sim 200$  GeV) falls within the band potentially observable by LISA. This is especially intriguing, as some models of baryogenesis require a strong phase transition at this scale; it is provocative to think that we could learn something significant about electroweak physics through a gravitational experiment.

## 7.8 ■ EXERCISES

1. Show that the Lagrangian (7.9) gives rise to the linearized version of Einstein's equation.
2. Consider a thin spherical shell of matter, with mass  $M$  and radius  $R$ , slowly rotating with an angular velocity  $\Omega$ .
  - (a) Show that the gravito-electric field  $\vec{G}$  vanishes, and calculate the gravito-magnetic field  $\vec{H}$  in terms of  $M$ ,  $R$ , and  $\Omega$ .
  - (b) The nonzero gravito-magnetic field caused by the shell leads to dragging of inertial frames, known as the **Lense-Thirring effect**. Calculate the rotation (relative to the inertial frame defined by the background Minkowski metric) of a freely-falling observer sitting at the center of the shell. In other words, calculate the precession of the spatial components of a parallel-transported vector located at the center.

3. Fermat's principle states that a light ray moves along a path of least time. For a medium with refractive index  $n(\mathbf{x})$ , this is equivalent to extremizing the time

$$t = \int n(\mathbf{x})[\delta_{ij}dx^i dx^j]^{1/2} \quad (7.204)$$

along the path. Show that Fermat's principle, with the refractive index given by  $n = 1 - 2\Phi$ , leads to the correct equation of motion for a photon in a spacetime perturbed by a Newtonian potential.

4. Show that the Lorenz gauge condition  $\partial_\mu \bar{h}^{\mu\nu} = 0$  is equivalent to the **harmonic gauge** condition. This gauge is defined by

$$\square x^\mu = 0, \quad (7.205)$$

where each coordinate  $x^\mu$  is thought of as a scalar function on spacetime. (Any function satisfying  $\square f = 0$  is known as an “harmonic function.”)

5. In the exercises for Chapter 3, we introduced the metric

$$ds^2 = -(du dv + dv du) + a^2(u)dx^2 + b^2(u)dy^2, \quad (7.206)$$

where  $a$  and  $b$  are unspecified functions of  $u$ . For appropriate functions  $a$  and  $b$ , this represents an *exact* gravitational plane wave.

- (a) Calculate the Christoffel symbols and Riemann tensor for this metric.
- (b) Use Einstein's equation in vacuum to derive equations obeyed by  $a(u)$  and  $b(u)$ .
- (c) Show that an exact solution can be found, in which both  $a$  and  $b$  are determined in terms of an *arbitrary* function  $f(u)$ .

6. Two objects of mass  $M$  have a head-on collision at event  $(0, 0, 0, 0)$ . In the distant past,  $t \rightarrow -\infty$ , the masses started at  $x \rightarrow \pm\infty$  with zero velocity.

- (a) Using Newtonian theory, show that  $x(t) = \pm(9Mt^2/8)^{1/3}$ .
- (b) For what separations is the Newtonian approximation reasonable?
- (c) Calculate  $h_{xx}^{\text{TT}}(t)$  at  $(x, y, z) = (0, R, 0)$ .

7. Gravitational waves can be detected by monitoring the distance between two free flying masses. If one of the masses is equipped with a laser and an accurate clock, and the other with a good mirror, the distance between the masses can be measured by timing how long it takes for a pulse of laser light to make the round-trip journey. How would you want your detector oriented to register the largest response from a plane wave of the form

$$ds^2 = -dt^2 + [1 + A \cos(\omega(t-z))]dx^2 + [1 - A \cos(\omega(t-z))]dy^2 + dz^2?$$

If the masses have a mean separation  $L$ , what is the largest change in the arrival time of the pulses caused by the wave? What frequencies  $\omega$  would go undetected?

8. The gravitational analog of *bremsstrahlung* radiation is produced when two masses scatter off each other. Consider what happens when a small mass  $m$  scatters off a large mass  $M$  with impact parameter  $b$  and total energy  $E = 0$ . Take  $M \gg m$  and  $M/b \ll 1$ . The motion of the small mass can be described by Newtonian physics, since  $M/b \ll 1$ . If the orbit lies in the  $(x, y)$  plane and if the large mass sits at

$(x, y, z) = (0, 0, 0)$ , calculate the gravitational wave amplitude for both polarizations at  $(x, y, z) = (0, 0, r)$ . Since the motion is not periodic, the gravitational waves will be burst-like and composed of many different frequencies. On physical grounds, what do you expect the dominant frequency to be? Estimate the total energy radiated by the system. How does this compare to the peak kinetic energy of the small mass?

*Hint:* The solution for the orbit can be found in Goldstein (2002). The solution is:

$$r = \frac{2b}{1 + \cos \theta},$$

$$t = \sqrt{\frac{2b^3}{M}} \left( \tan \frac{\theta}{2} + \frac{1}{3} \tan^3 \frac{\theta}{2} \right).$$

Time runs from  $t = (-\infty, \infty)$ . Rather than using the above implicit solution for  $\theta(t)$  you might want to use

$$\dot{\theta} = \sqrt{\frac{M}{8b^3}} (1 + \cos \theta)^2.$$

9. Verify that the expression (7.165) for the gravitational-wave energy-momentum tensor is invariant under gauge transformations  $h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$ .
10. Show that the integral expression (7.173) for the total energy in gravitational perturbations is independent of the spatial hypersurface  $\Sigma$ .

# CHAPTER

# 8

# Cosmology

## 8.1 ■ MAXIMALLY SYMMETRIC UNIVERSES

Contemporary cosmological models are based on the idea that the universe is pretty much the same everywhere—a stance sometimes known as the **Copernican principle**. On the face of it, such a claim seems crazy; the center of the sun, for example, bears little resemblance to the desolate cold of interstellar space. But we take the Copernican principle to apply only on the very largest scales, where local variations in density are averaged over. Its validity on such scales is manifested in a number of different observations, such as number counts of galaxies and observations of diffuse X-ray and  $\gamma$ -ray backgrounds, but is most clear in the 3K cosmic microwave background (CMB). Although we now know that the microwave background radiation is not perfectly smooth (and nobody ever expected that it was), the deviations from regularity are on the order of  $10^{-5}$  or less, certainly an adequate basis for an approximate description of spacetime on large scales.

The Copernican principle is related to two more mathematically precise properties that a manifold might have: isotropy and homogeneity. **Isotropy** applies at some specific point in the manifold, and states that the space looks the same no matter in what direction you look. More formally, a manifold  $M$  is isotropic around a point  $p$  if, for any two vectors  $V$  and  $W$  in  $T_p M$ , there is an isometry of  $M$  such that the pushforward of  $W$  under the isometry is parallel with  $V$  (not pushed forward). It is isotropy of space that is indicated by the observations of the microwave background.

**Homogeneity** is the statement that the metric is the same throughout the manifold. In other words, given any two points  $p$  and  $q$  in  $M$ , there is an isometry that takes  $p$  into  $q$ . Note that there is no necessary relationship between homogeneity and isotropy; a manifold can be homogeneous but nowhere isotropic (such as  $\mathbf{R} \times S^2$  in the usual metric), or it can be isotropic around a point without being homogeneous (such as a cone, which is isotropic around its vertex but certainly not homogeneous). On the other hand, if a space is isotropic *everywhere*, then it is homogeneous. Likewise if it is isotropic around one point and also homogeneous, it will be isotropic around every point. Since there is ample observational evidence for isotropy, and the Copernican principle would have us believe that we are not the center of the universe and therefore observers elsewhere should also observe isotropy, we will henceforth assume both homogeneity and isotropy.

The usefulness of homogeneity and isotropy is that they imply that a space is maximally symmetric. Think of isotropy as invariance under rotations, and homogeneity as invariance under translations, suitably generalized. Then homogeneity and isotropy together imply that a space has its maximum possible number of Killing vectors. An extreme application of the Copernican principle would be to insist that spacetime itself is maximally symmetric. In fact this will turn out not to be true; observationally we know that the universe is homogeneous and isotropic in *space*, but not in all of *spacetime*. However, it is interesting to begin by considering spacetimes that are maximally symmetric (which are, after all, special cases of the more general situation in which only space is maximally symmetric). As we shall see, there is a sense in which such universes are “ground states” of general relativity. This discussion is less relevant to the observed universe than subsequent parts of this chapter, and empirically-minded readers are welcome to skip ahead to the next section.

We mentioned in Chapter 3 that the Riemann tensor for a maximally symmetric  $n$ -dimensional manifold with metric  $g_{\mu\nu}$  can be written

$$R_{\rho\sigma\mu\nu} = \kappa(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}), \quad (8.1)$$

where  $\kappa$  is a normalized measure of the Ricci curvature,

$$\kappa = \frac{R}{n(n-1)}, \quad (8.2)$$

and the Ricci scalar  $R$  will be a constant over the manifold. Since at any single point we can always put the metric into its canonical form ( $g_{\mu\nu} = \eta_{\mu\nu}$ ), the kinds of maximally symmetric manifolds are characterized locally by the signature of the metric and the sign of the constant  $\kappa$ . The modifier “locally” is necessary to account for possible global differences, such as between the plane and the torus. We are interested in metrics of signature  $(-+++)$ . For vanishing curvature ( $\kappa = 0$ ) the maximally symmetric spacetime is well known; it is simply Minkowski space, with metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \quad (8.3)$$

The conformal diagram for Minkowski space is derived in Appendix H.

The maximally symmetric spacetime with positive curvature ( $\kappa > 0$ ) is called **de Sitter space**. Consider a five-dimensional Minkowski space with metric  $ds_5^2 = -du^2 + dx^2 + dy^2 + dz^2 + dw^2$ , and embed a hyperboloid given by

$$-u^2 + x^2 + y^2 + z^2 + w^2 = \alpha^2. \quad (8.4)$$

Now induce coordinates  $\{t, \chi, \theta, \phi\}$  on the hyperboloid via

$$u = \alpha \sinh(t/\alpha)$$

$$w = \alpha \cosh(t/\alpha) \cos \chi$$

$$\begin{aligned}x &= \alpha \cosh(t/\alpha) \sin \chi \cos \theta \\y &= \alpha \cosh(t/\alpha) \sin \chi \sin \theta \cos \phi \\z &= \alpha \cosh(t/\alpha) \sin \chi \sin \theta \sin \phi.\end{aligned}\quad (8.5)$$

The metric on the hyperboloid is then

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(t/\alpha) [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (8.6)$$

We recognize the expression in round parentheses as the metric on a two-sphere,  $d\Omega_2^2$ , and the expression in square brackets as the metric on a three-sphere,  $d\Omega_3^2$ . Thus, de Sitter space describes a spatial three-sphere that initially shrinks, reaching a minimum size at  $t = 0$ , and then re-expands. Of course this particular description is inherited from a certain coordinate system; we will see that there are equally valid alternative descriptions.

These coordinates cover the entire manifold. You can generally check this by, for example, following the behavior of geodesics near the edges of the coordinate system; if the coordinates were incomplete, geodesics would appear to terminate in finite affine parameter. The topology of de Sitter is thus  $\mathbf{R} \times S^3$ . This makes it very simple to derive the conformal diagram, since the important step in constructing conformal diagrams is to write the metric in a form in which it is conformally related to the Einstein static universe (a spacetime with topology  $\mathbf{R} \times S^3$ , describing a spatial three-sphere of constant radius through time). Consider the coordinate transformation from  $t$  to  $t'$  via

$$\cosh(t/\alpha) = \frac{1}{\cos(t')}. \quad (8.7)$$

The metric (8.6) now becomes

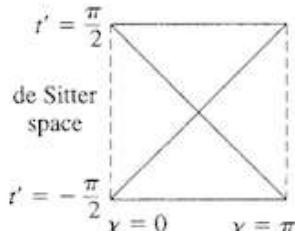
$$ds^2 = \frac{\alpha^2}{\cos^2(t')} d\bar{s}^2, \quad (8.8)$$

where  $d\bar{s}^2$  represents the metric on the Einstein static universe,

$$d\bar{s}^2 = -(dt')^2 + d\chi^2 + \sin^2 \chi d\Omega_2^2. \quad (8.9)$$

The range of the new time coordinate is

$$-\pi/2 < t' < \pi/2. \quad (8.10)$$



**FIGURE 8.1** Conformal diagram for de Sitter spacetime. Spacelike slices are three-spheres, so that points on the diagram represent two-spheres except for those at left and right edges, which are points.

The conformal diagram of de Sitter space will simply be a representation of the patch of the Einstein static universe to which de Sitter is conformally related. It looks like a square, as shown in Figure 8.1. A spacelike slice of constant  $t'$  represents a three-sphere; the dashed lines at the left and right edges are the north and south poles of this sphere. The diagonal lines represent null rays; a photon released at past infinity will get to precisely the antipodal point on the sphere at

future infinity. Keep in mind that the spacetime “ends” to the past and the future only through the magic of conformal transformations; the actual de Sitter space extends indefinitely into the future and past. Note also that two points can have future (or past) light cones that are completely disconnected; this reflects the fact that the spherical spatial sections are expanding so rapidly that light from one point can never come into contact with light from the other.

A similar hyperboloid construction reveals the  $\kappa < 0$  spacetime of maximal symmetry, known as **anti-de Sitter space**. Begin with a fictitious five-dimensional flat manifold with metric  $ds_5^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$ , and embed a hyperboloid given by

$$-u^2 - v^2 + x^2 + y^2 + z^2 = -\alpha^2. \quad (8.11)$$

Note all the minus signs. Then we can induce coordinates  $\{t', \rho, \theta, \phi\}$  on the hyperboloid via

$$\begin{aligned} u &= \alpha \sin(t') \cosh(\rho) \\ v &= \alpha \cos(t') \cosh(\rho) \\ x &= \alpha \sinh(\rho) \cos \theta \\ y &= \alpha \sinh(\rho) \sin \theta \cos \phi \\ z &= \alpha \sinh(\rho) \sin \theta \sin \phi, \end{aligned} \quad (8.12)$$

yielding a metric on this hyperboloid of the form

$$ds^2 = \alpha^2 (-\cosh^2(\rho) dt'^2 + d\rho^2 + \sinh^2(\rho) d\Omega_2^2). \quad (8.13)$$

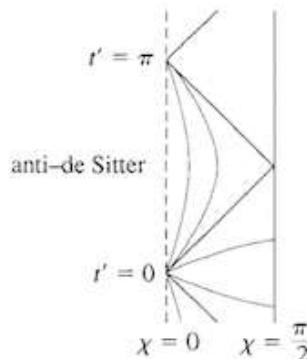
These coordinates have a strange feature, namely that  $t'$  is periodic. From (8.12),  $t'$  and  $t' + 2\pi$  represent the same place on the hyperboloid. Since  $\partial_{t'}$  is everywhere timelike, a curve with constant  $\{\rho, \theta, \phi\}$  as  $t'$  increases will be a closed timelike curve. However, this is not an intrinsic property of the spacetime, merely an artifact of how we have derived the metric from a particular embedding. We are welcome to consider the “covering space” of this manifold, the spacetime with metric given by (8.13) in which we allow  $t'$  to range from  $-\infty$  to  $\infty$ . There are no closed timelike curves in this space, which we will take to be the definition of anti-de Sitter space.

To derive the conformal diagram, perform a coordinate transformation analogous to that used for de Sitter, but now on the radial coordinate:

$$\cosh(\rho) = \frac{1}{\cos \chi}, \quad (8.14)$$

so that

$$ds^2 = \frac{\alpha^2}{\cos^2 \chi} d\tilde{s}^2, \quad (8.15)$$



**FIGURE 8.2** Conformal diagram for anti-de Sitter spacetime. Spacelike slices have the topology of  $\mathbf{R}^3$ , which we have represented in polar coordinates, so that points on the diagram stand for two-spheres except those at the left side, which stand for single points at the spatial origin. Infinity is a timelike surface at the right side.

where  $d\bar{s}^2$  represents the metric on the Einstein static universe (8.9). Unlike in de Sitter, the radial coordinate now appears in the conformal factor. In addition, for anti-de Sitter, the  $t'$  coordinate goes from minus infinity to plus infinity, while the range of the radial coordinate is

$$0 \leq \chi < \frac{\pi}{2}. \quad (8.16)$$

Thus, anti-de Sitter space is conformally related to half of the Einstein static universe. The conformal diagram is shown in Figure 8.2, which illustrates a few representative timelike and spacelike geodesics passing through the point  $t' = 0$ ,  $\chi = 0$ . Since  $\chi$  only goes to  $\pi/2$  rather than all the way to  $\pi$ , a spacelike slice of this spacetime has the topology of the interior of a hemisphere of  $S^3$ ; that is, it is topologically  $\mathbf{R}^3$  (and the entire spacetime therefore has the topology  $\mathbf{R}^4$ ). Note that we have drawn the diagram in polar coordinates, such that a point on the left side represents a point at the spatial origin, while one on the right side represents a two-sphere at spatial infinity. Another popular representation is to draw the spacetime in cross-section, so that the spatial origin lies in the middle and the right and left sides together comprise spatial infinity.

An interesting feature of anti-de Sitter is that infinity takes the form of a timelike hypersurface, defined by  $\chi = \pi/2$ . Because infinity is timelike, the space is not globally hyperbolic, we do not have a well-posed initial value problem in terms of information specified on a spacelike slice, since information can always “flow in from infinity.” Another interesting feature is that the exponential map is not onto the entire spacetime; geodesics, such as those drawn on the figure, which leave from a specified point do not cover the whole manifold. The future-pointing timelike geodesics, as indicated, can initially move radially outward from  $t' = 0$ ,  $\chi = 0$ , but eventually refocus to the point  $t' = \pi$ ,  $\chi = 0$  and will then move radially outward once again.

As an aside, it is irresistible to point out that the timelike nature of infinity enables a remarkable feature of string theory, the “AdS/CFT correspondence.” Here, AdS is of course the anti-de Sitter space we have been discussing, while CFT stands for a conformally-invariant field theory defined on the boundary [which is, for an  $n$ -dimensional AdS, an  $(n-1)$ -dimensional spacetime in its own right]. The AdS/CFT correspondence suggests that, in a certain limit, there is an equivalence between quantum gravity (or a supersymmetric version thereof) on an AdS background and a conformally-invariant nongravitational field theory defined on the boundary. Since we know a lot about nongravitational quantum field theory that we don’t know about quantum gravity, this correspondence (if it is true, which seems likely but remains unproven) reveals a great deal about what can happen in quantum gravity.<sup>1</sup>

So we have three spacetimes of maximal symmetry: Minkowski ( $\kappa = 0$ ), de Sitter ( $\kappa > 0$ ), and anti-de Sitter ( $\kappa < 0$ ). Are any one of these useful models for the real world? For that matter, are they solutions to Einstein’s equation? Start by taking the trace of the Riemann tensor as given by (8.1), specifying to four dimensions:

$$R_{\mu\nu} = 3\kappa g_{\mu\nu}, \quad R = 12\kappa. \quad (8.17)$$

So the Ricci tensor is proportional to the metric in a maximally symmetric space. A spacetime with this property is sometimes called an Einstein space; the Einstein static universe is *not* an example of an Einstein space, which can sometimes be confusing. What is worse, we will later encounter the Einstein-de Sitter cosmology, which is not related to Einstein spaces, the Einstein static universe, or to de Sitter space. The Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -3\kappa g_{\mu\nu}. \quad (8.18)$$

Therefore, Einstein’s equation  $G_{\mu\nu} = 8\pi GT_{\mu\nu}$  implies (in a maximally symmetric spacetime, not in general) that the energy-momentum tensor is proportional to the metric:

$$T_{\mu\nu} = -\frac{3\kappa}{8\pi G}g_{\mu\nu}. \quad (8.19)$$

Such an energy-momentum tensor corresponds to a vacuum energy or cosmological constant, as discussed in Chapter 4. The energy density and pressure are given by

$$\rho = -p = \frac{3\kappa}{8\pi G}. \quad (8.20)$$

If  $\rho$  is positive, we get a de Sitter solution; if  $\rho$  is negative, we get anti-de Sitter.

But in our universe, we have ordinary matter and radiation, as well as a possible vacuum energy. Our maximally symmetric spacetimes are not compatible

<sup>1</sup>For a comprehensive review article, see O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, and Y. Oz, *Phys. Rept.* 323, 183 (2000), <http://arxiv.org/hep-th/9905111>.

with a dynamically interesting amount of matter and/or radiation. Furthermore, since we observe the visible matter in the universe to be moving apart (the universe is expanding, as discussed below), the density of matter was higher in the past; so even if the matter contribution to the total energy were negligible today, it would have been appreciable in the earlier universe. The maximally symmetric spacetimes are therefore not reasonable models of the real world. They do, however, represent the (locally) unique solutions to Einstein's equation in the absence of any ordinary matter or gravitational radiation; it is in this sense that they may be thought of as ground states of general relativity.

## 8.2 ■ ROBERTSON-WALKER METRICS

To describe the real world, we are forced to give up the “perfect” Copernican principle, which implies symmetry throughout space and time, and postulate something more forgiving. It turns out to be straightforward, and consistent with observation, to posit that the universe is *spatially* homogeneous and isotropic, but evolving in time. In general relativity this translates into the statement that the universe can be foliated into spacelike slices such that each three-dimensional slice is maximally symmetric. We therefore consider our spacetime to be  $\mathbf{R} \times \Sigma$ , where  $\mathbf{R}$  represents the time direction and  $\Sigma$  is a maximally symmetric three-manifold. The spacetime metric thus takes the form

$$ds^2 = -dt^2 + R^2(t)d\sigma^2, \quad (8.21)$$

where  $t$  is the timelike coordinate,  $R(t)$  is a function known as the **scale factor**, and  $d\sigma^2$  is the metric on  $\Sigma$ , which can be expressed as

$$d\sigma^2 = \gamma_{ij}(u) du^i du^j, \quad (8.22)$$

where  $(u^1, u^2, u^3)$  are coordinates on  $\Sigma$  and  $\gamma_{ij}$  is a maximally symmetric three-dimensional metric. The scale factor tells us how big the spacelike slice  $\Sigma$  is at the moment  $t$ . (Don't confuse it with the curvature scalar.) The coordinates used here, in which the metric is free of cross terms  $dt du^i$  and coefficient of  $dt^2$  is independent of the  $u^i$ , are known as **comoving coordinates**, a special case of the Gaussian normal coordinates discussed in Appendix D. An observer who stays at constant  $u^i$  is also called “comoving.” Only a comoving observer will think that the universe looks isotropic; in fact on Earth we are not quite comoving, and as a result we see a dipole anisotropy in the cosmic microwave background as a result of the conventional Doppler effect.

Our interest is therefore in maximally symmetric Euclidean three-metrics  $\gamma_{ij}$ . We know that maximally symmetric metrics obey

$${}^{(3)}R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}), \quad (8.23)$$

where for future convenience we have introduced

$$k = {}^{(3)}R/6, \quad (8.24)$$

and we put a superscript <sup>(3)</sup> on the Riemann tensor to remind us that it is associated with the three-metric  $\gamma_{ij}$ , not the metric of the entire spacetime. The Ricci tensor is then

$${}^{(3)}R_{jl} = 2k\gamma_{jl}. \quad (8.25)$$

If the space is to be maximally symmetric, then it will certainly be spherically symmetric. We already know something about spherically symmetric spaces from our exploration of the Schwarzschild solution; the metric can be put in the form

$$d\sigma^2 = \gamma_{ij} du^i du^j = e^{2\beta(\bar{r})} d\bar{r}^2 + \bar{r}^2 d\Omega^2, \quad (8.26)$$

where  $\bar{r}$  is the radial coordinate and the metric on the two-sphere is  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$  as usual. The components of the Ricci tensor for such a metric can be obtained from (5.14), the Ricci tensor for a static, spherically symmetric spacetime, by setting  $\alpha = 0$  and  $r = \bar{r}$ , which gives

$$\begin{aligned} {}^{(3)}R_{11} &= \frac{2}{\bar{r}} \partial_1 \beta \\ {}^{(3)}R_{22} &= e^{-2\beta} (\bar{r} \partial_1 \beta - 1) + 1 \\ {}^{(3)}R_{33} &= [e^{-2\beta} (\bar{r} \partial_1 \beta - 1) + 1] \sin^2 \theta. \end{aligned} \quad (8.27)$$

We set these proportional to the metric using (8.25), and can solve for  $\beta(\bar{r})$ :

$$\beta = -\frac{1}{2} \ln(1 - k\bar{r}^2), \quad (8.28)$$

which yields the metric on the three-surface  $\Sigma$ ,

$$d\sigma^2 = \frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\Omega^2. \quad (8.29)$$

Notice from (8.24) that the value of  $k$  sets the curvature, and therefore the size, of the spatial surfaces. It is common to normalize this so that

$$k \in \{+1, 0, -1\}, \quad (8.30)$$

and absorb the physical size of the manifold into the scale factor  $R(t)$ .

The  $k = -1$  case corresponds to constant negative curvature on  $\Sigma$ , and is sometimes called **open**; the  $k = 0$  case corresponds to no curvature on  $\Sigma$ , and is called **flat**; the  $k = +1$  case corresponds to positive curvature on  $\Sigma$ , and is sometimes called **closed**. The physical interpretation of these cases is made more clear using an alternative form of the metric, obtained by introducing a new radial coordinate  $\chi$  defined by

$$d\chi = \frac{d\bar{r}}{\sqrt{1 - k\bar{r}^2}}. \quad (8.31)$$

This can be integrated to obtain

$$\bar{r} = S_k(\chi), \quad (8.32)$$

where

$$S_k(\chi) \equiv \begin{cases} \sin(\chi), & k = +1 \\ \chi, & k = 0 \\ \sinh(\chi), & k = -1, \end{cases} \quad (8.33)$$

so that

$$d\sigma^2 = d\chi^2 + S_k^2(\chi) d\Omega^2. \quad (8.34)$$

For the flat case  $k = 0$ , the metric on  $\Sigma$  becomes

$$\begin{aligned} d\sigma^2 &= d\chi^2 + \chi^2 d\Omega^2 \\ &= dx^2 + dy^2 + dz^2, \end{aligned} \quad (8.35)$$

which is simply flat Euclidean space. Globally, it could describe  $\mathbf{R}^3$  or a more complicated manifold, such as the three-torus  $S^1 \times S^1 \times S^1$ . For the closed case  $k = +1$  we have

$$d\sigma^2 = d\chi^2 + \sin^2 \chi d\Omega^2, \quad (8.36)$$

which is the metric of a three-sphere. In this case the only possible global structure is the complete three-sphere (except for the nonorientable manifold  $\mathbf{RP}^3$ , obtained by identifying antipodal points on  $S^3$ ). Finally in the open  $k = -1$  case we obtain

$$d\sigma^2 = d\chi^2 + \sinh^2 \chi d\Omega^2. \quad (8.37)$$

This is the metric for a three-dimensional space of constant negative curvature, a generalization of the hyperboloid discussed in Section 3.9. Globally such a space could extend forever (which is the origin of the word “open”), but it could also describe a nonsimply-connected compact space (so “open” is really not the most accurate description).

The metric on spacetime describes one of these maximally-symmetric hypersurfaces evolving in size, and can be written

$$ds^2 = -dt^2 + R^2(t) \left[ \frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\Omega^2 \right]. \quad (8.38)$$

This is the **Robertson–Walker (RW) metric**. We have not yet made use of Einstein’s equation; that will determine the behavior of the scale factor  $R(t)$ . Note that the substitutions

$$\begin{aligned} R &\rightarrow \lambda^{-1} R \\ \bar{r} &\rightarrow \lambda \bar{r} \\ k &\rightarrow \lambda^{-2} k \end{aligned} \tag{8.39}$$

leave (8.38) invariant. Therefore we can choose a convenient normalization. In the variables where the curvature  $k$  is normalized to  $\{+1, 0, -1\}$ , the scale factor has units of distance and the radial coordinate  $\bar{r}$  (or  $\chi$ ) is actually dimensionless; this is the most popular choice. We will flout the conventional wisdom and instead work with a dimensionless scale factor

$$a(t) = \frac{R(t)}{R_0}, \tag{8.40}$$

a coordinate with dimensions of distance

$$r = R_0 \bar{r}, \tag{8.41}$$

and a curvature parameter with dimensions of  $(\text{length})^{-2}$ ,

$$\kappa = \frac{k}{R_0^2}. \tag{8.42}$$

Note that  $\kappa$  can take on any value, not just  $\{+1, 0, -1\}$ . In these variables the Robertson–Walker metric is

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right]. \tag{8.43}$$

To convert to the more common notation, just plug in the relations (8.40), (8.41), and (8.42).

With the metric in hand, we can set about computing the connection coefficients and curvature tensor. Setting  $\dot{a} \equiv da/dt$ , the Christoffel symbols are given by

$$\begin{aligned} \Gamma_{11}^0 &= \frac{a\dot{a}}{1 - \kappa r^2} & \Gamma_{11}^1 &= \frac{\kappa r}{1 - \kappa r^2} \\ \Gamma_{22}^0 &= a\dot{a}r^2 & \Gamma_{33}^0 &= a\dot{a}r^2 \sin^2 \theta \\ \Gamma_{01}^1 &= \Gamma_{02}^2 & \Gamma_{03}^3 &= \frac{\dot{a}}{a} \\ \Gamma_{22}^1 &= -r(1 - \kappa r^2) & \Gamma_{33}^1 &= -r(1 - \kappa r^2) \sin^2 \theta \\ \Gamma_{12}^2 &= \Gamma_{13}^3 = \frac{1}{r} & & \\ \Gamma_{33}^2 &= -\sin \theta \cos \theta & \Gamma_{23}^3 &= \cot \theta, \end{aligned} \tag{8.44}$$

or related to these by symmetry. The nonzero components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1 - \kappa r^2} \\ R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa) \\ R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa) \sin^2 \theta, \end{aligned} \quad (8.45)$$

and the Ricci scalar is then

$$R = 6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{\kappa}{a^2} \right]. \quad (8.46)$$

### 8.3 ■ THE FRIEDMANN EQUATION

The RW metric is defined for any behavior of the scale factor  $a(t)$ ; our next step will be to plug it into Einstein's equation to derive the Friedmann equation(s) relating the scale factor to the energy-momentum of the universe. We will choose to model matter and energy by a perfect fluid. It is clear that, if a fluid that is isotropic in some frame leads to a metric that is isotropic in some frame, the two frames will coincide; that is, the fluid will be at rest in comoving coordinates. The four-velocity is then

$$U^\mu = (1, 0, 0, 0), \quad (8.47)$$

and the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \quad (8.48)$$

becomes

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix}. \quad (8.49)$$

With one index raised this takes the convenient form

$$T^\mu{}_\nu = \text{diag}(-\rho, p, p, p). \quad (8.50)$$

Note that the trace is given by

$$T = T^\mu{}_\mu = -\rho + 3p. \quad (8.51)$$

Before plugging in to Einstein's equation, it is educational to consider the zero component of the conservation of energy equation:

$$\begin{aligned}
 0 &= \nabla_\mu T^{\mu 0} \\
 &= \partial_\mu T^{\mu 0} + \Gamma_{\mu\lambda}^\mu T^{\lambda 0} - \Gamma_{\mu 0}^\lambda T^{\mu\lambda} \\
 &= -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p).
 \end{aligned} \tag{8.52}$$

To make progress we can choose an **equation of state**, a relationship between  $\rho$  and  $p$ . Often the perfect fluids relevant to cosmology obey the simple equation of state

$$p = w\rho, \tag{8.53}$$

where  $w$  is a constant independent of time. Of course we are free to define the parameter  $w = p/\rho$  whether or not it remains constant; if  $w$  varies, however, it is not really legitimate to call  $p = w\rho$  the “equation of state.” The conservation of energy equation becomes

$$\boxed{\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}} \tag{8.54}$$

If  $w$  is a constant, this can be integrated to obtain

$$\rho \propto a^{-3(1+w)}. \tag{8.55}$$

To get an idea about what values of  $w$  are allowed, refer to the discussion of energy conditions in Chapter 4. The Null Dominant Energy Condition, which allows for a vacuum energy of either sign but otherwise requires matter that cannot destabilize the vacuum, implies

$$|w| \leq 1. \tag{8.56}$$

While this requirement is by no means set in stone, it seems like a sensible conservative starting point for investigations of what might happen in the real world.

The two most popular examples of cosmological fluids are known as **matter** and **radiation**. Matter is any set of collisionless, nonrelativistic particles, which will have essentially zero pressure:

$$p_M = 0. \tag{8.57}$$

Examples include ordinary stars and galaxies, for which the pressure is negligible in comparison with the energy density. Matter is also known as *dust*, and universes whose energy density is mostly due to matter are known as **matter-dominated**. The energy density in matter falls off as

$$\rho_M \propto a^{-3}. \tag{8.58}$$

This is simply interpreted as the decrease in the number density of particles as the universe expands. For matter the energy density is dominated by the rest energy,

which is proportional to the number density. Radiation may be used to describe either actual electromagnetic radiation, or massive particles moving at relative velocities sufficiently close to the speed of light that they become indistinguishable from photons (at least as far as their equation of state is concerned). Although an isotropic gas of relativistic particles is a perfect fluid and thus has an energy-momentum tensor given by (8.48), we also know that  $T_{\mu\nu}$  for electromagnetism can be expressed in terms of the field strength as

$$T^{\mu\nu} = F^{\mu\lambda} F^\nu_\lambda - \frac{1}{4} g^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma}. \quad (8.59)$$

The trace of this is given by

$$T^\mu_\mu = F^{\mu\lambda} F_{\mu\lambda} - \frac{1}{4}(4)F^{\lambda\sigma} F_{\lambda\sigma} = 0. \quad (8.60)$$

But this must also equal (8.51), so the equation of state is

$$p_R = \frac{1}{3}\rho_R. \quad (8.61)$$

A universe in which most of the energy density is in the form of radiation is known as **radiation-dominated**. The energy density in radiation falls off as

$$\rho_R \propto a^{-4}. \quad (8.62)$$

Thus, the energy density in radiation falls off slightly faster than that in matter; this is because the number density of photons decreases in the same way as the number density of nonrelativistic particles, but individual photons also lose energy as  $a^{-1}$  as they redshift, which we will see later. Likewise, massive but relativistic particles will lose energy as they “slow down” in comoving coordinates. We believe that today the radiation energy density is much less than that of matter, with  $\rho_M/\rho_R \sim 10^3$ . However, in the past the universe was much smaller, and the energy density in radiation would have dominated at very early times.

As we have discussed, vacuum energy also takes the form of a perfect fluid, with an equation of state  $p_\Lambda = -\rho_\Lambda$ . The energy density is constant,

$$\rho_\Lambda \propto a^0. \quad (8.63)$$

Since the energy density in matter and radiation decreases as the universe expands, if there is a nonzero vacuum energy it tends to win out over the long term, as long as the universe doesn’t start contracting. If this happens, we say that the universe becomes **vacuum-dominated**. de Sitter and anti-de Sitter are vacuum-dominated solutions.

We now turn to Einstein’s equation. Recall that it can be written in the form (4.45):

$$R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right). \quad (8.64)$$

The  $\mu\nu = 00$  equation is

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p), \quad (8.65)$$

and the  $\mu\nu = ij$  equations give

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\kappa}{a^2} = 4\pi G(\rho - p). \quad (8.66)$$

There is only one distinct equation from  $\mu\nu = ij$ , due to isotropy. We can use (8.65) to eliminate second derivatives in (8.66), and do a little cleaning up to obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}, \quad (8.67)$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \quad (8.68)$$

Together these are known as the **Friedmann equations**, and metrics of the form (8.43) obey these equations define Friedmann–Robertson–Walker (FRW) universes. In fact, if we know the dependence of  $\rho$  on  $a$ , the first of these (8.67) is enough to solve for  $a(t)$ ; when you hear people refer to *the* Friedmann equation, this is the one to which they are referring, whereas (8.68) is sometimes called the *second* Friedmann equation.

A bunch of terminology is associated with the cosmological parameters, and we will just introduce the basics here. The rate of expansion is characterized by the **Hubble parameter**,

$$H = \frac{\dot{a}}{a}. \quad (8.69)$$

The value of the Hubble parameter at the present epoch is the Hubble constant,  $H_0$ . Current measurements lead us to believe that the Hubble constant is  $70 \pm 10$  km/sec/Mpc. (Mpc stands for megaparsec, which is  $3.09 \times 10^{24}$  cm.) Since there is still some uncertainty in this value, we often parameterize the Hubble constant as

$$H_0 = 100h \text{ km/sec/Mpc}, \quad (8.70)$$

so that  $h \approx 0.7$ . Typical cosmological scales are set by the **Hubble length**

$$\begin{aligned}
 d_H &= H_0^{-1}c \\
 &= 9.25 \times 10^{27} h^{-1} \text{ cm} \\
 &= 3.00 \times 10^3 h^{-1} \text{ Mpc},
 \end{aligned} \tag{8.71}$$

and the **Hubble time**

$$\begin{aligned}
 t_H &= H_0^{-1} \\
 &= 3.09 \times 10^{17} h^{-1} \text{ sec} \\
 &= 9.78 \times 10^9 h^{-1} \text{ yr.}
 \end{aligned} \tag{8.72}$$

Of course since we usually set  $c = 1$ , you will see  $H_0^{-1}$  referred to as both the Hubble length and the Hubble time. There is also the **deceleration parameter**,

$$q = -\frac{a\ddot{a}}{\dot{a}^2}, \tag{8.73}$$

which measures the rate of change of the rate of expansion.

Another useful quantity is the **density parameter**,

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{\text{crit}}}, \tag{8.74}$$

where the **critical density** is defined by

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G}. \tag{8.75}$$

This quantity, which will generally change with time, is called the *critical density* because the Friedmann equation (8.67) can be written

$$\Omega - 1 = \frac{\kappa}{H^2 a^2}. \tag{8.76}$$

The sign of  $\kappa$  is therefore determined by whether  $\Omega$  is greater than, equal to, or less than, one. We have

$$\begin{aligned}
 \rho < \rho_{\text{crit}} &\Leftrightarrow \Omega < 1 \Leftrightarrow \kappa < 0 \Leftrightarrow \text{open} \\
 \rho = \rho_{\text{crit}} &\Leftrightarrow \Omega = 1 \Leftrightarrow \kappa = 0 \Leftrightarrow \text{flat} \\
 \rho > \rho_{\text{crit}} &\Leftrightarrow \Omega > 1 \Leftrightarrow \kappa > 0 \Leftrightarrow \text{closed}.
 \end{aligned}$$

The density parameter, then, tells us which of the three Robertson–Walker geometries describes our universe. Determining it observationally is of crucial importance; recent measurements of the cosmic microwave background anisotropy lead us to believe that  $\Omega$  is very close to unity.

### 8.4 ■ EVOLUTION OF THE SCALE FACTOR

Given a specification of the amounts of energy density  $\rho_i$  in different species  $i$ , along with their equations of state  $p_i = p_i(\rho_i)$ , and the amount of spatial curvature  $\kappa$ , one can solve the Friedmann equation (8.67) to obtain a complete history of the evolution of the scale factor,  $a(t)$ . In general we simply numerically integrate the Friedmann equation (which is just a first-order differential equation), but it is useful to get a feeling for the types of solutions appropriate to different cosmological parameters.

To simplify our task, let us imagine that all of the different components of energy density evolve as power laws,

$$\rho_i = \rho_{i0} a^{-n_i}. \quad (8.77)$$

Comparing to (8.55), this is equivalent to positing that each equation-of-state parameter  $w_i = p_i/\rho_i$  is a constant equal to

$$w_i = \frac{1}{3} n_i - 1. \quad (8.78)$$

We can further streamline our expressions by treating the contribution of spatial curvature as a fictitious energy density

$$\rho_c \equiv -\frac{3\kappa}{8\pi G a^2}, \quad (8.79)$$

with a corresponding density parameter

$$\Omega_c = -\frac{\kappa}{H^2 a^2}. \quad (8.80)$$

It's *not* an energy density, of course, so don't forget that this is just notational sleight-of-hand. The behaviors of our favorite sources are summarized in the following table.

	$w_i$	$n_i$	
matter	0	3	
radiation	$\frac{1}{3}$	4	
curvature	$-\frac{1}{3}$	2	
vacuum	-1	0	

(8.81)

In these variables, the Friedmann equation (8.67) can be written

$$H^2 = \frac{8\pi G}{3} \sum_{i(c)} \rho_i, \quad (8.82)$$

where the notation  $\sum_{i(c)}$  indicates that we sum not only over all the actual components of energy density  $\rho_i$ , but also over the contribution of spatial curvature

$\rho_c$ . Note that if we divide both sides by  $H^2$ , we obtain

$$1 = \sum_{i(c)} \Omega_i. \quad (8.83)$$

The right-hand side is *not* the total density parameter  $\Omega$ , which only gets contributions from actual energy density (not curvature); we therefore have

$$\Omega_c = 1 - \Omega. \quad (8.84)$$

Let's begin by asking what can happen if all of the  $\rho_i$ 's (including  $\rho_c$ ) are non-negative. Because  $H^2$  is proportional to  $\sum_{i(c)} \rho_i$ , the universe will never undergo a transition from expanding to contracting so long as  $\sum_{i(c)} \rho_i \neq 0$ . We can also take the time derivative of the Hubble parameter,

$$\dot{H} = \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2, \quad (8.85)$$

and plug in the two Friedmann equations (8.67) and (8.68) to obtain

$$\dot{H} = -4\pi G \sum_{i(c)} (1 + w_i) \rho_i. \quad (8.86)$$

Since we are imagining that  $|w_i| \leq 1$ , when all the  $\rho_i$ 's are nonnegative we will always have  $\dot{H} \leq 0$ . In other words, the universe keeps expanding, but the expansion rate continually decreases (which suggests the excellent question, what made it so large in the first place?).

From (8.85) we see that  $\ddot{a}$  can be positive and  $\dot{H}$  be negative at the same time—the scale factor can be “accelerating” even though the expansion rate as measured by the Hubble parameter is decreasing (for example, if  $a \propto t^2$ ). This is an unavoidable subtlety of non-Euclidean geometry. The Hubble parameter and the derivative of the scale factor are the answers to two different questions. If we set two test particles at a fixed initial distance, and ask by how much they have separated a short time thereafter, the answer is given by the Hubble parameter. If, on the other hand, we pick some fixed source, and ask how it appears to move away from us with time, the answer is given by the change in the scale factor. There are consequently two very different and equally legitimate senses of “accelerating” (or “decelerating”). In practice, “accelerating” usually refers to a situation in which  $\ddot{a} > 0$ , even if  $\dot{H} < 0$ . This discussion is not completely academic; as we will see below, our current real universe seems to be of this type.

It is by no means necessary that each  $\rho_i$  should be nonnegative. Matter and radiation arise from dynamical particles and fields, and we consequently expect that their energy densities will never be negative; if they could be, empty space could decay into a collection of positive- and negative-energy fields. But vacuum and curvature are different stories. Vacuum energy is nondynamical, so a negative value cannot induce any instabilities, while curvature is simply a property of the spatial geometry, and can have either sign. If we therefore have either a

negative vacuum energy or a positive spatial curvature (remember  $\rho_c \propto -\kappa$ ), the Hubble parameter can vanish and even change sign. An example is provided by the de Sitter metric (8.6), which has a positive vacuum energy but also a positive spatial curvature; it describes a universe that initially collapses, reaches a turning point, and thereafter begins to expand.

The real world is an untidy place, consisting of numerous different kinds of energy density. Because different sources evolve at different rates, however, for long periods the energy density will be clearly dominated by one kind of source. It is therefore very useful to examine solutions to the Friedmann equation when there is only one kind of energy density  $\rho \propto a^{-n}$ . Because we are including spatial curvature as an effective energy source, this means we are considering either flat universes dominated by a single source, or completely empty universes with spatial curvature. The Friedmann equation then implies

$$\dot{a} \propto a^{1-n/2}. \quad (8.87)$$

This can be immediately integrated to obtain

$$a \propto t^{2/n} \quad (\text{for } \rho \propto a^{-n}). \quad (8.88)$$

Consider for example a flat universe dominated by matter,  $\Omega = \Omega_M = 1$ ; this is known as the Einstein-de Sitter model, and for a long time was the favorite (at least among theorists) to describe the real world. In an Einstein-de Sitter universe, the scale factor evolves as  $a \propto t^{2/3}$ . A flat radiation-dominated universe, meanwhile, evolves as  $a \propto t^{1/2}$ . The conformal diagram for any such universe with  $n > 2$  is derived in Appendix H. Even though we believe there are nonzero amounts of matter, radiation, and vacuum energy in the real universe, these solutions are still very useful; as we discuss later, the universe was radiation-dominated at early times, and was matter dominated as the universe expanded from  $a \sim 1/3000$  to  $a \sim 1/2$ .

These solutions all feature a singularity at  $a = 0$ , known as the **Big Bang**. It represents the creation of the universe from a singular state, not an explosion of matter into a pre-existing spacetime. It might be hoped that the perfect symmetry of our FRW universes is responsible for this singularity, but in fact that's not true; cosmological singularity theorems show that any universe with  $\rho > 0$  and  $p \geq 0$  must have begun at a singularity. Of course the energy density becomes arbitrarily high as  $a \rightarrow 0$ , and we don't expect classical general relativity to be an accurate description of nature in this regime; presumably quantum gravity becomes important, although it is unclear how at present.

Looking at (8.88), we see that a universe dominated by vacuum energy ( $n = 0$ ) is clearly a special case. The scale factor then expands as an exponential rather than a power law; the entire metric is

$$ds^2 = -dt^2 + e^{Ht}[dx^2 + dy^2 + dz^2], \quad (8.89)$$

where the Hubble parameter  $H$  is a constant. Of course, in Section 8.1 we already described a cosmological spacetime with a positive cosmological constant: de Sitter space, which featured  $\kappa > 0$  and  $a \propto \cosh(t/\alpha)$ . What is the relationship between that solution and the one here, with  $\kappa = 0$  and  $a \propto \exp(Ht)$ ? They are the same spacetime, represented in different coordinates. One way to verify this is to calculate the Riemann tensor for (8.89) and check that it has the characteristic form of a maximally symmetric spacetime, (8.1). Since maximally symmetric spacetimes with positive curvature are locally unique, the metrics (8.6) and (8.89) must describe the same manifold, or parts thereof. In fact, the coordinates of (8.89) only cover part of de Sitter; they are incomplete in the past. In the exercises you are asked to show that comoving geodesics in these coordinates reach  $t = -\infty$  in finite affine parameter; they run into the edge of the coordinates. In the conformal diagram of Figure 8.1, these coordinates cover the upper-right triangular portion of the square. See Hawking and Ellis (1973) for a more complete description of different coordinate systems on de Sitter and anti-de Sitter.

Another interesting special case is the completely empty universe, with  $\rho = 0$ , but with spatial curvature. The Friedmann equation becomes

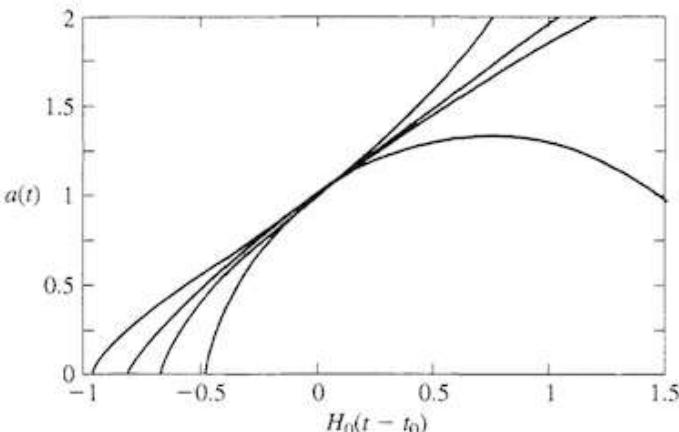
$$H^2 = -\frac{\kappa}{a^2}, \quad (8.90)$$

so the curvature  $\kappa$  must be negative. Thinking of curvature as a fictitious energy density  $\rho_c \propto a^{-2}$ , from (8.88) we know that such a universe will expand linearly,  $a \propto t$ . This spacetime is known as the **Milne universe**. However, just as with de Sitter, we know of another cosmological spacetime with  $\rho = 0$ —in this case, flat Minkowski space. Once again, the Milne spacetime is just a patch of Minkowski in a certain incomplete coordinate system. It can be thought of as the interior of the future light cone of some fixed point in Minkowski, foliated by negatively-curved hyperboloids. To check, it would suffice to calculate all of the components of the Riemann tensor, which turn out to vanish; any spacetime with vanishing Riemann curvature is locally Minkowski.

In contrast to these idealized solutions, a realistic cosmology will feature several forms of energy-momentum. In the current universe, we feel confident that the radiation density is significantly lower than the matter density, but that vacuum and matter are both dynamically important. It is therefore convenient to parameterize universes like ours by  $\Omega_M$  and  $\Omega_\Lambda$ , with the curvature fixed by  $\Omega_c = 1 - \Omega_M - \Omega_\Lambda$ . The expansion history of some particular examples of such universes is shown in Figure 8.3. As these universes expand, the relative influences of matter, curvature, and vacuum are altered, since the corresponding densities evolve at different rates:

$$\Omega_\Lambda \propto \Omega_c a^2 \propto \Omega_M a^3. \quad (8.91)$$

As  $a \rightarrow 0$  in the past, curvature and vacuum will be negligible, and the universe will behave as Einstein–de Sitter. As  $a \rightarrow \infty$  in the future, curvature and matter will be negligible, and the universe will asymptote to de Sitter; unless the scale



**FIGURE 8.3** Expansion histories for different values of  $\Omega_M$  and  $\Omega_\Lambda$ . From top to bottom, the curves describe  $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$ ,  $(0.3, 0.0)$ ,  $(1.0, 0.0)$ , and  $(4.0, 0.0)$ .

factor never reaches infinity, because the universe begins to recollapse at some finite time.

Recollapse will *always* occur if the vacuum energy is negative; as the universe expands, the vacuum energy eventually dominates, and the effect of  $\Omega_\Lambda < 0$  is to cause deceleration and recollapse (just as the effect of  $\Omega_\Lambda > 0$  is to push the universe apart). Recollapse is also possible with  $\Omega_\Lambda \geq 0$ , if  $\Omega_M$  is sufficiently large that it halts the universal expansion before  $\Omega_\Lambda$  has a chance to take over. The possibilities are expressed as different regions of the  $\Omega_M/\Omega_\Lambda$  parameter space in Figure 8.4. The diagonal line represents  $\Omega_{\text{total}} = 1$ , implying  $\kappa = 0$ .

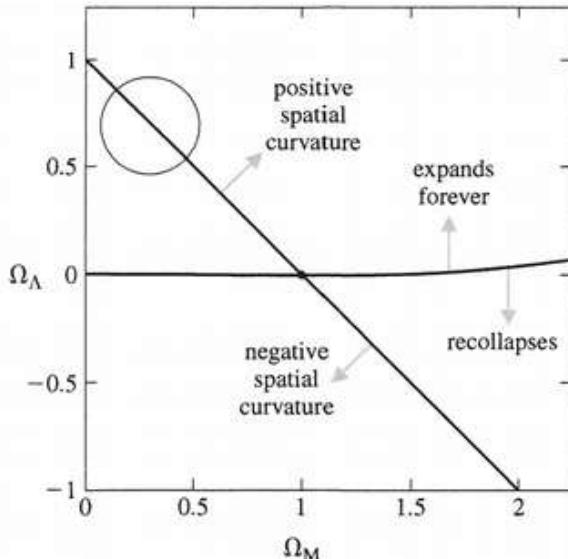
To determine the dividing line between perpetual expansion and eventual recollapse, note that collapse requires the Hubble parameter to pass through zero as it changes from positive to negative. The scale factor  $a_*$  at which this turnaround occurs can be found by setting  $H = 0$  in the Friedmann equation,

$$H^2 = 0 = \frac{8\pi G}{3} \left( \rho_{M0} a_*^{-3} + \rho_{\Lambda0} + \rho_{c0} a_*^{-2} \right). \quad (8.92)$$

We can divide this by  $H_0^2$ , use  $\Omega_{c0} = 1 - \Omega_{M0} - \Omega_{\Lambda0}$ , and rearrange a bit to obtain

$$\Omega_{\Lambda0} a_*^3 + (1 - \Omega_{M0} - \Omega_\Lambda) a_* + \Omega_{M0} = 0. \quad (8.93)$$

This is a cubic equation for  $a_*$ , the scale factor at turnaround. Of course we don't actually care very much about  $a_*$ ; what we care about are the values of  $\Omega_{\Lambda0}$ , given  $\Omega_{M0}$ , for which a real solution to (8.93) exists. Solving the cubic equation and doing some math, we find that the value of  $\Omega_{\Lambda0}$  for which the universe will expand forever is given by



**FIGURE 8.4** Properties of universes dominated by matter and vacuum energy, as a function of the density parameters  $\Omega_M$  and  $\Omega_\Lambda$ . The circular region in the upper-left corner represents roughly those values favored by experimental data (as of 2003).

$$\Omega_{\Lambda 0} \geq \begin{cases} 0 & 0 \leq \Omega_{M0} \leq 1 \\ 4\Omega_{M0} \cos^3 \left[ \frac{1}{3} \cos^{-1} \left( \frac{1 - \Omega_{M0}}{\Omega_{M0}} \right) + \frac{4\pi}{3} \right] & \Omega_{M0} > 1. \end{cases} \quad (8.94)$$

Note that, when  $\Omega_{\Lambda 0} = 0$ , open and flat universes ( $\Omega_0 = \Omega_{M0} \leq 1$ ) will expand forever, while closed universes ( $\Omega_0 = \Omega_{M0} > 1$ ) will recollapse. Traditional disdain for the cosmological constant has led to a folk belief that this is a necessary correspondence; once the possibility of vacuum energy is admitted, however, any combination of spatial geometry and eventual fate is possible.

In the upper-left corner of Figure 8.4, we have indicated the currently favored values of the cosmological parameters:  $\Omega_{M0} \sim 0.3$ ,  $\Omega_{\Lambda 0} \sim 0.7$ , as we will discuss in Section 8.7. This is well into the regime of perpetual expansion; if the vacuum energy remains truly constant (which it might not), our universe is fated to continue its expansion for all time.

We end this section by noting the difficulty of finding static solutions to the Friedmann equations. To be static, we must have not only  $\dot{a} = 0$ , but also  $\ddot{a} = 0$ . From (8.68), this can only happen if the pressure is

$$p = -\frac{1}{3}\rho, \quad (8.95)$$

and from (8.67), there must be a nonvanishing spatial curvature

$$\frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho. \quad (8.96)$$

Because the energy density and pressure must be of opposite sign, these conditions can't be fulfilled if we only invoke matter or radiation. When Einstein first looked for cosmological solutions in GR, astronomers had not yet discovered that the universe was expanding, so the lack of static solutions was considered problematic. This provided the motivation for Einstein to introduce the cosmological constant; the static conditions can be satisfied by a combination of matter and vacuum energy, with

$$\rho_\Lambda = \frac{1}{2} \rho_M, \quad (8.97)$$

along with the appropriate positive spatial curvature. These parameters describe the **Einstein static universe**. Today we know that the universe is expanding, so this solution is of little empirical interest; it is, however, extremely useful to theorists, providing the basis for the construction of conformal diagrams.

### 8.5 ■ REDSHIFTS AND DISTANCES

It is clear that we would like to determine a number of quantities observationally to decide which of the FRW models corresponds to our universe. Obviously we would like to determine  $H_0$ , since that is related to the age of the universe. We would also like to know  $\Omega$ , which determines  $\kappa$  through (8.76). To understand how these quantities might conceivably be measured, let's consider geodesic motion in an FRW universe. There are a number of spacelike Killing vectors, but no timelike Killing vector to give us a notion of conserved energy. There is, however, a Killing tensor. If  $U^\mu = (1, 0, 0, 0)$  is the four-velocity of comoving observers, then the tensor

$$K_{\mu\nu} = a^2(g_{\mu\nu} + U_\mu U_\nu) \quad (8.98)$$

satisfies  $\nabla_{(\sigma} K_{\mu\nu)} = 0$  (as you can check), and is therefore a Killing tensor. This means that if a particle has four-velocity  $V^\mu = dx^\mu/d\lambda$ , the quantity

$$K^2 = K_{\mu\nu} V^\mu V^\nu = a^2[V_\mu V^\mu + (U_\mu V^\mu)^2] \quad (8.99)$$

will be a constant along geodesics. Let's think about this, first for massive particles. Then we will have  $V_\mu V^\mu = -1$ , so

$$(V^0)^2 = 1 + |\vec{V}|^2, \quad (8.100)$$

where  $|\vec{V}|^2 = g_{ij} V^i V^j$ . We also have  $U_\mu V^\mu = -V^0$ , so (8.99) implies

$$|\vec{V}| = \frac{K}{a}. \quad (8.101)$$

The particle therefore "slows down" with respect to the comoving coordinates as the universe expands. In fact this is an actual slowing down, in the sense that a gas of particles with initially high relative velocities will cool down as the universe expands.

A similar thing happens to null geodesics. In this case  $V_\mu V^\mu = 0$ , and (8.99) implies

$$U_\mu V^\mu = \frac{K}{a}. \quad (8.102)$$

But the frequency of the photon as measured by a comoving observer is  $\omega = -U_\mu V^\mu$ . The frequency of the photon emitted with frequency  $\omega_{\text{em}}$  will therefore be observed with a lower frequency  $\omega_{\text{obs}}$  as the universe expands:

$$\frac{\omega_{\text{obs}}}{\omega_{\text{em}}} = \frac{a_{\text{em}}}{a_{\text{obs}}}. \quad (8.103)$$

Cosmologists like to speak of this in terms of the **redshift**  $z$  between the two events, defined by the fractional change in wavelength:

$$z_{\text{em}} = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}. \quad (8.104)$$

If the observation takes place today ( $a_{\text{obs}} = a_0 = 1$ ), this implies

$$a_{\text{em}} = \frac{1}{1 + z_{\text{em}}}. \quad (8.105)$$

So the redshift of an object tells us the scale factor when the photon was emitted.

Notice that this redshift is not the same as the conventional Doppler effect; it is the expansion of space, not the relative velocities of the observer and emitter, which leads to the redshift. Nevertheless, if we observe galaxies over distances that are small compared to the Hubble radius  $H_0^{-1}$  and the radius of spatial curvature  $\kappa^{-1/2}$ , the expansion of the universe looks very much like a set of galaxies moving apart from each other and the redshift looks very much like the Doppler effect. Consequently, astronomers often think of the redshift in terms of a “velocity”  $v = cz$ , where  $c$  is the speed of light. Even though we know you can’t really speak of the relative velocities between two objects at different points of a curved spacetime, the fiction works well over sufficiently short distances. Within this approximation, the “distance”  $d$  from us to a galaxy can be taken to be the **instantaneous physical distance**  $d_P$  (the distance, in physical units such as centimeters, between us and the location of the galaxy along our current spatial hypersurface). Let’s write the RW metric in the form

$$ds^2 = -dt^2 + a^2(t)R_0^2 \left[ d\chi^2 + S_k^2(\chi)d\Omega^2 \right], \quad (8.106)$$

where  $S_k(\chi)$  is defined by (8.33), and  $k \in \{+1, 0, -1\}$ . In this form, the instantaneous physical distance as measured at time  $t$  between us ( $\chi = 0$ ) and a galaxy at comoving radial coordinate  $\chi$  is

$$d_P(t) = a(t)R_0\chi, \quad (8.107)$$

where  $\chi$  remains constant because we assume both we and the observed galaxy are perfectly comoving. (They might not be, in which case it is trivial to include the corrections due to so-called “peculiar velocities.”) Of course “distance” is in quotes because there are several inequivalent useful notions of distance once we leave this approximation, but they all agree when  $d_P$  is small. Then the observed velocity (as inferred from the redshift) is simply

$$v = \dot{d}_P = \dot{a}R_0\chi = \frac{\dot{a}}{a}d_P. \quad (8.108)$$

Evaluated today, this becomes

$$v = H_0 d_P, \quad (8.109)$$

the famous **Hubble law**: the observed recession velocity is directly proportional to the distance, for galaxies that are not too far away.

If the redshift is not very small, we have to think more carefully about what we mean by “distance” in cosmology. The instantaneous physical distance is a convenient construct, but not itself observable, since observations always refer to events on our past light cone, not our current spatial hypersurface. In Euclidean space there are a number of different ways to infer the distance of an object; we could for example compare its apparent brightness to its intrinsic luminosity, or its apparent angular velocity to its intrinsic transverse speed, or its apparent angular size to its physical extent. For each of these cases, we can define a kind of distance that is what we *would* infer if space were Euclidean and the universe were not expanding.

Let’s start with the **luminosity distance**  $d_L$ , defined to satisfy

$$d_L^2 = \frac{L}{4\pi F}, \quad (8.110)$$

where  $L$  is the absolute luminosity of the source and  $F$  is the flux measured by the observer (the energy per unit time per unit area of some detector). This definition comes from the fact that in flat space, for a source at distance  $d$  the flux over the luminosity is just one over the area of a sphere centered around the source,  $F/L = 1/A(d) = 1/4\pi d^2$ . In an FRW universe, however, the flux will be diluted. Conservation of photons tells us that all of the photons emitted by the source will eventually pass through a sphere at comoving distance  $\chi$  from the emitter. But the flux is diluted by two additional effects: the individual photons redshift by a factor  $(1+z)$ , and the photons hit the sphere less frequently, since two photons emitted a time  $\delta t$  apart will be measured at a time  $(1+z)\delta t$  apart. Therefore we will have

$$\frac{F}{L} = \frac{1}{(1+z)^2 A}. \quad (8.111)$$

The area  $A$  of a sphere centered at comoving distance  $\chi$  can be derived from the coefficient of  $d\Omega^2$  in (8.106), yielding

$$A = 4\pi R_0^2 S_k^2(\chi), \quad (8.112)$$

where we have set  $a(t) = 1$  because we are observing the photons today. Putting it all together yields

$$d_L = (1+z)R_0S_k(\chi). \quad (8.113)$$

The luminosity distance  $d_L$  is something we might hope to measure, since there are some astrophysical sources whose absolute luminosities are known. But  $\chi$  is not observable, so we have to remove that from our equation. On a null geodesic (chosen to be radial for convenience) we have

$$0 = ds^2 = -dt^2 + a^2 R_0^2 d\chi^2, \quad (8.114)$$

or

$$\chi = R_0^{-1} \int \frac{dt}{a} = R_0^{-1} \int \frac{da}{a^2 H(a)}, \quad (8.115)$$

where we have used  $H = \dot{a}/a$ . It is conventional to convert the scale factor to redshift using  $a = 1/(1+z)$ , so we have

$$\chi(z) = R_0^{-1} \int_0^z \frac{dz'}{H(z')}. \quad (8.116)$$

In order to evaluate the Hubble parameter in this integral we use the Friedmann equation (8.67), which we write as in the previous section as

$$H^2 = \frac{8\pi G}{3} \sum_{i(c)} \rho_i. \quad (8.117)$$

To simplify things, we may again assume that each density component evolves as a power law,

$$\rho_i(z) = \rho_{i0} a^{-n_i} = \rho_{i0} (1+z)^{n_i}, \quad (8.118)$$

Then we can write

$$H(z) = H_0 E(z), \quad (8.119)$$

where

$$E(z) = \left[ \sum_{i(c)} \Omega_{i0} (1+z)^{n_i} \right]^{1/2}, \quad (8.120)$$

where the density parameters  $\Omega_i$  are defined by (8.74). The equations below involving  $E(z)$  will be true whether or not the energy sources evolve as power laws; if they do not, simply use  $E(z) = H(z)/H_0$  [where  $H(z)$  is determined by the Friedmann equation] rather than (8.120).

So the luminosity distance is

$$d_L(z) = (1+z)R_0 S_k \left[ R_0^{-1} H_0^{-1} \int \frac{dz'}{E(z')} \right]. \quad (8.121)$$

Note that  $R_0$  drops out when  $k = 0$ , which is good, because in that case it is a completely arbitrary parameter. Even when it is not arbitrary, it is still more common to speak in terms of  $\Omega_{c0} = -k/R_0^2 H_0^2$ , which can be measured either directly through determinations of the spatial curvature, or by measuring the density parameter and using  $\Omega_{c0} = 1 - \Omega_0$ . In terms of this parameter we have

$$R_0 = H_0^{-1} \sqrt{-k\Omega_{c0}} = \frac{H_0^{-1}}{\sqrt{|\Omega_{c0}|}}. \quad (8.122)$$

We therefore write the luminosity distance in terms of measurable cosmological parameters as

$$d_L(z) = (1+z) \frac{H_0^{-1}}{\sqrt{|\Omega_{c0}|}} S_k \left[ \sqrt{|\Omega_{c0}|} \int \frac{dz'}{E(z')} \right]. \quad (8.123)$$

Although it appears unwieldy, this equation is of central importance in cosmology. Given the observables  $H_0$  and  $\Omega_{c0}$ , we can straightforwardly calculate the luminosity distance to an object at any redshift  $z$ ; equally well, we can measure  $d_L(z)$  for objects at a range of redshifts, and from that information extract  $H_0$  and/or the  $\Omega_{i0}$ 's.

Along with the luminosity distance are two other related distance measures. Just as the luminosity distance is the distance we infer from the intrinsic and observed luminosity of the source if we were in flat space, the **proper motion distance**  $d_M$  is the distance we infer from the intrinsic and observed motion of the source. It is defined to be

$$d_M = \frac{u}{\dot{\theta}}, \quad (8.124)$$

where  $u$  is the proper transverse velocity (something you would measure, for example, in km/s) and  $\dot{\theta}$  is the observed angular velocity. The **angular diameter distance**, meanwhile, is the distance we infer from the intrinsic and observed size of the source; it is defined to be

$$d_A = \frac{R}{\dot{\theta}}, \quad (8.125)$$

where  $R$  is the proper size of the object and  $\theta$  is its observed angular diameter. In both cases we can derive formulas analogous to (8.123); fortunately, the unwieldy dependence on the cosmological parameters is common to all the distance measures, and we are left with a simple dependence on redshift:

$$d_L = (1+z)d_M = (1+z)^2 d_A. \quad (8.126)$$

as you are encouraged to check. So if we measure one such distance, we can easily convert to any other; or we can measure different distances independently and use (8.126) to test the consistency of the RW framework.

While we're contemplating distances, let's also consider the elapsed time between now and when the light from an object at redshift  $z$  was emitted. If the age of the universe today is  $t_0$  and the age when the photon was emitted is  $t_*$ , the **lookback time** is

$$\begin{aligned} t_0 - t_* &= \int_{t_*}^{t_0} dt \\ &= \int_{a_*}^1 \frac{da}{a H(a)} \\ &= H_0^{-1} \int_0^{z_*} \frac{dz'}{(1+z')E(z')}. \end{aligned} \quad (8.127)$$

For example, consider a flat ( $k = 0$ ) matter-dominated ( $\rho = \rho_M = \rho_{M0}a^{-3}$ ) universe. Then

$$E(z) = (1+z)^{3/2}, \quad (8.128)$$

so

$$\begin{aligned} t_0 - t_* &= H_0^{-1} \int_0^{z_*} \frac{dz'}{(1+z')^{5/2}} \\ &= \frac{2}{3} H_0^{-1} \left[ 1 - (1+z_*)^{-3/2} \right]. \end{aligned} \quad (8.129)$$

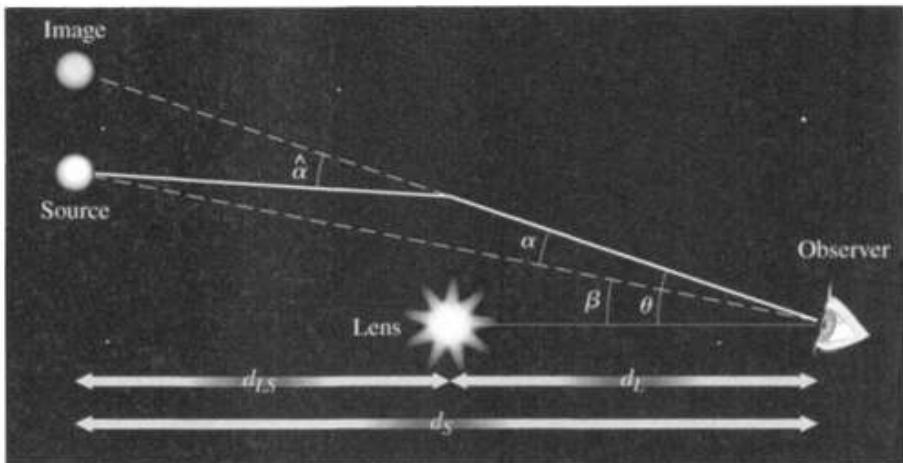
The total age of a matter-dominated universe is obtained by letting  $t_* \rightarrow 0$  ( $z_* \rightarrow \infty$ ),

$$t_0(\text{MD}) = \frac{2}{3} H_0^{-1}. \quad (8.130)$$

For universes that are not completely matter-dominated, the factor of  $\frac{2}{3}$  will be not quite right, but for reasonable values of the cosmological parameters we usually get  $t_0 \sim H_0^{-1}$ .

## 8.6 ■ GRAVITATIONAL LENSING

In Chapter 7 we introduced the concept of gravitational lensing: the deflection and time delay of light by a Newtonian gravitational field. In addition to providing a



**FIGURE 8.5** The geometry of gravitational lensing, encapsulated in the lens equation (8.132). The effect of the lens is to distort the angles  $\beta$  that would be observed in a flat Minkowski background into the angles  $\theta$ .

test of GR in the Solar System, lensing occurs in numerous astrophysical contexts, and has become an indispensable part of modern cosmology.<sup>2</sup>

Two important features distinguish cosmological lensing from the case we discussed earlier: a Robertson–Walker metric replaces the Minkowski background, and the lenses themselves are often more complex than simple point masses. A typical lensing geometry is portrayed in Figure 8.5. Throughout this discussion we will assume that the lens is “thin”—much smaller in spatial extent than the distances between the source, lens, and observer. In this case we can sensibly speak of a unique distance to the lens,  $d_L$ , and between the lens and the source,  $d_{LS}$ .

We describe a (possibly complicated) image on the sky by a set of angles between different components of the image. These angles can be thought of as two-dimensional vectors on the sky. The effect of the lens is to distort the angles that would be observed in the absence of any deflection, such as the angle  $\beta$  between the source and the lens, into a new image characterized by a set of angles  $\theta$ . We assume that the angles are small throughout. This map is described by the **reduced lensing angle**  $\vec{\alpha} = \vec{\theta} - \vec{\beta}$ . According to the geometry shown in Figure 8.5, it is related to the actual deflection angle  $\hat{\alpha}$  by

$$\vec{\alpha} = \frac{d_{LS}}{d_S} \hat{\alpha}. \quad (8.131)$$

<sup>2</sup>An excellent overview of gravitational lensing, from which our discussion borrows, can be found in R. Narayan and M. Bartelmann, “Lectures on Gravitational Lensing,” 13th Jerusalem Winter School in Theoretical Physics, <http://arxiv.org/astro-ph/9606001>.

We therefore get the **lens equation**

$$\vec{\beta} = \vec{\theta} - \frac{d_{LS}}{d_S} \hat{\alpha}. \quad (8.132)$$

The lens equation simply describes ray-tracing in a perturbed spacetime.

Of course, we should think carefully about the “distances”  $d_i$  portrayed in the figure. Lensing occurs in an expanding universe, which might also have spatial curvature. The lens equation will nevertheless hold if we *define* the distances  $d_i$  to be such that the geometrical relations described by the lens equation hold. In other words, these are the distances that we would infer, given the angles and transverse physical sizes, in a static Euclidean spatial background. But this is precisely the definition of the angular diameter distance (8.125). We therefore take all distances in this section to be angular-diameter distances. Note that angular-diameter distances do not necessarily add, so that  $d_S \neq d_L + d_{LS}$ .

As a simple example, consider a point mass lens. In our investigation of the Newtonian limit in Chapter 7, we found that the deflection angle for a photon traveling through a gravitational potential  $\Phi$  is given by

$$\hat{\alpha} = 2 \int \vec{\nabla}_\perp \Phi ds, \quad (8.133)$$

which for a point mass  $M$  at an impact parameter  $b$  becomes

$$\hat{\alpha} = \frac{4GM}{b}. \quad (8.134)$$

The impact parameter can be expressed as  $b = d_L \theta$ . The lens equation (8.132) becomes

$$\beta = \theta - \frac{d_{LS}}{d_S d_L} \frac{4GM}{\theta}. \quad (8.135)$$

It is illuminating to consider the simplest situation, in which the source and lens are collinear ( $\beta = 0$ ). In that case, the source will be lensed into an **Einstein ring** surrounding the lens, at an angular separation given by the **Einstein angle**:

$$\theta_E = \sqrt{\frac{4GM d_{LS}}{d_L d_S}}. \quad (8.136)$$

The Einstein angle sets a characteristic scale for lensing, even in more complicated configurations. We can also define an associated distance scale, the **Einstein radius**:

$$R_E = \sqrt{\frac{4GM d_L d_{LS}}{d_S}}, \quad (8.137)$$

When converting to centimeters or other physical units, don't forget that  $c = 1$  in all of our equations. To get a feeling for the amount of lensing in typical astrophysical situations, we can consider two common occurrences: "microlensing" by approximately solar-mass objects within our galaxy, and cosmological lensing by galaxies or clusters. In the former case the Einstein angle will be of order milliarcseconds, while in the latter case it will be of order arcseconds:

$$\begin{aligned}\theta_E &= 0.9 \sqrt{\left(\frac{M}{M_\odot}\right) \left(\frac{10 \text{ kpc}}{D}\right)} \text{ milliarcsecs} \\ &= 0.9 \sqrt{\left(\frac{M}{10^{11} M_\odot}\right) \left(\frac{\text{Gpc}}{D}\right)} \text{ arcsecs.}\end{aligned}\quad (8.138)$$

Sticking for the moment with the point-mass lens, most often we will not be lucky enough to have source and lens perfectly aligned, although a number of spectacular examples of Einstein rings have been observed. Then we can solve (8.135) to obtain two values of the image angle,

$$\theta_{\pm} = \frac{1}{2} \left( \beta \pm \sqrt{\beta^2 + 4\theta_E^2} \right). \quad (8.139)$$

The image at  $\theta_+$  will always be outside the Einstein angle, while  $\theta_-$  will be inside. In fact this formula is somewhat misleading, as there will always be an odd number of images; for a point mass lens, the third image would be located at the same position as the lens itself.

Now let's consider more general lenses than point masses. We know that the deflection angle will be given in terms of the Newtonian gravitational potential by (8.133). We can define the **lensing potential** by integrating over past-directed geodesic paths emanating from the observer, as

$$\psi(\vec{\theta}) = 2 \frac{d_L s}{d_L d_S} \int \Phi(d_L \vec{\theta}, s) ds. \quad (8.140)$$

In terms of the lensing potential, we can straightforwardly derive the reduced lensing angle by taking the gradient,

$$\begin{aligned}\vec{\alpha} &= \vec{\nabla}_{\theta} \psi \\ &= 2 \frac{d_L s}{d_S} \int \vec{\nabla}_\perp \Phi ds.\end{aligned}\quad (8.141)$$

Notice that the angular gradient  $\vec{\nabla}_\theta$  is related to  $\vec{\nabla}_\perp$ , the gradient with respect to transverse distance at the location of the lens, by a factor of  $d_L$ . The thin-lens approximation allows us to collapse the integral to quantities evaluated at the location of the lens. We can also take the (two-dimensional) Laplacian of the lensing potential to obtain the **convergence**  $\kappa$ , via

$$\begin{aligned}\kappa(\vec{\theta}) &\equiv \frac{1}{2} \nabla_{\theta}^2 \psi \\ &= \frac{d_L d_{LS}}{ds} \int \nabla^2 \Phi ds.\end{aligned}\quad (8.142)$$

The convergence can be thought of as a measure of the integrated mass density. We can invert the above expressions to write both the lensing potential and the reduced deflection angle in terms of the convergence, as

$$\psi(\vec{\theta}) = \frac{1}{\pi} \int \kappa(\vec{\theta}') \ln |\vec{\theta} - \vec{\theta}'| d^2 \theta' \quad (8.143)$$

and

$$\vec{\alpha}(\vec{\theta}) = \frac{1}{\pi} \int \kappa(\vec{\theta}') \frac{\vec{\theta} - \vec{\theta}'}{|\vec{\theta} - \vec{\theta}'|} d^2 \theta'. \quad (8.144)$$

To check these equations, remember that the vectors are defined only in the two transverse dimensions.

The convergence describes the focusing of light rays by the gravitational lens. This focusing causes the source to appear larger (just as in a magnifying glass). According to Liouville's theorem of conservation of phase-space density for the photons emitted by the source, the surface brightness of the source will be conserved under lensing; the increase in size therefore leads to magnification of the brightness. At the same time, we can have distortion caused by twisting of the light rays through the lens, which leads to shear of the shape of the image. To describe both phenomena, we consider the  $2 \times 2$  matrix of derivatives of the lens map,

$$A_{ij} \equiv \frac{\partial \beta^i}{\partial \theta^j}. \quad (8.145)$$

Note that there is no real distinction between upper and lower indices, as they are defined in a two-dimensional Euclidean plane. Since  $\vec{\beta} = \vec{\theta} - \vec{\alpha}$ , we have

$$\begin{aligned}A_{ij} &= \delta_{ij} - \frac{\partial \alpha^i}{\partial \theta^j} \\ &= \delta_{ij} - \psi_{ij},\end{aligned}\quad (8.146)$$

where we have introduced the notation

$$\psi_{ij} \equiv \frac{\partial^2 \psi}{\partial \theta^i \partial \theta^j}. \quad (8.147)$$

This matrix  $A$  encodes the local properties of the lensing map. Its inverse matrix is known as the *magnification tensor*,

$$M = \frac{\partial \vec{\theta}}{\partial \vec{\beta}} = A^{-1}. \quad (8.148)$$

Why does it get this name? The lens distorts an area element described by  $\vec{\beta}$  into one described by  $\vec{\theta}$ , and the change in area is described by the Jacobian of this map, which is simply the determinant of  $M$ . This determinant is defined as the **magnification**  $\mu$ ,

$$\mu = |M| = \frac{1}{|A|}. \quad (8.149)$$

The absolute magnitude of  $\mu$  tells us the actual change in brightness of the source;  $\mu$  may be negative, which means that the parity of the image has been flipped. We speak of magnification because lensing is only noticeable if the lens and source are near to each other on the sky, in which case the focusing effect leads only to increases in the apparent brightness; a lens far away from the source (in position on the sky) would lead to a minuscule decrease in the luminosity that will never be noticed. (If there are multiple images, the sum of the brightnesses of all the images will exceed that of the undistorted source.)

The components of  $A$  can be decomposed into the effects of convergence and shear. For the convergence, from  $\kappa = \frac{1}{2} \nabla_\theta^2 \psi$  we have

$$\kappa = \frac{1}{2}(\psi_{11} + \psi_{22}). \quad (8.150)$$

The **shear**, meanwhile, distorts the shape of the source; if an initially circular source is distorted into an ellipse of ellipticity  $\gamma$  and position angle  $\phi$ , we define the two components of the shear to be

$$\begin{aligned} \gamma_1 &= \gamma \cos(2\phi) \\ \gamma_2 &= \gamma \sin(2\phi), \end{aligned} \quad (8.151)$$

so that the total shear is  $\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$ . In terms of the lensing potential the components are given by

$$\begin{aligned} \gamma_1 &= \frac{1}{2}(\psi_{11} - \psi_{22}) \\ \gamma_2 &= \psi_{12} = \psi_{21}. \end{aligned} \quad (8.152)$$

Inverting these relationships to find the components of  $A$  yields

$$A = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}. \quad (8.153)$$

We can therefore express the magnification in terms of the convergence and shear, as

$$\mu = \frac{1}{(1 - \kappa)^2 - \gamma^2}. \quad (8.154)$$

These features of lensing are becoming increasingly important in observational cosmology. The obvious case of interest is so-called “strong lensing,” when the source is within the Einstein radius of the lens, and multiple images are possible. By observing several images of a single source, we can infer properties of the lens mass distribution (for example, to search for dark matter); we can also use the time delay along different paths to measure the Hubble constant, and the statistical frequency of lensing to constrain other cosmological parameters. However, lensing need not be strong to have an important effect. “Weak lensing,” when the source and lens are separated by more than an Einstein radius, will generally lead to small amounts of magnification and shear which are impossible to detect without a priori knowledge of the properties of the source. However, the shearing effect can be detected statistically, by looking at the shapes of thousands of galaxies that are assumed to be intrinsically random in their orientations. Shearing by weak lensing leads to correlated distortions in the shapes, which can reveal a great deal about the distribution of matter between the observer and the distant sources.

## 8.7 ■ OUR UNIVERSE

Throughout our discussion of the behavior of FRW cosmologies, we have alluded to the actual values of the cosmological parameters corresponding to the universe in which we live. Let us now be more systematic, and discuss both the universe we see today and a plausible extrapolation back to early times. Our discussion will necessarily be brief, both for reasons of space and because cosmology is an active area of research; look for recent review articles to get up-to-date descriptions of current views.

Many of our direct determinations of the expansion rate rely on the luminosity-distance formula (8.123) applied to some type of object whose intrinsic luminosity is assumed to be known, which we call **standard candles**. (Occasionally we measure the angular diameters of objects whose intrinsic size is assumed to be known: standard rulers.) The Hubble constant, for example, is measured with a variety of standard candles, and a consensus of different methods has converged on the value  $H_0 = 70 \pm 10 \text{ km/sec/Mpc}$ , mentioned above. Deviations at high redshift from the linear Hubble law (8.109) can yield information about the density parameters  $\Omega_{i0}$ , but only if we have very bright objects whose intrinsic luminosity is accurately known. These are provided by Type Ia supernovae, which are thought to be explosions of white dwarf stars that have accreted enough mass to surpass the Chandrasekhar limit. Since the Chandrasekhar limit is close to universal, the associated explosions are essentially of equal brightness (and some of the intrinsic variability can actually be accounted for by following the evolution of the brightness through time). It was measurements of SNe Ia at redshifts  $z > 0.3$  that provided the first direct evidence for a nonzero cosmological constant; these observations imply that  $\Omega_\Lambda$  is actually larger than  $\Omega_M$ . Recall that matter is pressureless,  $p_M = 0$ , whereas vacuum energy is associated with a negative pressure,  $p_\Lambda = -\rho_\Lambda$ . Plugging into the second Friedmann equation (8.68) we find that a

universe with both matter and  $\Lambda$  obeys

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_M - 2\rho_\Lambda). \quad (8.155)$$

Thus, if  $\rho_\Lambda$  is sufficiently large compared to  $\rho_M$  (as the supernova observations indicate), we can have  $\ddot{a} > 0$ , an accelerating universe (in the sense described in Section 8.4).

The matter density itself is measured by a variety of methods, often involving measuring the density  $\rho_M$  by looking for the gravitational effects of clustered matter and then extrapolating to large scales. Because  $\rho_M = (3H^2/8\pi G)\Omega_M$ , limits obtained in this way are often quoted in terms of  $\Omega_M h^2$ , where  $h$  is defined in (8.70). These days the uncertainty on  $H_0$  appears to be small enough that it is fairly safe to take  $h^2 \approx 0.5$ , which we do henceforth. Most contemporary methods are consistent with the result

$$\Omega_{M0} = 0.3 \pm 0.1. \quad (8.156)$$

Before there was good evidence for a cosmological constant, this low matter density was sometimes taken as an indication that space was negatively curved,  $\kappa < 0$ .

In addition to matter and cosmological constant, we also have radiation in the universe. Ordinary photons are the most obvious component of the radiation density, but any relativistic particle would contribute. For photons, most of the energy density resides in the cosmic microwave background, the leftover radiation from the Big Bang. Besides photons, the only obvious candidates for a radiation component are neutrinos. We expect that the number density of relic background neutrinos is comparable to that of photons; the photon density is likely to be somewhat larger, as photons can still be created after the number of neutrinos has become fixed. However, if the mass of the neutrinos is sufficiently large (greater than about  $10^{-4}$  eV), they will have become nonrelativistic today, and contribute to matter rather than to radiation. Current ideas about neutrino masses suggest that this probably is the case, but it is not perfectly clear. Furthermore, it is conceivable that there are as-yet-undetected massless particles in addition to the ones we know about (although they can't be too abundant, or they would suppress the formation of large-scale structure.) Altogether, it seems likely that the total radiation density is of the same order of magnitude as the photon density; in this case we would have

$$\Omega_{R0} \sim 10^{-4}. \quad (8.157)$$

As mentioned before, it is not surprising that the radiation density is lower than the matter density, as the former decays more rapidly as the universe expands. The radiation density goes as  $a^{-4}$ , while that in matter goes as  $a^{-3}$ ; so the epoch of matter-radiation equality occurred at a redshift

$$z_{eq} \approx \frac{\Omega_{M0}}{\Omega_{R0}} \sim 3 \times 10^3. \quad (8.158)$$

A further crucial constraint on the cosmological parameters comes from anisotropies in the temperature of the microwave background. The average temperature is  $T_{\text{CMB}} = 2.74\text{K}$ , but in 1992 the COBE satellite discovered fluctuations from place to place at a level of  $\Delta T/T \sim 10^{-5}$ . These anisotropies arise from a number of sources, including gravitational redshift/blueshift from photons moving out of potential wells at recombination (the Sachs-Wolfe effect, dominant on large angular scales), intrinsic temperature fluctuations at the surface of last scattering (dominant on small angular scales), and the Doppler effect from motions of the plasma. The physics describing the evolution of CMB anisotropies is outside the scope of this book. A map of the CMB temperature over the entire sky clearly contains a great deal of information, but no theory predicts what the temperature at any given point is supposed to be. Instead, modern theories generally predict the expectation value of the amount of anisotropy on any given angular scale. We therefore decompose the anisotropy field into spherical harmonics,

$$\frac{\Delta T}{T}(\theta, \phi) = \sum_{lm} a_{lm} Y_{lm}(\theta, \phi). \quad (8.159)$$

The expectation value of  $|a_{lm}|^2$  is likely to be independent of  $m$ ; otherwise the statistical characteristics of the anisotropy will change from place to place on the sky (although we should keep an open mind). The relevant parameters to be measured are therefore

$$C_l = \langle |a_{lm}|^2 \rangle. \quad (8.160)$$

Since for any fixed  $l$ , there are  $2l + 1$  possible values of  $m$  (from  $-l$  to  $l$ ), at all but the lowest  $l$ 's there are enough independent measurements of the  $a_{lm}$ 's to accurately determine their expectation values. The irreducible uncertainty at very small  $l$  is known as cosmic variance.

Numerous experiments have measured the  $C_l$ 's (the so-called CMB power spectrum), and improving these measurements is likely to be an important task for a number of years. (In addition to the temperature anisotropy, a great deal of information is contained in the polarization of the CMB, which is another target of considerable experimental effort.) To turn these observations into useful information, we need a specific theory to predict the CMB power spectrum as a function of the cosmological parameters. There are two leading possibilities (although one is much more leading than the other): either density perturbations are imprinted on all scales at extremely early times even modes for which the physical wavelength  $\lambda$  was much larger than the Hubble radius  $H^{-1}$ , or local dynamical mechanisms act as sources for anisotropies at all epochs. The latter possibility has essentially been ruled out by the CMB data; if anisotropies are produced continuously, we expect a relatively smooth, featureless spectrum of  $C_l$ 's, whereas the observations indicate a significant amount of structure. It is therefore much more popular to imagine a primordial source of perturbations, such as inflation (discussed in the next section). Inflationary perturbations are adiabatic—perturbations in the mat-

ter density are correlated with those in the radiation density—and of nearly equal magnitude at all scales. With this input, we can make definite predictions for the  $C_l$ 's as a function of all the cosmological parameters. Perhaps the most significant constraint obtained from experiments thus far is that universe is spatially flat, or nearly so;  $|\Omega_{\text{col}}| < 0.1$ . Combined with the measurements of the matter density  $\Omega_M \approx 0.3$ , we conclude that the vacuum energy density parameter should be

$$\Omega_{\Lambda 0} = 0.7 \pm 0.1. \quad (8.161)$$

This is nicely consistent with the Type Ia supernova results described above; the concordance picture described here is that indicated in Figure 8.4. Converting from density parameter to physical energy density using  $H_0 = 70 \text{ km/sec/Mpc}$  yields

$$\rho_{\text{vac}} \approx 10^{-8} \text{ erg/cm}^3, \quad (8.162)$$

as mentioned in our discussion of vacuum energy in Section 4.5.

One more remarkable feature completes our schematic picture of the present-day universe. We have mentioned that about 30% of the energy density in our universe consists of matter. But to a cosmologist, “matter” is any collection of nonrelativistic particles; the matter we infer from its gravitational influence need not be the same kind of ordinary matter we are familiar with from our experience on Earth. By **ordinary matter** we mean anything made from atoms and their constituents (protons, neutrons, and electrons); this would include all of the stars, planets, gas, and dust in the universe, immediately visible or otherwise. Occasionally such matter is referred to as *baryonic* matter, where baryons include protons, neutrons, and related particles (strongly interacting particles carrying a conserved quantum number known as baryon number). Of course electrons are conceptually an important part of ordinary matter, but by mass they are negligible compared to protons and neutrons:

$$\begin{aligned} m_p &= 0.938 \text{ GeV} \\ m_n &= 0.940 \text{ GeV} \\ m_e &= 0.511 \times 10^{-3} \text{ GeV}. \end{aligned} \quad (8.163)$$

In other words, the mass of ordinary matter comes overwhelmingly from baryons.

Ordinary baryonic matter, it turns out, is not nearly enough to account for the observed density  $\Omega_M \approx 0.3$ . Our current best estimates for the baryon density yield

$$\Omega_b = 0.04 \pm 0.02, \quad (8.164)$$

where these error bars are conservative by most standards. This determination comes from a variety of methods: direct counting of baryons (the least precise method), consistency with the CMB power spectrum (discussed above), and agreement with the predictions of the abundances of light elements for Big-Bang nucleosynthesis (discussed below). Most of the matter density must therefore be

in the form of **nonbaryonic dark matter**, which we will abbreviate to simply “dark matter.” (Baryons can be dark, but it is increasingly common to reserve the terminology for the nonbaryonic component.) Essentially every known particle in the Standard Model of particle physics has been ruled out as a candidate for this dark matter. Fortunately, there are a number of plausible candidates beyond the Standard Model, including neutralinos (the lightest of the additional stable particles predicted by supersymmetry, with masses  $\geq 100$  GeV) and axions (light pseudoscalar particles arising from spontaneous breakdown of a hypothetical Peccei-Quinn symmetry invoked to explain conservation of CP in the strong interactions, with masses  $\sim 10^{-4}$  eV). One of the few things we know about the dark matter is that it must be cold—not only is it nonrelativistic today, but it must have been that way for a very long time. If the dark matter were hot, it would have free-streamed out of overdense regions, suppressing the formation of galaxies. The other thing we know about cold dark matter (CDM) is that it should interact very weakly with ordinary matter, so as to have escaped detection thus far. Nevertheless, ambient dark matter particles may occasionally scatter off carefully shielded detectors in terrestrial laboratories; the attempt to directly detect dark matter by searching for the effects of such scatterings will be another significant experimental effort in the years to come.

The picture in which  $\Omega_M = 0.3$  and  $\Omega_\Lambda = 0.7$  seems to fit an impressive variety of observational data. The most surprising part of the picture is the cosmological constant. In Chapter 4 we mentioned that a naïve estimate of the vacuum energy yields a result many orders of magnitude larger than what has been measured. In fact there are three related puzzles: Why is the cosmological constant so much smaller than we expect? What is the origin of the small nonzero energy that comprises 70% of the current universe? And, why is the current value of the vacuum energy of the same order of magnitude as the matter density? The last problem is especially severe, as the vacuum energy and matter density evolve rapidly with respect to each other:

$$\frac{\Omega_\Lambda}{\Omega_M} \propto a^3. \quad (8.165)$$

If  $\Omega_M$  and  $\Omega_\Lambda$  are comparable today, in the past the vacuum energy would have been undetectably small, while in the future the matter density will be negligible. This “coincidence problem” has thus far proven to be a complete mystery. One suggested solution involves the “anthropic principle.” If there are many distinct parts of the universe (in space, or even in branches of the wavefunction) in which the cosmological constant takes on very different values, intelligent life is most likely to arise in those places where the absolute magnitude is not too large—a large positive  $\Lambda$  would tear particles apart before galaxies could form, while a large negative  $\Lambda$  would cause the universe to recollapse before life could evolve. The anthropic explanation of the observed vacuum energy provides a good fit to the data, although the need to invoke such an elaborate scheme to explain this one quantity strikes some as slightly extravagant.

Another possibility that may (or may not) bear on the coincidence problem is the idea that we have not detected a nonzero cosmological constant, but rather a dynamical component that closely mimics the properties of vacuum energy. Consideration of this possibility has led cosmologists to coin the term **dark energy** to describe whatever it is that has been detected, whether it is dynamical or turns out to be a cosmological constant after all. What we know about the dark energy is that it is relatively smoothly distributed through space (or it would have been detected through its local gravitational field, just like dark matter) and is evolving slowly with time (or it would not make the universe accelerate, as indicated by the supernova data). A simple candidate for a dynamical source of dark energy is provided by a slowly-rolling scalar field. Consider a field  $\phi$  with the usual action

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right], \quad (8.166)$$

for which the energy-momentum tensor is

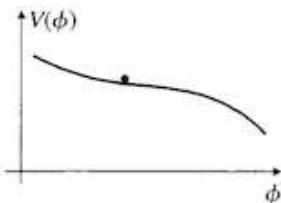
$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi + \left[ \frac{1}{2} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - V(\phi) \right] g_{\mu\nu} \quad (8.167)$$

and the equation of motion is

$$\square \phi - \frac{dV}{d\phi} = 0. \quad (8.168)$$

Assume that the field is completely homogeneous through space ( $\partial_i \phi = 0$ ). Then using the Christoffel symbols (8.44), we may express the d'Alembertian in terms of time derivatives and the Hubble constant to write (8.168) as

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \quad (8.169)$$



**FIGURE 8.6** Potential energy for a slowly-rolling scalar field.

We see that the Hubble parameter acts as a friction term; the field will tend to roll down the potential, but when  $H$  is too large the motion will be damped. Therefore, a scalar field with a sufficiently shallow potential (as portrayed in Figure 8.6) will roll very slowly, leading to a kinetic energy much smaller than the potential energy  $V(\phi)$ . The energy-momentum tensor is then

$$T_{\mu\nu} \approx -V(\phi)g_{\mu\nu}, \quad (8.170)$$

where  $\phi \approx \text{constant}$ . Comparing to (4.96), we see that the scalar field potential is mimicking a vacuum energy. As a simple example consider a quadratic potential,  $V(\phi) = \frac{1}{2}m^2\phi^2$ . Then (8.169) describes a damped harmonic oscillator, and overdamping will occur if  $H > m$ . But in particle-physics units, the Hubble constant today is  $H_0 \approx 10^{-33}$  eV, so the mass of this scalar field would have to be incredibly tiny compared to the masses of the familiar elementary particles in equation (8.163). This seems to be an unnatural fine-tuning. Nevertheless, models

of dynamical dark energy are being actively explored, partially in the hope that they will lead somehow to a solution of the coincidence problem.

With this view of the contemporary situation, we can imagine what the early universe must have been like to have produced what we see today. For purposes of physical intuition it is often more helpful to keep track of the era under consideration by indicating the temperature rather than the redshift or time since the Big Bang. The temperature today is

$$T_0 = 2.74 \text{ K} = 2.4 \times 10^{-4} \text{ eV}. \quad (8.171)$$

Of course, by “temperature” we mean the apparent blackbody temperature of the cosmic microwave background; in fact the CMB has not been in thermal equilibrium since recombination, so one should be careful in taking this concept too literally. Under adiabatic expansion, the temperature decreases as each relativistic particle redshifts, and we have  $T \propto a^{-1}$ . But there will be nonadiabatic phase transitions at specific moments in the early universe; in such circumstances the temperature doesn’t actually increase, but decreases more gradually. To help relate the temperature, density, and scale factor, we introduce two different measures of the **effective number of relativistic degrees of freedom**:  $g_*$  and  $g_{*S}$  (where  $S$  stands for entropy). Consider a set of bosonic and fermionic species, each with their own effective temperature  $T_i$ , and number of spin states  $g_i$ . For example, a massless photon has two spin states, so  $g_\gamma = 2$ ; a massive spin- $\frac{1}{2}$  fermion also has two spin states, so  $g_{e^-} = g_{e^+} = 2$ . The two different versions of the effective number of relativistic degrees of freedom obey

$$g_* = \sum_{\text{bosons}} g_i \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{\text{fermions}} g_i \left( \frac{T_i}{T} \right)^4 \quad (8.172)$$

and

$$g_{*S} = \sum_{\text{bosons}} g_i \left( \frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left( \frac{T_i}{T} \right)^3. \quad (8.173)$$

The mysterious factors of  $\frac{7}{8}$  arise from the difference between Bose and Fermi statistics when calculating the equilibrium distribution function. For any species in thermal equilibrium, the temperature  $T_i$  will be equal to the background temperature  $T$ ; but we might have decoupled species at a lower temperature, which contribute less to the effective number of relativistic degrees of freedom. The reason why we need to define two different measures is that they play different roles; the first relates the temperature to the energy density (in relativistic species) via

$$\rho_R = \frac{\pi^2}{30} g_* T^4, \quad (8.174)$$

while the second relates the temperature to the scale factor,

$$T \propto g_{*S}^{-1/3} a^{-1}, \quad (8.175)$$

In fact,  $g_*$  and  $g_{*S}$  are expected to be approximately equal so long as the relativistic degrees of freedom are those of the Standard Model of particle physics. A very rough guide is given by

$$g_* \approx g_{*S} \sim \begin{cases} 100 & T > 300 \text{ MeV} \\ 10 & 300 \text{ MeV} > T > 1 \text{ MeV} \\ 3 & T < 1 \text{ MeV}. \end{cases} \quad (8.176)$$

As we will discuss shortly, the events that change the effective number of relativistic degrees of freedom are the QCD phase transition at 300 MeV, and the annihilation of electron/positron pairs at 1 MeV.

With this background, let us consider the evolution of the universe from early times to today. To begin we imagine a Robertson–Walker metric with matter fields in thermal equilibrium at a temperature of  $1 \text{ TeV} = 1000 \text{ GeV}$ . The high-temperature plasma is a complicated mixture of elementary particles (quarks, leptons, gauge and Higgs bosons). The dominant form of energy density will be relativistic particles, so the early universe is radiation-dominated. It is also very close to flat, since the curvature term in the Friedmann equation evolves more slowly than the matter and radiation densities. The Friedmann equation is therefore

$$\begin{aligned} H^2 &= \frac{8\pi G}{3} \rho_R \\ &\approx 0.1 g_* \frac{T^4}{\bar{m}_P^2}, \end{aligned} \quad (8.177)$$

where the reduced Planck scale is  $\bar{m}_P = (8\pi G)^{-1/2} \approx 10^{18} \text{ GeV}$ . If the radiation-dominated phase extends back to very early times, the age of the universe will be approximately  $t \sim H^{-1}$ , or

$$t \sim \frac{\bar{m}_P}{T^2}. \quad (8.178)$$

In conventional units this becomes

$$t \sim 10^{-6} \left( \frac{\text{GeV}}{T} \right)^2 \text{ sec.} \quad (8.179)$$

Current experiments at particle accelerators have provided an accurate picture of what physics is like up to perhaps 100 GeV, so an additional order of magnitude is within the realm of reasonable extrapolation. At higher temperatures we are less sure what happens; there might be nothing very interesting between 1 TeV and the Planck scale, or this regime could be filled with all manner of surprises. Of course it is also conceivable that cosmology provides surprises at even lower temperatures, even though the Standard Model physics is well understood; in this section we are describing a conservative scenario, but as always it pays to keep an open mind.

A crucial feature of the Standard Model is the spontaneously broken symmetry of the electroweak sector. In cosmology, this symmetry breaking occurs at the electroweak phase transition, at  $T \sim 200$  GeV. Above this temperature the symmetry is unbroken, so that elementary fermions (quarks and leptons) and the weak interaction gauge bosons are all massless, while below this temperature we have the pattern of masses familiar from low-energy experiments. The electroweak phase transition is not expected to leave any discernible impact on the late universe; one possible exception is baryogenesis, discussed below.

At these temperatures the strong interactions described by quantum chromodynamics (QCD) are not so strong. At low energies/temperatures, QCD exhibits “confinement”—quarks and gluons are bound into composite particles such as baryons and mesons. But above the QCD scale  $\Lambda_{\text{QCD}} \sim 300$  MeV, quarks and gluons are free particles. As the universe expands and cools, the confinement of strongly-interacting particles into bound states is responsible for the first drop in the effective number of relativistic degrees of freedom noted in (8.176). The QCD phase transition is not expected to leave a significant imprint on the observable universe.

Just as the strong interactions are not very strong at high temperatures, the weak interactions are not as weak as you might think; they are still weak in the sense of being accurately described by perturbation theory, but they occur rapidly enough to keep weakly-interacting particles, such as neutrinos, in thermal equilibrium. This ceases to be the case when  $T \sim 1$  MeV. This is also approximately the temperature at which electrons and positrons become nonrelativistic and annihilate, decreasing the effective number of relativistic degrees of freedom, but the two events are unrelated. For temperatures below 1 MeV, we say the weak interactions are “frozen out”—the interaction rate drops below the expansion rate of the universe, so interactions happen too infrequently to keep particles in equilibrium. It may be the case that cold dark matter particles decouple from the plasma at this temperature. More confidently, we can infer that neutrons and protons cease to interconvert. The equilibrium abundance of neutrons at this temperature is about  $\frac{1}{6}$  the abundance of protons (due to the slightly larger neutron mass). The neutrons have a finite lifetime ( $\tau_n = 890$  sec) that is somewhat larger than the age of the universe at this epoch,  $t(1 \text{ MeV}) \approx 1$  sec, but they begin to gradually decay into protons and leptons. Soon thereafter, however, we reach a temperature somewhat below 100 keV, and **Big-Bang Nucleosynthesis** (BBN) begins.

The nuclear binding energy per nucleon is typically of order 1 MeV, so you might expect that nucleosynthesis would occur earlier; however, the large number of photons per nucleon prevents nucleosynthesis from taking place until the temperature drops below 100 keV. At that point the neutron/proton ratio is approximately  $\frac{1}{7}$ . Of all the light nuclei, it is energetically favorable for the nucleons to reside in  ${}^4\text{He}$ , and indeed that is what most of the free neutrons are converted into; for every two neutrons and fourteen protons, we end up with one helium nucleus and twelve protons. Thus, about 25% of the baryons by mass are converted to helium. In addition, there are trace amounts of deuterium (approximately  $10^{-5}$  deuterons per proton),  ${}^3\text{He}$  (also  $\sim 10^{-5}$ ), and  ${}^7\text{Li}$  ( $\sim 10^{-10}$ ).

Of course these numbers are predictions, which are borne out by observations of the primordial abundances of light elements. (Heavier elements are not synthesized in the Big Bang, but require stellar processes in the later universe.) We have glossed over numerous crucial details, especially those that explain how the different abundances depend on the cosmological parameters. For example, imagine that we deviate from the Standard Model by introducing more than three light neutrino species. This would increase the radiation energy density at a fixed temperature through (8.174), which in turn decreases the timescales associated with a given temperature (since  $t \sim H^{-1} \propto \rho_R^{-1/2}$ ). Nucleosynthesis would therefore happen somewhat earlier, resulting in a higher abundance of neutrons, and hence in a larger abundance of  $^4\text{He}$ . Observations of the primordial helium abundance, which are consistent with the Standard Model prediction, provided the first evidence that the number of light neutrinos is close to three. Similarly, all of the temperatures and timescales associated with nucleosynthesis depend on the baryon-to-photon ratio; agreement with the observed abundances requires that there be approximately  $5 \times 10^{-10}$  baryons per photon, which is the origin of the estimate (8.164) of the baryonic density parameter, and the associated need for nonbaryonic dark matter.

For our present purposes, perhaps the most profound feature of primordial nucleosynthesis is its sensitive dependence on the Friedmann relation between temperature and expansion rate, and hence on Einstein's equation. The success of BBN provides a stringent test of GR in a regime very far from our everyday experience. The fact that Einstein's theory, derived primarily from a need to reconcile gravitation with invariance under the Lorentz symmetries of electromagnetism, successfully describes the expansion of the universe when it was only one second old is a truly impressive accomplishment. To this day, BBN provides one of the most powerful constraints on alternative theories of gravity; in particular, it is the earliest epoch about which we have any direct observational signature.

Subsequent to nucleosynthesis, we have a plasma dominated by protons, electrons, and photons, with some helium and other nuclei. There is also dark matter, but it is assumed not to interact with the ordinary matter by this epoch. The next important event isn't until **recombination**, when electrons combine with protons (they combine with helium slightly earlier). Recombination happens at a temperature  $T \approx 0.3$  eV; at this point the universe is matter-dominated. Again, since the binding energy of hydrogen is 13.6 eV, you might expect recombination to occur earlier, but the large photon/baryon ratio delays it. The crucial importance of recombination is that it marks the epoch at which the universe becomes transparent. The ambient photons interact strongly with free electrons, so that the photon mean free path is very short prior to recombination, but it becomes essentially infinite once the electrons and protons combine into neutral hydrogen. These ambient photons are visible today as the cosmic microwave background, which provides a snapshot of the universe at  $T \approx 0.3$  eV, or a redshift  $z \approx 1200$ . Recombination is a somewhat gradual process, so any specification of when it happens is necessarily approximate.

Subsequent to recombination, the universe passes through a long period known as the “dark ages,” as galaxies are gradually assembled through gravitational instability, but there are as yet no visible stars to light up the universe. The dark ages are a mysterious time; the processes by which stars and galaxies form are highly complicated and nonlinear, and new kinds of observations will undoubtedly be necessary before this era is well understood.

Our story has now brought us to the present day, but there are a couple of missing points we should go back and fill in. One is the asymmetry between matter and antimatter in the universe. Essentially all of the visible matter in the universe seems to be composed of protons, neutrons, and electrons, rather than their antiparticles; if distant galaxies were primarily antimatter, we would expect to observe high-energy photons from the occasional annihilation of protons with antiprotons at the boundaries of the matter/antimatter domains. While it is possible to build in an asymmetry as an initial condition, this seems somehow unsatisfying, and most physicists would prefer to find a dynamical mechanism of baryogenesis by which an initially matter/antimatter symmetric state could evolve into our present universe. Such broken symmetries are common in particle physics, and indeed numerous mechanisms for baryogenesis have been proposed (generally at temperatures at or above the electroweak scale). None of these specific schemes, however, has proven sufficiently compelling to be adopted as a standard scenario. It seems probable that we will need a better understanding of physics beyond the Standard Model to understand the origin of the baryon asymmetry.

The other missing feature we need to mention is that the universe is not, of course, perfectly homogeneous and isotropic; the current large-scale structure in the universe seems to have evolved from adiabatic and nearly scale-free perturbations present at very early times at the level of  $\delta\rho/\rho \sim 10^{-5}$ . Evidence for the adiabatic and scale-free nature of these perturbations comes from a combination of observations of the CMB and large-scale structure. Both the high degree of isotropy and homogeneity, and the small deviations therefrom, are simply imposed as mysterious initial conditions in the conventional cosmology. A possible dynamical origin for both is provided by the inflationary-universe scenario, to which we now turn.

## 8.8 ■ INFLATION

In the conventional understanding of the Big-Bang model, the universe is taken to be radiation-dominated at early times and matter-dominated at late times, with, as we now suspect, a very late transition to vacuum-domination. This picture has met with great success in describing a wide variety of observational data; nevertheless, we may still ask whether the initial conditions giving rise such a universe seem natural. This is the kind of question one might ask in cosmology but not in other sciences. Typically, as physicists we look for laws of nature, and imagine that we are free to specify initial conditions and ask how they evolve under such laws. But the universe seems to have only one set of initial conditions, so it

seems sensible to wonder if they are relatively generic or finely-tuned. Within the conventional picture, the early universe is indeed finely tuned to incredible precision. In particular, two features of our universe seem highly nongeneric: its spatial flatness, and its high degree of isotropy and homogeneity. It might be that this is just the universe we are stuck with, and it makes no sense to ask about the likelihood of different initial conditions. Alternatively, it might be that these conditions are more likely than they appear at first, if there is some dynamical mechanism that can take a wide spectrum of initial conditions and evolve them toward flatness and homogeneity/isotropy. The inflationary universe scenario provides such a mechanism (and more, besides), and has become a central organizing principle of modern cosmology, even if we are still far from demonstrating its truth.

Before describing inflation, let's describe the two problems of unnaturalness it claims to solve: the flatness problem and the horizon problem associated with homogeneity/isotropy. The **flatness problem** comes from considering the Friedmann equation in a universe with matter and radiation but no vacuum energy, which for later convenience we write in terms of the reduced Planck mass  $\bar{m}_P = (8\pi G)^{-1/2}$  as

$$H^2 = \frac{1}{3\bar{m}_P^2}(\rho_M + \rho_R) - \frac{\kappa}{a^2}. \quad (8.180)$$

The curvature term  $-\kappa/a^2$  is proportional to  $a^{-2}$  (obviously), while the energy density terms fall off faster with increasing scale factor,  $\rho_M \propto a^{-3}$  and  $\rho_R \propto a^{-4}$ . This raises the question of why the ratio  $(\kappa a^{-2})/(\rho/3\bar{m}_P^2)$  isn't much larger than unity, given that  $a$  has increased by a factor of perhaps  $10^{30}$  since the Planck epoch. Said another way, the point  $\Omega = 1$  is a repulsive fixed point in a matter/radiation dominated universe—any deviation from this value will grow with time, so why do we observe  $\Omega \sim 1$  today?

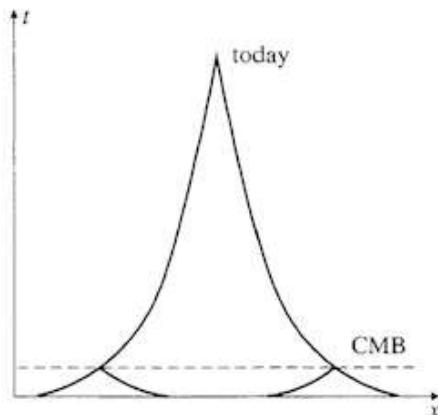
The **horizon problem** stems from the existence of particle horizons in FRW cosmologies, as illustrated in Figure 8.7. Horizons exist because there is only a finite amount of time since the Big Bang singularity, and thus only a finite distance that photons can travel within the age of the universe, as we briefly discussed in Chapter 2. Consider a photon moving along a radial trajectory in a flat universe (the generalization to nonflat universes is straightforward). A radial null path obeys

$$0 = ds^2 = -dt^2 + a^2 dr^2, \quad (8.181)$$

so the comoving (coordinate) distance traveled by such a photon between times  $t_1$  and  $t_2$  is

$$\Delta r = \int_{t_1}^{t_2} \frac{dt}{a(t)}. \quad (8.182)$$

To get the physical distance as it would be measured by an observer at any time  $t$ , simply multiply by  $a(t)$ . For simplicity let's imagine we are in a matter-dominated



**FIGURE 8.7** Past light cones in a universe expanding from a Big Bang singularity, illustrating particle horizons in cosmology. Points at recombination, observed today as parts of the cosmic microwave background on opposite sides of the sky, have nonoverlapping past light cones (in conventional cosmology); no causal signal could have influenced them to have the same temperature.

universe, for which

$$a = \left( \frac{t}{t_0} \right)^{2/3}. \quad (8.183)$$

Remember  $a_0 = 1$ . The Hubble parameter is therefore given by

$$\begin{aligned} H &= \frac{2}{3} t^{-1} \\ &= a^{-3/2} H_0. \end{aligned} \quad (8.184)$$

Then the photon travels a comoving distance

$$\Delta r = 2H_0^{-1} (\sqrt{a_2} - \sqrt{a_1}). \quad (8.185)$$

The comoving horizon size at any fixed value of the scale factor  $a = a_*$  is the distance a photon travels since the Big Bang,

$$r_{\text{hor}}(a_*) = 2H_0^{-1} \sqrt{a_*}. \quad (8.186)$$

The physical horizon size, as measured on the spatial hypersurface at  $a_*$ , is therefore simply

$$d_{\text{hor}}(a_*) = a_* r_{\text{hor}}(a_*) = 2H_*^{-1}. \quad (8.187)$$

Indeed, for any nearly-flat universe containing a mixture of matter and radiation, at any one epoch we will have

$$d_{\text{hor}}(a_*) \sim H_*^{-1} = d_H(a_*), \quad (8.188)$$

where the Hubble distance  $d_H$  was introduced in (8.71). This approximate equality leads to a strong temptation to use the terms “horizon distance” and “Hubble distance” interchangeably; this temptation should be resisted, since inflation can render the former much larger than the latter, as we will soon demonstrate.

The horizon problem is simply the fact that the CMB is isotropic to a high degree of precision, even though widely separated points on the last scattering surface are completely outside each others’ horizons. When we look at the CMB we are observing the universe at a scale factor  $a_{\text{CMB}} \approx 1/1200$ ; from (8.185), the comoving distance between a point on the CMB and an observer on Earth is

$$\begin{aligned}\Delta r &= 2H_0^{-1} (1 - \sqrt{a_{\text{CMB}}}) \\ &\approx 2H_0^{-1}.\end{aligned}\quad (8.189)$$

However, the comoving horizon distance for such a point is

$$\begin{aligned}r_{\text{hor}}(a_{\text{CMB}}) &= 2H_0^{-1} \sqrt{a_{\text{CMB}}} \\ &\approx 6 \times 10^{-2} H_0^{-1}.\end{aligned}\quad (8.190)$$

Hence, if we observe two widely-separated parts of the CMB, they will have nonoverlapping horizons; distinct patches of the CMB sky were causally disconnected at recombination. Nevertheless, they are observed to be at the same temperature to high precision. The question then is, how did they know ahead of time to coordinate their evolution in the right way, even though they were never in causal contact? We must somehow modify the causal structure of the conventional FRW cosmology.

Let’s consider modifying the conventional picture by positing a period of **inflation**: an era of acceleration ( $\ddot{a} > 0$ ) in the very early universe, driven by some component other than matter or radiation that redshifts away slowly as the universe expands. Then the flatness and horizon problems can be simultaneously solved. For simplicity consider the case where inflation is driven by a constant vacuum energy, leading to exponential expansion. Then, during the vacuum-dominated era,  $\rho/3\tilde{m}_p^2 \propto a^0$  grows rapidly with respect to  $-\kappa/a^2$ , so the universe becomes flatter with time ( $\Omega$  is driven to unity). If this process proceeds for a sufficiently long period, after which the vacuum energy is converted into matter and radiation, the density parameter will be sufficiently close to unity that it will not have had a chance to noticeably change into the present era. The horizon problem, meanwhile, can be traced to the fact that the physical distance between any two comoving objects grows as the scale factor, while the physical horizon size in a matter- or radiation-dominated universe grows more rapidly, as  $d_{\text{hor}} \sim a^{n/2} H_0^{-1}$ . This can again be solved by an early period of exponential expansion, in which the true horizon size grows to a fantastic amount, so that our horizon today is actually much larger than the naïve estimate that it is equal to the Hubble radius  $H_0^{-1}$ .

In fact, a truly exponential expansion is not necessary; for any accelerated expansion, the spatial curvature will diminish with respect to the energy density,

and the horizon distance will grow rapidly. Typically we require that this accelerated period be sustained for 60 or more *e*-folds (where the number of *e*-folds is  $N = \Delta \ln a$ ) which is what is needed to solve the horizon problem. It is easy to overshoot, and inflation generally makes the present-day universe spatially flat to incredible precision.

Now let's consider how we can get an inflationary phase in the early universe. The most straightforward way is to use the vacuum energy provided by the potential of a scalar field, the **inflaton**. Imagine a universe dominated by the energy of a spatially homogeneous scalar. The relevant equations of motion are precisely those of our discussion of dynamical dark energy in Section 8.7; the only difference is that the energy scale of inflation is much higher. We have the equation of motion for a scalar field in an RW metric,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad (8.191)$$

as well as the Friedmann equation,

$$H^2 = \frac{1}{3\tilde{m}_P^2} \left( \frac{1}{2}\dot{\phi}^2 + V(\phi) \right). \quad (8.192)$$

We have ignored the curvature term, since inflation will flatten the universe anyway. Inflation can occur if the evolution of the field is sufficiently gradual that the potential energy dominates the kinetic energy, and the second derivative of  $\phi$  is small enough to allow this state of affairs to be maintained for a sufficient period. Thus, we want

$$\begin{aligned} \dot{\phi}^2 &\ll V(\phi), \\ |\ddot{\phi}| &\ll |3H\dot{\phi}|, |V'|. \end{aligned} \quad (8.193)$$

Satisfying these conditions requires the smallness of two dimensionless quantities known as **slow-roll parameters**:

$$\begin{aligned} \epsilon &= \frac{1}{2}\tilde{m}_P^2 \left( \frac{V'}{V} \right)^2, \\ \eta &= \tilde{m}_P^2 \left( \frac{V''}{V} \right). \end{aligned} \quad (8.194)$$

Note that  $\epsilon \geq 0$ , while  $\eta$  can have either sign. Note also that these definitions are not universal; some people like to define them in terms of the Hubble parameter rather than the potential. Our choice describes whether a field has a chance to roll slowly for a while; the description in terms of the Hubble parameter describes whether the field actually is rolling slowly. When both of these quantities are small we can have a prolonged inflationary phase. They are not sufficient, however; no matter what the potential looks like, we can always choose initial conditions with  $|\dot{\phi}|$  so large that slow-roll is never applicable. However, most initial conditions are attracted to an inflationary phase if the slow-roll parameters are small.

It isn't hard to invent potentials that satisfy the slow-roll conditions. Consider perhaps the simplest possible example,<sup>3</sup>  $V(\phi) = \frac{1}{2}m^2\phi^2$ . In this case

$$\epsilon = \eta = \frac{2\bar{m}_P^2}{\phi^2}. \quad (8.195)$$

Clearly, for large enough  $\phi$ , we can get the slow-roll parameters to be as small as we like. However, we have the constraint that the energy density should not be as high as the Planck scale, so that our classical analysis makes sense; this implies  $\phi \ll \bar{m}_P^2/m$ . If we start the field at a value  $\phi_i$ , the number of  $e$ -folds before inflation ends (that is, before the slow-roll parameters become of order unity) will be

$$\begin{aligned} N &= \int_{t_i}^{t_e} H dt \\ &\approx -\bar{m}_P^{-2} \int_{\phi_i}^{\phi_e} \frac{V}{V'} d\phi \\ &\approx \frac{\phi_i^2}{4\bar{m}_P^2} - \frac{1}{2}. \end{aligned} \quad (8.196)$$

The first equality is always true, the second uses the slow-roll approximation, and the third is the result for this particular model. To get 60  $e$ -folds we therefore need  $\phi_i > 16\bar{m}_P$ . Together with the upper limit on the energy density, we find that there is an upper limit on the mass parameter,  $m \ll \bar{m}_P/16$ . In fact the size of the observed density fluctuations puts a more stringent upper limit on  $m$ , as we will discuss below. But there is no lower limit on  $m$ , so it is easy to obtain appropriate inflationary potentials only if we are willing to posit large hierarchies  $m \ll \bar{m}_P$ , or equivalently a small dimensionless number  $m/\bar{m}_P$ . Going through the same exercise with a  $\lambda\phi^4$  potential would have yielded a similar conclusion, that  $\lambda$  would have had to be quite small; we often say that the inflaton must be weakly coupled. Of course, there is a sense in which we are cheating, since for field values  $\phi > \bar{m}_P$  we should expect additional terms in the effective potential, of the form  $\bar{m}_P^{4-n}\phi^n$  with  $n > 4$ , to become important. So in a *realistic* model it can be quite hard to get an appropriate potential.

At some point inflation ends, and the energy in the inflaton potential is converted into a thermalized gas of matter and radiation, a process known as “reheating.” A proper understanding of the reheating process is of utmost importance, as it controls the production of various relics that we may or may not want in our universe. For example, one important beneficial aspect of inflation is that it can “inflate away” various relics that could be produced in the early universe, but are not observed today. A classic example occurs in the context of grand unified theories of particle physics, which generically predict the existence of super-

<sup>3</sup>We follow the exposition in A.R. Liddle, “An Introduction to Cosmological Inflation,” <http://arxiv.org/astro-ph/9901124>.

heavy magnetic monopoles, with an abundance many orders of magnitude greater than allowed by observations. Historically, the monopole problem was the primary motivation for the invention of inflation by Guth; solutions to the flatness and horizon problems were considered a bonus. Inflation can dilute the monopole abundance appropriately, but they will be produced anew if the universe reheats to above the temperature of the grand-unification phase transition; fortunately, this is not a stringent constraint on most models. Similar considerations apply to other unwanted relics; in supersymmetric models, an especially worrisome problem is raised by the abundance of gravitinos (supersymmetric partners of the graviton). At the same time, it is necessary to reheat to a sufficiently high temperature to allow for some sort of baryogenesis scenario. For any specific implementation of inflation within a particle-physics model, it is crucial to check that unwanted relics are dispersed while wanted relics (such as baryons) are preserved.

A crucial element of inflationary scenarios is the production of density perturbations, which may be the origin of the CMB temperature anisotropies and the large-scale structure in galaxies that we observe today. The idea behind density perturbations generated by inflation is fairly straightforward. Inflation will attenuate any ambient particle density rapidly to zero, leaving behind only the vacuum. But the vacuum state in an accelerating universe has a nonzero temperature, the Gibbons–Hawking temperature, analogous to the Hawking temperature of a black hole. We won’t be able to explore this subject in detail; here we simply outline the basic results.

For a universe dominated by a potential energy  $V$  the Gibbons–Hawking temperature is given by

$$T_{\text{GH}} = \frac{H}{2\pi} \sim \frac{V^{1/2}}{\hat{m}_P}. \quad (8.197)$$

Corresponding to this temperature are fluctuations in the inflaton field  $\phi$  at each wavenumber  $k$ , with magnitude

$$|\Delta\phi|_k = T_{\text{GH}}. \quad (8.198)$$

Since the potential is by hypothesis nearly flat, the fluctuations in  $\phi$  lead to small fluctuations in the energy density,

$$\delta\rho = V'(\phi)\delta\phi. \quad (8.199)$$

Inflation therefore produces density perturbations on every scale. The amplitude of the perturbations is nearly equal at each wavenumber, but there will be slight deviations due to the gradual change in  $V$  as the inflaton rolls. Describing the perturbations is a messy subject, involving countless different notations. A sensible place to start is root-mean-square (RMS) density fluctuation,

$$\left. \frac{\delta\rho}{\rho} \right|_{\text{rms}} = \sqrt{\left\langle \left( \frac{\delta\rho}{\rho} \right)^2 \right\rangle}. \quad (8.200)$$

where the angle brackets represent an average over spatial locations. For statistically isotropic perturbations (the expected amplitude is independent of direction), a bit of Fourier analysis allows us to write

$$\left(\frac{\delta\rho}{\rho}_{\text{rms}}\right)^2 = \int \Delta^2(k) d(\ln k), \quad (8.201)$$

where we have introduced the dimensionless power spectrum,

$$\Delta^2(k) \equiv \frac{k^3 |\delta_k|^2}{2\pi^2}, \quad (8.202)$$

and  $\delta_k$  is the expectation value of the Fourier transform of the fractional density perturbation,

$$\delta_{\mathbf{k}} = \frac{1}{(2\pi)^{3/2}} \int e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{\delta\rho}{\rho} d^3x, \quad (8.203)$$

which we've assumed to be isotropic. The dimensionless power spectrum is a function of time, as the amplitude for each mode evolves; it is most common to express the predictions of any specific model in terms of the amplitude of the perturbations at the moment when the physical wavelength of the mode,  $\lambda = a/k$ , is equal to the Hubble radius  $H^{-1}$ ,

$$A_S^2(k) \equiv \Delta^2(k) \Big|_{k=aH}. \quad (8.204)$$

Thus,  $A_S(k)$  measures the amplitude for different modes at different times. For inflation driven by a slowly-rolling scalar field,  $A_S(k)$  is related to the potential via

$$A_S^2(k) \sim \frac{V^3}{\bar{m}_P^6 (V')^2} \Bigg|_{k=aH} \sim \frac{V}{\bar{m}_P^4 \epsilon} \Bigg|_{k=aH}. \quad (8.205)$$

We have intentionally suppressed dimensionless numerical factors, which differ widely from reference to reference, in favor of highlighting the dependence on the potential.

The spectrum is given the subscript “S” because it describes scalar fluctuations in the metric. These are tied to the energy-momentum distribution, and the density fluctuations produced by inflation are adiabatic—fluctuations in the density of all species are correlated. The fluctuations are also Gaussian, in the sense that the phases of the Fourier modes describing fluctuations at different scales are uncorrelated. These aspects of inflationary perturbations—a nearly scale-free spectrum of adiabatic density fluctuations with a Gaussian distribution—are all consistent with current observations of the CMB and large-scale structure, and new data scheduled to be collected in years to come should greatly improve the precision of these tests.

It is not only the nearly-massless inflaton that is excited during inflation, but also any other nearly-massless particle. The other important example is the graviton, which corresponds to tensor perturbations in the metric (propagating excitations of the gravitational field). Tensor fluctuations have a spectrum

$$A_T^2(k) \sim \left. \frac{V}{m_P^4} \right|_{k=aH}. \quad (8.206)$$

Importantly, the tensor amplitude depends only on the potential, not on its derivatives; observations of tensor perturbations would therefore give direct information about the energy scale of inflation.

For purposes of understanding observations, it is useful to parameterize the perturbation spectra in terms of observable quantities. We therefore write

$$A_S^2(k) \propto k^{n_S - 1} \quad (8.207)$$

and

$$A_T^2(k) \propto k^{n_T}, \quad (8.208)$$

where  $n_S$  and  $n_T$  are the spectral indices. They are related to the slow-roll parameters of the potential by

$$n_S = 1 - 6\epsilon + 2\eta \quad (8.209)$$

and

$$n_T = -2\epsilon. \quad (8.210)$$

In models of the type we have considered (driven by single slowly-rolling scalar fields), there is a consistency relation relating the amplitudes and spectral indices of the scalar and tensor modes. It can be expressed in a convention-independent way as a relation between observable quantities, temperature fluctuations  $\Delta T$  due to the different perturbations, as

$$\frac{(\Delta T/T)_T^2}{(\Delta T/T)_S^2} = -7n_T. \quad (8.211)$$

The existence of tensor perturbations is a crucial prediction of inflation that may in principle be verifiable through observations of the polarization of the CMB. Polarization is also induced by ordinary density fluctuations, through the anisotropy of the Thompson scattering cross-section in an inhomogeneous plasma. Fortunately, we can imagine decomposing the polarization vector field on the sky into a curl-free part ( $E$ -modes) and a curl part ( $B$ -modes); the scalar perturbations lead to  $E$ -mode polarization, whereas tensor perturbations lead to  $B$ -modes (up to some inevitable processing in the post-recombination universe). CMB polarization has been detected; the challenge for the future will be to sepa-

rate out the scalar and tensor contributions, to test the prediction (8.211) of simple inflationary models. Of course this requires not only detecting the tensor-induced polarization, but measuring its spectral index with some precision.

Our current knowledge of the amplitude of the perturbations already gives us important information about the energy scale of inflation. The tensor perturbations depend on  $V$  alone, not its derivatives; if the CMB anisotropies seen by COBE are due to tensor fluctuations (possible, although unlikely), we can instantly derive  $V_{\text{inflation}} \sim (10^{16} \text{ GeV})^4$ . Here, the value of  $V$  being constrained is that which was responsible for creating the observed fluctuations; namely, 60  $e$ -folds before the end of inflation. This is remarkably reminiscent of the grand unification scale, which is very encouraging. Even in the more likely case that the perturbations observed in the CMB are scalar in nature, we can still write

$$V_{\text{inflation}}^{1/4} \sim \epsilon^{1/4} 10^{16} \text{ GeV}, \quad (8.212)$$

where  $\epsilon$  is the slow-roll parameter defined in (8.194). Although we expect  $\epsilon$  to be small, the  $1/4$  in the exponent means that the dependence on  $\epsilon$  is quite weak; unless this parameter is extraordinarily tiny, it is very likely that  $V_{\text{inflation}}^{1/4} \sim 10^{15}$ - $10^{16}$  GeV. The fact that we can have such information about such tremendous energy scales is a cause for great wonder.

## 8.9 ■ EXERCISES

- Consider an  $(N + n + 1)$ -dimensional spacetime with coordinates  $\{t, x^I, y^i\}$ , where  $I$  goes from 1 to  $N$  and  $i$  goes from 1 to  $n$ . Let the metric be

$$ds^2 = -dt^2 + a^2(t)\delta_{IJ}dx^I dx^J + b^2(t)\gamma_{ij}(y)dy^i dy^j, \quad (8.213)$$

where  $\delta_{IJ}$  is the usual Kronecker delta and  $\gamma_{ij}(y)$  is the metric on an  $n$ -dimensional maximally symmetric spatial manifold. Imagine that we normalize the metric  $\gamma$  such that the curvature parameter

$$k = \frac{R(\gamma)}{n(n-1)} \quad (8.214)$$

is either  $+1$ ,  $0$ , or  $-1$ , where  $R(\gamma)$  is the Ricci scalar corresponding to the metric  $\gamma_{ij}$ .

- (a) Calculate the Ricci tensor for this metric.
- (b) Define an energy-momentum tensor in terms of an energy density  $\rho$  and pressure in the  $x^I$  and  $y^i$  directions,  $p^{(N)}$  and  $p^{(n)}$ :

$$T_{00} = \rho \quad (8.215)$$

$$T_{IJ} = a^2 p^{(N)} \delta_{IJ} \quad (8.216)$$

$$T_{ij} = b^2 p^{(n)} \gamma_{ij}. \quad (8.217)$$

Plug the metric and  $T_{\mu\nu}$  into Einstein's equations to derive Friedmann-like equations for  $a$  and  $b$  (three independent equations in all).

- (c) Derive equations for the energy density and the two pressures at a static solution where  $\dot{a} = \dot{b} = \ddot{a} = \ddot{b} = 0$ , in terms of  $k$ ,  $n$ , and  $N$ . Use these to derive expressions for the equation-of-state parameters  $w^{(N)} = p^{(N)}/\rho$  and  $w^{(n)} = p^{(n)}/\rho$ , valid at the static solution.
2. Consider de Sitter space in coordinates where the metric takes the form
- $$ds^2 = -dt^2 + e^{2Ht}[dx^2 + dy^2 + dz^2]. \quad (8.218)$$
- Solve the geodesic equation for noncomoving observers ( $x^i$  not constant) to find the affine parameter as a function of  $t$ . Show that the geodesics reach  $t = -\infty$  in a finite affine parameter, demonstrating that these coordinates fail to cover the entire manifold.
3. In Appendix F we discuss Raychaudhuri's equation. Show that, applied to a Robertson-Walker cosmology, the Raychaudhuri equation is equivalent to the second Friedmann equation, (8.68).
4. Consider the best-fit universe, with density parameters  $\Omega_{R0} = 10^{-4}$ ,  $\Omega_{M0} = 0.3$ ,  $\Omega_{\Lambda0} = 0.7$ . Make a plot of the three  $\Omega_i$ 's as a function of the scale factor  $a$ , on a log scale, from  $a = 10^{-35}$  to  $a = 10^{35}$ . Indicate the Planck time, nucleosynthesis, and today.
5. In a flat spacetime, objects of a fixed physical size subtend smaller and smaller angles as they are further and further away; in an expanding universe this is not necessarily so. Consider the angular size  $\theta(z)$  of an object of physical size  $L$  at redshift  $z$ . In a matter-dominated flat universe, at what redshift is  $\theta(z)/L$  a minimum? If all galaxies are at least 10 kpc across (and always have been), what is the minimum angular size of a galaxy in such a universe? Express your result both in terms of  $H_0$ , and plugging in  $H_0 = 70 \text{ km/s/Mpc}$ .
6. In cosmology we tend to idealize nonrelativistic particles as having zero temperature  $T$  and pressure  $p$ . In reality, random motions will give them some temperature and pressure, satisfying  $p \propto T\rho$ .
- How does the pressure of a gas of massive particles decay as a function of the scale factor?
  - Suppose neutrinos have a mass  $m_\nu = 0.1 \text{ eV}$ , and a current temperature  $T_{\nu0} = 2 \text{ K}$ . At about what redshift did the neutrinos go from being relativistic to nonrelativistic?
7. Suppose that the universe started out in a state of equipartition at the Planck time (so that the energy density in matter and radiation are of order the Planck density, and the temporal and spatial curvature radii are of order the Planck length). Neglecting any spatial inhomogeneity, calculate how long a positively curved universe will last, and how old a negatively curved universe would be when the temperature reaches 3K. How old would a flat universe be when the temperature reaches 3K? How old would a flat universe be by the time the expansion rate slows to  $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ?

## 9.1 ■ INTRODUCTION

Nobody believes that general relativity is the final word as far as gravity is concerned. The singularity theorems provide internal evidence that the theory is somehow incomplete; more convincing, however, is the fact that GR is a classical theory, while the world is fundamentally quantum-mechanical. The search for a working theory of quantum gravity drives a great deal of research in theoretical physics today, and much has been learned along the way, but convincing success remains elusive.

There are two parts to general relativity: the framework of spacetime curvature and its influence on matter, and the dynamics of the metric in response to energy-momentum (as described by Einstein's equation). Lacking a true theory of quantum gravity, we may still take the first part of GR—the idea that matter fields propagate on a curved spacetime background—and consider the case where those matter fields are quantum-mechanical. In other words, we take the metric to be fixed, rather than obeying some dynamical equations, and study quantum field theory (QFT) in that curved spacetime.

The epochal event in the study of QFT in curved spacetime was Hawking's realization in 1976 that black holes are not really black, but instead emit thermal radiation at a **Hawking temperature** proportional to the surface gravity  $\kappa$ ,

$$T = \frac{\kappa}{2\pi}. \quad (9.1)$$

(Recall that our units set  $\hbar = c = k = 1$ ; the Hawking temperature is actually proportional to  $\hbar$  and inversely proportional to Boltzmann's constant  $k$ .) Since this remarkable discovery, QFT in curved spacetime has been put on a fairly rigorous theoretical footing, although its range of applicability is generally thought to be quite far away from any possible experimental probes. The Hawking temperature of a Schwarzschild black hole, for which  $\kappa = 1/4GM$ , can be written

$$T = \frac{1}{8\pi GM} = 1.2 \times 10^{26} K \left( \frac{1 \text{ g}}{M} \right) = 6.0 \times 10^{-8} K \left( \frac{M_\odot}{M} \right), \quad (9.2)$$

where  $M_\odot \sim 10^{33}$  g is the mass of the Sun. So the radiation from a realistic astrophysical black hole is at a much lower temperature even than the 3K cosmic microwave background, and thus would be hopelessly unobservable.

Recent observations in cosmology, however, have changed this situation somewhat. One example is the apparent discovery that the universe is accelerating, which is most readily interpreted as evidence for a nonzero vacuum energy (as discussed in Chapter 8). Although the magnitude of the vacuum energy remains a profound mystery, it seems clear that an understanding of how quantum-mechanical matter behaves in curved spacetime will play an important role in any eventual resolution to the puzzle. The other example comes from cosmological perturbations. Observations of the microwave background and large-scale structure provide strong evidence in favor of a nearly scale-free spectrum of primordial perturbations, including at wavelengths that would be much larger than the horizon size in a conventional cosmology. The leading theory for the origin of these perturbations comes from inflation. In the inflationary scenario, cosmological perturbations originate in the vacuum fluctuations of quantum fields in an inflating universe. If this picture is correct, what we are seeing in maps of the CMB is the imprint of primordial quantum fluctuations, greatly stretched by the expansion of the universe, and it is these fluctuations which eventually grew via gravitational instability into the galaxies and clusters we see today. At the very least, then, cosmological observations provide strong incentive for the study of QFT in curved spacetime.

Even without this empirical motivation, thought experiments based on QFT in curved spacetime have proven very fruitful in our tentative explorations of quantum gravity. In particular, the evaporation of black holes as predicted by Hawking radiation has led to the information-loss paradox, which we will discuss below. Since it is so difficult to do real experiments that bear directly on questions of quantum gravity, we must rely on thought experiments that focus on the tension between GR and quantum mechanics, much as Einstein used thought experiments in his attempts to reconcile classical dynamics with the Lorentz invariance of electromagnetism.

With these considerations in mind, the goal of the present chapter is to provide a brief introduction to some of the ideas and results of QFT in curved spacetime. Many introductory GR books do not cover this subject, usually because familiarity with ordinary QFT in flat spacetime should not be a prerequisite for studying GR. The happy fact is, however, that a familiarity with QFT in flat spacetime is by no means necessary for studying QFT in curved spacetime. This is because the features of QFT that are most interesting and useful in flat spacetime are almost completely distinct from those that are interesting and useful in curved spacetime. Deep down, a quantum field theory is simply an example of a quantum-mechanical system, just like a square well or a helium atom. Once a field theory is defined, applications in flat spacetime (to particle physics or condensed matter) will naturally focus on the issue of interactions between the various fields, often treated as perturbations around some natural vacuum state. In curved spacetime, however, we are generally interested in the effects of spacetime itself on the fields, for which the interactions are beside the point. We therefore can consider free (noninteracting) fields, but we will have to take great care in defining what an appropriate vacuum state should be. (Indeed, as we will see, almost all of

the states we deal with will be vacuum states!) Consequently, knowledge of QFT in flat spacetime is not only unnecessary for the present discussion, it probably won't even be of much help; the only prerequisite is a familiarity with the basics of ordinary quantum mechanics.

We will gradually work our way up to quantum field theory in curved spacetime, beginning with a review of the quantum mechanics of the system to which every physicist turns when the going gets rough: the simple harmonic oscillator. This is, of course, a paradigmatic example of the principles of the workings of quantum mechanics, but there is a bonus: When we next turn to field theory, we will find that the quantum mechanics of a free field in flat spacetime is precisely that of an infinite number of harmonic oscillators. (It is not that there is one oscillator at every point in space, but that each mode in the Fourier transform of the field acts like an harmonic oscillator.) The transition to field theory is then fairly straightforward. Once we grasp the basics of field theory, given our previous study of GR, it is not very difficult to generalize to curved spacetime, although a number of subtleties are encountered along the way. Our discussion will necessarily be somewhat superficial, focused on the goal of understanding the physical basis of Hawking radiation through an understanding of the Unruh effect in flat spacetime. In particular, we won't be discussing the important applications of QFT in curved spacetime to cosmology, nor will we be entering into detailed examination of renormalization and related issues. We will largely follow the discussion in Birrell and Davies (1982); look there or in Wald (1994) or in the review by Ford<sup>1</sup> for further discussion.

## 9.2 ■ QUANTUM MECHANICS

A quantum field theory is just a particular example of a quantum-mechanical system, so we can begin by reminding ourselves what that means. Of course, although the world is fundamentally quantum-mechanical, our intuition tends to align more readily with classical physics, so let's set the stage by thinking about classical mechanics. Any physical theory describing a certain system, classical or quantum, consists of the answers to three questions:

1. What are the possible states of the system? In classical mechanics, the space of states is typically given by a set of coordinates and momenta (what we might think of as "initial conditions" for the system). They can be specified exactly, and that is all there is to know about the state of the system.
2. What can we observe about the system? This question is often addressed only implicitly in classical mechanics, since the answer is trivial: any function of the coordinates and momenta qualifies as an observable.
3. How does the system evolve? This is usually expressed by a set of equations of motion. Given the state and the equations of motion, the subsequent

<sup>1</sup>L. H. Ford, "Quantum field theory in curved spacetime," (1997), <http://arxiv.org/gr-qc/9707062>.

evolution is uniquely defined; as a result, the space of initial conditions is equivalent to the space of classical solutions to the theory.

To make these ideas more concrete, and also because it will be directly relevant to our study of field theory, let's consider the simple harmonic oscillator. A simple harmonic oscillator may be thought of as a particle in one dimension subject to a quadratic potential. The state is specified by a single coordinate  $x$ , and a single momentum  $p$ . To get the equations of motion, we could start with the Lagrangian, which is written in terms of  $x$  and its time derivative  $\dot{x}$  as

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2x^2, \quad (9.3)$$

where we have set the mass of the oscillator to unity for convenience. We can immediately derive the equation of motion

$$\ddot{x} + \omega^2x = 0. \quad (9.4)$$

For the transition to quantum mechanics, however, it is more convenient to work in terms of the Hamiltonian, which is a function of  $x$  and  $p$  rather than  $x$  and  $\dot{x}$ . The Hamiltonian is related to the Lagrangian by a Legendre transformation,

$$H = p\dot{x} - L, \quad (9.5)$$

where the momentum satisfies

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x}. \quad (9.6)$$

We therefore have the Hamiltonian for the oscillator,

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2x^2, \quad (9.7)$$

and Hamilton's equations

$$\frac{dx}{dt} = \partial_p H = p, \quad \frac{dp}{dt} = -\partial_x H = -\omega^2x, \quad (9.8)$$

serve as equations of motion. The solutions are, of course, straightforward; it is useful to express them as complex numbers

$$x(t) = x_0 e^{i(\omega t + \alpha_0)}, \quad (9.9)$$

where  $x_0$  is the amplitude and  $\alpha_0$  is a phase. We can take the real part at the end of the day to get the physical answer.

Now we turn to quantum mechanics. Although quantum mechanics is profoundly different from classical mechanics, a given theory still consists of the answers to the same three questions listed above, with the answers taking somewhat different forms.

1. The state of the system is represented as an element of a **Hilbert space**.

Mathematically, a Hilbert space is just a complex vector space equipped with a complex-valued inner product with the property that taking the inner product of two states in the opposite order is equivalent to complex conjugation. We denote elements of the Hilbert space as  $|\psi\rangle$  and elements of the dual space as  $\langle\psi|$ , so that the inner product of  $|\psi_1\rangle$  and  $|\psi_2\rangle$  is  $\langle\psi_2|\psi_1\rangle$ , and obeys

$$\langle\psi_2|\psi_1\rangle^* = \langle\psi_1|\psi_2\rangle. \quad (9.10)$$

(We are glossing over technical requirements concerning completeness of the space.) In quantum mechanics the Hilbert spaces of interest are very often infinite-dimensional. For example, if a classical system is represented by coordinate  $x$  and momentum  $p$ , the Hilbert space could be taken to consist of all square-integrable complex-valued functions of  $x$ , or equivalently all square-integrable complex-valued functions of  $p$  (but not both at once).

2. Observables are represented by **self-adjoint operators** on the Hilbert space. The definition of “self-adjoint” is actually very subtle, but in simple circumstances amounts to our usual understanding of an Hermitian operator,

$$A^\dagger = A, \quad (9.11)$$

where  $A^\dagger$  obeys

$$\langle\psi_2|A\psi_1\rangle = \langle A^\dagger\psi_2|\psi_1\rangle \quad (9.12)$$

for all states  $|\psi_1\rangle$ ,  $|\psi_2\rangle$ . Of course many operators will not be Hermitian, but observables should have this property. In general such operators do not commute, so we cannot simultaneously specify the precise values of everything we might want to measure about the system; there will be a complete set of commuting observables that represents all we can say about a system at once.

3. Evolution of the system may be represented in one of two ways: as unitary evolution of the state vector in Hilbert space (the **Schrödinger picture**), or by keeping the state fixed and allowing the observables to evolve according to equations of motion (the **Heisenberg picture**).

Strictly speaking, quantum mechanics is just different from classical mechanics; it is by no means necessary to start with a classical model and “quantize” it. Nevertheless, we usually do exactly that. Even for simple classical models, there is more than one way to construct a quantized version; these include canonical quantization and path-integral quantization, as well as more exotic procedures. What is worse, there is no simple map between classical and quantum theories; there are classical theories with no well-defined quantum counterpart, classical theories with multiple quantum versions, and quantum theories without any classical

analogue. For our present purposes, we may blithely ignore all of these subtleties, and proceed directly with canonical quantization.

Once again, the simple harmonic oscillator provides a useful example. Consider first the familiar Schrödinger picture, in which states are represented by complex-valued wave functions that evolve with time, such as  $\psi(x, t)$ . The wave function is really the set of components of the state vector  $|\psi\rangle$ , expressed in the “delta-function position basis”  $|x\rangle$ , so that  $|\psi(t)\rangle = \int dx \psi(x, t)|x\rangle$ . Canonical quantization consists of imposing the canonical commutation relation,

$$[\hat{x}, \hat{p}] = i, \quad (9.13)$$

on the coordinate operator  $\hat{x}$  and its conjugate momentum  $\hat{p}$ . For states represented as wave functions depending on  $x$  and  $t$ ,  $\hat{x}$  is simply multiplication by  $x$ , so (9.13) can be implemented by setting

$$\hat{p} = -i\partial_x, \quad (9.14)$$

The Hamiltonian operator is

$$H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}\omega^2x^2, \quad (9.15)$$

and the equation of motion is the Schrödinger equation,

$$H\psi = i\partial_t\psi. \quad (9.16)$$

Since the Hamiltonian is time-independent, solutions to this equation separate into functions of space and functions of time,  $\psi(x, t) = f(t)g(x)$ . The solutions then come in a discrete set labeled by an integer  $n \geq 0$ , and we find (up to normalization)

$$\psi_n(x, t) = e^{-(1/2)\omega x^2} H_n(\sqrt{\omega}x) e^{-iE_n t}, \quad (9.17)$$

where  $H_n$  is a Hermite polynomial of degree  $n$ , and

$$E_n = \left(n + \frac{1}{2}\right)\omega. \quad (9.18)$$

These states are all eigenfunctions of  $H$ , and  $E_n$  is the energy eigenvalue. An arbitrary state of the oscillator will simply be a superposition of the energy eigenstates,

$$\psi(x, t) = \sum_n c_n \psi_n(x, t), \quad (9.19)$$

for some set of appropriately normalized coefficients  $c_n$ .

A number of important features of the quantum-mechanical oscillator are contained in this brief overview. There is a discrete spectrum of energy eigenstates; this is why it's called “quantum” mechanics (even though it is not hard to find

systems with continuous spectra). There is a ground state of lowest energy, plus a set of excited states uniquely labeled by their energy eigenvalue. The ground state has a nonvanishing energy,

$$E_0 = \frac{1}{2}\omega, \quad (9.20)$$

sometimes called the “zero-point” energy. It is interesting to note that the minimum energy of the classical system would have been zero, representing a particle with  $x = 0$  and  $p = 0$ . The quantum zero-point energy can be traced to the Heisenberg uncertainty principle, which forbids us from localizing a state simultaneously in both position and momentum; there is consequently a minimum amount of “jiggle” in the oscillator, leading to a nonzero ground-state energy. On the other hand, we could certainly have chosen to examine an oscillator with a potential given by  $V(x) = \frac{1}{2}\omega^2x^2 - \frac{1}{2}\omega$ ; our analysis would have been identical, except that the factor of  $\frac{1}{2}$  in (9.18) would have been missing, and the ground-state energy would have been zero. Quantum mechanics does not insist on a nonvanishing zero-point energy, it simply displaces the energy from the classical value.

An alternative way to solve the simple harmonic oscillator is to introduce creation and annihilation operators  $\hat{a}^\dagger$  and  $\hat{a}$  (often called raising and lowering operators), defined by

$$\hat{a} = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} - i\hat{p}), \quad (9.21)$$

so that

$$\hat{x} = \frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = -i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger). \quad (9.22)$$

Given our previous expressions for the commutation relations (9.13) and Hamiltonian (9.7), we can easily calculate the commutation relation for the creation and annihilation operators,

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad (9.23)$$

and the new expression for the Hamiltonian,

$$H = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\omega. \quad (9.24)$$

The creation/annihilation operators commute with the Hamiltonian via

$$\begin{aligned} [H, \hat{a}] &= -\omega\hat{a} \\ [H, \hat{a}^\dagger] &= \omega\hat{a}^\dagger. \end{aligned} \quad (9.25)$$

Comparing this version of the Hamiltonian to the energy eigenvalues (9.18), we are inspired to define a number operator

$$\hat{n} = \hat{a}^\dagger\hat{a}. \quad (9.26)$$

Let's think about why the creation/annihilation operators and the number operator deserve their names. Consider an eigenstate  $|n\rangle$  of the number operator,

$$\hat{n}|n\rangle = n|n\rangle, \quad (9.27)$$

where the  $\hat{n}$  on the left stands for the number operator, while the first  $n$  on the right stands for the actual number  $n$ . (This formula is the most charming in all of quantum mechanics.) By playing with the commutation relations, it is easy to show that

$$\begin{aligned}\hat{n}\hat{a}^\dagger|n\rangle &= (n+1)\hat{a}^\dagger|n\rangle \\ \hat{n}\hat{a}|n\rangle &= (n-1)\hat{a}|n\rangle.\end{aligned}\quad (9.28)$$

Thus, when  $\hat{a}^\dagger$  acts on  $|n\rangle$ , it gives another eigenstate of  $\hat{n}$  with eigenvalue raised by 1, while  $\hat{a}$  gives an eigenstate with eigenvalue lowered by 1. As before we can show that  $n$  takes integral values from 0 to  $\infty$ , so there must be a vacuum state  $|0\rangle$  satisfying

$$\hat{a}|0\rangle = 0. \quad (9.29)$$

From this state we can construct all of the eigenstates by successive operation by creation operators,

$$|n\rangle = \frac{1}{\sqrt{n!}} \left( \hat{a}^\dagger \right)^n |0\rangle. \quad (9.30)$$

The number operator counts the number of excitations above the ground state. The set of eigenstates  $|n\rangle$  acts as a basis; any state is an appropriate linear combination of these states. The creation and annihilation operators act on them according to

$$\begin{aligned}\hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle,\end{aligned}\quad (9.31)$$

and the energy of each state is of course given by (9.18). The basis states are taken to be time-independent, so a physical system obeying Schrödinger's equation will be described by a state

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t} |n\rangle, \quad (9.32)$$

where again the  $c_n$ 's are constant coefficients.

For purposes of smoothing the transition to field theory, it is useful to translate this Schrödinger-picture description into the Heisenberg picture, in which the states are fixed and the operators evolve with time. Given Schrödinger's equation (9.16), any state can be written formally as some fixed initial state acted on by a unitary time-evolution operator

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle, \quad (9.33)$$

where

$$U(t) = \mathcal{P} e^{-i \int H dt}, \quad (9.34)$$

(by unitary we mean  $U^\dagger U = 1$ .) The symbol  $\mathcal{P}$  stands for path-ordering, as discussed in Appendix I. If the Hamiltonian is time-independent, of course, we simply have  $U(t) = e^{-iHt}$ . The Schrödinger-picture expression for the matrix element of a time-independent operator  $A$  between time-dependent states  $|\psi_1(t)\rangle$  and  $|\psi_2(t)\rangle$  can then be written as a Heisenberg-picture expression in terms of a time-dependent operator  $A(t)$  and time-independent states as

$$\begin{aligned} \langle\psi_2(t)|A|\psi_1(t)\rangle &= \langle\psi_2(0)|U^\dagger(t)AU(t)|\psi_1(0)\rangle \\ &= \langle\psi_2|A(t)|\psi_1\rangle, \end{aligned} \quad (9.35)$$

where clearly the Heisenberg-picture operator is given by

$$A(t) = U^\dagger(t)AU(t). \quad (9.36)$$

Such an operator satisfies the **Heisenberg equation of motion**,

$$\frac{dA(t)}{dt} = i[H, A(t)], \quad (9.37)$$

which takes the place of Schrödinger's equation in this picture. For the harmonic oscillator, we would find

$$\frac{d\hat{a}}{dt} = -i\omega\hat{a}, \quad \frac{d\hat{a}^\dagger}{dt} = i\omega\hat{a}^\dagger, \quad (9.38)$$

with solutions

$$\hat{a}(t) = e^{-i\omega t}\hat{a}(0), \quad \hat{a}(t)^\dagger = e^{i\omega t}\hat{a}(0)^\dagger. \quad (9.39)$$

From this we immediately find

$$\hat{n}(t) = \hat{a}(t)^\dagger\hat{a}(t) = \hat{a}(0)^\dagger\hat{a}(0), \quad (9.40)$$

which reflects the fact that the number operator is conserved.

It is common to say that in the Heisenberg picture the states are time-independent; this is somewhat confusing, if nevertheless true. It might be better to say that the states extend throughout time, rather than only being defined at a fixed time. To make this more clear, consider a simple harmonic oscillator subject to an external influence, for example by simply adding a forcing term to the Hamiltonian,

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2x^2 + F(t), \quad (9.41)$$

where the function  $F(t)$  vanishes outside an interval,

$$F(t) = \begin{cases} 0 & t < t_1 \\ F(t) & t_1 \leq t \leq t_2 \\ 0 & t_2 < t. \end{cases} \quad (9.42)$$

We can think of someone coming along and shaking our oscillator for a short while, and then leaving it alone after that. In the Schrödinger picture, we would say that an oscillator that started in its ground state would be excited by the external force, and the final state would not be the ground state. In the Heisenberg picture, however, we take the state to be a solution to the equation of motion for all times, and say that the number operator went from being zero to some other value.

For the oscillator subject to a transient external force, there are clearly a set of states that look like energy eigenstates at early times, although they don't look that way in the future; we might call such states the "in states"  $|n_{\text{in}}\rangle$ , with the property that

$$\hat{n}(t < t_1)|n_{\text{in}}\rangle = n|n_{\text{in}}\rangle. \quad (9.43)$$

There is also a separate set of states that look like energy eigenstates at late times, correspondingly called "out states"  $|n_{\text{out}}\rangle$ , and obeying

$$\hat{n}(t > t_2)|n_{\text{out}}\rangle = n|n_{\text{out}}\rangle. \quad (9.44)$$

Both sets of states exist at all times, but they look like energy eigenstates only in the appropriate asymptotic regime. Either set forms a basis for the entire Hilbert space, so in particular we could decompose one set in terms of the other. For example, by multiplying by a complete set of in states, we can write

$$|n_{\text{out}}\rangle = \sum_m \langle m_{\text{in}}|n_{\text{out}}\rangle |m_{\text{in}}\rangle. \quad (9.45)$$

The complex numbers  $\langle m_{\text{in}}|n_{\text{out}}\rangle$  are matrix elements, which could, in principle, be calculated from the Hamiltonian (9.41); together they comprise the **S-matrix**. An observer equipped with a way to detect excitations of the oscillator would find that the number of excitations was changed by the applied force, and the S-matrix encodes the information necessary to characterize these changes between the asymptotic past and future. All of this discussion, needless to say, carries over essentially without modification to field theory. For particle physics, the role of the external force is played by the interactions between different particles, whereas for our purposes it will be played by the curvature of spacetime.

### 9.3 ■ QUANTUM FIELD THEORY IN FLAT SPACETIME

As we have already mentioned, quantum field theory is just a particular example of a quantum-mechanical system, in which we are quantizing a field (a function, or more generally some tensor field, defined on spacetime) rather than a single oscillator. We begin with the simplest possible example, of a free scalar field in flat spacetime; only a couple of generalizations are necessary to make the transition from a single oscillator to this field theory. Extending the theory to curved spacetime is straightforward as usual, involving writing the theory in a covariant form

and declaring it to be true. Once we lose the symmetries of Minkowski space, however, some of the ideas we think of as central in a quantum field theory will no longer seem so crucial; in particular, the notions of “vacuum” and “particles” will lose their privileged positions. (Expositions of quantum mechanics will occasionally make the point that waves and particles are complementary notions with different domains of validity, but don’t be misled; in quantum field theory it is the fields that are truly fundamental, while the particles are approximate notions useful in certain restricted circumstances.) In this section we study QFT in flat spacetime, before generalizing to curved spacetime in the next section.

We start with the classical theory, in this case a real scalar field  $\phi(x^\mu)$  in flat spacetime, just as we considered in Chapter 1, this time generalized to  $n$  dimensions. The action is the spacetime integral of the Lagrange density,  $S = \int d^n x \mathcal{L}$ ; we will consider the Klein–Gordon Lagrangian

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2. \quad (9.46)$$

It is not necessary to include the volume-element factor  $\sqrt{|g|}$ , since we are using inertial coordinates in Minkowski space, with metric

$$ds^2 = -dt^2 + (dx)^2. \quad (9.47)$$

The equation of motion is the Klein–Gordon equation,

$$\square\phi - m^2\phi = 0. \quad (9.48)$$

Translation into a Hamiltonian description for the field theory is straightforward. The conjugate momentum for a field is simply the derivative of the Lagrange density with respect to the time derivative of that field,

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}. \quad (9.49)$$

For the Klein–Gordon Lagrangian (9.46), this is

$$\pi = \dot{\phi}. \quad (9.50)$$

Of course, referring to the time derivative assumes that we have chosen a particular inertial frame; consequently, the Hamiltonian procedure necessarily violates manifest Lorentz invariance. If we are careful, however, observable quantities in the resulting theory will still be Lorentz-invariant. The Hamiltonian itself can be expressed as an integral over space of a Hamiltonian density,

$$H = \int d^{n-1}x \mathcal{H}, \quad (9.51)$$

which is related to the Lagrangian by a Legendre transformation,

$$\begin{aligned} \mathcal{H}(\phi, \pi) &= \pi\dot{\phi} - \mathcal{L}(\phi, \partial_\mu\phi) \\ &= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2, \end{aligned} \quad (9.52)$$

where  $(\nabla\phi)^2 = \delta^{ij}(\partial_i\phi)(\partial_j\phi)$ . The correspondence between this field theory and the harmonic oscillator should be clear: the field value  $\phi(x)$  plays the role of the coordinate  $x$ , with momentum field  $\pi(x)$  instead of a single momentum  $p$ . Instead of the state being specified by two numbers ( $x$  and  $p$ ) at some fixed time, we would have to give field values [ $\phi(x^i)$  and  $\pi(x^i)$ ] all over space at some fixed time as initial data, and there is an additional gradient term that was missing in the oscillator case; but otherwise the formalism is very similar.

We should emphasize that  $\phi(x^\mu)$  is *not* a wave function; it is a dynamical variable, generalizing the single degree of freedom  $x$  in the case of the harmonic oscillator. In a Schrödinger-picture quantization of the field theory, we would define a complex wave functional  $\Psi[\phi(x^\mu)]$ , which would represent the probability amplitude for finding the field in each configuration. Instead, however, we will use the Heisenberg picture, so that our primary concern will be to promote  $\phi$  to a quantum operator.

First, we should complete the classical analysis by actually solving this theory. It is not hard to write down solutions to the Klein–Gordon equation. One good example is a plane wave,

$$\phi(x^\mu) = \phi_0 e^{ik_\mu x^\mu} = \phi_0 e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}}, \quad (9.53)$$

where the wave vector has components

$$k^\mu = (\omega, \mathbf{k}), \quad (9.54)$$

and the frequency must satisfy the dispersion relation

$$\omega^2 = \mathbf{k}^2 + m^2. \quad (9.55)$$

There is a clear similarity between such a solution and that for the simple harmonic oscillator, given by (9.9). But there is also an important difference: For the oscillator, there is only one independent solution. Because the oscillator has a unique frequency, when we add two solutions with specified amplitude  $x_0$  and phase  $\alpha_0$ , they combine to give a third solution with the same frequency but different amplitude and phase. This is no longer true in field theory. Given (9.55), the frequency is determined by the spatial wave vector  $\mathbf{k}$ , at least up to sign. Therefore, instead of a single kind of solution, we have a set parameterized by  $\mathbf{k}$  and the sign of  $\omega$ .

However, we can still write down the most general solution by constructing a complete, orthonormal set of modes in terms of which any solution may be expressed. To make sense of “orthonormal,” we need to define an inner product on the space of solutions to the Klein–Gordon equation. Although the modes themselves are functions of spacetime, the appropriate inner product can be expressed as an integral over a constant-time hypersurface  $\Sigma_t$ ,

$$(\phi_1, \phi_2) = -i \int_{\Sigma_t} (\phi_1 \partial_t \phi_2^* - \phi_2^* \partial_t \phi_1) d^{n-1}x. \quad (9.56)$$

As we would hope, the inner product is actually *independent* of the hypersurface  $\Sigma_t$  over which the integral is taken, as you can easily check by using Stokes's theorem and the Klein–Gordon equation. Applying this inner product to two plane waves of different wave vectors gives

$$\begin{aligned} & \langle e^{ik_1^\mu x_\mu}, e^{ik_2^\nu x_\nu} \rangle \\ &= -i \int_{\Sigma_t} (e^{-i\omega_1 t + i\mathbf{k}_1 \cdot \mathbf{x}} \partial_t e^{i\omega_2 t - i\mathbf{k}_2 \cdot \mathbf{x}} - e^{i\omega_2 t - i\mathbf{k}_2 \cdot \mathbf{x}} \partial_t e^{-i\omega_1 t + i\mathbf{k}_1 \cdot \mathbf{x}}) d^{n-1}x \\ &= (\omega_2 + \omega_1) e^{-i(\omega_1 - \omega_2)t} \int_{\Sigma_t} e^{i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{x}} d^{n-1}x \\ &= (\omega_2 + \omega_1) e^{-i(\omega_1 - \omega_2)t} (2\pi)^{n-1} \delta^{(n-1)}(\mathbf{k}_1 - \mathbf{k}_2), \end{aligned} \quad (9.57)$$

where we have used

$$\int e^{i\mathbf{k} \cdot \mathbf{x}} d^{n-1}x = (2\pi)^{n-1} \delta^{(n-1)}(\mathbf{k}). \quad (9.58)$$

The inner product thus vanishes unless the spatial wave vectors  $\mathbf{k}$ , and hence the frequencies  $\omega$ , are equal for both modes. An orthonormal set of mode solutions is thus given by

$$f_{\mathbf{k}}(x^\mu) = \frac{e^{ik_\mu x^\mu}}{[(2\pi)^{n-1} 2\omega]^{1/2}}, \quad (9.59)$$

with  $k^\mu$  obeying (9.55), so that

$$(f_{\mathbf{k}_1}, f_{\mathbf{k}_2}) = \delta^{(n-1)}(\mathbf{k}_1 - \mathbf{k}_2). \quad (9.60)$$

Given the dispersion relation (9.55),  $\mathbf{k}$  only determines the frequency up to an overall sign. Our strategy will be to insist that  $\omega$  always be a positive number, and complete the set of modes by including the complex conjugates  $f_{\mathbf{k}}^*(x^\mu)$ . (Complex conjugation changes the sign of the  $\mathbf{k}$  term in the exponent as well as the  $\omega$  term, but the components of  $\mathbf{k}$  are defined from  $-\infty$  to  $\infty$  already.) The  $f_{\mathbf{k}}$  modes are said to be positive-frequency, meaning they satisfy

$$\partial_t f_{\mathbf{k}} = -i\omega f_{\mathbf{k}}, \quad \omega > 0, \quad (9.61)$$

while the  $f_{\mathbf{k}}^*$  modes are negative-frequency, satisfying

$$\partial_t f_{\mathbf{k}}^* = i\omega f_{\mathbf{k}}^*, \quad \omega > 0. \quad (9.62)$$

(Be careful; these modes are called negative-frequency even though  $\omega > 0$ , because the time derivative pulls down a factor  $+i\omega$  rather than  $-i\omega$ .) The complex conjugate modes are orthogonal to the original modes,

$$(f_{\mathbf{k}_1}, f_{\mathbf{k}_2}^*) = 0, \quad (9.63)$$

and orthonormal with each other but with a negative norm,

$$(f_{\mathbf{k}_1}^*, f_{\mathbf{k}_2}^*) = -\delta^{(n-1)}(\mathbf{k}_1 - \mathbf{k}_2). \quad (9.64)$$

Together, the modes  $f_{\mathbf{k}}$  and  $f_{\mathbf{k}}^*$  form a complete set, in terms of which we can expand any solution to the Klein–Gordon equation.

To canonically quantize this theory, we promote our classical variables (the fields and their conjugate momenta) to operators acting on a Hilbert space, and impose the canonical commutation relations on equal-time hypersurfaces:

$$\begin{aligned} [\phi(t, \mathbf{x}), \phi(t, \mathbf{x}')] &= 0 \\ [\pi(t, \mathbf{x}), \pi(t, \mathbf{x}')] &= 0 \\ [\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] &= i\delta^{(n-1)}(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (9.65)$$

In field theory we need to state explicitly that the field and its momentum commute with themselves throughout space; for a single oscillator this is implicit, since there is only a single coordinate and momentum, each of which will necessarily commute with itself. The delta function implies that operators at equal times commute everywhere except at coincident spatial points; this feature arises from the demands of causality (operators at spacelike separation cannot influence each other).

Just as classical solutions to the Klein–Gordon equation can be expanded in terms of the modes (9.59), so can the quantum operator field  $\phi(t, \mathbf{x})$ . Denoting the coefficients of the mode expansion of the field operator by  $\hat{a}_{\mathbf{k}}^\dagger$  and  $\hat{a}_{\mathbf{k}}$ , we have

$$\phi(t, \mathbf{x}) = \int d^{n-1}k [\hat{a}_{\mathbf{k}} f_{\mathbf{k}}(t, \mathbf{x}) + \hat{a}_{\mathbf{k}}^\dagger f_{\mathbf{k}}^*(t, \mathbf{x})]. \quad (9.66)$$

Plugging this expansion into (9.65), we find that the operators  $\hat{a}_{\mathbf{k}}^\dagger$  and  $\hat{a}_{\mathbf{k}}$  obey commutation relations

$$\begin{aligned} [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] &= 0 \\ [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger] &= 0 \\ [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] &= \delta^{(n-1)}(\mathbf{k} - \mathbf{k}'). \end{aligned} \quad (9.67)$$

These operators thus obey the commutation relations characteristic of creation and annihilation operators, familiar from (9.23) for the simple harmonic oscillator. The difference, of course, is that there are an infinite number of such operators, indexed by  $\mathbf{k}$ . We can see the relevance of dividing the modes into positive- and negative-frequency; the positive-frequency modes are coefficients of annihilation operators, while negative-frequency modes are coefficients of creation operators. The idea of positive- and negative-frequency modes will turn out to generalize to static spacetimes, although not to arbitrary spacetimes.

In the case of the harmonic oscillator, we used the creation and annihilation operators to define a basis for the Hilbert space in which the basis states were

eigenstates of the number operator. The same procedure works for the free scalar field, although now we have to keep track of separate numbers of excitations for each spatial wave vector  $\mathbf{k}$ . There will be a single vacuum state  $|0\rangle$ , characterized by the fact that it is annihilated by each  $\hat{a}_{\mathbf{k}}$ ,

$$\hat{a}_{\mathbf{k}}|0\rangle = 0 \quad \text{for all } \mathbf{k}. \quad (9.68)$$

A state with  $n_{\mathbf{k}}$  particles with identical momenta  $\mathbf{k}$  is created by repeated action by  $\hat{a}_{\mathbf{k}}^\dagger$ ,

$$|n_{\mathbf{k}}\rangle = \frac{1}{\sqrt{n_{\mathbf{k}}!}} \left( \hat{a}_{\mathbf{k}}^\dagger \right)^{n_{\mathbf{k}}} |0\rangle, \quad (9.69)$$

while a state with  $n_i$  excitations of various momenta  $\mathbf{k}_i$  would be

$$|n_1, n_2, \dots, n_j\rangle = \frac{1}{\sqrt{n_1! n_2! \cdots n_j!}} \left( \hat{a}_{\mathbf{k}_1}^\dagger \right)^{n_1} \left( \hat{a}_{\mathbf{k}_2}^\dagger \right)^{n_2} \cdots \left( \hat{a}_{\mathbf{k}_j}^\dagger \right)^{n_j} |0\rangle. \quad (9.70)$$

Acting on such a state, the creation and annihilation operators change the number of excitations, as expected:

$$\begin{aligned} \hat{a}_{\mathbf{k}_i} |n_1, n_2, \dots, n_i, \dots, n_j\rangle &= \sqrt{n_i} |n_1, n_2, \dots, n_i - 1, \dots, n_j\rangle \\ \hat{a}_{\mathbf{k}_i}^\dagger |n_1, n_2, \dots, n_i, \dots, n_j\rangle &= \sqrt{n_i + 1} |n_1, n_2, \dots, n_i + 1, \dots, n_j\rangle. \end{aligned} \quad (9.71)$$

We can define a number operator for each wave vector,

$$\hat{n}_{\mathbf{k}} = \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}, \quad (9.72)$$

which obeys

$$\hat{n}_{\mathbf{k}_i} |n_1, n_2, \dots, n_i, \dots, n_j\rangle = n_i |n_1, n_2, \dots, n_i, \dots, n_j\rangle. \quad (9.73)$$

The states that are eigenstates of the number operators form a basis for the entire Hilbert space, known as the **Fock basis**; the space constructed from this basis is often called “Fock space,” but of course it is just the original Hilbert space.

One thing we might want to investigate is how our Fock basis behaves under Lorentz transformations. We have clearly been taking advantage of the symmetries of Minkowski space, for example in using plane waves as a basis for solutions to the Klein–Gordon equation. The crucial aspect of these modes is our ability to distinguish between positive and negative frequencies, allowing for an interpretation of their coefficients in the mode expansion of  $\phi$  as annihilation and creation operators. Now consider a boost by velocity  $\mathbf{v} = d\mathbf{x}/dt$ , leading to new coordinates  $x^\mu'$  given by

$$t' = \gamma t - \gamma \mathbf{v} \cdot \mathbf{x}, \quad \mathbf{x}' = \gamma \mathbf{x} - \gamma \mathbf{v} t, \quad (9.74)$$

where  $\gamma = 1/\sqrt{1 - v^2}$ , and the inverse transformation is given by

$$t = \gamma t' + \gamma \mathbf{v} \cdot \mathbf{x}', \quad \mathbf{x} = \gamma \mathbf{x}' + \gamma \mathbf{v} t'. \quad (9.75)$$

The time derivative of our mode functions in the boosted frame is

$$\begin{aligned}\partial_{t'} f_{\mathbf{k}} &= \frac{\partial x^{\mu}}{\partial t'} \partial_{\mu} f_{\mathbf{k}} \\ &= \gamma(-i\omega) f_{\mathbf{k}} + \gamma \mathbf{v} \cdot (i\mathbf{k}) f_{\mathbf{k}} \\ &= -i\omega' f_{\mathbf{k}}\end{aligned}\quad (9.76)$$

where

$$\omega' = \gamma\omega - \gamma\mathbf{v} \cdot \mathbf{k} \quad (9.77)$$

is simply the frequency in the boosted frame. Clearly, then, a state describing a collection of particles with certain momenta is boosted into a state describing the same particles, but with boosted momenta. Thus, the total number operator in the two frames will coincide, and in particular the vacuum state will coincide. In this sense, our original choice of inertial frame was irrelevant. In the next section we will see that our ability to find positive- and negative-frequency solutions can be traced to the existence of a timelike Killing vector  $\partial_t$  in Minkowski spacetime, while the invariance of the Fock space under changes of basis can be traced to the fact that all such timelike Killing vectors are related by Lorentz transformations. Therefore, even if the frequency of a mode depends on the choice of inertial frame, the decomposition into positive and negative frequencies is invariant.

We would like to express the Hamiltonian

$$H = \int d^{n-1}x \left[ \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right] \quad (9.78)$$

in terms of the creation and annihilation operators, just as we did for the harmonic oscillator. We can analyze this expression term-by-term, starting with the  $\phi^2$  term for simplicity:

$$\begin{aligned}&\frac{1}{2}m^2 \int d^{n-1}x \phi^2 \\ &= \frac{1}{2}m^2 \int d^{n-1}x d^{n-1}k d^{n-1}k' (\hat{a}_{\mathbf{k}} f_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^\dagger f_{\mathbf{k}}^*) (\hat{a}_{\mathbf{k}'} f_{\mathbf{k}'} + \hat{a}_{\mathbf{k}'}^\dagger f_{\mathbf{k}'}^*) \\ &= \frac{1}{2}m^2 \int d^{n-1}x d^{n-1}k d^{n-1}k' (\hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'} f_{\mathbf{k}} f_{\mathbf{k}'} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}'}^\dagger f_{\mathbf{k}}^* f_{\mathbf{k}'}^* \\ &\quad + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'}^\dagger f_{\mathbf{k}} f_{\mathbf{k}'}^* + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}'} f_{\mathbf{k}}^* f_{\mathbf{k}'}^*).\end{aligned}\quad (9.79)$$

Zooming in on the first term in parentheses, and ignoring for the moment the integral over  $\mathbf{k}$ , we can plug in the explicit form of the mode functions (9.59) to obtain

$$\int d^{n-1}x d^{n-1}k' \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'} f_{\mathbf{k}} f_{\mathbf{k}'} = \int d^{n-1}x d^{n-1}k' \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'} \frac{e^{-i(\omega+\omega')t} e^{i(\mathbf{k}+\mathbf{k}') \cdot \mathbf{x}}}{2(2\pi)^{n-1} \sqrt{\omega\omega'}} \quad (9.79)$$

$$\begin{aligned}
&= \int d^{n-1}k' \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'} \frac{e^{-i(\omega+\omega')t}}{2\sqrt{\omega\omega'}} \delta^{(n-1)}(\mathbf{k} + \mathbf{k}') \\
&= \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} \frac{e^{-2i\omega t}}{2\omega}.
\end{aligned} \tag{9.80}$$

where we have used (9.58) again. Evaluating the other terms in (9.79) similarly, we find that the potential-energy contribution to the Hamiltonian therefore becomes

$$\begin{aligned}
\frac{1}{2}m^2 \int d^{n-1}x \phi^2 &= \frac{1}{2}m^2 \int d^{n-1}k \left(\frac{1}{2\omega}\right) \left[ \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} e^{-2i\omega t} \right. \\
&\quad \left. + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger e^{2i\omega t} \right].
\end{aligned} \tag{9.81}$$

For the kinetic-energy and gradient-energy pieces, the derivatives pull down factors of  $\omega$  and  $\mathbf{k}$  respectively; we obtain

$$\frac{1}{2} \int d^{n-1}x \dot{\phi}^2 = \frac{1}{2} \int d^{n-1}k \left(\frac{\omega}{2}\right) \left[ -\hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} e^{-2i\omega t} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger - \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger e^{2i\omega t} \right] \tag{9.82}$$

and

$$\begin{aligned}
\frac{1}{2} \int d^{n-1}x (\nabla\phi)^2 &= \frac{1}{2} \int d^{n-1}k \left(\frac{\mathbf{k}^2}{2\omega}\right) \left[ \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} e^{-2i\omega t} \right. \\
&\quad \left. + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger e^{2i\omega t} \right].
\end{aligned} \tag{9.83}$$

Using  $\omega^2 = \mathbf{k}^2 + m^2$ , we can put it all together to write the Hamiltonian for the scalar field theory as

$$\begin{aligned}
H &= \frac{1}{2} \int d^{n-1}k \left[ \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \right] \omega \\
&= \int d^{n-1}k \left[ \hat{n}_{\mathbf{k}} + \frac{1}{2} \delta^{(n-1)}(0) \right] \omega,
\end{aligned} \tag{9.84}$$

where the last step invokes the commutation relation (9.67) and the number operator  $\hat{n}_{\mathbf{k}} = \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$ . By similar logic, we can construct an operator corresponding to the spatial components of the total momentum, which works out to be

$$P^i = \int d^{n-1}k \hat{n}_{\mathbf{k}} k^i. \tag{9.85}$$

As we might expect, energy eigenstates will be those with fixed numbers of excitations, each of which carries an energy  $\omega$ . The excitations in the Fock basis

are interpreted as particles. This is how particles arise in a quantum field theory: energy eigenstates are collections of particles with definite momenta. Of course, our modes are plane waves that extend throughout space, not the localized tracks in bubble chambers that come to mind when we think of particles. What is worse, in a curved spacetime the wave equation will not have plane-wave solutions of definite frequency that we can interpret as particles. The solution to both issues is to think operationally, in terms of what would be observed by an experimental apparatus. The best strategy is to define a sensible notion of a particle detector that reduces to our intuitive picture in flat spacetime, and then define “particles” as “what a particle detector detects.” For a properly defined particle detector, our plane wave modes can be shown to “leave tracks” in the way we would hope; in an array of such detectors, if a plane wave sets off one detector, there is a high probability that it will set off other detectors along a path from the first one in a direction given by the wave vector. (We should point out that, if you visit an actual particle accelerator at a place like Fermilab or CERN, the detectors bear little resemblance to those invented by theorists studying quantum field theory in curved spacetime; deep down, however, there is a fundamental similarity.) For a discussion of particle detectors see Birrell and Davies (1982).

You might worry about the factor  $\delta^{(n-1)}(0)$  in the Hamiltonian (9.84), and well you should. It means that the Hamiltonian is infinite even when measured in the vacuum state  $|0\rangle$ . This term is the field-theory analogue of the harmonic-oscillator zero-point energy (9.20). In our discussion of the cosmological constant in Chapter 4, we mentioned that quantum fluctuations induced a formally infinite displacement of the classical vacuum energy; this infinite contribution to the scalar-field Hamiltonian will, when gravity is included, show up as a divergent cosmological constant. The fact that it is an integral over an infinite range of  $\mathbf{k}$  of the infinite quantity  $\delta^{(n-1)}(0)$  can be translated into the statement that the total energy is an integral over an infinitely big space of an infinite energy density. But the energy density contributed by high-frequency modes is the real problem, not the infinite volume; if we regularized the calculation by performing it in a box of volume  $L^{n-1}$ , we would find

$$\frac{1}{2} \int d^{n-1}k \delta^{(n-1)}(0) \omega \rightarrow \frac{1}{2} \left(\frac{L}{2\pi}\right)^{n-1} \sum_{\mathbf{k}} \omega, \quad (9.86)$$

which diverges even for finite  $L$ , since  $\mathbf{k}$  (and thus  $\omega$ ) can be arbitrarily large. Putting a cutoff at some high momentum  $k_{\max}$  would recover (4.104).

In the case of the simple harmonic oscillator, we pointed out that the zero-point energy could have been avoided had we chosen a classical potential with a negative minimum; the quantum-mechanical contribution does not necessarily represent the true answer, only the displacement of the energy from its classical value. The same holds in field theory; we are free to define our original classical scalar field theory so that the quantum-mechanical vacuum energy vanishes. However, we cannot simply subtract off a finite energy mode by mode, since our freedom is only to add a single constant to the potential, and thus to the Hamil-

tonian density (9.52). To obtain a finite Hamiltonian for the vacuum state, this constant would have to be infinite. There is nothing wrong with subtracting off an infinite constant; it is a venerable technique in quantum field theory, known as “renormalization.” At times renormalization can seem scary or somehow illegitimate, but in truth it is perfectly sensible; infinities only arise in the relationship between quantum theories and their classical counterparts, not in any observable quantities. Since Nature presumably doesn’t know or care about our fondness for classical mechanics, there should be nothing deeply disturbing about renormalization.

Of course, once we renormalize to obtain a finite vacuum energy, this energy could be anything we like; it is completely arbitrary. This continues to hold for quantum field theory in curved spacetime; we might not be able to decompose the field into modes of definite frequency, and it is therefore impossible to assign a vacuum energy contribution to each mode, but a careful analysis allows one to renormalize the vacuum energy to whatever number you like. Again, nothing profound has happened; the vacuum energy was completely arbitrary in our classical model in the first place, we simply chose it to be zero for convenience. The cosmological constant problem does not arise because quantum mechanics contributes a huge amount of vacuum energy, since this contribution can be straightforwardly renormalized away; the problem arises because there is no reason for the resulting arbitrary number to be close to zero. As discussed before, from the point of view of effective field theory the problem is somewhat sharper, since there is a logical expectation for the scale of the vacuum energy, namely the Planck scale at which unknown quantum-gravity effects should be contributing. Throughout this chapter, however, we will only be concerned with the propagation of quantum fields in fixed spacetime backgrounds, not in using the quantum energy-momentum tensor as a source for Einstein’s equation; we can therefore choose to ignore the cosmological constant problem.

#### 9.4 ■ QUANTUM FIELD THEORY IN CURVED SPACETIME

In Chapter 4 we discussed how easy it is to generalize physical theories from flat to curved spacetime—we simply express the theories in a coordinate-invariant form, and assert that they remain true when spacetime is curved. This procedure remains valid for quantum field theory, although we will need to give up on some of the concepts that seemed indispensable in flat spacetime.

We start with the Lagrange density of a scalar field in curved spacetime,

$$\mathcal{L} = \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 - \xi R \phi^2 \right). \quad (9.87)$$

Aside from the predictable appearance of the metric  $g_{\mu\nu}$  and its determinant, we have also included a direct coupling to the curvature scalar  $R$ , parameterized by a constant  $\xi$ . Since  $\xi$  is dimensionless, there is no reason to expect that it is small;

indeed, it should naturally be of order unity. In the literature there are two favorite choices for the value of  $\xi$ : **minimal coupling** simply turns off the direct interaction with  $R$ ,

$$\xi = 0, \quad (9.88)$$

while **conformal coupling** sets

$$\xi = \frac{(n-2)}{4(n-1)}, \quad (9.89)$$

which is  $\xi = \frac{1}{6}$  in four dimensions. Using the formulas in Appendix G, it is easy to check that when  $\xi$  takes on this value and  $m = 0$ , the scalar field theory is invariant under conformal transformations  $g_{\mu\nu} \rightarrow \omega^2(x)g_{\mu\nu}$ . In fact, there is no good reason to choose either minimal or conformal coupling in the real world; no symmetry is enhanced by minimal coupling, and conformal invariance is certainly not a symmetry of most physical theories. (Since conformal transformations are local changes of scale, theories characterized by dimensionful parameters such as masses will generally not be conformally invariant.) Even if a classical theory is conformally invariant, quantization can break this symmetry, which happens for example in the theory of quantum chromodynamics (QCD) coupled to massless quarks. Generally, in four dimensions it is difficult to find exactly conformally invariant interacting theories, although some models with high degrees of supersymmetry are known to be conformally invariant.

We may proceed to quantize the theory as before. The conjugate momentum is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\nabla_0\phi)}, \quad (9.90)$$

which for the Lagrangian (9.87) is

$$\pi = \sqrt{-g}\nabla_0\phi. \quad (9.91)$$

We can impose canonical commutation relations

$$\begin{aligned} [\phi(t, \mathbf{x}), \phi(t, \mathbf{x}')] &= 0 \\ [\pi(t, \mathbf{x}), \pi(t, \mathbf{x}')] &= 0 \\ [\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] &= \frac{i}{\sqrt{-g}}\delta^{(n-1)}(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (9.92)$$

The equation of motion for the scalar field is

$$\square\phi - m^2\phi - \xi R\phi = 0. \quad (9.93)$$

For a spacelike hypersurface  $\Sigma$  with induced metric  $\gamma_{ij}$  and unit normal vector  $n^\mu$ , the inner product on solutions to this equation is

$$(\phi_1, \phi_2) = -i \int_{\Sigma} (\phi_1 \nabla_{\mu} \phi_2^* - \phi_2^* \nabla_{\mu} \phi_1) n^{\mu} \sqrt{\gamma} d^{n-1}x, \quad (9.94)$$

which is independent of the choice of  $\Sigma$ .

So far, so good. To continue the steps we took in flat space, we would now introduce a set of positive- and negative-frequency modes forming a complete basis for solutions to (9.93), expand the field operator  $\phi$  in terms of these modes, and interpret the operator coefficients as creation and annihilation operators. It is at this point where our procedure breaks down. Since there will generically not be any timelike Killing vector, we will not in general be able to find solutions to the wave equation that separate into time-dependent and space-dependent factors, and correspondingly cannot classify modes as positive- or negative-frequency. We can find a set of basis modes, but the problem is that there will generally be many such sets, with no way to prefer one over any others, and the notion of a vacuum or number operator will depend sensitively on which set we choose.

Let's see what we can do. We will always be able to find a set of solutions  $f_i(x^{\mu})$  to (9.93) that are orthonormal,

$$(f_i, f_j) = \delta_{ij}, \quad (9.95)$$

and corresponding conjugate modes with negative norm,

$$(f_i^*, f_j^*) = -\delta_{ij}. \quad (9.96)$$

The index  $i$  may be continuous or discrete; for the moment we will adopt notation appropriate to the discrete case. These modes can be chosen to be a complete set, so that we may expand our field as

$$\phi = \sum_i (\hat{a}_i f_i + \hat{a}_i^\dagger f_i^*). \quad (9.97)$$

The coefficients  $\hat{a}_i$  and  $\hat{a}_i^\dagger$  have commutation relations

$$\begin{aligned} [\hat{a}_i, \hat{a}_j] &= 0 \\ [\hat{a}_i^\dagger, \hat{a}_j^\dagger] &= 0 \\ [\hat{a}_i, \hat{a}_j^\dagger] &= \delta_{ij}. \end{aligned} \quad (9.98)$$

There will be a vacuum state  $|0_f\rangle$  that is annihilated by all the annihilation operators,

$$\hat{a}_i |0_f\rangle = 0 \quad \text{for all } i. \quad (9.99)$$

From this vacuum state we can define an entire Fock basis for the Hilbert space. As before, a state with  $n_i$  excitations is created by repeated action by  $\hat{a}_i^\dagger$ ,

$$|n_i\rangle = \frac{1}{\sqrt{n_i!}} (\hat{a}_i^\dagger)^{n_i} |0_f\rangle, \quad (9.100)$$

and likewise for states with different kinds of excitations. We can even define a number operator for each mode,

$$\hat{n}_{fi} = \hat{a}_i^\dagger \hat{a}_i. \quad (9.101)$$

The subscript  $f$  on the vacuum state and the number operator reminds us that they are defined with respect to the set of modes  $f_i$ .

This apparatus seems quite similar to what we had in flat space; why can't we declare the excitations created by  $\hat{a}_i^\dagger$  to be particles and be done with it? We could, but we must face the fact that there are other choices we could have made; the basis modes  $f_i(x^\mu)$  are highly nonunique. Consider an alternative set of modes  $g_i(x^\mu)$  with all of the properties that our original modes possessed, including forming (along with conjugate modes  $g_i^*$ ) a complete basis with respect to which we can expand our field operator,

$$\phi = \sum_i (\hat{b}_i g_i + \hat{b}_i^\dagger g_i^*). \quad (9.102)$$

The annihilation and creation operators  $\hat{b}_i$  and  $\hat{b}_i^\dagger$  have commutation relations

$$\begin{aligned} [\hat{b}_i, \hat{b}_j] &= 0 \\ [\hat{b}_i^\dagger, \hat{b}_j^\dagger] &= 0 \\ [\hat{b}_i, \hat{b}_j^\dagger] &= \delta_{ij}, \end{aligned} \quad (9.103)$$

and there will be a vacuum state  $|0_g\rangle$  that is annihilated by all the annihilation operators,

$$\hat{b}_i |0_g\rangle = 0 \quad \text{for all } i. \quad (9.104)$$

We can construct a Fock basis by repeated application of creation operators on this vacuum, and define a number operator

$$\hat{n}_{gi} = \hat{b}_i^\dagger \hat{b}_i. \quad (9.105)$$

What we have lost in the transition from flat to curved spacetime is any reason to prefer one set of modes over any other. In flat spacetime, we were able to pick out a natural set of modes by demanding that they be positive-frequency with respect to the time coordinate, as defined by (9.61). The time coordinate is not unique, since we are free to perform Lorentz transformations; but we saw that the vacuum state and total number operators are invariant under such transformations. Thus, every inertial observer will agree on what is the vacuum state, and how many particles are around.

In the more general context we are considering now, if one observer defines particles with respect to a set of modes  $f_i$  and another observer uses a set of modes  $g_i$ , they will typically disagree on how many particles are observed (or even if particles are observed at all). To see this, it is convenient to expand each set of modes in terms of the other,

$$\begin{aligned} g_i &= \sum_j (\alpha_{ij} f_j + \beta_{ij} f_j^*) \\ f_i &= \sum_j (\alpha_{ji}^* g_j - \beta_{ji} g_j^*). \end{aligned} \quad (9.106)$$

The transformation from one set of basis modes into another is known as a **Bogoliubov transformation**, and the matrices  $\alpha_{ij}$  and  $\beta_{ij}$  implementing the transformation are Bogolubov coefficients. Using the orthonormality of the mode functions, they can be expressed as

$$\begin{aligned} \alpha_{ij} &= (g_i, f_j) \\ \beta_{ij} &= -(g_i, f_j^*). \end{aligned} \quad (9.107)$$

They satisfy their own normalization conditions,

$$\begin{aligned} \sum_j (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) &= \delta_{ij} \\ \sum_j (\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}) &= 0. \end{aligned} \quad (9.108)$$

As well as describing a transformation between modes, the Bogolubov coefficients can be used to transform between the operators,

$$\begin{aligned} \hat{a}_i &= \sum_j (\alpha_{ij} \hat{b}_j + \beta_{ij}^* \hat{b}_j^\dagger) \\ \hat{b}_i &= \sum_j (\alpha_{ij}^* \hat{a}_j - \beta_{ij}^* \hat{a}_j^\dagger). \end{aligned} \quad (9.109)$$

Now imagine that the system is in the  $f$ -vacuum  $|0_f\rangle$ , in which no  $f$ -particles would be observed; we would like to know how many particles are observed by an observer using the  $g$ -modes. We therefore calculate the expectation value of the  $g$  number operator in the  $f$ -vacuum:

$$\begin{aligned} \langle 0_f | \hat{n}_{gi} | 0_f \rangle &= \langle 0_f | \hat{b}_i^\dagger \hat{b}_i | 0_f \rangle \\ &= \left\langle 0_f \left| \sum_{jk} (\alpha_{ij} \hat{a}_j^\dagger - \beta_{ij} \hat{a}_j) (\alpha_{ik}^* \hat{a}_k - \beta_{ik}^* \hat{a}_k^\dagger) \right| 0_f \right\rangle \\ &= \sum_{jk} (-\beta_{ij}) (-\beta_{ik}^*) \langle 0_f | \hat{a}_j \hat{a}_k^\dagger | 0_f \rangle \end{aligned}$$

$$\begin{aligned}
 &= \sum_{jk} \beta_{ij} \beta_{ik}^* \langle 0_f | (\hat{a}_k^\dagger \hat{a}_j + \delta_{jk}) | 0_f \rangle \\
 &= \sum_{jk} \beta_{ij} \beta_{ik}^* \delta_{jk} \langle 0_f | 0_f \rangle \\
 &= \sum_j \beta_{ij} \beta_{ij}^*. \tag{9.110}
 \end{aligned}$$

The number of  $g$ -particles in the  $f$ -vacuum can thus be expressed in terms of the Bogolubov coefficients as

$$\langle 0_f | \hat{n}_{gi} | 0_f \rangle = \sum_j |\beta_{ij}|^2. \tag{9.111}$$

There is no reason for this to vanish; what looks like an empty vacuum from one perspective will be bubbling with particles according to another. If any of the  $\beta_{ij}$  are nonvanishing, the vacuum states will not coincide. We can understand why this is by looking at (9.109), where we see that  $\beta_{ij}$  describes the admixture of creation operators from one basis into the annihilation operators in the other basis.

This talk about modes and number operators may seem unnecessarily abstract; certainly, if an actual particle detector is traveling along some trajectory in a possibly-curved spacetime, it will either detect particles or not, without knowing what set of basis modes we are using for field theory. How do we know what definition of “particles” is actually being used by such a detector? The answer is that a detector measures the proper time  $\tau$  along its trajectory, and will define positive- and negative-frequency with respect to that proper time. Thus, if a set of modes  $f_i$  can be found that obey

$$\frac{D}{d\tau} f_i = -i\omega f_i, \tag{9.112}$$

we can use these modes to calculate how many particles the detector will see. Of course, it will generally not be possible to find such modes all over the spacetime. The one time that it might be possible is in a *static* spacetime, when we have a hypersurface-orthogonal timelike Killing vector  $K^\mu$ . In that case we can choose coordinates in which the metric components are independent of the time coordinate  $t$ , and there are no time-space cross terms:

$$\partial_0 g_{\mu\nu} = 0, \quad g_{0i} = 0. \tag{9.113}$$

(Indices  $i, j$  are now spatial components, not mode labels.) For such a metric, the d'Alembertian acting on some mode function  $f(t, \mathbf{x})$  works out to be

$$\square f = \left[ g^{00} \partial_0^2 + \tfrac{1}{2} g^{00} g^{ij} (\partial_i g_{00}) \partial_j + g^{ij} \partial_i \partial_j - g^{ij} \Gamma_{ij}^k \partial_k \right] f. \tag{9.114}$$

The equation of motion (9.93) can thus be written in the form

$$\partial_0^2 f = -\left(g^{00}\right)^{-1} \left[ g^{ij} \partial_i \partial_j + \tfrac{1}{2} g^{00} g^{ij} (\partial_i g_{00}) \partial_j - g^{ij} \Gamma_{ij}^k \partial_k - (m^2 + \xi R) \right] f. \quad (9.115)$$

The operator on the left is a pure time derivative, while the operator on the right involves only spatial derivatives and functions of space alone. We can therefore find separable solutions

$$f_\omega(t, \mathbf{x}) = e^{-i\omega t} \tilde{f}_\omega(\mathbf{x}), \quad (9.116)$$

which can be described as positive-frequency,

$$\partial_t f_\omega(t, \mathbf{x}) = -i\omega f_\omega(t, \mathbf{x}), \quad \omega > 0. \quad (9.117)$$

This relation can be recast in a coordinate-invariant form as

$$\mathcal{L}_K f_\omega = K^\mu \partial_\mu f_\omega = -i\omega f_\omega, \quad \omega > 0, \quad (9.118)$$

where  $\mathcal{L}_K f_\omega$  denotes the Lie derivative of  $f_\omega$  along  $K$ . There will also be negative-frequency conjugate modes,

$$\mathcal{L}_K f_\omega^* = K^\mu \partial_\mu f_\omega^* = i\omega f_\omega^*, \quad \omega > 0. \quad (9.119)$$

Together, the modes  $(f_\omega, f_\omega^*)$  will form a basis for solutions to the wave equation in a static background. The existence of such modes won't help us unless they are relevant for our detector; if the detector's trajectory follows along orbits of the Killing field (the four-velocity  $U^\mu = dx^\mu/d\tau$  is proportional to  $K^\mu$ ), the proper time will be proportional to the Killing time  $t$ , and modes that are positive-frequency with respect to this Killing vector will serve as a natural basis for describing Fock space. We will see this phenomenon at work in our discussion of the Unruh effect in the next section.

In the last section we mentioned the need to renormalize the vacuum energy in quantum field theory. This requirement still exists in curved spacetime, but an appropriate renormalization procedure is harder to construct, since there is no preferred mode basis. Nevertheless, algebraic methods have been developed to define a renormalized energy-momentum tensor rigorously, at least in certain cases; we won't delve into this subject in detail, but should at least present some of the underlying philosophy. The basic idea is that, even in the presence of curvature, spacetime should look Minkowskian on small enough scales. Because the vacuum-energy divergence we found in flat spacetime was due to short-wavelength modes, we should be able to match the behavior of fields in curved spacetime on very small scales to those in flat spacetime, and subtract off any divergences that appear. In particular, we consider the two-point function of a quantum field  $\phi$  in some state  $|\psi\rangle$ .

$$G(x_1, x_2) = \langle \psi | \phi(x_1) \phi(x_2) | \psi \rangle, \quad (9.120)$$

where  $x_1$  and  $x_2$  are two spacetime points. The two-point function in the Minkowski vacuum becomes singular as  $x_1$  and  $x_2$  are brought close to each other. We would like to characterize this singularity, and insist that it hold for any regular state in curved spacetime. By “brought close to each other” we mean that  $\sigma(x_1, x_2)$ , the squared distance along the shortest geodesic connecting the two points, goes to zero. In the limit as  $x_1$  and  $x_2$  are very close, the squared geodesic distance is simply

$$\sigma(x_1, x_2) = g_{\mu\nu}(x_1^\mu - x_2^\mu)(x_1^\nu - x_2^\nu), \quad x_1 \rightarrow x_2. \quad (9.121)$$

Of course, in a Lorentzian manifold, the geodesic distance will vanish when points are null separated, not only when they are coincident. We therefore include a small imaginary part and take the limit as it goes to zero, by defining

$$\sigma_\epsilon(x_1, x_2) = \sigma(x_1, x_2) + 2i\epsilon(t_1 - t_2) + \epsilon^2. \quad (9.122)$$

Here,  $t$  is the timelike coordinate, and the limit as  $\epsilon \rightarrow 0^+$  is assumed. (The manifest coordinate-dependence of this formula will be irrelevant in this limit.) Then it turns out that there is a unique singularity structure for the natural vacuum in Minkowski spacetime, such that the two-point function (in four dimensions) contains a leading singularity of the form  $1/(4\pi^2\sigma_\epsilon)$  and a subleading one proportional to  $\ln \sigma_\epsilon$ , with all other terms being regular. We therefore require that any physically reasonable quantum state in curved spacetime obey

$$G(x_1, x_2) = \frac{U(x_1, x_2)}{4\pi^2\sigma_\epsilon} + V(x_1, x_2) \ln \sigma_\epsilon + W(x_1, x_2), \quad (9.123)$$

where the functions  $U(x_1, x_2)$ ,  $V(x_1, x_2)$ , and  $W(x_1, x_2)$  are all regular at  $x_1 = x_2$ , and  $U(x, x) = 1$ . A state with this property is said to be a **Hadamard state**. It can be shown that the renormalized energy-momentum tensor is well-defined and nonsingular in all Hadamard states, and furthermore that it will be singular in any non-Hadamard state. If the Hadamard condition is obeyed on some partial Cauchy surface, it will also be obeyed everywhere in the domain of dependence; in other words, the energy-momentum tensor may become singular on a horizon, but not within the Cauchy development of some well-posed initial data. States of this form, therefore, seem appropriate for consideration in QFT on curved spacetime. For details see Wald (1994).

We see that QFT in curved spacetime shares most of the basic features of QFT in flat spacetime; the crucial difference involves what we cannot do, namely decide on a natural set of basis modes that all inertial observers would identify as particles. At the end of Section 9.2 we briefly discussed an oscillator subject to a transient force, and how to define an  $S$ -matrix relating number eigenstates at

early times to number eigenstates at late times. The same set of ideas translates directly to quantum field theory. If we have a situation in which spacetime is static in the asymptotic past and future, but with some disturbance in between, we can define in- and out-states that are energy eigenstates at early and late times, and a set of Bogolubov coefficients describing how the in-vacuum (for example) will be described as a multiparticle configuration in terms of the out-states. This phenomenon goes by the name of particle production by gravitational fields; relevant physical examples include the early universe and black holes.<sup>2</sup>

### 9.5 ■ THE UNRUH EFFECT

We must admit that, having put so much effort into understanding the basics of quantum field theory in curved spacetime, we won't actually do any detailed calculations in a curved background. Instead, we will investigate a phenomenon that relies on the ideas we have introduced, but is manifested even in flat spacetime: the Unruh effect, which states that an accelerating observer in the traditional Minkowski vacuum state will observe a thermal spectrum of particles. Historically, the Unruh effect was discovered in an attempt to understand the physics underlying the Hawking effect (thermal radiation in the presence of a black hole event horizon). Our strategy will be to carefully derive the Unruh effect, and in the next section argue under reasonable assumptions that this implies the Hawking effect, which is more difficult to derive directly just because it's harder to solve wave equations in curved spacetime than in flat spacetime.

The basic idea of the Unruh effect is simple: it is a manifestation of the idea that observers with different notions of positive- and negative-frequency modes will disagree on the particle content of a given state. For a uniformly accelerated observer in Minkowski space, the trajectory will move along orbits of a time-like Killing vector, but not that of the usual time-translation symmetry. We can therefore expand the field in modes appropriate to the accelerated observer, and calculate the number operator in the ordinary Minkowski vacuum, where we will find a thermal spectrum of particles. Different sets of explanatory words can be attached to this result; the basic lesson to learn is that what we think of as an inert vacuum actually has the character of a thermal state.

In the interest of discarding all possible complications to get at the underlying phenomenon, we consider a quantum field theory that is as simple as it can be without becoming completely trivial: a massless ( $m = 0$ ) scalar field in two spacetime dimensions ( $n = 2$ ). In two dimensions, conformal coupling and minimal coupling coincide, so we do not include any direct interaction with the curvature scalar. (We're in flat spacetime, so such a coupling wouldn't have any effect anyway.) The relevant wave equation is thus

$$\square\phi = 0. \tag{9.124}$$

<sup>2</sup>Interestingly, the first discussion of particle production in curved spacetime was given by Schrödinger himself; see E. Schrödinger (1939), *Physica (Utrecht)* 6, 899.

Before diving into the quantization of this field theory, let's think about two-dimensional Minkowski space as seen by a uniformly accelerating observer. We know that the metric can be written in inertial coordinates as

$$ds^2 = -dt^2 + dx^2. \quad (9.125)$$

Consider an observer moving at a uniform acceleration of magnitude  $\alpha$  in the  $x$ -direction. We claim that the resulting trajectory  $x^\mu(\tau)$  will be given by

$$\begin{aligned} t(\tau) &= \frac{1}{\alpha} \sinh(\alpha\tau) \\ x(\tau) &= \frac{1}{\alpha} \cosh(\alpha\tau). \end{aligned} \quad (9.126)$$

Let's verify that this path corresponds to constant acceleration. The acceleration two-vector is given in the globally inertial coordinate system by

$$a^\mu = \frac{D^2 x^\mu}{d\tau^2} = \frac{d^2 x^\mu}{d\tau^2}, \quad (9.127)$$

where the covariant derivative along the path is equal to the ordinary derivative because the Christoffel symbols vanish in these coordinates. The components of  $a^\mu$  are thus

$$\begin{aligned} a^t &= \alpha \sinh(\alpha\tau) \\ a^x &= \alpha \cosh(\alpha\tau), \end{aligned} \quad (9.128)$$

and the magnitude is

$$\sqrt{a_\mu a^\mu} = \sqrt{-\alpha^2 \sinh^2(\alpha\tau) + \alpha^2 \cosh^2(\alpha\tau)} = \alpha. \quad (9.129)$$

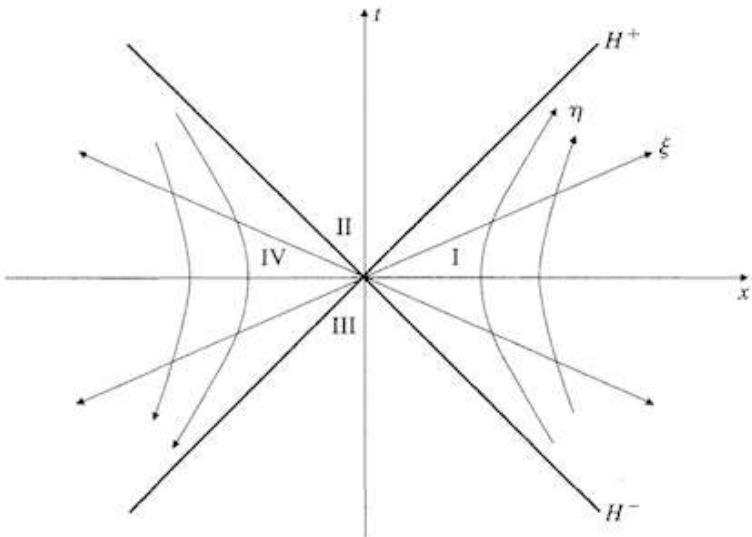
The path therefore corresponds to a constant acceleration of magnitude  $\alpha$ , as desired. The trajectory of our accelerated observer obeys the relation

$$x^2(\tau) = t^2(\tau) + 1/\alpha^2, \quad (9.130)$$

and thus describes an hyperboloid asymptoting to null paths  $x = -t$  in the past and  $x = t$  in the future. The accelerated observer travels from past null infinity to future null infinity, rather than timelike infinity as would be reached by geodesic observers.

We can choose new coordinates  $(\eta, \xi)$  on two-dimensional Minkowski space that are adapted to uniformly accelerated motion. Let

$$t = \frac{1}{\alpha} e^{a\xi} \sinh(a\eta), \quad x = \frac{1}{\alpha} e^{a\xi} \cosh(a\eta) \quad (x > |t|). \quad (9.131)$$



**FIGURE 9.1** Minkowski spacetime in Rindler coordinates. Region I is the region accessible to an observer undergoing constant acceleration in the  $+x$ -direction. The coordinates  $(\eta, \xi)$  can be used in region I, or separately in region IV, where they point in the opposite sense. The vector field  $\partial_\eta$  corresponds to the generator of Lorentz boost symmetry. The horizons  $H^\pm$  are Killing horizons for this vector field, and also represent boundaries of the past and future as witnessed by the Rindler observer.

The new coordinates have ranges

$$-\infty < \eta, \xi < +\infty, \quad (9.132)$$

and cover the wedge  $x > |t|$ , labeled as region I in Figure 9.1. In these coordinates, the constant-acceleration path (9.126) is given by

$$\begin{aligned} \eta(\tau) &= \frac{\alpha}{a}\tau \\ \xi(\tau) &= \frac{1}{a} \ln\left(\frac{a}{\alpha}\right), \end{aligned} \quad (9.133)$$

so that the proper time is proportional to  $\eta$  and the spatial coordinate  $\xi$  is constant. In particular, an observer with  $\alpha = a$  moves along the path

$$\eta = \tau, \quad \xi = 0. \quad (9.134)$$

The metric in these coordinates takes the form

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2). \quad (9.135)$$

Region I, with this metric, is known as **Rindler space**, even though it is obviously just a part of Minkowski space. A **Rindler observer** is one moving along

a constant-acceleration path, as in (9.133). The causal structure of Rindler space resembles the region  $r > 2GM$  of the maximally extended Schwarzschild solution of Figure 5.12. In particular, the null line  $x = t$ , labeled  $H^+$  in Figure 9.1, is a future Cauchy horizon for any  $\eta = \text{constant}$  spacelike hypersurface in region I; similarly,  $H^-$  is a past Cauchy horizon. These horizons are reminiscent of the event horizons in the Kruskal diagram, with static observers ( $r = \text{constant}$ ) in Schwarzschild being related to constant-acceleration paths in Rindler space.

The metric components in (9.135) are independent of  $\eta$ , so we immediately know that  $\partial_\eta$  is a Killing vector. But of course this is just Minkowski spacetime, so we think we know what all of the Killing vectors are. Indeed, if we express  $\partial_\eta$  in the  $(t, x)$  coordinates, we find

$$\begin{aligned}\partial_\eta &= \frac{\partial t}{\partial \eta} \partial_t + \frac{\partial x}{\partial \eta} \partial_x \\ &= e^{a\xi} [\cosh(a\eta)\partial_t + \sinh(a\eta)\partial_x] \\ &= a(x\partial_t + t\partial_x).\end{aligned}\quad (9.136)$$

This is nothing more or less than the Killing field associated with a boost in the  $x$ -direction. It is clear from this expression that this Killing field naturally extends throughout the spacetime; in regions II and III it is spacelike, while in region IV it is timelike but past-directed. The horizons we have identified are actually Killing horizons for  $\partial_\eta$ . The redshift factor, defined in (6.12) as the magnitude of the norm of the Killing vector, is

$$V = e^{a\xi}. \quad (9.137)$$

The surface gravity  $\kappa = \sqrt{\nabla_\mu V \nabla^\mu V}$  of this Killing horizon is thus

$$\kappa = a. \quad (9.138)$$

There is no real gravitational force, since we're in flat space; but this surface gravity characterizes the acceleration of Rindler observers.

We can also define coordinates  $(\eta, \xi)$  in region IV by flipping the signs in (9.131),

$$t = -\frac{1}{a}e^{a\xi} \sinh(a\eta), \quad x = -\frac{1}{a}e^{a\xi} \cosh(a\eta) \quad (x < |t|). \quad (9.139)$$

The sign guarantees that  $\partial_\eta$  and  $\partial_t$  point in opposite directions in region IV. Strictly speaking, we cannot use  $(\eta, \xi)$  simultaneously in regions I and IV, since the ranges of these coordinates are the same in each region, but we will be okay so long as we explicitly indicate to which region we are referring. The reason why it's better to use the same set of coordinate labels twice, rather than simply introducing new coordinates, is that the metric (9.135) will apply to both region I and region IV.

Along the surface  $t = 0$ ,  $\partial_\eta$  is a hypersurface-orthogonal timelike Killing vector, except for the single point  $x = 0$  where it vanishes. This vector can therefore be used to define a set of positive- and negative-frequency modes, on which we can build a Fock basis for the scalar-field Hilbert space. The massless Klein-Gordon equation in Rindler coordinates takes the form

$$\square\phi = e^{-2a\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi = 0. \quad (9.140)$$

A normalized plane wave  $g_k = (4\pi\omega)^{-1/2}e^{-i\omega\eta+ik\xi}$ , with  $\omega = |k|$ , solves this equation and apparently has positive frequency, in the sense that  $\partial_\eta g_k = -i\omega g_k$ . But we need our modes to be positive-frequency with respect to a future-directed Killing vector, and in region IV that role is played by  $\partial_{(-\eta)} = -\partial_\eta$  rather than  $\partial_\eta$ . To deal with this annoyance, we introduce two sets of modes, one with support in region I and the other in region IV:

$$g_k^{(1)} = \begin{cases} \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega\eta+ik\xi} & \text{I} \\ 0 & \text{IV} \end{cases} \quad (9.141)$$

$$g_k^{(2)} = \begin{cases} 0 & \text{I} \\ \frac{1}{\sqrt{4\pi\omega}}e^{+i\omega\eta+ik\xi} & \text{IV} \end{cases}$$

We take  $\omega = |k|$  in each case; in two dimensions, the spatial wave vector is just the single number  $k$ . Each set of modes is positive-frequency with respect to the appropriate future-directed timelike Killing vector,

$$\partial_\eta g_k^{(1)} = -i\omega g_k^{(1)} \quad (9.142)$$

$$\partial_{(-\eta)} g_k^{(2)} = -i\omega g_k^{(2)}, \quad \omega > 0.$$

These two sets, along with their conjugates, form a complete set of basis modes for any solutions to the wave equation throughout the spacetime. (The single point  $x = t = 0$  is a set of measure zero, so we shouldn't have to worry about it.) Both sets are nonvanishing in regions II and III of the Rindler diagram; this is obscured by writing them in terms of the coordinates  $\eta$  and  $\xi$ , but these functions can be analytically extended into the future and past regions. Denoting the associated annihilation operators as  $\hat{b}_k^{(1,2)}$ , we can write

$$\phi = \int dk \left( \hat{b}_k^{(1)} g_k^{(1)} + \hat{b}_k^{(1)\dagger} g_k^{(1)*} + \hat{b}_k^{(2)} g_k^{(2)} + \hat{b}_k^{(2)\dagger} g_k^{(2)*} \right). \quad (9.143)$$

This expansion is an alternative to our expression (9.66) in terms of the original Minkowski modes, which in two dimensions takes the form

$$\phi = \int dk \left( \hat{a}_k f_k + \hat{a}_k^\dagger f_k^* \right). \quad (9.144)$$

It is straightforward to check that the modes (9.141) are properly normalized with respect to the inner product (9.94). In the metric (9.135), the future-directed unit normal to the surface  $\eta = 0$  is normalized to

$$-1 = g_{\mu\nu} n^\mu n^\nu = -e^{2a\xi} (n^0)^2, \quad (9.145)$$

or

$$n^0 = e^{-a\xi}, \quad (9.146)$$

Meanwhile, the spatial metric determinant satisfies

$$\sqrt{\gamma} = e^{a\xi}. \quad (9.147)$$

We therefore have  $n^0 \sqrt{\gamma} = 1$ , and the calculation of the inner product of the Rindler modes follows precisely that of ordinary Minkowski modes. We end up with

$$\begin{aligned} (g_{k_1}^{(1)}, g_{k_2}^{(1)}) &= \delta(k_1 - k_2) \\ (g_{k_1}^{(2)}, g_{k_2}^{(2)}) &= \delta(k_1 - k_2) \\ (g_{k_1}^{(1)}, g_{k_2}^{(2)}) &= 0, \end{aligned} \quad (9.148)$$

and similarly for the conjugate modes.

There are thus two sets of modes, Minkowski and Rindler, with which we can expand solutions to the Klein–Gordon equation in a flat two-dimensional space-time. Although the Hilbert space for the theory is the same in either representation, its interpretation as a Fock space will be different; in particular, the vacuum states will be different. The Minkowski vacuum  $|0_M\rangle$ , satisfying

$$\hat{a}_k |0_M\rangle = 0, \quad (9.149)$$

will be described as a multi-particle state in the Rindler representation; likewise, the Rindler vacuum  $|0_R\rangle$ , satisfying

$$\hat{b}_k^{(1)} |0_R\rangle = \hat{b}_k^{(2)} |0_R\rangle = 0, \quad (9.150)$$

will be described as a multi-particle state in the Minkowski representation. At a practical level, the difference arises because an individual Rindler mode can never be written as a sum of positive-frequency Minkowski modes; at  $t = 0$  the Rindler modes only have support on the half-line, and such a function cannot be expanded in purely positive-frequency plane waves. Thus, the Rindler annihilation operators used to define  $|0_R\rangle$  are necessarily superpositions of Minkowski creation and annihilation operators, so the two vacua cannot coincide.

A Rindler observer will be static with respect to orbits of the boost Killing vector  $\partial_\eta$ . Such an observer in region I will therefore describe particles in terms

of the Rindler modes  $g_k^{(1)}$ , and in particular will observe a state in the Rindler vacuum to be devoid of particles, a state  $\hat{b}_k^{(1)\dagger} |0_R\rangle$  to contain a single particle of frequency  $\omega = |k|$ , and so on. Conversely, a Rindler observer traveling through the Minkowski vacuum state will detect a background of particles, even though an inertial observer would describe the state as being completely empty. What kind of particles would the Rindler observer detect? We know how to answer this question: Calculate the Bogolubov coefficients relating the Minkowski and Rindler modes, and use them to determine the expectation value of the Rindler number operator in the Minkowski vacuum. This is straightforward but tedious, so we will take a shortcut due to Unruh. We will find a set of modes that share the same vacuum state as the Minkowski modes (although the description of excited states may be different), but for which the overlap with the Rindler modes is more direct. The way to do this is to start with the Rindler modes, analytically extend them to the entire spacetime, and express this extension in terms of the original Rindler modes.

To see how this works, notice from (9.131) and (9.139) that we have the following relationships between the Minkowski coordinates  $(t, x)$  and Rindler coordinates  $(\eta, \xi)$  in regions I and IV:

$$\begin{aligned} e^{-a(\eta-\xi)} &= \begin{cases} a(-t+x) & \text{I} \\ a(t-x) & \text{IV} \end{cases} \\ e^{a(\eta+\xi)} &= \begin{cases} a(t+x) & \text{I} \\ a(-t-x) & \text{IV} \end{cases} \end{aligned} \quad (9.151)$$

We can therefore express the spacetime dependence of a mode  $g_k^{(1)}$  with  $k > 0$  (so  $\omega = k$ ) in terms of Minkowski coordinates in region I as

$$\begin{aligned} \sqrt{4\pi\omega} g_k^{(1)} &= e^{-i\omega\eta+ik\xi} \\ &= e^{-i\omega(\eta-\xi)} \\ &= a^{i\omega/a} (-t+x)^{i\omega/a}. \end{aligned} \quad (9.152)$$

The analytic extension of this function throughout spacetime is straightforward; we simply use this final expression for any values of  $(t, x)$ . But we would like to express the result in terms of the original Rindler modes everywhere; since the  $g_k^{(1)}$  modes vanish in region IV, we need to bring the modes  $g_k^{(2)}$  into play. When we express them in terms of the Minkowski coordinates in region IV, for  $k > 0$  we obtain

$$\begin{aligned} \sqrt{4\pi\omega} g_k^{(2)} &= e^{+i\omega\eta+ik\xi} \\ &= e^{+i\omega(\eta+\xi)} \\ &= a^{-i\omega/a} (-t-x)^{-i\omega/a}. \end{aligned} \quad (9.153)$$

This doesn't match the behavior of (9.152) that we want. But if we take the complex conjugate and reverse the wave number, we obtain

$$\begin{aligned}\sqrt{4\pi\omega}g_{-k}^{(2)*} &= e^{-i\omega\eta+ik\xi} \\ &= e^{-i\omega(\eta-\xi)} \\ &= a^{i\omega/a}(t-x)^{i\omega/a} \\ &= a^{i\omega/a}[e^{-i\pi}(-t+x)]^{i\omega/a} \\ &= a^{i\omega/a}e^{\pi\omega/a}(-t+x)^{i\omega/a}.\end{aligned}\quad (9.154)$$

The combination

$$\sqrt{4\pi\omega}\left(g_k^{(1)} + e^{-\pi\omega/a}g_{-k}^{(2)*}\right) = a^{i\omega/a}(-t+x)^{i\omega/a}\quad (9.155)$$

is therefore well-defined along the whole surface  $t = 0$ . We have explicitly examined the case  $k > 0$ , but an identical result obtains for  $k < 0$ .

A properly normalized version of this mode is given by

$$h_k^{(1)} = \frac{1}{\sqrt{2 \sinh(\frac{\pi\omega}{a})}} \left( e^{\pi\omega/2a} g_k^{(1)} + e^{-\pi\omega/2a} g_{-k}^{(2)*} \right). \quad (9.156)$$

This is an appropriate analytic extension of the  $g_k^{(1)}$  modes; to get a complete set, we need to include the extensions of the  $g_k^{(2)}$  modes, which by an analogous argument are given by

$$h_k^{(2)} = \frac{1}{\sqrt{2 \sinh(\frac{\pi\omega}{a})}} \left( e^{\pi\omega/2a} g_k^{(2)} + e^{-\pi\omega/2a} g_{-k}^{(1)*} \right). \quad (9.157)$$

To verify the normalization, for example for  $h_k^{(1)}$ , we use (9.148):

$$\begin{aligned}\left(h_{k_1}^{(1)}, h_{k_2}^{(1)}\right) &= \frac{1}{2\sqrt{\sinh(\frac{\pi\omega_1}{a})\sinh(\frac{\pi\omega_2}{a})}} \left[ e^{\pi(\omega_1+\omega_2)/2a} \left( g_{k_1}^{(1)}, g_{k_2}^{(1)} \right) \right. \\ &\quad \left. + e^{-\pi(\omega_1+\omega_2)/2a} \left( g_{-k_1}^{(2)*}, g_{-k_2}^{(2)*} \right) \right] \\ &= \frac{1}{2\sqrt{\sinh(\frac{\pi\omega_1}{a})\sinh(\frac{\pi\omega_2}{a})}} \left[ e^{\pi(\omega_1+\omega_2)/2a} \delta(k_1 - k_2) \right. \\ &\quad \left. + e^{-\pi(\omega_1+\omega_2)/2a} \delta(-k_1 + k_2) \right] \\ &= \frac{e^{\pi\omega_1/a} - e^{-\pi\omega_1/a}}{2\sinh(\frac{\pi\omega_1}{a})} \delta(k_1 - k_2)\end{aligned}$$

$$= \delta(k_1 - k_2), \quad (9.158)$$

just as we would like.

We can now expand our field in these modes,

$$\phi = \int dk \left( \hat{c}_k^{(1)} h_k^{(1)} + \hat{c}_k^{(1)\dagger} h_k^{(1)*} + \hat{c}_k^{(2)} h_k^{(2)} + \hat{c}_k^{(2)\dagger} h_k^{(2)*} \right). \quad (9.159)$$

From our discussion of Bogolubov transformations in Section 9.4, we know that the expressions (9.156) and (9.157) for the  $h_k^{(1,2)}$  modes in terms of the  $g_k^{(1,2)}$  modes implies corresponding expressions for the Rindler operators  $\hat{b}_k^{(1,2)}$  in terms of the operators  $\hat{c}_k^{(1,2)}$ , as

$$\begin{aligned} \hat{b}_k^{(1)} &= \frac{1}{\sqrt{2 \sinh(\frac{\pi\omega}{a})}} \left( e^{\pi\omega/2a} \hat{c}_k^{(1)} + e^{-\pi\omega/2a} \hat{c}_{-k}^{(2)\dagger} \right) \\ \hat{b}_k^{(2)} &= \frac{1}{\sqrt{2 \sinh(\frac{\pi\omega}{a})}} \left( e^{\pi\omega/2a} \hat{c}_k^{(2)} + e^{-\pi\omega/2a} \hat{c}_{-k}^{(1)\dagger} \right). \end{aligned} \quad (9.160)$$

We can therefore express the Rindler number operator in region I,

$$\hat{n}_R^{(1)}(k) = \hat{b}_k^{(1)\dagger} \hat{b}_k^{(1)}, \quad (9.161)$$

in terms of the new operators  $\hat{c}_k^{(1,2)}$ .

The original positive-frequency Minkowski plane-wave modes with  $k > 0$ ,  $f_k \propto e^{-i\omega(t-x)}$ , are analytic and bounded for complex  $(t, x)$  so long as  $\text{Im}(t-x) \leq 0$ . (Such modes are called “right-moving,” as they describe waves propagating to the right.) The same holds for our new modes  $h_k^{(1)}$  so long as we take the branch cut for the imaginary power to lie in the upper-half complex  $(t-x)$  plane, as we can see from examination of (9.152) and (9.154); this is consistent with our setting  $-1 = e^{-i\pi}$  in (9.154). Similar considerations apply to the  $h_k^{(2)}$  modes, which are analytic and bounded in the lower-half complex  $(t+x)$  plane, as are the positive-frequency Minkowski plane-wave modes with  $k < 0$  (left-moving). Consequently, unlike the original Rindler modes  $g_k^{(1,2)}$ , we know that the modes  $h_k^{(1,2)}$  can be expressed purely in terms of positive-frequency Minkowski modes  $f_k$ . They therefore share the same vacuum state  $|0_M\rangle$ , so that

$$\hat{c}_k^{(1)} |0_M\rangle = \hat{c}_k^{(2)} |0_M\rangle = 0. \quad (9.162)$$

The excited states will not coincide, but that won’t bother us, since we are interested in what a Rindler observer sees when the state is precisely in the Minkowski vacuum. An observer in region I, for example, will observe particles defined by the operators  $\hat{b}_k^{(1)}$ ; the expected number of such particles of frequency  $\omega$  will be given by

$$\langle 0_M | \hat{n}_R^{(1)}(k) | 0_M \rangle = \langle 0_M | \hat{b}_k^{(1)\dagger} \hat{b}_k^{(1)} | 0_M \rangle$$

$$\begin{aligned}
&= \frac{1}{2 \sinh(\frac{\pi \omega}{a})} \langle 0_M | e^{-\pi \omega/a} \hat{c}_{-k}^{(1)} \hat{c}_{-k}^{(1)\dagger} | 0_M \rangle \\
&= \frac{e^{-\pi \omega/a}}{2 \sinh(\frac{\pi \omega}{a})} \delta(0) \\
&= \frac{1}{e^{2\pi \omega/a} - 1} \delta(0),
\end{aligned} \tag{9.163}$$

where we have used the fact that a  $\hat{c}_k^{(1)\dagger} | 0_M \rangle$  is a normalized one-particle state,

$$\langle 0_M | \hat{c}_k^{(1)} \hat{c}_k^{(1)\dagger} | 0_M \rangle = \delta(0). \tag{9.164}$$

The delta function in (9.163) is merely an artifact of our use of (nonsquare-integrable) plane wave basis modes; had we constructed normalized wave packets, we would have obtained a finite result with an identical spectrum.

The result (9.163) is a Planck spectrum with temperature

$$T = \frac{a}{2\pi}. \tag{9.165}$$

Thus, *an observer moving with uniform acceleration through the Minkowski vacuum observes a thermal spectrum of particles*. This is the **Unruh effect**. Of course, there is more to thermal radiation than just the spectrum (9.163); to be truly thermal, we should check that there are no hidden correlations in the observed particles. This has been verified; the radiation detected by a Rindler observer is truly thermal. At the most basic level, the Unruh effect shows how two different sets of observers (inertial and Rindler) will describe the same state in very different terms; at a slightly deeper level, it reveals the essentially thermal nature of the vacuum in quantum field theory.

The temperature  $T = a/2\pi$  is what would be measured by an observer moving along the path  $\xi = 0$ , which feels an acceleration  $\alpha = a$ . Using (9.133), we know that any other path with  $\xi = \text{constant}$  feels an acceleration

$$\alpha = ae^{-a\xi} \tag{9.166}$$

and thus should measure thermal radiation with a temperature  $\alpha/2\pi$ . This is consistent with our discussion in Chapter 6 of the redshift witnessed by static observers moving along orbits of some Killing vector  $K^\mu$ ; we found that radiation emitted with frequency  $\omega_1$  at a point  $x_1$  would be observed at a point  $x_2$  with a frequency

$$\omega_2 = \frac{V_1}{V_2} \omega_1, \tag{9.167}$$

where the redshift factor  $V$  is the norm of the Killing vector. In (9.137) we found that the redshift factor associated with  $\partial_\eta$  is  $V = e^{a\xi}$ , so that

$$\omega_2 = e^{a(\xi_1 - \xi_2)} \omega_1. \tag{9.168}$$

Thus, if an observer at  $\xi_1 = 0$  detects a temperature  $T = a/2\pi$ , the observer at  $\xi_2 = \xi$  will see it to be redshifted to a temperature  $T = ae^{-a\xi}/2\pi$ , just as in (9.166). In particular, the temperature redshifts all the way to zero as  $\xi \rightarrow +\infty$ . This makes sense, since a Rindler observer at infinity will be nearly inertial, and will define the same notion of vacuum and particles as an ordinary Minkowski observer.

The Unruh effect tells us that an accelerated observer will detect particles in the Minkowski vacuum state. An inertial observer, of course, would describe the same state as being completely empty; indeed, the expectation value of the energy-momentum tensor would be  $\langle T_{\mu\nu} \rangle = 0$ . But if there is no energy-momentum, how can the Rindler observers detect particles? This is a subtle issue, but by no means a contradiction. If the Rindler observer is to detect background particles, she must carry a detector—some sort of apparatus coupled to the particles being detected. But if a detector is being maintained at constant acceleration, energy is not conserved; we need to do work constantly on the detector to keep it accelerating. From the point of view of the Minkowski observer, the Rindler detector *emits* as well as absorbs particles; once the coupling is introduced, the possibility of emission is unavoidable. When the detector registers a particle, the inertial observer would say that it had emitted a particle and felt a radiation-reaction force in response. Ultimately, then, the energy needed to excite the Rindler detector does not come from the background energy-momentum tensor, but from the energy we put into the detector to keep it accelerating.

## 9.6 ■ THE HAWKING EFFECT AND BLACK HOLE EVAPORATION

Even though it occurs in flat spacetime, the Unruh effect teaches us the most important lesson of QFT in curved spacetime, the idea that “vacuum” and “particles” are observer-dependent notions rather than fundamental concepts. In fact, given our understanding of the Unruh effect, we can see almost immediately how the Hawking effect arises. This should not be too surprising, as we have already noted the similarity between the causal structure of Rindler space and that of the maximally-extended Schwarzschild spacetime describing an eternal black hole. We will therefore be able to argue in favor of Hawking radiation without ever doing an explicit calculation in curved spacetime; of course, there are many features that you might like to investigate in more detail, for which the full power of the curved metric is necessary. In addition to Birrell and Davies (1982) and Wald (1994), there are good review articles where you can find a more full discussion of the issues discussed here.<sup>3</sup> Our derivation of Hawking radiation follows that of Jacobson.

<sup>3</sup>T.A. Jacobson, “Introductory Lectures on Black Hole Thermodynamics,” Lectures at University of Utrecht (1996), <http://www.fys.ruu.nl/~wwwthe/lectures/itfuu-0196.ps>; R.M. Wald, “The thermodynamics of black holes,” *Living Rev. Rel.* 4, 6 (2001), <http://arxiv.org/gr-qc/9912119>; J. Traschen, “An introduction to black hole evaporation” (2000), <http://arxiv.org/gr-qc/0010055>.

Consider a static observer at radius  $r_1 > 2GM$  outside a Schwarzschild black hole. Such an observer moves along orbits of the timelike Killing vector  $K = \partial_t$ . In Chapter 6 we showed that the redshift factor  $V = \sqrt{-K_\mu K^\mu}$  for static observers in Schwarzschild is given by

$$V = \sqrt{1 - \frac{2GM}{r}}, \quad (9.169)$$

with a corresponding magnitude of the acceleration given by

$$a = \frac{GM}{r\sqrt{r-2GM}}. \quad (9.170)$$

For observers very close to the event horizon,  $r_1 - 2GM \ll 2GM$ , this acceleration becomes very large compared to the scale set by the Schwarzschild radius,

$$a_1 \gg \frac{1}{2GM}. \quad (9.171)$$

The Schwarzschild radius in turn sets the radius of curvature of spacetime near the horizon. Therefore, as observed over length- and timescales set by  $a_1^{-1} \ll 2GM$ , spacetime looks essentially flat. Let us make the crucial assumption that the quantum state of some scalar field  $\phi$  looks like the Minkowski vacuum (free of any particles) as seen by *freely-falling* observers near the black hole. This assumption is reasonable, since the event horizon is not a local barrier; a freely-falling observer sees nothing special happen when crossing the horizon. Then the static observer looks just like a constant-acceleration observer in flat spacetime, and will detect Unruh radiation at a temperature  $T_1 = a_1/2\pi$ .

Now consider a static observer at infinity, or at least a distance  $r_2$  large compared to  $2GM$ . In that case there is no sense in which the spacetime curvature can be neglected over timescales  $a_2^{-1} \gg 2GM$ , so there is no reason to expect that they will see radiation with a temperature  $a_2/2\pi$ , where  $a_2$  is evaluated at  $r_2$ . But the radiation observed near the horizon will propagate to infinity with an appropriate redshift. We can apply the argument used at the end of the last section to determine what such an observer should see; they should detect thermal radiation redshifted to a temperature

$$T_2 = \frac{V_1}{V_2} T_1 = \frac{V_1}{V_2} \frac{a}{2\pi}. \quad (9.172)$$

At infinity we have  $V_2 \rightarrow 1$ , so the observed temperature is

$$T = \lim_{r_1 \rightarrow 2GM} \frac{V_1 a_1}{2\pi} = \frac{\kappa}{2\pi}, \quad (9.173)$$

where  $\kappa = \lim(Va)$  is the surface gravity; for Schwarzschild,  $\kappa = 1/4GM$ . Unlike for accelerating observers in flat spacetime, in Schwarzschild the static Killing vector has finite norm at infinity, and the radiation near the horizon redshifts to a finite value rather than all the way to zero. Observers far from the black

hole thus see a flux of thermal radiation emitted from the black hole at a temperature proportional to its surface gravity. This is the celebrated **Hawking effect**, and the radiation itself is known as Hawking radiation.

Despite its slickness, there is nothing dishonest about this derivation of the Hawking effect. In particular, the relation to acceleration makes it clear why the temperature is proportional to the black hole surface gravity (which continues to hold for more general black holes, not only Schwarzschild). However, we need to be clear about the assumption we made that the vacuum state near the horizon looks nonsingular to freely-falling observers. In technical terms, the renormalized energy-momentum tensor is taken to be finite at the horizon, or equivalently, the two-point function obeys the Hadamard condition (9.123).

The meaning of this assumption becomes more clear by considering possible vacuum states in the maximally extended Schwarzschild geometry. Such states are not necessarily physically relevant to a realistic black hole formed by gravitational collapse, but the possibilities that arise in the idealized case carry instructive lessons for the real world. We will only describe the states, not specify them quantitatively or derive any of their properties; for more details see the references above.

In searching for a vacuum state, we might begin by looking for a state that is regular [in the Hadamard sense, (9.123)] throughout spacetime. For maximally extended Schwarzschild, such a state was found by Hartle and Hawking, so we call it the **Hartle–Hawking vacuum**; indeed, this is the unique vacuum state that is regular everywhere and invariant under the Schwarzschild Killing vector  $\partial_t$ , representing time translations at infinity. In particular, recalling the conformal diagram of Schwarzschild shown in Figure 5.16, the Hartle–Hawking vacuum is regular on the past and future event horizons  $H^\pm$  at  $r = 2GM$ , and also on past and future null infinity  $\mathcal{I}^\pm$ . From the consideration of static observers as outlined above, we should then expect that the Hartle–Hawking vacuum features thermal radiation being emitted from the black hole, and indeed this turns out to be true. However, a close examination of this state reveals that there is an equal flux of thermal radiation coming in from past null infinity ( $\mathcal{I}^-$ ) toward the black hole; in other words, it represents a black hole in thermal equilibrium with its environment. This is not what we would use to model a realistic black hole in our universe. Another vacuum, more closely analogous to that of a black hole formed via gravitational collapse, is the **Unruh vacuum**, which is nonsingular on  $H^+$  (and therefore predicts outgoing Hawking radiation), but exhibits no incoming radiation from  $\mathcal{I}^-$ . The Unruh vacuum turns out to be singular on the past horizon  $H^-$  of Schwarzschild; this doesn't bother us if we are only using it as a model for realistic black holes, since a spacetime featuring gravitational collapse as in Figure 5.17 would not have a white hole or any past horizons. Finally, we might look for a vacuum state in which no particles come into the black hole, nor escape to infinity; in other words, vanishing flux at  $\mathcal{I}^\pm$ . There is such a state, called the **Boulware vacuum**. The existence of such a state seems to be in conflict with our argument for the Hawking effect from the Unruh effect, except that a careful analysis reveals that the Boulware vacuum is singular both on  $H^-$  and  $H^+$ . Thus,

the assumption that the vacuum is regular as seen by freely-falling observers near the horizon is violated in this state.

So a careful examination of vacuum states in an eternal Schwarzschild metric is consistent with our reasoning from the Unruh effect; states that are regular on  $H^+$  predict Hawking radiation of the expected form. Note that the existence of an event horizon is crucial to the argument; without such an horizon, the requirement that the state be regular on the horizon has no force. Consider for example a neutron star, whose radius may be close to the Schwarzschild radius but for which the spacetime is free of any horizons. Neutron stars do not emit any Hawking radiation. One way to understand this is to recognize that a static neutron-star metric features a Killing vector that is timelike everywhere, and can be used to define positive-frequency modes that extend throughout the spacetime and match the Minkowski modes at infinity. The resulting vacuum state would actually resemble the Boulware vacuum, free of flux at  $\mathcal{I}^\pm$ ; the fact that the full Boulware vacuum is singular on the horizon doesn't bother us in the neutron-star case, since there aren't any horizons.

To be absolutely sure that we have correctly chosen a vacuum state appropriate to realistic black holes, we should consider gravitational collapse in a spacetime that is nearly Minkowskian in the past and Schwarzschild in the future, as in Figure 5.17. If the vacuum takes the standard Minkowski form on  $\mathcal{I}^-$ , we can ask how the modes propagate through the collapse geometry to  $\mathcal{I}^+$ , defining an S-matrix as in (9.45) to determine what would be seen by asymptotic observers. This is in fact what Hawking did when he first discovered black hole radiation; the calculations involve some messy algebra but are basically straightforward, with the same answer for the temperature as we derived above.

Of course, from a complete calculation we can learn more than just the black-body temperature; we might ask, for example, what happens when the wavelength of the emitted radiation is comparable to the Schwarzschild radius, in which case our approximations clearly break down. If we were to carefully investigate the emission of arbitrary species of particles from any kind of black hole (that is, allowing for both charge and spin), we would find that the spectrum of emitted radiation takes the form

$$\langle \hat{n}_\omega \rangle = \frac{\Gamma(\omega)}{e^{2\pi(\omega-\mu)/\kappa} \pm 1}. \quad (9.174)$$

Here,  $\kappa$  is of course the surface gravity. The parameter  $\mu$  is a chemical potential, characterizing the tendency of the black hole to shed its conserved quantum numbers; a charged black hole preferentially emits particles with the same-sign charge as the hole, while a rotating black hole preferentially emits particles with the same-sign angular momentum as the hole. Hawking radiation therefore tends to bring black holes to a Schwarzschild state.  $\Gamma(\omega)$  is a greybody factor, which can be thought of as arising from backscattering of wavepackets off of the gravitational field and into the black hole. In the high-frequency limit the wavelength is very small and backscattering can be neglected; at very low frequencies the wavelength becomes greater than the Schwarzschild radius and backscattering

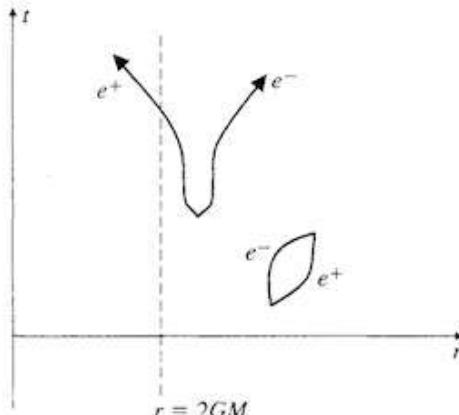
becomes important. Although an analytic expression for the greybody factor is hard to derive, in the limiting cases of large and small frequencies the greybody factor for a scalar field obeys

$$\begin{aligned}\Gamma(\omega) &\rightarrow 1, \quad \omega \gg \frac{1}{GM} \\ \Gamma(\omega) &\rightarrow \frac{A}{4\pi} \omega^2, \quad \omega \ll \frac{1}{GM},\end{aligned}\tag{9.175}$$

where  $A$  is the area of the black hole.

The discovery that black holes emit thermal radiation is certainly surprising from the point of view of classical general relativity, where we emphasized the impossibility of escape to infinity from points inside the event horizon. One picturesque way to understand what is going on is to think of vacuum fluctuations in terms of Feynman diagrams, with the fluctuations being represented by virtual particle/antiparticle pairs popping in and out of existence. This picture is also helpful, for example, in understanding observed phenomena such as the Lamb shift, in which atomic spectra are affected by the interaction of photons with virtual electron/positron pairs. Normally, the pairs will always annihilate, and their effect is only indirect, through a renormalization of processes coupled to the virtual particles. In the presence of an event horizon, however, occasionally one member of a virtual pair will fall into the black hole while its partner escapes to infinity, as depicted in Figure 9.2. In this picture, it is these escaping virtual particles that we observe as Hawking radiation. The total energy of the virtual pair must add to zero, but the infalling particle can have a negative energy as viewed from infinity, because the asymptotically-timelike Killing vector is spacelike inside the horizon. The picture is somewhat informal, but provides a useful heuristic for what is going on.

Once we know the formula for the temperature of a black hole we can fix the proportionality constants in the relationships between black hole parameters and



**FIGURE 9.2** Vacuum fluctuations occasionally result in one of a particle/antiparticle pair falling into the event horizon, and the other escaping to infinity as Hawking radiation.

thermodynamic variables, as listed in (6.118). Hawking radiation essentially consummates the marriage of black hole mechanics and thermodynamics; stationary black holes act just like bodies of energy  $E = M$  in thermal equilibrium with temperature  $T = \kappa/2\pi$  and entropy  $S = A/4G$ . This is a very large entropy indeed. For matter fields in the universe, the entropy is approximately equal to the number of relativistic particles; within one Hubble radius, this number works out to be

$$S_M \sim 10^{88}. \quad (9.176)$$

Meanwhile, the entropy of a black hole is the area of its horizon measured in Planck units (remember that we have been setting  $\hbar = 1$  all along). We can convert to astrophysical units to obtain

$$S_{BH} \sim 10^{90} \left( \frac{M}{10^6 M_\odot} \right)^2. \quad (9.177)$$

Thus, a single million-solar-mass black hole (such as can be found at the center of our galaxy, and many other galaxies) has more entropy than all of the matter in the visible universe. The total entropy of the universe is much smaller than we could make it, just by putting more mass into black holes. (When cosmologists say that the entropy  $S_M$  is large, they mean it is surprising that so much entropy is found within one curvature radius.) Presumably the reason why we are in such a low-entropy state has to do with initial conditions, and perhaps with inflation.

Coming back to black hole mechanics, we see a puzzle: The entropy of a macroscopic black hole will be huge, but from a statistical-mechanical point of view the entropy is supposed to measure the logarithm of the number of accessible states. A classical black hole is specified by a small number of parameters (mass, charge, and spin), so it is hard to know what those states could be. Nevertheless, we could take the attitude that this discrepancy doesn't really matter, since any information about the state of a black hole would presumably be hidden behind the event horizon.

The inclusion of quantum mechanics makes the puzzle worse rather than better, because black holes will not only radiate but also evaporate. When we started our investigation of QFT in curved spacetime, one of the rules we set was that we would assume a fixed background metric, and not worry about the effect of the energy-momentum tensor of the quantum fields themselves. Nevertheless, even in quantum mechanics we have conservation of energy (in the sense, for example, of a conserved ADM mass in an asymptotically flat spacetime). Hence, when Hawking radiation escapes to infinity, we may safely conclude that it will carry energy away from the black hole, which must therefore shrink in mass. (This phenomenon does not violate the area theorem, since the quantum field energy-momentum tensor will not obey the weak energy condition near the horizon.) As the mass shrinks, the surface gravity increases, and with it the temperature; there is a runaway process in which the entire mass evaporates away in a finite time.

Plugging in the numbers gives a lifetime of order

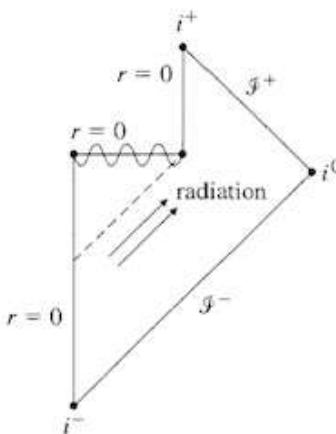
$$\tau_{\text{BH}} \sim \left( \frac{M}{m_p} \right)^3 t_p \sim \left( \frac{M}{M_\odot} \right)^3 \times 10^{71} \text{ sec}, \quad (9.178)$$

where  $m_p \sim 10^{-5}$  g is the Planck mass and  $t_p \sim 10^{-43}$  sec is the Planck time. Since the Hubble time is  $H_0^{-1} \sim 10^{18}$  sec, a solar-mass black hole has a lifetime of order  $10^{53}$  times the age of the universe. This seems like a long time, but we are speaking of questions of principle here.

You can see why the question of the black hole entropy has become so severe: Once the black hole has evaporated, we can no longer appeal to the event horizon as a way to hide purported states of the black hole. There is no black hole any more, just the Hawking radiation it produced. The fact that this radiation is supposed to be precisely thermal (no hidden correlations in the outgoing particles) means that it has no way of conveying the vast amount of information needed to specify the states implied by our entropy calculation. Thus, if we assemble two very different original states and collapse them into two black holes of the same mass, charge, and spin, they will radiate away into two indistinguishable clouds of Hawking particles. The information that went into the specification of the system before it became a black hole seems to have been erased; this is the **information loss paradox**. Both quantum field theory and general relativity feature unitary evolution—the information required to specify a state at early times is precisely equal to that needed to specify a state at later times, since they are connected by the equations of motion. But in the process of combining QFT with GR this unitarity has apparently been violated. It seems likely that we have made an inappropriate assumption somewhere in our argument, but it is hard to see where.

One way of conveying the essence of the information loss paradox is to consider a hypothetical conformal diagram for an evaporating black hole, shown in Figure 9.3. We don't really know what the full spacetime should look like, but here we have made the plausible assumptions that a singularity forms, along with an associated event horizon, both of which disappear when the black hole has fully evaporated, leaving behind a spacetime with a Minkowskian causal structure. The problem is then obvious if we think in terms of Cauchy surfaces. The future domain of dependence of an achronal surface stretching from spacelike infinity  $i^0$  to a point with  $r = 0$  to the past of the singularity would be the entire spacetime, so such a surface would be a Cauchy surface. But a similar surface stretching to a point with  $r = 0$  to the future of the singularity would not be a Cauchy surface, since the region behind the event horizon would not be in its domain of dependence. Thus, the past cannot be retrodicted from the future, due to the disappearance of information into the singularity. In other words, this process seems to be time-irreversible (in a microscopic sense, not merely a statistical sense), even though the dynamical laws that were used to predict it were fully invariant under time reversal.

In addressing the information loss paradox, keep in mind that our analysis of black-hole evaporation has only been in the context of a hybrid theory of quantum



**FIGURE 9.3** Hypothetical conformal diagram for an evaporating black hole. Energy is carried away by the Hawking radiation, so that the black hole eventually evaporates away entirely, leaving a future with the causal structure of Minkowski space. Information that falls past the event horizon into the singularity appears to be lost.

field theory coupled to general relativity, not in a realistic theory of quantum gravity. What might be going on in the real world? One possibility is that information really is lost, unitarity is violated, and we just have to learn to live with it. Many physicists find the introduction of such a fundamental breakdown of predictability to be unpalatable, and arguments have also been made that unitarity violations would necessarily lead to violations of energy conservation. Another possibility is that unitarity appears to be violated in our world, but only because the information that entered the black hole has somehow escaped to a disconnected region of space (a baby universe). General relativity predicts a singularity at the center of the black hole, not creation of a disconnected region, but clearly we are in a regime where quantum effects will dramatically alter our classical expectations, so we should keep an open mind.

Some evidence against information loss comes from string theory. String theory is naturally defined in 10 or 11 spacetime dimensions, and features not only one-dimensional extended objects (strings), but also various types of higher-dimensional extended objects known collectively as “branes.” A crucial aspect of string theory is a high degree of supersymmetry relating bosons to fermions. In the real world supersymmetry must be spontaneously broken if it exists at all, since we don’t observe a bosonic version of the electron with the same mass and charge. But as a tool for thought experiments, supersymmetry is invaluable. Supersymmetric configurations of strings and branes can be assembled that describe black hole geometries in various dimensions. In string theory there is a free parameter (really a scalar field), the string coupling, that controls the strength of gravity as well as the strength of other forces. If we consider a configuration describing a black hole at a certain value of the string coupling, as we decrease the coupling the Schwarzschild radius will eventually shrink below the size of

the configuration, which thus turns into a collection of weakly-coupled strings and branes. Due to the high degree of supersymmetry, we can be confident that various characteristics of the state remain unchanged as we vary the string coupling; in particular, we expect that the number of degrees of freedom (and thus the entropy) is unaltered. But in the weakly-coupled regime there is no black hole, we simply have a “gas” of conventional degrees of freedom (admittedly, of extended objects in higher dimensions), whose entropy we should be able to reliably calculate.

Strominger and Vafa considered this process for a particular type of five-dimensional supersymmetric black hole with different kinds of charges.<sup>4</sup> They found a remarkable result: the number of degrees of freedom of the system at weak coupling matches precisely that which would be predicted based on the entropy of the black hole at strong coupling. Since the black hole entropy depends nontrivially on the charges of the configuration, it seems unlikely that this agreement is simply an accident. Subsequent investigations have extended this analysis to other kinds of black holes, for which agreement continues to be found. Furthermore, we can even calculate the greybody factors expected for the black hole by considering scattering off of the weakly-coupled system; again, the result matches the strong-coupling expectation. Thus, in string theory at least, there is excellent reason to believe that the degrees of freedom implied by black hole radiation are really there.

Unfortunately, the string theory counting of states provides little direct understanding of how information about the black hole state could somehow be conveyed to the outgoing Hawking radiation. Nevertheless, we should certainly take seriously the possibility that this is what happens, even if there are severe difficulties in imagining how such a process might actually work. The difficulties arise when considering some information, perhaps in the form of a volume of an encyclopedia, being tossed into a large black hole, long before it has evaporated away. At this stage the black hole temperature is low, there is very little surface gravity, and the spacetime curvature near the event horizon is quite small. From the point of view of the encyclopedia, nothing special happens at the horizon, and we should expect it to fall through essentially unmolested. In particular, it is hard to imagine how the information in the encyclopedia can be transferred to the Hawking radiation being emitted at early times. In unitary evolution, the information cannot be duplicated; either it falls past the horizon with the encyclopedia, or it needs to be effectively extracted just before the horizon is crossed, which seems implausible. We might hope that the information accompanies the encyclopedia into a region near the singularity, and is somehow preserved there until late times when the hole is very small. But by then most of the radiating particles have already been emitted, and the number of states accessible to the final burst of radiation will generally be smaller than required to describe the different states that could have fallen into the hole.

<sup>4</sup>A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy,” *Phys. Lett. B* **379**, 99 (1996), <http://arxiv.org/hep-th/9601029>. For reviews see Johnson (2003) or A.W. Peet, “TASI lectures on black holes in string theory.” (2001), <http://arxiv.org/hep-th/0008241>.

To imagine that the information is somehow encoded in the outgoing radiation, it therefore seems necessary to encode correlations in the Hawking particles even at early times. We just argued that this is hard to do, given that the horizon is an unremarkable place when the black hole is large. One conceivable way out of this dilemma is to take the dramatic step of giving up on local quantum field theory. In other words, we have been making the implicit assumption that information can be sensibly described as being located in some region of space; this is an indisputable feature of ordinary quantum field theories. But perhaps quantum gravity is different, and the information contained in the black hole is somehow spread out nonlocally across the horizon. By itself this suggestion doesn't lead directly to a mechanism for getting the information into the outgoing Hawking radiation, but it does call into question some of the arguments we have given for why it would be difficult to do so.

A particular realization of nonlocality goes under the name of the **holographic principle**. This is the idea, suggested originally by 't Hooft and Susskind, that the number of degrees of freedom in a region of space is not proportional to the volume of the region (as would be expected in a local field theory), but rather to the area of the boundary of the region.<sup>5</sup> The inspiration comes of course from black hole entropy, which scales as the area of the event horizon; if the entropy counts the number of accessible states, holography would account for why it is the area rather than the enclosed volume that matters. You might worry about how to deal with closed universes, in which a region might consist of almost all of space but have a very small boundary, but a more covariant version of the holographic principle may be formulated by replacing the region of space by a set of "light-sheets" extending inward from the boundary. The great triumph of holography has been in the AdS/CFT correspondence, mentioned in Chapter 8. There, the physics of quantum gravity in an anti-de Sitter background is equivalent to a conformal field theory without gravity defined on the boundary of AdS, which has one lower dimension. One can imagine that all of the physical phenomena we observe in the universe could be described by the nonlocal holographic projection of some ordinary nongravitational theory defined in lower dimensions; it is by no means clear how we should go about constructing such a correspondence or connecting it with observations, but considerations of cosmology and the large-scale structure of the universe might be a promising place to start.

These remarks about black hole entropy, string theory, and holography are obviously not intended as a careful introduction to what is a very active area of research. Rather, they are meant to indicate some of the possibilities being explored at the forefront of gravitational physics. Classical general relativity is the most beautiful physical theory invented to date, but we have every right to expect that a synthesis of GR with other areas of physics will reveal layers of beauty we can only now imagine.

<sup>5</sup>For a review see R. Bousso, "The Holographic Principle" (2002), <http://arxiv.org/hep-th/0203101>.

## A

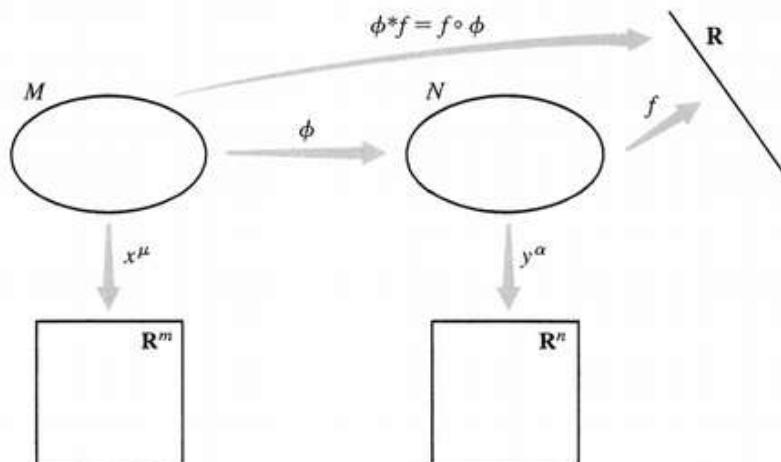
## Maps between Manifolds

When we discussed manifolds in Chapter 2, we introduced maps between two different manifolds and how maps could be composed. Here we will investigate such maps in much greater detail, focusing on the use of such maps in carrying along tensor fields from one manifold to another. The manifolds in question might end up being a submanifold and the bigger space in which it is embedded, or we might just have two different copies of the same abstract manifold being mapped to each other.

Consider two manifolds  $M$  and  $N$ , possibly of different dimension, with coordinate systems  $x^\mu$  and  $y^\alpha$ , respectively. We imagine that we have a map  $\phi : M \rightarrow N$  and a function  $f : N \rightarrow \mathbf{R}$ . Obviously we can compose  $\phi$  with  $f$  to construct a map  $(f \circ \phi) : M \rightarrow \mathbf{R}$ , which is simply a function on  $M$ . Such a construction is sufficiently useful that it gets its own name; we define the **pullback** of  $f$  by  $\phi$ , denoted  $\phi^* f$ , by

$$\phi^* f = (f \circ \phi). \quad (\text{A.1})$$

The name makes sense, since we think of  $\phi^*$  as “pulling back” the function  $f$  from  $N$  to  $M$  (see Figure A.1).



**FIGURE A.1** The pullback of a function  $f$  from  $N$  to  $M$  by a map  $\phi : M \rightarrow N$  is simply the composition of  $\phi$  with  $f$ .

We can pull functions back, but we cannot push them forward. If we have a function  $g : M \rightarrow \mathbf{R}$ , there is no way we can compose  $g$  with  $\phi$  to create a function on  $N$ ; the arrows don't fit together correctly. But recall that a vector can be thought of as a derivative operator that maps smooth functions to real numbers. This allows us to define the **pushforward** of a vector; if  $V(p)$  is a vector at a point  $p$  on  $M$ , we define the pushforward vector  $\phi_* V$  at the point  $\phi(p)$  on  $N$  by giving its action on functions on  $N$ :

$$(\phi_* V)(f) = V(\phi^* f). \quad (\text{A.2})$$

So to push forward a vector field we say “the action of  $\phi_* V$  on any function is simply the action of  $V$  on the pullback of that function.”<sup>1</sup>

This discussion is a little abstract, and it would be nice to have a more concrete description. We know that a basis for vectors on  $M$  is given by the set of partial derivatives  $\partial_\mu = \partial/\partial x^\mu$ , and a basis on  $N$  is given by the set of partial derivatives  $\partial_\alpha = \partial/\partial y^\alpha$ . Therefore we would like to relate the components of  $V = V^\mu \partial_\mu$  to those of  $(\phi_* V)^\alpha = (\phi_* V)^\alpha \partial_\alpha$ . We can find the sought-after relation by applying the pushed-forward vector to a test function and using the chain rule (2.12):

$$\begin{aligned} (\phi_* V)^\alpha \partial_\alpha f &= V^\mu \partial_\mu (\phi^* f) \\ &= V^\mu \partial_\mu (f \circ \phi) \\ &= V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha f. \end{aligned} \quad (\text{A.3})$$

This simple formula makes it irresistible to think of the pushforward operation  $\phi_*$  as a matrix operator,  $(\phi_* V)^\alpha = (\phi_*)^\alpha_\mu V^\mu$ , with the matrix being given by

$$(\phi_*)^\alpha_\mu = \frac{\partial y^\alpha}{\partial x^\mu}. \quad (\text{A.4})$$

The behavior of a vector under a pushforward thus bears an unmistakable resemblance to the vector transformation law under change of coordinates. In fact it is a generalization, since when  $M$  and  $N$  are the same manifold the constructions are (as we shall discuss) identical; but don't be fooled, since in general  $\mu$  and  $\alpha$  have different allowed values, and there is no reason for the matrix  $\partial y^\alpha/\partial x^\mu$  to be invertible.

It is a rewarding exercise to convince yourself that, although you can push vectors forward from  $M$  to  $N$  (given a map  $\phi : M \rightarrow N$ ), you cannot in general pull them back—just keep trying to invent an appropriate construction until the futility of the attempt becomes clear. Since one-forms are dual to vectors, you should not be surprised to hear that one-forms can be pulled back (but not in general pushed forward). To do this, remember that one-forms are linear maps from vectors to the real numbers. The pullback  $\phi^*\omega$  of a one-form  $\omega$  on  $N$  can

<sup>1</sup>Unfortunately the location of the asterisks is not completely standard; some references use a superscript \* for pushforward and a subscript \* for pullback, so be careful.

therefore be defined by its action on a vector  $V$  on  $M$ , by equating it with the action of  $\omega$  itself on the pushforward of  $V$ :

$$(\phi^*\omega)(V) = \omega(\phi_*V). \quad (\text{A.5})$$

Once again, there is a simple matrix description of the pullback operator on forms,  $(\phi^*\omega)_\mu = (\phi^*)_\mu^\alpha \omega_\alpha$ , which we can derive using the chain rule. It is given by

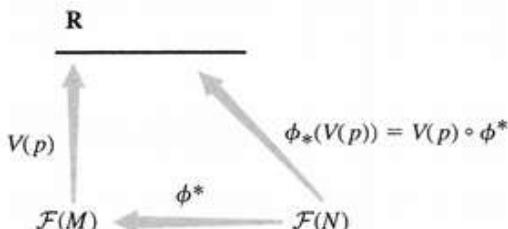
$$(\phi^*)_\mu^\alpha = \frac{\partial y^\alpha}{\partial x^\mu}. \quad (\text{A.6})$$

That is, it is the same matrix as the pushforward (A.4), but of course a different index is contracted when the matrix acts to pull back one-forms.

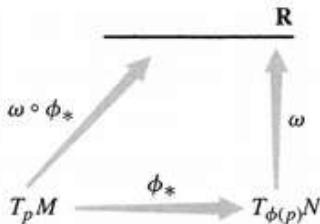
There is a way of thinking about why pullbacks and pushforwards work on some objects but not others, which may be helpful. If we denote the set of smooth functions on  $M$  by  $\mathcal{F}(M)$ , then a vector  $V(p)$  at a point  $p$  on  $M$  (that is, an element of the tangent space  $T_p M$ ) can be thought of as an operator from  $\mathcal{F}(M)$  to  $\mathbf{R}$ . But we already know that the pullback operator on functions maps  $\mathcal{F}(N)$  to  $\mathcal{F}(M)$ , just as  $\phi$  itself maps  $M$  to  $N$ , but in the opposite direction. Therefore we can define the pushforward  $\phi_*$  acting on vectors simply by composing maps, as we first defined the pullback of functions; this is shown in Figure A.2. Similarly, if  $T_q N$  is the tangent space at a point  $q$  on  $N$ , then a one-form  $\omega$  at  $q$  (that is, an element of the cotangent space  $T_q^* N$ ) can be thought of as an operator from  $T_q N$  to  $\mathbf{R}$ . Since the pushforward  $\phi_*$  maps  $T_p M$  to  $T_{\phi(p)} N$ , the pullback  $\phi^*$  of a one-form can also be thought of as mere composition of maps, as indicated in Figure A.3. If this is not helpful, don't worry about it. But do keep straight what exists and what doesn't; the actual concepts are simple, it's just forgetting which map goes what way that leads to confusion.

You will recall further that a  $(0, l)$  tensor—one with  $l$  lower indices and no upper ones—is a linear map from the direct product of  $l$  vectors to  $\mathbf{R}$ . We can therefore pull back not only one-forms, but tensors with an arbitrary number of lower indices. The definition is simply the action of the original tensor on the pushed-forward vectors:

$$(\phi^*T)(V^{(1)}, V^{(2)}, \dots, V^{(l)}) = T(\phi_*V^{(1)}, \phi_*V^{(2)}, \dots, \phi_*V^{(l)}), \quad (\text{A.7})$$



**FIGURE A.2** Pushing forward a vector, thought of as composition of a map between the spaces of functions on  $N$  and  $M$ , and a map from functions on  $M$  to  $\mathbf{R}$ .



**FIGURE A.3** Pulling back a one-form, thought of as composition of a map between tangent spaces  $T_p M$  and  $T_{\phi(p)} N$  and a map from  $T_{\phi(p)} N$  to  $\mathbf{R}$ .

where  $T_{\alpha_1 \dots \alpha_l}$  is a  $(0, l)$  tensor on  $N$ . We can similarly push forward any  $(k, 0)$  tensor  $S^{\mu_1 \dots \mu_k}$  by acting it on pulled-back one-forms:

$$(\phi_* S)(\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(k)}) = S(\phi^* \omega^{(1)}, \phi^* \omega^{(2)}, \dots, \phi^* \omega^{(k)}). \quad (\text{A.8})$$

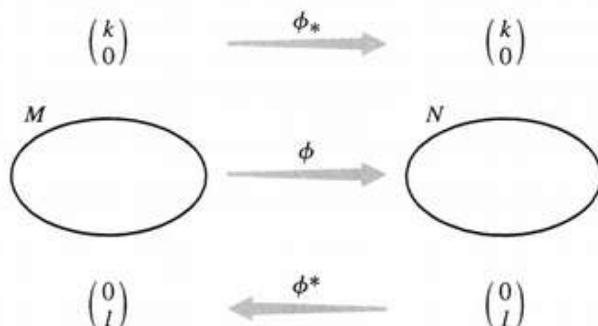
Fortunately, the matrix representations of the pushforward (A.4) and pullback (A.6) extend to the higher-rank tensors simply by assigning one matrix to each index; thus, for the pullback of a  $(0, l)$  tensor, we have

$$(\phi^* T)_{\mu_1 \dots \mu_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \dots \frac{\partial y^{\alpha_l}}{\partial x^{\mu_l}} T_{\alpha_1 \dots \alpha_l}, \quad (\text{A.9})$$

while for the pushforward of a  $(k, 0)$  tensor we have

$$(\phi_* S)^{\alpha_1 \dots \alpha_k} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \dots \frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}} S^{\mu_1 \dots \mu_k}. \quad (\text{A.10})$$

Our complete picture is therefore as portrayed in Figure A.4. Note that tensors with both upper and lower indices can generally be neither pushed forward nor pulled back.



**FIGURE A.4** A map  $\phi : M \rightarrow N$  allows us to pull back  $(0, l)$  tensors and push forward  $(k, 0)$  tensors.

This machinery becomes somewhat less imposing once we see it at work in a simple example. One common occurrence of a map between two manifolds is when  $M$  is actually a submanifold of  $N$ , which we will discuss more carefully in Appendix C. The basic idea is that there is a map from  $M$  to  $N$  that just takes an element of  $M$  to the “same” element of  $N$ . Consider the two-sphere embedded in  $\mathbf{R}^3$ , thought of as the locus of points a unit distance from the origin. If we put coordinates  $x^\mu = (\theta, \phi)$  on  $M = S^2$  and  $y^\alpha = (x, y, z)$  on  $N = \mathbf{R}^3$ , the map  $\phi : M \rightarrow N$  is given by

$$\phi(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \quad (\text{A.11})$$

Sticking the sphere into  $\mathbf{R}^3$  in this way induces a metric on  $S^2$ , which is just the pullback of the flat-space metric. The simple-minded way to find this is to start with the metric  $ds^2 = dx^2 + dy^2 + dz^2$  on  $\mathbf{R}^3$  and substitute (A.11) into this expression, yielding a metric  $d\theta^2 + \sin^2 \theta d\phi^2$  on  $S^2$ . Let’s see how this answer comes about using the more respectable formalism. (Of course it would be easier if we worked in spherical coordinates on  $\mathbf{R}^3$ , but doing it the hard way is more illustrative.) The matrix of partial derivatives is given by

$$\frac{\partial y^\alpha}{\partial x^\mu} = \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \theta \sin \phi & \sin \theta \cos \phi & 0 \end{pmatrix}. \quad (\text{A.12})$$

The metric on  $S^2$  is obtained by simply pulling back the metric from  $\mathbf{R}^3$ ,

$$\begin{aligned} (\phi^* g)_{\mu\nu} &= \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}, \end{aligned} \quad (\text{A.13})$$

as you can easily check. So the answer really is the same as you would get by naive substitution, but now we know why.

## B

## Diffeomorphisms and Lie Derivatives

In this Appendix we continue the explorations of the previous one, now focusing on the special case when the two manifolds are actually the same. Thus far, we have been careful to emphasize that a map  $\phi : M \rightarrow N$  can be used to pull certain things back (A.9) and push other things forward (A.10). The reason why it generally doesn't work both ways can be traced to the fact that  $\phi$  might not be invertible. If  $\phi$  is invertible (and both  $\phi$  and  $\phi^{-1}$  are smooth, which we always implicitly assume), then it defines a diffeomorphism between  $M$  and  $N$ . This can only be the case if  $M$  and  $N$  are actually the same abstract manifold; indeed, the existence of a diffeomorphism is the definition of two manifolds being the same. The beauty of diffeomorphisms is that we can use both  $\phi$  and  $\phi^{-1}$  to move tensors from  $M$  to  $N$ ; this will allow us to define the pushforward and pullback of arbitrary tensors. Specifically, for a  $(k, l)$  tensor field  $T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$  on  $M$ , we define the pushforward by

$$\begin{aligned} (\phi_* T)(\omega^{(1)}, \dots, \omega^{(k)}, V^{(1)}, \dots, V^{(l)}) \\ = T(\phi^* \omega^{(1)}, \dots, \phi^* \omega^{(k)}, [\phi^{-1}]_* V^{(1)}, \dots, [\phi^{-1}]_* V^{(l)}), \end{aligned} \quad (\text{B.1})$$

where the  $\omega^{(i)}$ 's are one-forms on  $N$  and the  $V^{(i)}$ 's are vectors on  $N$ . In components this becomes

$$(\phi_* T)^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \dots \frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial y^{\beta_1}} \dots \frac{\partial x^{\nu_l}}{\partial y^{\beta_l}} T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}. \quad (\text{B.2})$$

The appearance of the inverse matrix  $\partial x^\nu / \partial y^\beta$  is legitimate because  $\phi$  is invertible. Note that we could also define the pullback in the obvious way, but there is no need to write separate equations because the pullback  $\phi^*$  is the same as the pushforward via the inverse map,  $[\phi^{-1}]_*$ .

We are now in a position to explain the relationship between diffeomorphisms and coordinate transformations: they are two different ways of doing precisely the same thing. If you like, diffeomorphisms are “active coordinate transformations,” while traditional coordinate transformations are “passive.” Consider an  $n$ -dimensional manifold  $M$  with coordinate functions  $x^\mu : M \rightarrow \mathbf{R}^n$ . To change coordinates we can either simply introduce new functions  $y^\mu : M \rightarrow \mathbf{R}^n$  (“keep the manifold fixed, change the coordinate maps”), or we could just as well introduce a diffeomorphism  $\phi : M \rightarrow M$ , after which the coordinates would just be the pullbacks  $(\phi^* x)^\mu : M \rightarrow \mathbf{R}^n$  (“move the points on the manifold, and then

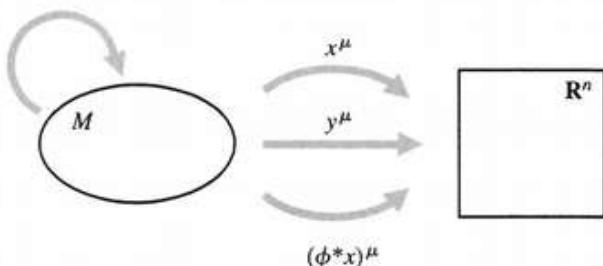
evaluate the coordinates of the new points”), as shown in Figure B.1. In this sense, (B.2) really is the tensor transformation law, just thought of from a different point of view.

Since a diffeomorphism allows us to pull back and push forward arbitrary tensors, it provides another way of comparing tensors at different points on a manifold. Given a diffeomorphism  $\phi : M \rightarrow M$  and a tensor field  $T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}(x)$ , we can form the difference between the value of the tensor at some point  $p$  and  $\phi^*[T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}(\phi(p))]$ , its value at  $\phi(p)$  pulled back to  $p$ . This suggests that we could define another kind of derivative operator on tensor fields, one that categorizes the rate of change of the tensor under the flow of the diffeomorphism. For that, however, a single discrete diffeomorphism is insufficient; we require a one-parameter family of diffeomorphisms,  $\phi_t$ . This family can be thought of as a smooth map  $\mathbf{R} \times M \rightarrow M$ , such that for each  $t \in \mathbf{R}$  we have a diffeomorphism  $\phi_t$ , satisfying  $\phi_s \circ \phi_t = \phi_{s+t}$ . This last condition implies that  $\phi_0$  is the identity map.

One-parameter families of diffeomorphisms can be thought of as arising from vector fields (and vice-versa). If we consider what happens to the point  $p$  under the entire family  $\phi_t$ , it is clear that it describes a curve in  $M$ ; since the same thing will be true of every point on  $M$ , these curves fill the manifold (although there can be degeneracies where the diffeomorphisms have fixed points). We can define a vector field  $V^\mu(x)$  to be the set of tangent vectors to each of these curves at every point, evaluated at  $t = 0$ . An example on  $S^2$  is provided by the diffeomorphism  $\phi_t(\theta, \phi) = (\theta, \phi + t)$ , shown in Figure B.2. We can reverse the construction to define a one-parameter family of diffeomorphisms from any vector field. Given a vector field  $V^\mu(x)$ , we define the **integral curves** of the vector field to be those curves  $x^\mu(t)$  that solve

$$\frac{dx^\mu}{dt} = V^\mu. \quad (\text{B.3})$$

Note that this familiar-looking equation is now to be interpreted in the opposite sense from our usual way; we are given the vectors, from which we define the curves. Solutions to (B.3) are guaranteed to exist as long as we don’t do anything silly like run into the edge of our manifold; the proof amounts to finding a coordinate system in which the problem reduces to the fundamental theorem



**FIGURE B.1** A coordinate change induced by the diffeomorphism  $\phi : M \rightarrow M$ .



**FIGURE B.2** A diffeomorphism on the two-sphere, given by a rotation about its axis.

of ordinary differential equations. Our diffeomorphisms  $\phi_t$  represent “flow down the integral curves,” and the associated vector field is referred to as the **generator** of the diffeomorphism. (Confusingly, vector fields and their integral curves also appear in the context of null hypersurfaces, where it is the curves rather than the vector fields that are called “generators.”) Integral curves are used all the time in elementary physics, just not given the name. The “lines of magnetic flux” traced out by iron filings in the presence of a magnet are simply the integral curves of the magnetic field vector  $\mathbf{B}$ .

Given a vector field  $V^\mu(x)$ , then, we have a family of diffeomorphisms parameterized by  $t$ , and we can ask how fast a tensor changes as we travel down the integral curves. For each  $t$  we can define this change as the difference between the pullback of the tensor to  $p$  and its original value at  $p$ ,

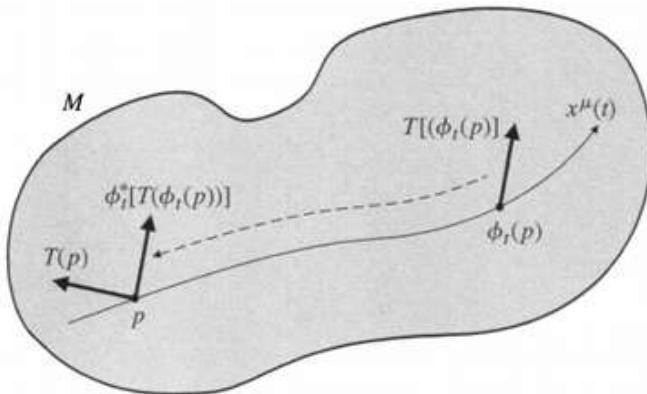
$$\Delta_t T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}(p) = \phi_t^*[T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}(\phi_t(p))] - T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}(p). \quad (\text{B.4})$$

Note that both terms on the right-hand side are tensors at  $p$ , as shown in Figure B.3. We then define the **Lie derivative** of the tensor along the vector field as

$$\mathcal{L}_V T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = \lim_{t \rightarrow 0} \left( \frac{\Delta_t T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}}{t} \right). \quad (\text{B.5})$$

The Lie derivative is a map from  $(k, l)$  tensor fields to  $(k, l)$  tensor fields, which is manifestly independent of coordinates. Since the definition essentially amounts to the conventional definition of an ordinary derivative applied to the component functions of the tensor, it should be clear that it is linear,

$$\mathcal{L}_V(aT + bS) = a\mathcal{L}_V T + b\mathcal{L}_V S, \quad (\text{B.6})$$



**FIGURE B.3** The rate of change of a tensor along the integral curves of a vector field is computed by comparing the original tensor  $T(p)$  at a point  $p$  to the value of  $T$  at a point  $\phi_t(p)$  by pulling  $T(\phi_t(p))$  back to  $p$ .

and obeys the Leibniz rule,

$$\mathcal{L}_V(T \otimes S) = (\mathcal{L}_V T) \otimes S + T \otimes (\mathcal{L}_V S), \quad (\text{B.7})$$

where  $S$  and  $T$  are tensors and  $a$  and  $b$  are constants. The Lie derivative is in fact a more primitive notion than the covariant derivative, since it does not require specification of a connection (although it does require a vector field, of course). A moment's reflection will convince you that it reduces to the ordinary directional derivative on functions,

$$\mathcal{L}_V f = V(f) = V^\mu \partial_\mu f. \quad (\text{B.8})$$

To discuss the action of the Lie derivative on tensors in terms of other operations we know, it is convenient to choose a coordinate system adapted to our problem. Specifically, we will work in coordinates  $x^\mu = (x^1, \dots, x^n)$ , such that  $x^1$  is the parameter along the integral curves and the other coordinates are chosen any way we like. Then the vector field takes the form  $V = \partial/\partial x^1$ ; that is, it has components  $V^\mu = (1, 0, 0, \dots, 0)$ . The magic of this coordinate system is that a diffeomorphism by  $t$  amounts to a coordinate transformation from  $x^\mu$  to  $y^\mu = (x^1 + t, x^2, \dots, x^n)$ . Thus, from (A.6) the pullback matrix is simply

$$(\phi_t^*)_\mu{}^\nu = \delta_\mu^\nu, \quad (\text{B.9})$$

and the components of the tensor pulled back from  $\phi_t(p)$  to  $p$  are simply

$$\phi_t^*[T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}(\phi_t(p))] = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}(x^1 + t, x^2, \dots, x^n). \quad (\text{B.10})$$

In this coordinate system, then, the Lie derivative becomes

$$\mathcal{L}_V T^{\mu_1 \dots \mu_k}{}_{v_1 \dots v_l} = \frac{\partial}{\partial x^1} T^{\mu_1 \dots \mu_k}{}_{v_1 \dots v_l}, \quad (\text{B.11})$$

and in particular the derivative of a vector field  $U^\mu(x)$  is

$$\mathcal{L}_V U^\mu = \frac{\partial U^\mu}{\partial x^1}. \quad (\text{B.12})$$

Although this expression is clearly not covariant, we know that the commutator  $[V, U]$  is a well-defined tensor, and in this coordinate system

$$\begin{aligned} [V, U]^\mu &= V^v \partial_v U^\mu - U^v \partial_v V^\mu \\ &= \frac{\partial U^\mu}{\partial x^1}. \end{aligned} \quad (\text{B.13})$$

Therefore the Lie derivative of  $U$  with respect to  $V$  has the same components in this coordinate system as the commutator of  $V$  and  $U$ ; but since both are vectors, they must be equal in any coordinate system:

$$\mathcal{L}_V U^\mu = [V, U]^\mu. \quad (\text{B.14})$$

As an immediate consequence, we have  $\mathcal{L}_V U = -\mathcal{L}_U V$ . It is because of (B.14) that the commutator is sometimes called the **Lie bracket**.

To derive the action of  $\mathcal{L}_V$  on a one-form  $\omega_\mu$ , begin by considering the action on the scalar  $\omega_\mu U^\mu$  for an arbitrary vector field  $U^\mu$ . First use the fact that the Lie derivative with respect to a vector field reduces to the action of the vector itself when applied to a scalar:

$$\begin{aligned} \mathcal{L}_V(\omega_\mu U^\mu) &= V(\omega_\mu U^\mu) \\ &= V^v \partial_v(\omega_\mu U^\mu) \\ &= V^v (\partial_v \omega_\mu) U^\mu + V^v \omega_\mu (\partial_v U^\mu). \end{aligned} \quad (\text{B.15})$$

Then use the Leibniz rule on the original scalar:

$$\begin{aligned} \mathcal{L}_V(\omega_\mu U^\mu) &= (\mathcal{L}_V \omega)_\mu U^\mu + \omega_\mu (\mathcal{L}_V U)^\mu \\ &= (\mathcal{L}_V \omega)_\mu U^\mu + \omega_\mu V^v \partial_v U^\mu - \omega_\mu U^v \partial_v V^\mu. \end{aligned} \quad (\text{B.16})$$

Setting these expressions equal to each other and requiring that equality hold for arbitrary  $U^\mu$ , we see that

$$\mathcal{L}_V \omega_\mu = V^v \partial_v \omega_\mu + (\partial_\mu V^v) \omega_v, \quad (\text{B.17})$$

which (like the definition of the commutator) is completely covariant, although not manifestly so.

By a similar procedure we can define the Lie derivative of an arbitrary tensor field. The answer can be written

$$\begin{aligned}\mathcal{L}_V T^{\mu_1 \mu_2 \cdots \mu_k}{}_{v_1 v_2 \cdots v_l} &= V^\sigma \partial_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}{}_{v_1 v_2 \cdots v_l} \\ &\quad - (\partial_\lambda V^{\mu_1}) T^{\lambda \mu_2 \cdots \mu_k}{}_{v_1 v_2 \cdots v_l} \\ &\quad - (\partial_\lambda V^{\mu_2}) T^{\mu_1 \lambda \cdots \mu_k}{}_{v_1 v_2 \cdots v_l} - \cdots \\ &\quad + (\partial_{v_1} V^\lambda) T^{\mu_1 \mu_2 \cdots \mu_k}{}_{\lambda v_2 \cdots v_l} \\ &\quad + (\partial_{v_2} V^\lambda) T^{\mu_1 \mu_2 \cdots \mu_k}{}_{v_1 \lambda \cdots v_l} + \cdots,\end{aligned}\tag{B.18}$$

Once again, this expression is covariant, despite appearances. It would undoubtedly be comforting, however, to have an equivalent expression that looked manifestly tensorial. In fact it turns out that we can write

$$\begin{aligned}\mathcal{L}_V T^{\mu_1 \mu_2 \cdots \mu_k}{}_{v_1 v_2 \cdots v_l} &= V^\sigma \nabla_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}{}_{v_1 v_2 \cdots v_l} \\ &\quad - (\nabla_\lambda V^{\mu_1}) T^{\lambda \mu_2 \cdots \mu_k}{}_{v_1 v_2 \cdots v_l} \\ &\quad - (\nabla_\lambda V^{\mu_2}) T^{\mu_1 \lambda \cdots \mu_k}{}_{v_1 v_2 \cdots v_l} - \cdots \\ &\quad + (\nabla_{v_1} V^\lambda) T^{\mu_1 \mu_2 \cdots \mu_k}{}_{\lambda v_2 \cdots v_l} \\ &\quad + (\nabla_{v_2} V^\lambda) T^{\mu_1 \mu_2 \cdots \mu_k}{}_{v_1 \lambda \cdots v_l} + \cdots,\end{aligned}\tag{B.19}$$

where  $\nabla_\mu$  represents *any* symmetric (torsion-free) covariant derivative (including, of course, one derived from a metric). You can check that all of the terms that would involve connection coefficients if we were to expand (B.19) would cancel, leaving only (B.18). Both versions of the formula for a Lie derivative are useful at different times. A particularly useful formula is for the Lie derivative of the metric:

$$\begin{aligned}\mathcal{L}_V g_{\mu\nu} &= V^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda) g_{\lambda\nu} + (\nabla_\nu V^\lambda) g_{\mu\lambda} \\ &= \nabla_\mu V_\nu + \nabla_\nu V_\mu,\end{aligned}\tag{B.20}$$

or

$$\mathcal{L}_V g_{\mu\nu} = 2\nabla_{(\mu} V_{\nu)},\tag{B.21}$$

where  $\nabla_\mu$  is the covariant derivative derived from  $g_{\mu\nu}$ .

Let's put some of these ideas into the context of general relativity. You will often hear it proclaimed that GR is a "diffeomorphism invariant" theory. What this means is that, if the universe is represented by a manifold  $M$  with metric  $g_{\mu\nu}$  and matter fields  $\psi$ , and  $\phi : M \rightarrow M$  is a diffeomorphism, then the sets  $(M, g_{\mu\nu}, \psi)$  and  $(M, \phi^* g_{\mu\nu}, \phi^* \psi)$  represent the same physical situation. Since diffeomorphisms are just active coordinate transformations, this is a highbrow

way of saying that the theory is coordinate invariant. Although such a statement is true, it is a source of great misunderstanding, for the simple fact that it conveys very little information. Any semi-respectable theory of physics is coordinate invariant, including those based on special relativity or Newtonian mechanics; GR is not unique in this regard. When people say that GR is diffeomorphism invariant, more likely than not they have one of two (closely related) concepts in mind: the theory is free of “prior geometry,” and there is no *preferred* coordinate system for spacetime. The first of these stems from the fact that the metric is a dynamical variable, and along with it the connection and volume element and so forth. Nothing is given to us ahead of time, unlike in classical mechanics or SR. As a consequence, there is no way to simplify life by sticking to a specific coordinate system adapted to some absolute elements of the geometry. This state of affairs forces us to be very careful; it is possible that two purportedly distinct configurations (of matter and metric) in GR are actually “the same,” related by a diffeomorphism. In a path integral approach to quantum gravity, where we would like to sum over all possible configurations, special care must be taken not to overcount by allowing physically indistinguishable configurations to contribute more than once. In SR or Newtonian mechanics, meanwhile, the existence of a preferred set of coordinates saves us from such ambiguities. The fact that GR has no preferred coordinate system is often garbled into the statement that it is coordinate invariant (or “generally covariant,” or “diffeomorphism invariant”); both things are true, but one has more content than the other.

On the other hand, the fact of diffeomorphism invariance can be put to good use. Recall that the complete action for gravity coupled to a set of matter fields  $\psi^i$  is given by a sum of the Hilbert action for GR plus the matter action,

$$S = \frac{1}{16\pi G} S_H[g_{\mu\nu}] + S_M[g_{\mu\nu}, \psi^i]. \quad (\text{B.22})$$

The Hilbert action  $S_H$  is diffeomorphism invariant when considered in isolation, so the matter action  $S_M$  must also be if the action as a whole is to be invariant. We can write the variation in  $S_M$  under a diffeomorphism as

$$\delta S_M = \int d^n x \frac{\delta S_M}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \int d^n x \frac{\delta S_M}{\delta \psi^i} \delta \psi^i. \quad (\text{B.23})$$

We are not considering arbitrary variations of the fields, only those that result from a diffeomorphism. Nevertheless, the matter equations of motion tell us that the variation of  $S_M$  with respect to  $\psi^i$  will vanish for any variation, since the gravitational part of the action doesn’t involve the matter fields. Hence, for a diffeomorphism invariant theory the first term on the right-hand side of (B.23) must also vanish. If the diffeomorphism is generated by an infinitesimal vector field  $V^\mu(x)$ , the infinitesimal change in the metric is simply given by its Lie derivative along  $V^\mu$ ; by (B.20) we have

$$\begin{aligned}\delta g_{\mu\nu} &= \mathcal{L}_V g_{\mu\nu} \\ &= 2\nabla_{(\mu} V_{\nu)}.\end{aligned}\tag{B.24}$$

Setting  $\delta S_M = 0$  then implies

$$\begin{aligned}0 &= \int d^nx \frac{\delta S_M}{\delta g_{\mu\nu}} \nabla_\mu V_\nu \\ &= - \int d^nx \sqrt{-g} V_\nu \nabla_\mu \left( \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \right),\end{aligned}\tag{B.25}$$

where we are able to drop the symmetrization of  $\nabla_{(\mu} V_{\nu)}$  since  $\delta S_M / \delta g_{\mu\nu}$  is already symmetric. Demanding that (B.25) hold for diffeomorphisms generated by arbitrary vector fields  $V^\mu$ , and using the definition (4.75) of the energy-momentum tensor, we obtain precisely the law of energy-momentum conservation,

$$\nabla_\mu T^{\mu\nu} = 0.\tag{B.26}$$

Conservation of  $T_{\mu\nu}$  is a powerful statement, and it might seem surprising that we derived it from as weak a requirement as diffeomorphism invariance. Actually we sneaked in a much stronger assumption, namely that there is a clean separation between the “matter” and “gravitational” actions (in the sense that no matter fields appeared in the gravitational action). If there were, for example, a scalar field multiplying the curvature scalar and also appearing in the matter action (as in the scalar-tensor theories discussed in Chapter 4), this assumption would have been violated, and  $T_{\mu\nu}$  would not be conserved by itself.

Recall that in Chapter 3 we spoke of symmetries and Killing vectors, with repeated appeals to look in the Appendices. Now that we understand more about diffeomorphisms, it is perfectly straightforward to understand symmetries. We say that a diffeomorphism  $\phi$  is a **symmetry** of some tensor  $T$  if the tensor is invariant after being pulled back under  $\phi$ :

$$\phi^* T = T.\tag{B.27}$$

Although symmetries may be discrete, it is also common to have a one-parameter family of symmetries  $\phi_t$ . If the family is generated by a vector field  $V^\mu(x)$ , then (B.27) amounts to

$$\mathcal{L}_V T = 0.\tag{B.28}$$

By (B.12), one implication of a symmetry is that, if  $T$  is symmetric under some one-parameter family of diffeomorphisms, we can always find a coordinate system in which the components of  $T$  are all independent of one of the coordinates (the integral curve coordinate of the vector field). The converse is also true; if all of the components are independent of one of the coordinates, then the partial

derivative vector field associated with that coordinate generates a symmetry of the tensor.

The most important symmetries are those of the metric, for which  $\phi^* g_{\mu\nu} = g_{\mu\nu}$ . A diffeomorphism of this type is called an isometry. If a one-parameter family of isometries is generated by a vector field  $K^\mu(x)$ , then  $K^\mu$  turns out to be a Killing vector field. The condition that  $K^\mu$  be a Killing vector is thus

$$\mathcal{L}_K g_{\mu\nu} = 0, \quad (\text{B.29})$$

or from (B.20),

$$\nabla_{(\mu} K_{\nu)} = 0. \quad (\text{B.30})$$

We recognize this last version as Killing's equation, (3.174). From our discussion in Chapter 3 we know that, if a spacetime has a Killing vector, we can find a coordinate system in which the metric is independent of one of the coordinates, and the quantity  $p_\mu K^\mu$  will be constant along geodesics with tangent vector  $p^\mu$ . Once we have set up the machinery of diffeomorphisms and Lie derivatives, the derivation of Killing vectors proceeds much more elegantly.

## B.1 ■ EXERCISES

1. In Euclidean three-space, find and draw the integral curves of the vector fields

$$A = \frac{y-x}{r} \frac{\partial}{\partial x} - \frac{x+y}{r} \frac{\partial}{\partial y}$$

and

$$B = xy \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}.$$

Calculate  $C = \mathcal{L}_A B$  and draw the integral curves of  $C$ .

## C

## Submanifolds

The notion of a submanifold, some subset of another manifold which might be (and usually is) of lower dimension, is intuitively straightforward; it should come as no surprise, however, to learn that a certain amount of formalism comes along for the ride. Submanifolds arise all the time in general relativity—as boundaries of spacetimes, hypersurfaces at fixed time, spaces into which larger spaces are foliated by the action of symmetries—so it is worth our effort to understand how they work.

Consider an  $n$ -dimensional manifold  $M$  and an  $m$ -dimensional manifold  $S$ , with  $m \leq n$ , and a map  $\phi : S \rightarrow M$ . If the map  $\phi$  is both  $C^\infty$  and one-to-one, and the inverse  $\phi^{-1} : \phi[S] \rightarrow S$  is also  $C^\infty$ , then we say that the image  $\phi[S]$  is an **embedded submanifold** of  $M$ . If  $\phi$  is one-to-one locally but not necessarily globally (that is, there may be self-intersections of  $\phi[S]$  in  $M$ ), then we say that  $\phi[S]$  is an **immersed submanifold** of  $M$ . When we speak of “submanifolds” without any particular modifier, we are imagining that they are embedded. An  $m$ -dimensional submanifold of an  $n$ -dimensional manifold is said to be of **codimension**  $n - m$ .

As discussed in Appendix A, the map  $\phi : S \rightarrow M$  can be used to push forward  $(k, 0)$  tensors from  $S$  to  $M$ , and to pull back  $(0, l)$  tensors from  $M$  to  $S$ . In particular, given a point  $q \in S$  and its image  $\phi(q) \in M$ , the tangent space  $T_{\phi(q)}\phi[S]$  is naturally identified as an  $m$ -dimensional subspace of the  $n$ -dimensional vector space  $T_{\phi(q)}M$ . If you think about the definition of a vector as the directional derivative along a curve, this makes perfect sense; any curve  $\gamma : \mathbf{R} \rightarrow S$  clearly defines a curve in  $M$  via composition  $(\phi \circ \gamma : \mathbf{R} \rightarrow M)$ , which in turn defines a directional derivative. Similarly, differential forms in  $M$  can be pulled back to  $S$  by restricting their action to vectors in the subspace  $T_{\phi(q)}\phi[S]$ .

Another way to define submanifolds is as places where a collection of functions takes on some specified fixed set of values. An  $m$ -dimensional submanifold of  $M$  can be specified in terms of  $n - m$  functions  $f^a(x)$ , where  $a$  runs from 1 to  $n - m$ , as the set of points  $x$ , where the  $f^a$ 's are equal to some constants  $f_*^a$ :

$$\begin{aligned} f^1(x) &= f_*^1 \\ f^2(x) &= f_*^2 \\ &\vdots \\ f^{n-m}(x) &= f_*^{n-m}. \end{aligned} \tag{C.1}$$

The functions should be nondegenerate, so that the submanifold really is of dimension  $m$ . Notice that the submanifold defined in this way is an actual subset of  $M$ ; it is equivalent to what we called  $\phi[S]$  in our previous definition. For convenience, we will henceforth tend to blur the distinction between the original space and its embedding as a submanifold, and simply refer to “the submanifold  $S$ .”

To see the relationship between the two definitions of a submanifold, imagine constructing a set of coordinates  $x^\mu = \{f^a, y^\alpha\}$  in a neighborhood of  $\phi[S] \subset M$ , consisting of the  $n - m$  functions  $f^a$  and an additional  $m$  functions  $y^\alpha$ . Then we can pull back the functions  $y^\alpha$  to serve as coordinates on  $S$ , and the map  $\phi : S \rightarrow M$  is simply given by

$$\phi : (y^\alpha) \rightarrow (f_*^a, y^\alpha). \quad (\text{C.2})$$

A simple example is the two-sphere  $S^2$ , which in fact we defined as the set of points a unit distance from the origin in  $\mathbf{R}^3$ . In polar coordinates  $(r, \theta, \phi)$ , this is equivalent to the requirement  $r = 1$ , so the coordinate  $r$  plays the role of the function  $f(x)$ , while  $\theta$  and  $\phi$  are induced coordinates on  $S^2$ .

We have already mentioned in (B.3) that specifying a single vector field leads to a family of integral curves, which are simply one-dimensional submanifolds. We might imagine generalizing this construction by using a set of several vector fields to define higher-dimensional submanifolds. Imagine we have an  $n$ -dimensional manifold  $M$ , an  $m$ -dimensional submanifold  $S$ , and a set of  $p$  linearly independent vector fields  $V_{(a)}^\mu$ , with  $p \geq m$ . Then the notion that these vector fields “fit together to define  $S$ ” means that each vector is tangent to  $S$  everywhere, so that the  $V_{(a)}^\mu$ ’s span each tangent space  $T_p S$ ; we say that  $S$  is an **integral submanifold** of the vector fields. However, any given set of vector fields may or may not actually fit together to define such submanifolds. Whether they do or not is revealed by **Frobenius’s theorem**: a set of vector fields  $V_{(a)}^\mu$  fit together to define integral submanifolds if and only if all of their commutators are in the space spanned by the  $V_{(a)}^\mu$ ’s; that is, if

$$[V_{(a)}, V_{(b)}]^\mu = \alpha^c V_{(c)}^\mu \quad (\text{C.3})$$

for some set of coefficients  $\alpha^c(x)$ . (In the language of group theory, this means that the vector fields form a Lie algebra.) We won’t provide a proof, but hopefully the result makes some mathematical sense. If the vector fields are going to fit together to form a submanifold  $S$ , they must remain tangent to  $S$  everywhere. But the commutator  $[V, W]$  is equivalent to the Lie derivative  $\mathcal{L}_V W$ , which measures how  $W$  changes as we travel along  $V$ . If this Lie derivative doesn’t remain in the space defined by the vectors, it means that  $W$  starts sticking out of the submanifold  $S$ . Examples of vector fields fitting together to form submanifolds are easy to come by; in Section 5.2 we discussed how the three Killing vectors associated with spherical symmetry define a foliation of a three-dimensional space into two-spheres. (Notice that the dimensionality of the integral submanifold can be less

than the number of vector fields.) For a discussion of Frobenius's theorem, see Schutz (1980).

An interesting alternative formulation of Frobenius's theorem uses differential forms. First notice that any set of  $p$  linearly independent one-forms  $\omega_\mu^{(a)}$  defines an  $(n-p)$ -dimensional vector subspace of  $T_p M$ , called the **annihilator** of the set of forms, consisting of those vectors  $V^\mu \in T_p M$  satisfying

$$\omega_\mu^{(a)} V^\mu = 0 \quad (\text{C.4})$$

for all  $\omega_\mu^{(a)}$ . So instead of asking whether a collection of vector fields fit together to define a set of submanifolds, we could ask whether a collection of one-forms  $\omega_\mu^{(a)}$  define a set of vector subspaces that fit together as tangent spaces to a set of submanifolds. To understand when this happens, recall the definition (C.1) of an  $m$ -dimensional submanifold as a place where a set of  $p = n - m$  functions  $f^a(x)$  are set equal to constants. A constant function is one for which the exterior derivative  $(df^a)_\mu = \nabla_\mu f^a$  vanishes; but if a function is constant only along some submanifold, that means that

$$df^a(V) = V^\mu \nabla_\mu f^a = 0 \quad (\text{C.5})$$

for all vectors  $V^\mu$  tangent to the submanifold,  $V^\mu \in T_p S$ . It also goes the other way; if a vector  $V^\mu$  is annihilated by all of the gradients  $\nabla_\mu f^a$ , it is necessarily tangent to the corresponding submanifold  $S$ . Therefore, if a set of one-forms are each exact,  $\omega_\mu^{(a)} = \nabla_\mu f^a$ , the vector spaces they annihilate will certainly define submanifolds, namely those along which the  $f^a$ 's are constant. But if a set of  $p$  one-forms annihilates a certain subspace, so will any other set of  $p$  one-forms that are linear combinations of the originals. We therefore say that a set of one-forms  $\omega_\mu^{(a)}$  is **surface-forming** if every member can be expressed as a linear combination of a set of exact forms; that is, if there exist functions  $g^a_b(x)$  and  $f^a(x)$  such that

$$\omega_\mu^{(a)} = \sum_b g^a_b \nabla_\mu f^b. \quad (\text{C.6})$$

Of course, when handed a set of forms, it might be hard to tell whether there exist functions such that this condition is satisfied; this is where the dual formulation of Frobenius's theorem comes in. This version of the theorem states that a set of one-forms  $\omega_\mu^{(a)}$  is surface-forming if and only if every pair of vectors in the annihilator of the set is also annihilated by the exterior derivatives  $d\omega^{(a)}$ . In other words, the set  $\omega_\mu^{(a)}$  will satisfy (C.6) if and only if, for every pair of vectors  $V^\mu$  and  $W^\mu$  satisfying  $\omega_\mu^{(a)} V^\mu = 0$  and  $\omega_\mu^{(a)} W^\mu = 0$  for all  $a$ , we also have

$$\nabla_{[\mu} \omega_{\nu]}^{(a)} V^\mu W^\nu = 0. \quad (\text{C.7})$$

A set of forms  $\omega_\mu^{(a)}$  satisfying this condition is sometimes called “closed,” which is obviously a generalization of the notion of a single form being closed (namely, that its exterior derivative vanishes). We won’t prove the equivalence of the dual formulation of Frobenius’ theorem with the vector-field version, but it clearly involves acting our set of forms on the vector-field commutator (C.3).

## D

## Hypersurfaces

A **hypersurface** is an  $(n - 1)$ -dimensional (codimension one) submanifold  $\Sigma$  of an  $n$ -dimensional manifold  $M$ . (Of course if  $n = 3$ ,  $\Sigma$  might as well just be called a “surface,” but we’ll continue to use “hyper-” for consistency.) Hypersurfaces are of great utility in general relativity, and a lot of formalism goes along with them. In this Appendix we collect a set of results in the study of hypersurfaces: normal vectors, generators of null hypersurfaces, Frobenius’s theorem for hypersurfaces, Gaussian normal coordinates, induced metrics, projection tensors, extrinsic curvature, and manifolds with boundary. It’s something of a smorgasbord, with all the messiness that implies, but hopefully appetizing and nutritious as well.

One way to specify a hypersurface  $\Sigma$  is by setting single function to a constant,

$$f(x) = f_*. \quad (\text{D.1})$$

The vector field

$$\xi^\mu = g^{\mu\nu} \nabla_\nu f \quad (\text{D.2})$$

will be normal to the surface, in the sense that it is orthogonal to all vectors in  $T_p \Sigma \subset T_p M$ . If  $\xi^\mu$  is timelike, the hypersurface is said to be spacelike; if  $\xi^\mu$  is spacelike the hypersurface is timelike, and if  $\xi^\mu$  is null the hypersurface is also null. Any vector field proportional to a normal vector field,

$$\xi^\mu = h(x) \nabla^\mu f \quad (\text{D.3})$$

for some function  $h(x)$ , will itself be a normal vector field; since the normal vector is unique up to scaling, any normal vector can be written in this form. For timelike and spacelike hypersurfaces, we can therefore define a normalized version of the normal vector,

$$n^\mu = \pm \frac{\xi^\mu}{|\xi_\mu \xi^\mu|^{1/2}}. \quad (\text{D.4})$$

Then  $n^\mu n_\mu = -1$  for spacelike surfaces and  $n^\mu n_\mu = +1$  for timelike surfaces; up to an overall orientation, such a normal vector field is unique. For spacelike surfaces the sign is typically chosen so as to make  $n^\mu$  be future-directed.

Null hypersurfaces have a special feature: they can be divided into a set of null geodesics, called **generators** of the hypersurface. Let’s see how this works.

Notice that the normal vector  $\zeta^\mu$  is tangent to  $\Sigma$  as well as normal to it, since null vectors are orthogonal to themselves. Therefore the integral curves  $x^\mu(\alpha)$ , satisfying

$$\zeta^\mu = \frac{dx^\mu}{d\alpha}, \quad (\text{D.5})$$

will be null curves contained in the hypersurface. These curves  $x^\mu(\alpha)$  necessarily turn out to be geodesics, although  $\alpha$  might not be an affine parameter. To verify this claim, recall that the general form of the geodesic equation can be expressed as

$$\zeta^\mu \nabla_\mu \zeta_\nu = \eta(\alpha) \zeta_\nu, \quad (\text{D.6})$$

where  $\eta(\alpha)$  is a function that will vanish if  $\alpha$  is an affine parameter. We simply plug in (D.2) and calculate:

$$\begin{aligned} \zeta^\mu \nabla_\mu \zeta_\nu &= \zeta^\mu \nabla_\mu \nabla_\nu f \\ &= \zeta^\mu \nabla_\nu \nabla_\mu f \\ &= \zeta^\mu \nabla_\nu \zeta_\mu \\ &= \frac{1}{2} \nabla_\nu (\zeta^\mu \zeta_\mu). \end{aligned} \quad (\text{D.7})$$

In the second line we used the torsion-free condition, that covariant derivatives acting on scalars commute. Note that, even though  $\zeta^\mu \zeta_\mu = 0$  on  $\Sigma$  itself, we can't be sure that  $\nabla_\nu (\zeta^\mu \zeta_\mu)$  vanishes, since  $\zeta^\mu \zeta_\mu$  might be nonzero off the hypersurface. If the gradient vanishes, (D.7) is the geodesic equation, and we're done. But if it doesn't vanish, we can use  $\zeta^\mu \zeta_\mu = 0$  as an alternative way to define the submanifold  $\Sigma$ , and its derivative defines a normal vector. Therefore, we must have

$$\nabla_\mu (\zeta^\nu \zeta_\nu) = g \nabla_\mu f = g \zeta_\mu, \quad (\text{D.8})$$

where  $g(x)$  is some scalar function. We then plug into (D.7) to get

$$\zeta^\mu \nabla_\mu \zeta_\nu = \frac{1}{2} g \zeta_\nu, \quad (\text{D.9})$$

which is equivalent to the geodesic equation (D.6). Of course, once we know that a path  $x^\mu(\alpha)$  is a geodesic, we are free to re-parameterize it with an affine parameter  $\lambda(\alpha)$ . Equivalently, we scale the normal vector field by a scalar function  $h(x)$ ,

$$\xi^\mu = h \zeta^\mu, \quad (\text{D.10})$$

such that  $\xi^\mu \nabla_\mu \xi^\nu = 0$ . It is conventional to do exactly this, and use the corresponding tangent vectors

$$\xi^\mu = \frac{dx^\mu}{d\lambda} \quad (\text{D.11})$$

as normal vectors to  $\Sigma$ . The null geodesics  $x^\mu(\lambda)$ , whose union is the null hypersurface  $\Sigma$ , are the generators of  $\Sigma$ .

From (D.3) we know that a vector field normal to a hypersurface can be written in the form  $\xi^\mu = h \nabla^\mu f$ . In the exercises for Chapter 4 you were asked to show that this implies

$$\xi_{[\mu} \nabla_{\nu} \xi_{\sigma]} = 0, \quad (\text{D.12})$$

or in differential forms notation,

$$\xi \wedge d\xi = 0. \quad (\text{D.13})$$

The converse, that any vector field satisfying this equation is orthogonal to a hypersurface, is harder to show from first principles, but is a direct consequence of the dual formulation of Frobenius's theorem. Imagine we have two vectors  $V^\mu$  and  $W^\mu$ , both of which are annihilated by a one-form  $\xi_\mu$  obeying (D.12). From Frobenius's theorem (C.7),  $\xi_\mu$  will define a hypersurface if and only if

$$\nabla_{[\mu} \xi_{\nu]} V^\mu W^\nu = 0. \quad (\text{D.14})$$

Applying the expression in (D.12) to  $V^\mu W^\nu$  and expanding the antisymmetrization brackets, we get

$$\begin{aligned} \xi_{[\mu} \nabla_{\nu} \xi_{\sigma]} V^\mu W^\nu &= \frac{1}{3} (\xi_\mu \nabla_{[\nu} \xi_{\sigma]} + \xi_\nu \nabla_{[\sigma} \xi_{\mu]} + \xi_\sigma \nabla_{[\mu} \xi_{\nu]}) V^\mu W^\nu + \frac{1}{3} \xi_\sigma \nabla_{[\mu} \xi_{\nu]} V^\mu W^\nu \\ &= \frac{1}{3} \xi_\sigma \nabla_{[\mu} \xi_{\nu]} V^\mu W^\nu, \end{aligned} \quad (\text{D.15})$$

where in the last line we used the fact that  $V^\mu$  and  $W^\mu$  are annihilated by  $\xi_\mu$ . But since  $\nabla_{[\mu} \xi_{\nu]} V^\mu W^\nu$  is a scalar and  $\xi_\sigma$  is a nonvanishing one-form, the only way (D.15) can vanish is if (D.14) holds. Therefore, (D.12) will be true if and only if  $\xi_\mu$  is hypersurface-orthogonal.

It is often convenient to put a coordinate system on a manifold (or part of it) that is naturally adapted to some hypersurface  $\Sigma$ ; Gaussian normal coordinates provide a convenient way to do just that. First choose coordinates  $y^i = \{y^1, \dots, y^{n-1}\}$  on  $\Sigma$ . At each point  $p \in \Sigma$ , construct the (unique) geodesic for which  $n^\mu$  is the tangent vector at  $p$ . Let  $z$  be the affine parameter on each geodesic. [This parameter is unique if  $n^\mu$  is normalized and  $z(p) = 0$ .] Any point  $q$  in a neighborhood of  $\Sigma$  lives on one such geodesic. To each such point we assign coordinates  $\{z, y^1, \dots, y^{n-1}\}$ , where the  $y^i$ 's are the coordinates of the point  $p$  connected to  $q$  by the geodesic we have constructed. These coordinates  $\{z, y^1, \dots, y^{n-1}\}$  are **Gaussian normal coordinates** (not to be confused with "Riemann normal coordinates," constructed by following geodesics in all directions from a single point  $p$ ). These coordinates will eventually fail to be well-defined if we reach a point where geodesics focus and intersect, but they will always exist in some region including  $\Sigma$ . All of our statements about Gaussian normal coordinates should be taken as applying in the region where they are well-defined.

Associated with the coordinate functions  $\{z, y^1, \dots, y^{n-1}\}$  are coordinate-basis vector fields  $\{\partial_z, \partial_1, \dots, \partial_{n-1}\}$ . For notational convenience let's label these vector fields

$$\begin{aligned}(\partial_z)^\mu &= n^\mu, \\ (\partial_i)^\mu &= Y_{(i)}^\mu,\end{aligned}\tag{D.16}$$

where the first line makes sense because  $\partial_z$  is simply the extension along the geodesics of the original normal vector  $n^\mu$ . With respect to these basis vectors, the metric takes on a simple form. To start, we know that

$$g_{zz} = ds^2(\partial_z, \partial_z) = n_\mu n^\mu = \pm 1,\tag{D.17}$$

since  $n^\mu$  is just the normalized tangent vector to the geodesics emanating from  $\Sigma$ . To encapsulate the sign ambiguity, let's label this  $\sigma$ :

$$\sigma = n_\mu n^\mu = \pm 1.\tag{D.18}$$

But it is also the case that  $g_{zi} = n_\mu Y_{(i)}^\mu = 0$ , as we can straightforwardly check. Start at the original surface  $\Sigma$ , where  $n_\mu Y_{(i)}^\mu = 0$  by hypothesis (since  $n^\mu$  is normal to  $\Sigma$ ). Then we calculate

$$\begin{aligned}\frac{D}{dz}(n_\mu Y_{(i)}^\mu) &= n^\nu \nabla_\nu(n_\mu Y_{(i)}^\mu) \\ &= n^\nu n_\mu \nabla_\nu Y_{(i)}^\mu \\ &= Y_{(i)}^\nu n_\mu \nabla_\nu n^\mu \\ &= \frac{1}{2} Y_{(i)}^\nu \nabla_\nu(n_\mu n^\mu) \\ &= 0.\end{aligned}\tag{D.19}$$

Let's explain this derivation line-by-line. The first line is simply the definition of the directional covariant derivative  $D/dz$ . The second uses the Leibniz rule, plus the fact that  $n_\mu$  is parallel-transported along the geodesic ( $n^\nu \nabla_\nu n_\mu = 0$ ). The third line uses the fact that  $n^\mu$  and  $Y_{(i)}^\nu$  are both coordinate basis vectors, so their Lie bracket vanishes:  $[n, Y_{(i)}]^\mu = n^\nu \nabla_\nu Y_{(i)}^\mu - Y_{(i)}^\nu \nabla_\nu n^\mu = 0$ . The fourth line again uses Leibniz and the fact that  $n_\mu$  is parallel-transported, while the fifth simply reflects the fact that  $n_\mu n^\mu = \sigma$  is a constant.

We can therefore write the metric in Gaussian normal coordinates as

$$ds^2 = \sigma dz^2 + \gamma_{ij} dy^i dy^j,\tag{D.20}$$

where  $\gamma_{ij} = g(\partial_i, \partial_j)$  will in general be a function of all the coordinates  $\{z, y^1, \dots, y^{n-1}\}$ . We haven't made any assumptions whatsoever about the geometry; we have simply chosen a coordinate system in which the metric takes a

certain form. Notice that setting  $z = \text{constant}$  defines a family of hypersurfaces diffeomorphic to the original surface  $\Sigma$ ; the lack of off-diagonal terms  $g_{zi}$  in (D.20) reflects the fact that the vector field  $n^\mu$  is orthogonal to all of these surfaces, not just the original one. Gaussian normal coordinates are by no means exotic; we use them all the time. Simple examples include inertial coordinates on Minkowski space,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad (\text{D.21})$$

or polar coordinates in Euclidean 3-space,

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (\text{D.22})$$

Ordinary Robertson–Walker coordinates in cosmology provide a slightly less trivial example,

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right]. \quad (\text{D.23})$$

Of course, the RW geometries are highly symmetric (homogeneous and isotropic). But, since we have just seen that Gaussian normal coordinates can always be defined, we know that we can describe a perfectly general geometry by altering the spatial components of the metric. This provides one popular way of describing cosmological perturbations; we define “synchronous gauge” for flat spatial sections as

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j, \quad (\text{D.24})$$

where  $h_{ij}(t, \mathbf{x})$  is the metric perturbation. (The generalization to curved spatial sections is immediate.) Again, we have not made any assumptions about the geometry, only chosen a potentially convenient coordinate system.

Recall that the map  $\phi : \Sigma \rightarrow M$  that embeds any submanifold allows us to pull back the metric from  $M$  to  $\Sigma$ . Given coordinates  $y^i$  on  $\Sigma$  and  $x^\mu$  on  $M$ , we define the **induced metric** on the submanifold as

$$(\phi^* g)_{ij} = \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} g_{\mu\nu}. \quad (\text{D.25})$$

In the case where the submanifold is a hypersurface, this induced metric is precisely the same as the  $\gamma_{ij}$  appearing in (D.20). To see this, notice that Gaussian normal coordinates are a special case of the natural embedding coordinates described by (C.2). We have a hypersurface  $\Sigma$  defined by  $z = z_*$  on  $M$ , with coordinates  $y^i$  defined on  $\Sigma$ , and a map  $\phi : \Sigma \rightarrow M$  given by

$$\phi : y^i \rightarrow x^\mu = (z_*, y^i). \quad (\text{D.26})$$

Given the form of the metric (D.20) on  $M$ , it is immediate that under this map the pullback (D.25) is simply

$$(\phi^* g)_{ij} = \gamma_{ij}. \quad (\text{D.27})$$

Keep in mind that this equation should only be evaluated in Gaussian normal coordinates; otherwise the right-hand side doesn't even make sense.

Along with an induced metric, submanifolds inherit an induced volume element from the manifold in which they are embedded. Recall that a volume element on an  $n$ -dimensional manifold with metric  $g_{\mu\nu}$  is given by the Levi–Civita tensor, which can be expressed as

$$\epsilon = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n. \quad (\text{D.28})$$

To get a volume element on a submanifold  $\Sigma$ , it is convenient to introduce Gaussian normal coordinates  $(z, y^1, \dots, y^{n-1})$ , in which the metric takes the form (D.20). The volume element  $\hat{\epsilon}$  on  $\Sigma$  will then take the form

$$\hat{\epsilon} = \sqrt{|\gamma|} dy^1 \wedge \cdots \wedge dy^{n-1}, \quad (\text{D.29})$$

(By choosing the first coordinate to be the one normal to the hypersurface, we have implicitly chosen a convention for how the orientation on  $M$  defines an orientation on  $\Sigma$ .) In these coordinates we have

$$\sqrt{|g|} = \sqrt{|\gamma|}, \quad (\text{D.30})$$

and the volume element on  $M$  therefore becomes

$$\epsilon = \sqrt{|\gamma|} dz \wedge dy^1 \wedge \cdots \wedge dy^{n-1}. \quad (\text{D.31})$$

We can relate the two volume elements by using the normal vector to  $\Sigma$ , which has components

$$n^\mu = (1, 0, \dots, 0). \quad (\text{D.32})$$

The contraction of  $\epsilon$  with  $n^\mu$  can be denoted

$$[\epsilon(n)]_{\mu_1 \dots \mu_{n-1}} = n^\lambda \epsilon_{\lambda \mu_1 \dots \mu_{n-1}}. \quad (\text{D.33})$$

It is then clear that, in these coordinates, we have

$$\begin{aligned} \epsilon(n) &= \sqrt{|\gamma|} dy^1 \wedge \cdots \wedge dy^{n-1} \\ &= \hat{\epsilon}. \end{aligned} \quad (\text{D.34})$$

Thus, the induced volume element has components

$$\hat{\epsilon}_{\mu_1 \dots \mu_{n-1}} = n^\lambda \epsilon_{\lambda \mu_1 \dots \mu_{n-1}}. \quad (\text{D.35})$$

But this is a relation between tensors, so will be true in any coordinate system. We can also reconstruct  $\epsilon$  from  $\hat{\epsilon}$  and  $n^\mu$ , via

$$\frac{1}{n} \epsilon_{v \mu_1 \dots \mu_{n-1}} = n_{[v} \hat{\epsilon}_{\mu_1 \dots \mu_{n-1}], \quad (\text{D.36})$$

as can easily be checked by contracting with  $n^\nu$ . The notion of a submanifold volume element will be crucial in our discussion of Stokes's theorem below.

Another concept closely related to the induced metric on a hypersurface is that of the **projection tensor** for a hypersurface  $\Sigma$  with unit normal vector  $n^\mu$ , given by

$$P_{\mu\nu} = g_{\mu\nu} - \sigma n_\mu n_\nu, \quad (\text{D.37})$$

where  $\sigma = n_\mu n^\mu$ . Let's collect some useful properties of this object. Given any vector  $V^\mu$  in  $T_p M$ ,  $P_{\mu\nu}$  will project it tangent to the hypersurface (that is, orthogonal to  $n^\mu$ ):

$$\begin{aligned} (P_{\mu\nu} V^\mu) n^\nu &= g_{\mu\nu} V^\mu n^\nu - \sigma n_\mu n_\nu V^\mu n^\nu \\ &= V^\mu n_\mu - \sigma^2 V^\mu n_\mu \\ &= 0. \end{aligned} \quad (\text{D.38})$$

Acting on any two vectors  $V^\mu$  and  $W^\nu$  that are already tangent to  $\Sigma$ , the projection tensor acts like the metric:

$$\begin{aligned} P_{\mu\nu} V^\mu W^\nu &= g_{\mu\nu} V^\mu W^\nu - \sigma n_\mu n_\nu V^\mu W^\nu \\ &= g_{\mu\nu} V^\mu W^\nu. \end{aligned} \quad (\text{D.39})$$

Finally, the projection tensor is idempotent; acting two (or more) times produces the same result as only acting once:

$$\begin{aligned} P^\mu{}_\lambda P^\lambda{}_\nu &= (\delta^\mu_\lambda - \sigma n^\mu n_\lambda)(\delta^\lambda_\nu - \sigma n^\lambda n_\nu) \\ &= \delta^\mu_\nu - \sigma n^\mu n_\nu - \sigma n^\mu n_\nu + \sigma^3 n^\mu n_\nu \\ &= P^\mu{}_\nu. \end{aligned} \quad (\text{D.40})$$

$P_{\mu\nu}$  is sometimes called the **first fundamental form** of the hypersurface. Because it really does act like the metric for vectors tangent to  $\Sigma$ , and hypersurfaces are often spacelike, you will sometimes see it referred to as the "spatial metric."

Long ago when we first spoke of manifolds and curvature, we were careful to distinguish between the "intrinsic" curvature of a space, as measured by the Riemann tensor, and the "extrinsic" curvature, which depends on how the space is embedded in some larger space. For example, a two-torus can have a flat metric, but any embedding in  $\mathbf{R}^3$  makes it look curved. We are now in a position to give a formal definition of this notion, which makes sense for hypersurfaces. Let's assume we have a family of hypersurfaces  $\Sigma$  with unit vector field  $n^\mu$ , and we extend  $n^\mu$  through a region (any way we like). Then the **extrinsic curvature** of  $\Sigma$  is simply given by the Lie derivative of the projection tensor along the normal vector field,

$$K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n P_{\mu\nu}. \quad (\text{D.41})$$

The extrinsic curvature, sometimes called the **second fundamental form** of the submanifold, is thus interpreted as the rate of change of the projection tensor (the spatial metric, if  $\Sigma$  is spacelike) as we travel along the normal vector field; it is independent of the extension of  $n^\mu$  away from  $\Sigma$ . It is the work of a few lines to show that this definition is equivalent to the projected Lie derivative of the metric itself,

$$K_{\mu\nu} = \frac{1}{2} P^\alpha{}_\mu P^\beta{}_\nu \mathcal{L}_n g_{\alpha\beta}. \quad (\text{D.42})$$

We know from (B.20) that the Lie derivative of  $g_{\mu\nu}$  is given by the symmetrized covariant derivative of the normal vector, so we have

$$K_{\mu\nu} = P^\alpha{}_\mu P^\beta{}_\nu \nabla_{(\alpha} n_{\beta)}. \quad (\text{D.43})$$

Since we are not assuming that the integral curves of  $n^\mu$  are geodesics, we can define the acceleration as

$$a^\mu = n^\nu \nabla_\nu n^\mu. \quad (\text{D.44})$$

Then it is the work of a few more lines to show that (D.43) is equivalent to

$$K_{\mu\nu} = \nabla_\mu n_\nu - \sigma n_\mu a_\nu. \quad (\text{D.45})$$

The extrinsic curvature has a number of nice properties. It is symmetric,

$$K_{\mu\nu} = K_{\nu\mu}, \quad (\text{D.46})$$

which looks obvious from (D.41), although not from (D.45). You can check that (D.45) really is symmetric, taking advantage of the fact that  $n^\mu$  is hypersurface-orthogonal. The extrinsic curvature is also orthogonal to the normal direction ("purely spatial"),

$$\begin{aligned} n^\mu K_{\mu\nu} &= n^\mu \nabla_\mu n_\nu - \sigma n^\mu n_\mu a_\nu \\ &= a_\nu - \sigma^2 a_\nu \\ &= 0. \end{aligned} \quad (\text{D.47})$$

We can define a covariant derivative acting along the hypersurface,  $\widehat{\nabla}_\mu$ , by taking an ordinary covariant derivative and projecting it. For example, on a  $(1, 1)$  tensor  $X^\mu{}_\nu$  we would have

$$\widehat{\nabla}_\sigma X^\mu{}_\nu = P^\alpha{}_\sigma P^\mu{}_\beta P^\gamma{}_\nu \nabla_\alpha X^\beta{}_\gamma. \quad (\text{D.48})$$

From this we can construct the curvature tensor on the hypersurface  $\widehat{R}^\rho{}_{\sigma\mu\nu}$ , for example by considering the commutator of covariant derivatives acting on a vector field  $V^\mu$ , which is tangent to the hypersurface ( $P^\mu{}_\nu V^\nu = V^\mu$ ),

$$[\widehat{\nabla}_\mu, \widehat{\nabla}_\nu] V^\rho = \widehat{R}^\rho{}_{\sigma\mu\nu} V^\sigma. \quad (\text{D.49})$$

Two important equations relate the  $n$ -dimensional Riemann curvature to the hypersurface Riemann curvature and the extrinsic curvature. **Gauss's equation** is

$$\widehat{R}^\rho{}_{\sigma\mu\nu} = P^\rho{}_\alpha P^\beta{}_\rho P^\gamma{}_\mu P^\delta{}_\nu R^\alpha{}_{\beta\gamma\delta} + \sigma(K^\rho{}_\mu K_{\sigma\nu} - K^\rho{}_\nu K_{\sigma\mu}). \quad (\text{D.50})$$

We can take the appropriate traces to get the hypersurface curvature scalar,

$$\widehat{R} = P^{\sigma\nu} \widehat{R}^\lambda{}_{\sigma\lambda\nu} = R - \sigma(2R_{\mu\nu} n^\mu n^\nu + K^2 - K^{\mu\nu} K_{\mu\nu}), \quad (\text{D.51})$$

where  $K = g^{\mu\nu} K_{\mu\nu}$ . We also have **Codazzi's equation**,

$$\widehat{\nabla}_{[\mu} K_{\nu]}{}^\mu = \frac{1}{2} P^\sigma{}_\nu R_{\rho\sigma} n^\rho. \quad (\text{D.52})$$

Together, (D.50) and (D.52) are, imaginatively enough, called the Gauss–Codazzi equations.

To stave off confusion, we should note that the definition of extrinsic curvature tends to vary from reference to reference. In some sources the normal vector field is taken to be geodesic everywhere ( $a^\mu = 0$ ); things then simplify considerably, and it's straightforward to show that in this case we have

$$\begin{aligned} K_{\mu\nu} &= \frac{1}{2} \mathcal{L}_n P_{\mu\nu} \\ &= \frac{1}{2} \mathcal{L}_n g_{\mu\nu} \\ &= \nabla_\mu n_\nu. \end{aligned} \quad (\text{D.53})$$

(If we are given an entire set of hypersurfaces ahead of time, we cannot simply assume that integral curves of the unit normal vector field are geodesics. However, we are often given just a single surface, in which case we are allowed to extend the normal vector field off the surface by solving the geodesic equation.) Other references prefer to think of the extrinsic curvature as a tensor  $\widehat{K}_{ij}$  living on  $\Sigma$  rather than in  $M$ . If we have an embedding  $\phi : y^i \rightarrow x^\mu$ , this version of the extrinsic curvature is given by the pullback,

$$\begin{aligned} \widehat{K}_{ij} &= (\phi^* K)_{ij} \\ &= \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} K_{\mu\nu}. \end{aligned} \quad (\text{D.54})$$

Finally, some sources like to define the extrinsic curvature to be minus our definition. It should be straightforward to convert back and forth between the different conventions.

To conclude our discussion, we mention that a very common appearance of hypersurfaces is as the **boundary** of a closed region  $N$  of a manifold  $M$ , conventionally denoted  $\partial N$ . If for example  $N$  consists of all the elements of  $\mathbf{R}^n$  that lie at a distance from the origin  $r \leq 1$ , the boundary  $\partial N$  is clearly the  $(n-1)$ -sphere defined by  $r = 1$ . We may extend this notion to cases where we are not considering a closed region, but an entire manifold with a boundary attached. A **manifold with**

**boundary** is a set equipped with an atlas of coordinate charts, exactly as in our definition of a manifold in Chapter 2, except that the charts are taken to be maps to the upper half of  $\mathbf{R}^n$ : the set of  $n$ -tuples  $\{x^1, \dots, x^n\}$  with  $x^1 \geq 0$ . The boundary  $\partial M$  is the set of points that are mapped to  $x^1 = 0$  by the charts. Then  $\partial M$  is naturally an  $(n - 1)$ -dimensional submanifold (without boundary). An example of a boundary of a manifold will appear in our later discussion of conformal diagrams, in which conformal infinity can be thought of as a boundary to spacetime. By continuity, we can treat the boundary as a hypersurface, including inducing metrics and so on; occasionally we need to be careful in taking derivatives on the boundary, but for the most part we can trust our intuition.

## E

## Stokes's Theorem

In Section 2.10 we introduced the idea that integration on a manifold maps  $n$ -form fields to the real numbers. This point of view leads to an elegant statement of one of the most powerful theorems of differential geometry: Stokes's theorem. This theorem is the generalization of the fundamental theorem of calculus,  $\int_b^a dx = a - b$ . Imagine that we have an  $n$ -dimensional region  $M$  (which might be an entire manifold) with boundary  $\partial M$ , and an  $(n - 1)$ -form  $\omega$  on  $M$ . We will soon explain what is meant by the boundary of a manifold. Then  $d\omega$  is an  $n$ -form, which can be integrated over  $M$ , while  $\omega$  itself can be integrated over  $\partial M$ . Stokes's theorem is simply

$$\int_M d\omega = \int_{\partial M} \omega. \quad (\text{E.1})$$

Different special cases of this theorem include not only the fundamental theorem of calculus, but also the theorems of Green, Gauss, and Stokes, familiar from vector calculus in three dimensions.

The presentation (E.1) of Stokes's theorem is extremely elegant, almost too elegant to be useful. We can, fortunately, recast it in pedestrian coordinate-and-index notation. It is convenient to first write the  $(n - 1)$ -form  $\omega$  as the Hodge dual of a one-form  $V$ ,

$$\omega = *V, \quad (\text{E.2})$$

with components

$$\begin{aligned} \omega_{\mu_1 \dots \mu_{n-1}} &= (*V)_{\mu_1 \dots \mu_{n-1}} \\ &= \epsilon^{\nu}_{\mu_1 \dots \mu_{n-1}} V_{\nu} \\ &= \epsilon_{\nu \mu_1 \dots \mu_{n-1}} V^{\nu}, \end{aligned} \quad (\text{E.3})$$

where  $\epsilon$  is the Levi-Civita  $n$ -form on  $M$  and we have raised the index on  $V$  in the last line. If we wanted to construct  $V$  from  $\omega$ , we apply the Hodge operator again to obtain

$$V = (-1)^{s+n-1} * *V = (-1)^{s+n-1} * \omega, \quad (\text{E.4})$$

where  $s$  equals  $-1$  for Lorentzian signatures and  $+1$  for Euclidean signatures. The exterior derivative of  $\omega = *V$  is an  $n$ -form, given by

$$\begin{aligned} (d\omega)_{\lambda\mu_1\cdots\mu_{n-1}} &= (d*V)_{\lambda\mu_1\cdots\mu_{n-1}} \\ &= n\nabla_{[\lambda}(\epsilon_{|\nu|\mu_1\cdots\mu_{n-1}\nu]}V^{\nu}) \\ &= n\epsilon_{\nu[\mu_1\cdots\mu_{n-1}}\nabla_{\lambda]}V^{\nu}, \end{aligned} \quad (\text{E.5})$$

where  $n$  is the dimensionality of the region, not to be confused with the normal vector  $n^\mu$  to the boundary. But any  $n$ -form can be written as a function  $f(x)$  times  $\epsilon$ , or equivalently as the Hodge dual of  $f(x)$ ,

$$d\omega = f\epsilon = *f. \quad (\text{E.6})$$

Taking the dual of both sides gives

$$f = (-1)^s * *f = (-1)^s * d\omega. \quad (\text{E.7})$$

In our case,

$$\begin{aligned} *d\omega &= *\epsilon * V \\ &= \frac{1}{n!}\epsilon^{\lambda\mu_1\cdots\mu_{n-1}}(n\epsilon_{\nu[\mu_1\cdots\mu_{n-1}}\nabla_{\lambda]}V^{\nu}) \\ &= \frac{1}{(n-1)!}(-1)^s(n-1)!\delta_{\nu}^{\lambda}\nabla_{\lambda}V^{\nu} \\ &= (-1)^s\nabla_{\nu}V^{\nu}. \end{aligned} \quad (\text{E.8})$$

Finally we recall that the Levi–Civita tensor is simply the volume element,

$$\begin{aligned} \epsilon &= \sqrt{|g|}dx^1\wedge\cdots\wedge dx^n \\ &= \sqrt{|g|}d^nx. \end{aligned} \quad (\text{E.9})$$

Putting it all together, we find

$$d\omega = \nabla_{\nu}V^{\nu}\sqrt{|g|}d^nx. \quad (\text{E.10})$$

So the exterior derivative of an  $(n-1)$ -form on an  $n$  manifold is just a slick way of representing the divergence of a vector (times the metric volume element).

To make sense of the right-hand side of (E.1), we recall from the previous Appendix that the induced volume element on a hypersurface (such as the boundary) is given by

$$\hat{\epsilon} = \sqrt{|\gamma|}d^{n-1}y, \quad (\text{E.11})$$

where  $\gamma_{ij}$  is the induced metric on the boundary in coordinates  $y^i$ . The components of  $\hat{\epsilon}$  in the  $x^\mu$  coordinates on  $M$  are

$$\hat{\epsilon}_{\mu_1\cdots\mu_{n-1}} = n^\lambda\epsilon_{\lambda\mu_1\cdots\mu_{n-1}}, \quad (\text{E.12})$$

where  $n^\mu$  is the unit normal to the boundary. For a general hypersurface, the sign of  $n^\mu$  is arbitrary; when the hypersurface is the boundary of a region, however, we have a notion of inward-pointing and outward-pointing. A crucial point is that, to correctly recover Stokes's theorem,  $n^\mu$  should be chosen to be inward-pointing if the boundary is timelike, and outward-pointing if it's spacelike. Since  $\omega$  is an  $(n - 1)$ -form, it must be proportional to  $\hat{\epsilon}$  when restricted to the  $(n - 1)$ -dimensional boundary. Following in the path of the previous paragraph, we derive

$$\omega = n_\mu V^\mu \sqrt{|\gamma|} d^{n-1} y. \quad (\text{E.13})$$

Stokes's theorem therefore relates the divergence of the vector field to its value on the boundary:

$$\boxed{\int_M d^n x \sqrt{|g|} \nabla_\mu V^\mu = \int_{\partial M} d^{n-1} y \sqrt{|\gamma|} n_\mu V^\mu.} \quad (\text{E.14})$$

This is the most common version of Stokes's theorem in general relativity.

You shouldn't get the impression that we need to descend to index notation to put Stokes's theorem to use. As a simple counterexample, let's show that the charge associated with a conserved current is "conserved" in a very general sense: Not only is it independent of time in some specific coordinate system, but also the charge passing through a spacelike hypersurface  $\Sigma$  is (under reasonable assumptions) completely independent of the choice of hypersurface. Start by imagining that we have a current  $J^\mu$  that is conserved, by which we mean

$$\nabla_\mu J^\mu = 0. \quad (\text{E.15})$$

In terms of the one-form  $J_\mu = g_{\mu\nu} J^\nu$ , we can translate the conservation condition into

$$d(*J) = 0. \quad (\text{E.16})$$

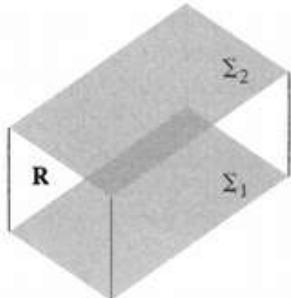
We then define the charge passing through a hypersurface  $\Sigma$  via

$$Q_\Sigma = - \int_\Sigma *J. \quad (\text{E.17})$$

Typically we will choose  $\Sigma$  to be a hypersurface of constant time, so that  $Q_\Sigma$  is the total charge throughout space at that moment in time; but the formula is applicable more generally. The minus sign is a convention, which can be understood by converting (temporarily) to components. Comparing to (E.2) and (E.13), we can turn (E.17) into

$$Q_\Sigma = - \int_\Sigma d^{n-1} y \sqrt{|\gamma|} n_\mu J^\mu. \quad (\text{E.18})$$

We see that the minus sign serves to compensate for the minus sign that the time component of  $n^\mu$  picks up when we lower the index, so that a positive charge



**FIGURE E.1** A region  $R$  of spacetime with spatial boundaries at infinity; the future and past boundaries include the two spatial hypersurfaces  $\Sigma_1$  and  $\Sigma_2$ .

density  $\rho = J^0$  yields a positive integrated total charge. Next imagine a four-dimensional spacetime region  $R$ , defined as the region between two spatial hypersurfaces  $\Sigma_1$  and  $\Sigma_2$ , as shown in Figure E.1; the part of the boundary connecting these two hypersurfaces is assumed to be off at infinity where all of the fields vanish, and can be ignored. The conservation law (E.16) and Stokes's theorem (E.1) then give us

$$\begin{aligned} 0 &= \int_R d(*J) \\ &= \int_{\partial R} *J \\ &= \int_{\Sigma_1} *J - \int_{\Sigma_2} *J \\ &= Q_1 - Q_2. \end{aligned} \tag{E.19}$$

The minus sign in the third line is due to the orientation on  $\Sigma_2$  inherited from  $R$ ; the normal vector is pointing inward, which is opposite from what would be the conventional choice in an integral over  $\Sigma_2$ . We see that  $Q_\Sigma$  will be the same over any spacelike hypersurface  $\Sigma$  chosen such that the current vanishes at infinity. Thus, Stokes's theorem shows how the existence of a divergenceless current implies the existence of a conserved charge.

Another use of Stokes's theorem (corresponding to the conventional use of Gauss's theorem in three-dimensional Euclidean space) is to actually calculate this charge  $Q$  by integrating over the hypersurface. Thinking momentarily about the real world, let's consider Maxwell's equations in a four-dimensional spacetime. These equations describe how the electromagnetic field strength tensor  $F_{\mu\nu}$  responds to the conserved current four-vector,

$$\nabla_\mu F^{\nu\mu} = J^\nu. \tag{E.20}$$

We can therefore plug  $\nabla_\mu F^{\nu\mu}$  into (E.18) to calculate the charge:

$$Q = - \int_{\Sigma} d^3y \sqrt{|\gamma|} n_\mu \nabla_\nu F^{\mu\nu}. \tag{E.21}$$

Whenever we are faced with the divergence of an antisymmetric tensor field  $F^{\mu\nu} = -F^{\nu\mu}$  integrated over a hypersurface  $\Sigma$ , we can follow similar steps to those used to arrive at (E.14), to relate the divergence to the value of  $F^{\mu\nu}$  on the boundary, this time at spatial infinity (if the hypersurface is timelike):

$$\int_{\Sigma} d^{n-1}y \sqrt{|\gamma|} n_\mu \nabla_\nu F^{\mu\nu} = \int_{\partial\Sigma} d^{n-2}z \sqrt{|\gamma^{(\partial\Sigma)}|} n_\mu \sigma_\nu F^{\mu\nu}, \tag{E.22}$$

where the  $z^a$ 's are coordinates on  $\partial\Sigma$ ,  $\gamma_{ab}^{(\partial\Sigma)}$  is the induced metric on  $\partial\Sigma$ , and  $\sigma^\mu$  is the unit normal to  $\partial\Sigma$ . You might worry about the integral over  $\partial\Sigma$ , since the

boundary of a boundary is zero; but  $\Sigma$  is not the entire boundary of any region, just a piece of one, so it can certainly have a boundary of its own.

Just to make sure we know what we're doing, let's verify that we can actually recover the charge of a point particle in Minkowski space. We write the metric in polar coordinates,

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (\text{E.23})$$

The electric field of a charge  $q$  in our units (Lorentz–Heaviside conventions, where there are no  $4\pi$ 's in Maxwell's equations) is

$$E^r = \frac{q}{4\pi r^2}, \quad (\text{E.24})$$

with other components vanishing; this is related to the field strength tensor by

$$F^{tr} = -F^{rt} = E^r. \quad (\text{E.25})$$

The unit normal vectors are

$$n^\mu = (1, 0, 0, 0), \quad \sigma^\mu = (0, 1, 0, 0), \quad (\text{E.26})$$

so that

$$n_\mu \sigma_\nu F^{\mu\nu} = -E^r = -\frac{q}{4\pi r^2}. \quad (\text{E.27})$$

The metric on the two-sphere at spatial infinity is

$$\gamma_{ab}^{(S^2)} dz^a dz^b = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (\text{E.28})$$

so the volume element is

$$d^2 z \sqrt{\gamma^{(S^2)}} = r^2 \sin \theta d\theta d\phi. \quad (\text{E.29})$$

Plugging (E.27), (E.29), and (E.21) into (E.22) gives

$$\begin{aligned} Q &= -\lim_{r \rightarrow \infty} \int_{S^2} d\theta d\phi r^2 \sin \theta \left( -\frac{q}{4\pi r^2} \right) \\ &= q, \end{aligned} \quad (\text{E.30})$$

which is just the answer we're looking for.

## F

## Geodesic Congruences

In Section 3.10 we derived the geodesic deviation equation, governing the evolution of a separation vector connecting a one-parameter family of neighboring geodesics. A more comprehensive picture of the behavior of neighboring geodesics comes from considering not just a one-parameter family, but an entire **congruence** of geodesics. A congruence is a set of curves in an open region of spacetime such that every point in the region lies on precisely one curve. We can think of a geodesic congruence as tracing the paths of a set of noninteracting particles moving through spacetime with nonintersecting paths. If the geodesics cross, the congruence necessarily comes to an end at that point. Clearly, in a multi-dimensional congruence there is a lot of information to keep track of; we will be interested in the local behavior in the neighborhood of a single geodesic, for which things become quite tractable.

Let  $U^\mu = dx^\mu/d\tau$  be the tangent vector field to a four-dimensional timelike geodesic congruence; equivalently, the four-velocity field of some pressureless fluid. (If the fluid were not pressureless, integral curves of  $U^\mu$  would not in general describe geodesics.) Null geodesics present special problems, which we will return to later; for now stick with the timelike case. For reference we recall that the tangent field is normalized and obeys the geodesic equation:

$$U_\mu U^\mu = -1, \quad U^\lambda \nabla_\lambda U^\mu = 0. \quad (\text{F.1})$$

When we discussed the geodesic deviation equation in Section 3.10, we considered a separation vector  $V^\mu$  pointing from one geodesic to a neighboring one, and found that it obeyed

$$\frac{DV^\mu}{d\tau} \equiv U^\nu \nabla_\nu V^\mu = B^\mu{}_\nu V^\nu, \quad (\text{F.2})$$

where

$$B^\mu{}_\nu = \nabla_\nu U^\mu. \quad (\text{F.3})$$

(In Chapter 3 we used  $T$  instead of  $U$ , and  $S$  instead of  $V$ .) The tensor  $B_{\mu\nu}$  therefore can be thought of as measuring the failure of  $V^\mu$  to be parallel-transported along the congruence; in other words, it describes the extent to which neighboring geodesics deviate from remaining perfectly parallel.

To deal with an entire congruence, rather than just a one-parameter family of curves, we can imagine setting up a set of three normal vectors orthogonal to our

timelike geodesics, and following their evolution. The failure of this set of vectors to be parallel-transported will tell us how nearby geodesics in the congruence are evolving. Equivalently, we can imagine a small sphere of test particles centered at some point, and we want to describe quantitatively the evolution of these particles with respect to their central geodesic. Fortunately, all we have to keep track of is the behavior of  $B_{\mu\nu}$ .

Given the vector field  $U^\mu$ , at each point  $p$  we consider the subspace of  $T_p M$  corresponding to vectors normal to  $U^\mu$ . Any vector in  $T_p M$  can be projected into this subspace via the projection tensor

$$P^\mu{}_\nu = \delta^\mu_\nu + U^\mu U_\nu, \quad (\text{F.4})$$

familiar from our discussion of submanifolds in Appendix D. In this case we are not projecting onto a submanifold, only onto a vector subspace of the tangent space, but the idea is the same. We notice that  $B_{\mu\nu}$  is already in the normal subspace, since

$$\begin{aligned} U^\mu B_{\mu\nu} &= U^\mu \nabla_\nu U_\mu = 0 \\ U^\nu B_{\mu\nu} &= U^\nu \nabla_\nu U_\mu = 0. \end{aligned} \quad (\text{F.5})$$

The first of these follows from  $\nabla_\nu(U^\mu U_\mu) = \nabla_\nu(-1) = 0$ , while the second follows from the geodesic equation. We should not confuse  $B_{\mu\nu}$  with the extrinsic curvature  $K_{\mu\nu}$  from (D.53); the difference is that our tangent vector field  $U^\mu$  will generally not be orthogonal to any hypersurface.

As a  $(0, 2)$  tensor,  $B_{\mu\nu}$  can be decomposed into symmetric and antisymmetric parts, and the symmetric part can further be decomposed into a trace and a trace-free part. Since  $B_{\mu\nu}$  is in the normal subspace, we can use  $P_{\mu\nu}$  to take the trace in this decomposition. The result can be written

$$B_{\mu\nu} = \frac{1}{3}\theta P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}. \quad (\text{F.6})$$

Here we have introduced three quantities describing the decomposition, starting with the **expansion**  $\theta$  of the congruence,

$$\theta = P^{\mu\nu} B_{\mu\nu} = \nabla_\mu U^\mu, \quad (\text{F.7})$$

which is simply the trace of  $B_{\mu\nu}$ . The expansion describes the change in volume of the sphere of test particles centered on our geodesic. It is clearly a scalar, which makes sense, since the overall expansion/contraction of the volume is described by a single number. The **shear**  $\sigma_{\mu\nu}$  is given by

$$\sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{1}{3}\theta P_{\mu\nu}. \quad (\text{F.8})$$

It is symmetric and traceless. The shear represents a distortion in the shape of our collection of test particles, from an initial sphere into an ellipsoid; symmetry represents the fact that elongation along (say) the  $x$ -direction is the same as

elongation along the  $-x$  direction. Finally we have the **rotation**  $\omega_{\mu\nu}$ , given by

$$\omega_{\mu\nu} = B_{[\mu\nu]}. \quad (\text{F.9})$$

It is an antisymmetric tensor, which also makes sense; the  $xy$  component (for example) describes a rotation about the  $z$  axis, while the  $yx$  component describes a rotation in the opposite sense around the same axis.

The evolution of our congruence is described by the covariant derivative of these quantities along the path,  $D/d\tau = U^\sigma \nabla_\sigma$ . We can straightforwardly calculate this for the entire tensor  $B_{\mu\nu}$ , and then take the appropriate decomposition. We have

$$\begin{aligned} \frac{DB_{\mu\nu}}{d\tau} &\equiv U^\sigma \nabla_\sigma B_{\mu\nu} = U^\sigma \nabla_\sigma \nabla_\nu U_\mu \\ &= U^\sigma \nabla_\nu \nabla_\sigma U_\mu + U^\sigma R^\lambda_{\mu\nu\sigma} U_\lambda \\ &= \nabla_\nu (U^\sigma \nabla_\sigma U_\mu) - (\nabla_\nu U^\sigma) (\nabla_\sigma U_\mu) - R_{\lambda\mu\nu\sigma} U^\sigma U^\lambda \\ &= -B^\sigma_{\nu} B_{\mu\sigma} - R_{\lambda\mu\nu\sigma} U^\sigma U^\lambda. \end{aligned} \quad (\text{F.10})$$

Taking the trace of this equation yields an evolution equation for the expansion,

$$\boxed{\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}U^\mu U^\nu.} \quad (\text{F.11})$$

This is **Raychaudhuri's equation**, and plays a crucial role in the proofs of the singularity theorems. [Sometimes the demand that the congruence obey the geodesic equation is dropped; this simply adds a term  $\nabla_\mu(U^\nu \nabla_\nu U^\mu)$  to the right-hand side.] Similarly, the symmetric trace-free part of (F.10) is

$$\begin{aligned} \frac{D\sigma_{\mu\nu}}{d\tau} &= -\frac{2}{3}\theta\sigma_{\mu\nu} - \sigma_{\mu\alpha}\sigma^\alpha_{\nu} - \omega_{\mu\rho}\omega^\rho_{\nu} + \frac{1}{3}P_{\mu\nu}(\sigma_{\alpha\beta}\sigma^{\alpha\beta} - \omega_{\alpha\beta}\omega^{\alpha\beta}) \\ &+ C_{\alpha\nu\mu\beta}U^\alpha U^\beta + \frac{1}{2}\bar{R}_{\mu\nu}, \end{aligned} \quad (\text{F.12})$$

where  $\bar{R}_{\mu\nu}$  is the spatially-projected trace-free part of  $R_{\mu\nu}$ ,

$$\bar{R}_{\mu\nu} = P^\alpha_{\mu} P^\beta_{\nu} R_{\alpha\beta} - \frac{1}{3}P_{\mu\nu} P^{\alpha\beta} R_{\alpha\beta}, \quad (\text{F.13})$$

and the antisymmetric part of (F.10) is

$$\frac{D\omega_{\mu\nu}}{d\tau} = -\frac{2}{3}\theta\omega_{\mu\nu} + \sigma_\mu^\alpha \omega_{\nu\alpha} - \sigma_\nu^\alpha \omega_{\mu\alpha}. \quad (\text{F.14})$$

These equations do not get used as frequently as Raychaudhuri's equation, but they're nice to have around.

Let's give a brief example of the way in which Raychaudhuri's equation gets used. First notice that, since the shear and rotation are both "spatial" tensors, we have

$$\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0, \quad \omega_{\mu\nu}\omega^{\mu\nu} \geq 0. \quad (\text{F.15})$$

Next, notice that the last term in (F.11) is just what appears if we combine Einstein's equation with the Strong Energy Condition; from Einstein's equation we know

$$R_{\mu\nu}U^\mu U^\nu = 8\pi G \left( T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu} \right) U^\mu U^\nu, \quad (\text{F.16})$$

and the SEC demands that the right-hand side of this expression be nonnegative for any timelike  $U^\mu$ . We therefore have

$$R_{\mu\nu}U^\mu U^\nu \geq 0 \quad (\text{F.17})$$

if the SEC holds. Finally, note that  $\omega_{\mu\nu} = 0$  if and only if the vector field  $U^\mu$  is orthogonal to a family of hypersurfaces. This follows straightforwardly from the facts that the rotation is a spatial tensor ( $U^\mu \omega_{\mu\nu} = 0$ ), and by Frobenius's theorem a necessary and sufficient condition for a vector field  $U^\mu$  to be hypersurface-orthogonal is  $U_{[\mu} \nabla_{\nu} U_{\rho]}$ ; the details are left for you to check. Therefore, if we have a congruence whose tangent field is hypersurface-orthogonal, in a spacetime obeying Einstein's equations and the SEC, Raychaudhuri's equation implies

$$\frac{d\theta}{d\tau} \leq -\frac{1}{3}\theta^2. \quad (\text{F.18})$$

This equation is easily integrated to obtain

$$\theta^{-1}(\tau) \geq \theta_0^{-1} + \frac{1}{3}\tau. \quad (\text{F.19})$$

Consider a hypersurface-orthogonal congruence, which is initially converging ( $\theta_0 < 0$ ) rather than expanding. Then (F.19) tells us convergence will continue, and we must hit a caustic (a place where geodesics cross) in a finite proper time  $\tau \leq -3\theta_0^{-1}$ . In other words, matter obeying the SEC can never begin to push geodesics apart, it can only increase the rate at which they are converging. Of course, this result only applies to some arbitrarily-chosen congruence, and the appearance of caustics certainly doesn't indicate any singularity in the spacetime (geodesics cross all the time, even in flat spacetime). But many of the proofs of the singularity theorems take advantage of this property of the Raychaudhuri equation to show that spacetime must be geodesically incomplete in some way.

We turn next to the behavior of congruences of null geodesics. These are trickier, essentially because our starting point (studying the evolution of vectors in a three-dimensional subspace normal to the tangent field) doesn't make as much sense, since the tangent vector of a null curve is normal to itself. Instead, in the null case what we care about is the evolution of vectors in a *two-dimensional* subspace of "spatial" vectors normal to the null tangent vector field  $k^\mu = dx^\mu/d\lambda$ .

Unfortunately, there is no unique way to define this subspace, as observers in different Lorentz frames will have different notions of what constitutes a spatial vector. Faced with this dilemma, we have two sensible approaches. A slick approach would be to define an abstract two-dimensional vector space by starting with the three-dimensional space of vectors orthogonal to  $k^\mu$ , and then taking equivalence classes where two vectors are equivalent if they differ by a multiple of  $k^\mu$ . The grungier approach, which we will follow, is simply to choose a second “auxiliary” null vector  $l^\mu$ , which (in some frame) points in the opposite spatial direction to  $k^\mu$ , normalized such that

$$l^\mu l_\mu = 0, \quad l^\mu k_\mu = -1. \quad (\text{F.20})$$

We furthermore demand that the auxiliary vector be parallel-transported,

$$k^\mu \nabla_\mu l^\nu = 0, \quad (\text{F.21})$$

which is compatible with (F.20) because parallel transport preserves inner products. The auxiliary null vector  $l^\mu$  is by no means unique, since as we've just noted the idea of pointing in opposite spatial directions is frame-dependent. Nevertheless, we can make a choice and hope that important quantities are independent of the arbitrary choice. Having done so, the two-dimensional space of normal vectors we are interested in, called  $T_\perp$ , consists simply of those vectors  $V^\mu$  that are orthogonal to both  $k^\mu$  and  $l^\mu$ ,

$$T_\perp = \{V^\mu | V^\mu k_\mu = 0, V^\mu l_\mu = 0\}. \quad (\text{F.22})$$

Our task now is to follow the evolution of deviation vectors living in this subspace, which represent a family of neighboring null geodesics.

Projecting into the normal subspace  $T_\perp$  requires a slightly modified definition of the projection tensor; it turns out that

$$Q_{\mu\nu} = g_{\mu\nu} + k_\mu l_\nu + k_\nu l_\mu \quad (\text{F.23})$$

does the trick. Namely,  $Q_{\mu\nu}$  will act like the metric when acting on vectors  $V^\mu$ ,  $W^\mu$  in  $T_\perp$ , while annihilating anything proportional to  $k^\mu$  or  $l^\mu$ . Some useful properties of this projection tensor include

$$\begin{aligned} Q_{\mu\nu} V^\nu W^\nu &= g_{\mu\nu} V^\mu W^\nu \\ Q^\mu{}_\nu V^\nu &= V^\mu \\ Q^\mu{}_\nu k^\nu &= 0 \\ Q^\mu{}_\nu l^\nu &= 0 \\ Q^\mu{}_\nu Q^\nu{}_\sigma &= Q^\mu{}_\sigma \\ k^\sigma \nabla_\sigma Q^\mu{}_\nu &= 0. \end{aligned} \quad (\text{F.24})$$

Just as for timelike geodesics, the failure of a normal deviation vector  $V^\mu$  to be parallel-propagated is governed by the tensor  $B^\mu{}_\nu = \nabla_\nu k^\mu$ , in the sense that

$$\frac{DV^\mu}{d\lambda} \equiv k^\nu \nabla_\nu V^\mu = B^\mu{}_\nu V^\nu. \quad (\text{F.25})$$

However, in the null case the tensor  $B_{\mu\nu}$  is actually more than we need; the relevant information is completely contained in the projected version,

$$\widehat{B}^\mu{}_\nu = Q^\mu{}_\alpha Q^\beta{}_\nu B^\alpha{}_\beta. \quad (\text{F.26})$$

To see this, we simply play around with (F.25), using the various properties in (F.24):

$$\begin{aligned} \frac{DV^\mu}{d\lambda} &= k^\nu \nabla_\nu V^\mu \\ &= k^\nu \nabla_\nu (Q^\mu{}_\rho V^\rho) \\ &= Q^\mu{}_\rho k^\nu \nabla_\nu V^\rho \\ &= Q^\mu{}_\rho B^\rho{}_\nu V^\nu \\ &= Q^\mu{}_\rho B^\rho{}_\nu Q^\nu{}_\sigma V^\sigma \\ &= \widehat{B}^\mu{}_\sigma V^\sigma. \end{aligned} \quad (\text{F.27})$$

So we only have to keep track of the evolution of this projected tensor, not the full  $B_{\mu\nu}$ .

To understand that evolution, we again decompose into the expansion, shear, and rotation:

$$\widehat{B}_{\mu\nu} = \frac{1}{2}\theta Q_{\mu\nu} + \widehat{\sigma}_{\mu\nu} + \widehat{\omega}_{\mu\nu}, \quad (\text{F.28})$$

where

$$\begin{aligned} \theta &= Q^{\mu\nu} \widehat{B}_{\mu\nu} = \widehat{B}^\mu{}_\mu \\ \widehat{\sigma}_{\mu\nu} &= \widehat{B}_{(\mu\nu)} - \frac{1}{2}\theta Q_{\mu\nu} \\ \widehat{\omega}_{\mu\nu} &= \widehat{B}_{[\mu\nu]}. \end{aligned} \quad (\text{F.29})$$

We find factors of  $\frac{1}{2}$  rather than  $\frac{1}{3}$  because our normal space  $T_\perp$  is two-dimensional, reflected in the fact that  $Q^{\mu\nu} Q_{\mu\nu} = 2$ . As in the timelike case,  $\widehat{\omega}_{\mu\nu} = 0$  is a necessary and sufficient condition for the congruence to be hypersurface-orthogonal. The evolution of  $\widehat{B}_{\mu\nu}$  along the path is given by

$$\begin{aligned} \frac{D\widehat{B}_{\mu\nu}}{d\lambda} &\equiv k^\sigma \nabla_\sigma \widehat{B}_{\mu\nu} = k^\sigma \nabla_\sigma (Q^\alpha{}_\mu Q^\beta{}_\nu \nabla_\alpha k_\beta) \\ &= Q^\alpha{}_\mu Q^\beta{}_\nu k^\sigma \nabla_\sigma \nabla_\alpha k_\beta \end{aligned}$$

$$\begin{aligned} &= -Q^\alpha_\mu Q^\beta_\nu (B_\alpha^\sigma B_{\beta\sigma} + R_{\alpha\lambda\beta\sigma} k^\lambda k^\sigma) \\ &= -\widehat{B}_\mu^\alpha \widehat{B}_{\nu\alpha} - Q^\alpha_\mu Q^\beta_\nu R_{\mu\lambda\nu\sigma} k^\lambda k^\sigma. \end{aligned} \quad (\text{F.30})$$

Continuing to follow our previous logic, we can take the trace of this equation to find an evolution equation for the expansion of null geodesics,

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \widehat{\sigma}_{\mu\nu}\widehat{\sigma}^{\mu\nu} + \widehat{\omega}_{\mu\nu}\widehat{\omega}^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu. \quad (\text{F.31})$$

Happily, this equation turns out to be completely independent of our arbitrarily chosen auxiliary vector  $l^\mu$ . First, the expansion itself is independent of  $l^\mu$ , as we easily verify:

$$\begin{aligned} \theta &= Q^{\mu\nu}\widehat{B}_{\mu\nu} \\ &= Q^{\mu\nu}B_{\mu\nu} \\ &= g^{\mu\nu}B_{\mu\nu}, \end{aligned} \quad (\text{F.32})$$

where the second line follows from  $Q^{\mu\nu}Q^\alpha_\nu = Q^{\mu\alpha}$ , and the third from  $k^\mu B_{\mu\nu} = k^\nu B_{\mu\nu} = 0$ . (This is why we never put a hat on  $\theta$  to begin with.) Second, both  $\widehat{\sigma}_{\mu\nu}\widehat{\sigma}^{\mu\nu}$  and  $\widehat{\omega}_{\mu\nu}\widehat{\omega}^{\mu\nu}$  are likewise independent of  $l^\mu$  (as you are welcome to verify), even though  $\widehat{\sigma}_{\mu\nu}$  and  $\widehat{\omega}_{\mu\nu}$  themselves are not. Finally, the projection tensors dropped out of the curvature-tensor piece when we took the trace. We therefore have a well-defined notion of the evolution of the expansion, independent of any arbitrary choices we made. Notice that, because  $k^\mu$  is null, Einstein's equation implies

$$\begin{aligned} R_{\mu\nu}k^\mu k^\nu &= 8\pi G \left( T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} \right) k^\mu k^\nu \\ &= 8\pi GT_{\mu\nu}k^\mu k^\nu. \end{aligned} \quad (\text{F.33})$$

For this to be nonnegative, we need only invoke the Null Energy Condition, which is the least restrictive of all the energy conditions we discussed in Chapter 3. Thus, null geodesics tend to converge to caustics under more general circumstances than timelike ones.

We can continue on to get evolution equations for the shear,

$$\frac{D\widehat{\sigma}_{\mu\nu}}{d\lambda} = -\theta\widehat{\sigma}_{\mu\nu} - Q^\alpha_\mu Q^\beta_\nu C_{\mu\lambda\nu\sigma} k^\lambda k^\sigma, \quad (\text{F.34})$$

and for the rotation,

$$\frac{D\widehat{\omega}_{\mu\nu}}{d\lambda} = -\theta\widehat{\omega}_{\mu\nu}. \quad (\text{F.35})$$

These equations are less natural than the one for the expansion, since the shear and rotation do depend on our choice of  $l^\mu$ ; nevertheless, they can be useful in specific circumstances.

# G

## Conformal Transformations

A **conformal transformation** is essentially a local change of scale. Since distances are measured by the metric, such transformations are implemented by multiplying the metric by a spacetime-dependent (nonvanishing) function:

$$\tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}, \quad (\text{G.1})$$

or equivalently

$$\tilde{ds}^2 = \omega^2(x) ds^2, \quad (\text{G.2})$$

for some nonvanishing function  $\omega(x)$ . (Here  $x$  is used to denote the collection of spacetime coordinates  $x^\mu$ .) Note that the inverse conformal transformation is trivial:  $g_{\mu\nu} = \omega^{-2} \tilde{g}_{\mu\nu}$ . Transformations of this sort have a number of uses in GR; our favorite purposes will be to change dynamical variables in scalar-tensor theories (as in Section 4.8), and to remap spacetimes into convenient conformal diagrams (as in the following Appendix).

We first mention one critical fact: *null curves are left invariant by conformal transformations*. By this we mean simply that, if  $x^\mu(\lambda)$  is a curve that is null with respect to  $g_{\mu\nu}$ , it will also be null with respect to  $\tilde{g}_{\mu\nu}$ . This follows immediately once we understand that a curve  $x^\mu(\lambda)$  is null if and only if its tangent vector  $dx^\mu/d\lambda$  is null,

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \quad (\text{G.3})$$

Then in the conformally-related metric we have

$$\tilde{g}_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \omega^2(x) g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \quad (\text{G.4})$$

Thus, curves that are null as defined by one metric will also be null as defined by any conformally-related metric. We may say that “conformal transformations leave light cones invariant.” (Indeed, you can check that they leave angles between any two four-vectors invariant, a feature that our conformal transformations share with the familiar conformal transformations of complex analysis.)

Let us next consider how geometrical quantities change under conformal transformations. A conformal transformation is not a change of coordinates, but an actual change of the geometry—timelike geodesics of  $\tilde{g}_{\mu\nu}$ , for example, will generally differ from timelike geodesics of  $g_{\mu\nu}$ . However, we can use conformal transformations to change our dynamical variables: anything that is a function of  $g_{\mu\nu}$

can be equally well thought of as a function of  $\tilde{g}_{\mu\nu}$  and  $\omega(x)$ . We then say that the quantities are expressed in the **conformal frame**. In this Appendix we collect some expressions for how quantities in the original metric  $g_{\mu\nu}$  are related to those in the conformal metric  $\tilde{g}_{\mu\nu}$ .

We begin by considering the Christoffel symbols. Because the connection coefficients are linear in derivatives of the metric and also linear in the inverse metric, the conformally-transformed connection takes the form

$$\tilde{\Gamma}_{\mu\nu}^{\rho} = \Gamma_{\mu\nu}^{\rho} + C^{\rho}_{\mu\nu}. \quad (\text{G.5})$$

$C^{\rho}_{\mu\nu}$  is clearly a tensor, as it is the difference of two connections. An explicit calculation reveals it to be given by

$$C^{\rho}_{\mu\nu} = \omega^{-1} (\delta_{\mu}^{\rho} \nabla_{\nu} \omega + \delta_{\nu}^{\rho} \nabla_{\mu} \omega - g_{\mu\nu} g^{\rho\lambda} \nabla_{\lambda} \omega). \quad (\text{G.6})$$

This formula immediately becomes useful when we consider how the Riemann tensor behaves under conformal transformations. In fact under any change of connection of the form (G.5), we have

$$\tilde{R}_{\sigma\mu\nu}^{\rho} = R_{\sigma\mu\nu}^{\rho} + \nabla_{\mu} C^{\rho}_{\nu\sigma} - \nabla_{\nu} C^{\rho}_{\mu\sigma} + C^{\rho}_{\mu\lambda} C^{\lambda}_{\nu\sigma} - C^{\rho}_{\nu\lambda} C^{\lambda}_{\mu\sigma}. \quad (\text{G.7})$$

Thus it is a matter of simply plugging in and grinding away to get

$$\begin{aligned} \tilde{R}_{\sigma\mu\nu}^{\rho} &= R_{\sigma\mu\nu}^{\rho} - 2 \left( \delta_{[\mu}^{\rho} \delta_{\nu]}^{\alpha} \delta_{\sigma}^{\beta} - g_{\sigma[\mu} \delta_{\nu]}^{\alpha} g^{\rho\beta} \right) \omega^{-1} (\nabla_{\alpha} \nabla_{\beta} \omega) \\ &\quad + 2 \left( 2 \delta_{[\mu}^{\rho} \delta_{\nu]}^{\alpha} \delta_{\sigma}^{\beta} - 2 g_{\sigma[\mu} \delta_{\nu]}^{\alpha} g^{\rho\beta} + g_{\sigma[\mu} \delta_{\nu]}^{\rho} g^{\alpha\beta} \right) \omega^{-2} (\nabla_{\alpha} \omega) (\nabla_{\beta} \omega). \end{aligned} \quad (\text{G.8})$$

Contracting the first and third indices yields the Ricci tensor,

$$\begin{aligned} \tilde{R}_{\sigma\nu} &= R_{\sigma\nu} - [(n-2) \delta_{\sigma}^{\alpha} \delta_{\nu}^{\beta} + g_{\sigma\nu} g^{\alpha\beta}] \omega^{-1} (\nabla_{\alpha} \nabla_{\beta} \omega) \\ &\quad + [2(n-2) \delta_{\sigma}^{\alpha} \delta_{\nu}^{\beta} - (n-3) g_{\sigma\nu} g^{\alpha\beta}] \omega^{-2} (\nabla_{\alpha} \omega) (\nabla_{\beta} \omega), \end{aligned} \quad (\text{G.9})$$

where  $n$  is the number of dimensions. Raising an index (with  $\tilde{g}^{\mu\nu} = \omega^{-2} g^{\mu\nu}$ ) and contracting again gets us the curvature scalar,

$$\tilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_{\alpha} \nabla_{\beta} \omega) - (n-1)(n-4) g^{\alpha\beta} \omega^{-4} (\nabla_{\alpha} \omega) (\nabla_{\beta} \omega). \quad (\text{G.10})$$

Another useful quantity is the covariant derivative of a scalar field  $\phi$ . The first covariant derivative is equal in the original or conformal frame, since they are both equal to the partial derivative:

$$\tilde{\nabla}_{\mu} \phi = \nabla_{\mu} \phi = \partial_{\mu} \phi. \quad (\text{G.11})$$

The second derivative, however, involves the Christoffel symbol, and therefore has a nontrivial transformation:

$$\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi = \nabla_\mu \nabla_\nu \phi - (\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha - g_{\mu\nu} g^{\alpha\beta}) \omega^{-1} (\nabla_\alpha \omega) (\nabla_\beta \phi). \quad (\text{G.12})$$

We can contract this with  $\tilde{g}^{\mu\nu}$  to obtain the D'Alembertian,

$$\tilde{\square} \phi = \omega^{-2} \square \phi + (n-2) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \omega) (\nabla_\beta \phi). \quad (\text{G.13})$$

Finally, we may want to go backward, and express quantities in the original metric in terms of the conformal metric. This is simply a matter of tedious computation, the answers to which are reproduced here for convenience. The curvature tensor and its contractions are

$$\begin{aligned} R^\rho{}_{\sigma\mu\nu} &= \tilde{R}^\rho{}_{\sigma\mu\nu} + 2 \left( \delta^\rho_{[\mu} \delta^\alpha_{\nu]} \delta^\beta_\sigma - \tilde{g}_{\sigma[\mu} \delta^\alpha_{\nu]} \tilde{g}^{\rho\beta} \right) \omega^{-1} (\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \omega) \\ &\quad + 2 \tilde{g}_{\sigma[\mu} \delta^\rho_{\nu]} \tilde{g}^{\alpha\beta} \omega^{-2} (\tilde{\nabla}_\alpha \omega) (\tilde{\nabla}_\beta \omega), \end{aligned} \quad (\text{G.14})$$

$$\begin{aligned} R_{\sigma\nu} &= \tilde{R}_{\sigma\nu} + [(n-2) \delta_\sigma^\alpha \delta_\nu^\beta + \tilde{g}_{\sigma\nu} \tilde{g}^{\alpha\beta}] \omega^{-1} (\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \omega) \\ &\quad - (n-1) \tilde{g}_{\sigma\nu} \tilde{g}^{\alpha\beta} \omega^{-2} (\tilde{\nabla}_\alpha \omega) (\tilde{\nabla}_\beta \omega), \end{aligned} \quad (\text{G.15})$$

and

$$R = \omega^2 \tilde{R} + 2(n-1) \tilde{g}^{\alpha\beta} \omega (\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \omega) - n(n-1) \tilde{g}^{\alpha\beta} (\tilde{\nabla}_\alpha \omega) (\tilde{\nabla}_\beta \omega), \quad (\text{G.16})$$

while the covariant derivatives of a scalar field are given by

$$\nabla_\mu \nabla_\nu \phi = \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi + (\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha - \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta}) \omega^{-1} (\tilde{\nabla}_\alpha \omega) (\tilde{\nabla}_\beta \phi) \quad (\text{G.17})$$

and

$$\square \phi = \omega^2 \tilde{\square} \phi - (n-2) \tilde{g}^{\alpha\beta} \omega (\tilde{\nabla}_\alpha \omega) (\tilde{\nabla}_\beta \phi). \quad (\text{G.18})$$

## G.1 ■ EXERCISES

1. Show that conformal transformations leave null geodesics invariant, that is, that the null geodesics of  $g_{\mu\nu}$  are the same as those of  $\omega^2 g_{\mu\nu}$ . (We already know that they leave null curves invariant; you have to show that the transformed curves still are geodesics.) What is the relationship between the affine parameters in the original and conformal metrics?
2. Show that in two dimensions, a conformal transformation can always be found (provided that the operator  $\nabla^\mu \nabla_\mu$  is invertible) such that the curvature of the transformed metric vanishes, at least in some coordinate chart. (It can't in general be done simultaneously over the entire manifold.) This means that any two-dimensional metric can be written locally as a flat metric multiplied by a conformal factor.
3. Suppose that two metrics are related by an overall conformal transformation of the form

$$\tilde{g}_{\mu\nu} = e^{\alpha(x)} g_{\mu\nu}. \quad (\text{G.19})$$

- (a) Show that if  $\xi^\mu$  is a Killing vector for the metric  $g_{\mu\nu}$ , then it is a conformal Killing vector for the metric  $\tilde{g}_{\mu\nu}$ . A **conformal Killing vector** obeys the equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = (\nabla_\lambda \alpha) \xi^\lambda g_{\mu\nu}. \quad (\text{G.20})$$

- (b) Show that  $\xi_\mu k^\mu$  is constant along photon geodesics in  $\tilde{g}_{\mu\nu}$ . Here  $k^\mu$  is the photon's 4-momentum.
- (c) Show that the conformal time  $\eta = \int dt/R(t)$  is associated with a conformal Killing vector  $\xi = \partial_\eta$ .
- (d) Use part (c) to rederive the relationship between the scale factor and redshift.

## H

## Conformal Diagrams

Curved spacetime manifolds can in principle be impossibly complex; fortunately, we may often approximate physically realistic situations by manifolds with high degrees of symmetry (especially spherical symmetry). Even symmetric spacetimes, however, can pose formidable challenges to our powers of visualization, if we try to imagine the global structure of such manifolds. It is therefore useful to be able to draw standardized representations of spacetime diagrams that capture the global properties and causal structure of sufficiently symmetric spacetimes. (By “causal structure” we mean the relationship between the past and future of different events, as defined by their light cones.) An elegant fulfillment of this wish is provided by **conformal diagrams** (or Carter–Penrose, or just Penrose diagrams).

A conformal diagram is simply an ordinary spacetime diagram for a metric on which we have performed a particularly clever coordinate transformation. Since our goal is to portray the causal structure of the spacetime, which is defined by its light cones, “clever” means that the new coordinates  $x^\mu$  have a “timelike” coordinate and a “radial” one, with the feature that radial light cones can be consistently portrayed at  $45^\circ$  on a spacetime diagram. In addition, we aim for coordinates in which “infinity” is only a finite coordinate value away, so that the structure of the entire spacetime is immediately apparent.

As explained in the previous Appendix, conformal transformations leave light cones invariant. Since we would like to find coordinates in which light cones are at  $45^\circ$ , we need only find coordinates in which the metric of interest is conformally related to a different metric for which we know that the light cones are at  $45^\circ$ . (Of course the angle at which our light cones are drawn depends on our units, or equivalently how we draw our axes; what we really mean is a set of coordinates  $T, R$  in which radial null rays satisfy  $dT/dR = \pm 1$ .)

Let’s begin with Minkowski space to see how the technique works. The Minkowski metric in polar coordinates is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \quad (\text{H.1})$$

where  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$  is the metric on a unit two-sphere. Here it is already true that we can draw light cones at  $45^\circ$  everywhere (the trajectories  $t = \pm r$  are null), but we would like to make the causal structure of the entire spacetime more transparent by switching to coordinates with finite ranges. Nothing unusual will happen to the  $\theta, \phi$  coordinates, but we will want to keep careful track of the ranges

of the other two coordinates. To start with of course we have

$$-\infty < t < \infty, \quad 0 \leq r < \infty. \quad (\text{H.2})$$

Technically, the worldline  $r = 0$  represents a coordinate singularity and should be covered by a different patch, but we all know what is going on so we'll just act like  $r = 0$  is well-behaved.

A first guess (which turns out not to work) might be simply to rescale the timelike and radial coordinates so that they cover a finite range. A good candidate is to use the arctangent, portrayed in Figure H.1, and define  $\tilde{t} = \arctan t$ ,  $\tilde{r} = \arctan r$ . The metric then would take the form [using  $d \tan x = (1/\cos^2 x)dx$ ]

$$ds^2 = -\frac{1}{\cos^4 \tilde{t}} d\tilde{t}^2 + \frac{1}{\cos^4 \tilde{r}} d\tilde{r}^2 + \tan^2 \tilde{r} d\Omega^2, \quad (\text{H.3})$$

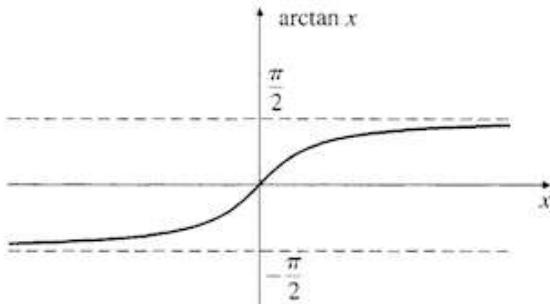
with

$$\begin{aligned} -\frac{\pi}{2} &< \tilde{t} < \frac{\pi}{2} \\ 0 &\leq \tilde{r} < \frac{\pi}{2}. \end{aligned} \quad (\text{H.4})$$

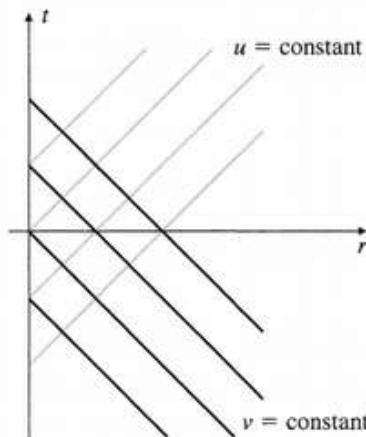
The good news is that the new coordinates have finite ranges; the bad news is that the slope of the light cones (given by  $d\tilde{t}/d\tilde{r} = \pm \cos^2 \tilde{t} / \cos^2 \tilde{r}$ ) is not equal to  $\pm 1$ , as we wished. If we were to draw the appropriate spacetime diagram (which you might want to do, just for fun), it would not be clear where null rays traveled, especially at the edges of the spacetime.

The way out of this cul-de-sac is, instead of straightforwardly manipulating the original coordinates  $t$  and  $r$ , to be even more clever and switch to null coordinates:

$$\begin{aligned} u &= t - r \\ v &= t + r, \end{aligned} \quad (\text{H.5})$$



**FIGURE H.1** The arctangent maps the real line to a finite interval.



**FIGURE H.2** Null radial coordinates on Minkowski space.

with corresponding ranges given by

$$-\infty < u < \infty, \quad -\infty < v < \infty, \quad u \leq v. \quad (\text{H.6})$$

These coordinates are as portrayed in Figure H.2, on which each point represents a 2-sphere of radius  $r = \frac{1}{2}(v - u)$ . The Minkowski metric in null coordinates is given by

$$ds^2 = -\frac{1}{2}(du dv + dv du) + \frac{1}{4}(v - u)^2 d\Omega^2. \quad (\text{H.7})$$

Now we use the arctangent to bring infinity into a finite coordinate value, letting

$$\begin{aligned} U &= \arctan u \\ V &= \arctan v, \end{aligned} \quad (\text{H.8})$$

with ranges

$$-\pi/2 < U < \pi/2, \quad -\pi/2 < V < \pi/2, \quad U \leq V. \quad (\text{H.9})$$

We then have

$$du dv + dv du = \frac{1}{\cos^2 U \cos^2 V} (dU dV + dV dU), \quad (\text{H.10})$$

and

$$\begin{aligned} (v - u)^2 &= (\tan V - \tan U)^2 = \frac{1}{\cos^2 U \cos^2 V} (\sin V \cos U - \cos V \sin U)^2 \\ &= \frac{1}{\cos^2 U \cos^2 V} \sin^2(V - U), \end{aligned} \quad (\text{H.11})$$

so that the metric (H.7) in these coordinates is

$$ds^2 = \frac{1}{4 \cos^2 U \cos^2 V} \left[ -2(dUdV + dVdU) + \sin^2(V - U) d\Omega^2 \right]. \quad (\text{H.12})$$

This form has a certain appeal, since the metric appears as a fairly simple expression multiplied by an overall factor. We can make it even better by transforming back to a timelike coordinate  $T$  and a radial coordinate  $R$ , via

$$T = V + U, \quad R = V - U, \quad (\text{H.13})$$

with ranges

$$0 \leq R < \pi, \quad |T| + R < \pi, \quad (\text{H.14})$$

Now the metric is

$$ds^2 = \omega^{-2}(T, R) \left( -dT^2 + dR^2 + \sin^2 R d\Omega^2 \right), \quad (\text{H.15})$$

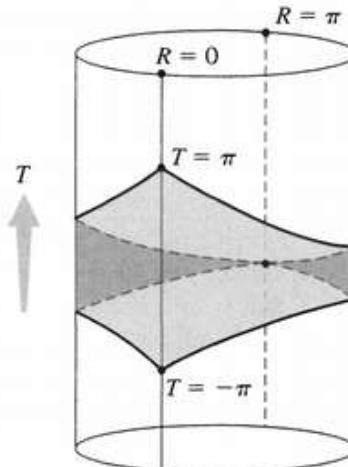
where

$$\begin{aligned} \omega &= 2 \cos U \cos V \\ &= 2 \cos \left[ \frac{1}{2}(T - R) \right] \cos \left[ \frac{1}{2}(T + R) \right] \\ &= \cos T + \cos R. \end{aligned} \quad (\text{H.16})$$

The original Minkowski metric, which we denoted  $ds^2$ , may therefore be thought of as related by a conformal transformation to the “unphysical” metric

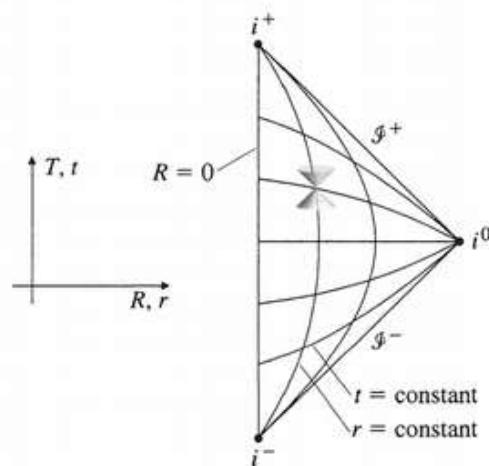
$$\begin{aligned} \tilde{ds}^2 &= \omega^2(T, R) ds^2 \\ &= -dT^2 + dR^2 + \sin^2 R d\Omega^2. \end{aligned} \quad (\text{H.17})$$

This describes the manifold  $\mathbf{R} \times S^3$ , where the 3-sphere is purely spacelike, perfectly round, and unchanging in time. There is curvature in this metric, unlike in Minkowski spacetime. This shouldn’t bother us, since it is unphysical; the true physical metric, obtained by a conformal transformation, is simply flat spacetime, no matter what coordinates we choose. In fact the metric (H.17) is that of the “Einstein static universe,” a static solution to Einstein’s equation with a perfect fluid and a cosmological constant (Figure H.3). Of course, the full range of coordinates on  $\mathbf{R} \times S^3$  would usually be  $-\infty < T < \infty$ ,  $0 \leq R \leq \pi$ , while Minkowski space is mapped into the subspace defined by (H.14). The entire  $\mathbf{R} \times S^3$  can be drawn as a cylinder, in which each circle of constant  $T$  represents a 3-sphere. The shaded region represents Minkowski space. We can unroll the shaded region to portray Minkowski space as a triangle, as shown in Figure H.4. This is the conformal diagram. Each point represents a two-sphere.



**FIGURE H.3** The Einstein static universe,  $\mathbf{R} \times S^3$ , portrayed as a cylinder. The shaded region is conformally related to Minkowski space.

In fact Minkowski space is only the *interior* of the above diagram (including  $R = 0$ ); the boundaries are not part of the original spacetime. The boundaries are referred to as **conformal infinity**, and the union of the original spacetime with conformal infinity is the **conformal compactification**, which is a manifold with boundary. The structure of the conformal diagram allows us to subdivide conformal infinity into a few different regions:



**FIGURE H.4** The conformal diagram of Minkowski space. Light cones are at  $\pm 45^\circ$  throughout the diagram.

$i^+$  = future timelike infinity ( $T = \pi$ ,  $R = 0$ )

$i^0$  = spatial infinity ( $T = 0$ ,  $R = \pi$ )

$i^-$  = past timelike infinity ( $T = -\pi$ ,  $R = 0$ )

$\mathcal{I}^+$  = future null infinity ( $T = \pi - R$ ,  $0 < R < \pi$ )

$\mathcal{I}^-$  = past null infinity ( $T = -\pi + R$ ,  $0 < R < \pi$ )

( $\mathcal{I}^+$  and  $\mathcal{I}^-$  are pronounced as “scri-plus” and “scri-minus,” respectively.) Note that  $i^+$ ,  $i^0$ , and  $i^-$  are actually *points*, since  $R = 0$  and  $R = \pi$  are the north and south poles of  $S^3$ . Meanwhile  $\mathcal{I}^+$  and  $\mathcal{I}^-$  are actually null surfaces, with the topology of  $\mathbf{R} \times S^2$ .

The conformal diagram for Minkowski spacetime contains a number of important features. Radial null geodesics are at  $\pm 45^\circ$  in the diagram. All timelike geodesics begin at  $i^-$  and end at  $i^+$ ; all null geodesics begin at  $\mathcal{I}^-$  and end at  $\mathcal{I}^+$ ; all spacelike geodesics both begin and end at  $i^0$ . On the other hand, there can be nongeodesic timelike curves that end at null infinity, if they become “asymptotically null.”

It is nice to be able to fit all of Minkowski space on a small piece of paper, but we don’t really learn much that we didn’t already know. Conformal diagrams are more useful when we want to represent slightly more complicated spacetimes, such as those for black holes. As discussed in Chapter 6, asymptotically flat spacetimes (or regions of a spacetime) are those that share the structure of  $\mathcal{I}^+$ ,  $i^0$ , and  $\mathcal{I}^-$  with Minkowski space. Equally importantly, the conformal diagram gives us an idea of the causal structure of the spacetime, for example, whether the past or future light cones of two specified points intersect. In Minkowski space this is always true for any two points, but curved spacetimes can be more interesting, as we saw for the case of an expanding universe in Chapter 2.

Let’s consider the conformal diagram for the cosmological spacetime introduced in Chapter 2, which provides a vivid illustration of the usefulness of this technique. When we put polar coordinates on space, the metric becomes

$$ds^2 = -dt^2 + t^{2q} \left( dr^2 + r^2 d\Omega^2 \right), \quad (\text{H.18})$$

where we have chosen to consider power-law behavior for the scale factor,  $a(t) = t^q$ , and  $0 < q < 1$ . A crucial difference between this metric and that of Minkowski space is the singularity at  $t = 0$ , which restricts the range of our coordinates:

$$0 < t < \infty \quad (\text{H.19})$$

$$0 \leq r < \infty. \quad (\text{H.20})$$

Other than this restricted coordinate range, our analysis follows almost precisely that of the case of flat spacetime. This is because we can bring the metric (H.18) to the form of flat spacetime times a conformal factor; once done, we need only to reproduce our previous coordinate transformations to express our expanding-universe metric as a conformal factor times the Einstein static universe.

We begin by choosing a new time coordinate  $\eta$ , sometimes called **conformal time**, which satisfies

$$dt^2 = t^{2q} d\eta^2, \quad (\text{H.21})$$

or

$$\eta = \frac{1}{1-q} t^{1-q}. \quad (\text{H.22})$$

This simple choice allows us to bring out the scale factor as an overall conformal factor,

$$ds^2 = [(1-q)\eta]^{2q/(1-q)} \left( -d\eta^2 + dr^2 + r^2 d\Omega^2 \right). \quad (\text{H.23})$$

The range of  $\eta$  is the same as that of  $t$ ,

$$0 < \eta < \infty. \quad (\text{H.24})$$

Note that  $\eta$  is a timelike coordinate [in the sense that the vector  $\partial_\eta$  is timelike,  $ds^2(\partial_\eta, \partial_\eta) < 0$ ], but it does not measure the proper time of a comoving clock (one with constant spatial coordinates). If we consider a trajectory  $x^\mu(\lambda) = (\eta(\lambda), 0, 0, 0)$ , and calculated the proper time  $\tau(\eta)$ , we would find that it was equal to our previous time coordinate but not our new one:  $\tau \propto t \propto \eta^{1/(1-q)}$ . So  $\eta$  is a timelike coordinate, but not the time that anyone would measure. This is perfectly okay, and simply serves as an illustration of the independence of the notions of observable quantities and spacetime coordinates.

Now that we have our expanding-universe metric in the form of a conformal factor times Minkowski, we can perform the same sequence of coordinate transformations—(H.5), (H.8), and (H.13)—where we allow  $\eta$  to take the place of  $t$ . These changes transform our coordinates from  $(\eta, r)$  to  $(T, R)$ , where the ranges are now

$$0 \leq R, \quad 0 < T, \quad T + R < \pi. \quad (\text{H.25})$$

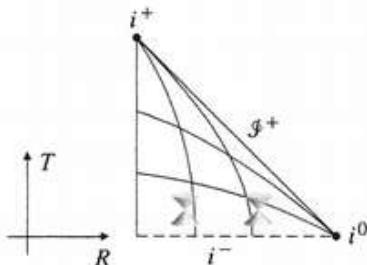
The metric (H.23) becomes

$$ds^2 = \omega^{-2}(T, R) \left( -dT^2 + dR^2 + \sin^2 R d\Omega^2 \right), \quad (\text{H.26})$$

where some heroic use of trigonometric identities reveals that the conformal factor is of the form

$$\omega(T, R) = \left( \frac{\cos T + \cos R}{2 \sin T} \right)^{2q} (\cos T + \cos R). \quad (\text{H.27})$$

The precise form of the conformal factor is actually not of primary importance; the crucial feature is that we have once again expressed our metric as a conformal factor times that of the Einstein static universe. The important distinction between



**FIGURE H.5** Conformal diagram for a Robertson–Walker universe with  $a(t) \propto t^q$  for  $0 < q < 1$ . The dashed line represents the singularity at  $t = 0$  (which also corresponds to  $T = 0$ ).

this case and that of flat spacetime is that the timelike coordinate ends at the singularity at  $T = 0$ ; otherwise the spacetime diagram is identical. We therefore have the conformal diagram of Figure H.5, which resembles the upper half of the Minkowski diagram (Figure H.4). Once again, light cones appear at  $45^\circ$ . We see how the conformal diagram makes the causal structure apparent; it is straightforward to choose two events in the spacetime with the property that their past light cones will hit the singularity before they intersect (while future light cones will always overlap). For more complicated geometries, this convenient way of representing a spacetime will be even more useful.



# The Parallel Propagator

The idea of parallel-transporting a tensor along a curve is obviously of central importance in GR. For a vector  $V^\mu$  being transported down a path  $x^\mu(\lambda)$ , the equation of parallel transport is

$$\frac{dx^\mu}{d\lambda} \nabla_\mu V^\nu \equiv \frac{dx^\mu}{d\lambda} \partial_\mu V^\nu + \frac{dx^\mu}{d\lambda} \Gamma_{\mu\sigma}^\nu V^\sigma = 0. \quad (\text{I.1})$$

It turns out to be possible to write down an explicit and general solution to this equation; it's somewhat formal, but interesting both in its own right and for its connections to techniques in quantum field theory.

We begin by noticing that for some path  $\gamma : \lambda \rightarrow x^\sigma(\lambda)$ , solving the parallel transport equation for a vector  $V^\mu$  amounts to finding a matrix  $P^\mu{}_\rho(\lambda, \lambda_0)$ , which relates the vector at its initial value  $V^\mu(\lambda_0)$  to its value somewhere later down the path:

$$V^\mu(\lambda) = P^\mu{}_\rho(\lambda, \lambda_0) V^\rho(\lambda_0). \quad (\text{I.2})$$

Of course the matrix  $P^\mu{}_\rho(\lambda, \lambda_0)$ , known as the **parallel propagator**, depends on the path  $\gamma$  (although it's hard to find a notation that indicates this without making  $\gamma$  look like an index). If we define

$$A^\mu{}_\rho(\lambda) = -\Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda}, \quad (\text{I.3})$$

where the quantities on the right-hand side are evaluated at  $x^\nu(\lambda)$ , then the parallel transport equation becomes

$$\frac{d}{d\lambda} V^\mu = A^\mu{}_\rho V^\rho. \quad (\text{I.4})$$

Since the parallel propagator must work for any vector, substituting (I.2) into (I.4) shows that  $P^\mu{}_\rho(\lambda, \lambda_0)$  also obeys this equation:

$$\frac{d}{d\lambda} P^\mu{}_\rho(\lambda, \lambda_0) = A^\mu{}_\sigma(\lambda) P^\sigma{}_\rho(\lambda, \lambda_0). \quad (\text{I.5})$$

To solve this equation, first integrate both sides:

$$P^\mu{}_\rho(\lambda, \lambda_0) = \delta_\rho^\mu + \int_{\lambda_0}^{\lambda} A^\mu{}_\sigma(\eta) P^\sigma{}_\rho(\eta, \lambda_0) d\eta. \quad (\text{I.6})$$

The Kronecker delta, it is easy to see, provides the correct normalization for  $\lambda = \lambda_0$ .

We can solve (I.6) by iteration, taking the right-hand side and plugging it into itself repeatedly, giving

$$P^\mu{}_\rho(\lambda, \lambda_0) = \delta^\mu_\rho + \int_{\lambda_0}^\lambda A^\mu{}_\rho(\eta) d\eta + \int_{\lambda_0}^\lambda \int_{\lambda_0}^\eta A^\mu{}_\sigma(\eta) A^\sigma{}_\rho(\eta') d\eta' d\eta + \dots \quad (\text{I.7})$$

The  $n$ th term in this series is an integral over an  $n$ -dimensional right triangle, or  $n$ -simplex:

$$\begin{aligned} & \int_{\lambda_0}^\lambda A(\eta_1) d\eta_1 \\ & \int_{\lambda_0}^\lambda \int_{\lambda_0}^{\eta_2} A(\eta_2) A(\eta_1) d\eta_1 d\eta_2 \\ & \int_{\lambda_0}^\lambda \int_{\lambda_0}^{\eta_3} \int_{\lambda_0}^{\eta_2} A(\eta_3) A(\eta_2) A(\eta_1) d^3\eta. \end{aligned}$$

See Figure I.1.

It would simplify things if we could consider such an integral to be over an  $n$ -cube instead of an  $n$ -simplex. Is there some way to do this? There are  $n!$  such simplices in each cube, so we would have to multiply by  $1/n!$  to compensate for this extra volume. But we also want to get the integrand right; using matrix notation, the integrand at  $n$ th order is  $A(\eta_n)A(\eta_{n-1}) \cdots A(\eta_1)$ , but with the special property that  $\eta_n \geq \eta_{n-1} \geq \cdots \geq \eta_1$ . We therefore define the **path-ordering symbol**,  $\mathcal{P}$ , to ensure that this condition holds. In other words, the expression

$$\mathcal{P}[A(\eta_n)A(\eta_{n-1}) \cdots A(\eta_1)] \quad (\text{I.8})$$

stands for the product of the  $n$  matrices  $A(\eta_i)$ , ordered in such a way that the largest value of  $\eta_i$  is on the left, and each subsequent value of  $\eta_i$  is less than or

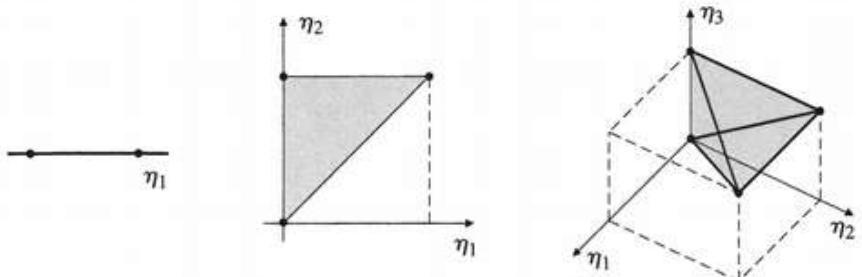


FIGURE I.1  $n$ -simplices ( $n$ -dimensional right triangles) for  $n = 1, 2, 3$ .

equal to the previous one. We then can express the  $n$ th-order term in (I.7) as

$$\begin{aligned} & \int_{\lambda_0}^{\lambda} \int_{\lambda_0}^{\eta_n} \cdots \int_{\lambda_0}^{\eta_2} A(\eta_n) A(\eta_{n-1}) \cdots A(\eta_1) d^n \eta \\ &= \frac{1}{n!} \int_{\lambda_0}^{\lambda} \int_{\lambda_0}^{\lambda} \cdots \int_{\lambda_0}^{\lambda} \mathcal{P}[A(\eta_n) A(\eta_{n-1}) \cdots A(\eta_1)] d^n \eta. \end{aligned} \quad (\text{I.9})$$

This expression contains no substantive statement about the matrices  $A(\eta_i)$ ; it is just notation. But we can now write (I.7) in matrix form as

$$P(\lambda, \lambda_0) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\lambda_0}^{\lambda} \mathcal{P}[A(\eta_n) A(\eta_{n-1}) \cdots A(\eta_1)] d^n \eta. \quad (\text{I.10})$$

This formula is just the series expression for an exponential; we therefore say that the parallel propagator is given by the path-ordered exponential

$$P(\lambda, \lambda_0) = \mathcal{P} \exp \left( \int_{\lambda_0}^{\lambda} A(\eta) d\eta \right), \quad (\text{I.11})$$

where once again this is just notation; the path-ordered exponential is defined to be the right-hand side of (I.10). We can write it more explicitly as

$$P^\mu{}_\nu(\lambda, \lambda_0) = \mathcal{P} \exp \left( - \int_{\lambda_0}^{\lambda} \Gamma^\mu_{\sigma\nu} \frac{dx^\sigma}{d\eta} d\eta \right). \quad (\text{I.12})$$

It's nice to have an explicit formula, even if it is rather abstract. The same kind of expression appears in quantum field theory as "Dyson's Formula," where it arises because the Schrödinger equation for the time-evolution operator has the same form as (I.5).

An especially interesting example of the parallel propagator occurs when the path is a loop, starting and ending at the same point. Then if the connection is metric-compatible, the resulting matrix will just be a Lorentz transformation on the tangent space at the point. This transformation is known as the "holonomy" of the loop. If you know the holonomy of every possible loop, that turns out to be equivalent to knowing the metric. One can then examine general relativity in the "loop representation," where the fundamental variables are holonomies rather than the explicit metric. A program called "loop quantum gravity" attempts to directly quantize general relativity in these variables (as opposed to something like string theory, in which GR falls out in some limit). A great deal of mathematical progress has been made in this direction, but fundamental obstacles remain.<sup>1</sup>

<sup>1</sup>For a review of this approach, see C. Rovelli, "Loop quantum gravity," *Living Rev. Rel.* 1, 1 (1998) <http://arxiv.org/gr-qc/9710008>.

## J

## Noncoordinate Bases

Early on in our study of manifolds, we made a decision to choose bases for our tangent spaces that were adapted to coordinates. For both aesthetic and pragmatic reasons, we should consider once again the formalism of connections and curvature, but this time using sets of basis vectors in the tangent space that are *not* derived from any coordinate system. It will turn out that this slight change in emphasis reveals a different point of view on the connection and curvature, one in which the relationship to gauge theories of particle physics is much more transparent. In fact the concepts to be introduced are very straightforward, but the subject is a notational nightmare, so it looks more difficult than it really is.

Until now we have been taking advantage of the fact that a natural basis for the tangent space  $T_p$  at a point  $p$  is given by the partial derivatives with respect to the coordinates at that point,  $\hat{e}_{(\mu)} = \partial_\mu$ . Similarly, a basis for the cotangent space  $T_p^*$  is given by the gradients of the coordinate functions,  $\hat{\theta}^{(\mu)} = dx^\mu$ . Nothing stops us, however, from setting up any bases we like. Let us therefore imagine that at each point in the manifold we introduce a set of basis vectors  $\hat{e}_{(a)}$  (indexed by a Latin letter rather than Greek, to remind us that they are not related to any coordinate system). We will choose these basis vectors to be “orthonormal,” in a sense that is appropriate to the signature of the manifold on which we are working. That is, if the canonical form of the metric is written  $\eta_{ab}$ , we demand that the inner product of our basis vectors be

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab}, \quad (\text{J.1})$$

where  $g(\cdot, \cdot)$  is the usual metric tensor. Thus, in a Lorentzian spacetime  $\eta_{ab}$  represents the Minkowski metric, while in a space with positive-definite metric it would represent the Euclidean metric. The set of vectors comprising an orthonormal basis is sometimes known as a **tetrad** (from Greek *tetras*, “a group of four”) or **vielbein** (from the German for “many legs”). In different numbers of dimensions it occasionally becomes a *vierbein* (four), *dreibein* (three), *zweibein* (two), and so on. Just as we cannot in general find coordinate charts that cover the entire manifold, we will often not be able to find a single set of smooth basis vector fields that are defined everywhere. As usual, we can overcome this problem by working in different patches and making sure things are well-behaved on the overlaps.

The point of having a basis is that any vector can be expressed as a linear combination of basis vectors. Specifically, we can express our old basis vectors

$\hat{e}_{(\mu)} = \partial_\mu$  in terms of the new ones:

$$\hat{e}_{(\mu)} = e_\mu^a \hat{e}_{(a)}. \quad (\text{J.2})$$

The components  $e_\mu^a$  form an  $n \times n$  invertible matrix. (In accord with our usual practice of blurring the distinction between objects and their components, we will refer to the  $e_\mu^a$  as the tetrad or vielbein, and often in the plural as “vielbeins.”) We denote their inverse by switching indices to obtain  $e^\mu_a$ , which satisfy

$$e^\mu_a e_\nu^a = \delta_\nu^\mu, \quad e_\mu^a e^\mu_b = \delta_b^a. \quad (\text{J.3})$$

These serve as the components of the vectors  $\hat{e}_{(a)}$  in the coordinate basis:

$$\hat{e}_{(a)} = e^\mu_a \hat{e}_{(\mu)}. \quad (\text{J.4})$$

In terms of the inverse vielbeins, (J.1) becomes

$$g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab}, \quad (\text{J.5})$$

or equivalently

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \quad (\text{J.6})$$

This last equation sometimes leads people to say that the vielbeins are the “square root” of the metric.

We can similarly set up an orthonormal basis of one-forms in  $T_p$ , which we denote  $\hat{\theta}^{(a)}$ . They may be chosen to be compatible with the basis vectors, in the sense that

$$\hat{\theta}^{(a)}(\hat{e}_{(b)}) = \delta_b^a. \quad (\text{J.7})$$

An immediate consequence is that the orthonormal one-forms are related to their coordinate-based cousins  $\hat{\theta}^{(\mu)} = dx^\mu$  by

$$\hat{\theta}^{(\mu)} = e^\mu_a \hat{\theta}^{(a)} \quad (\text{J.8})$$

and

$$\hat{\theta}^{(a)} = e_\mu^a \hat{\theta}^{(\mu)}. \quad (\text{J.9})$$

The vielbeins  $e_\mu^a$  thus serve double duty as the components of the coordinate basis vectors in terms of the orthonormal basis vectors, and as components of the orthonormal basis one-forms in terms of the coordinate basis one-forms; while the inverse vielbeins serve as the components of the orthonormal basis vectors in terms of the coordinate basis, and as components of the coordinate basis one-forms in terms of the orthonormal basis.

Any other vector can be expressed in terms of its components in the orthonormal basis. If a vector  $V$  is written in the coordinate basis as  $V^\mu \hat{e}_{(\mu)}$  and in the orthonormal basis as  $V^a \hat{e}_{(a)}$ , the sets of components will be related by

$$V^a = e_\mu{}^a V^\mu. \quad (\text{J.10})$$

So the vielbeins allow us to “switch from Latin to Greek indices and back.” The nice property of tensors, that there is usually only one sensible thing to do based on index placement, is of great help here. We can go on to refer to multi-index tensors in either basis, or even in terms of mixed components:

$$V^a{}_b = e_\mu{}^a V^\mu{}_b = e^\nu{}_b V^a{}_\nu = e_\mu{}^a e^\nu{}_b V^\mu{}_\nu. \quad (\text{J.11})$$

Looking back at (J.5), we see that the components of the metric tensor in the orthonormal basis are just those of the flat metric,  $\eta_{ab}$ . (For this reason the Greek indices are sometimes referred to as “curved” and the Latin ones as “flat.”) In fact we can go so far as to raise and lower the Latin indices using the flat metric and its inverse  $\eta^{ab}$ . You can check for yourself that everything works (for example, that the lowering an index with the metric commutes with changing from orthonormal to coordinate bases). In particular, our definition of the inverse vielbeins is consistent with our usual notion of raising and lowering indices,

$$e^\mu{}_a = g^{\mu\nu} \eta_{ab} e^\nu{}_b. \quad (\text{J.12})$$

We have introduced the vielbeins  $e_v{}^a$  as components of a set of basis vectors, evaluated in a different basis. This is equivalent to thinking of them as the components of a  $(1, 1)$  tensor,

$$e = e_v{}^a dx^\nu \otimes \hat{e}_{(a)}. \quad (\text{J.13})$$

But this is actually a tensor we already know and love: the identity map. If we act this tensor on a vector, we get back the same vector, just in a different basis; that’s the content of (J.10). Likewise, if we use the inverse vielbein  $e_a^\mu$  to convert the Latin index on  $e_v{}^a$  to a Greek index, according to (J.3) we get the Kronecker delta  $\delta_v^\mu$ , which of course is the identity map on vectors (or one-forms). This point is worth emphasizing because we could also choose to interpret  $e_v{}^a$  as a set of vector components (and some references do so), in which case the covariant derivative would look different. By introducing a new set of basis vectors and one-forms, we necessitate a return to our favorite topic of transformation properties. We’ve been careful all along to emphasize that the tensor transformation law was only an indirect outcome of a coordinate transformation; the real issue was a change of basis. Now that we have noncoordinate bases, these bases can be changed independently of the coordinates. The only restriction is that the orthonormality property (J.1) be preserved. But we know what kind of transformations preserve the flat metric—in a Euclidean signature metric they are orthogonal transfor-

tions, while in a Lorentzian signature metric they are Lorentz transformations. We therefore consider changes of basis of the form

$$\hat{e}_{(a)} \rightarrow \hat{e}_{(a')} = \Lambda^a{}_{a'}(x) \hat{e}_{(a)}, \quad (\text{J.14})$$

where the matrices  $\Lambda^a{}_{a'}(x)$  represent position-dependent transformations which (at each point) leave the canonical form of the metric unaltered:

$$\Lambda^a{}_{a'} \Lambda^b{}_{b'} \eta_{ab} = \eta_{a'b'}. \quad (\text{J.15})$$

In fact these matrices correspond to what in flat space we called the inverse Lorentz transformations (which operate on basis vectors); as before we also have ordinary Lorentz transformations  $\Lambda^{a'}{}_a$ , which transform the basis one-forms. As far as components are concerned, as before we transform upper indices with  $\Lambda^{a'}{}_a$  and lower indices with  $\Lambda^a{}_{a'}$ .

So we now have the freedom to perform a Lorentz transformation (or an ordinary Euclidean rotation, depending on the signature) at every point in space. These transformations are therefore called **local Lorentz transformations**, or LLT's. We still have our usual freedom to make changes in coordinates, which are called **general coordinate transformations**, or GCT's. Both can happen at the same time, resulting in a mixed tensor transformation law:

$$T^{a'\mu'}{}_{b'\nu'} = \Lambda^{a'}{}_a \frac{\partial x^\mu}{\partial x^a} \Lambda^b{}_{b'} \frac{\partial x^\nu}{\partial x^{b'}} T^{a\mu}{}_{b\nu}. \quad (\text{J.16})$$

Translating what we know about tensors into noncoordinate bases is for the most part merely a matter of sticking vielbeins in the right places. The crucial exception comes when we begin to differentiate things. In our ordinary formalism, the covariant derivative of a tensor is given by its partial derivative plus correction terms, one for each index, involving the tensor and the connection coefficients. The same procedure will continue to be true for the noncoordinate basis, but we replace the ordinary connection coefficients  $\Gamma^\lambda_{\mu\nu}$  by the **spin connection**, denoted  $\omega_\mu{}^a{}_b$ . Each Latin index gets a factor of the spin connection in the usual way:

$$\nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega_\mu{}^a{}_c X^c{}_b - \omega_\mu{}^c{}_b X^a{}_c. \quad (\text{J.17})$$

(The name “spin connection” comes from the fact that this can be used to take covariant derivatives of spinors, which is actually impossible using the conventional connection coefficients.) In the presence of mixed Latin and Greek indices we get terms of both kinds.

The usual demand that a tensor be independent of the way it is written allows us to derive a relationship between the spin connection, the vielbeins, and the  $\Gamma^\nu_{\mu\lambda}$ 's. Consider the covariant derivative of a vector  $X$ , first in a purely coordinate basis:

$$\begin{aligned} \nabla X &= (\nabla_\mu X^\nu) dx^\mu \otimes \partial_\nu \\ &= (\partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda) dx^\mu \otimes \partial_\nu. \end{aligned} \quad (\text{J.18})$$

Now find the same object in a mixed basis, and convert into the coordinate basis:

$$\begin{aligned}
 \nabla X &= (\nabla_\mu X^a) dx^\mu \otimes \hat{e}_{(a)} \\
 &= (\partial_\mu X^a + \omega_\mu{}^a{}_b X^b) dx^\mu \otimes \hat{e}_{(a)} \\
 &= (\partial_\mu (e_v{}^a X^v) + \omega_\mu{}^a{}_b e_\lambda{}^b X^\lambda) dx^\mu \otimes (e^\sigma{}_\sigma \partial_\sigma) \\
 &= e^\sigma{}_\sigma (e_v{}^a \partial_\mu X^v + X^v \partial_\mu e_v{}^a + \omega_\mu{}^a{}_b e_\lambda{}^b X^\lambda) dx^\mu \otimes \partial_\sigma \\
 &= (\partial_\mu X^v + e^v{}_\sigma \partial_\mu e_\lambda{}^a X^\lambda + e^v{}_\sigma e_\lambda{}^b \omega_\mu{}^a{}_b X^\lambda) dx^\mu \otimes \partial_v.
 \end{aligned} \tag{J.19}$$

Comparison with (J.18) reveals

$$\Gamma_{\mu\lambda}^v = e^v{}_\sigma \partial_\mu e_\lambda{}^a + e^v{}_\sigma e_\lambda{}^b \omega_\mu{}^a{}_b, \tag{J.20}$$

or equivalently

$$\omega_\mu{}^a{}_b = e_v{}^a e^\lambda{}_b \Gamma_{\mu\lambda}^v - e^\lambda{}_b \partial_\mu e_\lambda{}^a. \tag{J.21}$$

A bit of manipulation allows us to write this relation as the vanishing of the covariant derivative of the vielbein,

$$\begin{aligned}
 \nabla_\mu e_v{}^a &= \partial_\mu e_v{}^a - \Gamma_{\mu\nu}^\lambda e_\lambda{}^a + \omega_\mu{}^a{}_b e_v{}^b \\
 &= 0,
 \end{aligned} \tag{J.22}$$

which is sometimes known as the “tetrad postulate.” Note that this is always true; we did not need to assume anything about the connection in order to derive it. Specifically, we did not need to assume that the connection was metric compatible or torsion free. We did, however, implicitly take  $e_v{}^a$  to represent the  $(1, 1)$  tensor (J.13); since this tensor is the identity map, it is no surprise that its covariant derivative vanishes. (Not all references have this philosophy, so be careful.)

Since the connection may be thought of as something we need to introduce in order to fix up the transformation law of the covariant derivative, it should come as no surprise that the spin connection does not itself obey the tensor transformation law. Actually, under GCT’s the one lower Greek index does transform in the right way, as a one-form. But under LLT’s the spin connection transforms inhomogeneously, as

$$\omega_\mu{}^{a'}{}_b = \Lambda^{a'}{}_a \Lambda^b{}_b' \omega_\mu{}^a{}_b - \Lambda^c{}_b' \partial_\mu \Lambda^{a'}{}_c. \tag{J.23}$$

You are encouraged to check for yourself that this results in the proper transformation of the covariant derivative.

So far we have done nothing but empty formalism, translating things we already knew into a new notation. But the work we are doing does buy us two things. The first, which we already alluded to, is the ability to describe spinor fields on spacetime and take their covariant derivatives; we won’t explore this further here. The second is a change in viewpoint, in which we can think of various tensors as tensor-valued differential forms. For example, an object like  $X_\mu{}^a$ , which we

think of as a  $(1, 1)$  tensor written with mixed indices, can also be thought of as a “vector-valued one-form.” It has one lower Greek index, so we think of it as a one-form, but for each value of the lower index it is a vector. Similarly a tensor  $A_{\mu\nu}{}^a{}_b$ , antisymmetric in  $\mu$  and  $\nu$ , can be thought of as a “ $(1, 1)$ -tensor-valued two-form.” Thus, any tensor with some number of antisymmetric lower Greek indices and some number of Latin indices can be thought of as a differential form, but taking values in the tensor bundle. (Ordinary differential forms are simply scalar-valued forms.) The usefulness of this viewpoint comes when we consider exterior derivatives. If we want to think of  $X_\mu{}^a$  as a vector-valued one-form, we are tempted to take its exterior derivative:

$$(dX)_{\mu\nu}{}^a = \partial_\mu X_\nu{}^a - \partial_\nu X_\mu{}^a. \quad (\text{J.24})$$

It is easy to check that this object transforms like a two-form [that is, according to the transformation law for  $(0, 2)$  tensors] under GCT’s, but not as a vector under LLT’s (the Lorentz transformations depend on position, which introduces an inhomogeneous term into the transformation law). But we can fix this by judicious use of the spin connection, which can be thought of as a one-form, but not a tensor-valued one-form, due to the nontensorial transformation law (J.23). Thus, the object

$$(dX)_{\mu\nu}{}^a + (\omega \wedge X)_{\mu\nu}{}^a = \partial_\mu X_\nu{}^a - \partial_\nu X_\mu{}^a + \omega_\mu{}^a{}_b X_\nu{}^b - \omega_\nu{}^a{}_b X_\mu{}^b, \quad (\text{J.25})$$

as you can verify, transforms as a proper tensor.

An immediate application of this formalism is to the expressions for the torsion and curvature, the two tensors that characterize any given connection. The torsion, with two antisymmetric lower indices, can be thought of as a vector-valued two-form  $T_{\mu\nu}{}^a$ . The curvature, which is always antisymmetric in its last two indices, is a  $(1, 1)$ -tensor-valued two-form,  $R^a{}_b{}_{\mu\nu}$ . Using our freedom to suppress indices on differential forms, we can express these in terms of the basis one-forms

$$e^a = e_\mu{}^a dx^\mu \quad (\text{J.26})$$

and the spin-connection one-forms

$$\omega^a{}_b = \omega_\mu{}^a{}_b dx^\mu. \quad (\text{J.27})$$

Notice that we have switched notations, defining  $e^a \equiv \hat{\theta}^{(a)}$ . This is fairly conventional, as well as cleaner. The defining relations for the torsion and curvature are then

$$T^a = de^a + \omega^a{}_b \wedge e^b \quad (\text{J.28})$$

and

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \quad (\text{J.29})$$

Keep in mind that  $R^a{}_b$  represents the entire Riemann tensor, with Greek indices suppressed; don't confuse it with the Ricci tensor. These are known as the **Cartan structure equations**. They are equivalent to the usual definitions; let's go through the exercise of showing this for the torsion, and you can check the curvature for yourself. We have

$$\begin{aligned} T_{\mu\nu}{}^\lambda &= e^\lambda{}_a T_{\mu\nu}{}^a \\ &= e^\lambda{}_a (\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b) \\ &= \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda, \end{aligned} \quad (J.30)$$

which is just the original definition we gave. Here we have used (J.20), the expression for the  $\Gamma_{\mu\nu}^\lambda$ 's in terms of the vielbeins and spin connection. We can also express identities obeyed by these tensors as

$$dT^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b \quad (J.31)$$

and

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0. \quad (J.32)$$

The first of these is the generalization of  $R^\rho{}_{[\sigma\mu\nu]} = 0$ , while the second is the Bianchi identity  $\nabla_{[\lambda} R^\rho{}_{\sigma]\mu\nu]} = 0$ . (Sometimes both equations are called Bianchi identities.)

The form of these expressions leads to an almost irresistible temptation to define a “covariant-exterior derivative,” which acts on a tensor-valued form by taking the ordinary exterior derivative and then adding appropriate terms with the spin connection, one for each Latin index. Although we won't do that here, it is okay to give in to this temptation, and in fact the right-hand side of (J.28) and the left-hand sides of (J.31) and (J.32) can be thought of as just such covariant-exterior derivatives. But be careful, since (J.29) cannot be; you can't take any sort of covariant derivative of the spin connection, since it's not a tensor.

So far our equations have been true for general connections; let's see what we get for the Christoffel connection. The torsion-free requirement is just that (J.28) vanish; this does not lead immediately to any simple statement about the coefficients of the spin connection. Metric compatibility is expressed as the vanishing of the covariant derivative of the metric:  $\nabla g = 0$ . We can see what this leads to when we express the metric in the orthonormal basis, where its components are simply  $\eta_{ab}$ :

$$\begin{aligned} \nabla_\mu \eta_{ab} &= \partial_\mu \eta_{ab} - \omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac} \\ &= -\omega_{\mu ab} - \omega_{\mu ba}. \end{aligned} \quad (J.33)$$

Then setting this equal to zero implies

$$\omega_{\mu ab} = -\omega_{\mu ba}. \quad (J.34)$$

Thus, metric compatibility is equivalent to the antisymmetry of the spin connection in its Latin indices. (As before, such a statement is only sensible if both indices are either upstairs or downstairs.) These two conditions together allow us to express the spin connection in terms of the vielbeins. An explicit formula expresses this solution, but in practice it is easier to simply solve the torsion-free condition

$$\omega^a{}_b \wedge e^b = -de^a, \quad (J.35)$$

using the asymmetry of the spin connection, to find the individual components.

One of the best reasons for thinking about noncoordinate bases is that they actually lead to great simplifications in certain cases, including the calculation of the curvature tensor. Let's see how this works in a simple example, a spatially flat expanding universe, with metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \quad (J.36)$$

We will use the differential-forms notation of (J.26) and (J.27); calculations such as this are good evidence that this language is practically useful as well as elegant. The metric is thus written (for any geometry)

$$ds^2 = \eta_{ab}e^a \otimes e^b. \quad (J.37)$$

We need to choose basis one-forms  $e^a$  such that this matches our metric (J.36). There are many choices (related by local Lorentz transformations), but one obvious one:

$$\begin{aligned} e^0 &= dt \\ e^i &= adx^i. \end{aligned} \quad (J.38)$$

We would now like to solve for the spin connection using (J.35). The good news is that we basically can do it by guessing. First, by appropriately raising and lowering indices (with  $\eta^{ab}$  and  $\eta_{ab}$ ) we derive the consequences of the antisymmetry of  $\omega_{ab}$ :

$$\begin{aligned} \omega^0{}_0 &= 0 \\ \omega^0{}_j &= \omega^j{}_0 \\ \omega^i{}_j &= -\omega^j{}_i. \end{aligned} \quad (J.39)$$

We next calculate the right-hand side of (J.35),

$$\begin{aligned} de^0 &= 0 \\ de^i &= da \wedge dx^i = \dot{a}dt \wedge dx^i, \end{aligned} \quad (J.40)$$

and then the left,

$$\begin{aligned}\omega^0_b \wedge e^b &= \omega^0_j \wedge e^j = a\omega^0_j \wedge dx^j \\ \omega^i_b \wedge e^b &= \omega^i_0 \wedge e^0 + \omega^i_j \wedge e^j = \omega^i_0 \wedge dt + \omega^i_j \wedge dx^j.\end{aligned}\quad (J.41)$$

Plugging into (J.35) yields

$$\begin{aligned}\omega^0_j \wedge dx^j &= 0 \\ \omega^i_0 \wedge dt + \omega^i_j \wedge dx^j &= -\dot{a}dt \wedge dx^i.\end{aligned}\quad (J.42)$$

We would like to solve these equations for  $\omega^a_b$ . It is tempting to guess  $\omega^0_j = 0$ ; but then to solve the second equation we would require  $\omega^i_j = -\dot{a}\delta^i_j dt$ , which is incompatible with  $\omega^i_j = -\omega^j_i$  from (J.39). But we can solve the first equation by setting  $\omega^0_j$  proportional to  $dx^j$  (due to the antisymmetry of the wedge product). Indeed, if we choose

$$\omega^0_j = \dot{a}dx^j, \quad \omega^i_0 = \dot{a}dx^i, \quad (J.43)$$

we find that both equations in (J.42) are solved by setting

$$\omega^i_j = 0. \quad (J.44)$$

Now that we know the spin connection, we can easily get the curvature through

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b. \quad (J.45)$$

We first calculate the exterior derivative of the spin connection forms,

$$\begin{aligned}d\omega^i_0 &= \ddot{a}dt \wedge dx^i \\ d\omega^0_j &= \ddot{a}dt \wedge dx^j \\ d\omega^i_j &= 0,\end{aligned}\quad (J.46)$$

and then the wedge products,

$$\begin{aligned}\omega^0_c \wedge \omega^c_0 &= 0 \\ \omega^i_c \wedge \omega^c_0 &= 0 \\ \omega^i_c \wedge \omega^c_j &= \dot{a}^2 dx^i \wedge dx^j.\end{aligned}\quad (J.47)$$

We therefore obtain the curvature two-form,

$$\begin{aligned}R^0_0 &= 0 \\ R^0_j &= \ddot{a}dt \wedge dx^j \\ R^i_0 &= \ddot{a}dt \wedge dx^i \\ R^i_j &= \dot{a}^2 dx^i \wedge dx^j,\end{aligned}\quad (J.48)$$

For purposes of comparison, we can use vielbeins to convert  $R^a{}_{b\mu\nu}$  to our conventional expression  $R^\rho{}_{\sigma\mu\nu}$ , using

$$R^\rho{}_{\sigma\mu\nu} = e^\rho{}_a e_\sigma{}^b R^a{}_{b\mu\nu}. \quad (\text{J.49})$$

In component form the vielbeins (J.38) and their inverse are

$$e_\mu{}^a = \begin{pmatrix} 1 & & & \\ & a & & \\ & & a & \\ & & & a \end{pmatrix}, \quad e^\nu{}_b = \begin{pmatrix} 1 & & & \\ & a^{-1} & & \\ & & a^{-1} & \\ & & & a^{-1} \end{pmatrix}. \quad (\text{J.50})$$

We will also need to evaluate the components of the wedge products of basis forms, which is straightforward enough,

$$(dx^\alpha \wedge dx^\beta)_{\mu\nu} = \delta_\mu^\alpha \delta_\nu^\beta - \delta_\nu^\alpha \delta_\mu^\beta. \quad (\text{J.51})$$

Putting it all together yields the components  $R^\rho{}_{\sigma\mu\nu}$ ,

$$\begin{aligned} R^0{}_{j0l} &= a\ddot{a}\delta_{jl} \\ R^i{}_{0k0} &= -\frac{\ddot{a}}{a}\delta_k^i \\ R^i{}_{jkl} &= \dot{a}^2(\delta_k^i\delta_{jl} - \delta_l^i\delta_{jk}), \end{aligned} \quad (\text{J.52})$$

as well as ones obtained by antisymmetry in the last two indices. We may contract to get the components of the Ricci tensor  $R_{\sigma\nu} = R^\lambda{}_{\sigma\lambda\nu}$ ,

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{i0} &= 0 \\ R_{ij} &= (a\ddot{a} + 2\dot{a}^2)\delta_{ij}. \end{aligned} \quad (\text{J.53})$$

You can check that this agrees with our results from Chapter 8. Already in this simple example, the tetrad method was computationally simpler than the coordinate-basis method; in more complicated metrics the comparative advantage continues to grow.

In the language of noncoordinate bases, it is possible to compare the formalism of connections and curvature in Riemannian geometry to that of gauge theories in particle physics. In both situations, the fields of interest live in vector spaces that are assigned to each point in spacetime. In Riemannian geometry the vector spaces include the tangent space, the cotangent space, and the higher tensor spaces constructed from these. In gauge theories, on the other hand, we are concerned with “internal” vector spaces. The distinction is that the tangent space and its relatives are intimately associated with the manifold itself, and are naturally defined once the manifold is set up; the tangent space, for example, can be thought of as the space of directional derivatives at a point. In contrast, an internal vector

space can be of any dimension we like, and has to be defined as an independent addition to the manifold. In math jargon, the union of the base manifold with the internal vector spaces (defined at each point) is a **fiber bundle**, and each copy of the vector space is called the “fiber” (in accord with our definition of the tangent bundle).

Besides the base manifold (for us, spacetime) and the fibers, the other important ingredient in the definition of a fiber bundle is the “structure group,” a Lie group that acts on the fibers to describe how they are sewn together on overlapping coordinate patches. Without going into details, the structure group for the tangent bundle in a four-dimensional spacetime is generally  $GL(4, \mathbf{R})$ , the group of real invertible  $4 \times 4$  matrices; if we have a Lorentzian metric, this may be reduced to the Lorentz group  $SO(3, 1)$ . Now imagine that we introduce an internal three-dimensional vector space, and sew the fibers together with ordinary rotations; the structure group of this new bundle is then  $SO(3)$ . A field that lives in this bundle might be denoted  $\phi^A(x^\mu)$ , where  $A$  runs from one to three; it is a three-vector (an internal one, unrelated to spacetime) for each point on the manifold. We have freedom to choose the basis in the fibers in any way we wish; this means that “physical quantities” should be left invariant under local  $SO(3)$  transformations such as

$$\phi^A(x^\mu) \rightarrow \phi^{A'}(x^\mu) = O^{A'}_A(x^\mu)\phi^A(x^\mu), \quad (J.54)$$

where  $O^{A'}_A(x^\mu)$  is a matrix in  $SO(3)$  that depends on spacetime. Such transformations are known as **gauge transformations**, and theories invariant under them are called “gauge theories.”

For the most part it is not hard to arrange things such that physical quantities are invariant under gauge transformations. The one difficulty arises when we consider partial derivatives,  $\partial_\mu \phi^A$ . Because the matrix  $O^{A'}_A(x^\mu)$  depends on spacetime, it will contribute an unwanted term to the transformation of the partial derivative. By now you should be able to guess the solution: introduce a connection to correct for the inhomogeneous term in the transformation law. We therefore define a connection on the fiber bundle to be an object  $A_\mu{}^A{}_B$ , with two “group indices” and one spacetime index. Under GCT’s it transforms as a one-form, while under gauge transformations it transforms as

$$A_\mu{}^{A'}{}_{B'} = O^{A'}_A O^B{}_{B'} A_\mu{}^A{}_B - O^C{}_{B'} \partial_\mu O^{A'}{}_C. \quad (J.55)$$

(Beware: our conventions are different from those in the particle physics literature.) With this transformation law, the “gauge covariant derivative”

$$D_\mu \phi^A = \partial_\mu \phi^A + A_\mu{}^A{}_B \phi^B \quad (J.56)$$

transforms “tensorially” under gauge transformations, as you are welcome to check. [In ordinary electromagnetism the connection is just the conventional vector potential. No indices are necessary, because the structure group  $U(1)$  is one-dimensional.]

It is clear that this notion of a connection on an internal fiber bundle is very closely related to the connection on the tangent bundle, especially in the orthonormal-frame picture we have been discussing. The transformation law (J.55), for example, is exactly the same as the transformation law (J.23) for the spin connection. We can also define a curvature or “field strength” tensor which is a two-form

$$F^A{}_B = dA^A{}_B + A^A{}_C \wedge A^C{}_B, \quad (\text{J.57})$$

in exact correspondence with (J.29). We can parallel transport things along paths, and there is a construction analogous to the parallel propagator; the trace of the matrix obtained by parallel transporting a vector around a closed curve is called a “Wilson loop.”

We could go on in the development of the relationship between the tangent bundle and internal vector bundles, but that would be another book. Let us instead finish by emphasizing the important *difference* between the two constructions. The difference stems from the fact that the tangent bundle is closely related to the base manifold, while other fiber bundles are tacked on after the fact. It makes sense to say that a vector in the tangent space at  $p$  “points along a path” through  $p$ ; but this makes no sense for an internal vector bundle. There is therefore no analogue of the coordinate basis for an internal space—partial derivatives along curves have nothing to do with internal vectors. It follows in turn that there is nothing like the vielbeins, which relate orthonormal bases to coordinate bases. The torsion tensor, in particular, is only defined for a connection on the tangent bundle, not for any gauge theory connections; it can be thought of as the covariant exterior derivative of the vielbein, and no such construction is available on an internal bundle. You should appreciate the relationship between the different uses of the notion of a connection, without getting carried away.

## J.1 ■ EXERCISES

1. In (J.37) we mention that the metric in an orthonormal basis can be written

$$ds^2 = \eta_{ab} e^a \otimes e^b. \quad (\text{J.58})$$

How can this possibly be? If the components of the metric are  $\eta_{ab}$  everywhere, how can we know what the geometry is?

2. Calculate the connection one-forms, curvature two-forms, and hence the components of the Riemann tensor for the Mixmaster universe. The metric is given by

$$ds^2 = -dt \otimes dt + \alpha^2 \sigma^1 \otimes \sigma^1 + \beta^2 \sigma^2 \otimes \sigma^2 + \gamma^2 \sigma^3 \otimes \sigma^3.$$

Here  $\alpha, \beta, \gamma$  are functions of  $t$  only and the one-forms  $\sigma^i$  are given by

$$\sigma^1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi$$

$$\sigma^2 = \sin \psi d\theta - \cos \psi \sin \theta d\phi$$

$$\sigma^3 = d\psi + \cos \theta d\phi.$$

# Bibliography

I have made no attempt to provide careful citations to the original literature in the text. In keeping with the philosophy of focusing on pedagogy, I have included references to recent review articles where appropriate. Here I list a number of books that might be useful supplements to the one you are reading; the list is not meant to be comprehensive, and focuses on books that are in print and with which I happen to be familiar.

There are two websites that are invaluable resources for keeping up with recent work in gravitational physics. The first is the ArXiv e-print server for general relativity and quantum cosmology:

<http://arxiv.org/form/gr-qc/>

This is where researchers all over the world put their most recent papers, which can then be easily downloaded. There are similar servers for other areas of physics. The other website is for Living Reviews in Relativity:

<http://www.livingreviews.org/>

Living Reviews is an on-line journal specializing in review articles in all areas of gravitational physics. It is an excellent starting point for anyone interested in exploring recent work in a topic of current interest.

## Special Relativity

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## Undergraduate General Relativity

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- curved spacetimes and the behavior of particles in them, putting physics before formalism whenever possible.
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SEAN M. CARROLL is Research Professor of Physics at the California Institute of Technology. His research focuses on general relativity, cosmology, field theory, statistical mechanics, and quantum mechanics. He is the recipient of numerous awards, including the Gemant Award from the American Institute of Physics, the Winton Science Book Prize from the Royal Society, a Guggenheim fellowship, and teaching awards from MIT and the University of Chicago.

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DAVID DICKIE

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**Renegade:**

Adjective

'Having rejected tradition: Unconventional.'

**Merriam-Webster Dictionary**

## **Acquiescence to tyranny is the death of the spirit**

You may be 38 years old, as I happen to be. And one day, some great opportunity stands before you and calls you to stand up for some great principle, some great issue, some great cause. And you refuse to do it because you are afraid

... You refuse to do it because you want to live longer ...

You're afraid that you will lose your job, or you are afraid that you will be criticised or that you will lose your popularity, or you're afraid that somebody will stab you, or shoot at you or bomb your house; so you refuse to take the stand.

Well, you may go on and live until you are 90, but you're just as dead at 38 as you would be at 90. And the cessation of breathing in your life is but the belated announcement of an earlier death of the spirit.

**Martin Luther King**

**How the few control the many and always have – the many do  
whatever they're told**

'Forward, the Light Brigade!'  
Was there a man dismayed?  
Not though the soldier knew  
    Someone had blundered.  
Theirs not to make reply,  
Theirs not to reason why,  
Theirs but to do and die.  
    Into the valley of Death  
        Rode the six hundred.

Cannon to right of them,  
Cannon to left of them,  
Cannon in front of them  
    Volleyed and thundered;  
Stormed at with shot and shell,  
    Boldly they rode and well,  
        Into the jaws of Death,  
        Into the mouth of hell  
            Rode the six hundred

**Alfred Lord Tennyson (1809-1892)**

The mist is lifting slowly  
I can see the way ahead  
And I've left behind the empty streets  
That once inspired my life  
And the strength of the emotion  
Is like thunder in the air  
'Cos the promise that we made each other  
Haunts me to the end

The secret of your beauty  
And the mystery of your soul  
I've been searching for in everyone I meet  
And the times I've been mistaken  
It's impossible to say  
And the grass is growing  
Underneath our feet

The words that I remember  
From my childhood still are true  
That there's none so blind  
As those who will not see  
And to those who lack the courage  
And say it's dangerous to try  
Well they just don't know  
That love eternal will not be denied

I know you're out there somewhere  
Somewhere, somewhere  
I know you're out there somewhere

Somewhere you can hear my voice  
I know I'll find you somehow  
Somehow, somehow  
I know I'll find you somehow  
And somehow I'll return again to you

**The Moody Blues**

## **Are you a gutless wonder - or a Renegade Mind?**

Monuments put from pen to paper,  
Turns me into a gutless wonder,  
And if you tolerate this,  
Then your children will be next.  
Gravity keeps my head down,  
Or is it maybe shame ...

**Manic Street Preachers**

Rise like lions after slumber  
In unvanquishable number.  
Shake your chains to earth like dew  
Which in sleep have fallen on you.  
Ye are many – they are few.

**Percy Shelley**

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# CHAPTER ONE

## I'm thinking' – Oh, but are you?

*Think for yourself and let others enjoy the privilege of doing so too*  
Voltaire

French-born philosopher, mathematician and scientist René Descartes became famous for his statement in Latin in the 17th century which translates into English as: 'I think, therefore I am.'

On the face of it that is true. Thought reflects perception and perception leads to both behaviour and self-identity. In that sense 'we' are what we think. But who or what is doing the thinking and is thinking the only route to perception? Clearly, as we shall see, 'we' are not always the source of 'our' perception, indeed with regard to humanity as a whole this is rarely the case; and thinking is far from the only means of perception. Thought is the village idiot compared with other expressions of consciousness that we all have the potential to access and tap into. This has to be true when we *are* those other expressions of consciousness which are infinite in nature. We have forgotten this, or, more to the point, been manipulated to forget.

These are not just the esoteric musings of the navel. The whole foundation of human control and oppression is control of perception. Once perception is hijacked then so is behaviour which is dictated by perception. Collective perception becomes collective behaviour and collective behaviour is what we call human society. Perception is all and those behind human control know that which is

why perception is the target 24/7 of the psychopathic manipulators that I call the Global Cult. They know that if they dictate perception they will dictate behaviour and collectively dictate the nature of human society. They are further aware that perception is formed from information received and if they control the circulation of information they will to a vast extent direct human behaviour.

Censorship of information and opinion has become globally Nazi-like in recent years and never more blatantly than since the illusory ‘virus pandemic’ was triggered out of China in 2019 and across the world in 2020. Why have billions submitted to house arrest and accepted fascistic societies in a way they would have never believed possible? Those controlling the information spewing from government, mainstream media and Silicon Valley (all controlled by the same Global Cult networks) told them they were in danger from a ‘deadly virus’ and only by submitting to house arrest and conceding their most basic of freedoms could they and their families be protected. This monumental and provable lie became the *perception* of the billions and therefore the *behaviour* of the billions. In those few words you have the whole structure and modus operandi of human control. Fear is a perception – False Emotion Appearing Real – and fear is the currency of control. In short ... get them by the balls (or give them the impression that you have) and their hearts and minds will follow. Nothing grips the dangly bits and freezes the rear-end more comprehensively than fear.

## **World number 1**

There are two ‘worlds’ in what appears to be one ‘world’ and the prime difference between them is knowledge. First we have the mass of human society in which the population is maintained in coldly-calculated ignorance through control of information and the ‘education’ (indoctrination) system. That’s all you really need to control to enslave billions in a perceptual delusion in which what are perceived to be *their* thoughts and opinions are ever-repeated mantras that the system has been downloading all their lives through ‘education’, media, science, medicine, politics and academia

in which the personnel and advocates are themselves overwhelmingly the perceptual products of the same repetition. Teachers and academics in general are processed by the same programming machine as everyone else, but unlike the great majority they never leave the ‘education’ program. It gripped them as students and continues to grip them as programmers of subsequent generations of students. The programmed become the programmers – the programmed programmers. The same can largely be said for scientists, doctors and politicians and not least because as the American writer Upton Sinclair said: ‘It is difficult to get a man to understand something when his salary depends upon his not understanding it.’ If your career and income depend on thinking the way the system demands then you will – bar a few free-minded exceptions – concede your mind to the Perceptual Mainframe that I call the Postage Stamp Consensus. This is a tiny band of perceived knowledge and possibility ‘taught’ (downloaded) in the schools and universities, pounded out by the mainstream media and on which all government policy is founded. Try thinking, and especially speaking and acting, outside of the ‘box’ of consensus and see what that does for your career in the Mainstream Everything which bullies, harasses, intimidates and ridicules the population into compliance. Here we have the simple structure which enslaves most of humanity in a perceptual prison cell for an entire lifetime and I’ll go deeper into this process shortly. Most of what humanity is taught as fact is nothing more than programmed belief. American science fiction author Frank Herbert was right when he said: ‘Belief can be manipulated. Only knowledge is dangerous.’ In the ‘Covid’ age belief is promoted and knowledge is censored. It was always so, but never to the extreme of today.

## **World number 2**

A ‘number 2’ is slang for ‘doing a poo’ and how appropriate that is when this other ‘world’ is doing just that on humanity every minute of every day. World number 2 is a global network of secret societies and semi-secret groups dictating the direction of society via

governments, corporations and authorities of every kind. I have spent more than 30 years uncovering and exposing this network that I call the Global Cult and knowing its agenda is what has made my books so accurate in predicting current and past events. Secret societies are secret for a reason. They want to keep their hoarded knowledge to themselves and their chosen initiates and to hide it from the population which they seek through ignorance to control and subdue. The whole foundation of the division between World 1 and World 2 is *knowledge*. What number 1 knows number 2 must not. Knowledge they have worked so hard to keep secret includes (a) the agenda to enslave humanity in a centrally-controlled global dictatorship, and (b) the nature of reality and life itself. The latter (b) must be suppressed to allow the former (a) to prevail as I shall be explaining. The way the Cult manipulates and interacts with the population can be likened to a spider's web. The 'spider' sits at the centre in the shadows and imposes its will through the web with each strand represented in World number 2 by a secret society, satanic or semi-secret group, and in World number 1 – the world of the seen – by governments, agencies of government, law enforcement, corporations, the banking system, media conglomerates and Silicon Valley ([Fig 1](#) overleaf). The spider and the web connect and coordinate all these organisations to pursue the same global outcome while the population sees them as individual entities working randomly and independently. At the level of the web governments *are* the banking system *are* the corporations *are* the media *are* Silicon Valley *are* the World Health Organization working from their inner cores as one unit. Apparently unconnected countries, corporations, institutions, organisations and people are on the *same team* pursuing the same global outcome. Strands in the web immediately around the spider are the most secretive and exclusive secret societies and their membership is emphatically restricted to the Cult inner-circle emerging through the generations from particular bloodlines for reasons I will come to. At the core of the core you would get them in a single room. That's how many people are dictating the direction of human society and its transformation

through the ‘Covid’ hoax and other means. As the web expands out from the spider we meet the secret societies that many people will be aware of – the Freemasons, Knights Templar, Knights of Malta, Opus Dei, the inner sanctum of the Jesuit Order, and such like. Note how many are connected to the Church of Rome and there is a reason for that. The Roman Church was established as a revamp, a rebranding, of the relocated ‘Church’ of Babylon and the Cult imposing global tyranny today can be tracked back to Babylon and Sumer in what is now Iraq.



**Figure 1:** The global web through which the few control the many. (Image Neil Hague.)

Inner levels of the web operate in the unseen away from the public eye and then we have what I call the cusp organisations located at the point where the hidden meets the seen. They include a series of satellite organisations answering to a secret society founded in London in the late 19th century called the Round Table and among them are the Royal Institute of International Affairs (UK, founded in 1920); Council on Foreign Relations (US, 1921); Bilderberg Group (worldwide, 1954); Trilateral Commission (US/worldwide, 1972); and the Club of Rome (worldwide, 1968) which was created to exploit environmental concerns to justify the centralisation of global power to ‘save the planet’. The Club of Rome instigated with others the human-caused climate change hoax which has led to all the ‘green

new deals' demanding that very centralisation of control. Cusp organisations, which include endless 'think tanks' all over the world, are designed to coordinate a single global policy between political and business leaders, intelligence personnel, media organisations and anyone who can influence the direction of policy in their own sphere of operation. Major players and regular attenders will know what is happening – or some of it – while others come and go and are kept overwhelmingly in the dark about the big picture. I refer to these cusp groupings as semi-secret in that they can be publicly identified, but what goes on at the inner-core is kept very much 'in house' even from most of their members and participants through a fiercely-imposed system of compartmentalisation. Only let them know what they need to know to serve your interests and no more. The structure of secret societies serves as a perfect example of this principle. Most Freemasons never get higher than the bottom three levels of 'degree' (degree of knowledge) when there are 33 official degrees of the Scottish Rite. Initiates only qualify for the next higher 'compartment' or degree if those at that level choose to allow them. Knowledge can be carefully assigned only to those considered 'safe'. I went to my local Freemason's lodge a few years ago when they were having an 'open day' to show how cuddly they were and when I chatted to some of them I was astonished at how little the rank and file knew even about the most ubiquitous symbols they use. The mushroom technique – keep them in the dark and feed them bullshit – applies to most people in the web as well as the population as a whole. Sub-divisions of the web mirror in theme and structure transnational corporations which have a headquarters somewhere in the world dictating to all their subsidiaries in different countries. Subsidiaries operate in their methodology and branding to the same centrally-dictated plan and policy in pursuit of particular ends. The Cult web functions in the same way. Each country has its own web as a subsidiary of the global one. They consist of networks of secret societies, semi-secret groups and bloodline families and their job is to impose the will of the spider and the global web in their particular country. Subsidiary networks control and manipulate the national political system, finance, corporations, media, medicine, etc. to

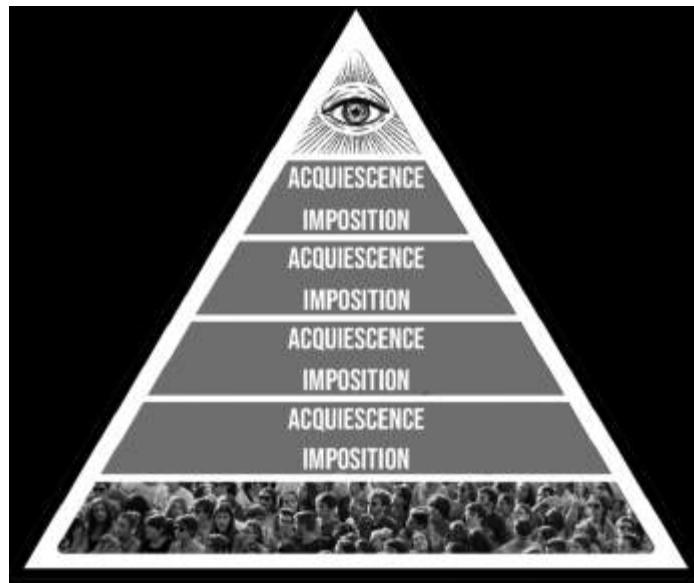
ensure that they follow the globally-dictated Cult agenda. These networks were the means through which the ‘Covid’ hoax could be played out with almost every country responding in the same way.

### **The ‘Yessir’ pyramid**

Compartmentalisation is the key to understanding how a tiny few can dictate the lives of billions when combined with a top-down sequence of imposition and acquiescence. The inner core of the Cult sits at the peak of the pyramidal hierarchy of human society ([Fig 2](#) overleaf). It imposes its will – its agenda for the world – on the level immediately below which acquiesces to that imposition. This level then imposes the Cult will on the level below them which acquiesces and imposes on the next level. Very quickly we meet levels in the hierarchy that have no idea there even is a Cult, but the sequence of imposition and acquiescence continues down the pyramid in just the same way. ‘I don’t know why we are doing this but the order came from “on-high” and so we better just do it.’ Alfred Lord Tennyson said of the cannon fodder levels in his poem *The Charge of the Light Brigade*: ‘Theirs not to reason why; theirs but to do and die.’ The next line says that ‘into the valley of death rode the six hundred’ and they died because they obeyed without question what their perceived ‘superiors’ told them to do. In the same way the population capitulated to ‘Covid’. The whole hierarchical pyramid functions like this to allow the very few to direct the enormous many.

Eventually imposition-acquiescence-imposition-acquiescence comes down to the mass of the population at the foot of the pyramid. If they acquiesce to those levels of the hierarchy imposing on them (governments/law enforcement/doctors/media) a circuit is completed between the population and the handful of super-psychopaths in the Cult inner core at the top of the pyramid. Without a circuit-breaking refusal to obey, the sequence of imposition and acquiescence allows a staggeringly few people to impose their will upon the entirety of humankind. We are looking at the very sequence that has subjugated billions since the start of 2020. Our freedom has not been taken from us. Humanity has given it

away. Fascists do not impose fascism because there are not enough of them. Fascism is imposed by the population acquiescing to fascism. Put another way allowing their perceptions to be programmed to the extent that leads to the population giving their freedom away by giving their perceptions – their mind – away. If this circuit is not broken by humanity ceasing to cooperate with their own enslavement then nothing can change. For that to happen people have to critically think and see through the lies and window dressing and then summon the backbone to act upon what they see. The Cult spends its days working to stop either happening and its methodology is systematic and highly detailed, but it can be overcome and that is what this book is all about.



**Figure 2:** The simple sequence of imposition and compliance that allows a handful of people at the peak of the pyramid to dictate the lives of billions.

## The Life Program

Okay, back to world number 1 or the world of the ‘masses’. Observe the process of what we call ‘life’ and it is a perceptual download from cradle to grave. The Cult has created a global structure in which perception can be programmed and the program continually topped-up with what appears to be constant confirmation that the program is indeed true reality. The important word here is ‘appears’.

This is the structure, the fly-trap, the Postage Stamp Consensus or Perceptual Mainframe, which represents that incredibly narrow band of perceived possibility delivered by the ‘education’ system, mainstream media, science and medicine. From the earliest age the download begins with parents who have themselves succumbed to the very programming their children are about to go through. Most parents don’t do this out of malevolence and mostly it is quite the opposite. They do what they believe is best for their children and that is what the program has told them is best. Within three or four years comes the major transition from parental programming to full-blown state (Cult) programming in school, college and university where perceptually-programmed teachers and academics pass on their programming to the next generations. Teachers who resist are soon marginalised and their careers ended while children who resist are called a problem child for whom Ritalin may need to be prescribed. A few years after entering the ‘world’ children are under the control of authority figures representing the state telling them when they have to be there, when they can leave and when they can speak, eat, even go to the toilet. This is calculated preparation for a lifetime of obeying authority in all its forms. Reflex-action fear of authority is instilled by authority from the start. Children soon learn the carrot and stick consequences of obeying or defying authority which is underpinned daily for the rest of their life. Fortunately I daydreamed through this crap and never obeyed authority simply because it told me to. This approach to my alleged ‘bettters’ continues to this day. There can be consequences of pursuing open-minded freedom in a world of closed-minded conformity. I spent a lot of time in school corridors after being ejected from the classroom for not taking some of it seriously and now I spend a lot of time being ejected from Facebook, YouTube and Twitter. But I can tell you that being true to yourself and not compromising your self-respect is far more exhilarating than bowing to authority for authority’s sake. You don’t have to be a sheep to the shepherd (authority) and the sheep dog (fear of not obeying authority).

The perceptual download continues throughout the formative years in school, college and university while script-reading ‘teachers’, ‘academics’ ‘scientists’, ‘doctors’ and ‘journalists’ insist that ongoing generations must be as programmed as they are. Accept the program or you will not pass your ‘exams’ which confirm your ‘degree’ of programming. It is tragic to think that many parents pressure their offspring to work hard at school to download the program and qualify for the next stage at college and university. The late, great, American comedian George Carlin said: ‘Here’s a bumper sticker I’d like to see: We are proud parents of a child who has resisted his teachers’ attempts to break his spirit and bend him to the will of his corporate masters.’ Well, the best of luck finding many of those, George. Then comes the moment to leave the formal programming years in academia and enter the ‘adult’ world of work. There you meet others in your chosen or prescribed arena who went through the same Postage Stamp Consensus program before you did. There is therefore overwhelming agreement between almost everyone on the basic foundations of Postage Stamp reality and the rejection, even contempt, of the few who have a mind of their own and are prepared to use it. This has two major effects. Firstly, the consensus confirms to the programmed that their download is really how things are. I mean, everyone knows that, right? Secondly, the arrogance and ignorance of Postage Stamp adherents ensure that anyone questioning the program will have unpleasant consequences for seeking their own truth and not picking their perceptions from the shelf marked: ‘Things you must believe without question and if you don’t you’re a dangerous lunatic conspiracy theorist and a harebrained nutter’.

Every government, agency and corporation is founded on the same Postage Stamp prison cell and you can see why so many people believe the same thing while calling it their own ‘opinion’. Fusion of governments and corporations in pursuit of the same agenda was the definition of fascism described by Italian dictator Benito Mussolini. The pressure to conform to perceptual norms downloaded for a lifetime is incessant and infiltrates society right

down to family groups that become censors and condemners of their own ‘black sheep’ for not, ironically, being sheep. We have seen an explosion of that in the ‘Covid’ era. Cult-owned global media unleashes its propaganda all day every day in support of the Postage Stamp and targets with abuse and ridicule anyone in the public eye who won’t bend their mind to the will of the tyranny. Any response to this is denied (certainly in my case). They don’t want to give a platform to expose official lies. Cult-owned-and-created Internet giants like Facebook, Google, YouTube and Twitter delete you for having an unapproved opinion. Facebook boasts that its AI censors delete 97-percent of ‘hate speech’ before anyone even reports it. Much of that ‘hate speech’ will simply be an opinion that Facebook and its masters don’t want people to see. Such perceptual oppression is widely known as fascism. Even Facebook executive Benny Thomas, a ‘CEO Global Planning Lead’, said in comments secretly recorded by investigative journalism operation Project Veritas that Facebook is ‘too powerful’ and should be broken up:

I mean, no king in history has been the ruler of two billion people, but Mark Zuckerberg is ... And he's 36. That's too much for a 36-year-old ... You should not have power over two billion people. I just think that's wrong.

Thomas said Facebook-owned platforms like Instagram, Oculus, and WhatsApp needed to be separate companies. ‘It’s too much power when they’re all one together’. That’s the way the Cult likes it, however. We have an executive of a Cult organisation in Benny Thomas that doesn’t know there is a Cult such is the compartmentalisation. Thomas said that Facebook and Google ‘are no longer companies, they’re countries’. Actually they are more powerful than countries on the basis that if you control information you control perception and control human society.

## **I love my oppressor**

Another expression of this psychological trickery is for those who realise they are being pressured into compliance to eventually

convince themselves to believe the official narratives to protect their self-respect from accepting the truth that they have succumbed to meek and subservient compliance. Such people become some of the most vehement defenders of the system. You can see them everywhere screaming abuse at those who prefer to think for themselves and by doing so reminding the compliers of their own capitulation to conformity. ‘You are talking dangerous nonsense you Covidiot!!’ Are you trying to convince me or yourself? It is a potent form of Stockholm syndrome which is defined as: ‘A psychological condition that occurs when a victim of abuse identifies and attaches, or bonds, positively with their abuser.’ An example is hostages bonding and even ‘falling in love’ with their kidnappers. The syndrome has been observed in domestic violence, abused children, concentration camp inmates, prisoners of war and many and various Satanic cults. These are some traits of Stockholm syndrome listed at [goodtherapy.org](http://goodtherapy.org):

- Positive regard towards perpetrators of abuse or captor [see ‘Covid’].
- Failure to cooperate with police and other government authorities when it comes to holding perpetrators of abuse or kidnapping accountable [or in the case of ‘Covid’ cooperating with the police to enforce and defend their captors’ demands].
- Little or no effort to escape [see ‘Covid’].
- Belief in the goodness of the perpetrators or kidnappers [see ‘Covid’].
- Appeasement of captors. This is a manipulative strategy for maintaining one’s safety. As victims get rewarded – perhaps with less abuse or even with life itself – their appeasing behaviours are reinforced [see ‘Covid’].
- Learned helplessness. This can be akin to ‘if you can’t beat ‘em, join ‘em’. As the victims fail to escape the abuse or captivity, they may start giving up and soon realize it’s just easier for everyone if they acquiesce all their power to their captors [see ‘Covid’].

- Feelings of pity toward the abusers, believing they are actually victims themselves. Because of this, victims may go on a crusade or mission to 'save' [protect] their abuser [see the venom unleashed on those challenging the official 'Covid' narrative].
- Unwillingness to learn to detach from their perpetrators and heal. In essence, victims may tend to be less loyal to themselves than to their abuser [*definitely* see 'Covid'].

Ponder on those traits and compare them with the behaviour of great swathes of the global population who have defended governments and authorities which have spent every minute destroying their lives and livelihoods and those of their children and grandchildren since early 2020 with fascistic lockdowns, house arrest and employment deletion to 'protect' them from a 'deadly virus' that their abusers' perceptually created to bring about this very outcome. We are looking at mass Stockholm syndrome. All those that agree to concede their freedom will believe those perceptions are originating in their own independent 'mind' when in fact by conceding their reality to Stockholm syndrome they have by definition conceded any independence of mind. Listen to the 'opinions' of the acquiescing masses in this 'Covid' era and what gushes forth is the repetition of the official version of everything delivered unprocessed, unfiltered and unquestioned. The whole programming dynamic works this way. I must be free because I'm told that I am and so I think that I am.

You can see what I mean with the chapter theme of 'I'm thinking – Oh, but *are you?*' The great majority are not thinking, let alone for themselves. They are repeating what authority has told them to believe which allows them to be controlled. Weaving through this mentality is the fear that the 'conspiracy theorists' are right and this again explains the often hysterical abuse that ensues when you dare to contest the official narrative of anything. Denial is the mechanism of hiding from yourself what you don't want to be true. Telling people what they want to hear is easy, but it's an infinitely greater challenge to tell them what they would rather not be happening.

One is akin to pushing against an open door while the other is met with vehement resistance no matter what the scale of evidence. I don't want it to be true so I'll convince myself that it's not. Examples are everywhere from the denial that a partner is cheating despite all the signs to the reflex-action rejection of any idea that world events in which country after country act in exactly the same way are centrally coordinated. To accept the latter is to accept that a force of unspeakable evil is working to destroy your life and the lives of your children with nothing too horrific to achieve that end. Who the heck wants that to be true? But if we don't face reality the end is duly achieved and the consequences are far worse and ongoing than breaking through the walls of denial today with the courage to make a stand against tyranny.

## **Connect the dots – but how?**

A crucial aspect of perceptual programming is to portray a world in which everything is random and almost nothing is connected to anything else. Randomness cannot be coordinated by its very nature and once you perceive events as random the idea they could be connected is waved away as the rantings of the tinfoil-hat brigade. You can't plan and coordinate random you idiot! No, you can't, but you can hide the coldly-calculated and long-planned behind the *illusion* of randomness. A foundation manifestation of the Renegade Mind is to scan reality for patterns that connect the apparently random and turn pixels and dots into pictures. This is the way I work and have done so for more than 30 years. You look for similarities in people, modus operandi and desired outcomes and slowly, then ever quicker, the picture forms. For instance: There would seem to be no connection between the 'Covid pandemic' hoax and the human-caused global-warming hoax and yet they are masks (appropriately) on the same face seeking the same outcome. Those pushing the global warming myth through the Club of Rome and other Cult agencies are driving the lies about 'Covid' – Bill Gates is an obvious one, but they are endless. Why would the same people be involved in both when they are clearly not connected? Oh, but they

are. Common themes with personnel are matched by common goals. The ‘solutions’ to both ‘problems’ are centralisation of global power to impose the will of the few on the many to ‘save’ humanity from ‘Covid’ and save the planet from an ‘existential threat’ (we need ‘zero Covid’ and ‘zero carbon emissions’). These, in turn, connect with the ‘dot’ of globalisation which was coined to describe the centralisation of global power in every area of life through incessant political and corporate expansion, trading blocks and superstates like the European Union. If you are the few and you want to control the many you have to centralise power and decision-making. The more you centralise power the more power the few at the centre will have over the many; and the more that power is centralised the more power those at the centre have to centralise even quicker. The momentum of centralisation gets faster and faster which is exactly the process we have witnessed. In this way the hoaxed ‘pandemic’ and the fakery of human-caused global warming serve the interests of globalisation and the seizure of global power in the hands of the Cult inner-circle which is behind ‘Covid’, ‘climate change’ and globalisation. At this point random ‘dots’ become a clear and obvious picture or pattern.

Klaus Schwab, the classic Bond villain who founded the Cult’s Gates-funded World Economic Forum, published a book in 2020, *The Great Reset*, in which he used the ‘problem’ of ‘Covid’ to justify a total transformation of human society to ‘save’ humanity from ‘climate change’. Schwab said: ‘The pandemic represents a rare but narrow window of opportunity to reflect, reimagine, and reset our world.’ What he didn’t mention is that the Cult he serves is behind both hoaxes as I show in my book *The Answer*. He and the Cult don’t have to reimagine the world. They know precisely what they want and that’s why they destroyed human society with ‘Covid’ to ‘build back better’ in their grand design. Their job is not to imagine, but to get humanity to imagine and agree with their plans while believing it’s all random. It must be pure coincidence that ‘The Great Reset’ has long been the Cult’s code name for the global imposition of fascism and replaced previous code-names of the ‘New World

'Order' used by Cult frontmen like Father George Bush and the 'New Order of the Ages' which emerged from Freemasonry and much older secret societies. New Order of the Ages appears on the reverse of the Great Seal of the United States as 'Novus ordo seclorum' underneath the Cult symbol used since way back of the pyramid and all seeing-eye ([Fig 3](#)). The pyramid is the hierarchy of human control headed by the illuminated eye that symbolises the force behind the Cult which I will expose in later chapters. The term 'Annuit Coeptis' translates as 'He favours our undertaking'. We are told the 'He' is the Christian god, but 'He' is not as I will be explaining.



**Figure 3:** The all-seeing eye of the Cult 'god' on the Freemason-designed Great Seal of the United States and also on the dollar bill.

## Having you on

Two major Cult techniques of perceptual manipulation that relate to all this are what I have called since the 1990s Problem-Reaction-Solution (PRS) and the Totalitarian Tiptoe (TT). They can be uncovered by the inquiring mind with a simple question: Who benefits? The answer usually identifies the perpetrators of a given action or happening through the concept of 'he who most benefits from a crime is the one most likely to have committed it'. The Latin 'Cue bono?' – Who benefits? – is widely attributed to the Roman orator and statesman Marcus Tullius Cicero. No wonder it goes back so far when the concept has been relevant to human behaviour since

history was recorded. Problem-Reaction-Solution is the technique used to manipulate us every day by covertly creating a problem (or the illusion of one) and offering the solution to the problem (or the illusion of one). In the first phase you create the problem and blame someone or something else for why it has happened. This may relate to a financial collapse, terrorist attack, war, global warming or pandemic, anything in fact that will allow you to impose the ‘solution’ to change society in the way you desire at that time. The ‘problem’ doesn’t have to be real. PRS is manipulation of perception and all you need is the population to believe the problem is real. Human-caused global warming and the ‘Covid pandemic’ only have to be *perceived* to be real for the population to accept the ‘solutions’ of authority. I refer to this technique as NO-Problem-Reaction-Solution. Billions did not meekly accept house arrest from early 2020 because there was a real deadly ‘Covid pandemic’ but because they perceived – believed – that to be the case. The antidote to Problem-Reaction-Solution is to ask who benefits from the proposed solution. Invariably it will be anyone who wants to justify more control through deletion of freedom and centralisation of power and decision-making.

The two world wars were Problem-Reaction-Solutions that transformed and realigned global society. Both were manipulated into being by the Cult as I have detailed in books since the mid-1990s. They dramatically centralised global power, especially World War Two, which led to the United Nations and other global bodies thanks to the overt and covert manipulations of the Rockefeller family and other Cult bloodlines like the Rothschilds. The UN is a stalking horse for full-blown world government that I will come to shortly. The land on which the UN building stands in New York was donated by the Rockefellers and the same Cult family was behind Big Pharma scalpel and drug ‘medicine’ and the creation of the World Health Organization as part of the UN. They have been stalwarts of the eugenics movement and funded Hitler’s race-purity expert Ernst Rudin. The human-caused global warming hoax has been orchestrated by the Club of Rome through the UN which is

manufacturing both the ‘problem’ through its Intergovernmental Panel on Climate Change and imposing the ‘solution’ through its Agenda 21 and Agenda 2030 which demand the total centralisation of global power to ‘save the world’ from a climate hoax the United Nations is itself perpetrating. What a small world the Cult can be seen to be particularly among the inner circles. The bedfellow of Problem-Reaction-Solution is the Totalitarian Tiptoe which became the Totalitarian Sprint in 2020. The technique is fashioned to hide the carefully-coordinated behind the cover of apparently random events. You start the sequence at ‘A’ and you know you are heading for ‘Z’. You don’t want people to know that and each step on the journey is presented as a random happening while all the steps strung together lead in the same direction. The speed may have quickened dramatically in recent times, but you can still see the incremental approach of the Tiptoe in the case of ‘Covid’ as each new imposition takes us deeper into fascism. Tell people they have to do this or that to get back to ‘normal’, then this and this and this. With each new demand adding to the ones that went before the population’s freedom is deleted until it disappears. The spider wraps its web around the flies more comprehensively with each new diktat. I’ll highlight this in more detail when I get to the ‘Covid’ hoax and how it has been pulled off. Another prime example of the Totalitarian Tiptoe is how the Cult-created European Union went from a ‘free-trade zone’ to a centralised bureaucratic dictatorship through the Tiptoe of incremental centralisation of power until nations became mere administrative units for Cult-owned dark suits in Brussels.

The antidote to ignorance is knowledge which the Cult seeks vehemently to deny us, but despite the systematic censorship to that end the Renegade Mind can overcome this by vociferously seeking out the facts no matter the impediments put in the way. There is also a method of thinking and perceiving – *knowing* – that doesn’t even need names, dates, place-type facts to identify the patterns that reveal the story. I’ll get to that in the final chapter. All you need to know about the manipulation of human society and to what end is still out there – *at the time of writing* – in the form of books, videos

and websites for those that really want to breach the walls of programmed perception. To access this knowledge requires the abandonment of the mainstream media as a source of information in the awareness that this is owned and controlled by the Cult and therefore promotes mass perceptions that suit the Cult. Mainstream media lies all day, every day. That is its function and very reason for being. Where it does tell the truth, here and there, is only because the truth and the Cult agenda very occasionally coincide. If you look for fact and insight to the BBC, CNN and virtually all the rest of them you are asking to be conned and perceptually programmed.

### **Know the outcome and you'll see the journey**

Events seem random when you have no idea where the world is being taken. Once you do the random becomes the carefully planned. Know the outcome and you'll see the journey is a phrase I have been using for a long time to give context to daily happenings that appear unconnected. Does a problem, or illusion of a problem, trigger a proposed 'solution' that further drives society in the direction of the outcome? Invariably the answer will be yes and the random – *abracadabra* – becomes the clearly coordinated. So what is this outcome that unlocks the door to a massively expanded understanding of daily events? I will summarise its major aspects – the fine detail is in my other books – and those new to this information will see that the world they thought they were living in is a very different place. The foundation of the Cult agenda is the incessant centralisation of power and all such centralisation is ultimately in pursuit of Cult control on a global level. I have described for a long time the planned world structure of top-down dictatorship as the Hunger Games Society. The term obviously comes from the movie series which portrayed a world in which a few living in military-protected hi-tech luxury were the overlords of a population condemned to abject poverty in isolated 'sectors' that were not allowed to interact. 'Covid' lockdowns and travel bans anyone? The 'Hunger Games' pyramid of structural control has the inner circle of the Cult at the top with pretty much the entire

population at the bottom under their control through dependency for survival on the Cult. The whole structure is planned to be protected and enforced by a military-police state ([Fig 4](#)).

Here you have the reason for the global lockdowns of the fake pandemic to coldly destroy independent incomes and livelihoods and make everyone dependent on the ‘state’ (the Cult that controls the ‘states’). I have warned in my books for many years about the plan to introduce a ‘guaranteed income’ – a barely survivable pittance – designed to impose dependency when employment was destroyed by AI technology and now even more comprehensively at great speed by the ‘Covid’ scam. Once the pandemic was played and lockdown consequences began to delete independent income the authorities began to talk right on cue about the need for a guaranteed income and a ‘Great Reset’. Guaranteed income will be presented as benevolent governments seeking to help a desperate people – desperate as a direct result of actions of the same governments. The truth is that such payments are a trap. You will only get them if you do exactly what the authorities demand including mass vaccination (genetic manipulation). We have seen this theme already in Australia where those dependent on government benefits have them reduced if parents don’t agree to have their children vaccinated according to an insane health-destroying government-dictated schedule. Calculated economic collapse applies to governments as well as people. The Cult wants rid of countries through the creation of a world state with countries broken up into regions ruled by a world government and super states like the European Union. Countries must be bankrupted, too, to this end and it’s being achieved by the trillions in ‘rescue packages’ and furlough payments, trillions in lost taxation, and money-no-object spending on ‘Covid’ including constant all-medium advertising (programming) which has made the media dependent on government for much of its income. The day of reckoning is coming – as planned – for government spending and given that it has been made possible by printing money and not by production/taxation there is inflation on the way that has the

potential to wipe out monetary value. In that case there will be no need for the Cult to steal your money. It just won't be worth anything (see the German Weimar Republic before the Nazis took over). Many have been okay with lockdowns while getting a percentage of their income from so-called furlough payments without having to work. Those payments are dependent, however, on people having at least a theoretical job with a business considered non-essential and ordered to close. As these business go under because they are closed by lockdown after lockdown the furlough stops and it will for everyone eventually. Then what? The 'then what?' is precisely the idea.



**Figure 4:** The Hunger Games Society structure I have long warned was planned and now the 'Covid' hoax has made it possible. This is the real reason for lockdowns.

## Hired hands

Between the Hunger Games Cult elite and the dependent population is planned to be a vicious military-police state (a fusion of the two into one force). This has been in the making for a long time with police looking ever more like the military and carrying weapons to match. The pandemic scam has seen this process accelerate so fast as

lockdown house arrest is brutally enforced by carefully recruited fascist minds and gormless system-servers. The police and military are planned to merge into a centrally-directed world army in a global structure headed by a world government which wouldn't be elected even by the election fixes now in place. The world army is not planned even to be human and instead wars would be fought, primarily against the population, using robot technology controlled by artificial intelligence. I have been warning about this for decades and now militaries around the world are being transformed by this very AI technology. The global regime that I describe is a particular form of fascism known as a technocracy in which decisions are not made by clueless and co-opted politicians but by unelected technocrats – scientists, engineers, technologists and bureaucrats. Cult-owned-and-controlled Silicon Valley giants are examples of technocracy and they already have far more power to direct world events than governments. They are with their censorship *selecting* governments. I know that some are calling the 'Great Reset' a Marxist communist takeover, but fascism and Marxism are different labels for the same tyranny. Tell those who lived in fascist Germany and Stalinist Russia that there was a difference in the way their freedom was deleted and their lives controlled. I could call it a fascist technocracy or a Marxist technocracy and they would be equally accurate. The Hunger Games society with its world government structure would oversee a world army, world central bank and single world cashless currency imposing its will on a microchipped population ([Fig 5](#)). Scan its different elements and see how the illusory pandemic is forcing society in this very direction at great speed. Leaders of 23 countries and the World Health Organization (WHO) backed the idea in March, 2021, of a global treaty for 'international cooperation' in 'health emergencies' and nations should 'come together as a global community for peaceful cooperation that extends beyond this crisis'. Cut the Orwellian bullshit and this means another step towards global government. The plan includes a cashless digital money system that I first warned about in 1993. Right at the start of 'Covid' the deeply corrupt Tedros

Adhanom Ghebreyesus, the crooked and merely gofer ‘head’ of the World Health Organization, said it was possible to catch the ‘virus’ by touching cash and it was better to use cashless means. The claim was ridiculous nonsense and like the whole ‘Covid’ mind-trick it was nothing to do with ‘health’ and everything to do with pushing every aspect of the Cult agenda. As a result of the Tedros lie the use of cash has plummeted. The Cult script involves a single world digital currency that would eventually be technologically embedded in the body. China is a massive global centre for the Cult and if you watch what is happening there you will know what is planned for everywhere. The Chinese government is developing a digital currency which would allow fines to be deducted immediately via AI for anyone caught on camera breaking its fantastic list of laws and the money is going to be programmable with an expiry date to ensure that no one can accrue wealth except the Cult and its operatives.



**Figure 5:** The structure of global control the Cult has been working towards for so long and this has been enormously advanced by the ‘Covid’ illusion.

## **Serfdom is so smart**

The Cult plan is far wider, extreme, and more comprehensive than even most conspiracy researchers appreciate and I will come to the true depths of deceit and control in the chapters ‘Who controls the

Cult?' and 'Escaping Wetiko'. Even the world that we know is crazy enough. We are being deluged with ever more sophisticated and controlling technology under the heading of 'smart'. We have smart televisions, smart meters, smart cards, smart cars, smart driving, smart roads, smart pills, smart patches, smart watches, smart skin, smart borders, smart pavements, smart streets, smart cities, smart communities, smart environments, smart growth, smart planet ... smart *everything* around us. Smart technologies and methods of operation are designed to interlock to create a global Smart Grid connecting the entirety of human society including human minds to create a centrally-dictated 'hive' mind. 'Smart cities' is code for densely-occupied megacities of total surveillance and control through AI. Ever more destructive frequency communication systems like 5G have been rolled out without any official testing for health and psychological effects (colossal). 5G/6G/7G systems are needed to run the Smart Grid and each one becomes more destructive of body and mind. Deleting independent income is crucial to forcing people into these AI-policed prisons by ending private property ownership (except for the Cult elite). The Cult's Great Reset now openly foresees a global society in which no one will own any possessions and everything will be rented while the Cult would own literally everything under the guise of government and corporations. The aim has been to use the lockdowns to destroy sources of income on a mass scale and when the people are destitute and in unrepayable amounts of debt (problem) Cult assets come forward with the pledge to write-off debt in return for handing over all property and possessions (solution). Everything – literally everything including people – would be connected to the Internet via AI. I was warning years ago about the coming Internet of Things (IoT) in which all devices and technology from your car to your fridge would be plugged into the Internet and controlled by AI. Now we are already there with much more to come. The next stage is the Internet of Everything (IoE) which is planned to include the connection of AI to the human brain and body to replace the human mind with a centrally-controlled AI mind. Instead of perceptions

being manipulated through control of information and censorship those perceptions would come direct from the Cult through AI. What do you think? You think whatever AI decides that you think. In human terms there would be no individual 'think' any longer. Too incredible? The ravings of a lunatic? Not at all. Cult-owned crazies in Silicon Valley have been telling us the plan for years without explaining the real motivation and calculated implications. These include Google executive and 'futurist' Ray Kurzweil who highlights the year 2030 for when this would be underway. He said:

Our thinking ... will be a hybrid of biological and non-biological thinking ... humans will be able to extend their limitations and 'think in the cloud' ... We're going to put gateways to the cloud in our brains ... We're going to gradually merge and enhance ourselves ... In my view, that's the nature of being human – we transcend our limitations.

As the technology becomes vastly superior to what we are then the small proportion that is still human gets smaller and smaller and smaller until it's just utterly negligible.

The sales-pitch of Kurzweil and Cult-owned Silicon Valley is that this would make us 'super-human' when the real aim is to make us post-human and no longer 'human' in the sense that we have come to know. The entire global population would be connected to AI and become the centrally-controlled 'hive-mind' of externally-delivered perceptions. The Smart Grid being installed to impose the Cult's will on the world is being constructed to allow particular locations – even one location – to control the whole global system. From these prime control centres, which absolutely include China and Israel, anything connected to the Internet would be switched on or off and manipulated at will. Energy systems could be cut, communication via the Internet taken down, computer-controlled driverless autonomous vehicles driven off the road, medical devices switched off, the potential is limitless given how much AI and Internet connections now run human society. We have seen nothing yet if we allow this to continue. Autonomous vehicle makers are working with law enforcement to produce cars designed to automatically pull over if they detect a police or emergency vehicle flashing from up to 100 feet away. At a police stop the car would be unlocked and the

window rolled down automatically. Vehicles would only take you where the computer (the state) allowed. The end of petrol vehicles and speed limiters on all new cars in the UK and EU from 2022 are steps leading to electric computerised transport over which ultimately you have no control. The picture is far bigger even than the Cult global network or web and that will become clear when I get to the nature of the ‘spider’. There is a connection between all these happenings and the instigation of DNA-manipulating ‘vaccines’ (which aren’t ‘vaccines’) justified by the ‘Covid’ hoax. That connection is the unfolding plan to transform the human body from a biological to a synthetic biological state and this is why synthetic biology is such a fast-emerging discipline of mainstream science. ‘Covid vaccines’ are infusing self-replicating synthetic genetic material into the cells to cumulatively take us on the Totalitarian Tiptoe from Human 1.0 to the synthetic biological Human 2.0 which will be physically and perceptually attached to the Smart Grid to one hundred percent control every thought, perception and deed.

Humanity needs to wake up and *fast*.

This is the barest explanation of where the ‘outcome’ is planned to go but it’s enough to see the journey happening all around us. Those new to this information will already see ‘Covid’ in a whole new context. I will add much more detail as we go along, but for the minutiae evidence see my mega-works, *The Answer*, *The Trigger* and *Everything You Need to Know But Have Never Been Told*.

Now – how does a Renegade Mind see the ‘world’?

## CHAPTER TWO

### Renegade Perception

*It is one thing to be clever and another to be wise*

George R.R. Martin

A simple definition of the difference between a programmed mind and a Renegade Mind would be that one sees only dots while the other connects them to see the picture. Reading reality with accuracy requires the observer to (a) know the planned outcome and (b) realise that everything, but *everything*, is connected.

The entirety of infinite reality is connected – that's its very nature – and with human society an expression of infinite reality the same must apply. Simple cause and effect is a connection. The effect is triggered by the cause and the effect then becomes the cause of another effect. Nothing happens in isolation because it *can't*. Life in whatever reality is simple choice and consequence. We make choices and these lead to consequences. If we don't like the consequences we can make different choices and get different consequences which lead to other choices and consequences. The choice and the consequence are not only connected they are indivisible. You can't have one without the other as an old song goes. A few cannot control the world unless those being controlled allow that to happen – cause and effect, choice and consequence. Control – who has it and who doesn't – is a two-way process, a symbiotic relationship, involving the controller and controlled. 'They took my freedom away!!' Well, yes, but you also gave it to them. Humanity is

subjected to mass control because humanity has acquiesced to that control. This is all cause and effect and literally a case of give and take. In the same way world events of every kind are connected and the Cult works incessantly to sell the illusion of the random and coincidental to maintain the essential (to them) perception of dots that hide the picture. Renegade Minds know this and constantly scan the world for patterns of connection. This is absolutely pivotal in understanding the happenings in the world and without that perspective clarity is impossible. First you know the planned outcome and then you identify the steps on the journey – the day-by-day apparently random which, when connected in relation to the outcome, no longer appear as individual events, but as the proverbial *chain* of events leading in the same direction. I'll give you some examples:

## **Political puppet show**

We are told to believe that politics is 'adversarial' in that different parties with different beliefs engage in an endless tussle for power. There may have been some truth in that up to a point – and only a point – but today divisions between 'different' parties are rhetorical not ideological. Even the rhetorical is fusing into one-speak as the parties eject any remaining free thinkers while others succumb to the ever-gathering intimidation of anyone with the 'wrong' opinion. The Cult is not a new phenomenon and can be traced back thousands of years as my books have documented. Its intergenerational initiates have been manipulating events with increasing effect the more that global power has been centralised. In ancient times the Cult secured control through the system of monarchy in which 'special' bloodlines (of which more later) demanded the right to rule as kings and queens simply by birthright and by vanquishing others who claimed the same birthright. There came a time, however, when people had matured enough to see the unfairness of such tyranny and demanded a say in who governed them. Note the word – *governed* them. Not served them – *governed* them, hence government defined as 'the political direction and control exercised over the

actions of the members, citizens, or inhabitants of communities, societies, and states; direction of the affairs of a state, community, etc.' Governments exercise control over rather than serve just like the monarchies before them. Bizarrely there are still countries like the United Kingdom which are ruled by a monarch *and* a government that officially answers to the monarch. The UK head of state and that of Commonwealth countries such as Canada, Australia and New Zealand is 'selected' by who in a *single family* had unprotected sex with whom and in what order. Pinch me it can't be true. Ouch! Shit, it is. The demise of monarchies in most countries offered a potential vacuum in which some form of free and fair society could arise and the Cult had that base covered. Monarchies had served its interests but they couldn't continue in the face of such widespread opposition and, anyway, replacing a 'royal' dictatorship that people could see with a dictatorship 'of the people' hiding behind the concept of 'democracy' presented far greater manipulative possibilities and ways of hiding coordinated tyranny behind the illusion of 'freedom'.

Democracy is quite wrongly defined as government selected by the population. This is not the case at all. It is government selected by *some* of the population (and then only in theory). This 'some' doesn't even have to be the majority as we have seen so often in first-past-the-post elections in which the so-called majority party wins fewer votes than the 'losing' parties combined. Democracy can give total power to a party in government from a minority of the votes cast. It's a sleight of hand to sell tyranny as freedom. Seventy-four million Trump-supporting Americans didn't vote for the 'Democratic' Party of Joe Biden in the distinctly dodgy election in 2020 and yet far from acknowledging the wishes and feelings of that great percentage of American society the Cult-owned Biden government set out from day one to destroy them and their right to a voice and opinion. Empty shell Biden and his Cult handlers said they were doing this to 'protect democracy'. Such is the level of lunacy and sickness to which politics has descended. Connect the dots and relate them to the desired outcome – a world government run by self-appointed technocrats and no longer even elected

politicians. While operating through its political agents in government the Cult is at the same time encouraging public distain for politicians by putting idiots and incompetents in theoretical power on the road to deleting them. The idea is to instil a public reaction that says of the technocrats: 'Well, they couldn't do any worse than the pathetic politicians.' It's all about controlling perception and Renegade Minds can see through that while programmed minds cannot when they are ignorant of both the planned outcome and the manipulation techniques employed to secure that end. This knowledge can be learned, however, and fast if people choose to get informed.

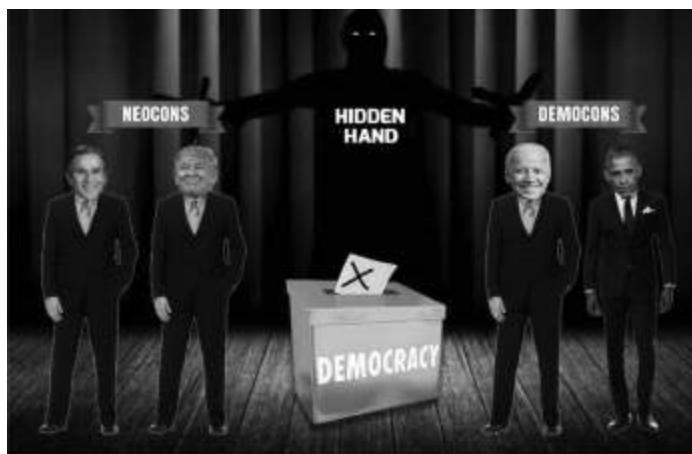
Politics may at first sight appear very difficult to control from a central point. I mean look at the 'different' parties and how would you be able to oversee them all and their constituent parts? In truth, it's very straightforward because of their structure. We are back to the pyramid of imposition and acquiescence. Organisations are structured in the same way as the system as a whole. Political parties are not open forums of free expression. They are hierarchies. I was a national spokesman for the British Green Party which claimed to be a different kind of politics in which influence and power was devolved; but I can tell you from direct experience – and it's far worse now – that Green parties are run as hierarchies like all the others however much they may try to hide that fact or kid themselves that it's not true. A very few at the top of all political parties are directing policy and personnel. They decide if you are elevated in the party or serve as a government minister and to do that you have to be a yes man or woman. Look at all the maverick political thinkers who never ascended the greasy pole. If you want to progress within the party or reach 'high-office' you need to fall into line and conform. Exceptions to this are rare indeed. Should you want to run for parliament or Congress you have to persuade the local or state level of the party to select you and for that you need to play the game as dictated by the hierarchy. If you secure election and wish to progress within the greater structure you need to go on conforming to what is acceptable to those running the hierarchy

from the peak of the pyramid. Political parties are perceptual gulags and the very fact that there are party 'Whips' appointed to 'whip' politicians into voting the way the hierarchy demands exposes the ridiculous idea that politicians are elected to serve the people they are supposed to represent. Cult operatives and manipulation has long seized control of major parties that have any chance of forming a government and at least most of those that haven't. A new party forms and the Cult goes to work to infiltrate and direct. This has reached such a level today that you see video compilations of 'leaders' of all parties whether Democrats, Republicans, Conservative, Labour and Green parroting the same Cult mantra of 'Build Back Better' and the 'Great Reset' which are straight off the Cult song-sheet to describe the transformation of global society in response to the Cult-instigated hoaxes of the 'Covid pandemic' and human-caused 'climate change'. To see Caroline Lucas, the Green Party MP that I knew when I was in the party in the 1980s, speaking in support of plans proposed by Cult operative Klaus Schwab representing the billionaire global elite is a real head-shaker.

## **Many parties – one master**

The party system is another mind-trick and was instigated to change the nature of the dictatorship by swapping 'royalty' for dark suits that people believed – though now ever less so – represented their interests. Understanding this trick is to realise that a single force (the Cult) controls all parties either directly in terms of the major ones or through manipulation of perception and ideology with others. You don't need to manipulate Green parties to demand your transformation of society in the name of 'climate change' when they are obsessed with the lie that this is essential to 'save the planet'. You just give them a platform and away they go serving your interests while believing they are being environmentally virtuous. America's political structure is a perfect blueprint for how the two or multi-party system is really a one-party state. The Republican Party is controlled from one step back in the shadows by a group made up of billionaires and their gofers known as neoconservatives or Neocons.

I have exposed them in fine detail in my books and they were the driving force behind the policies of the imbecilic presidency of Boy George Bush which included 9/11 (see *The Trigger* for a comprehensive demolition of the official story), the subsequent ‘war on terror’ (war of terror) and the invasions of Afghanistan and Iraq. The latter was a No-Problem-Reaction-Solution based on claims by Cult operatives, including Bush and British Prime Minister Tony Blair, about Saddam Hussein’s ‘weapons of mass destruction’ which did not exist as war criminals Bush and Blair well knew.



**Figure 6:** Different front people, different parties – same control system.

The Democratic Party has its own ‘Neocon’ group controlling from the background which I call the ‘Democons’ and here’s the penny-drop – the Neocons and Democons answer to the same masters one step further back into the shadows (Fig 6). At that level of the Cult the Republican and Democrat parties are controlled by the same people and no matter which is in power the Cult is in power. This is how it works in almost every country and certainly in Britain with Conservative, Labour, Liberal Democrat and Green parties now all on the same page whatever the rhetoric may be in their feeble attempts to appear different. Neocons operated at the time of Bush through a think tank called The Project for the New American Century which in September, 2000, published a document entitled *Rebuilding America’s Defenses: Strategies, Forces, and Resources*

*For a New Century* demanding that America fight ‘multiple, simultaneous major theatre wars’ as a ‘core mission’ to force regime-change in countries including Iraq, Libya and Syria. Neocons arranged for Bush (‘Republican’) and Blair (‘Labour Party’) to front-up the invasion of Iraq and when they departed the Democons orchestrated the targeting of Libya and Syria through Barack Obama (‘Democrat’) and British Prime Minister David Cameron (‘Conservative Party’). We have ‘different’ parties and ‘different’ people, but the same unfolding script. The more the Cult has seized the reigns of parties and personnel the more their policies have transparently pursued the same agenda to the point where the fascist ‘Covid’ impositions of the Conservative junta of Jackboot Johnson in Britain were opposed by the Labour Party because they were not fascist enough. The Labour Party is likened to the US Democrats while the Conservative Party is akin to a British version of the Republicans and on both sides of the Atlantic they all speak the same language and support the direction demanded by the Cult although some more enthusiastically than others. It’s a similar story in country after country because it’s all centrally controlled. Oh, but what about Trump? I’ll come to him shortly. Political ‘choice’ in the ‘party’ system goes like this: You vote for Party A and they get into government. You don’t like what they do so next time you vote for Party B and they get into government. You don’t like what they do when it’s pretty much the same as Party A and why wouldn’t that be with both controlled by the same force? Given that only two, sometimes three, parties have any chance of forming a government to get rid of Party B that you don’t like you have to vote again for Party A which ... you don’t like. This, ladies and gentlemen, is what they call ‘democracy’ which we are told – wrongly – is a term interchangeable with ‘freedom’.

## **The cult of cults**

At this point I need to introduce a major expression of the Global Cult known as Sabbatian-Frankism. Sabbatian is also spelt as Sabbatean. I will summarise here. I have published major exposés

and detailed background in other works. Sabbatian-Frankism combines the names of two frauds posing as 'Jewish' men, Sabbatai Zevi (1626-1676), a rabbi, black magician and occultist who proclaimed he was the Jewish messiah; and Jacob Frank (1726-1791), the Polish 'Jew', black magician and occultist who said he was the reincarnation of 'messiah' Zevi and biblical patriarch Jacob. They worked across two centuries to establish the Sabbatian-Frankist cult that plays a major, indeed central, role in the manipulation of human society by the Global Cult which has its origins much further back in history than Sabbatai Zevi. I should emphasise two points here in response to the shrill voices that will scream 'anti-Semitism': (1) Sabbatian-Frankists are NOT Jewish and only pose as such to hide their cult behind a Jewish façade; and (2) my information about this cult has come from Jewish sources who have long realised that their society and community has been infiltrated and taken over by interloper Sabbatian-Frankists. Infiltration has been the foundation technique of Sabbatian-Frankism from its official origin in the 17th century. Zevi's Sabbatian sect attracted a massive following described as the biggest messianic movement in Jewish history, spreading as far as Africa and Asia, and he promised a return for the Jews to the 'Promised Land' of Israel. Sabbatianism was not Judaism but an inversion of everything that mainstream Judaism stood for. So much so that this sinister cult would have a feast day when Judaism had a fast day and whatever was forbidden in Judaism the Sabbatians were encouraged and even commanded to do. This included incest and what would be today called Satanism. Members were forbidden to marry outside the sect and there was a system of keeping their children ignorant of what they were part of until they were old enough to be trusted not to unknowingly reveal anything to outsiders. The same system is employed to this day by the Global Cult in general which Sabbatian-Frankism has enormously influenced and now largely controls.

Zevi and his Sabbatians suffered a setback with the intervention by the Sultan of the Islamic Ottoman Empire in the Middle East and what is now the Republic of Turkey where Zevi was located. The

Sultan gave him the choice of proving his ‘divinity’, converting to Islam or facing torture and death. Funnily enough Zevi chose to convert or at least appear to. Some of his supporters were disillusioned and drifted away, but many did not with 300 families also converting – only in theory – to Islam. They continued behind this Islamic smokescreen to follow the goals, rules and rituals of Sabbatianism and became known as ‘crypto-Jews’ or the ‘Dönmeh’ which means ‘to turn’. This is rather ironic because they didn’t ‘turn’ and instead hid behind a fake Islamic persona. The process of appearing to be one thing while being very much another would become the calling card of Sabbatianism especially after Zevi’s death and the arrival of the Satanist Jacob Frank in the 18th century when the cult became Sabbatian-Frankism and plumbbed still new depths of depravity and infiltration which included – still includes – human sacrifice and sex with children. Wherever Sabbatians go paedophilia and Satanism follow and is it really a surprise that Hollywood is so infested with child abuse and Satanism when it was established by Sabbatian-Frankists and is still controlled by them? Hollywood has been one of the prime vehicles for global perceptual programming and manipulation. How many believe the version of ‘history’ portrayed in movies when it is a travesty and inversion (again) of the truth? Rabbi Marvin Antelman describes Frankism in his book, *To Eliminate the Opiate*, as ‘a movement of complete evil’ while Jewish professor Gershom Scholem said of Frank in *The Messianic Idea in Judaism*: ‘In all his actions [he was] a truly corrupt and degenerate individual ... one of the most frightening phenomena in the whole of Jewish history.’ Frank was excommunicated by traditional rabbis, as was Zevi, but Frank was undeterred and enjoyed vital support from the House of Rothschild, the infamous banking dynasty whose inner-core are Sabbatian-Frankists and not Jews. Infiltration of the Roman Church and Vatican was instigated by Frank with many Dönmeh ‘turning’ again to convert to Roman Catholicism with a view to hijacking the reins of power. This was the ever-repeating modus operandi and continues to be so. Pose as an advocate of the religion, culture or country that you want to control and then

manipulate your people into the positions of authority and influence largely as advisers, administrators and Svengalis for those that appear to be in power. They did this with Judaism, Christianity (Christian Zionism is part of this), Islam and other religions and nations until Sabbatian-Frankism spanned the world as it does today.

## **Sabbatian Saudis and the terror network**

One expression of the Sabbatian-Frankist Dönme within Islam is the ruling family of Saudi Arabia, the House of Saud, through which came the vile distortion of Islam known as Wahhabism. This is the violent creed followed by terrorist groups like Al-Qaeda and ISIS or Islamic State. Wahhabism is the hand-chopping, head-chopping ‘religion’ of Saudi Arabia which is used to keep the people in a constant state of fear so the interloper House of Saud can continue to rule. Al-Qaeda and Islamic State were lavishly funded by the House of Saud while being created and directed by the Sabbatian-Frankist network in the United States that operates through the Pentagon, CIA and the government in general of whichever ‘party’. The front man for the establishment of Wahhabism in the middle of the 18th century was a Sabbatian-Frankist ‘crypto-Jew’ posing as Islamic called Muhammad ibn Abd al-Wahhab. His daughter would marry the son of Muhammad bin Saud who established the first Saudi state before his death in 1765 with support from the British Empire. Bin Saud’s successors would establish modern Saudi Arabia in league with the British and Americans in 1932 which allowed them to seize control of Islam’s major shrines in Mecca and Medina. They have dictated the direction of Sunni Islam ever since while Iran is the major centre of the Shiite version and here we have the source of at least the public conflict between them. The Sabbatian network has used its Wahhabi extremists to carry out Problem-Reaction-Solution terrorist attacks in the name of ‘Al-Qaeda’ and ‘Islamic State’ to justify a devastating ‘war on terror’, ever-increasing surveillance of the population and to terrify people into compliance. Another insight of the Renegade Mind is the streetwise understanding that

just because a country, location or people are attacked doesn't mean that those apparently representing that country, location or people are not behind the attackers. Often they are *orchestrating* the attacks because of the societal changes that can be then justified in the name of 'saving the population from terrorists'.

I show in great detail in *The Trigger* how Sabbatian-Frankists were the real perpetrators of 9/11 and not '19 Arab hijackers' who were blamed for what happened. Observe what was justified in the name of 9/11 alone in terms of Middle East invasions, mass surveillance and control that fulfilled the demands of the Project for the New American Century document published by the Sabbatian Neocons. What appear to be enemies are on the deep inside players on the same Sabbatian team. Israel and Arab 'royal' dictatorships are all ruled by Sabbatians and the recent peace agreements between Israel and Saudi Arabia, the United Arab Emirates (UAE) and others are only making formal what has always been the case behind the scenes. Palestinians who have been subjected to grotesque tyranny since Israel was bombed and terrorised into existence in 1948 have never stood a chance. Sabbatian-Frankists have controlled Israel (so the constant theme of violence and war which Sabbatians love) and they have controlled the Arab countries that Palestinians have looked to for real support that never comes. 'Royal families' of the Arab world in Saudi Arabia, Bahrain, UAE, etc., are all Sabbatians with allegiance to the aims of the cult and not what is best for their Arabic populations. They have stolen the oil and financial resources from their people by false claims to be 'royal dynasties' with a genetic right to rule and by employing vicious militaries to impose their will.

## **Satanic 'illumination'**

The Satanist Jacob Frank formed an alliance in 1773 with two other Sabbatians, Mayer Amschel Rothschild (1744-1812), founder of the Rothschild banking dynasty, and Jesuit-educated fraudulent Jew, Adam Weishaupt, and this led to the formation of the Bavarian Illuminati, firstly under another name, in 1776. The Illuminati would

be the manipulating force behind the French Revolution (1789-1799) and was also involved in the American Revolution (1775-1783) before and after the Illuminati's official creation. Weishaupt would later become (in public) a Protestant Christian in archetypal Sabbatian style. I read that his name can be decoded as Adam-Weishaupt or 'the first man to lead those who know'. He wasn't a leader in the sense that he was a subordinate, but he did lead those below him in a crusade of transforming human society that still continues today. The theme was confirmed as early as 1785 when a horseman courier called Lanz was reported to be struck by lighting and extensive Illuminati documents were found in his saddlebags. They made the link to Weishaupt and detailed the plan for world takeover. Current events with 'Covid' fascism have been in the making for a very long time. Jacob Frank was jailed for 13 years by the Catholic Inquisition after his arrest in 1760 and on his release he headed for Frankfurt, Germany, home city and headquarters of the House of Rothschild where the alliance was struck with Mayer Amschel Rothschild and Weishaupt. Rothschild arranged for Frank to be given the title of Baron and he became a wealthy nobleman with a big following of Jews in Germany, the Austro-Hungarian Empire and other European countries. Most of them would have believed he was on their side.

The name 'Illuminati' came from the Zohar which is a body of works in the Jewish mystical 'bible' called the Kabbalah. 'Zohar' is the foundation of Sabbatian-Frankist belief and in Hebrew 'Zohar' means 'splendour', 'radiance', 'illuminated', and so we have 'Illuminati'. They claim to be the 'Illuminated Ones' from their knowledge systematically hidden from the human population and passed on through generations of carefully-chosen initiates in the global secret society network or Cult. Hidden knowledge includes an awareness of the Cult agenda for the world and the nature of our collective reality that I will explore later. Cult 'illumination' is symbolised by the torch held by the Statue of Liberty which was gifted to New York by French Freemasons in Paris who knew exactly what it represents. 'Liberty' symbolises the goddess worshipped in

Babylon as Queen Semiramis or Ishtar. The significance of this will become clear. Notice again the ubiquitous theme of inversion with the Statue of 'Liberty' really symbolising mass control ([Fig 7](#)). A mirror-image statute stands on an island in the River Seine in Paris from where New York Liberty originated ([Fig 8](#)). A large replica of the Liberty flame stands on top of the Pont de l'Alma tunnel in Paris where Princess Diana died in a Cult ritual described in *The Biggest Secret*. Lucifer 'the light bringer' is related to all this (and much more as we'll see) and 'Lucifer' is a central figure in Sabbatian-Frankism and its associated Satanism. Sabbatians reject the Jewish Torah, or Pentateuch, the 'five books of Moses' in the Old Testament known as Genesis, Exodus, Leviticus, Numbers, and Deuteronomy which are claimed by Judaism and Christianity to have been dictated by 'God' to Moses on Mount Sinai. Sabbatians say these do not apply to them and they seek to replace them with the Zohar to absorb Judaism and its followers into their inversion which is an expression of a much greater global inversion. They want to delete all religions and force humanity to worship a one-world religion – Sabbatian Satanism that also includes worship of the Earth goddess. Satanic themes are being more and more introduced into mainstream society and while Christianity is currently the foremost target for destruction the others are planned to follow.



**Figure 7:** The Cult goddess of Babylon disguised as the Statue of Liberty holding the flame of Lucifer the 'light bringer'.



**Figure 8:** Liberty's mirror image in Paris where the New York version originated.

## **Marx brothers**

Rabbi Marvin Antelman connects the Illuminati to the Jacobins in *To Eliminate the Opiate* and Jacobins were the force behind the French Revolution. He links both to the Bund der Gerechten, or League of the Just, which was the network that inflicted communism/Marxism on the world. Antelman wrote:

The original inner circle of the Bund der Gerechten consisted of born Catholics, Protestants and Jews [Sabbatian-Frankist infiltrators], and those representatives of respective subdivisions formulated schemes for the ultimate destruction of their faiths. The heretical Catholics laid plans which they felt would take a century or more for the ultimate destruction of the church; the apostate Jews for the ultimate destruction of the Jewish religion.

Sabbatian-created communism connects into this anti-religion agenda in that communism does not allow for the free practice of religion. The Sabbatian 'Bund' became the International Communist Party and Communist League and in 1848 'Marxism' was born with the Communist Manifesto of Sabbatian assets Karl Marx and Friedrich Engels. It is absolutely no coincidence that Marxism, just a different name for fascist and other centrally-controlled tyrannies, is being imposed worldwide as a result of the 'Covid' hoax and nor that Marxist/fascist China was the place where the hoax originated. The reason for this will become very clear in the chapter 'Covid: The calculated catastrophe'. The so-called 'Woke' mentality has hijacked

traditional beliefs of the political left and replaced them with far-right make-believe ‘social justice’ better known as Marxism. Woke will, however, be swallowed by its own perceived ‘revolution’ which is really the work of billionaires and billionaire corporations feigning being ‘Woke’. Marxism is being touted by Wokers as a replacement for ‘capitalism’ when we don’t have ‘capitalism’. We have cartelism in which the market is stitched up by the very Cult billionaires and corporations bankrolling Woke. Billionaires love Marxism which keeps the people in servitude while they control from the top.

Terminally naïve Wokers think they are ‘changing the world’ when it’s the Cult that is doing the changing and when they have played their vital part and become surplus to requirements they, too, will be targeted. The Illuminati-Jacobins were behind the period known as ‘The Terror’ in the French Revolution in 1793 and 1794 when Jacobin Maximillian de Robespierre and his Orwellian ‘Committee of Public Safety’ killed 17,000 ‘enemies of the Revolution’ who had once been ‘friends of the Revolution’. Karl Marx (1818-1883), whose Sabbatian creed of Marxism has cost the lives of at least 100 million people, is a hero once again to Wokers who have been systematically kept ignorant of real history by their ‘education’ programming. As a result they now promote a Sabbatian ‘Marxist’ abomination destined at some point to consume them. Rabbi Antelman, who spent decades researching the Sabbatian plot, said of the League of the Just and Karl Marx:

Contrary to popular opinion Karl Marx did not originate the Communist Manifesto. He was paid for his services by the League of the Just, which was known in its country of origin, Germany, as the Bund der Gaeachteten.

Antelman said the text attributed to Marx was the work of other people and Marx ‘was only repeating what others already said’. Marx was ‘a hired hack – lackey of the wealthy Illuminists’. Marx famously said that religion was the ‘opium of the people’ (part of the Sabbatian plan to demonise religion) and Antelman called his books, *To Eliminate the Opiate*. Marx was born Jewish, but his family converted to Christianity (Sabbatian modus operandi) and he

attacked Jews, not least in his book, *A World Without Jews*. In doing so he supported the Sabbatian plan to destroy traditional Jewishness and Judaism which we are clearly seeing today with the vindictive targeting of orthodox Jews by the Sabbatian government of Israel over 'Covid' laws. I don't follow any religion and it has done much damage to the world over centuries and acted as a perceptual straightjacket. Renegade Minds, however, are always asking *why* something is being done. It doesn't matter if they agree or disagree with what is happening – *why* is it happening is the question. The 'why?' can be answered with regard to religion in that religions create interacting communities of believers when the Cult wants to dismantle all discourse, unity and interaction (see 'Covid' lockdowns) and the ultimate goal is to delete all religions for a one-world religion of Cult Satanism worshipping their 'god' of which more later. We see the same 'why?' with gun control in America. I don't have guns and don't want them, but why is the Cult seeking to disarm the population at the same time that law enforcement agencies are armed to their molars and why has every tyrant in history sought to disarm people before launching the final takeover? They include Hitler, Stalin, Pol Pot and Mao who followed confiscation with violent seizing of power. You know it's a Cult agenda by the people who immediately race to the microphones to exploit dead people in multiple shootings. Ultra-Zionist Cult lackey Senator Chuck Schumer was straight on the case after ten people were killed in Boulder, Colorado in March, 2021. Simple rule ... if Schumer wants it the Cult wants it and the same with his ultra-Zionist mate the wild-eyed Senator Adam Schiff. At the same time they were calling for the disarmament of Americans, many of whom live a long way from a police response, Schumer, Schiff and the rest of these pampered clowns were sitting on Capitol Hill behind a razor-wired security fence protected by thousands of armed troops in addition to their own armed bodyguards. Mom and pop in an isolated home? They're just potential mass shooters.

## Zion Mainframe

Sabbatian-Frankists and most importantly the Rothschilds were behind the creation of 'Zionism', a political movement that demanded a Jewish homeland in Israel as promised by Sabbatai Zevi. The very symbol of Israel comes from the German meaning of the name Rothschild. Dynasty founder Mayer Amschel Rothschild changed the family name from Bauer to Rothschild, or 'Red-Shield' in German, in deference to the six-pointed 'Star of David' hexagram displayed on the family's home in Frankfurt. The symbol later appeared on the flag of Israel after the Rothschilds were centrally involved in its creation. Hexagrams are not a uniquely Jewish symbol and are widely used in occult ('hidden') networks often as a symbol for Saturn (see my other books for why). Neither are Zionism and Jewishness interchangeable. Zionism is a political movement and philosophy and not a 'race' or a people. Many Jews oppose Zionism and many non-Jews, including US President Joe Biden, call themselves Zionists as does Israel-centric Donald Trump. America's support for the Israel government is pretty much a gimme with ultra-Zionist billionaires and corporations providing fantastic and dominant funding for both political parties. Former Congresswoman Cynthia McKinney has told how she was approached immediately she ran for office to 'sign the pledge' to Israel and confirm that she would always vote in that country's best interests. All American politicians are approached in this way. Anyone who refuses will get no support or funding from the enormous and all-powerful Zionist lobby that includes organisations like mega-lobby group AIPAC, the American Israel Public Affairs Committee. Trump's biggest funder was ultra-Zionist casino and media billionaire Sheldon Adelson while major funders of the Democratic Party include ultra-Zionist George Soros and ultra-Zionist financial and media mogul, Haim Saban. Some may reel back at the suggestion that Soros is an Israel-firster (Sabbatian-controlled Israel-firster), but Renegade Minds watch the actions not the words and everywhere Soros donates his billions the Sabbatian agenda benefits. In the spirit of Sabbatian inversion Soros pledged \$1 billion for a new university network to promote 'liberal values and tackle intolerance'. He made the announcement during his annual speech

at the Cult-owned World Economic Forum in Davos, Switzerland, in January, 2020, after his ‘harsh criticism’ of ‘authoritarian rulers’ around the world. You can only laugh at such brazen mendacity. How *he* doesn’t laugh is the mystery. Translated from the Orwellian ‘liberal values and tackle intolerance’ means teaching non-white people to hate white people and for white people to loathe themselves for being born white. The reason for that will become clear.

### **The ‘Anti-Semitism’ fraud**

Zionists support the Jewish homeland in the land of Palestine which has been the Sabbatian-Rothschild goal for so long, but not for the benefit of Jews. Sabbatians and their global Anti-Semitism Industry have skewed public and political opinion to equate opposing the violent extremes of Zionism to be a blanket attack and condemnation of all Jewish people. Sabbatians and their global Anti-Semitism Industry have skewed public and political opinion to equate opposing the violent extremes of Zionism to be a blanket attack and condemnation of all Jewish people. This is nothing more than a Sabbatian protection racket to stop legitimate investigation and exposure of their agendas and activities. The official definition of ‘anti-Semitism’ has more recently been expanded to include criticism of Zionism – a *political movement* – and this was done to further stop exposure of Sabbatian infiltrators who created Zionism as we know it today in the 19th century. Renegade Minds will talk about these subjects when they know the shit that will come their way. People must decide if they want to know the truth or just cower in the corner in fear of what others will say. Sabbatians have been trying to label me as ‘anti-Semitic’ since the 1990s as I have uncovered more and more about their background and agendas. Useless, gutless, fraudulent ‘journalists’ then just repeat the smears without question and on the day I was writing this section a pair of unquestioning repeaters called Ben Quinn and Archie Bland (how appropriate) outright called me an ‘anti-Semite’ in the establishment propaganda sheet, the London *Guardian*, with no supporting evidence. The

Sabbatian Anti-Semitism Industry said so and who are they to question that? They wouldn't dare. Ironically 'Semitic' refers to a group of languages in the Middle East that are almost entirely Arabic. 'Anti-Semitism' becomes 'anti-Arab' which if the consequences of this misunderstanding were not so grave would be hilarious. Don't bother telling Quinn and Bland. I don't want to confuse them, bless 'em. One reason I am dubbed 'anti-Semitic' is that I wrote in the 1990s that Jewish operatives (Sabbatians) were heavily involved in the Russian Revolution when Sabbatians overthrew the Romanov dynasty. This apparently made me 'anti-Semitic'. Oh, really? Here is a section from *The Trigger*:

British journalist Robert Wilton confirmed these themes in his 1920 book *The Last Days of the Romanovs* when he studied official documents from the Russian government to identify the members of the Bolshevik ruling elite between 1917 and 1919. The Central Committee included 41 Jews among 62 members; the Council of the People's Commissars had 17 Jews out of 22 members; and 458 of the 556 most important Bolshevik positions between 1918 and 1919 were occupied by Jewish people. Only 17 were Russian. Then there were the 23 Jews among the 36 members of the vicious Cheka Soviet secret police established in 1917 who would soon appear all across the country.

Professor Robert Service of Oxford University, an expert on 20th century Russian history, found evidence that ['Jewish'] Leon Trotsky had sought to make sure that Jews were enrolled in the Red Army and were disproportionately represented in the Soviet civil bureaucracy that included the Cheka which performed mass arrests, imprisonment and executions of 'enemies of the people'. A US State Department Decimal File (861.00/5339) dated November 13th, 1918, names [Rothschild banking agent in America] Jacob Schiff and a list of ultra-Zionists as funders of the Russian Revolution leading to claims of a 'Jewish plot', but the key point missed by all is they were not 'Jews' – they were Sabbatian-Frankists.

Britain's Winston Churchill made the same error by mistake or otherwise. He wrote in a 1920 edition of the *Illustrated Sunday Herald* that those behind the Russian revolution were part of a 'worldwide conspiracy for the overthrow of civilisation and for the reconstitution of society on the basis of arrested development, of envious malevolence, and impossible equality' (see 'Woke' today because that has been created by the same network). Churchill said there was no need to exaggerate the part played in the creation of Bolshevism and in the actual bringing about of the Russian

Revolution 'by these international and for the most part atheistical Jews' ['atheistical Jews' = Sabbatians]. Churchill said it is certainly a very great one and probably outweighs all others: 'With the notable exception of Lenin, the majority of the leading figures are Jews.' He went on to describe, knowingly or not, the Sabbatian modus operandi of placing puppet leaders nominally in power while they control from the background:

Moreover, the principal inspiration and driving power comes from the Jewish leaders. Thus Tchitcherin, a pure Russian, is eclipsed by his nominal subordinate, Litvinoff, and the influence of Russians like Bukharin or Lunacharski cannot be compared with the power of Trotsky, or of Zinovieff, the Dictator of the Red Citadel (Petrograd), or of Krassin or Radek – all Jews. In the Soviet institutions the predominance of Jews is even more astonishing. And the prominent, if not indeed the principal, part in the system of terrorism applied by the Extraordinary Commissions for Combatting Counter-Revolution has been taken by Jews, and in some notable cases by Jewesses.

What I said about seriously disproportionate involvement in the Russian Revolution by Jewish 'revolutionaries' (Sabbatians) is provable fact, but truth is no defence against the Sabbatian Anti-Semitism Industry, its repeater parrots like Quinn and Bland, and the now breathtaking network of so-called 'Woke' 'anti-hate' groups with interlocking leaderships and funding which have the role of discrediting and silencing anyone who gets too close to exposing the Sabbatians. We have seen 'truth is no defence' confirmed in legal judgements with the Saskatchewan Human Rights Commission in Canada decreeing this: 'Truthful statements can be presented in a manner that would meet the definition of hate speech, and not all truthful statements must be free from restriction.' Most 'anti-hate' activists, who are themselves consumed by hatred, are too stupid and ignorant of the world to know how they are being used. They are far too far up their own virtue-signalling arses and it's far too dark for them to see anything.

## **The 'revolution' game**

The background and methods of the 'Russian' Revolution are straight from the Sabbatian playbook seen in the French Revolution

and endless others around the world that appear to start as a revolution of the people against tyrannical rule and end up with a regime change to more tyrannical rule overtly or covertly. Wars, terror attacks and regime overthrows follow the Sabbatian cult through history with its agents creating them as Problem-Reaction-Solutions to remove opposition on the road to world domination. Sabbatian dots connect the Rothschilds with the Illuminati, Jacobins of the French Revolution, the 'Bund' or League of the Just, the International Communist Party, Communist League and the Communist Manifesto of Karl Marx and Friedrich Engels that would lead to the Rothschild-funded Russian Revolution. The sequence comes under the heading of 'creative destruction' when you advance to your global goal by continually destroying the status quo to install a new status quo which you then also destroy. The two world wars come to mind. With each new status quo you move closer to your planned outcome. Wars and mass murder are to Sabbatians a collective blood sacrifice ritual. They are obsessed with death for many reasons and one is that death is an inversion of life. Satanists and Sabbatians are obsessed with death and often target churches and churchyards for their rituals. Inversion-obsessed Sabbatians explain the use of inverted symbolism including the *inverted* pentagram and *inverted* cross. The inversion of the cross has been related to targeting Christianity, but the cross was a religious symbol long before Christianity and its inversion is a statement about the Sabbatian mentality and goals more than any single religion.

Sabbatians operating in Germany were behind the rise of the occult-obsessed Nazis and the subsequent Jewish exodus from Germany and Europe to Palestine and the United States after World War Two. The Rothschild dynasty was at the forefront of this both as political manipulators and by funding the operation. Why would Sabbatians help to orchestrate the horrors inflicted on Jews by the Nazis and by Stalin after they organised the Russian Revolution? Sabbatians hate Jews and their religion, that's why. They pose as Jews and secure positions of control within Jewish society and play the 'anti-Semitism' card to protect themselves from exposure

through a global network of organisations answering to the Sabbatian-created-and-controlled globe-spanning intelligence network that involves a stunning web of military-intelligence operatives and operations for a tiny country of just nine million. Among them are Jewish assets who are not Sabbatians but have been convinced by them that what they are doing is for the good of Israel and the Jewish community to protect them from what they have been programmed since childhood to believe is a Jew-hating hostile world. The Jewish community is just a highly convenient cover to hide the true nature of Sabbatians. Anyone getting close to exposing their game is accused by Sabbatian place-people and gofers of 'anti-Semitism' and claiming that all Jews are part of a plot to take over the world. I am not saying that. I am saying that Sabbatians – the *real* Jew-haters – have infiltrated the Jewish community to use them both as a cover and an 'anti-Semitic' defence against exposure. Thus we have the Anti-Semitism Industry targeted researchers in this way and most Jewish people think this is justified and genuine. They don't know that their 'Jewish' leaders and institutions of state, intelligence and military are not controlled by Jews at all, but cultists and stooges of Sabbatian-Frankism. I once added my name to a pro-Jewish freedom petition online and the next time I looked my name was gone and text had been added to the petition blurb to attack me as an 'anti-Semite' such is the scale of perceptual programming.

## **Moving on America**

I tell the story in *The Trigger* and a chapter called 'Atlantic Crossing' how particularly after Israel was established the Sabbatians moved in on the United States and eventually grasped control of government administration, the political system via both Democrats and Republicans, the intelligence community like the CIA and National Security Agency (NSA), the Pentagon and mass media. Through this seriously compartmentalised network Sabbatians and their operatives in Mossad, Israeli Defense Forces (IDF) and US agencies pulled off 9/11 and blamed it on 19 'Al-Qaeda hijackers' dominated by men from, or connected to, Sabbatian-ruled Saudi

Arabia. The '19' were not even on the planes let alone flew those big passenger jets into buildings while being largely incompetent at piloting one-engine light aircraft. 'Hijacker' Hani Hanjour who is said to have flown American Airlines Flight 77 into the Pentagon with a turn and manoeuvre most professional pilots said they would have struggled to do was banned from renting a small plane by instructors at the Freeway Airport in Bowie, Maryland, just *six weeks* earlier on the grounds that he was an incompetent pilot. The Jewish population of the world is just 0.2 percent with even that almost entirely concentrated in Israel (75 percent Jewish) and the United States (around two percent). This two percent and globally 0.2 percent refers to *Jewish* people and not Sabbatian interlopers who are a fraction of that fraction. What a sobering thought when you think of the fantastic influence on world affairs of tiny Israel and that the Project for the New America Century (PNAC) which laid out the blueprint in September, 2000, for America's war on terror and regime change wars in Iraq, Libya and Syria was founded and dominated by Sabbatians known as 'Neocons'. The document conceded that this plan would not be supported politically or publicly without a major attack on American soil and a Problem-Reaction-Solution excuse to send troops to war across the Middle East. Sabbatian Neocons said:

... [The] process of transformation ... [war and regime change] ... is likely to be a long one, absent some catastrophic and catalysing event – like a new Pearl Harbor.

Four months later many of those who produced that document came to power with their inane puppet George Bush from the long-time Sabbatian Bush family. They included Sabbatian Dick Cheney who was officially vice-president, but really de-facto president for the entirety of the 'Bush' government. Nine months after the 'Bush' inauguration came what Bush called at the time 'the Pearl Harbor of the 21st century' and with typical Sabbatian timing and symbolism 2001 was the 60th anniversary of the attack in 1941 by the Japanese Air Force on Pearl Harbor, Hawaii, which allowed President Franklin Delano Roosevelt to take the United States into a Sabbatian-

instigated Second World War that he said in his election campaign that he never would. The evidence is overwhelming that Roosevelt and his military and intelligence networks knew the attack was coming and did nothing to stop it, but they did make sure that America's most essential naval ships were not in Hawaii at the time. Three thousand Americans died in the Pearl Harbor attacks as they did on September 11th. By the 9/11 year of 2001 Sabbatians had widely infiltrated the US government, military and intelligence operations and used their compartmentalised assets to pull off the 'Al-Qaeda' attacks. If you read *The Trigger* it will blow your mind to see the utterly staggering concentration of 'Jewish' operatives (Sabbatian infiltrators) in essential positions of political, security, legal, law enforcement, financial and business power before, during, and after the attacks to make them happen, carry them out, and then cover their tracks – and I do mean *staggering* when you think of that 0.2 percent of the world population and two percent of Americans which are Jewish while Sabbatian infiltrators are a fraction of that. A central foundation of the 9/11 conspiracy was the hijacking of government, military, Air Force and intelligence computer systems in real time through 'back-door' access made possible by Israeli (Sabbatian) 'cyber security' software. Sabbatian-controlled Israel is on the way to rivalling Silicon Valley for domination of cyberspace and is becoming the dominant force in cyber-security which gives them access to entire computer systems and their passcodes across the world. Then add to this that Zionists head (officially) Silicon Valley giants like Google (Larry Page and Sergey Brin), Google-owned YouTube (Susan Wojcicki), Facebook (Mark Zuckerberg and Sheryl Sandberg), and Apple (Chairman Arthur D. Levinson), and that ultra-Zionist hedge fund billionaire Paul Singer has a \$1 billion stake in Twitter which is only nominally headed by 'CEO' pothead Jack Dorsey. As cable news host Tucker Carlson said of Dorsey: 'There used to be debate in the medical community whether dropping a ton of acid had permanent effects and I think that debate has now ended.' Carlson made the comment after Dorsey told a hearing on Capitol Hill (if you cut through his bullshit) that he

believed in free speech so long as he got to decide what you can hear and see. These 'big names' of Silicon Valley are only front men and women for the Global Cult, not least the Sabbatians, who are the true controllers of these corporations. Does anyone still wonder why these same people and companies have been ferociously censoring and banning people (like me) for exposing any aspect of the Cult agenda and especially the truth about the 'Covid' hoax which Sabbatians have orchestrated?

The Jeffrey Epstein paedophile ring was a Sabbatian operation. He was officially 'Jewish' but he was a Sabbatian and women abused by the ring have told me about the high number of 'Jewish' people involved. The Epstein horror has Sabbatian written all over it and matches perfectly their modus operandi and obsession with sex and ritual. Epstein was running a Sabbatian blackmail ring in which famous people with political and other influence were provided with young girls for sex while everything was being filmed and recorded on hidden cameras and microphones at his New York house, Caribbean island and other properties. Epstein survivors have described this surveillance system to me and some have gone public. Once the famous politician or other figure knew he or she was on video they tended to do whatever they were told. Here we go again ...when you've got them by the balls their hearts and minds will follow. Sabbatians use this blackmail technique on a wide scale across the world to entrap politicians and others they need to act as demanded. Epstein's private plane, the infamous 'Lolita Express', had many well-known passengers including Bill Clinton while Bill Gates has flown on an Epstein plane and met with him four years after Epstein had been jailed for paedophilia. They subsequently met many times at Epstein's home in New York according to a witness who was there. Epstein's infamous side-kick was Ghislaine Maxwell, daughter of Mossad agent and ultra-Zionist mega-crooked British businessman, Bob Maxwell, who at one time owned the *Daily Mirror* newspaper. Maxwell was murdered at sea on his boat in 1991 by Sabbatian-controlled Mossad when he became a liability with his

business empire collapsing as a former Mossad operative has confirmed (see *The Trigger*).

### **Money, money, money, funny money ...**

Before I come to the Sabbatian connection with the last three US presidents I will lay out the crucial importance to Sabbatians of controlling banking and finance. Sabbatian Mayer Amschel Rothschild set out to dominate this arena in his family's quest for total global control. What is freedom? It is, in effect, choice. The more choices you have the freer you are and the fewer your choices the more you are enslaved. In the global structure created over centuries by Sabbatians the biggest decider and restrictor of choice is ... money. Across the world if you ask people what they would like to do with their lives and why they are not doing that they will reply 'I don't have the money'. This is the idea. A global elite of multi-billionaires are described as 'greedy' and that is true on one level; but control of money – who has it and who doesn't – is not primarily about greed. It's about control. Sabbatians have seized ever more control of finance and sucked the wealth of the world out of the hands of the population. We talk now, after all, about the 'One-percent' and even then the wealthiest are a lot fewer even than that. This has been made possible by a money scam so outrageous and so vast it could rightly be called the scam of scams founded on creating 'money' out of nothing and 'loaning' that with interest to the population. Money out of nothing is called 'credit'. Sabbatians have asserted control over governments and banking ever more completely through the centuries and secured financial laws that allow banks to lend hugely more than they have on deposit in a confidence trick known as fractional reserve lending. Imagine if you could lend money that doesn't exist and charge the recipient interest for doing so. You would end up in jail. Bankers by contrast end up in mansions, private jets, Malibu and Monaco.

Banks are only required to keep a fraction of their deposits and wealth in their vaults and they are allowed to lend 'money' they don't have called 'credit'. Go into a bank for a loan and if you succeed

the banker will not move any real wealth into your account. They will type into your account the amount of the agreed 'loan' – say £100,000. This is not wealth that really exists; it is non-existent, fresh-air, created-out-of-nothing 'credit' which has never, does not, and will never exist except in theory. Credit is backed by nothing except wind and only has buying power because people think that it has buying power and accept it in return for property, goods and services. I have described this situation as like those cartoon characters you see chasing each other and when they run over the edge of a cliff they keep running forward on fresh air until one of them looks down, realises what's happened, and they all crash into the ravine. The whole foundation of the Sabbatian financial system is to stop people looking down except for periodic moments when they want to crash the system (as in 2008 and 2020 ongoing) and reap the rewards from all the property, businesses and wealth their borrowers had signed over as 'collateral' in return for a 'loan' of fresh air. Most people think that money is somehow created by governments when it comes into existence from the start as a debt through banks 'lending' illusory money called credit. Yes, the very currency of exchange is a *debt* from day one issued as an interest-bearing loan. Why don't governments create money interest-free and lend it to their people interest-free? Governments are controlled by Sabbatians and the financial system is controlled by Sabbatians for whom interest-free money would be a nightmare come true. Sabbatians underpin their financial domination through their global network of central banks, including the privately-owned US Federal Reserve and Britain's Bank of England, and this is orchestrated by a privately-owned central bank coordination body called the Bank for International Settlements in Basle, Switzerland, created by the usual suspects including the Rockefellers and Rothschilds. Central bank chiefs don't answer to governments or the people. They answer to the Bank for International Settlements or, in other words, the Global Cult which is dominated today by Sabbatians.

## **Built-in disaster**

There are so many constituent scams within the overall banking scam. When you take out a loan of thin-air credit only the amount of that loan is theoretically brought into circulation to add to the amount in circulation; but you are paying back the principle plus interest. The additional interest is not created and this means that with every 'loan' there is a shortfall in the money in circulation between what is borrowed and what has to be paid back. There is never even close to enough money in circulation to repay all outstanding public and private debt including interest. Coldly weaved in the very fabric of the system is the certainty that some will lose their homes, businesses and possessions to the banking 'lender'. This is less obvious in times of 'boom' when the amount of money in circulation (and the debt) is expanding through more people wanting and getting loans. When a downturn comes and the money supply contracts it becomes painfully obvious that there is not enough money to service all debt and interest. This is less obvious in times of 'boom' when the amount of money in circulation (and the debt) is expanding through more people wanting and getting loans. When a downturn comes and the money supply contracts and it becomes painfully obvious – as in 2008 and currently – that there is not enough money to service all debt and interest.

Sabbatian banksters have been leading the human population through a calculated series of booms (more debt incurred) and busts (when the debt can't be repaid and the banks get the debtor's tangible wealth in exchange for non-existent 'credit'). With each 'bust' Sabbatian bankers have absorbed more of the world's tangible wealth and we end up with the One-percent. Governments are in bankruptcy levels of debt to the same system and are therefore owned by a system they do not control. The Federal Reserve, 'America's central bank', is privately-owned and American presidents only nominally appoint its chairman or woman to maintain the illusion that it's an arm of government. It's not. The 'Fed' is a cartel of private banks which handed billions to its associates and friends after the crash of 2008 and has been Sabbatian-controlled since it was manipulated into being in 1913 through the covert trickery of Rothschild banking agents Jacob Schiff and Paul

Warburg, and the Sabbatian Rockefeller family. Somehow from a Jewish population of two-percent and globally 0.2 percent (Sabbatian interlopers remember are far smaller) ultra-Zionists headed the Federal Reserve for 31 years between 1987 and 2018 in the form of Alan Greenspan, Bernard Bernanke and Janet Yellen (now Biden's Treasury Secretary) with Yellen's deputy chairman a Israeli-American dual citizen and ultra-Zionist Stanley Fischer, a former governor of the Bank of Israel. Ultra-Zionist Fed chiefs spanned the presidencies of Ronald Reagan ('Republican'), Father George Bush ('Republican'), Bill Clinton ('Democrat'), Boy George Bush ('Republican') and Barack Obama ('Democrat'). We should really add the pre-Greenspan chairman, Paul Adolph Volcker, 'appointed' by Jimmy Carter ('Democrat') who ran the Fed between 1979 and 1987 during the Carter and Reagan administrations before Greenspan took over. Volcker was a long-time associate and business partner of the Rothschilds. No matter what the 'party' officially in power the United States economy was directed by the same force. Here are members of the Obama, Trump and Biden administrations and see if you can make out a common theme.

## **Barack Obama ('Democrat')**

Ultra-Zionists Robert Rubin, Larry Summers, and Timothy Geithner ran the US Treasury in the Clinton administration and two of them reappeared with Obama. Ultra-Zionist Fed chairman Alan Greenspan had manipulated the crash of 2008 through deregulation and jumped ship just before the disaster to make way for ultra-Zionist Bernard Bernanke to hand out trillions to Sabbatian 'too big to fail' banks and businesses, including the ubiquitous ultra-Zionist Goldman Sachs which has an ongoing staff revolving door operation between itself and major financial positions in government worldwide. Obama inherited the fallout of the crash when he took office in January, 2009, and fortunately he had the support of his ultra-Zionist White House Chief of Staff Rahm Emmanuel, son of a terrorist who helped to bomb Israel into being in 1948, and his ultra-Zionist senior adviser David Axelrod, chief strategist in Obama's two

successful presidential campaigns. Emmanuel, later mayor of Chicago and former senior fundraiser and strategist for Bill Clinton, is an example of the Sabbatian policy after Israel was established of migrating insider families to America so their children would be born American citizens. ‘Obama’ chose this financial team throughout his administration to respond to the Sabbatian-instigated crisis:

Timothy Geithner (ultra-Zionist) Treasury Secretary; Jacob J. Lew, Treasury Secretary; Larry Summers (ultra-Zionist), director of the White House National Economic Council; Paul Adolph Volcker (Rothschild business partner), chairman of the Economic Recovery Advisory Board; Peter Orszag (ultra-Zionist), director of the Office of Management and Budget overseeing all government spending; Penny Pritzker (ultra-Zionist), Commerce Secretary; Jared Bernstein (ultra-Zionist), chief economist and economic policy adviser to Vice President Joe Biden; Mary Schapiro (ultra-Zionist), chair of the Securities and Exchange Commission (SEC); Gary Gensler (ultra-Zionist), chairman of the Commodity Futures Trading Commission (CFTC); Sheila Bair (ultra-Zionist), chair of the Federal Deposit Insurance Corporation (FDIC); Karen Mills (ultra-Zionist), head of the Small Business Administration (SBA); Kenneth Feinberg (ultra-Zionist), Special Master for Executive [bail-out] Compensation. Feinberg would be appointed to oversee compensation (with strings) to 9/11 victims and families in a campaign to stop them having their day in court to question the official story. At the same time ultra-Zionist Bernard Bernanke was chairman of the Federal Reserve and these are only some of the ultra-Zionists with allegiance to Sabbatian-controlled Israel in the Obama government. Obama’s biggest corporate donor was ultra-Zionist Goldman Sachs which had employed many in his administration.

## **Donald Trump ('Republican')**

Trump claimed to be an outsider (he wasn’t) who had come to ‘drain the swamp’. He embarked on this goal by immediately appointing ultra-Zionist Steve Mnuchin, a Goldman Sachs employee for 17

years, as his Treasury Secretary. Others included Gary Cohn (ultra-Zionist), chief operating officer of Goldman Sachs, his first Director of the National Economic Council and chief economic adviser, who was later replaced by Larry Kudlow (ultra-Zionist). Trump's senior adviser throughout his four years in the White House was his sinister son-in-law Jared Kushner, a life-long friend of Israel Prime Minister Benjamin Netanyahu. Kushner is the son of a convicted crook who was pardoned by Trump in his last days in office. Other ultra-Zionists in the Trump administration included: Stephen Miller, Senior Policy Adviser; Avrahm Berkowitz, Deputy Adviser to Trump and his Senior Adviser Jared Kushner; Ivanka Trump, Adviser to the President, who converted to Judaism when she married Jared Kushner; David Friedman, Trump lawyer and Ambassador to Israel; Jason Greenblatt, Trump Organization executive vice president and chief legal officer, who was made Special Representative for International Negotiations and the Israeli-Palestinian Conflict; Rod Rosenstein, Deputy Attorney General; Elliot Abrams, Special Representative for Venezuela, then Iran; John Eisenberg, National Security Council Legal Adviser and Deputy Council to the President for National Security Affairs; Anne Neuberger, Deputy National Manager, National Security Agency; Ezra Cohen-Watnick, Acting Under Secretary of Defense for Intelligence; Elan Carr, Special Envoy to monitor and combat anti-Semitism; Len Khodorkovsky, Deputy Special Envoy to monitor and combat anti-Semitism; Reed Cordish, Assistant to the President, Intragovernmental and Technology Initiatives. Trump Vice President Mike Pence and Secretary of State Mike Pompeo, both Christian Zionists, were also vehement supporters of Israel and its goals and ambitions.

Donald 'free-speech believer' Trump pardoned a number of financial and violent criminals while ignoring calls to pardon Julian Assange and Edward Snowden whose crimes are revealing highly relevant information about government manipulation and corruption and the widespread illegal surveillance of the American people by US 'security' agencies. It's so good to know that Trump is on the side of freedom and justice and not mega-criminals with

allegiance to Sabbatian-controlled Israel. These included a pardon for Israeli spy Jonathan Pollard who was jailed for life in 1987 under the Espionage Act. Aviem Sella, the Mossad agent who recruited Pollard, was also pardoned by Trump while Assange sat in jail and Snowden remained in exile in Russia. Sella had 'fled' (was helped to escape) to Israel in 1987 and was never extradited despite being charged under the Espionage Act. A Trump White House statement said that Sella's clemency had been 'supported by Benjamin Netanyahu, Ron Dermer, Israel's US Ambassador, David Friedman, US Ambassador to Israel and Miriam Adelson, wife of leading Trump donor Sheldon Adelson who died shortly before. Other friends of Jared Kushner were pardoned along with Sholom Weiss who was believed to be serving the longest-ever white-collar prison sentence of more than 800 years in 2000. The sentence was commuted of Ponzi-schemer Eliyahu Weinstein who defrauded Jews and others out of \$200 million. I did mention that Assange and Snowden were ignored, right? Trump gave Sabbatians almost everything they asked for in military and political support, moving the US Embassy from Tel Aviv to Jerusalem with its critical symbolic and literal implications for Palestinian statehood, and the 'deal of the Century' designed by Jared Kushner and David Friedman which gave the Sabbatian Israeli government the green light to substantially expand its already widespread program of building illegal Jewish-only settlements in the occupied land of the West Bank. This made a two-state 'solution' impossible by seizing all the land of a potential Palestinian homeland and that had been the plan since 1948 and then 1967 when the Arab-controlled Gaza Strip, West Bank, Sinai Peninsula and Syrian Golan Heights were occupied by Israel. All the talks about talks and road maps and delays have been buying time until the West Bank was physically occupied by Israeli real estate. Trump would have to be a monumentally ill-informed idiot not to see that this was the plan he was helping to complete. The Trump administration was in so many ways the Kushner administration which means the Netanyahu administration which means the Sabbatian administration. I understand why many opposing Cult fascism in all its forms gravitated to Trump, but he

was a crucial part of the Sabbatian plan and I will deal with this in the next chapter.

## **Joe Biden ('Democrat')**

A barely cognitive Joe Biden took over the presidency in January, 2021, along with his fellow empty shell, Vice-President Kamala Harris, as the latest Sabbatian gofers to enter the White House. Names on the door may have changed and the 'party' – the force behind them remained the same as Zionists were appointed to a stream of pivotal areas relating to Sabbatian plans and policy. They included: Janet Yellen, Treasury Secretary, former head of the Federal Reserve, and still another ultra-Zionist running the US Treasury after Mnuchin (Trump), Lew and Geithner (Obama), and Summers and Rubin (Clinton); Anthony Blinken, Secretary of State; Wendy Sherman, Deputy Secretary of State (so that's 'Biden's' Sabbatian foreign policy sorted); Jeff Zients, White House coronavirus coordinator; Rochelle Walensky, head of the Centers for Disease Control; Rachel Levine, transgender deputy health secretary (that's 'Covid' hoax policy under control); Merrick Garland, Attorney General; Alejandro Mayorkas, Secretary of Homeland Security; Cass Sunstein, Homeland Security with responsibility for new immigration laws; Avril Haines, Director of National Intelligence; Anne Neuberger, National Security Agency cybersecurity director (note, cybersecurity); David Cohen, CIA Deputy Director; Ronald Klain, Biden's Chief of Staff (see Rahm Emanuel); Eric Lander, a 'leading geneticist', Office of Science and Technology Policy director (see Smart Grid, synthetic biology agenda); Jessica Rosenworcel, acting head of the Federal Communications Commission (FCC) which controls Smart Grid technology policy and electromagnetic communication systems including 5G. How can it be that so many pivotal positions are held by two-percent of the American population and 0.2 percent of the world population administration after administration no matter who is the president and what is the party? It's a coincidence? Of course it's not and this is why Sabbatians have built their colossal global web of interlocking 'anti-

hate' hate groups to condemn anyone who asks these glaring questions as an 'anti-Semite'. The way that Jewish people horrifically abused in Sabbatian-backed Nazi Germany are exploited to this end is stomach-turning and disgusting beyond words.

## **Political fusion**

Sabbatian manipulation has reversed the roles of Republicans and Democrats and the same has happened in Britain with the Conservative and Labour Parties. Republicans and Conservatives were always labelled the 'right' and Democrats and Labour the 'left', but look at the policy positions now and the Democrat-Labour 'left' has moved further to the 'right' than Republicans and Conservatives under the banner of 'Woke', the Cult-created far-right tyranny. Where once the Democrat-Labour 'left' defended free speech and human rights they now seek to delete them and as I said earlier despite the 'Covid' fascism of the Jackboot Johnson Conservative government in the UK the Labour Party of leader Keir Starmer demanded even more extreme measures. The Labour Party has been very publicly absorbed by Sabbatians after a political and media onslaught against the previous leader, the weak and inept Jeremy Corbyn, over made-up allegations of 'anti-Semitism' both by him and his party. The plan was clear with this 'anti-Semite' propaganda and what was required in response was a swift and decisive 'fuck off' from Corbyn and a statement to expose the Anti-Semitism Industry (Sabbatian) attempt to silence Labour criticism of the Israeli government (Sabbatians) and purge the party of all dissent against the extremes of ultra-Zionism (Sabbatians). Instead Corbyn and his party fell to their knees and appeased the abusers which, by definition, is impossible. Appeasing one demand leads only to a new demand to be appeased until takeover is complete. Like I say – 'fuck off' would have been a much more effective policy and I have used it myself with great effect over the years when Sabbatians are on my case which is most of the time. I consider that fact a great compliment, by the way. The outcome of the Labour Party capitulation is that we now have a Sabbatian-controlled

Conservative Party ‘opposed’ by a Sabbatian-controlled Labour Party in a one-party Sabbatian state that hurtles towards the extremes of tyranny (the Sabbatian cult agenda). In America the situation is the same. Labour’s Keir Starmer spends his days on his knees with his tongue out pointing to Tel Aviv, or I guess now Jerusalem, while Boris Johnson has an ‘anti-Semitism czar’ in the form of former Labour MP John Mann who keeps Starmer company on his prayer mat.

Sabbatian influence can be seen in Jewish members of the Labour Party who have been ejected for criticism of Israel including those from families that suffered in Nazi Germany. Sabbatians despise real Jewish people and target them even more harshly because it is so much more difficult to dub them ‘anti-Semitic’ although in their desperation they do try.

## CHAPTER THREE

### The Pushbacker sting

*Until you realize how easy it is for your mind to be manipulated, you remain the puppet of someone else's game*

Evita Ochel

I will use the presidencies of Trump and Biden to show how the manipulation of the one-party state plays out behind the illusion of political choice across the world. No two presidencies could – on the face of it – be more different and apparently at odds in terms of direction and policy.

A Renegade Mind sees beyond the obvious and focuses on outcomes and consequences and not image, words and waffle. The Cult embarked on a campaign to divide America between those who blindly support its agenda (the mentality known as 'Woke') and those who are pushing back on where the Cult and its Sabbatians want to go. This presents infinite possibilities for dividing and ruling the population by setting them at war with each other and allows a perceptual ring fence of demonisation to encircle the Pushbackers in a modern version of the Little Big Horn in 1876 when American cavalry led by Lieutenant Colonel George Custer were drawn into a trap, surrounded and killed by Native American tribes defending their land of thousands of years from being seized by the government. In this modern version the roles are reversed and it's those defending themselves from the Sabbatian government who are surrounded and the government that's seeking to destroy them. This trap was set years ago and to explain how we must return to 2016

and the emergence of Donald Trump as a candidate to be President of the United States. He set out to overcome the best part of 20 other candidates in the Republican Party before and during the primaries and was not considered by many in those early stages to have a prayer of living in the White House. The Republican Party was said to have great reservations about Trump and yet somehow he won the nomination. When you know how American politics works – politics in general – there is no way that Trump could have become the party's candidate unless the Sabbatian-controlled 'Neocons' that run the Republican Party wanted that to happen. We saw the proof in emails and documents made public by WikiLeaks that the Democratic Party hierarchy, or Democons, systematically undermined the campaign of Bernie Sanders to make sure that Sabbatian gofer Hillary Clinton won the nomination to be their presidential candidate. If the Democons could do that then the Neocons in the Republican Party could have derailed Trump in the same way. But they didn't and at that stage I began to conclude that Trump could well be the one chosen to be president. If that was the case the 'why' was pretty clear to see – the goal of dividing America between Cult agenda-supporting Wokers and Pushbackers who gravitated to Trump because he was telling them what they wanted to hear. His constituency of support had been increasingly ignored and voiceless for decades and profoundly through the eight years of Sabbatian puppet Barack Obama. Now here was someone speaking their language of pulling back from the incessant globalisation of political and economic power, the exporting of American jobs to China and elsewhere by 'American' (Sabbatian) corporations, the deletion of free speech, and the mass immigration policies that had further devastated job opportunities for the urban working class of all races and the once American heartlands of the Midwest.

### **Beware the forked tongue**

Those people collectively sighed with relief that at last a political leader was apparently on their side, but another trait of the Renegade Mind is that you look even harder at people telling you

what you want to hear than those who are telling you otherwise. Obviously as I said earlier people wish what they want to hear to be true and genuine and they are much more likely to believe that than someone saying what they don't want to here and don't want to be true. Sales people are taught to be skilled in eliciting by calculated questioning what their customers want to hear and repeating that back to them as their own opinion to get their targets to like and trust them. Assets of the Cult are also sales people in the sense of selling perception. To read Cult manipulation you have to play the long and expanded game and not fall for the Vaudeville show of party politics. Both American parties are vehicles for the Cult and they exploit them in different ways depending on what the agenda requires at that moment. Trump and the Republicans were used to be the focus of dividing America and isolating Pushbackers to open the way for a Biden presidency to become the most extreme in American history by advancing the full-blown Woke (Cult) agenda with the aim of destroying and silencing Pushbackers now labelled Nazi Trump supporters and white supremacists.

Sabbatians wanted Trump in office for the reasons described by ultra-Zionist Saul Alinsky (1909-1972) who was promoting the Woke philosophy through 'community organising' long before anyone had heard of it. In those days it still went by its traditional name of Marxism. The reason for the manipulated Trump phenomenon was laid out in Alinsky's 1971 book, *Rules for Radicals*, which was his blueprint for overthrowing democratic and other regimes and replacing them with Sabbatian Marxism. Not surprisingly his to-do list was evident in the Sabbatian French and Russian 'Revolutions' and that in China which will become very relevant in the next chapter about the 'Covid' hoax. Among Alinsky's followers have been the deeply corrupt Barack Obama, House Speaker Nancy Pelosi and Hillary Clinton who described him as a 'hero'. All three are Sabbatian stooges with Pelosi personifying the arrogant corrupt idiocy that so widely fronts up for the Cult inner core. Predictably as a Sabbatian advocate of the 'light-bringer' Alinsky features Lucifer on the dedication page of his book as the original radical who gained

his own kingdom ('Earth' as we shall see). One of Alinsky's golden radical rules was to pick an individual and focus all attention, hatred and blame on them and not to target faceless bureaucracies and corporations. *Rules for Radicals* is really a Sabbatian handbook with its contents repeatedly employed all over the world for centuries and why wouldn't Sabbatians bring to power their designer-villain to be used as the individual on which all attention, hatred and blame was bestowed? This is what they did and the only question for me is how much Trump knew that and how much he was manipulated. A bit of both, I suspect. This was Alinsky's Trump technique from a man who died in 1972. The technique has spanned history:

Pick the target, freeze it, personalize it, polarize it. Don't try to attack abstract corporations or bureaucracies. Identify a responsible individual. Ignore attempts to shift or spread the blame.

From the moment Trump came to illusory power everything was about him. It wasn't about Republican policy or opinion, but all about Trump. Everything he did was presented in negative, derogatory and abusive terms by the Sabbatian-dominated media led by Cult operations such as CNN, MSNBC, *The New York Times* and the Jeff Bezos-owned *Washington Post* – 'Pick the target, freeze it, personalize it, polarize it.' Trump was turned into a demon to be vilified by those who hated him and a demi-god loved by those who worshipped him. This, in turn, had his supporters, too, presented as equally demonic in preparation for the punchline later down the line when Biden was about to take office. It was here's a Trump, there's a Trump, everywhere a Trump, Trump. Virtually every news story or happening was filtered through the lens of 'The Donald'. You loved him or hated him and which one you chose was said to define you as Satan's spawn or a paragon of virtue. Even supporting some Trump policies or statements and not others was enough for an assault on your character. No shades of grey were or are allowed. Everything is black and white (literally and figuratively). A Californian I knew had her head utterly scrambled by her hatred for Trump while telling people they should love each other. She was so totally consumed by

Trump Derangement Syndrome as it became to be known that this glaring contradiction would never have occurred to her. By definition anyone who criticised Trump or praised his opponents was a hero and this lady described Joe Biden as 'a kind, honest gentleman' when he's a provable liar, mega-crook and vicious piece of work to boot. Sabbatians had indeed divided America using Trump as the fall-guy and all along the clock was ticking on the consequences for his supporters.

### **In hock to his masters**

Trump gave Sabbatians via Israel almost everything they wanted in his four years. Ask and you shall receive was the dynamic between himself and Benjamin Netanyahu orchestrated by Trump's ultra-Zionist son-in-law Jared Kushner, his ultra-Zionist Ambassador to Israel, David Friedman, and ultra-Zionist 'Israel adviser', Jason Greenblatt. The last two were central to the running and protecting from collapse of his business empire, the Trump Organisation, and colossal business failures made him forever beholden to Sabbatian networks that bailed him out. By the start of the 1990s Trump owed \$4 billion to banks that he couldn't pay and almost \$1 billion of that was down to him personally and not his companies. This mega-disaster was the result of building two new casinos in Atlantic City and buying the enormous Taj Mahal operation which led to crippling debt payments. He had borrowed fantastic sums from 72 banks with major Sabbatian connections and although the scale of debt should have had him living in a tent alongside the highway they never foreclosed. A plan was devised to lift Trump from the mire by BT Securities Corporation and Rothschild Inc. and the case was handled by Wilber Ross who had worked for the Rothschilds for 27 years. Ross would be named US Commerce Secretary after Trump's election. Another crucial figure in saving Trump was ultra-Zionist 'investor' Carl Icahn who bought the Taj Mahal casino. Icahn was made special economic adviser on financial regulation in the Trump administration. He didn't stay long but still managed to find time to make a tidy sum of a reported \$31.3 million when he sold his

holdings affected by the price of steel three days before Trump imposed a 235 percent tariff on steel imports. What amazing bits of luck these people have. Trump and Sabbatian operatives have long had a close association and his mentor and legal adviser from the early 1970s until 1986 was the dark and genetically corrupt ultra-Zionist Roy Cohn who was chief counsel to Senator Joseph McCarthy's 'communist' witch-hunt in the 1950s. *Esquire* magazine published an article about Cohn with the headline 'Don't mess with Roy Cohn'. He was described as the most feared lawyer in New York and 'a ruthless master of dirty tricks ... [with] ... more than one Mafia Don on speed dial'. Cohn's influence, contacts, support and protection made Trump a front man for Sabbatians in New York with their connections to one of Cohn's many criminal employers, the 'Russian' Sabbatian Mafia. Israel-centric media mogul Rupert Murdoch was introduced to Trump by Cohn and they started a long friendship. Cohn died in 1986 weeks after being disbarred for unethical conduct by the Appellate Division of the New York State Supreme Court. The wheels of justice do indeed run slow given the length of Cohn's crooked career.

## **QAnon-sense**

We are asked to believe that Donald Trump with his fundamental connections to Sabbatian networks and operatives has been leading the fight to stop the Sabbatian agenda for the fascistic control of America and the world. Sure he has. A man entrapped during his years in the White House by Sabbatian operatives and whose biggest financial donor was casino billionaire Sheldon Adelson who was Sabbatian to his DNA?? Oh, do come on. Trump has been used to divide America and isolate Pushbackers on the Cult agenda under the heading of 'Trump supporters', 'insurrectionists' and 'white supremacists'. The US Intelligence/Mossad Psyop or psychological operation known as QAnon emerged during the Trump years as a central pillar in the Sabbatian campaign to lead Pushbackers into the trap set by those that wished to destroy them. I knew from the start that QAnon was a scam because I had seen the same scenario many

times before over 30 years under different names and I had written about one in particular in the books. ‘Not again’ was my reaction when QAnon came to the fore. The same script is pulled out every few years and a new name added to the letterhead. The story always takes the same form: ‘Insiders’ or ‘the good guys’ in the government-intelligence-military ‘Deep State’ apparatus were going to instigate mass arrests of the ‘bad guys’ which would include the Rockefellers, Rothschilds, Barack Obama, Hillary Clinton, George Soros, etc., etc. Dates are given for when the ‘good guys’ are going to move in, but the dates pass without incident and new dates are given which pass without incident. The central message to Pushbackers in each case is that they don’t have to do anything because there is ‘a plan’ and it is all going to be sorted by the ‘good guys’ on the inside. ‘Trust the plan’ was a QAnon mantra when the only plan was to misdirect Pushbackers into putting their trust in a Psyop they believed to be real. Beware, beware, those who tell you what you want to hear and always check it out. Right up to Biden’s inauguration QAnon was still claiming that ‘the Storm’ was coming and Trump would stay on as president when Biden and his cronies were arrested and jailed. It was never going to happen and of course it didn’t, but what did happen as a result provided that punchline to the Sabbatian Trump/QAnon Psyop.

On January 6th, 2021, a very big crowd of Trump supporters gathered in the National Mall in Washington DC down from the Capitol Building to protest at what they believed to be widespread corruption and vote fraud that stopped Trump being re-elected for a second term as president in November, 2020. I say as someone that does not support Trump or Biden that the evidence is clear that major vote-fixing went on to favour Biden, a man with cognitive problems so advanced he can often hardly string a sentence together without reading the words written for him on the Teleprompter. Glaring ballot discrepancies included serious questions about electronic voting machines that make vote rigging a comparative cinch and hundreds of thousands of paper votes that suddenly appeared during already advanced vote counts and virtually all of

them for Biden. Early Trump leads in crucial swing states suddenly began to close and disappear. The pandemic hoax was used as the excuse to issue almost limitless numbers of mail-in ballots with no checks to establish that the recipients were still alive or lived at that address. They were sent to streams of people who had not even asked for them. Private organisations were employed to gather these ballots and who knows what they did with them before they turned up at the counts. The American election system has been manipulated over decades to become a sick joke with more holes than a Swiss cheese for the express purpose of dictating the results. Then there was the criminal manipulation of information by Sabbatian tech giants like Facebook, Twitter and Google-owned YouTube which deleted pro-Trump, anti-Biden accounts and posts while everything in support of Biden was left alone. Sabbatians wanted Biden to win because after the dividing of America it was time for full-on Woke and every aspect of the Cult agenda to be unleashed.

### **Hunter gatherer**

Extreme Silicon Valley bias included blocking information by the *New York Post* exposing a Biden scandal that should have ended his bid for president in the final weeks of the campaign. Hunter Biden, his monumentally corrupt son, is reported to have sent a laptop to be repaired at a local store and failed to return for it. Time passed until the laptop became the property of the store for non-payment of the bill. When the owner saw what was on the hard drive he gave a copy to the FBI who did nothing even though it confirmed widespread corruption in which the Joe Biden family were using his political position, especially when he was vice president to Obama, to make multiple millions in countries around the world and most notably Ukraine and China. Hunter Biden's one-time business partner Tony Bobulinski went public when the story broke in the *New York Post* to confirm the corruption he saw and that Joe Biden not only knew what was going on he also profited from the spoils. Millions were handed over by a Chinese company with close

connections – like all major businesses in China – to the Chinese communist party of President Xi Jinping. Joe Biden even boasted at a meeting of the Cult's World Economic Forum that as vice president he had ordered the government of Ukraine to fire a prosecutor. What he didn't mention was that the same man just happened to be investigating an energy company which was part of Hunter Biden's corrupt portfolio. The company was paying him big bucks for no other reason than the influence his father had. Overnight Biden's presidential campaign should have been over given that he had lied publicly about not knowing what his son was doing. Instead almost the entire Sabbatian-owned mainstream media and Sabbatian-owned Silicon Valley suppressed circulation of the story. This alone went a mighty way to rigging the election of 2020. Cult assets like Mark Zuckerberg at Facebook also spent hundreds of millions to be used in support of Biden and vote 'administration'.

The Cult had used Trump as the focus to divide America and was now desperate to bring in moronic, pliable, corrupt Biden to complete the double-whammy. No way were they going to let little things like the will of the people thwart their plan. Silicon Valley widely censored claims that the election was rigged because it *was* rigged. For the same reason anyone claiming it was rigged was denounced as a 'white supremacist' including the pathetically few Republican politicians willing to say so. Right across the media where the claim was mentioned it was described as a 'false claim' even though these excuses for 'journalists' would have done no research into the subject whatsoever. Trump won seven million more votes than any sitting president had ever achieved while somehow a cognitively-challenged soon to be 78-year-old who was hidden away from the public for most of the campaign managed to win more votes than any presidential candidate in history. It makes no sense. You only had to see election rallies for both candidates to witness the enthusiasm for Trump and the apathy for Biden. Tens of thousands would attend Trump events while Biden was speaking in empty car parks with often only television crews attending and framing their shots to hide the fact that no one was there. It was pathetic to see

footage come to light of Biden standing at a podium making speeches only to TV crews and party fixers while reading the words written for him on massive Teleprompter screens. So, yes, those protestors on January 6th had a point about election rigging, but some were about to walk into a trap laid for them in Washington by the Cult Deep State and its QAnon Psyop. This was the Capitol Hill riot ludicrously dubbed an ‘insurrection’.

## **The spider and the fly**

Renegade Minds know there are not two ‘sides’ in politics, only one side, the Cult, working through all ‘sides’. It’s a stage show, a puppet show, to direct the perceptions of the population into focusing on diversions like parties and candidates while missing the puppeteers with their hands holding all the strings. The Capitol Hill ‘insurrection’ brings us back to the Little Big Horn. Having created two distinct opposing groupings – Woke and Pushbackers – the trap was about to be sprung. Pushbackers were to be encircled and isolated by associating them all in the public mind with Trump and then labelling Trump as some sort of Confederate leader. I knew immediately that the Capitol riot was a set-up because of two things. One was how easy the rioters got into the building with virtually no credible resistance and secondly I could see – as with the ‘Covid’ hoax in the West at the start of 2020 – how the Cult could exploit the situation to move its agenda forward with great speed. My experience of Cult techniques and activities over more than 30 years has showed me that while they do exploit situations they haven’t themselves created this never happens with events of fundamental agenda significance. Every time major events giving cultists the excuse to rapidly advance their plan you find they are manipulated into being for the specific reason of providing that excuse – Problem-Reaction-Solution. Only a tiny minority of the huge crowd of Washington protestors sought to gain entry to the Capitol by smashing windows and breaching doors. That didn’t matter. The whole crowd and all Pushbackers, even if they did not support Trump, were going to be lumped together as dangerous

insurrectionists and conspiracy theorists. The latter term came into widespread use through a CIA memo in the 1960s aimed at discrediting those questioning the nonsensical official story of the Kennedy assassination and it subsequently became widely employed by the media. It's still being used by inept 'journalists' with no idea of its origin to discredit anyone questioning anything that authority claims to be true. When you are perpetrating a conspiracy you need to discredit the very word itself even though the dictionary definition of conspiracy is merely 'the activity of secretly planning with other people to do something bad or illegal' and 'a general agreement to keep silent about a subject for the purpose of keeping it secret'. On that basis there are conspiracies almost wherever you look. For obvious reasons the Cult and its lapdog media have to claim there are no conspiracies even though the word appears in state laws as with conspiracy to defraud, to murder, and to corrupt public morals.

Agent provocateurs are widely used by the Cult Deep State to manipulate genuine people into acting in ways that suit the desired outcome. By genuine in this case I mean protestors genuinely supporting Trump and claims that the election was stolen. In among them, however, were agents of the state wearing the garb of Trump supporters and QAnon to pump-prime the Capitol riot which some genuine Trump supporters naively fell for. I described the situation as 'Come into my parlour said the spider to the fly'. Leaflets appeared through the Woke paramilitary arm Antifa, the anti-fascist fascists, calling on supporters to turn up in Washington looking like Trump supporters even though they hated him. Some of those arrested for breaching the Capitol Building were sourced to Antifa and its stable mate Black Lives Matter. Both organisations are funded by Cult billionaires and corporations. One man charged for the riot was according to his lawyer a former FBI agent who had held top secret security clearance for 40 years. Attorney Thomas Plofchan said of his client, 66-year-old Thomas Edward Caldwell:

He has held a Top Secret Security Clearance since 1979 and has undergone multiple Special Background Investigations in support of his clearances. After retiring from the Navy, he

worked as a section chief for the Federal Bureau of Investigation from 2009-2010 as a GS-12 [mid-level employee].

He also formed and operated a consulting firm performing work, often classified, for U.S government customers including the US Drug Enforcement Agency, Department of Housing and Urban Development, the US Coast Guard, and the US Army Personnel Command.

A judge later released Caldwell pending trial in the absence of evidence about a conspiracy or that he tried to force his way into the building. *The New York Post* reported a 'law enforcement source' as saying that 'at least two known Antifa members were spotted' on camera among Trump supporters during the riot while one of the rioters arrested was John Earle Sullivan, a seriously extreme Black Lives Matter Trump-hater from Utah who was previously arrested and charged in July, 2020, over a BLM-Antifa riot in which drivers were threatened and one was shot. Sullivan is the founder of Utah-based Insurgence USA which is an affiliate of the Cult-created-and-funded Black Lives Matter movement. Footage appeared and was then deleted by Twitter of Trump supporters calling out Antifa infiltrators and a group was filmed changing into pro-Trump clothing before the riot. Security at the building was *pathetic* – as planned. Colonel Leroy Fletcher Prouty, a man with long experience in covert operations working with the US security apparatus, once described the tell-tale sign to identify who is involved in an assassination. He said:

No one has to direct an assassination – it happens. The active role is played secretly by permitting it to happen. This is the greatest single clue. Who has the power to call off or reduce the usual security precautions?

This principle applies to many other situations and certainly to the Capitol riot of January 6th, 2021.

## **The sting**

With such a big and potentially angry crowd known to be gathering near the Capitol the security apparatus would have had a major police detail to defend the building with National Guard troops on

standby given the strength of feeling among people arriving from all over America encouraged by the QAnon Psyop and statements by Donald Trump. Instead Capitol Police ‘security’ was flimsy, weak, and easily breached. The same number of officers was deployed as on a regular day and that is a blatant red flag. They were not staffed or equipped for a possible riot that had been an obvious possibility in the circumstances. No protective and effective fencing worth the name was put in place and there were no contingency plans. The whole thing was basically a case of standing aside and waving people in. Once inside police mostly backed off apart from one Capitol police officer who ridiculously shot dead unarmed Air Force veteran protestor Ashli Babbitt without a warning as she climbed through a broken window. The ‘investigation’ refused to name or charge the officer after what must surely be considered a murder in the circumstances. They just lifted a carpet and swept. The story was endlessly repeated about five people dying in the ‘armed insurrection’ when there was no report of rioters using weapons. Apart from Babbitt the other four died from a heart attack, strokes and apparently a drug overdose. Capitol police officer Brian Sicknick was reported to have died after being bludgeoned with a fire extinguisher when he was alive after the riot was over and died later of what the Washington Medical Examiner’s Office said was a stroke. Sicknick had no external injuries. The lies were delivered like rapid fire. There was a narrative to build with incessant repetition of the lie until the lie became the accepted ‘everybody knows that’ truth. The ‘Big Lie’ technique of Nazi Propaganda Minister Joseph Goebbels is constantly used by the Cult which was behind the Nazis and is today behind the ‘Covid’ and ‘climate change’ hoaxes. Goebbels said:

If you tell a lie big enough and keep repeating it, people will eventually come to believe it. The lie can be maintained only for such time as the State can shield the people from the political, economic and/or military consequences of the lie. It thus becomes vitally important for the State to use all of its powers to repress dissent, for the truth is the mortal enemy of the lie, and thus by extension, the truth is the greatest enemy of the State.

Most protestors had a free run of the Capitol Building. This allowed pictures to be taken of rioters in iconic parts of the building including the Senate chamber which could be used as propaganda images against all Pushbackers. One Congresswoman described the scene as ‘the worst kind of non-security anybody could ever imagine’. Well, the first part was true, but someone obviously did imagine it and made sure it happened. Some photographs most widely circulated featured people wearing QAnon symbols and now the Psyop would be used to dub all QAnon followers with the ubiquitous fit-all label of ‘white supremacist’ and ‘insurrectionists’. When a Muslim extremist called Noah Green drove his car at two police officers at the Capitol Building killing one in April, 2021, there was no such political and media hysteria. They were just disappointed he wasn’t white.

## **The witch-hunt**

Government prosecutor Michael Sherwin, an aggressive, dark-eyed, professional Rottweiler led the ‘investigation’ and to call it over the top would be to underestimate reality a thousand fold. Hundreds were tracked down and arrested for the crime of having the wrong political views and people were jailed who had done nothing more than walk in the building, committed no violence or damage to property, took a few pictures and left. They were labelled a ‘threat to the Republic’ while Biden sat in the White House signing executive orders written for him that were dismantling ‘the Republic’. Even when judges ruled that a mother and son should not be in jail the government kept them there. Some of those arrested have been badly beaten by prison guards in Washington and lawyers for one man said he suffered a fractured skull and was made blind in one eye. Meanwhile a woman is shot dead for no reason by a Capitol Police officer and we are not allowed to know who he is never mind what has happened to him although that will be *nothing*. The Cult’s QAnon/Trump sting to identify and isolate Pushbackers and then target them on the road to crushing and deleting them was a resounding success. You would have thought the Russians had

invaded the building at gunpoint and lined up senators for a firing squad to see the political and media reaction. Congresswoman Alexandria Ocasio-Cortez is a child in a woman's body, a terrible-twins, me, me, me, Woker narcissist of such proportions that words have no meaning. She said she thought she was going to die when 'insurrectionists' banged on her office door. It turned out she wasn't even in the Capitol Building when the riot was happening and the 'banging' was a Capitol Police officer. She referred to herself as a 'survivor' which is an insult to all those true survivors of violent and sexual abuse while she lives her pampered and privileged life talking drivel for a living. Her Woke colleague and fellow mega-narcissist Rashida Tlaib broke down describing the devastating effect on her, too, of *not being* in the building when the rioters were there. Ocasio-Cortez and Tlaib are members of a fully-Woke group of Congresswomen known as 'The Squad' along with Ilhan Omar and Ayanna Pressley. The Squad from what I can see can be identified by its vehement anti-white racism, anti-white men agenda, and, as always in these cases, the absence of brain cells on active duty.

The usual suspects were on the riot case immediately in the form of Democrat ultra-Zionist senators and operatives Chuck Schumer and Adam Schiff demanding that Trump be impeached for 'his part in the insurrection'. The same pair of prats had led the failed impeachment of Trump over the invented 'Russia collusion' nonsense which claimed Russia had helped Trump win the 2016 election. I didn't realise that Tel Aviv had been relocated just outside Moscow. I must find an up-to-date map. The Russia hoax was a Sabbatian operation to keep Trump occupied and impotent and to stop any rapport with Russia which the Cult wants to retain as a perceptual enemy to be pulled out at will. Puppet Biden began attacking Russia when he came to office as the Cult seeks more upheaval, division and war across the world. A two-year stage show 'Russia collusion inquiry' headed by the not-very-bright former 9/11 FBI chief Robert Mueller, with support from 19 lawyers, 40 FBI agents plus intelligence analysts, forensic accountants and other

staff, devoured tens of millions of dollars and found no evidence of Russia collusion which a ten-year-old could have told them on day one. Now the same moronic Schumer and Schiff wanted a second impeachment of Trump over the Capitol ‘insurrection’ (riot) which the arrested development of Schumer called another ‘Pearl Harbor’ while others compared it with 9/11 in which 3,000 died and, in the case of CNN, with the Rwandan genocide in the 1990s in which an estimated 500,000 to 600,000 were murdered, between 250, 000 and 500,000 women were raped, and populations of whole towns were hacked to death with machetes. To make those comparisons purely for Cult political reasons is beyond insulting to those that suffered and lost their lives and confirms yet again the callous inhumanity that we are dealing with. Schumer is a monumental idiot and so is Schiff, but they serve the Cult agenda and do whatever they’re told so they get looked after. Talking of idiots – another inane man who spanned the Russia and Capitol impeachment attempts was Senator Eric Swalwell who had the nerve to accuse Trump of collusion with the Russians while sleeping with a Chinese spy called Christine Fang or ‘Fang Fang’ which is straight out of a Bond film no doubt starring Klaus Schwab as the bloke living on a secret island and controlling laser weapons positioned in space and pointing at world capitals. Fang Fang plays the part of Bond’s infiltrator girlfriend which I’m sure she would enjoy rather more than sharing a bed with the brainless Swalwell, lying back and thinking of China. The FBI eventually warned Swalwell about Fang Fang which gave her time to escape back to the Chinese dictatorship. How very thoughtful of them. The second Trump impeachment also failed and hardly surprising when an impeachment is supposed to remove a sitting president and by the time it happened Trump was no longer president. These people are running your country America, well, officially anyway. Terrifying isn’t it?

## **Outcomes tell the story - always**

The outcome of all this – and it’s the *outcome* on which Renegade Minds focus, not the words – was that a vicious, hysterical and

obviously pre-planned assault was launched on Pushbackers to censor, silence and discredit them and even targeted their right to earn a living. They have since been condemned as 'domestic terrorists' that need to be treated like Al-Qaeda and Islamic State. 'Domestic terrorists' is a label the Cult has been trying to make stick since the period of the Oklahoma bombing in 1995 which was blamed on 'far-right domestic terrorists'. If you read *The Trigger* you will see that the bombing was clearly a Problem-Reaction-Solution carried out by the Deep State during a Bill Clinton administration so corrupt that no dictionary definition of the term would even nearly suffice. Nearly 30,000 troops were deployed from all over America to the empty streets of Washington for Biden's inauguration. Ten thousand of them stayed on with the pretext of protecting the capital from insurrectionists when it was more psychological programming to normalise the use of the military in domestic law enforcement in support of the Cult plan for a police-military state. Biden's fascist administration began a purge of 'wrong-thinkers' in the military which means anyone that is not on board with Woke. The Capitol Building was surrounded by a fence with razor wire and the Land of the Free was further symbolically and literally dismantled. The circle was completed with the installation of Biden and the exploitation of the QAnon Psyop.

America had never been so divided since the civil war of the 19th century, Pushbackers were isolated and dubbed terrorists and now, as was always going to happen, the Cult immediately set about deleting what little was left of freedom and transforming American society through a swish of the hand of the most controlled 'president' in American history leading (officially at least) the most extreme regime since the country was declared an independent state on July 4th, 1776. Biden issued undebated, dictatorial executive orders almost by the hour in his opening days in office across the whole spectrum of the Cult wish-list including diluting controls on the border with Mexico allowing thousands of migrants to illegally enter the United States to transform the demographics of America and import an election-changing number of perceived Democrat

voters. Then there were Biden deportation amnesties for the already illegally resident (estimated to be as high as 20 or even 30 million). A bill before Congress awarded American citizenship to anyone who could prove they had worked in agriculture for just 180 days in the previous two years as 'Big Ag' secured its slave labour long-term. There were the plans to add new states to the union such as Puerto Rico and making Washington DC a state. They are all parts of a plan to ensure that the Cult-owned Woke Democrats would be permanently in power.

## **Border – what border?**

I have exposed in detail in other books how mass immigration into the United States and Europe is the work of Cult networks fuelled by the tens of billions spent to this and other ends by George Soros and his global Open Society (open borders) Foundations. The impact can be seen in America alone where the population has increased by *100 million* in little more than 30 years mostly through immigration. I wrote in *The Answer* that the plan was to have so many people crossing the southern border that the numbers become unstoppable and we are now there under Cult-owned Biden. El Salvador in Central America puts the scale of what is happening into context. A third of the population now lives in the United States, much of it illegally, and many more are on the way. The methodology is to crush Central and South American countries economically and spread violence through machete-wielding psychopathic gangs like MS-13 based in El Salvador and now operating in many American cities. Biden-imposed lax security at the southern border means that it is all but open. He said before his 'election' that he wanted to see a surge towards the border if he became president and that was the green light for people to do just that after election day to create the human disaster that followed for both America and the migrants. When that surge came the imbecilic Alexandria Ocasio-Cortez said it wasn't a 'surge' because they are 'children, not insurgents' and the term 'surge' (used by Biden) was a claim of 'white supremacists'.

This disingenuous lady may one day enter the realm of the most basic intelligence, but it won't be any time soon.

Sabbatians and the Cult are in the process of destroying America by importing violent people and gangs in among the genuine to terrorise American cities and by overwhelming services that cannot cope with the sheer volume of new arrivals. Something similar is happening in Europe as Western society in general is targeted for demographic and cultural transformation and upheaval. The plan demands violence and crime to create an environment of intimidation, fear and division and Soros has been funding the election of district attorneys across America who then stop prosecuting many crimes, reduce sentences for violent crimes and free as many violent criminals as they can. Sabbatians are creating the chaos from which order – their order – can respond in a classic Problem-Reaction-Solution. A Freemasonic moto says ‘Ordo Ab Chao’ (Order out of Chaos) and this is why the Cult is constantly creating chaos to impose a new ‘order’. Here you have the reason the Cult is constantly creating chaos. The ‘Covid’ hoax can be seen with those entering the United States by plane being forced to take a ‘Covid’ test while migrants flooding through southern border processing facilities do not. Nothing is put in the way of mass migration and if that means ignoring the government’s own ‘Covid’ rules then so be it. They know it’s all bullshit anyway. Any pushback on this is denounced as ‘racist’ by Wokers and Sabbatian fronts like the ultra-Zionist Anti-Defamation League headed by the appalling Jonathan Greenblatt which at the same time argues that Israel should not give citizenship and voting rights to more Palestinian Arabs or the ‘Jewish population’ (in truth the Sabbatian network) will lose control of the country.

## **Society-changing numbers**

Biden’s masters have declared that countries like El Salvador are so dangerous that their people must be allowed into the United States for humanitarian reasons when there are fewer murders in large parts of many Central American countries than in US cities like

Baltimore. That is not to say Central America cannot be a dangerous place and Cult-controlled American governments have been making it so since way back, along with the dismantling of economies, in a long-term plan to drive people north into the United States. Parts of Central America are very dangerous, but in other areas the story is being greatly exaggerated to justify relaxing immigration criteria. Migrants are being offered free healthcare and education in the United States as another incentive to head for the border and there is no requirement to be financially independent before you can enter to prevent the resources of America being drained. You can't blame migrants for seeking what they believe will be a better life, but they are being played by the Cult for dark and nefarious ends. The numbers since Biden took office are huge. In February, 2021, more than 100,000 people were known to have tried to enter the US illegally through the southern border (it was 34,000 in the same month in 2020) and in March it was 170,000 – a 418 percent increase on March, 2020. These numbers are only known people, not the ones who get in unseen. The true figure for migrants illegally crossing the border in a single month was estimated by one congressman at 250,000 and that number will only rise under Biden's current policy. Gangs of murdering drug-running thugs that control the Mexican side of the border demand money – thousands of dollars – to let migrants cross the Rio Grande into America. At the same time gun battles are breaking out on the border several times a week between rival Mexican drug gangs (which now operate globally) who are equipped with sophisticated military-grade weapons, grenades and armoured vehicles. While the Capitol Building was being 'protected' from a non-existent 'threat' by thousands of troops, and others were still deployed at the time in the Cult Neocon war in Afghanistan, the southern border of America was left to its fate. This is not incompetence, it is cold calculation.

By March, 2021, there were 17,000 unaccompanied children held at border facilities and many of them are ensnared by people traffickers for paedophile rings and raped on their journey north to America. This is not conjecture – this is fact. Many of those designated

children are in reality teenage boys or older. Meanwhile Wokers posture their self-purity for encouraging poor and tragic people to come to America and face this nightmare both on the journey and at the border with the disgusting figure of House Speaker Nancy Pelosi giving disingenuous speeches about caring for migrants. The woman's evil. Wokers condemned Trump for having children in cages at the border (so did Obama, *Shhhh*), but now they are sleeping on the floor without access to a shower with one border facility 729 percent over capacity. The Biden insanity even proposed flying migrants from the southern border to the northern border with Canada for 'processing'. The whole shambles is being overseen by ultra-Zionist Secretary of Homeland Security, the moronic liar Alejandro Mayorkas, who banned news cameras at border facilities to stop Americans seeing what was happening. Mayorkas said there was not a ban on news crews; it was just that they were not allowed to film. Alongside him at Homeland Security is another ultra-Zionist Cass Sunstein appointed by Biden to oversee new immigration laws. Sunstein despises conspiracy researchers to the point where he suggests they should be banned or *taxed* for having such views. The man is not bonkers or anything. He's perfectly well-adjusted, but adjusted to what is the question. Criticise what is happening and you are a 'white supremacist' when earlier non-white immigrants also oppose the numbers which effect their lives and opportunities. Black people in poor areas are particularly damaged by uncontrolled immigration and the increased competition for work opportunities with those who will work for less. They are also losing voting power as Hispanics become more dominant in former black areas. It's a downward spiral for them while the billionaires behind the policy drone on about how much they care about black people and 'racism'. None of this is about compassion for migrants or black people – that's just wind and air. Migrants are instead being mercilessly exploited to transform America while the countries they leave are losing their future and the same is true in Europe. Mass immigration may now be the work of Woke Democrats, but it can be traced back to the 1986 Immigration Reform and Control Act (it

wasn't) signed into law by Republican hero President Ronald Reagan which gave amnesty to millions living in the United States illegally and other incentives for people to head for the southern border. Here we have the one-party state at work again.

## **Save me syndrome**

Almost every aspect of what I have been exposing as the Cult agenda was on display in even the first days of 'Biden' with silencing of Pushbackers at the forefront of everything. A Renegade Mind will view the Trump years and QAnon in a very different light to their supporters and advocates as the dots are connected. The QAnon/Trump Psyop has given the Cult all it was looking for. We may not know how much, or little, that Trump realised he was being used, but that's a side issue. This pincer movement produced the desired outcome of dividing America and having Pushbackers isolated. To turn this around we have to look at new routes to empowerment which do not include handing our power to other people and groups through what I will call the 'Save Me Syndrome' – 'I want someone else to do it so that I don't have to'. We have seen this at work throughout human history and the QAnon/Trump Psyop is only the latest incarnation alongside all the others. Religion is an obvious expression of this when people look to a 'god' or priest to save them or tell them how to be saved and then there are 'save me' politicians like Trump. Politics is a diversion and not a 'saviour'. It is a means to block positive change, not make it possible.

Save Me Syndrome always comes with the same repeating theme of handing your power to whom or what you believe will save you while your real 'saviour' stares back from the mirror every morning. Renegade Minds are constantly vigilant in this regard and always asking the question 'What can I do?' rather than 'What can someone else do for me?' Gandhi was right when he said: 'You must be the change you want to see in the world.' We are indeed the people we have been waiting for. We are presented with a constant raft of reasons to concede that power to others and forget where the real power is. Humanity has the numbers and the Cult does not. It has to

use diversion and division to target the unstoppable power that comes from unity. Religions, governments, politicians, corporations, media, QAnon, are all different manifestations of this power-diversion and dilution. Refusing to give your power to governments and instead handing it to Trump and QAnon is not to take a new direction, but merely to recycle the old one with new names on the posters. I will explore this phenomenon as we proceed and how to break the cycles and recycles that got us here through the mists of repeating perception and so repeating history.

For now we shall turn to the most potent example in the entire human story of the consequences that follow when you give your power away. I am talking, of course, of the 'Covid' hoax.

## CHAPTER FOUR

### 'Covid': Calculated catastrophe

*Facts are threatening to those invested in fraud*  
DaShanne Stokes

We can easily unravel the real reason for the 'Covid pandemic' hoax by employing the Renegade Mind methodology that I have outlined this far. We'll start by comparing the long-planned Cult outcome with the 'Covid pandemic' outcome. Know the outcome and you'll see the journey.

I have highlighted the plan for the Hunger Games Society which has been in my books for so many years with the very few controlling the very many through ongoing dependency. To create this dependency it is essential to destroy independent livelihoods, businesses and employment to make the population reliant on the state (the Cult) for even the basics of life through a guaranteed pittance income. While independence of income remained these Cult ambitions would be thwarted. With this knowledge it was easy to see where the 'pandemic' hoax was going once talk of 'lockdowns' began and the closing of all but perceived 'essential' businesses to 'save' us from an alleged 'deadly virus'. Cult corporations like Amazon and Walmart were naturally considered 'essential' while mom and pop shops and stores had their doors closed by fascist decree. As a result with every new lockdown and new regulation more small and medium, even large businesses not owned by the Cult, went to the wall while Cult giants and their frontmen and women grew financially fatter by the second. Mom and pop were

denied an income and the right to earn a living and the wealth of people like Jeff Bezos (Amazon), Mark Zuckerberg (Facebook) and Sergei Brin and Larry Page (Google/Alphabet) have reached record levels. The Cult was increasing its own power through further dramatic concentrations of wealth while the competition was being destroyed and brought into a state of dependency. Lockdowns have been instigated to secure that very end and were never anything to do with health. My brother Paul spent 45 years building up a bus repair business, but lockdowns meant buses were running at a fraction of normal levels for months on end. Similar stories can told in their hundreds of millions worldwide. Efforts of a lifetime coldly destroyed by Cult multi-billionaires and their lackeys in government and law enforcement who continued to earn their living from the taxation of the people while denying the right of the same people to earn theirs. How different it would have been if those making and enforcing these decisions had to face the same financial hardships of those they affected, but they never do.

## **Gates of Hell**

Behind it all in the full knowledge of what he is doing and why is the psychopathic figure of Cult operative Bill Gates. His puppet Tedros at the World Health Organization declared 'Covid' a pandemic in March, 2020. The WHO had changed the definition of a 'pandemic' in 2009 just a month before declaring the 'swine flu pandemic' which would not have been so under the previous definition. The same applies to 'Covid'. The definition had included... 'an infection by an infectious agent, occurring simultaneously in different countries, with a significant mortality rate relative to the proportion of the population infected'. The new definition removed the need for 'significant mortality'. The 'pandemic' has been fraudulent even down to the definition, but Gates demanded economy-destroying lockdowns, school closures, social distancing, mandatory masks, a 'vaccination' for every man, woman and child on the planet and severe consequences and restrictions for those that refused. Who gave him this power? The

Cult did which he serves like a little boy in short trousers doing what his daddy tells him. He and his psychopathic missus even smiled when they said that much worse was to come (what they knew was planned to come). Gates responded in the matter-of-fact way of all psychopaths to a question about the effect on the world economy of what he was doing:

Well, it won't go to zero but it will shrink. Global GDP is probably going to take the biggest hit ever [Gates was smiling as he said this] ... in my lifetime this will be the greatest economic hit. But you don't have a choice. People act as if you have a choice. People don't feel like going to the stadium when they might get infected ... People are deeply affected by seeing these stats, by knowing they could be part of the transmission chain, old people, their parents and grandparents, could be affected by this, and so you don't get to say ignore what is going on here.

There will be the ability to open up, particularly in rich countries, if things are done well over the next few months, but for the world at large normalcy only returns when we have largely vaccinated the entire population.

The man has no compassion or empathy. How could he when he's a psychopath like all Cult players? My own view is that even beyond that he is very seriously mentally ill. Look in his eyes and you can see this along with his crazy flailing arms. You don't do what he has done to the world population since the start of 2020 unless you are mentally ill and at the most extreme end of psychopathic. You especially don't do it when to you know, as we shall see, that cases and deaths from 'Covid' are fakery and a product of monumental figure massaging. 'These stats' that Gates referred to are based on a 'test' that's not testing for the 'virus' as he has known all along. He made his fortune with big Cult support as an infamously ruthless software salesman and now buys global control of 'health' (death) policy without the population he affects having any say. It's a breathtaking outrage. Gates talked about people being deeply affected by fear of 'Covid' when that was because of *him* and his global network lying to them minute-by-minute supported by a lying media that he seriously influences and funds to the tune of hundreds of millions. He's handed big sums to media operations including the BBC, NBC, Al Jazeera, Univision, *PBS NewsHour*,

*ProPublica, National Journal, The Guardian, The Financial Times, The Atlantic, Texas Tribune, USA Today* publisher Gannett, Washington Monthly, Le Monde, Center for Investigative Reporting, Pulitzer Center on Crisis Reporting, National Press Foundation, International Center for Journalists, Solutions Journalism Network, the Poynter Institute for Media Studies, and many more. Gates is everywhere in the ‘Covid’ hoax and the man must go to prison – or a mental facility – for the rest of his life and his money distributed to those he has taken such enormous psychopathic pleasure in crushing.

## **The Muscle**

The Hunger Games global structure demands a police-military state – a fusion of the two into one force – which viciously imposes the will of the Cult on the population and protects the Cult from public rebellion. In that regard, too, the ‘Covid’ hoax just keeps on giving. Often unlawful, ridiculous and contradictory ‘Covid’ rules and regulations have been policed across the world by moronic automatons and psychopaths made faceless by face-nappy masks and acting like the Nazi SS and fascist blackshirts and brownshirts of Hitler and Mussolini. The smallest departure from the rules decreed by the psychos in government and their clueless gofers were jumped upon by the face-nappy fascists. Brutality against public protestors soon became commonplace even on girls, women and old people as the brave men with the batons – the Face-Nappies as I call them – broke up peaceful protests and handed out fines like confetti to people who couldn’t earn a living let alone pay hundreds of pounds for what was once an accepted human right. Robot Face-Nappies of Nottingham police in the English East Midlands fined one group £11,000 for attending a child’s birthday party. For decades I charted the transformation of law enforcement as genuine, decent officers were replaced with psychopaths and the brain dead who would happily and brutally do whatever their masters told them. Now they were let loose on the public and I would emphasise the point that none of this just happened. The step-by-step change in the dynamic between police and public was orchestrated from the shadows by

those who knew where this was all going and the same with the perceptual reframing of those in all levels of authority and official administration through ‘training courses’ by organisations such as Common Purpose which was created in the late 1980s and given a massive boost in Blair era Britain until it became a global phenomenon. Supposed public ‘servants’ began to view the population as the enemy and the same was true of the police. This was the start of the explosion of behaviour manipulation organisations and networks preparing for the all-war on the human psyche unleashed with the dawn of 2020. I will go into more detail about this later in the book because it is a core part of what is happening.

Police desecrated beauty spots to deter people gathering and arrested women for walking in the countryside alone ‘too far’ from their homes. We had arrogant, clueless sergeants in the Isle of Wight police where I live posting on Facebook what they insisted the population must do or else. A schoolmaster sergeant called Radford looked young enough for me to ask if his mother knew he was out, but he was posting what he *expected* people to do while a Sergeant Wilkinson boasted about fining lads for meeting in a McDonald’s car park where they went to get a lockdown takeaway. Wilkinson added that he had even cancelled their order. What a pair of prats these people are and yet they have increasingly become the norm among Jackboot Johnson’s Yellowshirts once known as the British police. This was the theme all over the world with police savagery common during lockdown protests in the United States, the Netherlands, and the fascist state of Victoria in Australia under its tyrannical and again moronic premier Daniel Andrews. Amazing how tyrannical and moronic tend to work as a team and the same combination could be seen across America as arrogant, narcissistic Woke governors and mayors such as Gavin Newsom (California), Andrew Cuomo (New York), Gretchen Whitmer (Michigan), Lori Lightfoot (Chicago) and Eric Garcetti (Los Angeles) did their Nazi and Stalin impressions with the full support of the compliant brutality of their enforcers in uniform as they arrested small business owners defying

fascist shutdown orders and took them to jail in ankle shackles and handcuffs. This happened to bistro owner Marlena Pavlos-Hackney in Gretchen Whitmer's fascist state of Michigan when police arrived to enforce an order by a state-owned judge for 'putting the community at risk' at a time when other states like Texas were dropping restrictions and migrants were pouring across the southern border without any 'Covid' questions at all. I'm sure there are many officers appalled by what they are ordered to do, but not nearly enough of them. If they were truly appalled they would not do it. As the months passed every opportunity was taken to have the military involved to make their presence on the streets ever more familiar and 'normal' for the longer-term goal of police-military fusion.

Another crucial element to the Hunger Games enforcement network has been encouraging the public to report neighbours and others for 'breaking the lockdown rules'. The group faced with £11,000 in fines at the child's birthday party would have been dobbed-in by a neighbour with a brain the size of a pea. The technique was most famously employed by the Stasi secret police in communist East Germany who had public informants placed throughout the population. A police chief in the UK says his force doesn't need to carry out 'Covid' patrols when they are flooded with so many calls from the public reporting other people for visiting the beach. Dorset police chief James Vaughan said people were so enthusiastic about snitching on their fellow humans they were now operating as an auxiliary arm of the police: 'We are still getting around 400 reports a week from the public, so we will respond to reports ... We won't need to be doing hotspot patrols because people are very quick to pick the phone up and tell us.' Vaughan didn't say that this is a pillar of all tyrannies of whatever complexion and the means to hugely extend the reach of enforcement while spreading distrust among the people and making them wary of doing anything that might get them reported. Those narcissistic Isle of Wight sergeants Radford and Wilkinson never fail to add a link to their Facebook posts where the public can inform on their fellow slaves.

Neither would be self-aware enough to realise they were imitating the Stasi which they might well never have heard of. Government psychologists that I will expose later laid out a policy to turn communities against each other in the same way.

## **A coincidence? Yep, and I can knit fog**

I knew from the start of the alleged pandemic that this was a Cult operation. It presented limitless potential to rapidly advance the Cult agenda and exploit manipulated fear to demand that every man, woman and child on the planet was ‘vaccinated’ in a process never used on humans before which infuses self-replicating *synthetic* material into human cells. Remember the plan to transform the human body from a biological to a synthetic biological state. I’ll deal with the ‘vaccine’ (that’s not actually a vaccine) when I focus on the genetic agenda. Enough to say here that mass global ‘vaccination’ justified by this ‘new virus’ set alarms ringing after 30 years of tracking these people and their methods. The ‘Covid’ hoax officially beginning in China was also a big red flag for reasons I will be explaining. The agenda potential was so enormous that I could dismiss any idea that the ‘virus’ appeared naturally. Major happenings with major agenda implications never occur without Cult involvement in making them happen. My questions were twofold in early 2020 as the media began its campaign to induce global fear and hysteria: Was this alleged infectious agent released on purpose by the Cult or did it even exist at all? I then did what I always do in these situations. I sat, observed and waited to see where the evidence and information would take me. By March and early April synchronicity was strongly – and ever more so since then – pointing me in the direction of *there is no ‘virus’*. I went public on that with derision even from swathes of the alternative media that voiced a scenario that the Chinese government released the ‘virus’ in league with Deep State elements in the United States from a top-level bio-lab in Wuhan where the ‘virus’ is said to have first appeared. I looked at that possibility, but I didn’t buy it for several reasons. Deaths from the ‘virus’ did not in any way match what they

would have been with a ‘deadly bioweapon’ and it is much more effective if you sell the *illusion* of an infectious agent rather than having a real one unless you can control through injection who has it and who doesn’t. Otherwise you lose control of events. A made-up ‘virus’ gives you a blank sheet of paper on which you can make it do whatever you like and have any symptoms or mutant ‘variants’ you choose to add while a real infectious agent would limit you to what it actually does. A phantom disease allows you to have endless ludicrous ‘studies’ on the ‘Covid’ dollar to widen the perceived impact by inventing ever more ‘at risk’ groups including one study which said those who walk slowly may be almost four times more likely to die from the ‘virus’. People are in psychiatric wards for less.

A real ‘deadly bioweapon’ can take out people in the hierarchy that are not part of the Cult, but essential to its operation. Obviously they don’t want that. Releasing a real disease means you immediately lose control of it. Releasing an illusory one means you don’t. Again it’s vital that people are extra careful when dealing with what they want to hear. A bioweapon unleashed from a Chinese laboratory in collusion with the American Deep State may fit a conspiracy narrative, but is it true? Would it not be far more effective to use the excuse of a ‘virus’ to justify the real bioweapon – the ‘vaccine’? That way your disease agent does not have to be transmitted and arrives directly through a syringe. I saw a French virologist Luc Montagnier quoted in the alternative media as saying he had discovered that the alleged ‘new’ severe acute respiratory syndrome coronavirus , or SARS-CoV-2, was made artificially and included elements of the human immunodeficiency ‘virus’ (HIV) and a parasite that causes malaria. SARS-CoV-2 is alleged to trigger an alleged illness called Covid-19. I remembered Montagnier’s name from my research years before into claims that an HIV ‘retrovirus’ causes AIDS – claims that were demolished by Berkeley virologist Peter Duesberg who showed that no one had ever proved that HIV causes acquired immunodeficiency syndrome or AIDS. Claims that become accepted as fact, publicly and medically, with no proof whatsoever are an ever-recurring story that profoundly applies to

'Covid'. Nevertheless, despite the lack of proof, Montagnier's team at the Pasteur Institute in Paris had a long dispute with American researcher Robert Gallo over which of them discovered and isolated the HIV 'virus' and with *no evidence* found it to cause AIDS. You will see later that there is also no evidence that any 'virus' causes any disease or that there is even such a thing as a 'virus' in the way it is said to exist. The claim to have 'isolated' the HIV 'virus' will be presented in its real context as we come to the shocking story – and it is a story – of SARS-CoV-2 and so will Montagnier's assertion that he identified the full SARS-CoV-2 genome.

## **Hoax in the making**

We can pick up the 'Covid' story in 2010 and the publication by the Rockefeller Foundation of a document called 'Scenarios for the Future of Technology and International Development'. The inner circle of the Rockefeller family has been serving the Cult since John D. Rockefeller (1839-1937) made his fortune with Standard Oil. It is less well known that the same Rockefeller – the Bill Gates of his day – was responsible for establishing what is now referred to as 'Big Pharma', the global network of pharmaceutical companies that make outrageous profits dispensing scalpel and drug 'medicine' and are obsessed with pumping vaccines in ever-increasing number into as many human arms and backsides as possible. John D. Rockefeller was the driving force behind the creation of the 'education' system in the United States and elsewhere specifically designed to program the perceptions of generations thereafter. The Rockefeller family donated exceptionally valuable land in New York for the United Nations building and were central in establishing the World Health Organization in 1948 as an agency of the UN which was created from the start as a Trojan horse and stalking horse for world government. Now enter Bill Gates. His family and the Rockefellers have long been extremely close and I have seen genealogy which claims that if you go back far enough the two families fuse into the same bloodline. Gates has said that the Bill and Melinda Gates Foundation was inspired by the Rockefeller Foundation and why not

when both are serving the same Cult? Major tax-exempt foundations are overwhelmingly criminal enterprises in which Cult assets fund the Cult agenda in the guise of 'philanthropy' while avoiding tax in the process. Cult operatives can become mega-rich in their role of front men and women for the psychopaths at the inner core and they, too, have to be psychopaths to knowingly serve such evil. Part of the deal is that a big percentage of the wealth gleaned from representing the Cult has to be spent advancing the ambitions of the Cult and hence you have the Rockefeller Foundation, Bill and Melinda Gates Foundation (and so many more) and people like George Soros with his global Open Society Foundations spending their billions in pursuit of global Cult control. Gates is a global public face of the Cult with his interventions in world affairs including Big Tech influence; a central role in the 'Covid' and 'vaccine' scam; promotion of the climate change shakedown; manipulation of education; geoengineering of the skies; and his food-control agenda as the biggest owner of farmland in America, his GMO promotion and through other means. As one writer said: 'Gates monopolizes or wields disproportionate influence over the tech industry, global health and vaccines, agriculture and food policy (including biopiracy and fake food), weather modification and other climate technologies, surveillance, education and media.' The almost limitless wealth secured through Microsoft and other not-allowed-to-fail ventures (including vaccines) has been ploughed into a long, long list of Cult projects designed to enslave the entire human race. Gates and the Rockefellers have been working as one unit with the Rockefeller-established World Health Organization leading global 'Covid' policy controlled by Gates through his mouth-piece Tedros. Gates became the WHO's biggest funder when Trump announced that the American government would cease its donations, but Biden immediately said he would restore the money when he took office in January, 2021. The Gates Foundation (the Cult) owns through limitless funding the world health system and the major players across the globe in the 'Covid' hoax.

Okay, with that background we return to that Rockefeller Foundation document of 2010 headed ‘Scenarios for the Future of Technology and International Development’ and its ‘imaginary’ epidemic of a virulent and deadly influenza strain which infected 20 percent of the global population and killed eight million in seven months. The Rockefeller scenario was that the epidemic destroyed economies, closed shops, offices and other businesses and led to governments imposing fierce rules and restrictions that included mandatory wearing of face masks and body-temperature checks to enter communal spaces like railway stations and supermarkets. The document predicted that even after the height of the Rockefeller-envisioned epidemic the authoritarian rule would continue to deal with further pandemics, transnational terrorism, environmental crises and rising poverty. Now you may think that the Rockefellers are our modern-day seers or alternatively, and rather more likely, that they well knew what was planned a few years further on. Fascism had to be imposed, you see, to ‘protect citizens from risk and exposure’. The Rockefeller scenario document said:

During the pandemic, national leaders around the world flexed their authority and imposed airtight rules and restrictions, from the mandatory wearing of face masks to body-temperature checks at the entries to communal spaces like train stations and supermarkets. Even after the pandemic faded, this more authoritarian control and oversight of citizens and their activities stuck and even intensified. In order to protect themselves from the spread of increasingly global problems – from pandemics and transnational terrorism to environmental crises and rising poverty – leaders around the world took a firmer grip on power.

At first, the notion of a more controlled world gained wide acceptance and approval. Citizens willingly gave up some of their sovereignty – and their privacy – to more paternalistic states in exchange for greater safety and stability. Citizens were more tolerant, and even eager, for top-down direction and oversight, and national leaders had more latitude to impose order in the ways they saw fit.

In developed countries, this heightened oversight took many forms: biometric IDs for all citizens, for example, and tighter regulation of key industries whose stability was deemed vital to national interests. In many developed countries, enforced cooperation with a suite of new regulations and agreements slowly but steadily restored both order and, importantly, economic growth.

There we have the prophetic Rockefellers in 2010 and three years later came their paper for the Global Health Summit in Beijing, China, when government representatives, the private sector, international organisations and groups met to discuss the next 100 years of 'global health'. The Rockefeller Foundation-funded paper was called 'Dreaming the Future of Health for the Next 100 Years' and more prophecy ensued as it described a dystopian future: 'The abundance of data, digitally tracking and linking people may mean the 'death of privacy' and may replace physical interaction with transient, virtual connection, generating isolation and raising questions of how values are shaped in virtual networks.' Next in the 'Covid' hoax preparation sequence came a 'table top' simulation in 2018 for another 'imaginary' pandemic of a disease called Clade X which was said to kill 900 million people. The exercise was organised by the Gates-funded Johns Hopkins University's Center for Health Security in the United States and this is the very same university that has been compiling the disgustingly and systematically erroneous global figures for 'Covid' cases and deaths. Similar Johns Hopkins health crisis scenarios have included the Dark Winter exercise in 2001 and Atlantic Storm in 2005.

## **Nostradamus 201**

For sheer predictive genius look no further prophecy-watchers than the Bill Gates-funded Event 201 held only six weeks before the 'coronavirus pandemic' is supposed to have broken out in China and Event 201 was based on a scenario of a global 'coronavirus pandemic'. Melinda Gates, the great man's missus, told the BBC that he had 'prepared for years' for a coronavirus pandemic which told us what we already knew. Nostradamugates had predicted in a TED talk in 2015 that a pandemic was coming that would kill a lot of people and demolish the world economy. My god, the man is a machine – possibly even literally. Now here he was only weeks before the real thing funding just such a simulated scenario and involving his friends and associates at Johns Hopkins, the World Economic Forum Cult-front of Klaus Schwab, the United Nations,

Johnson & Johnson, major banks, and officials from China and the Centers for Disease Control in the United States. What synchronicity – Johns Hopkins would go on to compile the fraudulent ‘Covid’ figures, the World Economic Forum and Schwab would push the ‘Great Reset’ in response to ‘Covid’, the Centers for Disease Control would be at the forefront of ‘Covid’ policy in the United States, Johnson & Johnson would produce a ‘Covid vaccine’, and everything would officially start just weeks later in China. Spooky, eh? They were even accurate in creating a simulation of a ‘virus’ pandemic because the ‘real thing’ would also be a simulation. Event 201 was not an exercise preparing for something that might happen; it was a rehearsal for what those in control knew was *going* to happen and very shortly. Hours of this simulation were posted on the Internet and the various themes and responses mirrored what would soon be imposed to transform human society. News stories were inserted and what they said would be commonplace a few weeks later with still more prophecy perfection. Much discussion focused on the need to deal with misinformation and the ‘anti-vax movement’ which is exactly what happened when the ‘virus’ arrived – was said to have arrived – in the West.

Cult-owned social media banned criticism and exposure of the official ‘virus’ narrative and when I said there *was* no ‘virus’ in early April, 2020, I was banned by one platform after another including YouTube, Facebook and later Twitter. The mainstream broadcast media in Britain was in effect banned from interviewing me by the Tony-Blair-created government broadcasting censor Ofcom headed by career government bureaucrat Melanie Dawes who was appointed just as the ‘virus’ hoax was about to play out in January, 2020. At the same time the Ickonic media platform was using Vimeo, another ultra-Zionist-owned operation, while our own player was being created and they deleted in an instant hundreds of videos, documentaries, series and shows to confirm their unbelievable vindictiveness. We had copies, of course, and they had to be restored one by one when our player was ready. These people have no class. Sabbatian Facebook promised free advertisements for the Gates-

controlled World Health Organization narrative while deleting ‘false claims and conspiracy theories’ to stop ‘misinformation’ about the alleged coronavirus. All these responses could be seen just a short while earlier in the scenarios of Event 201. Extreme censorship was absolutely crucial for the Cult because the official story was so ridiculous and unsupportable by the evidence that it could never survive open debate and the free-flow of information and opinion. If you can’t win a debate then don’t have one is the Cult’s approach throughout history. Facebook’s little boy front man – front boy – Mark Zuckerberg equated ‘credible and accurate information’ with official sources and exposing their lies with ‘misinformation’.

## **Silencing those that can see**

The censorship dynamic of Event 201 is now the norm with an army of narrative-supporting ‘fact-checker’ organisations whose entire reason for being is to tell the public that official narratives are true and those exposing them are lying. One of the most appalling of these ‘fact-checkers’ is called NewsGuard founded by ultra-Zionist Americans Gordon Crovitz and Steven Brill. Crovitz is a former publisher of *The Wall Street Journal*, former Executive Vice President of Dow Jones, a member of the Council on Foreign Relations (CFR), and on the board of the American Association of Rhodes Scholars. The CFR and Rhodes Scholarships, named after Rothschild agent Cecil Rhodes who plundered the gold and diamonds of South Africa for his masters and the Cult, have featured widely in my books. NewsGuard don’t seem to like me for some reason – I really can’t think why – and they have done all they can to have me censored and discredited which is, to quote an old British politician, like being savaged by a dead sheep. They are, however, like all in the censorship network, very well connected and funded by organisations themselves funded by, or connected to, Bill Gates. As you would expect with anything associated with Gates NewsGuard has an offshoot called HealthGuard which ‘fights online health care hoaxes’. How very kind. Somehow the NewsGuard European Managing Director Anna-Sophie Harling, a remarkably young-

looking woman with no broadcasting experience and little hands-on work in journalism, has somehow secured a position on the ‘Content Board’ of UK government broadcast censor Ofcom. An executive of an organisation seeking to discredit dissidents of the official narratives is making decisions for the government broadcast ‘regulator’ about content?? Another appalling ‘fact-checker’ is Full Fact funded by George Soros and global censors Google and Facebook.

It’s amazing how many activists in the ‘fact-checking’, ‘anti-hate’, arena turn up in government-related positions – people like UK Labour Party activist Imran Ahmed who heads the Center for Countering Digital Hate founded by people like Morgan McSweeney, now chief of staff to the Labour Party’s hapless and useless ‘leader’ Keir Starmer. Digital Hate – which is what it really is – uses the American spelling of Center to betray its connection to a transatlantic network of similar organisations which in 2020 shapeshifted from attacking people for ‘hate’ to attacking them for questioning the ‘Covid’ hoax and the dangers of the ‘Covid vaccine’. It’s just a coincidence, you understand. This is one of Imran Ahmed’s hysterical statements: ‘I would go beyond calling anti-vaxxers conspiracy theorists to say they are an extremist group that pose a national security risk.’ No one could ever accuse this prat of understatement and he’s including in that those parents who are now against vaccines after their children were damaged for life or killed by them. He’s such a nice man. Ahmed does the rounds of the Woke media getting soft-ball questions from spineless ‘journalists’ who never ask what right he has to campaign to destroy the freedom of speech of others while he demands it for himself. There also seems to be an overrepresentation in Ofcom of people connected to the narrative-worshipping BBC. This incredible global network of narrative-support was super-vital when the ‘Covid’ hoax was played in the light of the mega-whopper lies that have to be defended from the spotlight cast by the most basic intelligence.

## **Setting the scene**

The Cult plays the long game and proceeds step-by-step ensuring that everything is in place before major cards are played and they don't come any bigger than the 'Covid' hoax. The psychopaths can't handle events where the outcome isn't certain and as little as possible – preferably nothing – is left to chance. Politicians, government and medical officials who would follow direction were brought to illusory power in advance by the Cult web whether on the national stage or others like state governors and mayors of America. For decades the dynamic between officialdom, law enforcement and the public was changed from one of service to one of control and dictatorship. Behaviour manipulation networks established within government were waiting to impose the coming 'Covid' rules and regulations specifically designed to subdue and rewire the psyche of the people in the guise of protecting health. These included in the UK the Behavioural Insights Team part-owned by the British government Cabinet Office; the Scientific Pandemic Insights Group on Behaviours (SPI-B); and a whole web of intelligence and military groups seeking to direct the conversation on social media and control the narrative. Among them are the cyberwarfare (on the people) 77th Brigade of the British military which is also coordinated through the Cabinet Office as civilian and military leadership continues to combine in what they call the Fusion Doctrine. The 77th Brigade is a British equivalent of the infamous Israeli (Sabbatian) military cyberwarfare and Internet manipulation operation Unit 8200 which I expose at length in *The Trigger*. Also carefully in place were the medical and science advisers to government – many on the payroll past or present of Bill Gates – and a whole alternative structure of unelected government stood by to take control when elected parliaments were effectively closed down once the 'Covid' card was slammed on the table. The structure I have described here and so much more was installed in every major country through the Cult networks. The top-down control hierarchy looks like this: The Cult – Cult-owned Gates – the World Health Organization and Tedros – Gates-funded or controlled chief medical officers and science 'advisers' (dictators) in each country –

political ‘leaders’ – law enforcement – The People. Through this simple global communication and enforcement structure the policy of the Cult could be imposed on virtually the entire human population so long as they acquiesced to the fascism. With everything in place it was time for the button to be pressed in late 2019/early 2020.

These were the prime goals the Cult had to secure for its will to prevail:

- 1) Locking down economies, closing all but designated ‘essential’ businesses (Cult-owned corporations were ‘essential’), and putting the population under house arrest was an imperative to destroy independent income and employment and ensure dependency on the Cult-controlled state in the Hunger Games Society. Lockdowns had to be established as the global blueprint from the start to respond to the ‘virus’ and followed by pretty much the entire world.
- 2) The global population had to be terrified into believing in a deadly ‘virus’ that didn’t actually exist so they would unquestioningly obey authority in the belief that authority must know how best to protect them and their families. Software salesman Gates would suddenly morph into the world’s health expert and be promoted as such by the Cult-owned media.
- 3) A method of testing that wasn’t testing for the ‘virus’, but was only claimed to be, had to be in place to provide the illusion of ‘cases’ and subsequent ‘deaths’ that had a very different cause to the ‘Covid-19’ that would be scribbled on the death certificate.
- 4) Because there was no ‘virus’ and the great majority testing positive with a test not testing for the ‘virus’ would have no symptoms of anything the lie had to be sold that people without symptoms (without the ‘virus’) could still pass it on to others. This was crucial to justify for the first time quarantining – house arresting – healthy people. Without this the economy-destroying lockdown of *everybody* could not have been credibly sold.
- 5) The ‘saviour’ had to be seen as a vaccine which beyond evil drug companies were working like angels of mercy to develop as quickly as possible, with all corners cut, to save the day. The public must absolutely not know that the ‘vaccine’ had nothing to do with a ‘virus’ or that the contents were ready and waiting with a very different motive long before the ‘Covid’ card was even lifted from the pack.

I said in March, 2020, that the ‘vaccine’ would have been created way ahead of the ‘Covid’ hoax which justified its use and the following December an article in the New York *Intelligencer* magazine said the Moderna ‘vaccine’ had been ‘designed’ by

January, 2020. This was ‘before China had even acknowledged that the disease could be transmitted from human to human, more than a week before the first confirmed coronavirus case in the United States’. The article said that by the time the first American death was announced a month later ‘the vaccine had already been manufactured and shipped to the National Institutes of Health for the beginning of its Phase I clinical trial’. The ‘vaccine’ was actually ‘designed’ long before that although even with this timescale you would expect the article to ask how on earth it could have been done that quickly. Instead it asked why the ‘vaccine’ had not been rolled out then and not months later. Journalism in the mainstream is truly dead. I am going to detail in the next chapter why the ‘virus’ has never existed and how a hoax on that scale was possible, but first the foundation on which the Big Lie of ‘Covid’ was built.

## **The test that doesn’t test**

Fraudulent ‘testing’ is the bottom line of the whole ‘Covid’ hoax and was the means by which a ‘virus’ that did not exist *appeared* to exist. They could only achieve this magic trick by using a test not testing for the ‘virus’. To use a test that *was* testing for the ‘virus’ would mean that every test would come back negative given there was no ‘virus’. They chose to exploit something called the RT-PCR test invented by American biochemist Kary Mullis in the 1980s who said publicly that his PCR test … *cannot detect infectious disease*. Yes, the ‘test’ used worldwide to detect infectious ‘Covid’ to produce all the illusory ‘cases’ and ‘deaths’ compiled by Johns Hopkins and others *cannot detect infectious disease*. This fact came from the mouth of the man who invented PCR and was awarded the Nobel Prize in Chemistry in 1993 for doing so. Sadly, and incredibly conveniently for the Cult, Mullis died in August, 2019, at the age of 74 just before his test would be fraudulently used to unleash fascism on the world. He was said to have died from pneumonia which was an irony in itself. A few months later he would have had ‘Covid-19’ on his death certificate. I say the timing of his death was convenient because had he lived Mullis, a brilliant, honest and decent man, would have been

vociferously speaking out against the use of his test to detect 'Covid' when it was never designed, or able, to do that. I know that to be true given that Mullis made the same point when his test was used to 'detect' – not detect – HIV. He had been seriously critical of the Gallo/Montagnier claim to have isolated the HIV 'virus' and shown it to cause AIDS for which Mullis said there was no evidence. AIDS is actually not a disease but a series of diseases from which people die all the time. When they die from those *same diseases* after a positive 'test' for HIV then AIDS goes on their death certificate. I think I've heard that before somewhere. Countries instigated a policy with 'Covid' that anyone who tested positive with a test not testing for the 'virus' and died of any other cause within 28 days and even longer 'Covid-19' had to go on the death certificate. Cases have come from the test that can't test for infectious disease and the deaths are those who have died of *anything* after testing positive with a test not testing for the 'virus'. I'll have much more later about the death certificate scandal.

Mullis was deeply dismissive of the now US 'Covid' star Anthony Fauci who he said was a liar who didn't know anything about anything – 'and I would say that to his face – nothing.' He said of Fauci: 'The man thinks he can take a blood sample, put it in an electron microscope and if it's got a virus in there you'll know it – he doesn't understand electron microscopy and he doesn't understand medicine and shouldn't be in a position like he's in.' That position, terrifyingly, has made him the decider of 'Covid' fascism policy on behalf of the Cult in his role as director since 1984 of the National Institute of Allergy and Infectious Diseases (NIAID) while his record of being wrong is laughable; but being wrong, so long as it's the *right kind* of wrong, is why the Cult loves him. He'll say anything the Cult tells him to say. Fauci was made Chief Medical Adviser to the President immediately Biden took office. Biden was installed in the White House by Cult manipulation and one of his first decisions was to elevate Fauci to a position of even more control. This is a coincidence? Yes, and I identify as a flamenco dancer called Lola. How does such an incompetent criminal like Fauci remain in that

pivotal position in American health since *the 1980s*? When you serve the Cult it looks after you until you are surplus to requirements. Kary Mullis said prophetically of Fauci and his like: ‘Those guys have an agenda and it’s not an agenda we would like them to have ... they make their own rules, they change them when they want to, and Tony Fauci does not mind going on television in front of the people who pay his salary and lie directly into the camera.’ Fauci has done that almost daily since the ‘Covid’ hoax began. Lying is in Fauci’s DNA. To make the situation crystal clear about the PCR test this is a direct quote from its inventor Kary Mullis:

It [the PCR test] doesn’t tell you that you’re sick and doesn’t tell you that the thing you ended up with was really going to hurt you ...’

Ask yourself why governments and medical systems the world over have been using this very test to decide who is ‘infected’ with the SARS-CoV-2 ‘virus’ and the alleged disease it allegedly causes, ‘Covid-19’. The answer to that question will tell you what has been going on. By the way, here’s a little show-stopper – the ‘new’ SARS-CoV-2 ‘virus’ was ‘identified’ as such right from the start using ... *the PCR test not testing for the ‘virus’*. If you are new to this and find that shocking then stick around. I have hardly started yet. Even worse, other ‘tests’, like the ‘Lateral Flow Device’ (LFD), are considered so useless that they have to be *confirmed* by the PCR test! Leaked emails written by Ben Dyson, adviser to UK ‘Health’ Secretary Matt Hancock, said they were ‘dangerously unreliable’. Dyson, executive director of strategy at the Department of Health, wrote: ‘As of today, someone who gets a positive LFD result in (say) London has at best a 25 per cent chance of it being a true positive, but if it is a self-reported test potentially as low as 10 per cent (on an optimistic assumption about specificity) or as low as 2 per cent (on a more pessimistic assumption).’ These are the ‘tests’ that schoolchildren and the public are being urged to have twice a week or more and have to isolate if they get a positive. Each fake positive goes in the statistics as a ‘case’ no matter how ludicrously inaccurate and the

'cases' drive lockdown, masks and the pressure to 'vaccinate'. The government said in response to the email leak that the 'tests' were accurate which confirmed yet again what shocking bloody liars they are. The real false positive rate is *100 percent* as we'll see. In another 'you couldn't make it up' the UK government agreed to pay £2.8 billion to California's Innova Medical Group to supply the irrelevant lateral flow tests. The company's primary test-making centre is in China. Innova Medical Group, established in March, 2020, is owned by Pasaca Capital Inc, chaired by Chinese-American millionaire Charles Huang who was born in Wuhan.

## **How it works – and how it doesn't**

The RT-PCR test, known by its full title of Polymerase chain reaction, is used across the world to make millions, even billions, of copies of a DNA/RNA genetic information sample. The process is called 'amplification' and means that a tiny sample of genetic material is amplified to bring out the detailed content. I stress that it is not testing for an infectious disease. It is simply amplifying a sample of genetic material. In the words of Kary Mullis: 'PCR is ... just a process that's used to make a whole lot of something out of something.' To emphasise the point companies that make the PCR tests circulated around the world to 'test' for 'Covid' warn on the box that it can't be used to detect 'Covid' or infectious disease and is for research purposes only. It's okay, rest for a minute and you'll be fine. This is the test that produces the 'cases' and 'deaths' that have been used to destroy human society. All those global and national medical and scientific 'experts' demanding this destruction to 'save us' KNOW that the test is not testing for the 'virus' and the cases and deaths they claim to be real are an almost unimaginable fraud. Every one of them and so many others including politicians and psychopaths like Gates and Tedros must be brought before Nuremberg-type trials and jailed for the rest of their lives. The more the genetic sample is amplified by PCR the more elements of that material become sensitive to the test and by that I don't mean sensitive for a 'virus' but for elements of the genetic material which

is naturally in the body or relates to remnants of old conditions of various kinds lying dormant and causing no disease. Once the amplification of the PCR reaches a certain level *everyone* will test positive. So much of the material has been made sensitive to the test that everyone will have some part of it in their body. Even lying criminals like Fauci have said that once PCR amplifications pass 35 cycles everything will be a false positive that cannot be trusted for the reasons I have described. I say, like many proper doctors and scientists, that 100 percent of the ‘positives’ are false, but let’s just go with Fauci for a moment.

He says that any amplification over 35 cycles will produce false positives and yet the US Centers for Disease Control (CDC) and Food and Drug Administration (FDA) have recommended up to 40 cycles and the National Health Service (NHS) in Britain admitted in an internal document for staff that it was using 45 cycles of amplification. A long list of other countries has been doing the same and at least one ‘testing’ laboratory has been using 50 cycles. Have you ever heard a doctor, medical ‘expert’ or the media ask what level of amplification has been used to claim a ‘positive’. The ‘test’ comes back ‘positive’ and so you have the ‘virus’, end of story. Now we can see how the government in Tanzania could send off samples from a goat and a pawpaw fruit under human names and both came back positive for ‘Covid-19’. Tanzania president John Magufuli mocked the ‘Covid’ hysteria, the PCR test and masks and refused to import the DNA-manipulating ‘vaccine’. The Cult hated him and an article sponsored by the Bill Gates Foundation appeared in the London *Guardian* in February, 2021, headed ‘It’s time for Africa to rein in Tanzania’s anti-vaxxer president’. Well, ‘reined in’ he shortly was. Magufuli appeared in good health, but then, in March, 2021, he was dead at 61 from ‘heart failure’. He was replaced by Samia Hassan Suhulu who is connected to Klaus Schwab’s World Economic Forum and she immediately reversed Magufuli’s ‘Covid’ policy. A sample of cola tested positive for ‘Covid’ with the PCR test in Germany while American actress and singer-songwriter Erykah Badu tested positive in one nostril and negative in the other. Footballer Ronaldo called

the PCR test ‘bullshit’ after testing positive three times and being forced to quarantine and miss matches when there was nothing wrong with him. The mantra from Tedros at the World Health Organization and national governments (same thing) has been test, test, test. They know that the more tests they can generate the more fake ‘cases’ they have which go on to become ‘deaths’ in ways I am coming to. The UK government has its Operation Moonshot planned to test multiple millions every day in workplaces and schools with free tests for everyone to use twice a week at home in line with the Cult plan from the start to make testing part of life. A government advertisement for an ‘Interim Head of Asymptomatic Testing Communication’ said the job included responsibility for delivering a ‘communications strategy’ (propaganda) ‘to support the expansion of asymptomatic testing that *“normalises testing as part of everyday life”*. More tests means more fake ‘cases’, ‘deaths’ and fascism. I have heard of, and from, many people who booked a test, couldn’t turn up, and yet got a positive result through the post for a test they’d never even had. The whole thing is crazy, but for the Cult there’s method in the madness. Controlling and manipulating the level of amplification of the test means the authorities can control whenever they want the number of apparent ‘cases’ and ‘deaths’. If they want to justify more fascist lockdown and destruction of livelihoods they keep the amplification high. If they want to give the illusion that lockdowns and the ‘vaccine’ are working then they lower the amplification and ‘cases’ and ‘deaths’ will appear to fall. In January, 2021, the Cult-owned World Health Organization suddenly warned laboratories about over-amplification of the test and to lower the threshold. Suddenly headlines began appearing such as: ‘Why ARE “Covid” cases plummeting?’ This was just when the vaccine rollout was underway and I had predicted months before they would make cases appear to fall through amplification tampering when the ‘vaccine’ came. These people are so predictable.

## Cow vaccines?

The question must be asked of what is on the test swabs being poked far up the nose of the population to the base of the brain? A nasal swab punctured one woman's brain and caused it to leak fluid. Most of these procedures are being done by people with little training or medical knowledge. Dr Lorraine Day, former orthopaedic trauma surgeon and Chief of Orthopaedic Surgery at San Francisco General Hospital, says the tests are really a '*vaccine*'. Cows have long been vaccinated this way. She points out that masks have to cover the nose and the mouth where it is claimed the 'virus' exists in saliva. Why then don't they take saliva from the mouth as they do with a DNA test instead of pushing a long swab up the nose towards the brain? The ethmoid bone separates the nasal cavity from the brain and within that bone is the cribriform plate. Dr Day says that when the swab is pushed up against this plate and twisted the procedure is 'depositing things back there'. She claims that among these 'things' are nanoparticles that can enter the brain. Researchers have noted that a team at the Gates-funded Johns Hopkins have designed tiny, star-shaped micro-devices that can latch onto intestinal mucosa and release drugs into the body. Mucosa is the thin skin that covers the inside surface of parts of the body such as *the nose* and mouth and produces mucus to protect them. The Johns Hopkins micro-devices are called 'theragrippers' and were 'inspired' by a parasitic worm that digs its sharp teeth into a host's intestines. Nasal swabs are also coated in the sterilisation agent ethylene oxide. The US National Cancer Institute posts this explanation on its website:

At room temperature, ethylene oxide is a flammable colorless gas with a sweet odor. It is used primarily to produce other chemicals, including antifreeze. In smaller amounts, ethylene oxide is used as a pesticide and a sterilizing agent. The ability of ethylene oxide to damage DNA makes it an effective sterilizing agent but also accounts for its cancer-causing activity.

The Institute mentions lymphoma and leukaemia as cancers most frequently reported to be associated with occupational exposure to ethylene oxide along with stomach and breast cancers. How does anyone think this is going to work out with the constant testing

regime being inflicted on adults and children at home and at school that will accumulate in the body anything that's on the swab?

## **Doctors know best**

It is vital for people to realise that 'hero' doctors 'know' only what the Big Pharma-dominated medical authorities tell them to 'know' and if they refuse to 'know' what they are told to 'know' they are out the door. They are mostly not physicians or healers, but repeaters of the official narrative – or else. I have seen alleged professional doctors on British television make shocking statements that we are supposed to take seriously. One called 'Dr' Amir Khan, who is actually telling patients how to respond to illness, said that men could take the birth pill to 'help slow down the effects of Covid-19'. In March, 2021, another ridiculous 'Covid study' by an American doctor proposed injecting men with the female sex hormone progesterone as a 'Covid' treatment. British doctor Nighat Arif told the BBC that face coverings were now going to be part of ongoing normal. Yes, the vaccine protects you, she said (evidence?) ... but the way to deal with viruses in the community was always going to come down to hand washing, face covering and keeping a physical distance. That's not what we were told before the 'vaccine' was circulating. Arif said she couldn't imagine ever again going on the underground or in a lift without a mask. I was just thanking my good luck that she was not my doctor when she said – in March, 2021 – that if 'we are *behaving* and we are doing all the right things' she thought we could 'have our nearest and dearest around us at home ... around *Christmas* and *New Year!*' Her patronising delivery was the usual school teacher talking to six-year-olds as she repeated every government talking point and probably believed them all. If we have learned anything from the 'Covid' experience surely it must be that humanity's perception of doctors needs a fundamental rethink. NHS 'doctor' Sara Kayat told her television audience that the 'Covid vaccine' would '100 percent prevent hospitalisation and death'. Not even Big Pharma claimed that. We have to stop taking 'experts' at their word without question when so many of them are

clueless and only repeating the party line on which their careers depend. That is not to say there are not brilliant doctors – there are and I have spoken to many of them since all this began – but you won't see them in the mainstream media or quoted by the psychopaths and yes-people in government.

## **Remember the name – Christian Drosten**

German virologist Christian Drosten, Director of Charité Institute of Virology in Berlin, became a national star after the pandemic hoax began. He was feted on television and advised the German government on 'Covid' policy. Most importantly to the wider world Drosten led a group that produced the 'Covid' testing protocol for the PCR test. What a remarkable feat given the PCR cannot test for infectious disease and even more so when you think that Drosten said that his method of testing for SARS-CoV-2 was developed 'without having virus material available'. *He developed a test for a 'virus' that he didn't have and had never seen.* Let that sink in as you survey the global devastation that came from what he did. The whole catastrophe of Drosten's 'test' was based on the alleged genetic sequence published by Chinese scientists on the Internet. We will see in the next chapter that this alleged 'genetic sequence' has never been produced by China or anyone and cannot be when there is no SARS-CoV-2. Drosten, however, doesn't seem to let little details like that get in the way. He was the lead author with Victor Corman from the same Charité Hospital of the paper 'Detection of 2019 novel coronavirus (2019-nCoV) by real-time PCR' published in a magazine called *Eurosurveillance*. This became known as the Corman-Drosten paper. In November, 2020, with human society devastated by the effects of the Corman-Drosten test baloney, the protocol was publicly challenged by 22 international scientists and independent researchers from Europe, the United States, and Japan. Among them were senior molecular geneticists, biochemists, immunologists, and microbiologists. They produced a document headed 'External peer review of the RTPCR test to detect SARS-Cov-2 Reveals 10 Major Flaws At The Molecular and Methodological Level: Consequences

For False-Positive Results'. The flaws in the Corman-Drosten test included the following:

- The test is non-specific because of erroneous design
- Results are enormously variable
- The test is unable to discriminate between the whole 'virus' and viral fragments
- It doesn't have positive or negative controls
- The test lacks a standard operating procedure
- It is unsupported by proper peer view

The scientists said the PCR 'Covid' testing protocol was not founded on science and they demanded the Corman-Drosten paper be retracted by *Eurosurveillance*. They said all present and previous Covid deaths, cases, and 'infection rates' should be subject to a massive retroactive inquiry. Lockdowns and travel restrictions should be reviewed and relaxed and those diagnosed through PCR to have 'Covid-19' should not be forced to isolate. Dr Kevin Corbett, a health researcher and nurse educator with a long academic career producing a stream of peer-reviewed publications at many UK universities, made the same point about the PCR test debacle. He said of the scientists' conclusions: 'Every scientific rationale for the development of that test has been totally destroyed by this paper. It's like Hiroshima/Nagasaki to the Covid test.' He said that China hadn't given them an isolated 'virus' when Drosten developed the test. Instead they had developed the test from *a sequence in a gene bank.*' Put another way ... *they made it up!* The scientists were supported in this contention by a Portuguese appeals court which ruled in November, 2020, that PCR tests are unreliable and it is unlawful to quarantine people based solely on a PCR test. The point about China not providing an isolated virus must be true when the 'virus' has never been isolated to this day and the consequences of that will become clear. Drosten and company produced this useless 'protocol' right on cue in January, 2020, just as the 'virus' was said to

be moving westward and it somehow managed to successfully pass a peer-review in 24 hours. In other words there was no peer-review for a test that would be used to decide who had 'Covid' and who didn't across the world. The Cult-created, Gates-controlled World Health Organization immediately recommended all its nearly 200 member countries to use the Drosten PCR protocol to detect 'cases' and 'deaths'. The sting was underway and it continues to this day.

So who is this Christian Drosten that produced the means through which death, destruction and economic catastrophe would be justified? His education background, including his doctoral thesis, would appear to be somewhat shrouded in mystery and his track record is dire as with another essential player in the 'Covid' hoax, the Gates-funded Professor Neil Ferguson at the Gates-funded Imperial College in London of whom more shortly. Drosten predicted in 2003 that the alleged original SARS 'virus' (SARS-1') was an epidemic that could have serious effects on economies and an effective vaccine would take at least two years to produce. Drosten's answer to every alleged 'outbreak' is a vaccine which you won't be shocked to know. What followed were just 774 official deaths worldwide and none in Germany where there were only nine cases. That is even if you believe there ever was a SARS 'virus' when the evidence is zilch and I will expand on this in the next chapter. Drosten claims to be co-discoverer of 'SARS-1' and developed a test for it in 2003. He was screaming warnings about 'swine flu' in 2009 and how it was a widespread infection far more severe than any dangers from a vaccine could be and people should get vaccinated. It would be helpful for Drosten's vocal chords if he simply recorded the words 'the virus is deadly and you need to get vaccinated' and copies could be handed out whenever the latest made-up threat comes along. Drosten's swine flu epidemic never happened, but Big Pharma didn't mind with governments spending hundreds of millions on vaccines that hardly anyone bothered to use and many who did wished they hadn't. A study in 2010 revealed that the risk of dying from swine flu, or H1N1, was no higher than that of the annual seasonal flu which is what at least most of 'it' really was as in

the case of ‘Covid-19’. A media investigation into Drosten asked how with such a record of inaccuracy he could be *the* government adviser on these issues. The answer to that question is the same with Drosten, Ferguson and Fauci – they keep on giving the authorities the ‘conclusions’ and ‘advice’ they want to hear. Drosten certainly produced the goods for them in January, 2020, with his PCR protocol garbage and provided the foundation of what German internal medicine specialist Dr Claus Köhnlein, co-author of *Virus Mania*, called the ‘test pandemic’. The 22 scientists in the *Eurosurveillance* challenge called out conflicts of interest within the Drosten ‘protocol’ group and with good reason. Olfert Landt, a regular co-author of Drosten ‘studies’, owns the biotech company TIB Molbiol Syntheselabor GmbH in Berlin which manufactures and sells the tests that Drosten and his mates come up with. They have done this with SARS, Enterotoxigenic E. coli (ETEC), MERS, Zika ‘virus’, yellow fever, and now ‘Covid’. Landt told the *Berliner Zeitung* newspaper:

The testing, design and development came from the Charité [Drosten and Corman]. We simply implemented it immediately in the form of a kit. And if we don’t have the virus, which originally only existed in Wuhan, we can make a synthetic gene to simulate the genome of the virus. That’s what we did very quickly.

This is more confirmation that the Drosten test was designed without access to the ‘virus’ and only a synthetic simulation which is what SARS-CoV-2 really is – a computer-generated synthetic fiction. It’s quite an enterprise they have going here. A Drosten team decides what the test for something should be and Landt’s biotech company flogs it to governments and medical systems across the world. His company must have made an absolute fortune since the ‘Covid’ hoax began. Dr Reiner Fuellmich, a prominent German consumer protection trial lawyer in Germany and California, is on Drosten’s case and that of Tedros at the World Health Organization for crimes against humanity with a class-action lawsuit being prepared in the United States and other legal action in Germany.

## **Why China?**

Scamming the world with a ‘virus’ that doesn’t exist would seem impossible on the face of it, but not if you have control of the relatively few people that make policy decisions and the great majority of the global media. Remember it’s not about changing ‘real’ reality it’s about controlling *perception* of reality. You don’t have to make something happen you only have to make people *believe* that it’s happening. Renegade Minds understand this and are therefore much harder to swindle. ‘Covid-19’ is not a ‘real’ ‘virus’. It’s a mind virus, like a computer virus, which has infected the minds, not the bodies, of billions. It all started, publically at least, in China and that alone is of central significance. The Cult was behind the revolution led by its asset Mao Zedong, or Chairman Mao, which established the People’s Republic of China on October 1st, 1949. It should have been called The Cult’s Republic of China, but the name had to reflect the recurring illusion that vicious dictatorships are run by and for the people (see all the ‘Democratic Republics’ controlled by tyrants). In the same way we have the ‘Biden’ Democratic Republic of America officially ruled by a puppet tyrant (at least temporarily) on behalf of Cult tyrants. The creation of Mao’s merciless communist/fascist dictatorship was part of a frenzy of activity by the Cult at the conclusion of World War Two which, like the First World War, it had instigated through its assets in Germany, Britain, France, the United States and elsewhere. Israel was formed in 1948; the Soviet Union expanded its ‘Iron Curtain’ control, influence and military power with the Warsaw Pact communist alliance in 1955; the United Nations was formed in 1945 as a Cult precursor to world government; and a long list of world bodies would be established including the World Health Organization (1948), World Trade Organization (1948 under another name until 1995), International Monetary Fund (1945) and World Bank (1944). Human society was redrawn and hugely centralised in the global Problem-Reaction-Solution that was World War Two. All these changes were significant. Israel would become the headquarters of the Sabbatians

and the revolution in China would prepare the ground and control system for the events of 2019/2020.

Renegade Minds know there are no borders except for public consumption. The Cult is a seamless, borderless global entity and to understand the game we need to put aside labels like borders, nations, countries, communism, fascism and democracy. These delude the population into believing that countries are ruled within their borders by a government of whatever shade when these are mere agencies of a global power. America's illusion of democracy and China's communism/fascism are subsidiaries – vehicles – for the same agenda. We may hear about conflict and competition between America and China and on the lower levels that will be true; but at the Cult level they are branches of the same company in the way of the McDonald's example I gave earlier. I have tracked in the books over the years support by US governments of both parties for Chinese Communist Party infiltration of American society through allowing the sale of land, even military facilities, and the acquisition of American business and university influence. All this is underpinned by the infamous stealing of intellectual property and technological know-how. Cult-owned Silicon Valley corporations waive their fraudulent 'morality' to do business with human-rights-free China; Cult-controlled Disney has become China's PR department; and China in effect owns 'American' sports such as basketball which depends for much of its income on Chinese audiences. As a result any sports player, coach or official speaking out against China's horrific human rights record is immediately condemned or fired by the China-worshipping National Basketball Association. One of the first acts of China-controlled Biden was to issue an executive order telling federal agencies to stop making references to the 'virus' by the 'geographic location of its origin'. Long-time Congressman Jerry Nadler warned that criticising China, America's biggest rival, leads to hate crimes against Asian people in the United States. So shut up you bigot. China is fast closing in on Israel as a country that must not be criticised which is apt, really, given that Sabbatians control them both. The two countries have

developed close economic, military, technological and strategic ties which include involvement in China's 'Silk Road' transport and economic initiative to connect China with Europe. Israel was the first country in the Middle East to recognise the establishment of Mao's tyranny in 1950 months after it was established.

### **Project Wuhan – the 'Covid' Psyop**

I emphasise again that the Cult plays the long game and what is happening to the world today is the result of centuries of calculated manipulation following a script to take control step-by-step of every aspect of human society. I will discuss later the common force behind all this that has spanned those centuries and thousands of years if the truth be told. Instigating the Mao revolution in China in 1949 with a 2020 'pandemic' in mind is not only how they work – the 71 years between them is really quite short by the Cult's standards of manipulation preparation. The reason for the Cult's Chinese revolution was to create a fiercely-controlled environment within which an extreme structure for human control could be incubated to eventually be unleashed across the world. We have seen this happen since the 'pandemic' emerged from China with the Chinese control-structure founded on AI technology and tyrannical enforcement sweep across the West. Until the moment when the Cult went for broke in the West and put its fascism on public display Western governments had to pay some lip-service to freedom and democracy to not alert too many people to the tyranny-in-the-making. Freedoms were more subtly eroded and power centralised with covert government structures put in place waiting for the arrival of 2020 when that smokescreen of 'freedom' could be dispensed with. The West was not able to move towards tyranny before 2020 anything like as fast as China which was created as a tyranny and had no limits on how fast it could construct the Cult's blueprint for global control. When the time came to impose that structure on the world it was the same Cult-owned Chinese communist/fascist government that provided the excuse – the 'Covid pandemic'. It was absolutely crucial to the Cult plan for the Chinese response to the 'pandemic' –

draconian lockdowns of the entire population – to become the blueprint that Western countries would follow to destroy the livelihoods and freedom of their people. This is why the Cult-owned, Gates-owned, WHO Director-General Tedros said early on:

The Chinese government is to be congratulated for the extraordinary measures it has taken to contain the outbreak. China is actually setting a new standard for outbreak response and it is not an exaggeration.

*Forbes* magazine said of China: ‘... those measures protected untold millions from getting the disease’. The Rockefeller Foundation ‘epidemic scenario’ document in 2010 said ‘prophetically’:

However, a few countries did fare better – China in particular. The Chinese government’s quick imposition and enforcement of mandatory quarantine for all citizens, as well as its instant and near-hermetic sealing off of all borders, saved millions of lives, stopping the spread of the virus far earlier than in other countries and enabling a swifter post-pandemic recovery.

Once again – *spooky*.

The first official story was the ‘bat theory’ or rather the bat diversion. The source of the ‘virus outbreak’ we were told was a “wet market” in Wuhan where bats and other animals are bought and eaten in horrifically unhygienic conditions. Then another story emerged through the alternative media that the ‘virus’ had been released on purpose or by accident from a BSL-4 (biosafety level 4) laboratory in Wuhan not far from the wet market. The lab was reported to create and work with lethal concoctions and bioweapons. Biosafety level 4 is the highest in the World Health Organization system of safety and containment. Renegade Minds are aware of what I call designer manipulation. The ideal for the Cult is for people to buy its prime narrative which in the opening salvos of the ‘pandemic’ was the wet market story. It knows, however, that there is now a considerable worldwide alternative media of researchers sceptical of anything governments say and they are often given a version of events in a form they can perceive as credible while misdirecting them from the real truth. In this case let them

think that the conspiracy involved is a ‘bioweapon virus’ released from the Wuhan lab to keep them from the real conspiracy – *there is no ‘virus’*. The WHO’s current position on the source of the outbreak at the time of writing appears to be: ‘We haven’t got a clue, mate.’ This is a good position to maintain mystery and bewilderment. The inner circle will know where the ‘virus’ came from – *nowhere*. The bottom line was to ensure the public believed there *was* a ‘virus’ and it didn’t much matter if they thought it was natural or had been released from a lab. The belief that there was a ‘deadly virus’ was all that was needed to trigger global panic and fear. The population was terrified into handing their power to authority and doing what they were told. They had to or they were ‘all gonna die’.

In March, 2020, information began to come my way from real doctors and scientists and my own additional research which had my intuition screaming: ‘Yes, that’s it! *There is no virus.*’ The ‘bioweapon’ was not the ‘virus’; it was the ‘vaccine’ already being talked about that would be the bioweapon. My conclusion was further enhanced by happenings in Wuhan. The ‘virus’ was said to be sweeping the city and news footage circulated of people collapsing in the street (which they’ve never done in the West with the same ‘virus’). The Chinese government was building ‘new hospitals’ in a matter of ten days to ‘cope with demand’ such was the virulent nature of the ‘virus’. Yet in what seemed like no time the ‘new hospitals’ closed – even if they even opened – and China declared itself ‘virus-free’. It was back to business as usual. This was more propaganda to promote the Chinese draconian lockdowns in the West as the way to ‘beat the virus’. Trouble was that we subsequently had lockdown after lockdown, but never business as usual. As the people of the West and most of the rest of the world were caught in an ever-worsening spiral of lockdown, social distancing, masks, isolated old people, families forced apart, and livelihood destruction, it was party-time in Wuhan. Pictures emerged of thousands of people enjoying pool parties and concerts. It made no sense until you realised there never was a ‘virus’ and the

whole thing was a Cult set-up to transform human society out of one its major global strongholds – China.

How is it possible to deceive virtually the entire world population into believing there is a deadly virus when there is not even a ‘virus’ let alone a deadly one? It’s nothing like as difficult as you would think and that’s clearly true because it happened.

**Postscript:** See end of book Postscript for more on the ‘Wuhan lab virus release’ story which the authorities and media were pushing heavily in the summer of 2021 to divert attention from the truth that the ‘Covid virus’ is pure invention.

## CHAPTER FIVE

### **There is no ‘virus’**

***You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time***

**Abraham Lincoln**

The greatest form of mind control is repetition. The more you repeat the same mantra of alleged ‘facts’ the more will accept them to be true. It becomes an ‘everyone knows that, mate’. If you can also censor any other version or alternative to your alleged ‘facts’ you are pretty much home and cooking.

By the start of 2020 the Cult owned the global mainstream media almost in its entirety to spew out its ‘Covid’ propaganda and ignore or discredit any other information and view. Cult-owned social media platforms in Cult-owned Silicon Valley were poised and ready to unleash a campaign of ferocious censorship to obliterate all but the official narrative. To complete the circle many demands for censorship by Silicon Valley were led by the mainstream media as ‘journalists’ became full-out enforcers for the Cult both as propagandists and censors. Part of this has been the influx of young people straight out of university who have become ‘journalists’ in significant positions. They have no experience and a headful of programmed perceptions from their years at school and university at a time when today’s young are the most perceptually-targeted generations in known human history given the insidious impact of technology. They enter the media perceptually prepared and ready to repeat the narratives of the system that programmed them to

repeat its narratives. The BBC has a truly pathetic ‘specialist disinformation reporter’ called Marianna Spring who fits this bill perfectly. She is clueless about the world, how it works and what is really going on. Her role is to discredit anyone doing the job that a proper journalist would do and system-serving hacks like Spring wouldn’t dare to do or even see the need to do. They are too busy licking the arse of authority which can never be wrong and, in the case of the BBC propaganda programme, *Panorama*, contacting payments systems such as PayPal to have a donations page taken down for a film company making documentaries questioning vaccines. Even the BBC soap opera *EastEnders* included a disgracefully biased scene in which an inarticulate white working class woman was made to look foolish for questioning the ‘vaccine’ while a well-spoken black man and Asian woman promoted the government narrative. It ticked every BBC box and the fact that the black and minority community was resisting the ‘vaccine’ had nothing to do with the way the scene was written. The BBC has become a disgusting tyrannical propaganda and censorship operation that should be defunded and disbanded and a free media take its place with a brief to stop censorship instead of demanding it. A BBC ‘interview’ with Gates goes something like: ‘Mr Gates, sir, if I can call you sir, would you like to tell our audience why you are such a great man, a wonderful humanitarian philanthropist, and why you should absolutely be allowed as a software salesman to decide health policy for approaching eight billion people? Thank you, sir, please sir.’ Propaganda programming has been incessant and merciless and when all you hear is the same story from the media, repeated by those around you who have only heard the same story, is it any wonder that people on a grand scale believe absolute mendacious garbage to be true? You are about to see, too, why this level of information control is necessary when the official ‘Covid’ narrative is so nonsensical and unsupportable by the evidence.

## **Structure of Deceit**

The pyramid structure through which the ‘Covid’ hoax has been manifested is very simple and has to be to work. As few people as possible have to be involved with full knowledge of what they are doing – and why – or the real story would get out. At the top of the pyramid are the inner core of the Cult which controls Bill Gates who, in turn, controls the World Health Organization through his pivotal funding and his puppet Director-General mouthpiece, Tedros.

Before he was appointed Tedros was chair of the Gates-founded Global Fund to ‘fight against AIDS, tuberculosis and malaria’, a board member of the Gates-funded ‘vaccine alliance’ GAVI, and on the board of another Gates-funded organisation. Gates owns him and picked him for a specific reason – Tedros is a crook and worse. ‘Dr’ Tedros (he’s not a medical doctor, the first WHO chief not to be) was a member of the tyrannical Marxist government of Ethiopia for decades with all its human rights abuses. He has faced allegations of corruption and misappropriation of funds and was exposed three times for covering up cholera epidemics while Ethiopia’s health minister. Tedros appointed the mass-murdering genocidal Zimbabwe dictator Robert Mugabe as a WHO goodwill ambassador for public health which, as with Tedros, is like appointing a psychopath to run a peace and love campaign. The move was so ridiculous that he had to drop Mugabe in the face of widespread condemnation. American economist David Steinman, a Nobel peace prize nominee, lodged a complaint with the International Criminal Court in The Hague over alleged genocide by Tedros when he was Ethiopia’s foreign minister. Steinman says Tedros was a ‘crucial decision maker’ who directed the actions of Ethiopia’s security forces from 2013 to 2015 and one of three officials in charge when those security services embarked on the ‘killing’ and ‘torturing’ of Ethiopians. You can see where Tedros is coming from and it’s sobering to think that he has been the vehicle for Gates and the Cult to direct the global response to ‘Covid’. Think about that. A psychopathic Cult dictates to psychopath Gates who dictates to psychopath Tedros who dictates how countries of the world must respond to a ‘Covid virus’ never scientifically shown to exist. At the same time psychopathic Cult-owned Silicon Valley information

giants like Google, YouTube, Facebook and Twitter announced very early on that they would give the Cult/Gates/Tedros/WHO version of the narrative free advertising and censor those who challenged their intelligence-insulting, mendacious story.

The next layer in the global ‘medical’ structure below the Cult, Gates and Tedros are the chief medical officers and science ‘advisers’ in each of the WHO member countries which means virtually all of them. Medical officers and arbiters of science (they’re not) then take the WHO policy and recommended responses and impose them on their country’s population while the political ‘leaders’ say they are deciding policy (they’re clearly not) by ‘following the science’ on the advice of the ‘experts’ – the same medical officers and science ‘advisers’ (dictators). In this way with the rarest of exceptions the entire world followed the same policy of lockdown, people distancing, masks and ‘vaccines’ dictated by the psychopathic Cult, psychopathic Gates and psychopathic Tedros who we are supposed to believe give a damn about the health of the world population they are seeking to enslave. That, amazingly, is all there is to it in terms of crucial decision-making. Medical staff in each country then follow like sheep the dictates of the shepherds at the top of the national medical hierarchies – chief medical officers and science ‘advisers’ who themselves follow like sheep the shepherds of the World Health Organization and the Cult. Shepherds at the national level often have major funding and other connections to Gates and his Bill and Melinda Gates Foundation which carefully hands out money like confetti at a wedding to control the entire global medical system from the WHO down.

## **Follow the money**

Christopher Whitty, Chief Medical Adviser to the UK Government at the centre of ‘virus’ policy, a senior adviser to the government’s Scientific Advisory Group for Emergencies (SAGE), and Executive Board member of the World Health Organization, was gifted a grant of \$40 million by the Bill and Melinda Gates Foundation for malaria research in Africa. The BBC described the unelected Whitty as ‘the

official who will probably have the greatest impact on our everyday lives of any individual policymaker in modern times' and so it turned out. What Gates and Tedros have said Whitty has done like his equivalents around the world. Patrick Vallance, co-chair of SAGE and the government's Chief Scientific Adviser, is a former executive of Big Pharma giant GlaxoSmithKline with its fundamental financial and business connections to Bill Gates. In September, 2020, it was revealed that Vallance owned a deferred bonus of shares in GlaxoSmithKline worth £600,000 while the company was 'developing' a 'Covid vaccine'. Move along now – nothing to see here – what could possibly be wrong with that? Imperial College in London, a major player in 'Covid' policy in Britain and elsewhere with its 'Covid-19' Response Team, is funded by Gates and has big connections to China while the now infamous Professor Neil Ferguson, the useless 'computer modeller' at Imperial College is also funded by Gates. Ferguson delivered the dramatically inaccurate excuse for the first lockdowns (much more in the next chapter). The Institute for Health Metrics and Evaluation (IHME) in the United States, another source of outrageously false 'Covid' computer models to justify lockdowns, is bankrolled by Gates who is a vehement promotor of lockdowns. America's version of Whitty and Vallance, the again now infamous Anthony Fauci, has connections to 'Covid vaccine' maker Moderna as does Bill Gates through funding from the Bill and Melinda Gates Foundation. Fauci is director of the National Institute of Allergy and Infectious Diseases (NIAID), a major recipient of Gates money, and they are very close. Deborah Birx who was appointed White House Coronavirus Response Coordinator in February, 2020, is yet another with ties to Gates. Everywhere you look at the different elements around the world behind the coordination and decision making of the 'Covid' hoax there is Bill Gates and his money. They include the World Health Organization; Centers for Disease Control (CDC) in the United States; National Institutes of Health (NIH) of Anthony Fauci; Imperial College and Neil Ferguson; the London School of Hygiene where Chris Whitty worked; Regulatory agencies like the UK Medicines & Healthcare products Regulatory Agency (MHRA)

which gave emergency approval for ‘Covid vaccines’; Wellcome Trust; GAVI, the Vaccine Alliance; the Coalition for Epidemic Preparedness Innovations (CEPI); Johns Hopkins University which has compiled the false ‘Covid’ figures; and the World Economic Forum. A [Nationalfile.com](#) article said:

Gates has a lot of pull in the medical world, he has a multi-million dollar relationship with Dr. Fauci, and Fauci originally took the Gates line supporting vaccines and casting doubt on [the drug hydroxychloroquine]. Coronavirus response team member Dr. Deborah Birx, appointed by former president Obama to serve as United States Global AIDS Coordinator, also sits on the board of a group that has received billions from Gates’ foundation, and Birx reportedly used a disputed Bill Gates-funded model for the White House’s Coronavirus effort. Gates is a big proponent for a population lockdown scenario for the Coronavirus outbreak.

Another funder of Moderna is the Defense Advanced Research Projects Agency (DARPA), the technology-development arm of the Pentagon and one of the most sinister organisations on earth. DARPA had a major role with the CIA covert technology-funding operation In-Q-Tel in the development of Google and social media which is now at the centre of global censorship. Fauci and Gates are extremely close and openly admit to talking regularly about ‘Covid’ policy, but then why wouldn’t Gates have a seat at every national ‘Covid’ table after his Foundation committed \$1.75 billion to the ‘fight against Covid-19’. When passed through our Orwellian Translation Unit this means that he has bought and paid for the Cult-driven ‘Covid’ response worldwide. Research the major ‘Covid’ response personnel in your own country and you will find the same Gates funding and other connections again and again. Medical and science chiefs following World Health Organization ‘policy’ sit atop a medical hierarchy in their country of administrators, doctors and nursing staff. These ‘subordinates’ are told they must work and behave in accordance with the policy delivered from the ‘top’ of the national ‘health’ pyramid which is largely the policy delivered by the WHO which is the policy delivered by Gates and the Cult. The whole ‘Covid’ narrative has been imposed on medical staff by a climate of fear although great numbers don’t even need that to comply. They do so through breathtaking levels of ignorance and

include doctors who go through life simply repeating what Big Pharma and their hierarchical masters tell them to say and believe. No wonder Big Pharma ‘medicine’ is one of the biggest killers on Planet Earth.

The same top-down system of intimidation operates with regard to the Cult Big Pharma cartel which also dictates policy through national and global medical systems in this way. The Cult and Big Pharma agendas are the same because the former controls and owns the latter. ‘Health’ administrators, doctors, and nursing staff are told to support and parrot the dictated policy or they will face consequences which can include being fired. How sad it’s been to see medical staff meekly repeating and imposing Cult policy without question and most of those who can see through the deceit are only willing to speak anonymously off the record. They know what will happen if their identity is known. This has left the courageous few to expose the lies about the ‘virus’, face masks, overwhelmed hospitals that aren’t, and the dangers of the ‘vaccine’ that isn’t a vaccine. When these medical professionals and scientists, some renowned in their field, have taken to the Internet to expose the truth their articles, comments and videos have been deleted by Cult-owned Facebook, Twitter and YouTube. What a real head-shaker to see YouTube videos with leading world scientists and highly qualified medical specialists with an added link underneath to the notorious Cult propaganda website *Wikipedia* to find the ‘facts’ about the same subject.

### **HIV – the ‘Covid’ trial-run**

I’ll give you an example of the consequences for health and truth that come from censorship and unquestioning belief in official narratives. The story was told by PCR inventor Kary Mullis in his book *Dancing Naked in the Mind Field*. He said that in 1984 he accepted as just another scientific fact that Luc Montagnier of France’s Pasteur Institute and Robert Gallo of America’s National Institutes of Health had independently discovered that a ‘retrovirus’ dubbed HIV (human immunodeficiency virus) caused AIDS. They

were, after all, Mullis writes, specialists in retroviruses. This is how the medical and science pyramids work. Something is announced or *assumed* and then becomes an everybody-knows-that purely through repetition of the assumption as if it is fact. Complete crap becomes accepted truth with no supporting evidence and only repetition of the crap. This is how a 'virus' that doesn't exist became the 'virus' that changed the world. The HIV-AIDS fairy story became a multi-billion pound industry and the media poured out propaganda terrifying the world about the deadly HIV 'virus' that caused the lethal AIDS. By then Mullis was working at a lab in Santa Monica, California, to detect retroviruses with his PCR test in blood donations received by the Red Cross. In doing so he asked a virologist where he could find a reference for HIV being the cause of AIDS. 'You don't need a reference,' the virologist said ... '*Everybody knows it.*' Mullis said he wanted to quote a reference in the report he was doing and he said he felt a little funny about not knowing the source of such an important discovery when everyone else seemed to. The virologist suggested he cite a report by the Centers for Disease Control and Prevention (CDC) on morbidity and mortality. Mullis read the report, but it only said that an organism had been identified and did not say how. The report did not identify the original scientific work. Physicians, however, *assumed* (key recurring theme) that if the CDC was convinced that HIV caused AIDS then proof must exist. Mullis continues:

I did computer searches. Neither Montagnier, Gallo, nor anyone else had published papers describing experiments which led to the conclusion that HIV probably caused AIDS. I read the papers in Science for which they had become well known as AIDS doctors, but all they had said there was that they had found evidence of a past infection by something which was probably HIV in some AIDS patients.

They found antibodies. Antibodies to viruses had always been considered evidence of past disease, not present disease. Antibodies signaled that the virus had been defeated. The patient had saved himself. There was no indication in these papers that this virus caused a disease. They didn't show that everybody with the antibodies had the disease. In fact they found some healthy people with antibodies.

Mullis asked why their work had been published if Montagnier and Gallo hadn't really found this evidence, and why had they been fighting so hard to get credit for the discovery? He says he was hesitant to write 'HIV is the probable cause of AIDS' until he found published evidence to support that. 'Tens of thousands of scientists and researchers were spending billions of dollars a year doing research based on this idea,' Mullis writes. 'The reason had to be there somewhere; otherwise these people would not have allowed their research to settle into one narrow channel of investigation.' He said he lectured about PCR at numerous meetings where people were always talking about HIV and he asked them how they knew that HIV was the cause of AIDS:

Everyone said something. Everyone had the answer at home, in the office, in some drawer. They all knew, and they would send me the papers as soon as they got back. But I never got any papers. Nobody ever sent me the news about how AIDS was caused by HIV.

Eventually Mullis was able to ask Montagnier himself about the reference proof when he lectured in San Diego at the grand opening of the University of California AIDS Research Center. Mullis says this was the last time he would ask his question without showing anger. Montagnier said he should reference the CDC report. 'I read it', Mullis said, and it didn't answer the question. 'If Montagnier didn't know the answer who the hell did?' Then one night Mullis was driving when an interview came on National Public Radio with Peter Duesberg, a prominent virologist at Berkeley and a California Scientist of the Year. Mullis says he finally understood why he could not find references that connected HIV to AIDS – *there weren't any!* No one had ever proved that HIV causes AIDS even though it had spawned a multi-billion pound global industry and the media was repeating this as fact every day in their articles and broadcasts terrifying the shit out of people about AIDS and giving the impression that a positive test for HIV (see 'Covid') was a death sentence. Duesberg was a threat to the AIDS gravy train and the agenda that underpinned it. He was therefore abused and castigated after he told the Proceedings of the National Academy of Sciences

there was no good evidence implicating the new ‘virus’. Editors rejected his manuscripts and his research funds were deleted. Mullis points out that the CDC has defined AIDS as one of more than 30 diseases *if accompanied* by a positive result on a test that detects antibodies to HIV; but those same diseases are not defined as AIDS cases when antibodies are not detected:

If an HIV-positive woman develops uterine cancer, for example, she is considered to have AIDS. If she is not HIV positive, she simply has uterine cancer. An HIV-positive man with tuberculosis has AIDS; if he tests negative he simply has tuberculosis. If he lives in Kenya or Colombia, where the test for HIV antibodies is too expensive, he is simply presumed to have the antibodies and therefore AIDS, and therefore he can be treated in the World Health Organization’s clinic. It’s the only medical help available in some places. And it’s free, because the countries that support WHO are worried about AIDS.

Mullis accuses the CDC of continually adding new diseases (see ever more ‘Covid symptoms’) to the grand AIDS definition and of virtually doctoring the books to make it appear as if the disease continued to spread. He cites how in 1993 the CDC enormously broadened its AIDS definition and county health authorities were delighted because they received \$2,500 per year from the Federal government for every reported AIDS case. Ladies and gentlemen, I have just described, via Kary Mullis, the ‘Covid pandemic’ of 2020 and beyond. Every element is the same and it’s been pulled off in the same way by the same networks.

### **The ‘Covid virus’ exists? Okay – prove it. Er ... still waiting**

What Kary Mullis described with regard to ‘HIV’ has been repeated with ‘Covid’. A claim is made that a new, or ‘novel’, infection has been found and the entire medical system of the world repeats that as fact exactly as they did with HIV and AIDS. No one in the mainstream asks rather relevant questions such as ‘How do you know?’ and ‘Where is your proof?’ The SARS-CoV-2 ‘virus’ and the ‘Covid-19 disease’ became an overnight ‘everybody-knows-that’. The origin could be debated and mulled over, but what you could not suggest was that ‘SARS-CoV-2’ didn’t exist. That would be

ridiculous. ‘Everybody knows’ the ‘virus’ exists. Well, I didn’t for one along with American proper doctors like Andrew Kaufman and Tom Cowan and long-time American proper journalist Jon Rappaport. We dared to pursue the obvious and simple question: ‘Where’s the evidence?’ The overwhelming majority in medicine, journalism and the general public did not think to ask that. After all, *everyone knew* there was a new ‘virus’. Everyone was saying so and I heard it on the BBC. Some would eventually argue that the ‘deadly virus’ was nothing like as deadly as claimed, but few would venture into the realms of its very existence. Had they done so they would have found that the evidence for that claim had gone AWOL as with HIV causes AIDS. In fact, not even that. For something to go AWOL it has to exist in the first place and scientific proof for a ‘SARS-Cov-2’ can be filed under nothing, nowhere and zilch.

Dr Andrew Kaufman is a board-certified forensic psychiatrist in New York State, a Doctor of Medicine and former Assistant Professor and Medical Director of Psychiatry at SUNY Upstate Medical University, and Medical Instructor of Hematology and Oncology at the Medical School of South Carolina. He also studied biology at the Massachusetts Institute of Technology (MIT) and trained in Psychiatry at Duke University. Kaufman is retired from allopathic medicine, but remains a consultant and educator on natural healing, I saw a video of his very early on in the ‘Covid’ hoax in which he questioned claims about the ‘virus’ in the absence of any supporting evidence and with plenty pointing the other way. I did everything I could to circulate his work which I felt was asking the pivotal questions that needed an answer. I can recommend an excellent pull-together interview he did with the website The Last Vagabond entitled *Dr Andrew Kaufman: Virus Isolation, Terrain Theory and Covid-19* and his website is [andrewkaufmanmd.com](http://andrewkaufmanmd.com). Kaufman is not only a forensic psychiatrist; he is forensic in all that he does. He always reads original scientific papers, experiments and studies instead of second-third-fourth-hand reports about the ‘virus’ in the media which are repeating the repeated repetition of the narrative. When he did so with the original Chinese ‘virus’ papers Kaufman

realised that there was no evidence of a ‘SARS-Cov-2’. They had never – from the start – shown it to exist and every repeat of this claim worldwide was based on the accepted existence of proof that was nowhere to be found – see Kary Mullis and HIV. Here we go again.

## **Let's postulate**

Kaufman discovered that the Chinese authorities immediately concluded that the cause of an illness that broke out among about 200 initial patients in Wuhan was a ‘new virus’ when there were no grounds to make that conclusion. The alleged ‘virus’ was not isolated from other genetic material in their samples and then shown through a system known as Koch’s postulates to be the causative agent of the illness. The world was told that the SARS-Cov-2 ‘virus’ caused a disease they called ‘Covid-19’ which had ‘flu-like’ symptoms and could lead to respiratory problems and pneumonia. If it wasn’t so tragic it would almost be funny. *‘Flu-like’ symptoms?* *Pneumonia? Respiratory disease?* What in CHINA and particularly in Wuhan, one of the most polluted cities in the world with a resulting epidemic of respiratory disease?? Three hundred thousand people get pneumonia in China every year and there are nearly a billion cases worldwide of ‘flu-like symptoms’. These have a whole range of causes – including pollution in Wuhan – but no other possibility was credibly considered in late 2019 when the world was told there was a new and deadly ‘virus’. The global prevalence of pneumonia and ‘flu-like systems’ gave the Cult networks unlimited potential to re-diagnose these other causes as the mythical ‘Covid-19’ and that is what they did from the very start. Kaufman revealed how Chinese medical and science authorities (all subordinates to the Cult-owned communist government) took genetic material from the lungs of only a few of the first patients. The material contained their own cells, bacteria, fungi and other microorganisms living in their bodies. The only way you could prove the existence of the ‘virus’ and its responsibility for the alleged ‘Covid-19’ was to isolate the virus from all the other material – a process also known as ‘purification’ – and

then follow the postulates sequence developed in the late 19th century by German physician and bacteriologist Robert Koch which became the ‘gold standard’ for connecting an alleged causation agent to a disease:

1. The microorganism (bacteria, fungus, virus, etc.) must be present in every case of the disease and all patients must have the same symptoms. It must also *not be present in healthy individuals*.
2. The microorganism must be isolated from the host with the disease. If the microorganism is a bacteria or fungus it must be grown in a pure culture. If it is a virus, it must be purified (i.e. containing no other material except the virus particles) from a clinical sample.
3. The specific disease, with all of its characteristics, must be reproduced when the infectious agent (the purified virus or a pure culture of bacteria or fungi) is inoculated into a healthy, susceptible host.
4. The microorganism must be recoverable from the experimentally infected host as in step 2.

*Not one* of these criteria has been met in the case of ‘SARS-Cov-2’ and ‘Covid-19’. Not ONE. EVER. Robert Koch refers to bacteria and not viruses. What are called ‘viral particles’ are so minute (hence masks are useless by any definition) that they could only be seen after the invention of the electron microscope in the 1930s and can still only be observed through that means. American bacteriologist and virologist Thomas Milton Rivers, the so-called ‘Father of Modern Virology’ who was very significantly director of the Rockefeller Institute for Medical Research in the 1930s, developed a less stringent version of Koch’s postulates to identify ‘virus’ causation known as ‘Rivers criteria’. ‘Covid’ did not pass that process either. Some even doubt whether any ‘virus’ can be isolated from other particles containing genetic material in the Koch method. Freedom of Information requests in many countries asking for scientific proof that the ‘Covid virus’ has been purified and isolated and shown to exist have all come back with a ‘we don’t have that’ and when this happened with a request to the UK Department of Health they added this comment:

However, outside of the scope of the [Freedom of Information Act] and on a discretionary basis, the following information has been advised to us, which may be of interest. Most infectious diseases are caused by viruses, bacteria or fungi. Some bacteria or fungi have the capacity to grow on their own in isolation, for example in colonies on a petri dish. Viruses are different in that they are what we call 'obligate pathogens' – that is, they cannot survive or reproduce without infecting a host ...

... For some diseases, it is possible to establish causation between a microorganism and a disease by isolating the pathogen from a patient, growing it in pure culture and reintroducing it to a healthy organism. These are known as 'Koch's postulates' and were developed in 1882. However, as our understanding of disease and different disease-causing agents has advanced, these are no longer the method for determining causation [Andrew Kaufman asks why in that case are there two published articles falsely claiming to satisfy Koch's postulates].

It has long been known that viral diseases cannot be identified in this way as viruses cannot be grown in 'pure culture'. When a patient is tested for a viral illness, this is normally done by looking for the presence of antigens, or viral genetic code in a host with molecular biology techniques [Kaufman asks how you could know the origin of these chemicals without having a pure culture for comparison].

For the record 'antigens' are defined so:

Invading microorganisms have antigens on their surface that the human body can recognise as being foreign – meaning not belonging to it. When the body recognises a foreign antigen, lymphocytes (white blood cells) produce antibodies, which are complementary in shape to the antigen.

Notwithstanding that this is open to question in relation to 'SARS-CoV-2' the presence of 'antibodies' can have many causes and they are found in people that are perfectly well. Kary Mullis said: 'Antibodies ... had always been considered evidence of past disease, not present disease.'

## **'Covid' really is a computer 'virus'**

Where the UK Department of Health statement says 'viruses' are now 'diagnosed' through a 'viral genetic code in a host with molecular biology techniques', they mean ... *the PCR test* which its inventor said cannot test for infectious disease. They have no credible method of connecting a 'virus' to a disease and we will see that there is no scientific proof that any 'virus' causes any disease or there is any such thing as a 'virus' in the way that it is described. Tenacious Canadian researcher Christine Massey and her team made

some 40 Freedom of Information requests to national public health agencies in different countries asking for proof that SARS-CoV-2 has been isolated and not one of them could supply that information. Massey said of her request in Canada: 'Freedom of Information reveals Public Health Agency of Canada has no record of 'SARS-CoV-2' isolation performed by anyone, anywhere, ever.' If you accept the comment from the UK Department of Health it's because they can't isolate a 'virus'. Even so many 'science' papers claimed to have isolated the 'Covid virus' until they were questioned and had to admit they hadn't. A reply from the Robert Koch Institute in Germany was typical: 'I am not aware of a paper which purified isolated SARS-CoV-2.' So what the hell was Christian Drosten and his gang using to design the 'Covid' testing protocol that has produced all the illusory Covid' cases and 'Covid' deaths when the head of the Chinese version of the CDC admitted there was a problem right from the start in that the 'virus' had never been isolated/purified? Breathe deeply: What they are calling 'Covid' is actually created by a *computer program* i.e. *they made it up* – er, that's it. They took lung fluid, with many sources of genetic material, from one single person alleged to be infected with Covid-19 by a PCR test which they *claimed*, without clear evidence, contained a 'virus'. They used several computer programs to create a model of a theoretical virus genome sequence from more than fifty-six million small sequences of RNA, each of an unknown source, assembling them like a puzzle with no known solution. The computer filled in the gaps with sequences from bits in the gene bank to make it look like a bat SARS-like coronavirus! A wave of the magic wand and poof, an *in silico* (computer-generated) genome, a scientific fantasy, was created. UK health researcher Dr Kevin Corbett made the same point with this analogy:

... It's like giving you a few bones and saying that's your fish. It could be any fish. Not even a skeleton. Here's a few fragments of bones. That's your fish ... It's all from gene bank and the bits of the virus sequence that weren't there they made up.

They synthetically created them to fill in the blanks. That's what genetics is; it's a code. So it's ABBBCCDDDD and you're missing some what you think is EEE so you put it in. It's all

synthetic. You just manufacture the bits that are missing. This is the end result of the geneticization of virology. This is basically a computer virus.

Further confirmation came in an email exchange between British citizen journalist Frances Leader and the government's Medicines & Healthcare Products Regulatory Agency (the Gates-funded MHRA) which gave emergency permission for untested 'Covid vaccines' to be used. The agency admitted that the 'vaccine' is not based on an isolated 'virus', but comes from a *computer-generated model*. Frances Leader was naturally banned from Cult-owned fascist Twitter for making this exchange public. The process of creating computer-generated alleged 'viruses' is called 'in silico' or 'in silicon' – computer chips – and the term 'in silico' is believed to originate with biological experiments using only a computer in 1989. 'Vaccines' involved with 'Covid' are also produced 'in silico' or by computer not a natural process. If the original 'virus' is nothing more than a made-up computer model how can there be 'new variants' of something that never existed in the first place? They are not new 'variants'; they are new *computer models* only minutely different to the original program and designed to further terrify the population into having the 'vaccine' and submitting to fascism. You want a 'new variant'? Click, click, enter – there you go. Tell the medical profession that you have discovered a 'South African variant', 'UK variants' or a 'Brazilian variant' and in the usual HIV-causes-AIDS manner they will unquestioningly repeat it with no evidence whatsoever to support these claims. They will go on television and warn about the dangers of 'new variants' while doing nothing more than repeating what they have been told to be true and knowing that any deviation from that would be career suicide. Big-time insiders will know it's a hoax, but much of the medical community is clueless about the way they are being played and themselves play the public without even being aware they are doing so. What an interesting 'coincidence' that AstraZeneca and Oxford University were conducting 'Covid vaccine trials' in the three countries – the UK, South Africa and Brazil – where the first three 'variants' were claimed to have 'broken out'.

## **Here's your 'virus' – it's a unicorn**

Dr Andrew Kaufman presented a brilliant analysis describing how the 'virus' was imagined into fake existence when he dissected an article published by *Nature* and written by 19 authors detailing *alleged* 'sequencing of a complete viral genome' of the 'new SARS-CoV-2 virus'. This computer-modelled *in silico* genome was used as a template for all subsequent genome sequencing experiments that resulted in the so-called variants which he said now number more than 6,000. The fake genome was constructed from more than 56 million individual short strands of RNA. Those little pieces were assembled into longer pieces by finding areas of overlapping sequences. The computer programs created over two million possible combinations from which the authors simply chose the longest one. They then compared this to a 'bat virus' and the computer 'alignment' rearranged the sequence and filled in the gaps! They called this computer-generated abomination the 'complete genome'. Dr Tom Cowan, a fellow medical author and collaborator with Kaufman, said such computer-generation constitutes scientific fraud and he makes this superb analogy:

Here is an equivalency: A group of researchers claim to have found a unicorn because they found a piece of a hoof, a hair from a tail, and a snippet of a horn. They then add that information into a computer and program it to re-create the unicorn, and they then claim this computer re-creation is the real unicorn. Of course, they had never actually seen a unicorn so could not possibly have examined its genetic makeup to compare their samples with the actual unicorn's hair, hooves and horn.

The researchers claim they decided which is the real genome of SARS-CoV-2 by 'consensus', sort of like a vote. Again, different computer programs will come up with different versions of the imaginary 'unicorn', so they come together as a group and decide which is the real imaginary unicorn.

This is how the 'virus' that has transformed the world was brought into fraudulent 'existence'. Extraordinary, yes, but as the Nazis said the bigger the lie the more will believe it. Cowan, however, wasn't finished and he went on to identify what he called the real blockbuster in the paper. He quotes this section from a paper written

by virologists and published by the CDC and then explains what it means:

Therefore, we examined the capacity of SARS-CoV-2 to infect and replicate in several common primate and human cell lines, including human adenocarcinoma cells (A549), human liver cells (HUH 7.0), and human embryonic kidney cells (HEK-293T). In addition to Vero E6 and Vero CCL81 cells. ... Each cell line was inoculated at high multiplicity of infection and examined 24h post-infection.

No CPE was observed in any of the cell lines except in Vero cells, which grew to greater than 10 to the 7th power at 24 h post-infection. In contrast, HUH 7.0 and 293T showed only modest viral replication, and A549 cells were incompatible with SARS CoV-2 infection.

Cowan explains that when virologists attempt to prove infection they have three possible 'hosts' or models on which they can test. The first was humans. Exposure to humans was generally not done for ethical reasons and has never been done with SARS-CoV-2 or any coronavirus. The second possible host was animals. Cowan said that forgetting for a moment that they never actually use purified virus when exposing animals they do use solutions that they *claim* contain the virus. Exposure to animals has been done with SARS-CoV-2 in an experiment involving mice and this is what they found: *None of the wild (normal) mice got sick*. In a group of genetically-modified mice, a statistically insignificant number lost weight and had slightly bristled fur, but they experienced nothing like the illness called 'Covid-19'. Cowan said the third method – the one they mostly rely on – is to inoculate solutions they *say* contain the virus onto a variety of tissue cultures. This process had never been shown to kill tissue *unless* the sample material was starved of nutrients and poisoned as *part of the process*. Yes, incredibly, in tissue experiments designed to show the 'virus' is responsible for killing the tissue they starve the tissue of nutrients and add toxic drugs including antibiotics and they do not have control studies to see if it's the starvation and poisoning that is degrading the tissue rather than the 'virus' they allege to be in there somewhere. You want me to pinch you? Yep, I understand. Tom Cowan said this about the whole nonsensical farce as he explains what that quote from the CDC paper really means:

The shocking thing about the above quote is that using their own methods, the virologists found that solutions containing SARS-CoV-2 – even in high amounts – were NOT, I repeat NOT, infective to any of the three human tissue cultures they tested. In plain English, this means they proved, on their terms, that this ‘new coronavirus’ is not infectious to human beings. It is ONLY infective to monkey kidney cells, and only then when you add two potent drugs (gentamicin and amphotericin), known to be toxic to kidneys, to the mix.

My friends, read this again and again. These virologists, published by the CDC, performed a clear proof, on their terms, showing that the SARS-CoV-2 virus is harmless to human beings. That is the only possible conclusion, but, unfortunately, this result is not even mentioned in their conclusion. They simply say they can provide virus stocks cultured only on monkey Vero cells, thanks for coming.

Cowan concluded: ‘If people really understood how this “science” was done, I would hope they would storm the gates and demand honesty, transparency and truth.’ Dr Michael Yeadon, former Vice President and Chief Scientific Adviser at drug giant Pfizer has been a vocal critic of the ‘Covid vaccine’ and its potential for multiple harm. He said in an interview in April, 2021, that ‘not one [vaccine] has the virus. He was asked why vaccines normally using a ‘dead’ version of a disease to activate the immune system were not used for ‘Covid’ and instead we had the synthetic methods of the ‘mRNA Covid vaccine’. Yeadon said that to do the former ‘you’d have to have some of [the virus] wouldn’t you?’ He added: ‘No-one’s got any – seriously.’ Yeadon said that surely they couldn’t have fooled the whole world for a year without having a virus, ‘but oddly enough ask around – no one’s got it’. He didn’t know why with all the ‘great labs’ around the world that the virus had not been isolated – ‘Maybe they’ve been too busy running bad PCR tests and vaccines that people don’t need.’ What is today called ‘science’ is not ‘science’ at all. Science is no longer what is, but whatever people can be manipulated to *believe* that it is. Real science has been hijacked by the Cult to dispense and produce the ‘expert scientists’ and contentions that suit the agenda of the Cult. How big-time this has happened with the ‘Covid’ hoax which is entirely based on fake science delivered by fake ‘scientists’ and fake ‘doctors’. The human-caused climate change hoax is also entirely based on fake science delivered by fake ‘scientists’ and fake ‘climate experts’. In both cases real

scientists, climate experts and doctors have their views suppressed and deleted by the Cult-owned science establishment, media and Silicon Valley. This is the ‘science’ that politicians claim to be ‘following’ and a common denominator of ‘Covid’ and climate are Cult psychopaths Bill Gates and his mate Klaus Schwab at the Gates-funded World Economic Forum. But, don’t worry, it’s all just a coincidence and absolutely nothing to worry about. Zzzzzzzz.

## **What is a ‘virus’ REALLY?**

Dr Tom Cowan is one of many contesting the very existence of viruses let alone that they cause disease. This is understandable when there is no scientific evidence for a disease-causing ‘virus’. German virologist Dr Stefan Lanka won a landmark case in 2017 in the German Supreme Court over his contention that there is no such thing as a measles virus. He had offered a big prize for anyone who could prove there is and Lanka won his case when someone sought to claim the money. There is currently a prize of more than 225,000 euros on offer from an Isolate Truth Fund for anyone who can prove the isolation of SARS-CoV-2 and its genetic substance. Lanka wrote in an article headed ‘The Misconception Called Virus’ that scientists think a ‘virus’ is causing tissue to become diseased and degraded when in fact it is the *processes they are using* which do that – not a ‘virus’. Lanka has done an important job in making this point clear as Cowan did in his analysis of the CDC paper. Lanka says that all claims about viruses as disease-causing pathogens are wrong and based on ‘easily recognisable, understandable and verifiable misinterpretations.’ Scientists believed they were working with ‘viruses’ in their laboratories when they were really working with ‘typical particles of specific dying tissues or cells ...’ Lanka said that the tissue decaying process claimed to be caused by a ‘virus’ still happens when no alleged ‘virus’ is involved. It’s the *process* that does the damage and not a ‘virus’. The genetic sample is deprived of nutrients, removed from its energy supply through removal from the body and then doused in toxic antibiotics to remove any bacteria. He confirms again that establishment scientists do not (pinch me)

conduct control experiments to see if this is the case and if they did they would see the claims that 'viruses' are doing the damage is nonsense. He adds that during the measles 'virus' court case he commissioned an independent laboratory to perform just such a control experiment and the result was that the tissues and cells died in the exact same way as with alleged 'infected' material. This is supported by a gathering number of scientists, doctors and researchers who reject what is called 'germ theory' or the belief in the body being infected by contagious sources emitted by other people. Researchers Dawn Lester and David Parker take the same stance in their highly-detailed and sourced book *What Really Makes You Ill – Why everything you thought you knew about disease is wrong* which was recommended to me by a number of medical professionals genuinely seeking the truth. Lester and Parker say there is no provable scientific evidence to show that a 'virus' can be transmitted between people or people and animals or animals and people:

The definition also claims that viruses are the cause of many diseases, as if this has been definitively proven. But this is not the case; there is no original scientific evidence that definitively demonstrates that any virus is the cause of any disease. The burden of proof for any theory lies with those who proposed it; but none of the existing documents provides 'proof' that supports the claim that 'viruses' are pathogens.

Dr Tom Cowan employs one of his clever analogies to describe the process by which a 'virus' is named as the culprit for a disease when what is called a 'virus' is only material released by cells detoxing themselves from infiltration by chemical or radiation poisoning. The tidal wave of technologically-generated radiation in the 'smart' modern world plus all the toxic food and drink are causing this to happen more than ever. Deluded 'scientists' misread this as a gathering impact of what they wrongly label 'viruses'.

## Paper can infect houses

Cowan said in an article for [davidicke.com](http://davidicke.com) – with his tongue only mildly in his cheek – that he believed he had made a tremendous

discovery that may revolutionise science. He had discovered that small bits of paper are alive, ‘well alive-ish’, can ‘infect’ houses, and then reproduce themselves inside the house. The result was that this explosion of growth in the paper inside the house causes the house to explode, blowing it to smithereens. His evidence for this new theory is that in the past months he had carefully examined many of the houses in his neighbourhood and found almost no scraps of paper on the lawns and surrounds of the house. There was an occasional stray label, but nothing more. Then he would return to these same houses a week or so later and with a few, not all of them, particularly the old and decrepit ones, he found to his shock and surprise they were littered with stray bits of paper. He knew then that the paper had infected these houses, made copies of itself, and blew up the house. A young boy on a bicycle at one of the sites told him he had seen a demolition crew using dynamite to explode the house the previous week, but Cowan dismissed this as the idle thoughts of silly boys because ‘I was on to something big’. He was on to how ‘scientists’ mistake genetic material in the detoxifying process for something they call a ‘virus’. Cowan said of his house and paper story:

If this sounds crazy to you, it’s because it should. This scenario is obviously nuts. But consider this admittedly embellished, for effect, current viral theory that all scientists, medical doctors and virologists currently believe.

He takes the example of the ‘novel SARS-Cov2’ virus to prove the point. First they take someone with an undefined illness called ‘Covid-19’ and don’t even attempt to find any virus in their sputum. Never mind the scientists still describe how this ‘virus’, which they have not located attaches to a cell receptor, injects its genetic material, in ‘Covid’s’ case, RNA, into the cell. The RNA once inserted exploits the cell to reproduce itself and makes ‘thousands, nay millions, of copies of itself ... Then it emerges victorious to claim its next victim’:

If you were to look in the scientific literature for proof, actual scientific proof, that uniform SARS-CoV2 viruses have been properly isolated from the sputum of a sick person, that actual spike proteins could be seen protruding from the virus (which has not been found), you would find that such evidence doesn't exist.

If you go looking in the published scientific literature for actual pictures, proof, that these spike proteins or any viral proteins are ever attached to any receptor embedded in any cell membrane, you would also find that no such evidence exists. If you were to look for a video or documented evidence of the intact virus injecting its genetic material into the body of the cell, reproducing itself and then emerging victorious by budding off the cell membrane, you would find that no such evidence exists.

The closest thing you would find is electron micrograph pictures of cellular particles, possibly attached to cell debris, both of which to be seen were stained by heavy metals, a process that completely distorts their architecture within the living organism. This is like finding bits of paper stuck to the blown-up bricks, thereby proving the paper emerged by taking pieces of the bricks on its way out.

## **The Enders baloney**

Cowan describes the 'Covid' story as being just as make-believe as his paper story and he charts back this fantasy to a Nobel Prize winner called John Enders (1897-1985), an American biomedical scientist who has been dubbed 'The Father of Modern Vaccines'. Enders is claimed to have 'discovered' the process of the viral culture which 'proved' that a 'virus' caused measles. Cowan explains how Enders did this 'by using the EXACT same procedure that has been followed by every virologist to find and characterize every new virus since 1954'. Enders took throat swabs from children with measles and immersed them in 2ml of milk. Penicillin (100u/ml) and the antibiotic streptomycin (50,g/ml) were added and the whole mix was centrifuged – rotated at high speed to separate large cellular debris from small particles and molecules as with milk and cream, for example. Cowan says that if the aim is to find little particles of genetic material ('viruses') in the snot from children with measles it would seem that the last thing you would do is mix the snot with other material – milk –that also has genetic material. 'How are you ever going to know whether whatever you found came from the snot or the milk?' He points out that streptomycin is a 'nephrotoxic' or poisonous-to-the-kidney drug. You will see the relevance of that

shortly. Cowan says that it gets worse, much worse, when Enders describes the culture medium upon which the virus 'grows': 'The culture medium consisted of bovine amniotic fluid (90%), beef embryo extract (5%), horse serum (5%), antibiotics and phenol red as an indicator of cell metabolism.' Cowan asks incredulously: 'Did he just say that the culture medium also contained fluids and tissues that are themselves rich sources of genetic material?' The genetic cocktail, or 'medium', is inoculated onto tissue and cells from rhesus monkey *kidney* tissue. This is where the importance of streptomycin comes in and currently-used antimicrobials and other drugs that are *poisonous to kidneys* and used in ALL modern viral cultures (e.g. gentamicin, streptomycin, and amphotericin). Cowan asks: 'How are you ever going to know from this witch's brew where any genetic material comes from as we now have five different sources of rich genetic material in our mix?' Remember, he says, that all genetic material, whether from monkey kidney tissues, bovine serum, milk, etc., is made from the exact same components. The same central question returns: 'How are you possibly going to know that it was the virus that killed the kidney tissue and not the toxic antibiotic and starvation rations on which you are growing the tissue?' John Enders answered the question himself – *you can't*:

A second agent was obtained from an uninoculated culture of monkey kidney cells. The cytopathic changes [death of the cells] it induced in the unstained preparations could not be distinguished with confidence from the viruses isolated from measles.

The death of the cells ('cytopathic changes') happened in exactly the same manner, whether they inoculated the kidney tissue with the measles snot or not, Cowan says. 'This is evidence that the destruction of the tissue, the very proof of viral causation of illness, was not caused by anything in the snot because they saw the same destructive effect when the snot was not even used ... the cytopathic, i.e., cell-killing, changes come from the process of the culture itself, not from any virus in any snot, period.' Enders quotes in his 1957 paper a virologist called Ruckle as reporting similar findings 'and in addition has isolated an agent from monkey kidney tissue that is so

far indistinguishable from human measles virus'. In other words, Cowan says, these particles called 'measles viruses' are simply and clearly breakdown products of the starved and poisoned tissue. For measles 'virus' see all 'viruses' including the so-called 'Covid virus'. Enders, the 'Father of Modern Vaccines', also said:

There is a potential risk in employing cultures of primate cells for the production of vaccines composed of attenuated virus, since the presence of other agents possibly latent in primate tissues cannot be definitely excluded by any known method.

Cowan further quotes from a paper published in the journal *Viruses* in May, 2020, while the 'Covid pandemic' was well underway in the media if not in reality. 'EVs' here refers to particles of genetic debris from our own tissues, such as exosomes of which more in a moment: 'The remarkable resemblance between EVs and viruses has caused quite a few problems in the studies focused on the analysis of EVs released during viral infections.' Later the paper adds that to date a reliable method that can actually guarantee a complete separation (of EVs from viruses) DOES NOT EXIST. This was published at a time when a fairy tale 'virus' was claimed in total certainty to be causing a fairy tale 'viral disease' called 'Covid-19' – a fairy tale that was already well on the way to transforming human society in the image that the Cult has worked to achieve for so long. Cowan concludes his article:

To summarize, there is no scientific evidence that pathogenic viruses exist. What we think of as 'viruses' are simply the normal breakdown products of dead and dying tissues and cells. When we are well, we make fewer of these particles; when we are starved, poisoned, suffocated by wearing masks, or afraid, we make more.

There is no engineered virus circulating and making people sick. People in laboratories all over the world are making genetically modified products to make people sick. These are called vaccines. There is no virome, no 'ecosystem' of viruses, viruses are not 8%, 50% or 100 % of our genetic material. These are all simply erroneous ideas based on the misconception called a virus.

## **What is 'Covid'? Load of bollocks**

The background described here by Cowan and Lanka was emphasised in the first video presentation that I saw by Dr Andrew Kaufman when he asked whether the ‘Covid virus’ was in truth a natural defence mechanism of the body called ‘exosomes’. These are released by cells when in states of toxicity – see the same themes returning over and over. They are released ever more profusely as chemical and radiation toxicity increases and think of the potential effect therefore of 5G alone as its destructive frequencies infest the human energetic information field with a gathering pace (5G went online in Wuhan in 2019 as the ‘virus’ emerged). I’ll have more about this later. Exosomes transmit a warning to the rest of the body that ‘Houston, we have a problem’. Kaufman presented images of exosomes and compared them with ‘Covid’ under an electron microscope and the similarity was remarkable. They both attach to the same cell receptors (*claimed* in the case of ‘Covid’), contain the same genetic material in the form of RNA or ribonucleic acid, and both are found in ‘viral cell cultures’ with damaged or dying cells. James Hildreth MD, President and Chief Executive Officer of the Meharry Medical College at Johns Hopkins, said: ‘The virus is fully an exosome in every sense of the word.’ Kaufman’s conclusion was that there is no ‘virus’: ‘This entire pandemic is a completely manufactured crisis … there is no evidence of anyone dying from [this] illness.’ Dr Tom Cowan and Sally Fallon Morell, authors of *The Contagion Myth*, published a statement with Dr Kaufman in February, 2021, explaining why the ‘virus’ does not exist and you can read it that in full in the Appendix.

‘Virus’ theory can be traced to the ‘cell theory’ in 1858 of German physician Rudolf Virchow (1821-1920) who contended that disease originates from a single cell infiltrated by a ‘virus’. Dr Stefan Lanka said that findings and insights with respect to the structure, function and central importance of tissues in the creation of life, which were already known in 1858, comprehensively refute the cell theory. Virchow ignored them. We have seen the part later played by John Enders in the 1950s and Lanka notes that infection theories were only established as a global dogma through the policies and

eugenics of the Third Reich in Nazi Germany (creation of the same Sabbatian cult behind the ‘Covid’ hoax). Lanka said: ‘Before 1933, scientists dared to contradict this theory; after 1933, these critical scientists were silenced’. Dr Tom Cowan’s view is that ill-health is caused by too much of something, too little of something, or toxification from chemicals and radiation – not contagion. We must also highlight as a major source of the ‘virus’ theology a man still called the ‘Father of Modern Virology’ – Thomas Milton Rivers (1888-1962). There is no way given the Cult’s long game policy that it was a coincidence for the ‘Father of Modern Virology’ to be director of the Rockefeller Institute for Medical Research from 1937 to 1956 when he is credited with making the Rockefeller Institute a leader in ‘viral research’. Cult Rockefellers were the force behind the creation of Big Pharma ‘medicine’, established the World Health Organisation in 1948, and have long and close associations with the Gates family that now runs the WHO during the pandemic hoax through mega-rich Cult gofer and psychopath Bill Gates.

Only a Renegade Mind can see through all this bullshit by asking the questions that need to be answered, not taking ‘no’ or prevarication for an answer, and certainly not hiding from the truth in fear of speaking it. Renegade Minds have always changed the world for the better and they will change this one no matter how bleak it may currently appear to be.

## CHAPTER SIX

### Sequence of deceit

*If you tell the truth, you don't have to remember anything*

Mark Twain

**A**gainst the background that I have laid out this far the sequence that took us from an invented 'virus' in Cult-owned China in late 2019 to the fascist transformation of human society can be seen and understood in a whole new context.

We were told that a deadly disease had broken out in Wuhan and the world media began its campaign (coordinated by behavioural psychologists as we shall see) to terrify the population into unquestioning compliance. We were shown images of Chinese people collapsing in the street which never happened in the West with what was supposed to be the same condition. In the earliest days when alleged cases and deaths were few the fear register was hysterical in many areas of the media and this would expand into the common media narrative across the world. The real story was rather different, but we were never told that. The Chinese government, one of the Cult's biggest centres of global operation, said they had discovered a new illness with flu-like and pneumonia-type symptoms in a city with such toxic air that it is overwhelmed with flu-like symptoms, pneumonia and respiratory disease. Chinese scientists said it was a new – 'novel' – coronavirus which they called Sars-Cov-2 and that it caused a disease they labelled 'Covid-19'. There was no evidence for this and the 'virus' has never to this day been isolated, purified and its genetic code established from that. It

was from the beginning a computer-generated fiction. Stories of Chinese whistleblowers saying the number of deaths was being suppressed or that the ‘new disease’ was related to the Wuhan bio-lab misdirected mainstream and alternative media into cul-de-sacs to obscure the real truth – there was no ‘virus’.

Chinese scientists took genetic material from the lung fluid of just a few people and said they had found a ‘new’ disease when this material had a wide range of content. There was no evidence for a ‘virus’ for the very reasons explained in the last two chapters. The ‘virus’ has never been shown to (a) exist and (b) cause any disease. People were diagnosed on symptoms that are so widespread in Wuhan and polluted China and with a PCR test that can’t detect infectious disease. On this farce the whole global scam was sold to the rest of the world which would also diagnose respiratory disease as ‘Covid-19’ from symptoms alone or with a PCR test not testing for a ‘virus’. Flu miraculously disappeared *worldwide* in 2020 and into 2021 as it was redesignated ‘Covid-19’. It was really the same old flu with its ‘flu-like’ symptoms attributed to ‘flu-like’ ‘Covid-19’. At the same time with very few exceptions the Chinese response of draconian lockdown and fascism was the chosen weapon to respond across the West as recommended by the Cult-owned Tedros at the Cult-owned World Health Organization run by the Cult-owned Gates. All was going according to plan. Chinese scientists – everything in China is controlled by the Cult-owned government – compared their contaminated RNA lung-fluid material with other RNA sequences and said it appeared to be just under 80 percent identical to the SARS-CoV-1 ‘virus’ claimed to be the cause of the SARS (severe acute respiratory syndrome) ‘outbreak’ in 2003. They decreed that because of this the ‘new virus’ had to be related and they called it SARS-CoV-2. There are some serious problems with this assumption and *assumption* was all it was. Most ‘factual’ science turns out to be assumptions repeated into everyone-knows-that. A match of under 80-percent is meaningless. Dr Kaufman makes the point that there’s a 96 percent genetic correlation between humans and chimpanzees, but ‘no one would say our genetic material is part

of the chimpanzee family'. Yet the Chinese authorities were claiming that a much lower percentage, less than 80 percent, proved the existence of a new 'coronavirus'. For goodness sake human DNA is 60 percent similar to a *banana*.

## **You are feeling sleepy**

The entire 'Covid' hoax is a global Psyop, a psychological operation to program the human mind into believing and fearing a complete fantasy. A crucial aspect of this was what *appeared* to happen in Italy. It was all very well streaming out daily images of an alleged catastrophe in Wuhan, but to the Western mind it was still on the other side of the world in a very different culture and setting. A reaction of 'this could happen to me and my family' was still nothing like as intense enough for the mind-doctors. The Cult needed a Western example to push people over that edge and it chose Italy, one of its major global locations going back to the Roman Empire. An Italian 'Covid' crisis was manufactured in a particular area called Lombardy which just happens to be notorious for its toxic air and therefore respiratory disease. Wuhan, China, *déjà vu*. An hysterical media told horror stories of Italians dying from 'Covid' in their droves and how Lombardy hospitals were being overrun by a tidal wave of desperately ill people needing treatment after being struck down by the 'deadly virus'. Here was the psychological turning point the Cult had planned. Wow, if this is happening in Italy, the Western mind concluded, this indeed could happen to me and my family. Another point is that Italian authorities responded by following the Chinese blueprint so vehemently recommended by the Cult-owned World Health Organization. They imposed fascistic lockdowns on the whole country viciously policed with the help of surveillance drones sweeping through the streets seeking out anyone who escaped from mass house arrest. Livelihoods were destroyed and psychology unravelled in the way we have witnessed since in all lockdown countries. Crucial to the plan was that Italy responded in this way to set the precedent of suspending freedom and imposing fascism in a 'Western liberal democracy'. I emphasised in an

animated video explanation on [davidicke.com](http://davidicke.com) posted in the summer of 2020 how important it was to the Cult to expand the Chinese lockdown model across the West. Without this, and the bare-faced lie that non-symptomatic people could still transmit a ‘disease’ they didn’t have, there was no way locking down the whole population, sick and not sick, could be pulled off. At just the right time and with no evidence Cult operatives and gofers claimed that people without symptoms could pass on the ‘disease’. In the name of protecting the ‘vulnerable’ like elderly people, who lockdowns would kill by the tens of thousands, we had for the first time healthy people told to isolate as well as the sick. The great majority of people who tested positive had no symptoms because there was nothing wrong with them. It was just a trick made possible by a test not testing for the ‘virus’.

Months after my animated video the Gates-funded Professor Neil Ferguson at the Gates-funded Imperial College confirmed that I was right. He didn’t say it in those terms, naturally, but he did say it. Ferguson will enter the story shortly for his outrageously crazy ‘computer models’ that led to Britain, the United States and many other countries following the Chinese and now Italian methods of response. Put another way, following the Cult script. Ferguson said that SAGE, the UK government’s scientific advisory group which has controlled ‘Covid’ policy from the start, wanted to follow the Chinese lockdown model (while they all continued to work and be paid), but they wondered if they could possibly, in Ferguson’s words, ‘get away with it in Europe’. ‘Get away with it’? Who the hell do these moronic, arrogant people think they are? This appalling man Ferguson said that once Italy went into national lockdown they realised they, too, could mimic China:

It’s a communist one-party state, we said. We couldn’t get away with it in Europe, we thought ... and then Italy did it. And we realised we could. Behind this garbage from Ferguson is a simple fact: Doing the same as China in every country was the plan from the start and Ferguson’s ‘models’ would play a central role in achieving that. It’s just a coincidence, of course, and absolutely nothing to worry your little head about.

## **Oops, sorry, our mistake**

Once the Italian segment of the Psyop had done the job it was designed to do a very different story emerged. Italian authorities revealed that 99 percent of those who had 'died from Covid-19' in Italy had one, two, three, or more 'co-morbidities' or illnesses and health problems that could have ended their life. The US Centers for Disease Control and Prevention (CDC) published a figure of 94 percent for Americans dying of 'Covid' while having other serious medical conditions – on average two to three (some five or six) other potential causes of death. In terms of death from an unproven 'virus' I say it is 100 percent. The other one percent in Italy and six percent in the US would presumably have died from 'Covid's' flu-like symptoms with a range of other possible causes in conjunction with a test not testing for the 'virus'. Fox News reported that even more startling figures had emerged in one US county in which 410 of 422 deaths attributed to 'Covid-19' had other potentially deadly health conditions. The Italian National Health Institute said later that the average age of people dying with a 'Covid-19' diagnosis in Italy was about 81. Ninety percent were over 70 with ten percent over 90. In terms of other reasons to die some 80 percent had two or more chronic diseases with half having three or more including cardiovascular problems, diabetes, respiratory problems and cancer. Why is the phantom 'Covid-19' said to kill overwhelmingly old people and hardly affect the young? Old people continually die of many causes and especially respiratory disease which you can re-diagnose 'Covid-19' while young people die in tiny numbers by comparison and rarely of respiratory disease. Old people 'die of Covid' because they die of other things that can be redesignated 'Covid' and it really is that simple.

## **Flu has flown**

The blueprint was in place. Get your illusory 'cases' from a test not testing for the 'virus' and redesignate other causes of death as 'Covid-19'. You have an instant 'pandemic' from something that is nothing more than a computer-generated fiction. With near-on a

billion people having ‘flu-like’ symptoms every year the potential was limitless and we can see why flu quickly and apparently miraculously disappeared *worldwide* by being diagnosed ‘Covid-19’. The painfully bloody obvious was explained away by the childlike media in headlines like this in the UK *‘Independent’*: ‘Not a single case of flu detected by Public Health England this year as Covid restrictions suppress virus’. I kid you not. The masking, social distancing and house arrest that did not make the ‘Covid virus’ disappear somehow did so with the ‘flu virus’. Even worse the article, by a bloke called Samuel Lovett, suggested that maybe the masking, sanitising and other ‘Covid’ measures should continue to keep the flu away. With a ridiculousness that disturbs your breathing (it’s ‘Covid-19’) the said Lovett wrote: ‘With widespread social distancing and mask-wearing measures in place throughout the UK, the usual routes of transmission for influenza have been blocked.’ He had absolutely no evidence to support that statement, but look at the consequences of him acknowledging the obvious. With flu not disappearing at all and only being relabelled ‘Covid-19’ he would have to contemplate that ‘Covid’ was a hoax on a scale that is hard to imagine. You need guts and commitment to truth to even go there and that’s clearly something Samuel Lovett does not have in abundance. He would never have got it through the editors anyway.

Tens of thousands die in the United States alone every winter from flu including many with pneumonia complications. CDC figures record *45 million* Americans diagnosed with flu in 2017-2018 of which 61,000 died and some reports claim 80,000. Where was the same hysteria then that we have seen with ‘Covid-19’? Some 250,000 Americans are admitted to hospital with pneumonia every year with about 50,000 cases proving fatal. About 65 million suffer respiratory disease every year and three million deaths makes this the third biggest cause of death worldwide. You only have to redesignate a portion of all these people ‘Covid-19’ and you have an instant global pandemic or the *appearance* of one. Why would doctors do this? They are told to do this and all but a few dare not refuse those who must be obeyed. Doctors in general are not researching their own

knowledge and instead take it direct and unquestioned from the authorities that own them and their careers. The authorities say they must now diagnose these symptoms ‘Covid-19’ and not flu, or whatever, and they do it. Dark suits say put ‘Covid-19’ on death certificates no matter what the cause of death and the doctors do it. Renegade Minds don’t fall for the illusion that doctors and medical staff are all highly-intelligent, highly-principled, seekers of medical truth. *Some are*, but not the majority. They are repeaters, gofers, and yes sir, no sir, purveyors of what the system demands they purvey. The ‘Covid’ con is not merely confined to diseases of the lungs. Instructions to doctors to put ‘Covid-19’ on death certificates for anyone dying of *anything* within 28 days (or much more) of a positive test not testing for the ‘virus’ opened the floodgates. The term dying *with* ‘Covid’ and not *of* ‘Covid’ was coined to cover the truth. Whether it was a *with* or an *of* they were all added to the death numbers attributed to the ‘deadly virus’ compiled by national governments and globally by the Gates-funded Johns Hopkins operation in the United States that was so involved in those ‘pandemic’ simulations. Fraudulent deaths were added to the ever-growing list of fraudulent ‘cases’ from false positives from a false test. No wonder Professor Walter Ricciardi, scientific advisor to the Italian minister of health, said after the Lombardy hysteria had done its job that ‘Covid’ death rates were due to Italy having the second oldest population in the world and to *how hospitals record deaths*:

The way in which we code deaths in our country is very generous in the sense that all the people who die in hospitals with the coronavirus are deemed to be dying of the coronavirus. On re-evaluation by the National Institute of Health, only 12 per cent of death certificates have shown a direct causality from coronavirus, while 88 per cent of patients who have died have at least one pre-morbidity – many had two or three.

This is extraordinary enough when you consider the propaganda campaign to use Italy to terrify the world, but how can they even say twelve percent were genuine when the ‘virus’ has not been shown to exist, its ‘code’ is a computer program, and diagnosis comes from a test not testing for it? As in China, and soon the world, ‘Covid-19’ in

Italy was a redesignation of diagnosis. Lies and corruption were to become the real ‘pandemic’ fuelled by a pathetically-compliant medical system taking its orders from the tiny few at the top of their national hierarchy who answered to the World Health Organization which answers to Gates and the Cult. Doctors were told – ordered – to diagnose a particular set of symptoms ‘Covid-19’ and put that on the death certificate for any cause of death if the patient had tested positive with a test not testing for the virus or had ‘Covid’ symptoms like the flu. The United States even introduced big financial incentives to manipulate the figures with hospitals receiving £4,600 from the Medicare system for diagnosing someone with regular pneumonia, \$13,000 if they made the diagnosis from the same symptoms ‘Covid-19’ pneumonia, and \$39, 000 if they put a ‘Covid’ diagnosed patient on a ventilator that would almost certainly kill them. A few – painfully and pathetically few – medical whistleblowers revealed (before Cult-owned YouTube deleted their videos) that they had been instructed to ‘let the patient crash’ and put them straight on a ventilator instead of going through a series of far less intrusive and dangerous methods as they would have done before the pandemic hoax began and the financial incentives kicked in. We are talking cold-blooded murder given that ventilators are so damaging to respiratory systems they are usually the last step before heaven awaits. Renegade Minds never fall for the belief that people in white coats are all angels of mercy and cannot be full-on psychopaths. I have explained in detail in *The Answer* how what I am describing here played out across the world coordinated by the World Health Organization through the medical hierarchies in almost every country.

## **Medical scientist calls it**

Information about the non-existence of the ‘virus’ began to emerge for me in late March, 2020, and mushroomed after that. I was sent an email by Sir Julian Rose, a writer, researcher, and organic farming promotor, from a medical scientist friend of his in the United States. Even at that early stage in March the scientist was able to explain

how the ‘Covid’ hoax was being manipulated. He said there were no reliable tests for a specific ‘Covid-19 virus’ and nor were there any reliable agencies or media outlets for reporting numbers of actual ‘Covid-19’ cases. We have seen in the long period since then that he was absolutely right. ‘Every action and reaction to Covid-19 is based on totally flawed data and we simply cannot make accurate assessments,’ he said. Most people diagnosed with ‘Covid-19’ were showing nothing more than cold and flu-like symptoms ‘because most coronavirus strains *are* nothing more than cold/flu-like symptoms’. We had farcical situations like an 84-year-old German man testing positive for ‘Covid-19’ and his nursing home ordered to quarantine only for him to be found to have a common cold. The scientist described back then why PCR tests and what he called the ‘Mickey Mouse test kits’ were useless for what they were claimed to be identifying. ‘The idea these kits can isolate a specific virus like Covid-19 is nonsense,’ he said. Significantly, he pointed out that ‘if you want to create a totally false panic about a totally false pandemic – pick a coronavirus’. This is exactly what the Cult-owned Gates, World Economic Forum and Johns Hopkins University did with their Event 201 ‘simulation’ followed by their real-life simulation called the ‘pandemic’. The scientist said that all you had to do was select the sickest of people with respiratory-type diseases in a single location – ‘say Wuhan’ – and administer PCR tests to them. You can then claim that anyone showing ‘viral sequences’ similar to a coronavirus ‘which will inevitably be quite a few’ is suffering from a ‘new’ disease:

Since you already selected the sickest flu cases a fairly high proportion of your sample will go on to die. You can then say this ‘new’ virus has a CFR [case fatality rate] higher than the flu and use this to infuse more concern and do more tests which will of course produce more ‘cases’, which expands the testing, which produces yet more ‘cases’ and so on and so on. Before long you have your ‘pandemic’, and all you have done is use a simple test kit trick to convert the worst flu and pneumonia cases into something new that doesn’t ACTUALLY EXIST [my emphasis].

He said that you then ‘just run the same scam in other countries’ and make sure to keep the fear message running high ‘so that people

will feel panicky and less able to think critically'. The only problem to overcome was the fact *there is no* actual new deadly pathogen and only regular sick people. This meant that deaths from the 'new deadly pathogen' were going to be way too low for a real new deadly virus pandemic, but he said this could be overcome in the following ways – all of which would go on to happen:

1. You can claim this is just the beginning and more deaths are imminent [you underpin this with fantasy 'computer projections']. Use this as an excuse to quarantine everyone and then claim the quarantine prevented the expected millions of dead.
2. You can [say that people] 'minimizing' the dangers are irresponsible and bully them into not talking about numbers.
3. You can talk crap about made up numbers hoping to blind people with pseudoscience.
4. You can start testing well people (who, of course, will also likely have shreds of coronavirus [RNA] in them) and thus inflate your 'case figures' with 'asymptomatic carriers' (you will of course have to spin that to sound deadly even though any virologist knows the more symptom-less cases you have the less deadly is your pathogen).

The scientist said that if you take these simple steps 'you can have your own entirely manufactured pandemic up and running in weeks'. His analysis made so early in the hoax was brilliantly prophetic of what would actually unfold. Pulling all the information together in these recent chapters we have this is simple 1, 2, 3, of how you can delude virtually the entire human population into believing in a 'virus' that doesn't exist:

- A 'Covid case' is someone who tests positive with a test not testing for the 'virus'.
- A 'Covid death' is someone who dies of *any cause* within 28 days (or much longer) of testing positive with a test not testing for the 'virus'.
- Asymptomatic means there is nothing wrong with you, but they claim you can pass on what you don't have to justify locking

down (quarantining) healthy people in totality.

The foundations of the hoax are that simple. A study involving ten million people in Wuhan, published in November, 2020, demolished the whole lie about those without symptoms passing on the ‘virus’. They found ‘300 asymptomatic cases’ and traced their contacts to find that not one of them was detected with the ‘virus’.

‘Asymptomatic’ patients and their contacts were isolated for no less than two weeks and nothing changed. I know it’s all crap, but if you are going to claim that those without symptoms can transmit ‘the virus’ then you must produce evidence for that and they never have. Even World Health Organization official Dr Maria Van Kerkhove, head of the emerging diseases and zoonosis unit, said as early as June, 2020, that she doubted the validity of asymptomatic transmission. She said that ‘from the data we have, it still seems to be rare that an asymptomatic person actually transmits onward to a secondary individual’ and by ‘rare’ she meant that she couldn’t cite any case of asymptomatic transmission.

## **The Ferguson factor**

The problem for the Cult as it headed into March, 2020, when the script had lockdown due to start, was that despite all the manipulation of the case and death figures they still did not have enough people alleged to have died from ‘Covid’ to justify mass house arrest. This was overcome in the way the scientist described: ‘You can claim this is just the beginning and more deaths are imminent ... Use this as an excuse to quarantine everyone and then claim the quarantine prevented the expected millions of dead.’ Enter one Professor Neil Ferguson, the Gates-funded ‘epidemiologist’ at the Gates-funded Imperial College in London. Ferguson is Britain’s Christian Drosten in that he has a dire record of predicting health outcomes, but is still called upon to advise government on the next health outcome when another ‘crisis’ comes along. This may seem to be a strange and ridiculous thing to do. Why would you keep turning for policy guidance to people who have a history of being

monumentally wrong? Ah, but it makes sense from the Cult point of view. These ‘experts’ keep on producing predictions that suit the Cult agenda for societal transformation and so it was with Neil Ferguson as he revealed his horrific (and clearly insane) computer model predictions that allowed lockdowns to be imposed in Britain, the United States and many other countries. Ferguson does not have even an A-level in biology and would appear to have no formal training in computer modelling, medicine or epidemiology, according to Derek Winton, an MSc in Computational Intelligence. He wrote an article somewhat aghast at what Ferguson did which included taking no account of respiratory disease ‘seasonality’ which means it is far worse in the winter months. Who would have thought that respiratory disease could be worse in the winter? Well, certainly not Ferguson.

The massively China-connected Imperial College and its bizarre professor provided the excuse for the long-incubated Chinese model of human control to travel westward at lightning speed. Imperial College confirms on its website that it collaborates with the Chinese Research Institute; publishes more than 600 research papers every year with Chinese research institutions; has 225 Chinese staff; 2,600 Chinese students – the biggest international group; 7,000 former students living in China which is the largest group outside the UK; and was selected for a tour by China’s President Xi Jinping during his state visit to the UK in 2015. The college takes major donations from China and describes itself as the UK’s number one university collaborator with Chinese research institutions. The China communist/fascist government did not appear phased by the woeful predictions of Ferguson and Imperial when during the lockdown that Ferguson induced the college signed a five-year collaboration deal with China tech giant Huawei that will have Huawei’s indoor 5G network equipment installed at the college’s West London tech campus along with an ‘AI cloud platform’. The deal includes Chinese sponsorship of Imperial’s Venture Catalyst entrepreneurship competition. Imperial is an example of the enormous influence the Chinese government has within British and North American

universities and research centres – and further afield. Up to 200 academics from more than a dozen UK universities are being investigated on suspicion of ‘unintentionally’ helping the Chinese government build weapons of mass destruction by ‘transferring world-leading research in advanced military technology such as aircraft, missile designs and cyberweapons’. Similar scandals have broken in the United States, but it’s all a coincidence. Imperial College serves the agenda in many other ways including the promotion of every aspect of the United Nations Agenda 21/2030 (the Great Reset) and produced computer models to show that human-caused ‘climate change’ is happening when in the real world it isn’t. Imperial College is driving the climate agenda as it drives the ‘Covid’ agenda (both Cult hoaxes) while Patrick Vallance, the UK government’s Chief Scientific Adviser on ‘Covid’, was named Chief Scientific Adviser to the UN ‘climate change’ conference known as COP26 hosted by the government in Glasgow, Scotland. ‘Covid’ and ‘climate’ are fundamentally connected.

## **Professor Woeful**

From Imperial’s bosom came Neil Ferguson still advising government despite his previous disasters and it was announced early on that he and other key people like UK Chief Medical Adviser Chris Whitty had caught the ‘virus’ as the propaganda story was being sold. Somehow they managed to survive and we had Prime Minister Boris Johnson admitted to hospital with what was said to be a severe version of the ‘virus’ in this same period. His whole policy and demeanour changed when he returned to Downing Street. It’s a small world with these government advisors – especially in their communal connections to Gates – and Ferguson had partnered with Whitty to write a paper called ‘Infectious disease: Tough choices to reduce Ebola transmission’ which involved another scare-story that didn’t happen. Ferguson’s ‘models’ predicted that up to 150, 000 could die from ‘mad cow disease’, or BSE, and its version in sheep if it was transmitted to humans. BSE was not transmitted and instead triggered by an organophosphate pesticide used to treat a pest on

cows. Fewer than 200 deaths followed from the human form. Models by Ferguson and his fellow incompetents led to the unnecessary culling of millions of pigs, cattle and sheep in the foot and mouth outbreak in 2001 which destroyed the lives and livelihoods of farmers and their families who had often spent decades building their herds and flocks. Vast numbers of these animals did not have foot and mouth and had no contact with the infection. Another ‘expert’ behind the cull was Professor Roy Anderson, a computer modeller at Imperial College specialising in the epidemiology of *human*, not animal, disease. Anderson has served on the Bill and Melinda Gates Grand Challenges in Global Health advisory board and chairs another Gates-funded organisation. Gates is everywhere.

In a precursor to the ‘Covid’ script Ferguson backed closing schools ‘for prolonged periods’ over the swine flu ‘pandemic’ in 2009 and said it would affect a third of the world population if it continued to spread at the speed he claimed to be happening. His mates at Imperial College said much the same and a news report said: ‘One of the authors, the epidemiologist and disease modeller Neil Ferguson, who sits on the World Health Organisation’s emergency committee for the outbreak, said the virus had “full pandemic potential”.’ Professor Liam Donaldson, the Chris Whitty of his day as Chief Medical Officer, said the worst case could see 30 percent of the British people infected by swine flu with 65,000 dying. Ferguson and Donaldson were indeed proved correct when at the end of the year the number of deaths attributed to swine flu was 392. The term ‘expert’ is rather liberally applied unfortunately, not least to complete idiots. Swine flu ‘projections’ were great for GlaxoSmithKline (GSK) as millions rolled in for its Pandemrix influenza vaccine which led to brain damage with children most affected. The British government (taxpayers) paid out more than £60 million in compensation after GSK was given immunity from prosecution. Yet another ‘Covid’ déjà vu. Swine flu was supposed to have broken out in Mexico, but Dr Wolfgang Wodarg, a German doctor, former member of parliament and critic of the ‘Covid’ hoax, observed ‘the spread of swine flu’ in Mexico City at the time. He

said: 'What we experienced in Mexico City was a very mild flu which did not kill more than usual – which killed even fewer people than usual.' Hyping the fear against all the facts is not unique to 'Covid' and has happened many times before. Ferguson is reported to have over-estimated the projected death toll of bird flu (H5N1) by some three million-fold, but bird flu vaccine makers again made a killing from the scare. This is some of the background to the Neil Ferguson who produced the perfectly-timed computer models in early 2020 predicting that half a million people would die in Britain without draconian lockdown and 2.2 million in the United States. Politicians panicked, people panicked, and lockdowns of alleged short duration were instigated to 'flatten the curve' of cases gleaned from a test not testing for the 'virus'. I said at the time that the public could forget the 'short duration' bit. This was an agenda to destroy the livelihoods of the population and force them into mass control through dependency and there was going to be nothing 'short' about it. American researcher Daniel Horowitz described the consequences of the 'models' spewed out by Gates-funded Ferguson and Imperial College:

What led our government and the governments of many other countries into panic was a single Imperial College of UK study, funded by global warming activists, that predicted 2.2 million deaths if we didn't lock down the country. In addition, the reported 8-9% death rate in Italy scared us into thinking there was some other mutation of this virus that they got, which might have come here.

Together with the fact that we were finally testing and had the ability to actually report new cases, we thought we were headed for a death spiral. But again ... we can't flatten a curve if we don't know when the curve started.

How about it *never* started?

## **Giving them what they want**

An investigation by German news outlet *Welt Am Sonntag* (*World on Sunday*) revealed how in March, 2020, the German government gathered together 'leading scientists from several research institutes and universities' and 'together, they were to produce a [modelling]

paper that would serve as legitimization for further tough political measures'. The Cult agenda was justified by computer modelling not based on evidence or reality; it was specifically constructed to justify the Cult demand for lockdowns all over the world to destroy the independent livelihoods of the global population. All these modellers and everyone responsible for the 'Covid' hoax have a date with a trial like those in Nuremberg after World War Two when Nazis faced the consequences of their war crimes. These corrupt-beyond-belief 'modellers' wrote the paper according to government instructions and it said that if lockdown measures were lifted then up to one million Germans would die from 'Covid-19' adding that some would die 'agonizingly at home, gasping for breath' unable to be treated by hospitals that couldn't cope. All lies. No matter – it gave the Cult all that it wanted. What did long-time government 'modeller' Neil Ferguson say? If the UK and the United States didn't lockdown half a million would die in Britain and 2.2 million Americans. Anyone see a theme here? 'Modellers' are such a crucial part of the lockdown strategy that we should look into their background and follow the money. Researcher Rosemary Frei produced an excellent article headlined 'The Modelling-paper Mafiosi'. She highlights a guy called John Edmunds, a British epidemiologist, and professor in the Faculty of Epidemiology and Population Health at the London School of Hygiene & Tropical Medicine. He studied at Imperial College. Edmunds is a member of government 'Covid' advisory bodies which have been dictating policy, the New and Emerging Respiratory Virus Threats Advisory Group (NERVTAG) and the Scientific Advisory Group for Emergencies (SAGE).

Ferguson, another member of NERVTAG and SAGE, led the way with the original 'virus' and Edmunds has followed in the 'variant' stage and especially the so-called UK or Kent variant known as the 'Variant of Concern' (VOC) B.1.1.7. He said in a co-written report for the Centre for Mathematical modelling of Infectious Diseases at the London School of Hygiene and Tropical Medicine, with input from the Centre's 'Covid-19' Working Group, that there was 'a realistic

possibility that VOC B.1.1.7 is associated with an increased risk of death compared to non-VOC viruses'. Fear, fear, fear, get the vaccine, fear, fear, fear, get the vaccine. Rosemary Frei reveals that almost all the paper's authors and members of the modelling centre's 'Covid-19' Working Group receive funding from the Bill and Melinda Gates Foundation and/or the associated Gates-funded Wellcome Trust. The paper was published by e-journal *Medr* <sup>xiv</sup> which only publishes papers not peer-reviewed and the journal was established by an organisation headed by Facebook's Mark Zuckerberg and his missus. What a small world it is. Frei discovered that Edmunds is on the Scientific Advisory Board of the Coalition for Epidemic Preparedness Innovations (CEPI) which was established by the Bill and Melinda Gates Foundation, Klaus Schwab's Davos World Economic Forum and Big Pharma giant Wellcome. CEPI was 'launched in Davos [in 2017] to develop vaccines to stop future epidemics', according to its website. 'Our mission is to accelerate the development of vaccines against emerging infectious diseases and enable equitable access to these vaccines for people during outbreaks.' What kind people they are. Rosemary Frei reveals that Public Health England (PHE) director Susan Hopkins is an author of her organisation's non-peer-reviewed reports on 'new variants'. Hopkins is a professor of infectious diseases at London's Imperial College which is gifted tens of millions of dollars a year by the Bill and Melinda Gates Foundation. Gates-funded modelling disaster Neil Ferguson also co-authors Public Health England reports and he spoke in December, 2020, about the potential danger of the B.1.1.7. 'UK variant' promoted by Gates-funded modeller John Edmunds. When I come to the 'Covid vaccines' the 'new variants' will be shown for what they are – bollocks.

## **Connections, connections**

All these people and modellers are lockdown-obsessed or, put another way, they demand what the Cult demands. Edmunds said in January, 2021, that to ease lockdowns too soon would be a disaster and they had to 'vaccinate much, much, much more widely than the

elderly'. Rosemary Frei highlights that Edmunds is married to Jeanne Pimenta who is described in a LinkedIn profile as director of epidemiology at GlaxoSmithKline (GSK) and she held shares in the company. Patrick Vallance, co-chair of SAGE and the government's Chief Scientific Adviser, is a former executive of GSK and has a deferred bonus of shares in the company worth £600,000. GSK has serious business connections with Bill Gates and is collaborating with mRNA-'vaccine' company CureVac to make 'vaccines' for the new variants that Edmunds is talking about. GSK is planning a 'Covid vaccine' with drug giant Sanofi. Puppet Prime Minister Boris Johnson announced in the spring of 2021 that up to 60 million vaccine doses were to be made at the GSK facility at Barnard Castle in the English North East. Barnard Castle, with a population of just 6,000, was famously visited in breach of lockdown rules in April, 2020, by Johnson aide Dominic Cummings who said that he drove there 'to test his eyesight' before driving back to London. Cummings would be better advised to test his integrity – not that it would take long. The GSK facility had nothing to do with his visit then although I'm sure Patrick Vallance would have been happy to arrange an introduction and some tea and biscuits. Ruthless psychopath Gates has made yet another fortune from vaccines in collaboration with Big Pharma companies and gushes at the phenomenal profits to be made from vaccines – more than a 20-to-1 return as he told one interviewer. Gates also tweeted in December, 2019, with the foreknowledge of what was coming: 'What's next for our foundation? I'm particularly excited about what the next year could mean for one of the best buys in global health: vaccines.'

Modeller John Edmunds is a big promotor of vaccines as all these people appear to be. He's the dean of the London School of Hygiene & Tropical Medicine's Faculty of Epidemiology and Population Health which is primarily funded by the Bill and Melinda Gates Foundation and the Gates-established and funded GAVI vaccine alliance which is the Gates vehicle to vaccinate the world. The organisation Doctors Without Borders has described GAVI as being 'aimed more at supporting drug-industry desires to promote new

products than at finding the most efficient and sustainable means for fighting the diseases of poverty'. But then that's why the psychopath Gates created it. John Edmunds said in a video that the London School of Hygiene & Tropical Medicine is involved in every aspect of vaccine development including large-scale clinical trials. He contends that mathematical modelling can show that vaccines protect individuals and society. That's on the basis of shit in and shit out, I take it. Edmunds serves on the UK Vaccine Network as does Ferguson and the government's foremost 'Covid' adviser, the grim-faced, dark-eyed Chris Whitty. The Vaccine Network says it works 'to support the government to identify and shortlist targeted investment opportunities for the most promising vaccines and vaccine technologies that will help combat infectious diseases with epidemic potential, and to address structural issues related to the UK's broader vaccine infrastructure'. Ferguson is acting Director of the Imperial College Vaccine Impact Modelling Consortium which has funding from the Bill and Melina Gates Foundation and the Gates-created GAVI 'vaccine alliance'. Anyone wonder why these characters see vaccines as the answer to every problem? Ferguson is wildly enthusiastic in his support for GAVI's campaign to vaccine children en masse in poor countries. You would expect someone like Gates who has constantly talked about the need to reduce the population to want to fund vaccines to keep more people alive. I'm sure that's why he does it. The John Edmunds London School of Hygiene & Tropical Medicine (LSHTM) has a Vaccines Manufacturing Innovation Centre which develops, tests and commercialises vaccines. Rosemary Frei writes:

The vaccines centre also performs affiliated activities like combating 'vaccine hesitancy'. The latter includes the Vaccine Confidence Project. The project's stated purpose is, among other things, 'to provide analysis and guidance for early response and engagement with the public to ensure sustained confidence in vaccines and immunisation'. The Vaccine Confidence Project's director is LSHTM professor Heidi Larson. For more than a decade she's been researching how to combat vaccine hesitancy.

How the bloody hell can blokes like John Edmunds and Neil Ferguson with those connections and financial ties model 'virus' case

and death projections for the government and especially in a way that gives their paymasters like Gates exactly what they want? It's insane, but this is what you find throughout the world.

## **'Covid' is not dangerous, oops, wait, yes it is**

Only days before Ferguson's nightmare scenario made Jackboot Johnson take Britain into a China-style lockdown to save us from a deadly 'virus' the UK government website gov.uk was reporting something very different to Ferguson on a page of official government guidance for 'high consequence infectious diseases (HCID)'. It said this about 'Covid-19':

As of 19 March 2020, COVID-19 *is no longer considered to be a high consequence infectious diseases (HCID) in the UK* [my emphasis]. The 4 nations public health HCID group made an interim recommendation in January 2020 to classify COVID-19 as an HCID. This was based on consideration of the UK HCID criteria about the virus and the disease with information available during the early stages of the outbreak.

Now that more is known about COVID-19, the public health bodies in the UK have reviewed the most up to date information about COVID-19 against the UK HCID criteria. They have determined that several features have now changed; in particular, more information is available about mortality rates (low overall), and there is now greater clinical awareness and a specific and sensitive laboratory test, the availability of which continues to increase. The Advisory Committee on Dangerous Pathogens (ACDP) is also of the opinion that COVID-19 should no longer be classified as an HCID.

Soon after the government had been exposed for downgrading the risk they upgraded it again and everyone was back to singing from the same Cult hymn book. Ferguson and his fellow Gates clones indicated that lockdowns and restrictions would have to continue until a Gates-funded vaccine was developed. Gates said the same because Ferguson and his like were repeating the Gates script which is the Cult script. 'Flatten the curve' became an ongoing nightmare of continuing lockdowns with periods in between of severe restrictions in pursuit of destroying independent incomes and had nothing to do with protecting health about which the Cult gives not a shit. Why wouldn't Ferguson be pushing a vaccine 'solution' when he's owned by vaccine-obsessive Gates who makes a fortune from them and

when Ferguson heads the Vaccine Impact Modelling Consortium at Imperial College funded by the Gates Foundation and GAVI, the ‘vaccine alliance’, created by Gates as his personal vaccine promotion operation? To compound the human catastrophe that Ferguson’s ‘models’ did so much to create he was later exposed for breaking his own lockdown rules by having sexual liaisons with his married girlfriend Antonia Staats at his home while she was living at another location with her husband and children. Staats was a ‘climate’ activist and senior campaigner at the Soros-funded Avaaz which I wouldn’t trust to tell me that grass is green. Ferguson had to resign as a government advisor over this hypocrisy in May, 2020, but after a period of quiet he was back being quoted by the ridiculous media on the need for more lockdowns and a vaccine rollout. Other government-advising ‘scientists’ from Imperial College held the fort in his absence and said lockdown could be indefinite until a vaccine was found. The Cult script was being sung by the payrolled choir. I said there was no intention of going back to ‘normal’ when the ‘vaccine’ came because the ‘vaccine’ is part of a very different agenda that I will discuss in Human 2.0. Why would the Cult want to let the world go back to normal when destroying that normal forever was the whole point of what was happening? House arrest, closing businesses and schools through lockdown, (un)social distancing and masks all followed the Ferguson fantasy models. Again as I predicted (these people are so predictable) when the ‘vaccine’ arrived we were told that house arrest, lockdown, (un)social distancing and masks would still have to continue. I will deal with the masks in the next chapter because they are of fundamental importance.

## **Where's the 'pandemic'?**

Any mildly in-depth assessment of the figures revealed what was really going on. Cult-funded and controlled organisations still have genuine people working within them such is the number involved. So it is with Genevieve Briand, assistant program director of the Applied Economics master’s degree program at Johns Hopkins

University. She analysed the impact that 'Covid-19' had on deaths from *all* causes in the United States using official data from the CDC for the period from early February to early September, 2020. She found that allegedly 'Covid' *related*-deaths exceeded those from heart disease which she found strange with heart disease always the biggest cause of fatalities. Her research became even more significant when she noted the sudden decline in 2020 of *all* non-'Covid' deaths: 'This trend is completely contrary to the pattern observed in all previous years ... the total decrease in deaths by other causes almost exactly equals the increase in deaths by Covid-19.' This was such a game, set and match in terms of what was happening that Johns Hopkins University deleted the article on the grounds that it 'was being used to support false and dangerous inaccuracies about the impact of the pandemic'. No – because it exposed the scam from official CDC figures and this was confirmed when those figures were published in January, 2021. Here we can see the effect of people dying from heart attacks, cancer, road accidents and gunshot wounds – *anything* – having 'Covid-19' on the death certificate along with those diagnosed from 'symptoms' who had even not tested positive with a test not testing for the 'virus'. I am not kidding with the gunshot wounds, by the way. Brenda Bock, coroner in Grand County, Colorado, revealed that two gunshot victims tested positive for the 'virus' within the previous 30 days and were therefore classified as 'Covid deaths'. Bock said: 'These two people had tested positive for Covid, but that's not what killed them. A gunshot wound is what killed them.' She said she had not even finished her investigation when the state listed the gunshot victims as deaths due to the 'virus'. The death and case figures for 'Covid-19' are an absolute joke and yet they are repeated like parrots by the media, politicians and alleged medical 'experts'. The official Cult narrative is the only show in town.

Genevieve Briand found that deaths from all causes were not exceptional in 2020 compared with previous years and a Spanish magazine published figures that said the same about Spain which was a 'Covid' propaganda hotspot at one point. *Discovery Salud*, a

health and medicine magazine, quoted government figures which showed how 17,000 *fewer* people died in Spain in 2020 than in 2019 and more than 26,000 fewer than in 2018. The age-standardised mortality rate for England and Wales when age distribution is taken into account was significantly lower in 2020 than the 1970s, 80s and 90s, and was only the ninth highest since 2000. Where is the ‘pandemic’?

Post mortems and autopsies virtually disappeared for ‘Covid’ deaths amid claims that ‘virus-infected’ bodily fluids posed a risk to those carrying out the autopsy. This was rejected by renowned German pathologist and forensic doctor Klaus Püschel who said that he and his staff had by then done 150 autopsies on ‘Covid’ patients with no problems at all. He said they were needed to know why some ‘Covid’ patients suffered blood clots and not severe respiratory infections. The ‘virus’ is, after all, called SARS or ‘severe acute respiratory syndrome’. I highlighted in the spring of 2020 this phenomenon and quoted New York intensive care doctor Cameron Kyle-Sidell who posted a soon deleted YouTube video to say that they had been told to prepare to treat an infectious disease called ‘Covid-19’, but that was not what they were dealing with. Instead he likened the lung condition of the most severely ill patients to what you would expect with cabin depressurisation in a plane at 30,000 feet or someone dropped on the top of Everest without oxygen or acclimatisation. I have never said this is not happening to a small minority of alleged ‘Covid’ patients – I am saying this is not caused by a phantom ‘contagious virus’. Indeed Kyle-Sidell said that ‘Covid-19’ was not the disease they were told was coming their way. ‘We are operating under a medical paradigm that is untrue,’ he said, and he believed they were treating the wrong disease: ‘These people are being slowly starved of oxygen.’ Patients would take off their oxygen masks in a state of fear and stress and while they were blue in the face on the brink of death. They did not look like patients dying of pneumonia. You can see why they don’t want autopsies when their virus doesn’t exist and there is another condition in some people that they don’t wish to be uncovered. I should add here that

the 5G system of millimetre waves was being rapidly introduced around the world in 2020 and even more so now as they fire 5G at the Earth from satellites. At 60 gigahertz within the 5G range that frequency interacts with the oxygen molecule and stops people breathing in sufficient oxygen to be absorbed into the bloodstream. They are installing 5G in schools and hospitals. The world is not mad or anything. 5G can cause major changes to the lungs and blood as I detail in *The Answer* and these consequences are labelled 'Covid-19', the alleged symptoms of which can be caused by 5G and other electromagnetic frequencies as cells respond to radiation poisoning.

## **The 'Covid death' scam**

Dr Scott Jensen, a Minnesota state senator and medical doctor, exposed 'Covid' Medicare payment incentives to hospitals and death certificate manipulation. He said he was sent a seven-page document by the US Department of Health 'coaching' him on how to fill out death certificates which had never happened before. The document said that he didn't need to have a laboratory test for 'Covid-19' to put that on the death certificate and that shocked him when death certificates are supposed to be about facts. Jensen described how doctors had been 'encouraged, if not pressured' to make a diagnosis of 'Covid-19' if they thought it was probable or '*presumed*'. No positive test was necessary – not that this would have mattered anyway. He said doctors were told to diagnose 'Covid' by symptoms when these were the same as colds, allergies, other respiratory problems, and certainly with influenza which 'disappeared' in the 'Covid' era. A common sniffle was enough to get the dreaded verdict. Ontario authorities decreed that a single care home resident with *one* symptom from a long list must lead to the isolation of the entire home. Other courageous doctors like Jensen made the same point about death figure manipulation and how deaths by other causes were falling while 'Covid-19 deaths' were rising at the same rate due to re-diagnosis. Their videos rarely survive long on YouTube with its Cult-supporting algorithms courtesy of CEO Susan Wojcicki and her bosses at Google. Figure-tampering was so glaring

and ubiquitous that even officials were letting it slip or outright saying it. UK chief scientific adviser Patrick Vallance said on one occasion that ‘Covid’ on the death certificate doesn’t mean ‘Covid’ was the cause of death (so why the hell is it there?) and we had the rare sight of a BBC reporter telling the truth when she said: ‘Someone could be successfully treated for Covid, in say April, discharged, and then in June, get run over by a bus and die ... That person would still be counted as a Covid death in England.’ Yet the BBC and the rest of the world media went on repeating the case and death figures as if they were real. Illinois Public Health Director Dr Ngozi Ezike revealed the deceit while her bosses must have been clenching their buttocks:

If you were in a hospice and given a few weeks to live and you were then found to have Covid that would be counted as a Covid death. [There might be] a clear alternate cause, but it is still listed as a Covid death. So everyone listed as a Covid death doesn’t mean that was the cause of the death, but that they had Covid at the time of death.

Yes, a ‘Covid virus’ never shown to exist and tested for with a test not testing for the ‘virus’. In the first period of the pandemic hoax through the spring of 2020 the process began of designating almost everything a ‘Covid’ death and this has continued ever since. I sat in a restaurant one night listening to a loud conversation on the next table where a family was discussing in bewilderment how a relative who had no symptoms of ‘Covid’, and had died of a long-term problem, could have been diagnosed a death by the ‘virus’. I could understand their bewilderment. If they read this book they will know why this medical fraud has been perpetrated the world over.

## **Some media truth shock**

The media ignored the evidence of death certificate fraud until eventually one columnist did speak out when she saw it first-hand. Bel Mooney is a long-time national newspaper journalist in Britain currently working for the *Daily Mail*. Her article on February 19th, 2021, carried this headline: ‘My dad Ted passed three Covid tests

and died of a chronic illness yet he's officially one of Britain's 120,000 victims of the virus and is far from alone ... so how many more are there?' She told how her 99-year-old father was in a care home with a long-standing chronic obstructive pulmonary disease and vascular dementia. Maybe, but he was still aware enough to tell her from the start that there was no 'virus' and he refused the 'vaccine' for that reason. His death was not unexpected given his chronic health problems and Mooney said she was shocked to find that 'Covid-19' was declared the cause of death on his death certificate. She said this was a 'bizarre and unacceptable untruth' for a man with long-time health problems who had tested negative twice at the home for the 'virus'. I was also shocked by this story although not by what she said. I had been highlighting the death certificate manipulation for ten months. It was the confirmation that a professional full-time journalist only realised this was going on when it affected her directly and neither did she know that whether her dad tested positive or negative was irrelevant with the test not testing for the 'virus'. Where had she been? She said she did not believe in 'conspiracy theories' without knowing I'm sure that this and 'conspiracy theorists' were terms put into widespread circulation by the CIA in the 1960s to discredit those who did not accept the ridiculous official story of the Kennedy assassination. A blanket statement of 'I don't believe in conspiracy theories' is always bizarre. The dictionary definition of the term alone means the world is drowning in conspiracies. What she said was even more daft when her dad had just been affected by the 'Covid' conspiracy. Why else does she think that 'Covid-19' was going on the death certificates of people who died of something else?

To be fair once she saw from personal experience what was happening she didn't mince words. Mooney was called by the care home on the morning of February 9th to be told her father had died in his sleep. When she asked for the official cause of death what came back was 'Covid-19'. Mooney challenged this and was told there had been deaths from Covid on the dementia floor (confirmed by a test not testing for the 'virus') so they considered it 'reasonable

to assume'. 'But doctor,' Mooney rightly protested, 'an assumption isn't a diagnosis.' She said she didn't blame the perfectly decent and sympathetic doctor – 'he was just doing his job'. Sorry, but that's *bullshit*. He wasn't doing his job at all. He was putting a false cause of death on the death certificate and that is a criminal offence for which he should be brought to account and the same with the millions of doctors worldwide who have done the same. They were not doing their job they were following orders and that must not wash at new Nuremberg trials any more than it did at the first ones. Mooney's doctor was 'assuming' (presuming) as he was told to, but 'just following orders' makes no difference to his actions. A doctor's job is to serve the patient and the truth, not follow orders, but that's what they have done all over the world and played a central part in making the 'Covid' hoax possible with all its catastrophic consequences for humanity. Shame on them and they must answer for their actions. Mooney said her disquiet worsened when she registered her father's death by telephone and was told by the registrar there had been very many other cases like hers where 'the deceased' had not tested positive for 'Covid' yet it was recorded as the cause of death. The test may not matter, but those involved at their level *think* it matters and it shows a callous disregard for accurate diagnosis. The pressure to do this is coming from the top of the national 'health' pyramids which in turn obey the World Health Organization which obeys Gates and the Cult. Mooney said the registrar agreed that this must distort the national figures adding that 'the strangest thing is that every winter we record countless deaths from flu, and this winter there have been none. Not one!' She asked if the registrar thought deaths from flu were being misdiagnosed and lumped together with 'Covid' deaths. The answer was a 'puzzled yes'. Mooney said that the funeral director said the same about 'Covid' deaths which had nothing to do with 'Covid'. They had lost count of the number of families upset by this and other funeral companies in different countries have had the same experience. Mooney wrote:

The nightly shroud-waving and shocking close-ups of pain imposed on us by the TV news bewildered and terrified the population into eager compliance with lockdowns. We were invited to ‘save the NHS’ and to grieve for strangers – the real-life loved ones behind those shocking death counts. Why would the public imagine what I now fear, namely that the way Covid-19 death statistics are compiled might make the numbers seem greater than they are?

Oh, just a little bit – like 100 percent.

## **Do the maths**

Mooney asked why a country would wish to skew its mortality figures by wrongly certifying deaths? What had been going on? Well, if you don’t believe in conspiracies you will never find the answer which is that *it’s a conspiracy*. She did, however, describe what she had discovered as a ‘national scandal’. In reality it’s a global scandal and happening everywhere. Pillars of this conspiracy were all put into place before the button was pressed with the Drosten PCR protocol and high amplifications to produce the cases and death certificate changes to secure illusory ‘Covid’ deaths.

Mooney notes that normally two doctors were needed to certify a death, with one having to know the patient, and how the rules were changed in the spring of 2020 to allow one doctor to do this. In the same period ‘Covid deaths’ were decreed to be all cases where Covid-19 was put on the death certificate even without a positive test or any symptoms. Mooney asked: ‘How many of the 30,851 (as of January 15) care home resident deaths with Covid-19 on the certificate (32.4 per cent of all deaths so far) were based on an assumption, like that of my father? And what has that done to our national psyche?’ All of them is the answer to the first question and it has devastated and dismantled the national psyche, actually the global psyche, on a colossal scale. In the UK case and death data is compiled by organisations like Public Health England (PHE) and the Office for National Statistics (ONS). Mooney highlights the insane policy of counting a death from any cause as ‘Covid-19’ if this happens within 28 days of a positive test (with a test not testing for the ‘virus’) and she points out that ONS statistics reflect deaths ‘involving Covid’ ‘or due to Covid’ which meant in practice any

death where ‘Covid-19’ was mentioned on the death certificate. She described the consequences of this fraud:

Most people will accept the narrative they are fed, so panicky governments here and in Europe witnessed the harsh measures enacted in totalitarian China and jumped into lockdown. Headlines about Covid deaths tolled like the knell that would bring doomsday to us all. Fear stalked our empty streets. Politicians parroted the frankly ridiculous aim of ‘zero Covid’ and shut down the economy, while most British people agreed that lockdown was essential and (astonishingly to me, as a patriotic Brit) even wanted more restrictions.

For what? Lies on death certificates? Never mind the grim toll of lives ruined, suicides, schools closed, rising inequality, depression, cancelled hospital treatments, cancer patients in a torture of waiting, poverty, economic devastation, loneliness, families kept apart, and so on. How many lives have been lost as a direct result of lockdown?

She said that we could join in a national chorus of shock and horror at reaching the 120,000 death toll which was surely certain to have been totally skewed all along, but what about the human cost of lockdown justified by these ‘death figures’? *The British Medical Journal* had reported a 1,493 percent increase in cases of children taken to Great Ormond Street Hospital with abusive head injuries alone and then there was the effect on families:

Perhaps the most shocking thing about all this is that families have been kept apart – and obeyed the most irrational, changing rules at the whim of government – because they believed in the statistics. They succumbed to fear, which his generation rejected in that war fought for freedom. Dad (God rest his soul) would be angry. And so am I.

Another theme to watch is that in the winter months when there are more deaths from all causes they focus on ‘Covid’ deaths and in the summer when the British Lung Foundation says respiratory disease plummets by 80 percent they rage on about ‘cases’. Either way fascism on population is always the answer.

## **Nazi eugenics in the 21st century**

Elderly people in care homes have been isolated from their families month after lonely month with no contact with relatives and grandchildren who were banned from seeing them. We were told

that lockdown fascism was to ‘protect the vulnerable’ like elderly people. At the same time Do Not Resuscitate (DNR) orders were placed on their medical files so that if they needed resuscitation it wasn’t done and ‘Covid-19’ went on their death certificates. Old people were not being ‘protected’ they were being culled – murdered in truth. DNR orders were being decreed for disabled and young people with learning difficulties or psychological problems. The UK Care Quality Commission, a non-departmental body of the Department of Health and Social Care, found that 34 percent of those working in health and social care were pressured into placing ‘do not attempt cardiopulmonary resuscitation’ orders on ‘Covid’ patients who suffered from disabilities and learning difficulties without involving the patient or their families in the decision. UK judges ruled that an elderly woman with dementia should have the DNA-manipulating ‘Covid vaccine’ against her son’s wishes and that a man with severe learning difficulties should have the jab despite his family’s objections. Never mind that many had already died. The judiciary always supports doctors and government in fascist dictatorships. They wouldn’t dare do otherwise. A horrific video was posted showing fascist officers from Los Angeles police forcibly giving the ‘Covid’ shot to women with special needs who were screaming that they didn’t want it. The same fascists are seen giving the jab to a sleeping elderly woman in a care home. This is straight out of the Nazi playbook. Hitler’s Nazis committed mass murder of the mentally ill and physically disabled throughout Germany and occupied territories in the programme that became known as Aktion T4, or just T4. Sabbatian-controlled Hitler and his grotesque crazies set out to kill those they considered useless and unnecessary. The Reich Committee for the Scientific Registering of Hereditary and Congenital Illnesses registered the births of babies identified by physicians to have ‘defects’. By 1941 alone more than 5,000 children were murdered by the state and it is estimated that in total the number of innocent people killed in Aktion T4 was between 275,000 and 300,000. Parents were told their children had been sent away for ‘special treatment’ never to return. It is rather pathetic to see claims about plans for new extermination camps being dismissed today

when the same force behind current events did precisely that 80 years ago. Margaret Sanger was a Cult operative who used 'birth control' to sanitise her programme of eugenics. Organisations she founded became what is now Planned Parenthood. Sanger proposed that 'the whole dysgenic population would have its choice of segregation or sterilization'. These included epileptics, 'feeble-minded', and prostitutes. Sanger opposed charity because it perpetuated 'human waste'. She reveals the Cult mentality and if anyone thinks that extermination camps are a 'conspiracy theory' their naivety is touching if breathtakingly stupid.

If you don't believe that doctors can act with callous disregard for their patients it is worth considering that doctors and medical staff agreed to put government-decreed DNR orders on medical files and do nothing when resuscitation is called for. I don't know what you call such people in your house. In mine they are Nazis from the Josef Mengele School of Medicine. Phenomenal numbers of old people have died worldwide from the effects of lockdown, depression, lack of treatment, the 'vaccine' (more later) and losing the will to live. A common response at the start of the manufactured pandemic was to remove old people from hospital beds and transfer them to nursing homes. The decision would result in a mass cull of elderly people in those homes through lack of treatment – *not* 'Covid'. Care home whistleblowers have told how once the 'Covid' era began doctors would not come to their homes to treat patients and they were begging for drugs like antibiotics that often never came. The most infamous example was ordered by New York governor Andrew Cuomo, brother of a moronic CNN host, who amazingly was given an Emmy Award for his handling of the 'Covid crisis' by the ridiculous Wokers that hand them out. Just how ridiculous could be seen in February, 2021, when a Department of Justice and FBI investigation began into how thousands of old people in New York died in nursing homes after being discharged from hospital to make way for 'Covid' patients on Cuomo's say-so – and how he and his staff covered up these facts. This couldn't have happened to a nicer psychopath. Even then there was a 'Covid' spin. Reports said that

thousands of old people who tested positive for ‘Covid’ in hospital were transferred to nursing homes to both die of ‘Covid’ and transmit it to others. No – they were in hospital because they were ill and the fact that they tested positive with a test not testing for the ‘virus’ is irrelevant. They were ill often with respiratory diseases ubiquitous in old people near the end of their lives. Their transfer out of hospital meant that their treatment stopped and many would go on to die.

### **They're old. Who gives a damn?**

I have exposed in the books for decades the Cult plan to cull the world’s old people and even to introduce at some point what they call a ‘demise pill’ which at a certain age everyone would take and be out of here by law. In March, 2021, Spain legalised euthanasia and assisted suicide following the Netherlands, Belgium, Luxembourg and Canada on the Tiptoe to the demise pill. Treatment of old people by many ‘care’ homes has been a disgrace in the ‘Covid’ era. There are many, many, caring staff – I know some. There have, however, been legions of stories about callous treatment of old people and their families. Police were called when families came to take their loved ones home in the light of isolation that was killing them. They became prisoners of the state. Care home residents in insane, fascist Ontario, Canada, were not allowed to leave their *room* once the ‘Covid’ hoax began. UK staff have even wheeled elderly people away from windows where family members were talking with them. Oriana Criscuolo from Stockport in the English North West dropped off some things for her 80-year-old father who has Parkinson’s disease and dementia and she wanted to wave to him through a ground-floor window. She was told that was ‘illegal’. When she went anyway they closed the curtains in the middle of the day. Oriana said:

It’s just unbelievable. I cannot understand how care home staff – people who are being paid to care – have become so uncaring. Their behaviour is inhumane and cruel. It’s beyond belief.

She was right and this was not a one-off. What a way to end your life in such loveless circumstances. UK registered nurse Nicky Millen, a proper old school nurse for 40 years, said that when she started her career care was based on dignity, choice, compassion and empathy. Now she said ‘the things that are important to me have gone out of the window.’ She was appalled that people were dying without their loved ones and saying goodbye on iPads. Nicky described how a distressed 89-year-old lady stroked her face and asked her ‘how many paracetamol would it take to finish me off’. Life was no longer worth living while not seeing her family. Nicky said she was humiliated in front of the ward staff and patients for letting the lady stroke her face and giving her a cuddle. Such is the dehumanisation that the ‘Covid’ hoax has brought to the surface. Nicky worked in care homes where patients told her they were being held prisoner. ‘I want to live until I die’, one said to her. ‘I had a lady in tears because she hadn’t seen her great-grandson.’ Nicky was compassionate old school meeting psychopathic New Normal. She also said she had worked on a ‘Covid’ ward with no ‘Covid’ patients. Jewish writer Shai Held wrote an article in March, 2020, which was headlined ‘The Staggering, Heartless Cruelty Toward the Elderly’. What he described was happening from the earliest days of lockdown. He said ‘the elderly’ were considered a group and not unique individuals (the way of the Woke). Shai Held said:

Notice how the all-too-familiar rhetoric of dehumanization works: ‘The elderly’ are bunched together as a faceless mass, all of them considered culprits and thus effectively deserving of the suffering the pandemic will inflict upon them. Lost entirely is the fact that the elderly are individual human beings, each with a distinctive face and voice, each with hopes and dreams, memories and regrets, friendships and marriages, loves lost and loves sustained.

‘The elderly’ have become another dehumanised group for which anything goes and for many that has resulted in cold disregard for their rights and their life. The distinctive face that Held talks about is designed to be deleted by masks until everyone is part of a faceless mass.

## **'War-zone' hospitals myth**

Again and again medical professionals have told me what was really going on and how hospitals 'overrun like war zones' according to the media were virtually empty. The mantra from medical whistleblowers was please don't use my name or my career is over. Citizen journalists around the world sneaked into hospitals to film evidence exposing the 'war-zone' lie. They really *were* largely empty with closed wards and operating theatres. I met a hospital worker in my town on the Isle of Wight during the first lockdown in 2020 who said the only island hospital had never been so quiet. Lockdown was justified by the psychopaths to stop hospitals being overrun. At the same time that the island hospital was near-empty the military arrived here to provide *extra beds*. It was all propaganda to ramp up the fear to ensure compliance with fascism as were never-used temporary hospitals with thousands of beds known as Nightingales and never-used make-shift mortuaries opened by the criminal UK government. A man who helped to install those extra island beds attributed to the army said they were never used and the hospital was empty. Doctors and nurses 'stood around talking or on their phones, wandering down to us to see what we were doing'. There were no masks or social distancing. He accused the useless local island paper, the *County Press*, of 'pumping the fear as if our hospital was overrun and we only have one so it should have been'. He described ambulances parked up with crews outside in deck chairs. When his brother called an ambulance he was told there was a two-hour backlog which he called 'bullshit'. An old lady on the island fell 'and was in a bad way', but a caller who rang for an ambulance was told the situation wasn't urgent enough. Ambulance stations were working under capacity while people would hear ambulances with sirens blaring driving through the streets. When those living near the stations realised what was going on they would follow them as they left, circulated around an urban area with the sirens going, and then came back without stopping. All this was to increase levels of fear and the same goes for the 'ventilator shortage crisis' that cost tens of millions for hastily produced ventilators never to be used.

Ambulance crews that agreed to be exploited in this way for fear propaganda might find themselves a mirror. I wish them well with that. Empty hospitals were the obvious consequence of treatment and diagnoses of non-'Covid' conditions cancelled and those involved handed a death sentence. People have been dying at home from undiagnosed and untreated cancer, heart disease and other life-threatening conditions to allow empty hospitals to deal with a 'pandemic' that wasn't happening.

## **Death of the innocent**

'War-zones' have been laying off nursing staff, even doctors where they can. There was no work for them. Lockdown was justified by saving lives and protecting the vulnerable they were actually killing with DNR orders and preventing empty hospitals being 'overrun'. In Britain the mantra of stay at home to 'save the NHS' was everywhere and across the world the same story was being sold when it was all lies. Two California doctors, Dan Erickson and Artin Massihi at Accelerated Urgent Care in Bakersfield, held a news conference in April, 2020, to say that intensive care units in California were 'empty, essentially', with hospitals shutting floors, not treating patients and laying off doctors. The California health system was working at minimum capacity 'getting rid of doctors because we just don't have the volume'. They said that people with conditions such as heart disease and cancer were not coming to hospital out of fear of 'Covid-19'. Their video was deleted by Susan Wojcicki's Cult-owned YouTube after reaching five million views. Florida governor Ron Desantis, who rejected the severe lockdowns of other states and is being targeted for doing so, said that in March, 2020, every US governor was given models claiming they would run out of hospital beds in days. That was never going to happen and the 'modellers' knew it. Deceit can be found at every level of the system. Urgent children's operations were cancelled including fracture repairs and biopsies to spot cancer. Eric Nicholls, a consultant paediatrician, said 'this is obviously concerning and we need to return to normal operating and to increase capacity as soon as possible'. Psychopaths

in power were rather less concerned *because* they are psychopaths. Deletion of urgent care and diagnosis has been happening all over the world and how many kids and others have died as a result of the actions of these cold and heartless lunatics dictating ‘health’ policy? The number must be stratospheric. Richard Sullivan, professor of cancer and global health at King’s College London, said people feared ‘Covid’ more than cancer such was the campaign of fear. ‘Years of lost life will be quite dramatic’, Sullivan said, with ‘a huge amount of avoidable mortality’. Sarah Woolnough, executive director for policy at Cancer Research UK, said there had been a 75 percent drop in urgent referrals to hospitals by family doctors of people with suspected cancer. Sullivan said that ‘a lot of services have had to scale back – we’ve seen a dramatic decrease in the amount of elective cancer surgery’. Lockdown deaths worldwide has been absolutely fantastic with the *New York Post* reporting how data confirmed that ‘lockdowns end more lives than they save’:

There was a sharp decline in visits to emergency rooms and an increase in fatal heart attacks because patients didn’t receive prompt treatment. Many fewer people were screened for cancer. Social isolation contributed to excess deaths from dementia and Alzheimer’s.

Researchers predicted that the social and economic upheaval would lead to tens of thousands of “deaths of despair” from drug overdoses, alcoholism and suicide. As unemployment surged and mental-health and substance-abuse treatment programs were interrupted, the reported levels of anxiety, depression and suicidal thoughts increased dramatically, as did alcohol sales and fatal drug overdoses.

This has been happening while nurses and other staff had so much time on their hands in the ‘war-zones’ that Tic-Tok dancing videos began appearing across the Internet with medical staff dancing around in empty wards and corridors as people died at home from causes that would normally have been treated in hospital.

## **Mentions in dispatches**

One brave and truth-committed whistleblower was Louise Hampton, a call handler with the UK NHS who made a viral Internet video saying she had done ‘fuck all’ during the ‘pandemic’

which was ‘a load of bollocks’. She said that ‘Covid-19’ was rebranded flu and of course she lost her job. This is what happens in the medical and endless other professions now when you tell the truth. Louise filmed inside ‘war-zone’ accident and emergency departments to show they were empty and I mean *empty* as in no one there. The mainstream media could have done the same and blown the gaff on the whole conspiracy. They haven’t to their eternal shame. Not that most ‘journalists’ seem capable of manifesting shame as with the psychopaths they slavishly repeat without question. The relative few who were admitted with serious health problems were left to die alone with no loved ones allowed to see them because of ‘Covid’ rules and they included kids dying without the comfort of mum and dad at their bedside while the evil behind this couldn’t give a damn. It was all good fun to them. A Scottish NHS staff nurse publicly quit in the spring of 2021 saying: ‘I can no longer be part of the lies and the corruption by the government.’ She said hospitals ‘aren’t full, the beds aren’t full, beds have been shut, wards have been shut’. Hospitals were never busy throughout ‘Covid’. The staff nurse said that Nicola Sturgeon, tragically the leader of the Scottish government, was on television saying save the hospitals and the NHS – ‘but the beds are empty’ and ‘we’ve not seen flu, we always see flu every year’. She wrote to government and spoke with her union Unison (the unions are Cult-compromised and *useless*, but nothing changed. Many of her colleagues were scared of losing their jobs if they spoke out as they wanted to. She said nursing staff were being affected by wearing masks all day and ‘my head is splitting every shift from wearing a mask’. The NHS is part of the fascist tyranny and must be dismantled so we can start again with human beings in charge. (Ironically, hospitals were reported to be busier again when official ‘Covid’ cases *fell* in spring/summer of 2021 and many other conditions required treatment at the same time as *the fake vaccine rollout*.)

I will cover the ‘Covid vaccine’ scam in detail later, but it is another indicator of the sickening disregard for human life that I am highlighting here. The DNA-manipulating concoctions do not fulfil

the definition of a 'vaccine', have never been used on humans before and were given only emergency approval because trials were not completed and they continued using the unknowing public. The result was what a NHS senior nurse with responsibility for 'vaccine' procedure said was 'genocide'. She said the 'vaccines' were not 'vaccines'. They had not been shown to be safe and claims about their effectiveness by drug companies were 'poetic licence'. She described what was happening as a 'horrid act of human annihilation'. The nurse said that management had instigated a policy of not providing a Patient Information Leaflet (PIL) before people were 'vaccinated' even though health care professionals are supposed to do this according to protocol. Patients should also be told that they are taking part in an ongoing clinical trial. Her challenges to what is happening had seen her excluded from meetings and ridiculed in others. She said she was told to 'watch my step ... or I would find myself surplus to requirements'. The nurse, who spoke anonymously in fear of her career, said she asked her NHS manager why he/she was content with taking part in genocide against those having the 'vaccines'. The reply was that everyone had to play their part and to 'put up, shut up, and get it done'. Government was 'leaning heavily' on NHS management which was clearly leaning heavily on staff. This is how the global 'medical' hierarchy operates and it starts with the Cult and its World Health Organization.

She told the story of a doctor who had the Pfizer jab and when questioned had no idea what was in it. The doctor had never read the literature. We have to stop treating doctors as intellectual giants when so many are moral and medical pygmies. The doctor did not even know that the 'vaccines' were not fully approved or that their trials were ongoing. They were, however, asking their patients if they minded taking part in follow-ups for research purposes – yes, the *ongoing clinical trial*. The nurse said the doctor's ignorance was not rare and she had spoken to a hospital consultant who had the jab without any idea of the background or that the 'trials' had not been completed. Nurses and pharmacists had shown the same ignorance.

'My NHS colleagues have forsaken their duty of care, broken their code of conduct – Hippocratic Oath – and have been brainwashed just the same as the majority of the UK public through propaganda ...' She said she had not been able to recruit a single NHS colleague, doctor, nurse or pharmacist to stand with her and speak out. Her union had refused to help. She said that if the genocide came to light she would not hesitate to give evidence at a Nuremberg-type trial against those in power who could have affected the outcomes but didn't.

## **And all for what?**

To put the nonsense into perspective let's say the 'virus' does exist and let's go completely crazy and accept that the official manipulated figures for cases and deaths are accurate. *Even then* a study by Stanford University epidemiologist Dr John Ioannidis published on the World Health Organization website produced an average infection to fatality rate of ... 0.23 percent! Ioannidis said: 'If one could sample equally from all locations globally, the median infection fatality rate might even be substantially lower than the 0.23% observed in my analysis.' For healthy people under 70 it was ... 0.05 percent! This compares with the 3.4 percent claimed by the Cult-owned World Health Organization when the hoax was first played and maximum fear needed to be generated. An updated Stanford study in April, 2021, put the 'infection' to 'fatality' rate at just 0.15 percent. Another team of scientists led by Megan O'Driscoll and Henrik Salje studied data from 45 countries and published their findings on the Nature website. For children and young people the figure is so small it virtually does not register although authorities will be hyping dangers to the young when they introduce DNA-manipulating 'vaccines' for children. The O'Driscoll study produced an average infection-fatality figure of 0.003 for children from birth to four; 0.001 for 5 to 14; 0.003 for 15 to 19; and it was still only 0.456 up to 64. To claim that children must be 'vaccinated' to protect them from 'Covid' is an obvious lie and so there must be another reason and there is. What's more the average age of a 'Covid' death is akin

to the average age that people die in general. The average age of death in England is about 80 for men and 83 for women. The average age of death from alleged 'Covid' is between 82 and 83. California doctors, Dan Erickson and Artin Massihi, said at their April media conference that projection models of millions of deaths had been 'woefully inaccurate'. They produced detailed figures showing that Californians had a 0.03 chance of dying from 'Covid' based on the number of people who tested positive (with a test not testing for the 'virus'). Erickson said there was a 0.1 percent chance of dying from 'Covid' in the *state* of New York, not just the city, and a 0.05 percent chance in Spain, a centre of 'Covid-19' hysteria at one stage. The Stanford studies supported the doctors' data with fatality rate estimates of 0.23 and 0.15 percent. How close are these figures to my estimate of *zero*? Death-rate figures claimed by the World Health Organization at the start of the hoax were some 15 times higher. The California doctors said there was no justification for lockdowns and the economic devastation they caused. Everything they had ever learned about quarantine was that you quarantine the *sick* and not the healthy. They had never seen this before and it made no medical sense.

Why in the light of all this would governments and medical systems the world over say that billions must go under house arrest; lose their livelihood; in many cases lose their mind, their health and their life; force people to wear masks dangerous to health and psychology; make human interaction and even family interaction a criminal offence; ban travel; close restaurants, bars, watching live sport, concerts, theatre, and any activity involving human togetherness and discourse; and closing schools to isolate children from their friends and cause many to commit suicide in acts of hopelessness and despair? The California doctors said lockdown consequences included increased child abuse, partner abuse, alcoholism, depression, and other impacts they were seeing every day. Who would do that to the entire human race if not mentally-ill psychopaths of almost unimaginable extremes like Bill Gates? We must face the reality of what we are dealing with and come out of

denial. Fascism and tyranny are made possible only by the target population submitting and acquiescing to fascism and tyranny. The whole of human history shows that to be true. Most people naively and unquestioning believed what they were told about a ‘deadly virus’ and meekly and weakly submitted to house arrest. Those who didn’t believe it – at least in total – still submitted in fear of the consequences of not doing so. For the rest who wouldn’t submit draconian fines have been imposed, brutal policing by psychopaths *for* psychopaths, and condemnation from the meek and weak who condemn the Pushbackers on behalf of the very force that has them, too, in its gunsights. ‘Pathetic’ does not even begin to suffice.

Britain’s brainless ‘Health’ Secretary Matt Hancock warned anyone lying to border officials about returning from a list of ‘hotspot’ countries could face a jail sentence of up to ten years which is more than for racially-aggravated assault, incest and attempting to have sex with a child under 13. Hancock is a lunatic, but he has the state apparatus behind him in a Cult-led chain reaction and the same with UK ‘Vaccine Minister’ Nadhim Zahawi, a prominent member of the mega-Cult secret society, Le Cercle, which featured in my earlier books. The Cult enforces its will on governments and medical systems; government and medical systems enforce their will on business and police; business enforces its will on staff who enforce it on customers; police enforce the will of the Cult on the population and play their essential part in creating a world of fascist control that their own children and grandchildren will have to live in their entire lives. It is a hierarchical pyramid of imposition and acquiescence and, yes indeedy, of clinical insanity.

Does anyone bright enough to read this book have to ask what the answer is? I think not, but I will reveal it anyway in the fewest of syllables: Tell the psychos and their moronic lackeys to fuck off and let’s get on with our lives. We are many – They are few.

## CHAPTER SEVEN

### War on your mind

***One believes things because one has been conditioned to believe them***

**Aldous Huxley, *Brave New World***

I have described the ‘Covid’ hoax as a ‘Psyop’ and that is true in every sense and on every level in accordance with the definition of that term which is psychological warfare. Break down the ‘Covid pandemic’ to the foundation themes and it is psychological warfare on the human individual and collective mind.

The same can be said for the entire human belief system involving every subject you can imagine. Huxley was right in his contention that people believe what they are conditioned to believe and this comes from the repetition throughout their lives of the same falsehoods. They spew from government, corporations, media and endless streams of ‘experts’ telling you what the Cult wants you to believe and often believing it themselves (although *far* from always). ‘Experts’ are rewarded with ‘prestigious’ jobs and titles and as agents of perceptual programming with regular access to the media. The Cult has to control the narrative – control *information* – or they lose control of the vital, crucial, without-which-they-cannot-prevail public perception of reality. The foundation of that control today is the Internet made possible by the Defense Advanced Research Projects Agency (DARPA), the incredibly sinister technological arm of the Pentagon. The Internet is the result of military technology.

DARPA openly brags about establishing the Internet which has been a long-term project to lasso the minds of the global population. I have said for decades the plan is to control information to such an extreme that eventually no one would see or hear anything that the Cult does not approve. We are closing in on that end with ferocious censorship since the ‘Covid’ hoax began and in my case it started back in the 1990s in terms of books and speaking venues. I had to create my own publishing company in 1995 precisely because no one else would publish my books even then. I think they’re all still running.

## **Cult Internet**

To secure total control of information they needed the Internet in which pre-programmed algorithms can seek out ‘unclean’ content for deletion and even stop it being posted in the first place. The Cult had to dismantle print and non-Internet broadcast media to ensure the transfer of information to the appropriate-named ‘Web’ – a critical expression of the *Cult* web. We’ve seen the ever-quickenning demise of traditional media and control of what is left by a tiny number of corporations operating worldwide. Independent journalism in the mainstream is already dead and never was that more obvious than since the turn of 2020. The Cult wants all information communicated via the Internet to globally censor and allow the plug to be pulled any time. Lockdowns and forced isolation has meant that communication between people has been through electronic means and no longer through face-to-face discourse and discussion. Cult psychopaths have targeted the bars, restaurants, sport, venues and meeting places in general for this reason. None of this is by chance and it’s to stop people gathering in any kind of privacy or number while being able to track and monitor all Internet communications and block them as necessary. Even private messages between individuals have been censored by these fascists that control Cult fronts like Facebook, Twitter, Google and YouTube which are all officially run by Sabbatian place-people and from the background by higher-level Sabbatian place people.

Facebook, Google, Amazon and their like were seed-funded and supported into existence with money-no-object infusions of funds either directly or indirectly from DARPA and CIA technology arm In-Q-Tel. The Cult plays the long game and prepares very carefully for big plays like 'Covid'. Amazon is another front in the psychological war and pretty much controls the global market in book sales and increasingly publishing. Amazon's limitless funds have deleted fantastic numbers of independent publishers to seize global domination on the way to deciding which books can be sold and circulated and which cannot. Moves in that direction are already happening. Amazon's leading light Jeff Bezos is the grandson of Lawrence Preston Gise who worked with DARPA predecessor ARPA. Amazon has big connections to the CIA and the Pentagon. The plan I have long described went like this:

1. Employ military technology to establish the Internet.
2. Sell the Internet as a place where people can freely communicate without censorship and allow that to happen until the Net becomes the central and irreversible pillar of human society. If the Internet had been highly censored from the start many would have rejected it.
3. Fund and manipulate major corporations into being to control the circulation of information on your Internet using cover stories about geeks in garages to explain how they came about. Give them unlimited funds to expand rapidly with no need to make a profit for years while non-Cult companies who need to balance the books cannot compete. You know that in these circumstances your Googles, YouTubes, Facebooks and Amazons are going to secure near monopolies by either crushing or buying up the opposition.
4. Allow freedom of expression on both the Internet and communication platforms to draw people in until the Internet is the central and irreversible pillar of human society and your communication corporations have reached a stage of near monopoly domination.
5. Then unleash your always-planned frenzy of censorship on the basis of 'where else are you going to go?' and continue to expand that until nothing remains that the Cult does not want its human targets to see.

The process was timed to hit the 'Covid' hoax to ensure the best chance possible of controlling the narrative which they knew they had to do at all costs. They were, after all, about to unleash a 'deadly virus' that didn't really exist. If you do that in an environment of free-flowing information and opinion you would be dead in the

water before you could say Gates is a psychopath. The network was in place through which the Cult-created-and-owned World Health Organization could dictate the ‘Covid’ narrative and response policy slavishly supported by Cult-owned Internet communication giants and mainstream media while those telling a different story were censored. Google, YouTube, Facebook and Twitter openly announced that they would do this. What else would we expect from Cult-owned operations like Facebook which former executives have confirmed set out to make the platform more addictive than cigarettes and coldly manipulates emotions of its users to sow division between people and groups and scramble the minds of the young? If Zuckerberg lives out the rest of his life without going to jail for crimes against humanity, and most emphatically against the young, it will be a travesty of justice. Still, no matter, cause and effect will catch up with him eventually and the same with Sergey Brin and Larry Page at Google with its CEO Sundar Pichai who fix the Google search results to promote Cult narratives and hide the opposition. Put the same key words into Google and other search engines like DuckDuckGo and you will see how different results can be. Wikipedia is another intensely biased ‘encyclopaedia’ which skews its content to the Cult agenda. YouTube links to Wikipedia’s version of ‘Covid’ and ‘climate change’ on video pages in which experts in their field offer a different opinion (even that is increasingly rare with Wojcicki censorship). Into this ‘Covid’ silence-them network must be added government media censors, sorry ‘regulators’, such as Ofcom in the UK which imposed tyrannical restrictions on British broadcasters that had the effect of banning me from ever appearing. Just to debate with me about my evidence and views on ‘Covid’ would mean breaking the fascistic impositions of Ofcom and its CEO career government bureaucrat Melanie Dawes. Gutless British broadcasters tremble at the very thought of fascist Ofcom.

## **Psychos behind ‘Covid’**

The reason for the ‘Covid’ catastrophe in all its facets and forms can be seen by whom and what is driving the policies worldwide in such a coordinated way. Decisions are not being made to protect health, but to target psychology. The dominant group guiding and ‘advising’ government policy are not medical professionals. They are psychologists and behavioural scientists. Every major country has its own version of this phenomenon and I’ll use the British example to show how it works. In many ways the British version has been affecting the wider world in the form of the huge behaviour manipulation network in the UK which operates in other countries. The network involves private companies, government, intelligence and military. The Cabinet Office is at the centre of the government ‘Covid’ Psyop and part-owns, with ‘innovation charity’ Nesta, the Behavioural Insights Team (BIT) which claims to be independent of government but patently isn’t. The BIT was established in 2010 and its job is to manipulate the psyche of the population to acquiesce to government demands and so much more. It is also known as the ‘Nudge Unit’, a name inspired by the 2009 book by two ultra-Zionists, Cass Sunstein and Richard Thaler, called *Nudge: Improving Decisions About Health, Wealth, and Happiness*. The book, as with the Behavioural Insights Team, seeks to ‘nudge’ behaviour (manipulate it) to make the public follow patterns of action and perception that suit those in authority (the Cult). Sunstein is so skilled at this that he advises the World Health Organization and the UK Behavioural Insights Team and was Administrator of the White House Office of Information and Regulatory Affairs in the Obama administration. Biden appointed him to the Department of Homeland Security – another ultra-Zionist in the fold to oversee new immigration laws which is another policy the Cult wants to control. Sunstein is desperate to silence anyone exposing conspiracies and co-authored a 2008 report on the subject in which suggestions were offered to ban ‘conspiracy theorizing’ or impose ‘some kind of tax, financial or otherwise, on those who disseminate such theories’. I guess a psychiatrist’s chair is out of the question?

Sunstein's mate Richard Thaler, an 'academic affiliate' of the UK Behavioural Insights Team, is a proponent of 'behavioural economics' which is defined as the study of 'the effects of psychological, cognitive, emotional, cultural and social factors on the decisions of individuals and institutions'. Study the effects so they can be manipulated to be what you want them to be. Other leading names in the development of behavioural economics are ultra-Zionists Daniel Kahneman and Robert J. Shiller and they, with Thaler, won the Nobel Memorial Prize in Economic Sciences for their work in this field. The Behavioural Insights Team is operating at the heart of the UK government and has expanded globally through partnerships with several universities including Harvard, Oxford, Cambridge, University College London (UCL) and Pennsylvania. They claim to have 'trained' (reframed) 20,000 civil servants and run more than 750 projects involving 400 randomised controlled trials in dozens of countries' as another version of mind reframers Common Purpose. BIT works from its office in New York with cities and their agencies, as well as other partners, across the United States and Canada – this is a company part-owned by the British government Cabinet Office. An executive order by President Cult-servant Obama established a US Social and Behavioral Sciences Team in 2015. They all have the same reason for being and that's to brainwash the population directly and by brainwashing those in positions of authority.

### **'Covid' mind game**

Another prime aspect of the UK mind-control network is the 'independent' [joke] Scientific Pandemic Insights Group on Behaviours (SPI-B) which 'provides behavioural science advice aimed at anticipating and helping people adhere to interventions that are recommended by medical or epidemiological experts'. That means manipulating public perception and behaviour to do whatever government tells them to do. It's disgusting and if they really want the public to be 'safe' this lot should all be under lock and key. According to the government website SPI-B consists of

'behavioural scientists, health and social psychologists, anthropologists and historians' and advises the Whitty-Vallance-led Scientific Advisory Group for Emergencies (SAGE) which in turn advises the government on 'the science' (it doesn't) and 'Covid' policy. When politicians say they are being guided by 'the science' this is the rabble in each country they are talking about and that 'science' is dominated by behaviour manipulators to enforce government fascism through public compliance. The Behaviour Insight Team is headed by psychologist David Solomon Halpern, a visiting professor at King's College London, and connects with a national and global web of other civilian and military organisations as the Cult moves towards its goal of fusing them into one fascistic whole in every country through its 'Fusion Doctrine'. The behaviour manipulation network involves, but is not confined to, the Foreign Office; National Security Council; government communications headquarters (GCHQ); MI5; MI6; the Cabinet Office-based Media Monitoring Unit; and the Rapid Response Unit which 'monitors digital trends to spot emerging issues; including misinformation and disinformation; and identifies the best way to respond'.

There is also the 77th Brigade of the UK military which operates like the notorious Israeli military's Unit 8200 in manipulating information and discussion on the Internet by posing as members of the public to promote the narrative and discredit those who challenge it. Here we have the military seeking to manipulate *domestic* public opinion while the Nazis in government are fine with that. Conservative Member of Parliament Tobias Ellwood, an advocate of lockdown and control through 'vaccine passports', is a Lieutenant Colonel reservist in the 77th Brigade which connects with the military operation jHub, the 'innovation centre' for the Ministry of Defence and Strategic Command. jHub has also been involved with the civilian National Health Service (NHS) in 'symptom tracing' the population. The NHS is a key part of this mind control network and produced a document in December, 2020, explaining to staff how to use psychological manipulation with different groups and ages to get them to have the DNA-manipulating 'Covid vaccine'

that's designed to cumulatively rewrite human genetics. The document, called 'Optimising Vaccination Roll Out – Do's and Dont's for all messaging, documents and "communications" in the widest sense', was published by NHS England and the NHS Improvement *Behaviour Change Unit* in partnership with Public Health England and Warwick Business School. I hear the mantra about 'save the NHS' and 'protect the NHS' when we need to scrap the NHS and start again. The current version is far too corrupt, far too anti-human and totally compromised by Cult operatives and their assets. UK government broadcast media censor Ofcom will connect into this web – as will the BBC with its tremendous Ofcom influence – to control what the public see and hear and dictate mass perception. Nuremberg trials must include personnel from all these organisations.

## **The fear factor**

The 'Covid' hoax has led to the creation of the UK Cabinet Office-connected Joint Biosecurity Centre (JBC) which is officially described as providing 'expert advice on pandemics' using its independent [all Cult operations are 'independent'] analytical function to provide real-time analysis about infection outbreaks to identify and respond to outbreaks of Covid-19'. Another role is to advise the government on a response to spikes in infections – 'for example by closing schools or workplaces in local areas where infection levels have risen'. Put another way, promoting the Cult agenda. The Joint Biosecurity Centre is modelled on the Joint Terrorism Analysis Centre which analyses intelligence to set 'terrorism threat levels' and here again you see the fusion of civilian and military operations and intelligence that has led to military intelligence producing documents about 'vaccine hesitancy' and how it can be combated. Domestic civilian matters and opinions should not be the business of the military. The Joint Biosecurity Centre is headed by Tom Hurd, director general of the Office for Security and Counter-Terrorism from the establishment-to-its-fingertips Hurd family. His father is former Foreign Secretary Douglas Hurd. How coincidental that Tom

Hurd went to the elite Eton College and Oxford University with Boris Johnson. Imperial College with its ridiculous computer modeller Neil Ferguson will connect with this gigantic web that will itself interconnect with similar set-ups in other major and not so major countries. Compared with this Cult network the politicians, be they Boris Johnson, Donald Trump or Joe Biden, are bit-part players ‘following the science’. The network of psychologists was on the ‘Covid’ case from the start with the aim of generating maximum fear of the ‘virus’ to ensure compliance by the population. A government behavioural science group known as SPI-B produced a paper in March, 2020, for discussion by the main government science advisory group known as SAGE. It was headed ‘Options for increasing adherence to social distancing measures’ and it said the following in a section headed ‘Persuasion’:

- A substantial number of people still do not feel sufficiently personally threatened; it could be that they are reassured by the low death rate in their demographic group, although levels of concern may be rising. Having a good understanding of the risk has been found to be positively associated with adoption of COVID-19 social distancing measures in Hong Kong.
- The perceived level of personal threat needs to be increased among those who are complacent, using hard-hitting evaluation of options for increasing social distancing emotional messaging. To be effective this must also empower people by making clear the actions they can take to reduce the threat.
- Responsibility to others: There seems to be insufficient understanding of, or feelings of responsibility about, people’s role in transmitting the infection to others ... Messaging about actions need to be framed positively in terms of protecting oneself and the community, and increase confidence that they will be effective.
- Some people will be more persuaded by appeals to play by the rules, some by duty to the community, and some to personal risk.

All these different approaches are needed. The messaging also needs to take account of the realities of different people's lives. Messaging needs to take account of the different motivational levers and circumstances of different people.

All this could be achieved the SPI-B psychologists said by *using the media to increase the sense of personal threat* which translates as terrify the shit out of the population, including children, so they all do what we want. That's not happened has it? Those excuses for 'journalists' who wouldn't know journalism if it bit them on the arse (the great majority) have played their crucial part in serving this Cult-government Psyop to enslave their own kids and grandkids. How they live with themselves I have no idea. The psychological war has been underpinned by constant government 'Covid' propaganda in almost every television and radio ad break, plus the Internet and print media, which has pounded out the fear with taxpayers footing the bill for their own programming. The result has been people terrified of a 'virus' that doesn't exist or one with a tiny fatality rate even if you believe it does. People walk down the street and around the shops wearing face-nappies damaging their health and psychology while others report those who refuse to be that naïve to the police who turn up in their own face-nappies. I had a cameraman come to my flat and he was so frightened of 'Covid' he came in wearing a mask and refused to shake my hand in case he caught something. He had – naïveitis – and the thought that he worked in the mainstream media was both depressing and made his behaviour perfectly explainable. The fear which has gripped the minds of so many and frozen them into compliance has been carefully cultivated by these psychologists who are really psychopaths. If lives get destroyed and a lot of young people commit suicide it shows our plan is working. SPI-B then turned to compulsion on the public to comply. 'With adequate preparation, rapid change can be achieved', it said. Some countries had introduced mandatory self-isolation on a wide scale without evidence of major public unrest and a large majority of the UK's population appeared to be supportive of more coercive measures with 64 percent of adults saying they would

support putting London under a lockdown (watch the ‘polls’ which are designed to make people believe that public opinion is in favour or against whatever the subject in hand).

For ‘aggressive protective measures’ to be effective, the SPI-B paper said, special attention should be devoted to those population groups that are more at risk. Translated from the Orwellian this means making the rest of population feel guilty for not protecting the ‘vulnerable’ such as old people which the Cult and its agencies were about to kill on an industrial scale with lockdown, lack of treatment and the Gates ‘vaccine’. Psychopath psychologists sold their guilt-trip so comprehensively that Los Angeles County Supervisor Hilda Solis reported that children were apologising (from a distance) to their parents and grandparents for bringing ‘Covid’ into their homes and getting them sick. ‘... These apologies are just some of the last words that loved ones will ever hear as they die alone,’ she said. Gut-wrenchingly Solis then used this childhood tragedy to tell children to stay at home and ‘keep your loved ones alive’. Imagine heaping such potentially life-long guilt on a kid when it has absolutely nothing to do with them. These people are deeply disturbed and the psychologists behind this even more so.

## **Uncivil war – divide and rule**

Professional mind-controllers at SPI-B wanted the media to increase a sense of responsibility to others (do as you’re told) and promote ‘positive messaging’ for those actions while in contrast to invoke ‘social disapproval’ by the unquestioning, obedient, community of anyone with a mind of their own. Again the compliant Goebbels-like media obliged. This is an old, old, trick employed by tyrannies the world over throughout human history. You get the target population to keep the target population in line – *your* line. SPI-B said this could ‘play an important role in preventing anti-social behaviour or discouraging failure to enact pro-social behaviour’. For ‘anti-social’ in the Orwellian parlance of SPI-B see any behaviour that government doesn’t approve. SPI-B recommendations said that ‘social disapproval’ should be accompanied by clear messaging and

promotion of strong collective identity – hence the government and celebrity mantra of ‘we’re all in this together’. Sure we are. The mind doctors have such contempt for their targets that they think some clueless comedian, actor or singer telling them to do what the government wants will be enough to win them over. We have had UK comedian Lenny Henry, actor Michael Caine and singer Elton John wheeled out to serve the propagandists by urging people to have the DNA-manipulating ‘Covid’ non-‘vaccine’. The role of Henry and fellow black celebrities in seeking to coax a ‘vaccine’ reluctant black community into doing the government’s will was especially stomach-turning. An emotion-manipulating script and carefully edited video featuring these black ‘celebs’ was such an insult to the intelligence of black people and where’s the self-respect of those involved selling their souls to a fascist government agenda? Henry said he heard black people’s ‘legitimate worries and concerns’, but people must ‘trust the facts’ when they were doing exactly that by not having the ‘vaccine’. They had to include the obligatory reference to Black Lives Matter with the line ... ‘Don’t let coronavirus cost even more black lives – because we matter’. My god, it was pathetic. ‘I know the vaccine is safe and what it does.’ How? ‘I’m a comedian and it says so in my script.’

SPI-B said social disapproval needed to be carefully managed to avoid victimisation, scapegoating and misdirected criticism, but they knew that their ‘recommendations’ would lead to exactly that and the media were specifically used to stir-up the divide-and-conquer hostility. Those who conform like good little baa, baas, are praised while those who have seen through the tidal wave of lies are ‘Covidiots’. The awake have been abused by the fast asleep for not conforming to fascism and impositions that the awake know are designed to endanger their health, dehumanise them, and tear asunder the very fabric of human society. We have had the curtain-twitchers and morons reporting neighbours and others to the face-nappied police for breaking ‘Covid rules’ with fascist police delighting in posting links and phone numbers where this could be done. The Cult cannot impose its will without a compliant police

and military or a compliant population willing to play their part in enslaving themselves and their kids. The words of a pastor in Nazi Germany are so appropriate today:

First they came for the socialists and I did not speak out because I was not a socialist.

Then they came for the trade unionists and I did not speak out because I was not a trade unionist.

Then they came for the Jews and I did not speak out because I was not a Jew.

Then they came for me and there was no one left to speak for me.

Those who don't learn from history are destined to repeat it and so many are.

### **'Covid' rules: Rewiring the mind**

With the background laid out to this gigantic national and global web of psychological manipulation we can put 'Covid' rules into a clear and sinister perspective. Forget the claims about protecting health. 'Covid' rules are about dismantling the human mind, breaking the human spirit, destroying self-respect, and then putting Humpty Dumpty together again as a servile, submissive slave. Social isolation through lockdown and distancing have devastating effects on the human psyche as the psychological psychopaths well know and that's the real reason for them. Humans need contact with each other, discourse, closeness and touch, or they eventually, and literally, go crazy. Masks, which I will address at some length, fundamentally add to the effects of isolation and the Cult agenda to dehumanise and de-individualise the population. To do this while knowing – in fact *seeking* – this outcome is the very epitome of evil and psychologists involved in this *are* the epitome of evil. They must like all the rest of the Cult demons and their assets stand trial for crimes against humanity on a scale that defies the imagination. Psychopaths in uniform use isolation to break enemy troops and agents and make them subservient and submissive to tell what they know. The technique is rightly considered a form of torture and

torture is most certainly what has been imposed on the human population.

Clinically-insane American psychologist Harry Harlow became famous for his isolation experiments in the 1950s in which he separated baby monkeys from their mothers and imprisoned them for months on end in a metal container or ‘pit of despair’. They soon began to show mental distress and depression as any idiot could have predicted. Harlow put other monkeys in steel chambers for three, six or twelve months while denying them any contact with animals or humans. He said that the effects of total social isolation for six months were ‘so devastating and debilitating that we had assumed initially that twelve months of isolation would not produce any additional decrement’; but twelve months of isolation ‘almost obliterated the animals socially’. This is what the Cult and its psychopaths are doing to you and your children. Even monkeys in partial isolation in which they were not allowed to form relationships with other monkeys became ‘aggressive and hostile, not only to others, but also towards their own bodies’. We have seen this in the young as a consequence of lockdown. UK government psychopaths launched a public relations campaign telling people not to hug each other even after they received the ‘Covid-19 vaccine’ which we were told with more lies would allow a return to ‘normal life’. A government source told *The Telegraph*: ‘It will be along the lines that it is great that you have been vaccinated, but if you are going to visit your family and hug your grandchildren there is a chance you are going to infect people you love.’ The source was apparently speaking from a secure psychiatric facility. Janet Lord, director of Birmingham University’s Institute of Inflammation and Ageing, said that parents and grandparents should avoid hugging their children. Well, how can I put it, Ms Lord? Fuck off. Yep, that’ll do.

## **Destroying the kids – where are the parents?**

Observe what has happened to people enslaved and isolated by lockdown as suicide and self-harm has soared worldwide,

particularly among the young denied the freedom to associate with their friends. A study of 49,000 people in English-speaking countries concluded that almost half of young adults are at clinical risk of mental health disorders. A national survey in America of 1,000 currently enrolled high school and college students found that 5 percent reported attempting suicide during the pandemic. Data from the US CDC's National Syndromic Surveillance Program from January 1st to October 17th, 2020, revealed a 31 percent increase in mental health issues among adolescents aged 12 to 17 compared with 2019. The CDC reported that America in general suffered the biggest drop in life expectancy since World War Two as it fell by a year in the first half of 2020 as a result of 'deaths of despair' – overdoses and suicides. Deaths of despair have leapt by more than 20 percent during lockdown and include the highest number of fatal overdoses ever recorded in a single year – 81,000. Internet addiction is another consequence of being isolated at home which lowers interest in physical activities as kids fall into inertia and what's the point? Children and young people are losing hope and giving up on life, sometimes literally. A 14-year-old boy killed himself in Maryland because he had 'given up' when his school district didn't reopen; an 11-year-old boy shot himself during a zoom class; a teenager in Maine succumbed to the isolation of the 'pandemic' when he ended his life after experiencing a disrupted senior year at school. Children as young as nine have taken their life and all these stories can be repeated around the world. Careers are being destroyed before they start and that includes those in sport in which promising youngsters have not been able to take part. The plan of the psycho-psychologists is working all right. Researchers at Cambridge University found that lockdowns cause significant harm to children's mental health. Their study was published in the *Archives of Disease in Childhood*, and followed 168 children aged between 7 and 11. The researchers concluded:

During the UK lockdown, children's depression symptoms have increased substantially, relative to before lockdown. The scale of this effect has direct relevance for the continuation of different elements of lockdown policy, such as complete or partial school closures ...

... Specifically, we observed a statistically significant increase in ratings of depression, with a medium-to-large effect size. Our findings emphasise the need to incorporate the potential impact of lockdown on child mental health in planning the ongoing response to the global pandemic and the recovery from it.

Not a chance when the Cult's psycho-psychologists were getting exactly what they wanted. The UK's Royal College of Paediatrics and Child Health has urged parents to look for signs of eating disorders in children and young people after a three to four fold increase. Specialists say the 'pandemic' is a major reason behind the rise. You don't say. The College said isolation from friends during school closures, exam cancellations, loss of extra-curricular activities like sport, and an increased use of social media were all contributory factors along with fears about the virus (psycho-psychologists again), family finances, and students being forced to quarantine. Doctors said young people were becoming severely ill by the time they were seen with 'Covid' regulations reducing face-to-face consultations. Nor is it only the young that have been devastated by the psychopaths. Like all bullies and cowards the Cult is targeting the young, elderly, weak and infirm. A typical story was told by a British lady called Lynn Parker who was not allowed to visit her husband in 2020 for the last ten and half months of his life 'when he needed me most' between March 20th and when he died on December 19th. This vacates the criminal and enters the territory of evil. The emotional impact on the immune system alone is immense as are the number of people of all ages worldwide who have died as a result of Cult-demanded, Gates-demanded, lockdowns.

## **Isolation is torture**

The experience of imposing solitary confinement on millions of prisoners around the world has shown how a large percentage become 'actively psychotic and/or acutely suicidal'. Social isolation has been found to trigger 'a specific psychiatric syndrome, characterized by hallucinations; panic attacks; overt paranoia; diminished impulse control; hypersensitivity to external stimuli; and difficulties with thinking, concentration and memory'. Juan Mendez,

a United Nations rapporteur (investigator), said that isolation is a form of torture. Research has shown that even after isolation prisoners find it far more difficult to make social connections and I remember chatting to a shop assistant after one lockdown who told me that when her young son met another child again he had no idea how to act or what to do. Hannah Flanagan, Director of Emergency Services at Journey Mental Health Center in Dane County, Wisconsin, said: ‘The specificity about Covid social distancing and isolation that we’ve come across as contributing factors to the suicides are really new to us this year.’ But they are not new to those that devised them. They are getting the effect they want as the population is psychologically dismantled to be rebuilt in a totally different way. Children and the young are particularly targeted. They will be the adults when the full-on fascist AI-controlled technocracy is planned to be imposed and they are being prepared to meekly submit. At the same time older people who still have a memory of what life was like before – and how fascist the new normal really is – are being deleted. You are going to see efforts to turn the young against the old to support this geriatric genocide. Hannah Flanagan said the big increase in suicide in her county proved that social isolation is not only harmful, but deadly. Studies have shown that isolation from others is one of the main risk factors in suicide and even more so with women. Warnings that lockdown could create a ‘perfect storm’ for suicide were ignored. After all this was one of the *reasons* for lockdown. Suicide, however, is only the most extreme of isolation consequences. There are many others. Dr Dhruv Khullar, assistant professor of healthcare policy at Weill Cornell Medical College, said in a *New York Times* article in 2016 long before the fake ‘pandemic’:

A wave of new research suggests social separation is bad for us. Individuals with less social connection have disrupted sleep patterns, altered immune systems, more inflammation and higher levels of stress hormones. One recent study found that isolation increases the risk of heart disease by 29 percent and stroke by 32 percent. Another analysis that pooled data from 70 studies and 3.4 million people found that socially isolated individuals had a 30 percent higher risk of dying in the next seven years, and that this effect was largest in middle age.

Loneliness can accelerate cognitive decline in older adults, and isolated individuals are twice as likely to die prematurely as those with more robust social interactions. These effects start early: Socially isolated children have significantly poorer health 20 years later, even after controlling for other factors. All told, loneliness is as important a risk factor for early death as obesity and smoking.

There you have proof from that one article alone four years before 2020 that those who have enforced lockdown, social distancing and isolation knew what the effect would be and that is even more so with professional psychologists that have been driving the policy across the globe. We can go back even further to the years 2000 and 2003 and the start of a major study on the effects of isolation on health by Dr Janine Gronewold and Professor Dirk M. Hermann at the University Hospital in Essen, Germany, who analysed data on 4,316 people with an average age of 59 who were recruited for the long-term research project. They found that socially isolated people are more than 40 percent more likely to have a heart attack, stroke, or other major cardiovascular event and nearly 50 percent more likely to die from any cause. Given the financial Armageddon unleashed by lockdown we should note that the study found a relationship between increased cardiovascular risk and lack of financial support. After excluding other factors social isolation was still connected to a 44 percent increased risk of cardiovascular problems and a 47 percent increased risk of death by any cause. Lack of financial support was associated with a 30 percent increase in the risk of cardiovascular health events. Dr Gronewold said it had been known for some time that feeling lonely or lacking contact with close friends and family can have an impact on physical health and the study had shown that having strong social relationships is of high importance for heart health. Gronewold said they didn't understand yet why people who are socially isolated have such poor health outcomes, but this was obviously a worrying finding, particularly during these times of prolonged social distancing. Well, it can be explained on many levels. You only have to identify the point in the body where people feel loneliness and missing people they are parted from – it's in the centre of the chest where they feel the ache of loneliness and the ache of missing people. 'My heart aches for

you' ... 'My heart aches for some company.' I will explain this more in the chapter Escaping Wetiko, but when you realise that the body is the mind – they are expressions of each other – the reason why state of the mind dictates state of the body becomes clear.

American psychologist Ranjit Powar was highlighting the effects of lockdown isolation as early as April, 2020. She said humans have evolved to be social creatures and are wired to live in interactive groups. Being isolated from family, friends and colleagues could be unbalancing and traumatic for most people and could result in short or even long-term psychological and physical health problems. An increase in levels of anxiety, aggression, depression, forgetfulness and hallucinations were possible psychological effects of isolation. 'Mental conditions may be precipitated for those with underlying pre-existing susceptibilities and show up in many others without any pre-condition.' Powar said personal relationships helped us cope with stress and if we lost this outlet for letting off steam the result can be a big emotional void which, for an average person, was difficult to deal with. 'Just a few days of isolation can cause increased levels of anxiety and depression' – so what the hell has been the effect on the global population of *18 months* of this at the time of writing? Powar said: 'Add to it the looming threat of a dreadful disease being repeatedly hammered in through the media and you have a recipe for many shades of mental and physical distress.' For those with a house and a garden it is easy to forget that billions have had to endure lockdown isolation in tiny overcrowded flats and apartments with nowhere to go outside. The psychological and physical consequences of this are unimaginable and with lunatic and abusive partners and parents the consequences have led to tremendous increases in domestic and child abuse and alcoholism as people seek to shut out the horror. Ranjit Powar said:

Staying in a confined space with family is not all a rosy picture for everyone. It can be extremely oppressive and claustrophobic for large low-income families huddled together in small single-room houses. Children here are not lucky enough to have many board/electronic games or books to keep them occupied.

Add to it the deep insecurity of running out of funds for food and basic necessities. On the other hand, there are people with dysfunctional family dynamics, such as domineering, abusive or alcoholic partners, siblings or parents which makes staying home a period of trial. Incidence of suicide and physical abuse against women has shown a worldwide increase. Heightened anxiety and depression also affect a person's immune system, making them more susceptible to illness.

To think that Powar's article was published on April 11th, 2020.

## **Six-feet fantasy**

Social (unsocial) distancing demanded that people stay six feet or two metres apart. UK government advisor Robert Dingwall from the New and Emerging Respiratory Virus Threats Advisory Group said in a radio interview that the two-metre rule was 'conjured up out of nowhere' and was not based on science. No, it was not based on *medical* science, but it didn't come out of nowhere. The distance related to *psychological* science. Six feet/two metres was adopted in many countries and we were told by people like the criminal Anthony Fauci and his ilk that it was founded on science. Many schools could not reopen because they did not have the space for six-feet distancing. Then in March, 2021, after a year of six-feet 'science', a study published in the *Journal of Infectious Diseases* involving more than 500,000 students and almost 100,000 staff over 16 weeks revealed no significant difference in 'Covid' cases between six feet and three feet and Fauci changed his tune. Now three feet was okay. There is no difference between six feet and three *inches* when there is no 'virus' and they got away with six feet for psychological reasons for as long as they could. I hear journalists and others talk about 'unintended consequences' of lockdown. They are not *unintended* at all; they have been coldly-calculated for a specific outcome of human control and that's why super-psychopaths like Gates have called for them so vehemently. Super-psychopath psychologists have demanded them and psychopathic or clueless, spineless, politicians have gone along with them by 'following the science'. But it's not science at all. 'Science' is not what is; it's only what people can be manipulated to believe it is. The whole 'Covid' catastrophe is

founded on mind control. Three word or three statement mantras issued by the UK government are a well-known mind control technique and so we've had 'Stay home/protect the NHS/save lives', 'Stay alert/control the virus/save lives' and 'hands/face/space'. One of the most vocal proponents of extreme 'Covid' rules in the UK has been Professor Susan Michie, a member of the British Communist Party, who is not a medical professional. Michie is the director of the Centre for Behaviour Change at University College London. She is a *behavioural psychologist* and another filthy rich 'Marxist' who praised China's draconian lockdown. She was known by fellow students at Oxford University as 'Stalin's nanny' for her extreme Marxism. Michie is an influential member of the UK government's Scientific Advisory Group for Emergencies (SAGE) and behavioural manipulation groups which have dominated 'Covid' policy. She is a consultant adviser to the World Health Organization on 'Covid-19' and behaviour. Why the hell are lockdowns anything to do with her when they are claimed to be about health? Why does a behavioural psychologist from a group charged with changing the behaviour of the public want lockdown, human isolation and mandatory masks? Does that question really need an answer? Michie *absolutely* has to explain herself before a Nuremberg court when humanity takes back its world again and even more so when you see the consequences of masks that she demands are compulsory. This is a Michie classic:

The benefits of getting primary school children to wear masks is that regardless of what little degree of transmission is occurring in those age groups it could help normalise the practice. Young children wearing masks may be more likely to get their families to accept masks.

Those words alone should carry a prison sentence when you ponder on the callous disregard for children involved and what a statement it makes about the mind and motivations of Susan Michie. What a lovely lady and what she said there encapsulates the mentality of the psychopaths behind the 'Covid' horror. Let us compare what Michie said with a countrywide study in Germany published at [researchsquare.com](https://www.researchsquare.com) involving 25,000 school children and 17,854 health complaints submitted by parents. Researchers

found that masks are harming children physically, psychologically, and behaviourally with 24 health issues associated with mask wearing. They include: shortness of breath (29.7%); dizziness (26.4%); increased headaches (53%); difficulty concentrating (50%); drowsiness or fatigue (37%); and malaise (42%). Nearly a third of children experienced more sleep issues than before and a quarter developed new fears. Researchers found health issues and other impairments in 68 percent of masked children covering their faces for an average of 4.5 hours a day. Hundreds of those taking part experienced accelerated respiration, tightness in the chest, weakness, and short-term impairment of consciousness. A reminder of what Michie said again:

The benefits of getting primary school children to wear masks is that regardless of what little degree of transmission is occurring in those age groups it could help normalise the practice. Young children wearing masks may be more likely to get their families to accept masks.

Psychopaths in government and psychology now have children and young people – plus all the adults – wearing masks for hours on end while clueless teachers impose the will of the psychopaths on the young they should be protecting. What the hell are parents doing?

## **Cult lab rats**

We have some schools already imposing on students microchipped buzzers that activate when they get ‘too close’ to their pals in the way they do with lab rats. How apt. To the Cult and its brain-dead servants our children *are* lab rats being conditioned to be unquestioning, dehumanised slaves for the rest of their lives. Children and young people are being weaned and frightened away from the most natural human instincts including closeness and touch. I have tracked in the books over the years how schools were banning pupils from greeting each other with a hug and the whole Cult-induced Me Too movement has terrified men and boys from a relaxed and natural interaction with female friends and work colleagues to the point where many men try never to be in a room

alone with a woman that's not their partner. Airhead celebrities have as always played their virtue-signalling part in making this happen with their gross exaggeration. For every monster like Harvey Weinstein there are at least tens of thousands of men that don't treat women like that; but everyone must be branded the same and policy changed for them as well as the monster. I am going to be using the word 'dehumanise' many times in this chapter because that is what the Cult is seeking to do and it goes very deep as we shall see. Don't let them kid you that social distancing is planned to end one day. That's not the idea. We are seeing more governments and companies funding and producing wearable gadgets to keep people apart and they would not be doing that if this was meant to be short-term. A tech start-up company backed by GCHQ, the British Intelligence and military surveillance headquarters, has created a social distancing wrist sensor that alerts people when they get too close to others. The CIA has also supported tech companies developing similar devices. The wearable sensor was developed by Tended, one of a number of start-up companies supported by GCHQ (see the CIA and DARPA). The device can be worn on the wrist or as a tag on the waistband and will vibrate whenever someone wearing the device breaches social distancing and gets anywhere near natural human contact. The company had a lucky break in that it was developing a distancing sensor when the 'Covid' hoax arrived which immediately provided a potentially enormous market. How fortunate. The government in big-time Cult-controlled Ontario in Canada is investing \$2.5 million in wearable contact tracing technology that 'will alert users if they may have been exposed to the Covid-19 in the workplace and will beep or vibrate if they are within six feet of another person'. Facedrive Inc., the technology company behind this, was founded in 2016 with funding from the Ontario Together Fund and obviously they, too, had a prophet on the board of directors. The human surveillance and control technology is called TraceSCAN and would be worn by the human cyborgs in places such as airports, workplaces, construction sites, care homes and ... schools.

I emphasise schools with children and young people the prime targets. You know what is planned for society as a whole if you keep your eyes on the schools. They have always been places where the state program the next generation of slaves to be its compliant worker-ants – or Woker-ants these days; but in the mist of the ‘Covid’ madness they have been transformed into mind laboratories on a scale never seen before. Teachers and head teachers are just as programmed as the kids – often more so. Children are kept apart from human interaction by walk lanes, classroom distancing, staggered meal times, masks, and the rolling-out of buzzer systems. Schools are now physically laid out as a laboratory maze for lab-rats. Lunatics at a school in Anchorage, Alaska, who should be prosecuted for child abuse, took away desks and forced children to kneel (know your place) on a mat for five hours a day while wearing a mask and using their chairs as a desk. How this was supposed to impact on a ‘virus’ only these clinically insane people can tell you and even then it would be clap-trap. The school banned recess (interaction), art classes (creativity), and physical exercise (getting body and mind moving out of inertia). Everyone behind this outrage should be in jail or better still a mental institution. The behavioural manipulators are all for this dystopian approach to schools.

Professor Susan Michie, the mind-doctor and British Communist Party member, said it was wrong to say that schools were safe. They had to be made so by ‘distancing’, masks and ventilation (sitting all day in the cold). I must ask this lady round for dinner on a night I know I am going to be out and not back for weeks. She probably wouldn’t be able to make it, anyway, with all the visits to her own psychologist she must have block-booked.

## **Masking identity**

I know how shocking it must be for you that a behaviour manipulator like Michie wants everyone to wear masks which have long been a feature of mind-control programs like the infamous MKUltra in the United States, but, there we are. We live and learn. I spent many years from 1996 to right across the millennium

researching mind control in detail on both sides of the Atlantic and elsewhere. I met a large number of mind-control survivors and many had been held captive in body and mind by MKUltra. MK stands for mind-control, but employs the German spelling in deference to the Nazis spirited out of Germany at the end of World War Two by Operation Paperclip in which the US authorities, with help from the Vatican, transported Nazi mind-controllers and engineers to America to continue their work. Many of them were behind the creation of NASA and they included Nazi scientist and SS officer Wernher von Braun who swapped designing V-2 rockets to bombard London with designing the Saturn V rockets that powered the NASA moon programme's Apollo craft. I think I may have mentioned that the Cult has no borders. Among Paperclip escapees was Josef Mengele, the Angel of Death in the Nazi concentration camps where he conducted mind and genetic experiments on children often using twins to provide a control twin to measure the impact of his 'work' on the other. If you want to observe the Cult mentality in all its extremes of evil then look into the life of Mengele. I have met many people who suffered mercilessly under Mengele in the United States where he operated under the name Dr Greene and became a stalwart of MKUltra programming and torture. Among his locations was the underground facility in the Mojave Desert in California called the China Lake Naval Weapons Station which is almost entirely below the surface. My books *The Biggest Secret*, *Children of the Matrix* and *The Perception Deception* have the detailed background to MKUltra.

The best-known MKUltra survivor is American Cathy O'Brien. I first met her and her late partner Mark Phillips at a conference in Colorado in 1996. Mark helped her escape and deprogram from decades of captivity in an offshoot of MKUltra known as Project Monarch in which 'sex slaves' were provided for the rich and famous including Father George Bush, Dick Cheney and the Clintons. Read Cathy and Mark's book *Trance-Formation of America* and if you are new to this you will be shocked to the core. I read it in 1996 shortly before, with the usual synchronicity of my life, I found

myself given a book table at the conference right next to hers. MKUltra never ended despite being very publicly exposed (only a small part of it) in the 1970s and continues in other guises. I am still in touch with Cathy. She contacted me during 2020 after masks became compulsory in many countries to tell me how they were used as part of MKUltra programming. I had been observing 'Covid regulations' and the relationship between authority and public for months. I saw techniques that I knew were employed on individuals in MKUltra being used on the global population. I had read many books and manuals on mind control including one called *Silent Weapons for Quiet Wars* which came to light in the 1980s and was a guide on how to perceptually program on a mass scale. 'Silent Weapons' refers to mind-control. I remembered a line from the manual as governments, medical authorities and law enforcement agencies have so obviously talked to – or rather at – the adult population since the 'Covid' hoax began as if they are children. The document said:

If a person is spoken to by a T.V. advertiser as if he were a twelve-year-old, then, due to suggestibility, he will, with a certain probability, respond or react to that suggestion with the uncritical response of a twelve-year-old and will reach in to his economic reservoir and deliver its energy to buy that product on impulse when he passes it in the store.

That's why authority has spoken to adults like children since all this began.

### **Why did Michael Jackson wear masks?**

Every aspect of the 'Covid' narrative has mind-control as its central theme. Cathy O'Brien wrote an article for [davidicke.com](http://davidicke.com) about the connection between masks and mind control. Her daughter Kelly who I first met in the 1990s was born while Cathy was still held captive in MKUltra. Kelly was forced to wear a mask as part of her programming from the age of *two* to dehumanise her, target her sense of individuality and reduce the amount of oxygen her brain and body received. *Bingo*. This is the real reason for compulsory

masks, why they have been enforced en masse, and why they seek to increase the number they demand you wear. First one, then two, with one disgraceful alleged ‘doctor’ recommending four which is nothing less than a death sentence. Where and how often they must be worn is being expanded for the purpose of mass mind control and damaging respiratory health which they can call ‘Covid-19’. Canada’s government headed by the man-child Justin Trudeau, says it’s fine for children of two and older to wear masks. An insane ‘study’ in Italy involving just 47 children concluded there was no problem for babies as young as *four months* wearing them. Even after people were ‘vaccinated’ they were still told to wear masks by the criminal that is Anthony Fauci. Cathy wrote that mandating masks is allowing the authorities literally to control the air we breathe which is what was done in MKUltra. You might recall how the singer Michael Jackson wore masks and there is a reason for that. He was subjected to MKUltra mind control through Project Monarch and his psyche was scrambled by these simpletons. Cathy wrote:

In MKUltra Project Monarch mind control, Michael Jackson had to wear a mask to silence his voice so he could not reach out for help. Remember how he developed that whisper voice when he wasn’t singing? Masks control the mind from the outside in, like the redefining of words is doing. By controlling what we can and cannot say for fear of being labeled racist or beaten, for example, it ultimately controls thought that drives our words and ultimately actions (or lack thereof).

Likewise, a mask muffles our speech so that we are not heard, which controls voice ... words ... mind. This is Mind Control. Masks are an obvious mind control device, and I am disturbed so many people are complying on a global scale. Masks depersonalize while making a person feel as though they have no voice. It is a barrier to others. People who would never choose to comply but are forced to wear a mask in order to keep their job, and ultimately their family fed, are compromised. They often feel shame and are subdued. People have stopped talking with each other while media controls the narrative.

The ‘no voice’ theme has often become literal with train passengers told not to speak to each other in case they pass on the ‘virus’, singing banned for the same reason and bonkers California officials telling people riding roller coasters that they cannot shout and scream. Cathy said she heard every day from healed MKUltra survivors who cannot wear a mask without flashing back on ways

their breathing was controlled – ‘from ball gags and penises to water boarding’. She said that through the years when she saw images of people in China wearing masks ‘due to pollution’ that it was really to control their oxygen levels. ‘I knew it was as much of a population control mechanism of depersonalisation as are burkas’, she said. Masks are another Chinese communist/fascist method of control that has been swept across the West as the West becomes China at lightning speed since we entered 2020.

## **Mask-19**

There are other reasons for mandatory masks and these include destroying respiratory health to call it ‘Covid-19’ and stunting brain development of children and the young. Dr Margarite Griesz-Brisson MD, PhD, is a Consultant Neurologist and Neurophysiologist and the Founder and Medical Director of the London Neurology and Pain Clinic. Her CV goes down the street and round the corner. She is clearly someone who cares about people and won’t parrot the propaganda. Griesz-Brisson has a PhD in pharmacology, with special interest in neurotoxicology, environmental medicine, neuroregeneration and neuroplasticity (the way the brain can change in the light of information received). She went public in October, 2020, with a passionate warning about the effects of mask-wearing laws:

The reinhalation of our exhaled air will without a doubt create oxygen deficiency and a flooding of carbon dioxide. We know that the human brain is very sensitive to oxygen deprivation. There are nerve cells for example in the hippocampus that can’t be longer than 3 minutes without oxygen – they cannot survive. The acute warning symptoms are headaches, drowsiness, dizziness, issues in concentration, slowing down of reaction time – reactions of the cognitive system.

Oh, I know, let’s tell bus, truck and taxi drivers to wear them and people working machinery. How about pilots, doctors and police? Griesz-Brisson makes the important point that while the symptoms she mentions may fade as the body readjusts this does not alter the fact that people continue to operate in oxygen deficit with long list of

potential consequences. She said it was well known that neurodegenerative diseases take years or decades to develop. 'If today you forget your phone number, the breakdown in your brain would have already started 20 or 30 years ago.' She said degenerative processes in your brain are getting amplified as your oxygen deprivation continues through wearing a mask. Nerve cells in the brain are unable to divide themselves normally in these circumstances and lost nerve cells will no longer be regenerated. 'What is gone is gone.' Now consider that people like shop workers and *schoolchildren* are wearing masks for hours every day. What in the name of sanity is going to be happening to them? 'I do not wear a mask, I need my brain to think', Griesz-Brisson said, 'I want to have a clear head when I deal with my patients and not be in a carbon dioxide-induced anaesthesia'. If you are told to wear a mask anywhere ask the organisation, police, store, whatever, for their risk assessment on the dangers and negative effects on mind and body of enforcing mask-wearing. They won't have one because it has never been done not even by government. All of them must be subject to class-action lawsuits as the consequences come to light. They don't do mask risk assessments for an obvious reason. They know what the conclusions would be and independent scientific studies that *have* been done tell a horror story of consequences.

### **'Masks are criminal'**

Dr Griesz-Brisson said that for children and adolescents, masks are an absolute no-no. They had an extremely active and adaptive immune system and their brain was incredibly active with so much to learn. 'The child's brain, or the youth's brain, is thirsting for oxygen.' The more metabolically active an organ was, the more oxygen it required; and in children and adolescents every organ was metabolically active. Griesz-Brisson said that to deprive a child's or adolescent's brain of oxygen, or to restrict it in any way, was not only dangerous to their health, it was absolutely criminal. 'Oxygen deficiency inhibits the development of the brain, and the damage that has taken place as a result CANNOT be reversed.' Mind

manipulators of MKUltra put masks on two-year-olds they wanted to neurologically rewire and you can see why. Griesz-Brisson said a child needs the brain to learn and the brain needs oxygen to function. 'We don't need a clinical study for that. This is simple, indisputable physiology.' Consciously and purposely induced oxygen deficiency was an absolutely deliberate health hazard, and an absolute medical contraindication which means that 'this drug, this therapy, this method or measure should not be used, and is not allowed to be used'. To coerce an entire population to use an absolute medical contraindication by force, she said, there had to be definite and serious reasons and the reasons must be presented to competent interdisciplinary and independent bodies to be verified and authorised. She had this warning of the consequences that were coming if mask wearing continued:

When, in ten years, dementia is going to increase exponentially, and the younger generations couldn't reach their god-given potential, it won't help to say 'we didn't need the masks'. I know how damaging oxygen deprivation is for the brain, cardiologists know how damaging it is for the heart, pulmonologists know how damaging it is for the lungs. Oxygen deprivation damages every single organ. Where are our health departments, our health insurance, our medical associations? It would have been their duty to be vehemently against the lockdown and to stop it and stop it from the very beginning.

Why do the medical boards issue punishments to doctors who give people exemptions? Does the person or the doctor seriously have to prove that oxygen deprivation harms people? What kind of medicine are our doctors and medical associations representing? Who is responsible for this crime? The ones who want to enforce it? The ones who let it happen and play along, or the ones who don't prevent it?

All of the organisations and people she mentions there either answer directly to the Cult or do whatever hierarchical levels above them tell them to do. The outcome of both is the same. 'It's not about masks, it's not about viruses, it's certainly not about your health', Griesz-Brisson said. 'It is about much, much more. I am not participating. I am not afraid.' They were taking our air to breathe and there was no unfounded medical exemption from face masks. Oxygen deprivation was dangerous for every single brain. It had to be the free decision of every human being whether they want to

wear a mask that was absolutely ineffective to protect themselves from a virus. She ended by rightly identifying where the responsibility lies for all this:

The imperative of the hour is personal responsibility. We are responsible for what we think, not the media. We are responsible for what we do, not our superiors. We are responsible for our health, not the World Health Organization. And we are responsible for what happens in our country, not the government.

Halle-bloody-lujah.

## **But surgeons wear masks, right?**

Independent studies of mask-wearing have produced a long list of reports detailing mental, emotional and physical dangers. What a definition of insanity to see police officers imposing mask-wearing on the public which will cumulatively damage their health while the police themselves wear masks that will cumulatively damage *their* health. It's utter madness and both public and police do this because 'the government says so' – yes a government of brain-donor idiots like UK Health Secretary Matt Hancock reading the 'follow the science' scripts of psychopathic, lunatic psychologists. The response you get from Stockholm syndrome sufferers defending the very authorities that are destroying them and their families is that 'surgeons wear masks'. This is considered the game, set and match that they must work and don't cause oxygen deficit. Well, actually, scientific studies have shown that they *do* and oxygen levels are monitored in operating theatres to compensate. Surgeons wear masks to stop spittle and such like dropping into open wounds – not to stop 'viral particles' which are so minuscule they can only be seen through an electron microscope. Holes in the masks are significantly bigger than 'viral particles' and if you sneeze or cough they will breach the mask. I watched an incredibly disingenuous 'experiment' that claimed to prove that masks work in catching 'virus' material from the mouth and nose. They did this with a slow motion camera and the mask did block big stuff which stayed inside the mask and

against the face to be breathed in or cause infections on the face as we have seen with many children. ‘Viral particles’, however, would never have been picked up by the camera as they came through the mask when they are far too small to be seen. The ‘experiment’ was therefore disingenuous *and* useless.

Studies have concluded that wearing masks in operating theatres (and thus elsewhere) make no difference to preventing infection while the opposite is true with toxic shite building up in the mask and this had led to an explosion in tooth decay and gum disease dubbed by dentists ‘mask mouth’. You might have seen the Internet video of a furious American doctor urging people to take off their masks after a four-year-old patient had been rushed to hospital the night before and nearly died with a lung infection that doctors sourced to mask wearing. A study in the journal *Cancer Discovery* found that inhalation of harmful microbes can contribute to advanced stage lung cancer in adults and long-term use of masks can help breed dangerous pathogens. Microbiologists have said frequent mask wearing creates a moist environment in which microbes can grow and proliferate before entering the lungs. The Canadian Agency for Drugs and Technologies in Health, or CADTH, a Canadian national organisation that provides research and analysis to healthcare decision-makers, said this as long ago as 2013 in a report entitled ‘Use of Surgical Masks in the Operating Room: A Review of the Clinical Effectiveness and Guidelines’. It said:

- No evidence was found to support the use of surgical face masks to reduce the frequency of surgical site infections
- No evidence was found on the effectiveness of wearing surgical face masks to protect staff from infectious material in the operating room.
- Guidelines recommend the use of surgical face masks by staff in the operating room to protect both operating room staff and patients (despite the lack of evidence).

We were told that the world could go back to ‘normal’ with the arrival of the ‘vaccines’. When they came, fraudulent as they are, the story changed as I knew that it would. We are in the midst of transforming ‘normal’, not going back to it. Mary Ramsay, head of immunisation at Public Health England, echoed the words of US criminal Anthony Fauci who said masks and other regulations must stay no matter if people are vaccinated. The Fauci idiot continued to wear two masks – different colours so both could be clearly seen – after he *claimed* to have been vaccinated. Senator Rand Paul told Fauci in one exchange that his double-masks were ‘theatre’ and he was right. It’s all theatre. Mary Ramsay back-tracked on the vaccine-return-to-normal theme when she said the public may need to wear masks and social-distance for years despite the jabs. ‘People have got used to those lower-level restrictions now, and [they] can live with them’, she said telling us what the idea has been all along. ‘The vaccine does not give you a pass, even if you have had it, you must continue to follow all the guidelines’ said a Public Health England statement which reneged on what we had been told before and made having the ‘vaccine’ irrelevant to ‘normality’ even by the official story. Spain’s fascist government trumped everyone by passing a law mandating the wearing of masks on the beach and even when swimming in the sea. The move would have devastated what’s left of the Spanish tourist industry, posed potential breathing dangers to swimmers and had Northern European sunbathers walking around with their forehead brown and the rest of their face white as a sheet. The ruling was so crazy that it had to be retracted after pressure from public and tourist industry, but it confirmed where the Cult wants to go with masks and how clinically insane authority has become. The determination to make masks permanent and hide the serious dangers to body and mind can be seen in the censorship of scientist Professor Denis Rancourt by Bill Gates-funded academic publishing website ResearchGate over his papers exposing the dangers and uselessness of masks. Rancourt said:

ResearchGate today has permanently locked my account, which I have had since 2015. Their reasons graphically show the nature of their attack against democracy, and their corruption of

science ... By their obscene non-logic, a scientific review of science articles reporting on harms caused by face masks has a 'potential to cause harm'. No criticism of the psychological device (face masks) is tolerated, if the said criticism shows potential to influence public policy.

This is what happens in a fascist world.

## **Where are the 'greens' (again)?**

Other dangers of wearing masks especially regularly relate to the inhalation of minute plastic fibres into the lungs and the deluge of discarded masks in the environment and oceans. Estimates predicted that more than 1.5 billion disposable masks will end up in the world's oceans every year polluting the water with tons of plastic and endangering marine wildlife. Studies project that humans are using 129 billion face masks each month worldwide – about three million a minute. Most are disposable and made from plastic, non-biodegradable microfibers that break down into smaller plastic particles that become widespread in ecosystems. They are littering cities, clogging sewage channels and turning up in bodies of water. I have written in other books about the immense amounts of microplastics from endless sources now being absorbed into the body. Rolf Halden, director of the Arizona State University (ASU) Biodesign Center for Environmental Health Engineering, was the senior researcher in a 2020 study that analysed 47 human tissue samples and found microplastics in all of them. 'We have detected these chemicals of plastics in every single organ that we have investigated', he said. I wrote in *The Answer* about the world being deluged with microplastics. A study by the Worldwide Fund for Nature (WWF) found that people are consuming on average every week some 2,000 tiny pieces of plastic mostly through water and also through marine life and the air. Every year humans are ingesting enough microplastics to fill a heaped dinner plate and in a life-time of 79 years it is enough to fill two large waste bins. Marco Lambertini, WWF International director general said: 'Not only are plastics polluting our oceans and waterways and killing marine life – it's in all of us and we can't escape consuming plastics,' American

geologists found tiny plastic fibres, beads and shards in rainwater samples collected from the remote slopes of the Rocky Mountain National Park near Denver, Colorado. Their report was headed: 'It is raining plastic.' Rachel Adams, senior lecturer in Biomedical Science at Cardiff Metropolitan University, said that among health consequences are internal inflammation and immune responses to a 'foreign body'. She further pointed out that microplastics become carriers of toxins including mercury, pesticides and dioxins (a known cause of cancer and reproductive and developmental problems). These toxins accumulate in the fatty tissues once they enter the body through microplastics. Now this is being compounded massively by people putting plastic on their face and throwing it away.

Workers exposed to polypropylene plastic fibres known as 'flock' have developed 'flock worker's lung' from inhaling small pieces of the flock fibres which can damage lung tissue, reduce breathing capacity and exacerbate other respiratory problems. Now ... commonly used surgical masks have three layers of melt-blown textiles made of ... polypropylene. We have billions of people putting these microplastics against their mouth, nose and face for hours at a time day after day in the form of masks. How does anyone think that will work out? I mean – what could possibly go wrong? We posted a number of scientific studies on this at [davidicke.com](http://davidicke.com), but when I went back to them as I was writing this book the links to the science research website where they were hosted were dead. Anything that challenges the official narrative in any way is either censored or vilified. The official narrative is so unsupportable by the evidence that only deleting the truth can protect it. A study by Chinese scientists still survived – with the usual twist which it why it was still active, I guess. Yes, they found that virtually all the masks they tested increased the daily intake of microplastic fibres, but people should still wear them because the danger from the 'virus' was worse said the crazy 'team' from the Institute of Hydrobiology in Wuhan. Scientists first discovered microplastics in lung tissue of some patients who died of lung cancer

in the 1990s. Subsequent studies have confirmed the potential health damage with the plastic degrading slowly and remaining in the lungs to accumulate in volume. Wuhan researchers used a machine simulating human breathing to establish that masks shed up to nearly 4,000 microplastic fibres in a month with reused masks producing more. Scientists said some masks are laced with toxic chemicals and a variety of compounds seriously restricted for both health and environmental reasons. They include cobalt (used in blue dye) and formaldehyde known to cause watery eyes, burning sensations in the eyes, nose, and throat, plus coughing, wheezing and nausea. No – that must be 'Covid-19'.

### **Mask 'worms'**

There is another and potentially even more sinister content of masks. Mostly new masks of different makes filmed under a microscope around the world have been found to contain strange black fibres or 'worms' that appear to move or 'crawl' by themselves and react to heat and water. The nearest I have seen to them are the self-replicating fibres that are pulled out through the skin of those suffering from Morgellons disease which has been connected to the phenomena of 'chemtrails' which I will bring into the story later on. Morgellons fibres continue to grow outside the body and have a form of artificial intelligence. Black 'worm' fibres in masks have that kind of feel to them and there is a nanotechnology technique called 'worm micelles' which carry and release drugs or anything else you want to deliver to the body. For sure the suppression of humanity by mind altering drugs is the Cult agenda big time and the more excuses they can find to gain access to the body the more opportunities there are to make that happen whether through 'vaccines' or masks pushed against the mouth and nose for hours on end.

So let us summarise the pros and cons of masks:

**Against masks:** Breathing in your own carbon dioxide; depriving the body and brain of sufficient oxygen; build-up of toxins in the mask that can be breathed into the lungs and cause rashes on the face and ‘mask-mouth’; breathing microplastic fibres and toxic chemicals into the lungs; dehumanisation and deleting individualisation by literally making people faceless; destroying human emotional interaction through facial expression and deleting parental connection with their babies which look for guidance to their facial expression.

**For masks:** They don’t protect you from a ‘virus’ that doesn’t exist and even if it did ‘viral’ particles are so minute they are smaller than the holes in the mask.

Governments, police, supermarkets, businesses, transport companies, and all the rest who seek to impose masks have done no risk assessment on their consequences for health and psychology and are now open to group lawsuits when the impact becomes clear with a cumulative epidemic of respiratory and other disease. Authorities will try to exploit these effects and hide the real cause by dubbing them ‘Covid-19’. Can you imagine setting out to force the population to wear health-destroying masks without doing any assessment of the risks? It is criminal and it is evil, but then how many people targeted in this way, who see their children told to wear them all day at school, have asked for a risk assessment? Billions can’t be imposed upon by the few unless the billions allow it. Oh, yes, with just a tinge of irony, 85 percent of all masks made worldwide come from *China*.

## **Wash your hands in toxic shite**

‘Covid’ rules include the use of toxic sanitisers and again the health consequences of constantly applying toxins to be absorbed through the skin is obvious to any level of Renegade Mind. America’s Food and Drug Administration (FDA) said that sanitisers are drugs and issued a warning about 75 dangerous brands which contain

methanol used in antifreeze and can cause death, kidney damage and blindness. The FDA circulated the following warning even for those brands that it claims to be safe:

Store hand sanitizer out of the reach of pets and children, and children should use it only with adult supervision. Do not drink hand sanitizer. This is particularly important for young children, especially toddlers, who may be attracted by the pleasant smell or brightly colored bottles of hand sanitizer.

Drinking even a small amount of hand sanitizer can cause alcohol poisoning in children. (However, there is no need to be concerned if your children eat with or lick their hands after using hand sanitizer.) During this coronavirus pandemic, poison control centers have had an increase in calls about accidental ingestion of hand sanitizer, so it is important that adults monitor young children's use.

Do not allow pets to swallow hand sanitizer. If you think your pet has eaten something potentially dangerous, call your veterinarian or a pet poison control center right away. Hand sanitizer is flammable and should be stored away from heat and flames. When using hand sanitizer, rub your hands until they feel completely dry before performing activities that may involve heat, sparks, static electricity, or open flames.

There you go, perfectly safe, then, and that's without even a mention of the toxins absorbed through the skin. Come on kids – sanitise your hands everywhere you go. It will save you from the 'virus'. Put all these elements together of the 'Covid' normal and see how much health and psychology is being cumulatively damaged, even devastated, to 'protect your health'. Makes sense, right? They are only imposing these things because they care, right? *Right?*

## **Submitting to insanity**

Psychological reframing of the population goes very deep and is done in many less obvious ways. I hear people say how contradictory and crazy 'Covid' rules are and how they are ever changing. This is explained away by dismissing those involved as idiots. It is a big mistake. The Cult is delighted if its cold calculation is perceived as incompetence and idiocy when it is anything but. Oh, yes, there are idiots within the system – lots of them – but they are *administering* the Cult agenda, mostly unknowingly. They are not deciding and dictating it. The bulwark against tyranny is self-

respect, always has been, always will be. It is self-respect that has broken every tyranny in history. By its very nature self-respect will not bow to oppression and its perpetrators. There is so little self-respect that it's always the few that overturn dictators. Many may eventually follow, but the few with the iron spines (self-respect) kick it off and generate the momentum. The Cult targets self-respect in the knowledge that once this has gone only submission remains. Crazy, contradictory, ever-changing 'Covid' rules are systematically applied by psychologists to delete self-respect. They *want* you to see that the rules make no sense. It is one thing to decide to do something when *you* have made the choice based on evidence and logic. You still retain your self-respect. It is quite another when you can see what you are being told to do is insane, ridiculous and makes no sense, and *yet you still do it*. Your self-respect is extinguished and this has been happening as ever more obviously stupid and nonsensical things have been demanded and the great majority have complied even when they can see they are stupid and nonsensical.

People walk around in face-nappies knowing they are damaging their health and make no difference to a 'virus'. They do it in fear of not doing it. I know it's daft, but I'll do it anyway. When that happens something dies inside of you and submissive reframing has begun. Next there's a need to hide from yourself that you have conceded your self-respect and you convince yourself that you have not really submitted to fear and intimidation. You begin to believe that you are complying with craziness because it's the right thing to do. When first you concede your self-respect of  $2+2 = 4$  to  $2+2 = 5$  you *know* you are compromising your self-respect. Gradually to avoid facing that fact you begin to *believe* that  $2+2=5$ . You have been reframed and I have been watching this process happening in the human psyche on an industrial scale. The Cult is working to break your spirit and one of its major tools in that war is humiliation. I read how former American soldier Bradley Manning (later Chelsea Manning after a sex-change) was treated after being jailed for supplying WikiLeaks with documents exposing the enormity of

government and elite mendacity. Manning was isolated in solitary confinement for eight months, put under 24-hour surveillance, forced to hand over clothing before going to bed, and stand naked for every roll call. This is systematic humiliation. The introduction of anal swab 'Covid' tests in China has been done for the same reason to delete self-respect and induce compliant submission. Anal swabs are mandatory for incoming passengers in parts of China and American diplomats have said they were forced to undergo the indignity which would have been calculated humiliation by the Cult-owned Chinese government that has America in its sights.

### **Government-people: An abusive relationship**

Spirit-breaking psychological techniques include giving people hope and apparent respite from tyranny only to take it away again. This happened in the UK during Christmas, 2020, when the psycho-psychologists and their political lackeys announced an easing of restrictions over the holiday only to reimpose them almost immediately on the basis of yet another lie. There is a big psychological difference between getting used to oppression and being given hope of relief only to have that dashed. Psychologists know this and we have seen the technique used repeatedly. Then there is traumatising people before you introduce more extreme regulations that require compliance. A perfect case was the announcement by the dark and sinister Whitty and Vallance in the UK that 'new data' predicted that 4,000 could die every day over the winter of 2020/2021 if we did not lockdown again. I think they call it lying and after traumatising people with that claim out came Jackboot Johnson the next day with new curbs on human freedom. Psychologists know that a frightened and traumatised mind becomes suggestable to submission and behaviour reframing. Underpinning all this has been to make people fearful and suspicious of each other and see themselves as a potential danger to others. In league with deleted self-respect you have the perfect psychological recipe for self-loathing. The relationship between authority and public is now demonstrably the same as that of

subservience to an abusive partner. These are signs of an abusive relationship explained by psychologist Leslie Becker-Phelps:

**Psychological and emotional abuse:** Undermining a partner's self-worth with verbal attacks, name-calling, and belittling. Humiliating the partner in public, unjustly accusing them of having an affair, or interrogating them about their every behavior. Keeping partner confused or off balance by saying they were just kidding or blaming the partner for 'making' them act this way ... Feigning in public that they care while turning against them in private. This leads to victims frequently feeling confused, incompetent, unworthy, hopeless, and chronically self-doubting. [Apply these techniques to how governments have treated the population since New Year, 2020, and the parallels are obvious.]

**Physical abuse:** The abuser might physically harm their partner in a range of ways, such as grabbing, hitting, punching, or shoving them. They might throw objects at them or harm them with a weapon. [Observe the physical harm imposed by masks, lockdown, and so on.]

**Threats and intimidation:** One way abusers keep their partners in line is by instilling fear. They might be verbally threatening, or give threatening looks or gestures. Abusers often make it known that they are tracking their partner's every move. They might destroy their partner's possessions, threaten to harm them, or threaten to harm their family members. Not surprisingly, victims of this abuse often feel anxiety, fear, and panic. [No words necessary.]

**Isolation:** Abusers often limit their partner's activities, forbidding them to talk or interact with friends or family. They might limit access to a car or even turn off their phone. All of this might be done by physically holding them against their will, but is often accomplished through psychological abuse and intimidation. The more isolated a person feels, the fewer resources they have to help gain perspective on their situation and to escape from it. [No words necessary.]

**Economic abuse:** Abusers often make their partners beholden to them for money by controlling access to funds of any kind. They might prevent their partner from getting a job or withhold access to money they earn from a job. This creates financial dependency that makes leaving the relationship very difficult. [See destruction of livelihoods and the proposed meagre 'guaranteed income' so long as you do whatever you are told.]

**Using children:** An abuser might disparage their partner's parenting skills, tell their children lies about their partner, threaten to take custody of their children, or threaten to harm their children. These tactics instil fear and often elicit compliance. [See reframed social service mafia and how children are being mercilessly abused by the state over 'Covid' while their parents look on too frightened to do anything.]

A further recurring trait in an abusive relationship is the abused blaming themselves for their abuse and making excuses for the abuser. We have the public blaming each other for lockdown abuse by government and many making excuses for the government while attacking those who challenge the government. How often we have heard authorities say that rules are being imposed or reimposed only because people have refused to 'behave' and follow the rules. We don't want to do it – it's *you*.

Renegade Minds are an antidote to all of these things. They will never concede their self-respect no matter what the circumstances. Even when apparent humiliation is heaped upon them they laugh in its face and reflect back the humiliation on the abuser where it belongs. Renegade Minds will never wear masks they know are only imposed to humiliate, suppress and damage both physically and psychologically. Consequences will take care of themselves and they will never break their spirit or cause them to concede to tyranny. UK newspaper columnist Peter Hitchens was one of the few in the mainstream media to speak out against lockdowns and forced vaccinations. He then announced he had taken the jab. He wanted to see family members abroad and he believed vaccine passports were inevitable even though they had not yet been introduced. Hitchens

has a questioning and critical mind, but not a Renegade one. If he had no amount of pressure would have made him concede. Hitchens excused his action by saying that the battle has been lost. Renegade Minds never accept defeat when freedom is at stake and even if they are the last one standing the self-respect of not submitting to tyranny is more important than any outcome or any consequence.

That's why Renegade Minds are the only minds that ever changed anything worth changing.

## CHAPTER EIGHT

### 'Reframing' insanity

*Insanity is relative. It depends on who has who locked in what cage*

Ray Bradbury

'Reframing' a mind means simply to change its perception and behaviour. This can be done subconsciously to such an extent that subjects have no idea they have been 'reframed' while to any observer changes in behaviour and attitudes are obvious.

Human society is being reframed on a ginormous scale since the start of 2020 and here we have the reason why psychologists rather than doctors have been calling the shots. Ask most people who have succumbed to 'Covid' reframing if they have changed and most will say 'no'; but they *have* and fundamentally. The Cult's long-game has been preparing for these times since way back and crucial to that has been to prepare both population and officialdom mentally and emotionally. To use the mind-control parlance they had to reframe the population with a mentality that would submit to fascism and reframe those in government and law enforcement to impose fascism or at least go along with it. The result has been the fact-deleted mindlessness of 'Wokeness' and officialdom that has either enthusiastically or unquestioningly imposed global tyranny demanded by reframed politicians on behalf of psychopathic and deeply evil cultists. 'Cognitive reframing' identifies and challenges the way someone sees the world in the form of situations, experiences and emotions and then restructures those perceptions to view the same set of circumstances in a different way. This can have

benefits if the attitudes are personally destructive while on the other side it has the potential for individual and collective mind control which the subject has no idea has even happened.

Cognitive therapy was developed in the 1960s by Aaron T. Beck who was born in Rhode Island in 1921 as the son of Jewish immigrants from the Ukraine. He became interested in the techniques as a treatment for depression. Beck's daughter Judith S. Beck is prominent in the same field and they founded the Beck Institute for Cognitive Behavior Therapy in Philadelphia in 1994. Cognitive reframing, however, began to be used worldwide by those with a very dark agenda. The Cult reframes politicians to change their attitudes and actions until they are completely at odds with what they once appeared to stand for. The same has been happening to government administrators at all levels, law enforcement, military and the human population. Cultists love mind control for two main reasons: It allows them to control what people think, do and say to secure agenda advancement and, by definition, it calms their legendary insecurity and fear of the unexpected. I have studied mind control since the time I travelled America in 1996. I may have been talking to next to no one in terms of an audience in those years, but my goodness did I gather a phenomenal amount of information and knowledge about so many things including the techniques of mind control. I have described this in detail in other books going back to *The Biggest Secret* in 1998. I met a very large number of people recovering from MKUltra and its offshoots and successors and I began to see how these same techniques were being used on the population in general. This was never more obvious than since the 'Covid' hoax began.

## **Reframing the enforcers**

I have observed over the last two decades and more the very clear transformation in the dynamic between the police, officialdom and the public. I tracked this in the books as the relationship mutated from one of serving the public to seeing them as almost the enemy and certainly a lower caste. There has always been a class divide

based on income and always been some psychopathic, corrupt, and big-I-am police officers. This was different. Wholesale change was unfolding in the collective dynamic; it was less about money and far more about position and perceived power. An us-and-them was emerging. Noses were lifted skyward by government administration and law enforcement and their attitude to the public they were *supposed* to be serving changed to one of increasing contempt, superiority and control. The transformation was so clear and widespread that it had to be planned. Collective attitudes and dynamics do not change naturally and organically that quickly on that scale. I then came across an organisation in Britain called Common Purpose created in the late 1980s by Julia Middleton who would work in the office of Deputy Prime Minister John Prescott during the long and disastrous premiership of war criminal Tony Blair. When Blair speaks the Cult is speaking and the man should have been in jail a long time ago. Common Purpose proclaims itself to be one of the biggest 'leadership development' organisations in the world while functioning as a *charity* with all the financial benefits which come from that. It hosts 'leadership development' courses and programmes all over the world and claims to have 'brought together' what it calls 'leaders' from more than 100 countries on six continents. The modus operandi of Common Purpose can be compared with the work of the UK government's reframing network that includes the Behavioural Insights Team 'nudge unit' and 'Covid' reframing specialists at SPI-B. WikiLeaks described Common Purpose long ago as 'a hidden virus in our government and schools' which is unknown to the general public: 'It recruits and trains "leaders" to be loyal to the directives of Common Purpose and the EU, instead of to their own departments, which they then undermine or subvert, the NHS [National Health Service] being an example.' This is a vital point to understand the 'Covid' hoax. The NHS, and its equivalent around the world, has been utterly reframed in terms of administrators and much of the medical personnel with the transformation underpinned by recruitment policies. The outcome has been the criminal and psychopathic behaviour of the

NHS over ‘Covid’ and we have seen the same in every other major country. WikiLeaks said Common Purpose trainees are ‘learning to rule without regard to democracy’ and to usher in a police state (current events explained). Common Purpose operated like a ‘glue’ and had members in the NHS, BBC, police, legal profession, church, many of Britain’s 7,000 quangos, local councils, the Civil Service, government ministries and Parliament, and controlled many RDA’s (Regional Development Agencies). Here we have one answer for how and why British institutions and their like in other countries have changed so negatively in relation to the public. This further explains how and why the beyond-disgraceful reframed BBC has become a propaganda arm of ‘Covid’ fascism. They are all part of a network pursuing the same goal.

By 2019 Common Purpose was quoting a figure of 85,000 ‘leaders’ that had attended its programmes. These ‘students’ of all ages are known as Common Purpose ‘graduates’ and they consist of government, state and local government officials and administrators, police chiefs and officers, and a whole range of others operating within the national, local and global establishment. Cressida Dick, Commissioner of the London Metropolitan Police, is the Common Purpose graduate who was the ‘Gold Commander’ that oversaw what can only be described as the murder of Brazilian electrician Jean Charles de Menezes in 2005. He was held down by psychopathic police and shot seven times in the head by a psychopathic lunatic after being mistaken for a terrorist when he was just a bloke going about his day. Dick authorised officers to pursue and keep surveillance on de Menezes and ordered that he be stopped from entering the underground train system. Police psychopaths took her at her word clearly. She was ‘disciplined’ for this outrage by being *promoted* – eventually to the top of the ‘Met’ police where she has been a disaster. Many Chief Constables controlling the police in different parts of the UK are and have been Common Purpose graduates. I have heard the ‘graduate’ network described as a sort of Mafia or secret society operating within the fabric of government at all levels pursuing a collective policy

ingrained at Common Purpose training events. Founder Julia Middleton herself has said:

Locally and internationally, Common Purpose graduates will be 'lighting small fires' to create change in their organisations and communities ... The Common Purpose effect is best illustrated by the many stories of small changes brought about by leaders, who themselves have changed.

A Common Purpose mission statement declared:

Common Purpose aims to improve the way society works by expanding the vision, decision-making ability and influence of all kinds of leaders. The organisation runs a variety of educational programmes for leaders of all ages, backgrounds and sectors, in order to provide them with the inspirational, information and opportunities they need to change the world.

Yes, but into what? Since 2020 the answer has become clear.

### **NLP and the Delphi technique**

Common Purpose would seem to be a perfect name or would common programming be better? One of the foundation methods of reaching 'consensus' (group think) is by setting the agenda theme and then encouraging, cajoling or pressuring everyone to agree a 'consensus' in line with the core theme promoted by Common Purpose. The methodology involves the 'Delphi technique', or an adaption of it, in which opinions are expressed that are summarised by a 'facilitator or change agent' at each stage. Participants are 'encouraged' to modify their views in the light of what others have said. Stage by stage the former individual opinions are merged into group consensus which just happens to be what Common Purpose wants them to believe. A key part of this is to marginalise anyone refusing to concede to group think and turn the group against them to apply pressure to conform. We are seeing this very technique used on the general population to make 'Covid' group-thinkers hostile to those who have seen through the bullshit. People can be reframed by using perception manipulation methods such as Neuro-Linguistic Programming (NLP) in which you change perception with the use of

carefully constructed language. An NLP website described the technique this way:

... A method of influencing brain behaviour (the 'neuro' part of the phrase) through the use of language (the 'linguistic' part) and other types of communication to enable a person to 'recode' the way the brain responds to stimuli (that's the 'programming') and manifest new and better behaviours. Neuro-Linguistic Programming often incorporates hypnosis and self-hypnosis to help achieve the change (or 'programming') that is wanted.

British alternative media operation UKColumn has done very detailed research into Common Purpose over a long period. I quoted co-founder and former naval officer Brian Gerrish in my book *Remember Who You Are*, published in 2011, as saying the following years before current times:

It is interesting that many of the mothers who have had children taken by the State speak of the Social Services people being icily cool, emotionless and, as two ladies said in slightly different words, '... like little robots'. We know that NLP is cumulative, so people can be given small imperceptible doses of NLP in a course here, another in a few months, next year etc. In this way, major changes are accrued in their personality, but the day by day change is almost unnoticeable.

In these and other ways 'graduates' have had their perceptions uniformly reframed and they return to their roles in the institutions of government, law enforcement, legal profession, military, 'education', the UK National Health Service and the whole swathe of the establishment structure to pursue a common agenda preparing for the 'post-industrial', 'post-democratic' society. I say 'preparing' but we are now there. 'Post-industrial' is code for the Great Reset and 'post-democratic' is 'Covid' fascism. UKColumn has spoken to partners of those who have attended Common Purpose 'training'. They have described how personalities and attitudes of 'graduates' changed very noticeably for the worse by the time they had completed the course. They had been 'reframed' and told they are the 'leaders' – the special ones – who know better than the population. There has also been the very demonstrable recruitment of psychopaths and narcissists into government administration at all

levels and law enforcement. If you want psychopathy hire psychopaths and you get a simple cause and effect. If you want administrators, police officers and 'leaders' to perceive the public as lesser beings who don't matter then employ narcissists. These personalities are identified using 'psychometrics' that identifies knowledge, abilities, attitudes and personality traits, mostly through carefully-designed questionnaires and tests. As this policy has passed through the decades we have had power-crazy, power-trippers appointed into law enforcement, security and government administration in preparation for current times and the dynamic between public and law enforcement/officialdom has been transformed. UKColumn's Brian Gerrish said of the narcissistic personality:

Their love of themselves and power automatically means that they will crush others who get in their way. I received a major piece of the puzzle when a friend pointed out that when they made public officials re-apply for their own jobs several years ago they were also required to do psychometric tests. This was undoubtedly the start of the screening process to get 'their' sort of people in post.

How obvious that has been since 2020 although it was clear what was happening long before if people paid attention to the changing public-establishment dynamic.

## **Change agents**

At the centre of events in 'Covid' Britain is the National Health Service (NHS) which has behaved disgracefully in slavishly following the Cult agenda. The NHS management structure is awash with Common Purpose graduates or 'change agents' working to a common cause. Helen Bevan, a Chief of Service Transformation at the NHS Institute for Innovation and Improvement, co-authored a document called 'Towards a million change agents, a review of the social movements literature: implications for large scale change in the NHS'. The document compared a project management approach to that of change and social movements where 'people change

themselves and each other – peer to peer’. Two definitions given for a ‘social movement’ were:

*A group of people who consciously attempt to build a radically new social order; involves people of a broad range of social backgrounds; and deploys politically confrontational and socially disruptive tactics* – Cyrus Zirakzadeh 1997

*Collective challenges, based on common purposes and social solidarities, in sustained interaction with elites, opponents, and authorities* – Sidney Tarrow 1994

Helen Bevan wrote another NHS document in which she defined ‘framing’ as ‘the process by which leaders construct, articulate and put across their message in a powerful and compelling way in order to win people to their cause and call them to action’. I think I could come up with another definition that would be rather more accurate. The National Health Service and institutions of Britain and the wider world have been taken over by reframed ‘change agents’ and that includes everything from the United Nations to national governments, local councils and social services which have been kidnapping children from loving parents on an extraordinary and gathering scale on the road to the end of parenthood altogether. Children from loving homes are stolen and kidnapped by the state and put into the ‘care’ (inversion) of the local authority through council homes, foster parents and forced adoption. At the same time children are allowed to be abused without response while many are under council ‘care’. UKColumn highlighted the Common Purpose connection between South Yorkshire Police and Rotherham council officers in the case of the scandal in that area of the sexual exploitation of children to which the authorities turned not one blind eye, but both:

We were alarmed to discover that the Chief Executive, the Strategic Director of Children and Young People's Services, the Manager for the Local Strategic Partnership, the Community Cohesion Manager, the Cabinet Member for Cohesion, the Chief Constable and his predecessor had all attended Leadership training courses provided by the pseudo-charity Common Purpose.

Once 'change agents' have secured positions of hire and fire within any organisation things start to move very quickly. Personnel are then hired and fired on the basis of whether they will work towards the agenda the change agent represents. If they do they are rapidly promoted even though they may be incompetent. Those more qualified and skilled who are pre-Common Purpose 'old school' see their careers stall and even disappear. This has been happening for decades in every institution of state, police, 'health' and social services and all of them have been transformed as a result in their attitudes to their jobs and the public. Medical professions, including nursing, which were once vocations for the caring now employ many cold, callous and couldn't give a shit personality types. The UKColumn investigation concluded:

By blurring the boundaries between people, professions, public and private sectors, responsibility and accountability, Common Purpose encourages 'graduates' to believe that as new selected leaders, they can work together, outside of the established political and social structures, to achieve a paradigm shift or CHANGE – so called 'Leading Beyond Authority'. In doing so, the allegiance of the individual becomes 'reframed' on CP colleagues and their NETWORK.

## **Reframing the Face-Nappies**

Nowhere has this process been more obvious than in the police where recruitment of psychopaths and development of unquestioning mind-controlled group-thinkers have transformed law enforcement into a politically-correct 'Woke' joke and a travesty of what should be public service. Today they wear their face-nappies like good little gofers and enforce 'Covid' rules which are fascism under another name. Alongside the specifically-recruited psychopaths we have software minds incapable of free thought. Brian Gerrish again:

An example is the policeman who would not get on a bike for a press photo because he had not done the cycling proficiency course. Normal people say this is political correctness gone mad. Nothing could be further from the truth. The policeman has been reframed, and in his reality it is perfect common sense not to get on the bike ‘because he hasn’t done the cycling course’.

Another example of this is where the police would not rescue a boy from a pond until they had taken advice from above on the ‘risk assessment’. A normal person would have arrived, perhaps thought of the risk for a moment, and dived in. To the police now ‘reframed’, they followed ‘normal’ procedure.

There are shocking cases of reframed ambulance crews doing the same. Sheer unthinking stupidity of London Face-Nappies headed by Common Purpose graduate Cressida Dick can be seen in their behaviour at a vigil in March, 2021, for a murdered woman, Sarah Everard. A police officer had been charged with the crime. Anyone with a brain would have left the vigil alone in the circumstances. Instead they ‘manhandled’ women to stop them breaking ‘Covid rules’ to betray classic reframing. Minds in the thrall of perception control have no capacity for seeing a situation on its merits and acting accordingly. ‘Rules is rules’ is their only mind-set. My father used to say that rules and regulations are for the guidance of the intelligent and the blind obedience of the idiot. Most of the intelligent, decent, coppers have gone leaving only the other kind and a few old school for whom the job must be a daily nightmare. The combination of psychopaths and rule-book software minds has been clearly on public display in the ‘Covid’ era with automaton robots in uniform imposing fascistic ‘Covid’ regulations on the population without any personal initiative or judging situations on their merits. There are thousands of examples around the world, but I’ll make my point with the infamous Derbyshire police in the English East Midlands – the ones who think pouring dye into beauty spots and using drones to track people walking in the countryside away from anyone is called ‘policing’. To them there are rules decreed by the government which they have to enforce and in their bewildered state a group gathering in a closed space and someone walking alone in the countryside are the same thing. It is beyond idiocy and enters the realm of clinical insanity.

Police officers in Derbyshire said they were ‘horrified’ – *horrified* – to find 15 to 20 ‘irresponsible’ kids playing a football match at a closed leisure centre ‘in breach of coronavirus restrictions’. When they saw the police the kids ran away leaving their belongings behind and the reframed men and women of Derbyshire police were seeking to establish their identities with a view to fining their parents. The most natural thing for youngsters to do – kicking a ball about – is turned into a criminal activity and enforced by the moronic software programs of Derbyshire police. You find the same mentality in every country. These barely conscious ‘horrified’ officers said they had to take action because ‘we need to ensure these rules are being followed’ and ‘it is of the utmost importance that you ensure your children are following the rules and regulations for Covid-19’. Had any of them done ten seconds of research to see if this parroting of their masters’ script could be supported by any evidence? Nope. Reframed people don’t think – others think for them and that’s the whole idea of reframing. I have seen police officers one after the other repeating without question word for word what officialdom tells them just as I have seen great swathes of the public doing the same. Ask either for ‘their’ opinion and out spews what they have been told to think by the official narrative. Police and public may seem to be in different groups, but their mentality is the same. Most people do whatever they are told in fear not doing so or because they believe what officialdom tells them; almost the entirety of the police do what they are told for the same reason. Ultimately it’s the tiny inner core of the global Cult that’s telling both what to do.

So Derbyshire police were ‘horrified’. Oh, really? Why did they think those kids were playing football? It was to relieve the psychological consequences of lockdown and being denied human contact with their friends and interaction, touch and discourse vital to human psychological health. Being denied this month after month has dismantled the psyche of many children and young people as depression and suicide have exploded. Were Derbyshire police *horrified by that?* Are you kidding? Reframed people don’t have those

mental and emotional processes that can see how the impact on the psychological health of youngsters is far more dangerous than any 'virus' even if you take the mendacious official figures to be true. The reframed are told (programmed) how to act and so they do. The Derbyshire Chief Constable in the first period of lockdown when the black dye and drones nonsense was going on was Peter Goodman. He was the man who severed the connection between his force and the Derbyshire Constabulary *Male Voice* Choir when he decided that it was not inclusive enough to allow women to join. The fact it was a male voice choir making a particular sound produced by male voices seemed to elude a guy who terrifyingly ran policing in Derbyshire. He retired weeks after his force was condemned as disgraceful by former Supreme Court Justice Jonathan Sumption for their behaviour over extreme lockdown impositions. Goodman was replaced by his deputy Rachel Swann who was in charge when her officers were 'horrified'. The police statement over the boys committing the hanging-offence of playing football included the line about the youngsters being 'irresponsible in the times we are all living through' missing the point that the real relevance of the 'times we are all living through' is the imposition of fascism enforced by psychopaths and reframed minds of police officers playing such a vital part in establishing the fascist tyranny that their own children and grandchildren will have to live in their entire lives. As a definition of insanity that is hard to beat although it might be run close by imposing masks on people that can have a serious effect on their health while wearing a face nappy all day themselves. Once again public and police do it for the same reason – the authorities tell them to and who are they to have the self-respect to say no?

## **Wokers in uniform**

How reframed do you have to be to arrest a *six-year-old* and take him to court for *picking a flower* while waiting for a bus? Brain dead police and officialdom did just that in North Carolina where criminal proceedings happen regularly for children under nine. Attorney Julie Boyer gave the six-year-old crayons and a colouring book

during the ‘flower’ hearing while the ‘adults’ decided his fate. County Chief District Court Judge Jay Corpening asked: ‘Should a child that believes in Santa Claus, the Easter Bunny and the tooth fairy be making life-altering decisions?’ Well, of course not, but common sense has no meaning when you have a common purpose and a reframed mind. Treating children in this way, and police operating in American schools, is all part of the psychological preparation for children to accept a police state as normal all their adult lives. The same goes for all the cameras and biometric tracking technology in schools. Police training is focused on reframing them as snowflake Wokers and this is happening in the military. Pentagon top brass said that ‘training sessions on extremism’ were needed for troops who asked why they were so focused on the Capitol Building riot when Black Lives Matter riots were ignored. What’s the difference between them some apparently and rightly asked. Actually, there is a difference. Five people died in the Capitol riot, only one through violence, and that was a police officer shooting an unarmed protestor. BLM riots killed at least 25 people and cost billions. Asking the question prompted the psychopaths and reframed minds that run the Pentagon to say that more ‘education’ (programming) was needed. Troop training is all based on psychological programming to make them fodder for the Cult – ‘Military men are just dumb, stupid animals to be used as pawns in foreign policy’ as Cult-to-his-DNA former Secretary of State Henry Kissinger famously said. Governments see the police in similar terms and it’s time for those among them who can see this to defend the people and stop being enforcers of the Cult agenda upon the people.

The US military, like the country itself, is being targeted for destruction through a long list of Woke impositions. Cult-owned gaga ‘President’ Biden signed an executive order when he took office to allow taxpayer money to pay for transgender surgery for active military personnel and veterans. Are you a man soldier? No, I’m a LGBTQIA+ with a hint of Skoliosexual and Spectrasexual. Oh, good man. Bad choice of words you bigot. The Pentagon announced in March, 2021, the appointment of the first ‘diversity and inclusion

officer' for US Special Forces. Richard Torres-Estrada arrived with the publication of a 'D&I Strategic Plan which will guide the enterprise-wide effort to institutionalize and sustain D&I'. If you think a Special Forces 'Strategic Plan' should have something to do with defending America you haven't been paying attention.

Defending Woke is now the military's new role. Torres-Estrada has posted images comparing Donald Trump with Adolf Hitler and we can expect no bias from him as a representative of the supposedly non-political Pentagon. Cable news host Tucker Carlson said: 'The Pentagon is now the Yale faculty lounge but with cruise missiles.' Meanwhile Secretary of Defense Lloyd Austin, a board member of weapons-maker Raytheon with stock and compensation interests in October, 2020, worth \$1.4 million, said he was purging the military of the 'enemy within' – anyone who isn't Woke and supports Donald Trump. Austin refers to his targets as 'racist extremists' while in true Woke fashion being himself a racist extremist. Pentagon documents pledge to 'eradicate, eliminate and conquer all forms of racism, sexism and homophobia'. The definitions of these are decided by 'diversity and inclusion committees' peopled by those who see racism, sexism and homophobia in every situation and opinion. Woke (the Cult) is dismantling the US military and purging testosterone as China expands its military and gives its troops 'masculinity training'. How do we think that is going to end when this is all Cult coordinated? The US military, like the British military, is controlled by Woke and spineless top brass who just go along with it out of personal career interests.

## **'Woke' means fast asleep**

Mind control and perception manipulation techniques used on individuals to create group-think have been unleashed on the global population in general. As a result many have no capacity to see the obvious fascist agenda being installed all around them or what 'Covid' is really all about. Their brains are firewalled like a computer system not to process certain concepts, thoughts and realisations that are bad for the Cult. The young are most targeted as the adults they

will be when the whole fascist global state is planned to be fully implemented. They need to be prepared for total compliance to eliminate all pushback from entire generations. The Cult has been pouring billions into taking complete control of 'education' from schools to universities via its operatives and corporations and not least Bill Gates as always. The plan has been to transform 'education' institutions into programming centres for the mentality of 'Woke'. James McConnell, professor of psychology at the University of Michigan, wrote in *Psychology Today* in 1970:

The day has come when we can combine sensory deprivation with drugs, hypnosis, and astute manipulation of reward and punishment, to gain almost absolute control over an individual's behaviour. It should then be possible to achieve a very rapid and highly effective type of brainwashing that would allow us to make dramatic changes in a person's behaviour and personality ...

... We should reshape society so that we all would be trained from birth to want to do what society wants us to do. We have the techniques to do it... no-one owns his own personality you acquired, and there's no reason to believe you should have the right to refuse to acquire a new personality if your old one is anti-social.

This was the potential for mass brainwashing in 1970 and the mentality there displayed captures the arrogant psychopathy that drives it forward. I emphasise that not all young people have succumbed to Woke programming and those that haven't are incredibly impressive people given that today's young are the most perceptually-targeted generations in history with all the technology now involved. Vast swathes of the young generations, however, have fallen into the spell – and that's what it is – of Woke. The Woke mentality and perceptual program is founded on *inversion* and you will appreciate later why that is so significant. Everything with Woke is inverted and the opposite of what it is claimed to be. Woke was a term used in African-American culture from the 1900s and referred to an awareness of social and racial justice. This is not the meaning of the modern version or 'New Woke' as I call it in *The Answer*. Oh, no, Woke today means something very different no matter how much Wokers may seek to hide that and insist Old Woke and New

Woke are the same. See if you find any 'awareness of social justice' here in the modern variety:

- Woke demands 'inclusivity' while excluding anyone with a different opinion and calls for mass censorship to silence other views.
- Woke claims to stand against oppression when imposing oppression is the foundation of all that it does. It is the driver of political correctness which is nothing more than a Cult invention to manipulate the population to silence itself.
- Woke believes itself to be 'liberal' while pursuing a global society that can only be described as fascist (see 'anti-fascist' fascist Antifa).
- Woke calls for 'social justice' while spreading injustice wherever it goes against the common 'enemy' which can be easily identified as a differing view.
- Woke is supposed to be a metaphor for 'awake' when it is solid-gold asleep and deep in a Cult-induced coma that meets the criteria for 'off with the fairies'.

I state these points as obvious facts if people only care to look. I don't do this with a sense of condemnation. We need to appreciate that the onslaught of perceptual programming on the young has been incessant and merciless. I can understand why so many have been reframed, or, given their youth, framed from the start to see the world as the Cult demands. The Cult has had access to their minds day after day in its 'education' system for their entire formative years. Perception is formed from information received and the Cult-created system is a life-long download of information delivered to elicit a particular perception, thus behaviour. The more this has expanded into still new extremes in recent decades and ever-increasing censorship has deleted other opinions and information why wouldn't that lead to a perceptual reframing on a mass scale? I

have described already cradle-to-grave programming and in more recent times the targeting of young minds from birth to adulthood has entered the stratosphere. This has taken the form of skewing what is ‘taught’ to fit the Cult agenda and the omnipresent techniques of group-think to isolate non-believers and pressure them into line. There has always been a tendency to follow the herd, but we really are in a new world now in relation to that. We have parents who can see the ‘Covid’ hoax told by their children not to stop them wearing masks at school, being ‘Covid’ tested or having the ‘vaccine’ in fear of the peer-pressure consequences of being different. What is ‘peer-pressure’ if not pressure to conform to group-think? Renegade Minds never group-think and always retain a set of perceptions that are unique to them. Group-think is always underpinned by consequences for not group-thinking. Abuse now aimed at those refusing DNA-manipulating ‘Covid vaccines’ are a potent example of this. The biggest pressure to conform comes from the very group which is itself being manipulated. ‘I am programmed to be part of a hive mind and so you must be.’

Woke control structures in ‘education’ now apply to every mainstream organisation. Those at the top of the ‘education’ hierarchy (the Cult) decide the policy. This is imposed on governments through the Cult network; governments impose it on schools, colleges and universities; their leadership impose the policy on teachers and academics and they impose it on children and students. At any level where there is resistance, perhaps from a teacher or university lecturer, they are targeted by the authorities and often fired. Students themselves regularly demand the dismissal of academics (increasingly few) at odds with the narrative that the students have been programmed to believe in. It is quite a thought that students who are being targeted by the Cult become so consumed by programmed group-think that they launch protests and demand the removal of those who are trying to push back against those targeting the students. Such is the scale of perceptual inversion. We see this with ‘Covid’ programming as the Cult imposes the rules via psycho-psychologists and governments on

shops, transport companies and businesses which impose them on their staff who impose them on their customers who pressure Pushbackers to conform to the will of the Cult which is in the process of destroying them and their families. Scan all aspects of society and you will see the same sequence every time.

## **Fact free Woke and hijacking the 'left'**

There is no more potent example of this than 'Woke', a mentality only made possible by the deletion of factual evidence by an 'education' system seeking to produce an ever more uniform society. Why would you bother with facts when you don't know any? Deletion of credible history both in volume and type is highly relevant. Orwell said: 'Who controls the past controls the future: who controls the present controls the past.' They who control the perception of the past control the perception of the future and they who control the present control the perception of the past through the writing and deleting of history. Why would you oppose the imposition of Marxism in the name of Wokeism when you don't know that Marxism cost at least 100 million lives in the 20th century alone? Watch videos and read reports in which Woker generations are asked basic historical questions – it's mind-blowing. A survey of 2,000 people found that six percent of millennials (born approximately early 1980s to early 2000s) believed the Second World War (1939-1945) broke out with the assassination of President Kennedy (in 1963) and one in ten thought Margaret Thatcher was British Prime Minister at the time. She was in office between 1979 and 1990. We are in a post-fact society. Provable facts are no defence against the fascism of political correctness or Silicon Valley censorship. Facts don't matter anymore as we have witnessed with the 'Covid' hoax. Sacrificing uniqueness to the Woke group-think religion is all you are required to do and that means thinking for yourself is the biggest Woke no, no. All religions are an expression of group-think and censorship and Woke is just another religion with an orthodoxy defended by group-think and censorship. Burned at

the stake becomes burned on Twitter which leads back eventually to burned at the stake as Woke humanity regresses to ages past.

The biggest Woke inversion of all is its creators and funders. I grew up in a traditional left of centre political household on a council estate in Leicester in the 1950s and 60s – you know, the left that challenged the power of wealth-hoarding elites and threats to freedom of speech and opinion. In those days students went on marches defending freedom of speech while today's Wokers march for its deletion. What on earth could have happened? Those very elites (collectively the Cult) that we opposed in my youth and early life have funded into existence the antithesis of that former left and hijacked the 'brand' while inverting everything it ever stood for. We have a mentality that calls itself 'liberal' and 'progressive' while acting like fascists. Cult billionaires and their corporations have funded themselves into control of 'education' to ensure that Woke programming is unceasing throughout the formative years of children and young people and that non-Wokers are isolated (that word again) whether they be students, teachers or college professors. The Cult has funded into existence the now colossal global network of Woke organisations that have spawned and promoted all the 'causes' on the Cult wish-list for global transformation and turned Wokers into demanders of them. Does anyone really think it's a coincidence that the Cult agenda for humanity is a carbon (sorry) copy of the societal transformations desired by Woke?? These are only some of them:

**Political correctness:** The means by which the Cult deletes all public debates that it knows it cannot win if we had the free-flow of information and evidence.

**Human-caused 'climate change':** The means by which the Cult seeks to transform society into a globally-controlled dictatorship imposing its will over the fine detail of everyone's lives 'to save the planet' which doesn't actually need saving.

**Transgender obsession:** Preparing collective perception to accept the ‘new human’ which would not have genders because it would be created technologically and not through procreation. I’ll have much more on this in Human 2.0.

**Race obsession:** The means by which the Cult seeks to divide and rule the population by triggering racial division through the perception that society is more racist than ever when the opposite is the case. Is it perfect in that regard? No. But to compare today with the racism of apartheid and segregation brought to an end by the civil rights movement in the 1960s is to insult the memory of that movement and inspirations like Martin Luther King. Why is the ‘anti-racism’ industry (which it is) so dominated by privileged white people?

**White supremacy:** This is a label used by privileged white people to demonise poor and deprived white people pushing back on tyranny to marginalise and destroy them. White people are being especially targeted as the dominant race by number within Western society which the Cult seeks to transform in its image. If you want to change a society you must weaken and undermine its biggest group and once you have done that by using the other groups you next turn on them to do the same ... ‘Then they came for the Jews and I was not a Jew so I did nothing.’

**Mass migration:** The mass movement of people from the Middle East, Africa and Asia into Europe, from the south into the United States and from Asia into Australia are another way the Cult seeks to dilute the racial, cultural and political influence of white people on Western society. White people ask why their governments appear to be working against them while being politically and culturally biased towards incoming cultures. Well, here’s your answer. In the same way sexually ‘straight’ people, men and women, ask why the

authorities are biased against them in favour of other sexualities. The answer is the same – that's the way the Cult wants it to be for very sinister motives.

These are all central parts of the Cult agenda and central parts of the Woke agenda and Woke was created and continues to be funded to an immense degree by Cult billionaires and corporations. If anyone begins to say 'coincidence' the syllables should stick in their throat.

### **Billionaire 'social justice warriors'**

Joe Biden is a 100 percent-owned asset of the Cult and the Wokers' man in the White House whenever he can remember his name and for however long he lasts with his rapidly diminishing cognitive function. Even walking up the steps of an aircraft without falling on his arse would appear to be a challenge. He's not an empty-shell puppet or anything. From the minute Biden took office (or the Cult did) he began his executive orders promoting the Woke wish-list. You will see the Woke agenda imposed ever more severely because it's really the *Cult* agenda. Woke organisations and activist networks spawned by the Cult are funded to the extreme so long as they promote what the Cult wants to happen. Woke is funded to promote 'social justice' by billionaires who become billionaires by destroying social justice. The social justice mantra is only a cover for dismantling social justice and funded by billionaires that couldn't give a damn about social justice. Everything makes sense when you see that. One of Woke's premier funders is Cult billionaire financier George Soros who said: 'I am basically there to make money, I cannot and do not look at the social consequences of what I do.' This is the same Soros who has given more than \$32 billion to his Open Society Foundations global Woke network and funded Black Lives Matter, mass immigration into Europe and the United States, transgender activism, climate change activism, political correctness and groups targeting 'white supremacy' in the form of privileged white thugs that dominate Antifa. What a scam it all is and when

you are dealing with the unquestioning fact-free zone of Woke scamming them is child's play. All you need to pull it off in all these organisations are a few in-the-know agents of the Cult and an army of naïve, reframed, uninformed, narcissistic, know-nothings convinced of their own self-righteousness, self-purity and virtue.

Soros and fellow billionaires and billionaire corporations have poured hundreds of millions into Black Lives Matter and connected groups and promoted them to a global audience. None of this is motivated by caring about black people. These are the billionaires that have controlled and exploited a system that leaves millions of black people in abject poverty and deprivation which they do absolutely nothing to address. The same Cult networks funding BLM were behind the *slave trade!* Black Lives Matter hijacked a phrase that few would challenge and they have turned this laudable concept into a political weapon to divide society. You know that BLM is a fraud when it claims that *All Lives Matter*, the most inclusive statement of all, is 'racist'. BLM and its Cult masters don't want to end racism. To them it's a means to an end to control all of humanity never mind the colour, creed, culture or background. What has destroying the nuclear family got to do with ending racism? Nothing – but that is one of the goals of BLM and also happens to be a goal of the Cult as I have been exposing in my books for decades. Stealing children from loving parents and giving schools ever more power to override parents is part of that same agenda. BLM is a Marxist organisation and why would that not be the case when the Cult created Marxism *and* BLM? Patrisse Cullors, a BLM co-founder, said in a 2015 video that she and her fellow organisers, including co-founder Alicia Garza, are 'trained Marxists'. The lady known after marriage as Patrisse Khan-Cullors bought a \$1.4 million home in 2021 in one of the whitest areas of California with a black population of just 1.6 per cent and has so far bought *four* high-end homes for a total of \$3.2 million. How very Marxist. There must be a bit of spare in the BLM coffers, however, when Cult corporations and billionaires have handed over the best part of \$100 million. Many black people can see that Black Lives Matter is not

working for them, but against them, and this is still more confirmation. Black journalist Jason Whitlock, who had his account suspended by Twitter for simply linking to the story about the ‘Marxist’s’ home buying spree, said that BLM leaders are ‘making millions of dollars off the backs of these dead black men who they wouldn’t spit on if they were on fire and alive’.

## **Black Lies Matter**

Cult assets and agencies came together to promote BLM in the wake of the death of career criminal George Floyd who had been jailed a number of times including for forcing his way into the home of a black woman with others in a raid in which a gun was pointed at her stomach. Floyd was filmed being held in a Minneapolis street in 2020 with the knee of a police officer on his neck and he subsequently died. It was an appalling thing for the officer to do, but the same technique has been used by police on peaceful protestors of lockdown without any outcry from the Woke brigade. As unquestioning supporters of the Cult agenda Wokers have supported lockdown and all the ‘Covid’ claptrap while attacking anyone standing up to the tyranny imposed in its name. Court documents would later include details of an autopsy on Floyd by County Medical Examiner Dr Andrew Baker who concluded that Floyd had taken a fatal level of the drug fentanyl. None of this mattered to fact-free, question-free, Woke. Floyd’s death was followed by worldwide protests against police brutality amid calls to defund the police. Throwing babies out with the bathwater is a Woke speciality. In the wake of the murder of British woman Sarah Everard a Green Party member of the House of Lords, Baroness Jones of Moulsecoomb (Nincompoopia would have been better), called for a 6pm curfew for all men. This would be in breach of the Geneva Conventions on war crimes which ban collective punishment, but that would never have crossed the black and white Woke mind of Baroness Nincompoopia who would have been far too convinced of her own self-righteousness to compute such details. Many American cities did defund the police in the face of Floyd riots

and after \$15 million was deleted from the police budget in Washington DC under useless Woke mayor Muriel Bowser car-jacking alone rose by 300 percent and within six months the US capital recorded its highest murder rate in 15 years. The same happened in Chicago and other cities in line with the Cult/Soros plan to bring fear to streets and neighbourhoods by reducing the police, releasing violent criminals and not prosecuting crime. This is the mob-rule agenda that I have warned in the books was coming for so long. Shootings in the area of Minneapolis where Floyd was arrested increased by 2,500 percent compared with the year before. Defunding the police over George Floyd has led to a big increase in dead people with many of them black. Police protection for politicians making these decisions stayed the same or increased as you would expect from professional hypocrites. The Cult doesn't actually want to abolish the police. It wants to abolish local control over the police and hand it to federal government as the psychopaths advance the Hunger Games Society. Many George Floyd protests turned into violent riots with black stores and businesses destroyed by fire and looting across America fuelled by Black Lives Matter. Woke doesn't do irony. If you want civil rights you must loot the liquor store and the supermarket and make off with a smart TV. It's the only way.

### **It's not a race war – it's a class war**

Black people are patronised by privileged blacks and whites alike and told they are victims of white supremacy. I find it extraordinary to watch privileged blacks supporting the very system and bloodline networks behind the slave trade and parroting the same Cult-serving manipulative crap of their privileged white, often billionaire, associates. It is indeed not a race war but a class war and colour is just a diversion. Black Senator Cory Booker and black Congresswoman Maxine Waters, more residents of Nincompoopia, personify this. Once you tell people they are victims of someone else you devalue both their own responsibility for their plight and the power they have to impact on their reality and experience. Instead

we have: 'You are only in your situation because of whitey – turn on them and everything will change.' It won't change. Nothing changes in our lives unless *we* change it. Crucial to that is never seeing yourself as a victim and always as the creator of your reality. Life is a simple sequence of choice and consequence. Make different choices and you create different consequences. *You* have to make those choices – not Black Lives Matter, the Woke Mafia and anyone else that seeks to dictate your life. Who are they these Wokers, an emotional and psychological road traffic accident, to tell you what to do? Personal empowerment is the last thing the Cult and its Black Lives Matter want black people or anyone else to have. They claim to be defending the underdog while *creating* and perpetuating the underdog. The Cult's worst nightmare is human unity and if they are going to keep blacks, whites and every other race under economic servitude and control then the focus must be diverted from what they have in common to what they can be manipulated to believe divides them. Blacks have to be told that their poverty and plight is the fault of the white bloke living on the street in the same poverty and with the same plight they are experiencing. The difference is that your plight black people is due to him, a white supremacist with 'white privilege' living on the street. Don't unite as one human family against your mutual oppressors and suppressors – fight the oppressor with the white face who is as financially deprived as you are. The Cult knows that as its 'Covid' agenda moves into still new levels of extremism people are going to respond and it has been spreading the seeds of disunity everywhere to stop a united response to the evil that targets *all of us*.

Racist attacks on 'whiteness' are getting ever more outrageous and especially through the American Democratic Party which has an appalling history for anti-black racism. Barack Obama, Joe Biden, Hillary Clinton and Nancy Pelosi all eulogised about Senator Robert Byrd at his funeral in 2010 after a nearly 60-year career in Congress. Byrd was a brutal Ku Klux Klan racist and a violent abuser of Cathy O'Brien in MKUltra. He said he would never fight in the military 'with a negro by my side' and 'rather I should die a thousand times,

and see Old Glory trampled in the dirt never to rise again, than to see this beloved land of ours become degraded by race mongrels, a throwback to the blackest specimen from the wilds'. Biden called Byrd a 'very close friend and mentor'. These 'Woke' hypocrites are not anti-racist they are anti-poor and anti-people not of their perceived class. Here is an illustration of the scale of anti-white racism to which we have now descended. Seriously Woke and moronic *New York Times* contributor Damon Young described whiteness as a 'virus' that 'like other viruses will not die until there are no bodies left for it to infect'. He went on: '... the only way to stop it is to locate it, isolate it, extract it, and kill it.' Young can say that as a black man with no consequences when a white man saying the same in reverse would be facing a jail sentence. *That's* racism. We had super-Woke numbskull senators Tammy Duckworth and Mazie Hirono saying they would object to future Biden Cabinet appointments if he did not nominate more Asian Americans and Pacific Islanders. Never mind the ability of the candidate what do they look like? Duckworth said: 'I will vote for racial minorities and I will vote for LGBTQ, but anyone else I'm not voting for.' Appointing people on the grounds of race is illegal, but that was not a problem for this ludicrous pair. They were on-message and that's a free pass in any situation.

## Critical race racism

White children are told at school they are intrinsically racist as they are taught the divisive 'critical race theory'. This claims that the law and legal institutions are inherently racist and that race is a socially constructed concept used by white people to further their economic and political interests at the expense of people of colour. White is a 'virus' as we've seen. Racial inequality results from 'social, economic, and legal differences that white people create between races to maintain white interests which leads to poverty and criminality in minority communities'. I must tell that to the white guy sleeping on the street. The principal of East Side Community School in New York sent white parents a manifesto that called on

them to become ‘white traitors’ and advocate for full ‘white abolition’. These people are teaching your kids when they urgently need a psychiatrist. The ‘school’ included a chart with ‘eight white identities’ that ranged from ‘white supremacist’ to ‘white abolition’ and defined the behaviour white people must follow to end ‘the regime of whiteness’. Woke blacks and their privileged white associates are acting exactly like the slave owners of old and Ku Klux Klan racists like Robert Byrd. They are too full of their own self-purity to see that, but it’s true. Racism is not a body type; it’s a state of mind that can manifest through any colour, creed or culture.

Another racial fraud is ‘*equity*’. Not equality of treatment and opportunity – equity. It’s a term spun as equality when it means something very different. Equality in its true sense is a raising up while ‘*equity*’ is a race to the bottom. Everyone in the same level of poverty is ‘*equity*’. Keep everyone down – that’s equity. The Cult doesn’t want anyone in the human family to be empowered and BLM leaders, like all these ‘anti-racist’ organisations, continue their privileged, pampered existence by perpetuating the perception of gathering racism. When is the last time you heard an ‘anti-racist’ or ‘anti-Semitism’ organisation say that acts of racism and discrimination have *fallen*? It’s not in the interests of their fund-raising and power to influence and the same goes for the professional soccer anti-racism operation, Kick It Out. Two things confirmed that the Black Lives Matter riots in the summer of 2020 were Cult creations. One was that while anti-lockdown protests were condemned in this same period for ‘transmitting ‘Covid’ the authorities supported mass gatherings of Black Lives Matter supporters. I even saw self-deluding people claiming to be doctors say the two types of protest were not the same. No – the non-existent ‘Covid’ was in favour of lockdowns and attacked those that protested against them while ‘Covid’ supported Black Lives Matter and kept well away from its protests. The whole thing was a joke and as lockdown protestors were arrested, often brutally, by reframed Face-Nappies we had the grotesque sight of police officers taking the knee to Black Lives Matter, a Cult-funded Marxist

organisation that supports violent riots and wants to destroy the nuclear family and white people.

## **He's not white? Shucks!**

Woke obsession with race was on display again when ten people were shot dead in Boulder, Colorado, in March, 2021. Cult-owned Woke TV channels like CNN said the shooter appeared to be a white man and Wokers were on Twitter condemning 'violent white men' with the usual mantras. Then the shooter's name was released as Ahmad Al Aliwi Alissa, an anti-Trump Arab-American, and the sigh of disappointment could be heard five miles away. Never mind that ten people were dead and what that meant for their families. Race baiting was all that mattered to these sick Cult-serving people like Barack Obama who exploited the deaths to further divide America on racial grounds which is his job for the Cult. This is the man that 'racist' white Americans made the first black president of the United States and then gave him a second term. Not-very-bright Obama has become filthy rich on the back of that and today appears to have a big influence on the Biden administration. Even so he's still a downtrodden black man and a victim of white supremacy. This disingenuous fraud reveals the contempt he has for black people when he puts on a Deep South Alabama accent whenever he talks to them, no, *at* them.

Another BLM red flag was how the now fully-Woke (fully-Cult) and fully-virtue-signalled professional soccer authorities had their teams taking the knee before every match in support of Marxist Black Lives Matter. Soccer authorities and clubs displayed 'Black Lives Matter' on the players' shirts and flashed the name on electronic billboards around the pitch. Any fans that condemned what is a Freemasonic taking-the-knee ritual were widely condemned as you would expect from the Woke virtue-signallers of professional sport and the now fully-Woke media. We have reverse racism in which you are banned from criticising any race or culture except for white people for whom anything goes – say what you like, no problem. What has this got to do with racial harmony and

equality? We've had black supremacists from Black Lives Matter telling white people to fall to their knees in the street and apologise for their white supremacy. Black supremacists acting like white supremacist slave owners of the past couldn't breach their self-obsessed, race-obsessed sense of self-purity. Joe Biden appointed a race-obsessed black supremacist Kristen Clarke to head the Justice Department Civil Rights Division. Clarke claimed that blacks are endowed with 'greater mental, physical and spiritual abilities' than whites. If anyone reversed that statement they would be vilified. Clarke is on-message so no problem. She's never seen a black-white situation in which the black figure is anything but a virtuous victim and she heads the Civil Rights Division which should treat everyone the same or it isn't civil rights. Another perception of the Renegade Mind: If something or someone is part of the Cult agenda they will be supported by Woke governments and media no matter what. If they're not, they will be condemned and censored. It really is that simple and so racist Clarke prospers despite (make that because of) her racism.

## **The end of culture**

Biden's administration is full of such racial, cultural and economic bias as the Cult requires the human family to be divided into warring factions. We are now seeing racially-segregated graduations and everything, but everything, is defined through the lens of perceived 'racism. We have 'racist' mathematics, 'racist' food and even 'racist' *plants*. World famous Kew Gardens in London said it was changing labels on plants and flowers to tell its pre-'Covid' more than two million visitors a year how racist they are. Kew director Richard Deverell said this was part of an effort to 'move quickly to decolonise collections' after they were approached by one Ajay Chhabra 'an actor with an insight into how sugar cane was linked to slavery'. They are *plants* you idiots. 'Decolonisation' in the Woke manual really means colonisation of society with its mentality and by extension colonisation by the Cult. We are witnessing a new Chinese-style 'Cultural Revolution' so essential to the success of all

Marxist takeovers. Our cultural past and traditions have to be swept away to allow a new culture to be built-back-better. Woke targeting of long-standing Western cultural pillars including historical monuments and cancelling of historical figures is what happened in the Mao revolution in China which ‘purged remnants of capitalist and traditional elements from Chinese society’ and installed Maoism as the dominant ideology’. For China see the Western world today and for ‘dominant ideology’ see Woke. Better still see Marxism or Maoism. The ‘Covid’ hoax has specifically sought to destroy the arts and all elements of Western culture from people meeting in a pub or restaurant to closing theatres, music venues, sports stadiums, places of worship and even banning *singing*. Destruction of Western society is also why criticism of any religion is banned except for Christianity which again is the dominant religion as white is the numerically-dominant race. Christianity may be fading rapidly, but its history and traditions are weaved through the fabric of Western society. Delete the pillars and other structures will follow until the whole thing collapses. I am not a Christian defending that religion when I say that. I have no religion. It’s just a fact. To this end Christianity has itself been turned Woke to usher its own downfall and its ranks are awash with ‘change agents’ – knowing and unknowing – at every level including Pope Francis (*definitely* knowing) and the clueless Archbishop of Canterbury Justin Welby (possibly not, but who can be sure?). Woke seeks to coordinate attacks on Western culture, traditions, and ways of life through ‘intersectionality’ defined as ‘the complex, cumulative way in which the effects of multiple forms of discrimination (such as racism, sexism, and classism) combine, overlap, or intersect especially in the experiences of marginalised individuals or groups’. Wade through the Orwellian Woke-speak and this means coordinating disparate groups in a common cause to overthrow freedom and liberal values.

The entire structure of public institutions has been infested with Woke – government at all levels, political parties, police, military, schools, universities, advertising, media and trade unions. This abomination has been achieved through the Cult web by appointing

Wokers to positions of power and battering non-Wokers into line through intimidation, isolation and threats to their job. Many have been fired in the wake of the empathy-deleted, vicious hostility of 'social justice' Wokers and the desire of gutless, spineless employers to virtue-signal their Wokeness. Corporations are filled with Wokers today, most notably those in Silicon Valley. Ironically at the top they are not Woke at all. They are only exploiting the mentality their Cult masters have created and funded to censor and enslave while the Wokers cheer them on until it's their turn. Thus the Woke 'liberal left' is an inversion of the traditional liberal left. Campaigning for justice on the grounds of power and wealth distribution has been replaced by campaigning for identity politics. The genuine traditional left would never have taken money from today's billionaire abusers of fairness and justice and nor would the billionaires have wanted to fund that genuine left. It would not have been in their interests to do so. The division of opinion in those days was between the haves and have nots. This all changed with Cult manipulated and funded identity politics. The division of opinion today is between Wokers and non-Wokers and not income brackets. Cult corporations and their billionaires may have taken wealth disparity to cataclysmic levels of injustice, but as long as they speak the language of Woke, hand out the dosh to the Woke network and censor the enemy they are 'one of us'. Billionaires who don't give a damn about injustice are laughing at them till their bellies hurt. Wokers are not even close to self-aware enough to see that. The transformed 'left' dynamic means that Wokers who drone on about 'social justice' are funded by billionaires that have destroyed social justice the world over. It's *why* they are billionaires.

## **The climate con**

Nothing encapsulates what I have said more comprehensively than the hoax of human-caused global warming. I have detailed in my books over the years how Cult operatives and organisations were the pump-primers from the start of the climate con. A purpose-built vehicle for this is the Club of Rome established by the Cult in 1968

with the Rockefellers and Rothschilds centrally involved all along. Their gofer frontman Maurice Strong, a Canadian oil millionaire, hosted the Earth Summit in Rio de Janeiro, Brazil, in 1992 where the global ‘green movement’ really expanded in earnest under the guiding hand of the Cult. The Earth Summit established Agenda 21 through the Cult-created-and-owned United Nations to use the illusion of human-caused climate change to justify the transformation of global society to save the world from climate disaster. It is a No-Problem-Reaction-Solution sold through governments, media, schools and universities as whole generations have been terrified into believing that the world was going to end in their lifetimes unless what old people had inflicted upon them was stopped by a complete restructuring of how everything is done. Chill, kids, it’s all a hoax. Such restructuring is precisely what the Cult agenda demands (purely by coincidence of course). Today this has been given the codename of the Great Reset which is only an updated term for Agenda 21 and its associated Agenda 2030. The latter, too, is administered through the UN and was voted into being by the General Assembly in 2015. Both 21 and 2030 seek centralised control of all resources and food right down to the raindrops falling on your own land. These are some of the demands of Agenda 21 established in 1992. See if you recognise this society emerging today:

- End national sovereignty
- State planning and management of all land resources, ecosystems, deserts, forests, mountains, oceans and fresh water; agriculture; rural development; biotechnology; and ensuring ‘*equity*’
- The state to ‘define the role’ of business and financial resources
- Abolition of private property
- ‘Restructuring’ the family unit (see BLM)
- Children raised by the state
- People told what their job will be
- Major restrictions on movement
- Creation of ‘human settlement zones’

- Mass resettlement as people are forced to vacate land where they live
- Dumbing down education
- Mass global depopulation in pursuit of all the above

The United Nations was created as a Trojan horse for world government. With the climate con of critical importance to promoting that outcome you would expect the UN to be involved. Oh, it's involved all right. The UN is promoting Agenda 21 and Agenda 2030 justified by 'climate change' while also driving the climate hoax through its Intergovernmental Panel on Climate Change (IPCC), one of the world's most corrupt organisations. The IPCC has been lying ferociously and constantly since the day it opened its doors with the global media hanging unquestioningly on its every mendacious word. The Green movement is entirely Woke and has long lost its original environmental focus since it was co-opted by the Cult. An obsession with 'global warming' has deleted its values and scrambled its head. I experienced a small example of what I mean on a beautiful country walk that I have enjoyed several times a week for many years. The path merged into the fields and forests and you felt at one with the natural world. Then a 'Green' organisation, the Hampshire and Isle of Wight Wildlife Trust, took over part of the land and proceeded to cut down a large number of trees, including mature ones, to install a horrible big, bright steel 'this-is-ours-stay-out' fence that destroyed the whole atmosphere of this beautiful place. No one with a feel for nature would do that. Day after day I walked to the sound of chainsaws and a magnificent mature weeping willow tree that I so admired was cut down at the base of the trunk. When I challenged a Woke young girl in a green shirt (of course) about this vandalism she replied: 'It's a weeping willow – it will grow back.' This is what people are paying for when they donate to the Hampshire and Isle of Wight Wildlife Trust and many other 'green' organisations today. It is not the environmental movement that I knew and instead has become a support-system – as with Extinction Rebellion – for a very dark agenda.

## **Private jets for climate justice**

The Cult-owned, Gates-funded, World Economic Forum and its founder Klaus Schwab were behind the emergence of Greta Thunberg to harness the young behind the climate agenda and she was invited to speak to the world at ... the UN. Schwab published a book, *Covid-19: The Great Reset* in 2020 in which he used the 'Covid' hoax and the climate hoax to lay out a new society straight out of Agenda 21 and Agenda 2030. Bill Gates followed in early 2021 when he took time out from destroying the world to produce a book in his name about the way to save it. Gates flies across the world in private jets and admitted that 'I probably have one of the highest greenhouse gas footprints of anyone on the planet ... my personal flying alone is gigantic.' He has also bid for the planet's biggest private jet operator. Other climate change saviours who fly in private jets include John Kerry, the US Special Presidential Envoy for Climate, and actor Leonardo DiCaprio, a 'UN Messenger of Peace with special focus on climate change'. These people are so full of bullshit they could corner the market in manure. We mustn't be sceptical, though, because the Gates book, *How to Avoid a Climate Disaster: The Solutions We Have and the Breakthroughs We Need*, is a genuine attempt to protect the world and not an obvious pile of excrement attributed to a mega-psychopath aimed at selling his masters' plans for humanity. The Gates book and the other shite-pile by Klaus Schwab could have been written by the same person and may well have been. Both use 'climate change' and 'Covid' as the excuses for their new society and by coincidence the Cult's World Economic Forum and Bill and Melinda Gates Foundation promote the climate hoax and hosted Event 201 which pre-empted with a 'simulation' the very 'coronavirus' hoax that would be simulated for real on humanity within weeks. The British 'royal' family is promoting the 'Reset' as you would expect through Prince 'climate change caused the war in Syria' Charles and his hapless son Prince William who said that we must 'reset our relationship with nature and our trajectory as a species' to avoid a climate disaster. Amazing how many promoters of the 'Covid' and 'climate change' control

systems are connected to Gates and the World Economic Forum. A 'study' in early 2021 claimed that carbon dioxide emissions must fall by the equivalent of a global lockdown roughly every two years for the next decade to save the planet. The 'study' appeared in the same period that the Schwab mob claimed in a video that lockdowns destroying the lives of billions are good because they make the earth 'quieter' with less 'ambient noise'. They took down the video amid a public backlash for such arrogant, empathy-deleted stupidity You see, however, where they are going with this. Corinne Le Quéré, a professor at the Tyndall Centre for Climate Change Research, University of East Anglia, was lead author of the climate lockdown study, and she writes for ... the World Economic Forum. Gates calls in 'his' book for changing 'every aspect of the economy' (long-time Cult agenda) and for humans to eat synthetic 'meat' (predicted in my books) while cows and other farm animals are eliminated.

Australian TV host and commentator Alan Jones described what carbon emission targets would mean for farm animals in Australia alone if emissions were reduced as demanded by 35 percent by 2030 and zero by 2050:

Well, let's take agriculture, the total emissions from agriculture are about 75 million tonnes of carbon dioxide, equivalent. Now reduce that by 35 percent and you have to come down to 50 million tonnes, I've done the maths. So if you take for example 1.5 million cows, you're going to have to reduce the herd by 525,000 [by] 2030, nine years, that's 58,000 cows a year. The beef herd's 30 million, reduce that by 35 percent, that's 10.5 million, which means 1.2 million cattle have to go every year between now and 2030. This is insanity!

There are 75 million sheep. Reduce that by 35 percent, that's 26 million sheep, that's almost 3 million a year. So under the Paris Agreement over 30 million beasts. dairy cows, cattle, pigs and sheep would go. More than 8,000 every minute of every hour for the next decade, do these people know what they're talking about?

Clearly they don't at the level of campaigners, politicians and administrators. The Cult *does* know; that's the outcome it wants. We are faced with not just a war on humanity. Animals and the natural world are being targeted and I have been saying since the 'Covid' hoax began that the plan eventually was to claim that the 'deadly virus' is able to jump from animals, including farm animals and

domestic pets, to humans. Just before this book went into production came this story: 'Russia registers world's first Covid-19 vaccine for cats & dogs as makers of Sputnik V warn pets & farm animals could spread virus'. The report said 'top scientists warned that the deadly pathogen could soon begin spreading through homes and farms' and 'the next stage is the infection of farm and domestic animals'. Know the outcome and you'll see the journey. Think what that would mean for animals and keep your eye on a term called zoonosis or zoonotic diseases which transmit between animals and humans. The Cult wants to break the connection between animals and people as it does between people and people. Farm animals fit with the Cult agenda to transform food from natural to synthetic.

### **The gas of life is killing us**

There can be few greater examples of Cult inversion than the condemnation of carbon dioxide as a dangerous pollutant when it is the gas of life. Without it the natural world would be dead and so we would all be dead. We breathe in oxygen and breathe out carbon dioxide while plants produce oxygen and absorb carbon dioxide. It is a perfect symbiotic relationship that the Cult wants to dismantle for reasons I will come to in the final two chapters. Gates, Schwab, other Cult operatives and mindless repeaters, want the world to be 'carbon neutral' by at least 2050 and the earlier the better. 'Zero carbon' is the cry echoed by lunatics calling for 'Zero Covid' when we already have it. These carbon emission targets will deindustrialise the world in accordance with Cult plans – the post-industrial, post-democratic society – and with so-called renewables like solar and wind not coming even close to meeting human energy needs blackouts and cold are inevitable. Texans got the picture in the winter of 2021 when a snow storm stopped wind turbines and solar panels from working and the lights went down along with water which relies on electricity for its supply system. Gates wants everything to be powered by electricity to ensure that his masters have the kill switch to stop all human activity, movement, cooking, water and warmth any time they like. The climate lie is so

stupendously inverted that it claims we must urgently reduce carbon dioxide when we *don't have enough*.

Co<sub>2</sub> in the atmosphere is a little above 400 parts per million when the optimum for plant growth is 2,000 ppm and when it falls anywhere near 150 ppm the natural world starts to die and so do we. It fell to as low as 280 ppm in an 1880 measurement in Hawaii and rose to 413 ppm in 2019 with industrialisation which is why the planet has become *greener* in the industrial period. How insane then that psychopathic madman Gates is not satisfied only with blocking the rise of Co<sub>2</sub>. He's funding technology to suck it out of the atmosphere. The reason why will become clear. The industrial era is not destroying the world through Co<sub>2</sub> and has instead turned around a potentially disastrous ongoing fall in Co<sub>2</sub>. Greenpeace co-founder and scientist Patrick Moore walked away from Greenpeace in 1986 and has exposed the green movement for fear-mongering and lies. He said that 500 million years ago there was *17 times* more Co<sub>2</sub> in the atmosphere than we have today and levels have been falling for hundreds of millions of years. In the last 150 million years Co<sub>2</sub> levels in Earth's atmosphere had reduced by *90 percent*. Moore said that by the time humanity began to unlock carbon dioxide from fossil fuels we were at '38 seconds to midnight' and in that sense: 'Humans are [the Earth's] salvation.' Moore made the point that only half the Co<sub>2</sub> emitted by fossil fuels stays in the atmosphere and we should remember that all pollution pouring from chimneys that we are told is carbon dioxide is in fact nothing of the kind. It's pollution. Carbon dioxide is an invisible gas.

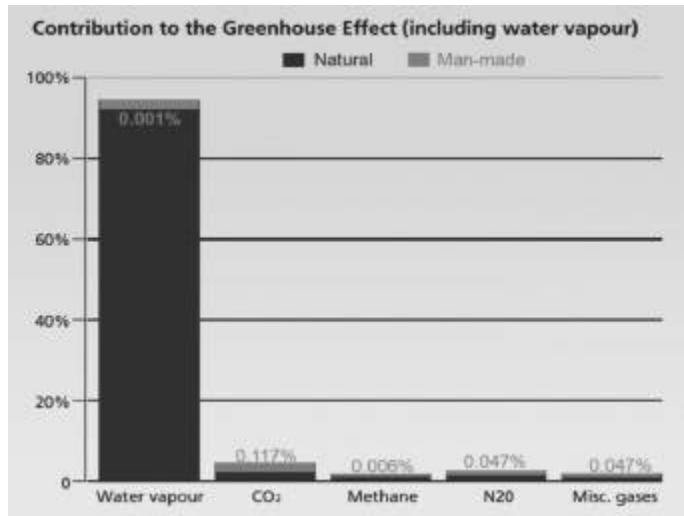
William Happer, Professor of Physics at Princeton University and long-time government adviser on climate, has emphasised the Co<sub>2</sub> deficiency for maximum growth and food production. Greenhouse growers don't add carbon dioxide for a bit of fun. He said that most of the warming in the last 100 years, after the earth emerged from the super-cold period of the 'Little Ice Age' into a natural warming cycle, was over by 1940. Happer said that a peak year for warming in 1988 can be explained by a 'monster El Nino' which is a natural and cyclical warming of the Pacific that has nothing to do with 'climate

change'. He said the effect of Co2 could be compared to painting a wall with red paint in that once two or three coats have been applied it didn't matter how much more you slapped on because the wall will not get much redder. Almost all the effect of the rise in Co2 has already happened, he said, and the volume in the atmosphere would now have to *double* to increase temperature by a single degree. Climate hoaxers know this and they have invented the most ridiculously complicated series of 'feedback' loops to try to overcome this rather devastating fact. You hear puppet Greta going on cluelessly about feedback loops and this is why.

### **The Sun affects temperature? No you *climate denier***

Some other nonsense to contemplate: Climate graphs show that rises in temperature do not follow rises in Co2 – *it's the other way round* with a lag between the two of some 800 years. If we go back 800 years from present time we hit the Medieval Warm Period when temperatures were higher than now without any industrialisation and this was followed by the Little Ice Age when temperatures plummeted. The world was still emerging from these centuries of serious cold when many climate records began which makes the ever-repeated line of the 'hottest year since records began' meaningless when you are not comparing like with like. The coldest period of the Little Ice Age corresponded with the lowest period of sunspot activity when the Sun was at its least active. Proper scientists will not be at all surprised by this when it confirms the obvious fact that earth temperature is affected by the scale of Sun activity and the energetic power that it subsequently emits; but when is the last time you heard a climate hoaxter talking about the Sun as a source of earth temperature?? Everything has to be focussed on Co2 which makes up just 0.117 percent of so-called greenhouse gases and only a fraction of even that is generated by human activity. The rest is natural. More than 90 percent of those greenhouse gases are water vapour and clouds ([Fig 9](#)). Ban moisture I say. Have you noticed that the climate hoaxers no longer use the polar bear as their promotion image? That's because far from becoming extinct polar

bear communities are stable or thriving. Joe Bastardi, American meteorologist, weather forecaster and outspoken critic of the climate lie, documents in his book *The Climate Chronicles* how weather patterns and events claimed to be evidence of climate change have been happening since long before industrialisation: 'What happened before naturally is happening again, as is to be expected given the cyclical nature of the climate due to the design of the planet.' If you read the detailed background to the climate hoax in my other books you will shake your head and wonder how anyone could believe the crap which has spawned a multi-trillion dollar industry based on absolute garbage (see HIV causes AIDS and Sars-Cov-2 causes 'Covid-19'). Climate and 'Covid' have much in common given they have the same source. They both have the contradictory *everything* factor in which everything is explained by reference to them. It's hot – 'it's climate change'. It's cold – 'it's climate change'. I got a sniffle – 'it's Covid'. I haven't got a sniffle – 'it's Covid'. Not having a sniffle has to be a symptom of 'Covid'. Everything is and not having a sniffle is especially dangerous if you are a slow walker. For sheer audacity I offer you a Cambridge University 'study' that actually linked 'Covid' to 'climate change'. It had to happen eventually. They concluded that climate change played a role in 'Covid-19' spreading from animals to humans because ... wait for it ... I kid you not ... *the two groups were forced closer together as populations grow.* Er, that's it. The whole foundation on which this depended was that 'Bats are the likely zoonotic origin of SARS-CoV-1 and SARS-CoV-2'. Well, they are not. They are nothing to do with it. Apart from bats not being the origin and therefore 'climate change' effects on bats being irrelevant I am in awe of their academic insight. Where would we be without them? Not where we are that's for sure.



**Figure 9:** The idea that the gas of life is disastrously changing the climate is an insult to brain cell activity.

One other point about the weather is that climate modification is now well advanced and not every major weather event is natural – or earthquake come to that. I cover this subject at some length in other books. China is openly planning a rapid expansion of its weather modification programme which includes changing the climate in an area more than one and a half times the size of India. China used weather manipulation to ensure clear skies during the 2008 Olympics in Beijing. I have quoted from US military documents detailing how to employ weather manipulation as a weapon of war and they did that in the 1960s and 70s during the conflict in Vietnam with Operation Popeye manipulating monsoon rains for military purposes. Why would there be international treaties on weather modification if it wasn't possible? Of course it is. Weather is energetic information and it can be changed.

## How was the climate hoax pulled off? See 'Covid'

If you can get billions to believe in a 'virus' that doesn't exist you can get them to believe in human-caused climate change that doesn't exist. Both are being used by the Cult to transform global society in the way it has long planned. Both hoaxes have been achieved in pretty much the same way. First you declare a lie is a fact. There's a

'virus' you call SARS-Cov-2 or humans are warming the planet with their behaviour. Next this becomes, via Cult networks, the foundation of government, academic and science policy and belief. Those who parrot the mantra are given big grants to produce research that confirms the narrative is true and ever more 'symptoms' are added to make the 'virus'/'climate change' sound even more scary. Scientists and researchers who challenge the narrative have their grants withdrawn and their careers destroyed. The media promote the lie as the unquestionable truth and censor those with an alternative view or evidence. A great percentage of the population believe what they are told as the lie becomes an everybody-knows-that and the believing-masses turn on those with a mind of their own. The technique has been used endlessly throughout human history. Wokers are the biggest promotorrs of the climate lie *and* 'Covid' fascism because their minds are owned by the Cult; their sense of self-righteous self-purity knows no bounds; and they exist in a bubble of reality in which facts are irrelevant and only get in the way of looking without seeing.

Running through all of this like veins in a blue cheese is control of information, which means control of perception, which means control of behaviour, which collectively means control of human society. The Cult owns the global media and Silicon Valley fascists for the simple reason that it *has* to. Without control of information it can't control perception and through that human society. Examine every facet of the Cult agenda and you will see that anything supporting its introduction is never censored while anything pushing back is always censored. I say again: Psychopaths that know why they are doing this must go before Nuremberg trials and those that follow their orders must trot along behind them into the same dock. 'I was just following orders' didn't work the first time and it must not work now. Nuremberg trials must be held all over the world before public juries for politicians, government officials, police, compliant doctors, scientists and virologists, and all Cult operatives such as Gates, Tedros, Fauci, Vallance, Whitty, Ferguson, Zuckerberg, Wojcicki, Brin, Page, Dorsey, the whole damn lot of

them – including, no *especially*, the psychopath psychologists. Without them and the brainless, gutless excuses for journalists that have repeated their lies, none of this could be happening. Nobody can be allowed to escape justice for the psychological and economic Armageddon they are all responsible for visiting upon the human race.

As for the compliant, unquestioning, swathes of humanity, and the self-obsessed, all-knowing ignorance of the Wokers ... don't start me. God help their kids. God help their grandkids. God *help them*.

## CHAPTER NINE

### We must have it? So what is it?

*Well I won't back down. No, I won't back down. You can stand me up at the Gates of Hell. But I won't back down*

Tom Petty

I will now focus on the genetically-manipulating ‘Covid vaccines’ which do not meet this official definition of a vaccine by the US Centers for Disease Control (CDC): ‘A product that stimulates a person’s immune system to produce immunity to a specific disease, protecting the person from that disease.’ On that basis ‘Covid vaccines’ are not a vaccine in that the makers don’t even claim they stop infection or transmission.

They are instead part of a multi-levelled conspiracy to change the nature of the human body and what it means to be ‘human’ and to depopulate an enormous swathe of humanity. What I shall call Human 1.0 is on the cusp of becoming Human 2.0 and for very sinister reasons. Before I get to the ‘Covid vaccine’ in detail here’s some background to vaccines in general. Government regulators do not test vaccines – the makers do – and the makers control which data is revealed and which isn’t. Children in America are given 50 vaccine doses by age six and 69 by age 19 and the effect of the whole combined schedule has never been tested. Autoimmune diseases when the immune system attacks its own body have soared in the mass vaccine era and so has disease in general in children and the young. Why wouldn’t this be the case when vaccines target the *immune system*? The US government gave Big Pharma drug

companies immunity from prosecution for vaccine death and injury in the 1986 National Childhood Vaccine Injury Act (NCVIA) and since then the government (taxpayer) has been funding compensation for the consequences of Big Pharma vaccines. The criminal and satanic drug giants can't lose and the vaccine schedule has increased dramatically since 1986 for this reason. There is no incentive to make vaccines safe and a big incentive to make money by introducing ever more. Even against a ridiculously high bar to prove vaccine liability, and with the government controlling the hearing in which it is being challenged for compensation, the vaccine court has so far paid out more than \$4 billion. These are the vaccines we are told are safe and psychopaths like Zuckerberg censor posts saying otherwise. The immunity law was even justified by a ruling that vaccines by their nature were 'unavoidably unsafe'.

Check out the ingredients of vaccines and you will be shocked if you are new to this. *They put that in children's bodies?? What??* Try aluminium, a brain toxin connected to dementia, aborted foetal tissue and formaldehyde which is used to embalm corpses. World-renowned aluminium expert Christopher Exley had his research into the health effect of aluminium in vaccines shut down by Keele University in the UK when it began taking funding from the Bill and Melinda Gates Foundation. Research when diseases 'eradicated' by vaccines began to decline and you will find the fall began long *before* the vaccine was introduced. Sometimes the fall even plateaued after the vaccine. Diseases like scarlet fever for which there was no vaccine declined in the same way because of environmental and other factors. A perfect case in point is the polio vaccine. Polio began when lead arsenate was first sprayed as an insecticide and residues remained in food products. Spraying started in 1892 and the first US polio epidemic came in Vermont in 1894. The simple answer was to stop spraying, but Rockefeller-created Big Pharma had a better idea. Polio was decreed to be caused by the *poliovirus* which 'spreads from person to person and can infect a person's spinal cord'. Lead arsenate was replaced by the lethal DDT which had the same effect of causing paralysis by damaging the brain and central nervous

system. Polio plummeted when DDT was reduced and then banned, but the vaccine is still given the credit for something it didn't do. Today by far the biggest cause of polio is the vaccines promoted by Bill Gates. Vaccine justice campaigner Robert Kennedy Jr, son of assassinated (by the Cult) US Attorney General Robert Kennedy, wrote:

In 2017, the World Health Organization (WHO) reluctantly admitted that the global explosion in polio is predominantly vaccine strain. The most frightening epidemics in Congo, Afghanistan, and the Philippines, are all linked to vaccines. In fact, by 2018, 70% of global polio cases were vaccine strain.

Vaccines make fortunes for Cult-owned Gates and Big Pharma while undermining the health and immune systems of the population. We had a glimpse of the mentality behind the Big Pharma cartel with a report on WION (World is One News), an international English language TV station based in India, which exposed the extraordinary behaviour of US drug company Pfizer over its 'Covid vaccine'. The WION report told how Pfizer had made fantastic demands of Argentina, Brazil and other countries in return for its 'vaccine'. These included immunity from prosecution, even for Pfizer negligence, government insurance to protect Pfizer from law suits and handing over as collateral sovereign assets of the country to include Argentina's bank reserves, military bases and embassy buildings. Pfizer demanded the same of Brazil in the form of waiving sovereignty of its assets abroad; exempting Pfizer from Brazilian laws; and giving Pfizer immunity from all civil liability. This is a 'vaccine' developed with government funding. Big Pharma is evil incarnate as a creation of the Cult and all must be handed tickets to Nuremberg.

### **Phantom 'vaccine' for a phantom 'disease'**

I'll expose the 'Covid vaccine' fraud and then go on to the wider background of why the Cult has set out to 'vaccinate' every man, woman and child on the planet for an alleged 'new disease' with a survival rate of 99.77 percent (or more) even by the grotesquely-

manipulated figures of the World Health Organization and Johns Hopkins University. The ‘infection’ to ‘death’ ratio is 0.23 to 0.15 percent according to Stanford epidemiologist Dr John Ioannidis and while estimates vary the danger remains tiny. I say that if the truth be told the fake infection to fake death ratio is zero. Never mind all the evidence I have presented here and in *The Answer* that there is no ‘virus’ let us just focus for a moment on that death-rate figure of say 0.23 percent. The figure includes all those worldwide who have tested positive with a test not testing for the ‘virus’ and then died within 28 days or even longer of any other cause – *any other cause*. Now subtract all those illusory ‘Covid’ deaths on the global data sheets from the 0.23 percent. What do you think you would be left with? *Zero*. A vaccination has never been successfully developed for a so-called coronavirus. They have all failed at the animal testing stage when they caused hypersensitivity to what they were claiming to protect against and made the impact of a disease far worse. Cult-owned vaccine corporations got around that problem this time by bypassing animal trials, going straight to humans and making the length of the ‘trials’ before the public rollout as short as they could get away with. Normally it takes five to ten years or more to develop vaccines that still cause demonstrable harm to many people and that’s without including the long-term effects that are never officially connected to the vaccination. ‘Covid’ non-vaccines have been officially produced and approved in a matter of months from a standing start and part of the reason is that (a) they were developed before the ‘Covid’ hoax began and (b) they are based on computer programs and not natural sources. Official non-trials were so short that government agencies gave *emergency*, not full, approval. ‘Trials’ were not even completed and full approval cannot be secured until they are. Public ‘Covid vaccination’ is actually a *continuation of the trial*. Drug company ‘trials’ are not scheduled to end until 2023 by which time a lot of people are going to be dead. Data on which government agencies gave this emergency approval was supplied by the Big Pharma corporations themselves in the form of Pfizer/BioNTech, AstraZeneca, Moderna, Johnson & Johnson, and

others, and this is the case with all vaccines. By its very nature *emergency* approval means drug companies do not have to prove that the ‘vaccine’ is ‘safe and effective’. How could they with trials way short of complete? Government regulators only have to *believe* that they *could* be safe and effective. It is criminal manipulation to get products in circulation with no testing worth the name. Agencies giving that approval are infested with Big Pharma-connected place-people and they act in the interests of Big Pharma (the Cult) and not the public about whom they do not give a damn.

## **More human lab rats**

‘Covid vaccines’ produced in record time by Pfizer/BioNTech and Moderna employ a technique *never approved before for use on humans*. They are known as mRNA ‘vaccines’ and inject a synthetic version of ‘viral’ mRNA or ‘messenger RNA’. The key is in the term ‘messenger’. The body works, or doesn’t, on the basis of information messaging. Communications are constantly passing between and within the genetic system and the brain. Change those messages and you change the state of the body and even its very nature and you can change psychology and behaviour by the way the brain processes information. I think you are going to see significant changes in personality and perception of many people who have had the ‘Covid vaccine’ synthetic potions. Insider Aldous Huxley predicted the following in 1961 and mRNA ‘vaccines’ can be included in the term ‘pharmacological methods’:

There will be, in the next generation or so, a pharmacological method of making people love their servitude, and producing dictatorship without tears, so to speak, producing a kind of painless concentration camp for entire societies, so that people will in fact have their own liberties taken away from them, but rather enjoy it, because they will be distracted from any desire to rebel by propaganda or brainwashing, or brainwashing enhanced by pharmacological methods. And this seems to be the final revolution.

Apologists claim that mRNA synthetic ‘vaccines’ don’t change the DNA genetic blueprint because RNA does not affect DNA only the other way round. This is so disingenuous. A process called ‘reverse

'transcription' can convert RNA into DNA and be integrated into DNA in the cell nucleus. This was highlighted in December, 2020, by scientists at Harvard and Massachusetts Institute of Technology (MIT). Geneticists report that more than 40 percent of mammalian genomes results from reverse transcription. On the most basic level if messaging changes then that sequence must lead to changes in DNA which is receiving and transmitting those communications. How can introducing synthetic material into cells not change the cells where DNA is located? The process is known as transfection which is defined as 'a technique to insert foreign nucleic acid (DNA or RNA) into a cell, typically with the intention of altering the properties of the cell'. Researchers at the Sloan Kettering Institute in New York found that changes in messenger RNA can deactivate tumour-suppressing proteins and thereby promote cancer. This is what happens when you mess with messaging. 'Covid vaccine' maker Moderna was founded in 2010 by Canadian stem cell biologist Derrick J. Rossi after his breakthrough discovery in the field of transforming and reprogramming stem cells. These are neutral cells that can be programmed to become any cell including sperm cells. Moderna was therefore founded on the principle of genetic manipulation and has never produced any vaccine or drug before its genetically-manipulating synthetic 'Covid' shite. Look at the name – Mode-RNA or Modify-RNA. Another important point is that the US Supreme Court has ruled that genetically-modified DNA, or complementary DNA (cDNA) synthesized in the laboratory from messenger RNA, can be patented and owned. These psychopaths are doing this to the human body.

Cells replicate synthetic mRNA in the 'Covid vaccines' and in theory the body is tricked into making antigens which trigger antibodies to target the 'virus spike proteins' which as Dr Tom Cowan said have *never been seen*. Cut the crap and these 'vaccines' deliver *self-replicating* synthetic material to the cells with the effect of changing human DNA. The more of them you have the more that process is compounded while synthetic material is all the time self-replicating. 'Vaccine'-maker Moderna describes mRNA as 'like

software for the cell' and so they are messing with the body's software. What happens when you change the software in a computer? Everything changes. For this reason the Cult is preparing a production line of mRNA 'Covid vaccines' and a long list of excuses to use them as with all the 'variants' of a 'virus' never shown to exist. The plan is further to transfer the mRNA technique to other vaccines mostly given to children and young people. The cumulative consequences will be a transformation of human DNA through a constant infusion of synthetic genetic material which will kill many and change the rest. Now consider that governments that have given emergency approval for a vaccine that's not a vaccine; never been approved for humans before; had no testing worth the name; and the makers have been given immunity from prosecution for any deaths or adverse effects suffered by the public. The UK government awarded *permanent legal indemnity* to itself and its employees for harm done when a patient is being treated for 'Covid-19' or 'suspected Covid-19'. That is quite a thought when these are possible 'side-effects' from the 'vaccine' (they are not 'side', they are effects) listed by the US Food and Drug Administration:

Guillain-Barre syndrome; acute disseminated encephalomyelitis; transverse myelitis; encephalitis; myelitis; encephalomyelitis; meningoencephalitis; meningitis; encephalopathy; convulsions; seizures; stroke; narcolepsy; cataplexy; anaphylaxis; acute myocardial infarction (heart attack); myocarditis; pericarditis; autoimmune disease; death; implications for pregnancy, and birth outcomes; other acute demyelinating diseases; non anaphylactic allergy reactions; thrombocytopenia ; disseminated intravascular coagulation; venous thromboembolism; arthritis; arthralgia; joint pain; Kawasaki disease; multisystem inflammatory syndrome in children; vaccine enhanced disease. The latter is the way the 'vaccine' has the potential to make diseases far worse than they would otherwise be.

UK doctor and freedom campaigner Vernon Coleman described the conditions in this list as 'all unpleasant, most of them very serious, and you can't get more serious than death'. The thought that anyone at all has had the 'vaccine' in these circumstances is testament to the potential that humanity has for clueless, unquestioning, stupidity and for many that programmed stupidity has already been terminal.

## An insider speaks

Dr Michael Yeadon is a former Vice President, head of research and Chief Scientific Adviser at vaccine giant Pfizer. Yeadon worked on the inside of Big Pharma, but that did not stop him becoming a vocal critic of 'Covid vaccines' and their potential for multiple harms, including infertility in women. By the spring of 2021 he went much further and even used the no, no, term 'conspiracy'. When you begin to see what is going on it is impossible not to do so. Yeadon spoke out in an interview with freedom campaigner James Delingpole and I mentioned earlier how he said that no one had samples of 'the virus'. He explained that the mRNA technique originated in the anti-cancer field and ways to turn on and off certain genes which could be advantageous if you wanted to stop cancer growing out of control. 'That's the origin of them. They are a very unusual application, really.' Yeadon said that treating a cancer patient with an aggressive procedure might be understandable if the alternative was dying, but it was quite another thing to use the same technique as a public health measure. Most people involved wouldn't catch the infectious agent you were vaccinating against and if they did they probably wouldn't die:

If you are really using it as a public health measure you really want to as close as you can get to zero side-effects ... I find it odd that they chose techniques that were really cutting their teeth in the field of oncology and I'm worried that in using gene-based vaccines that have to be injected in the body and spread around the body, get taken up into some cells, and the regulators haven't quite told us which cells they get taken up into ... you are going to be generating a wide range of responses ... with multiple steps each of which could go well or badly.

I doubt the Cult intends it to go well. Yeadon said that you can put any gene you like into the body through the 'vaccine'. 'You can certainly give them a gene that would do them some harm if you wanted.' I was intrigued when he said that when used in the cancer field the technique could turn genes on and off. I explore this process in *The Answer* and with different genes having different functions you could create mayhem – physically and psychologically – if you turned the wrong ones on and the right ones off. I read reports of an experiment by researchers at the University of Washington's school of computer science and engineering in which they encoded DNA to infect computers. The body is itself a biological computer and if human DNA can inflict damage on a computer why can't the computer via synthetic material mess with the human body? It can. The Washington research team said it was possible to insert malicious malware into 'physical DNA strands' and corrupt the computer system of a gene sequencing machine as it 'reads gene letters and stores them as binary digits 0 and 1'. They concluded that hackers could one day use blood or spit samples to access computer systems and obtain sensitive data from police forensics labs or infect genome files. It is at this level of digital interaction that synthetic 'vaccines' need to be seen to get the full picture and that will become very clear later on. Michael Yeadon said it made no sense to give the 'vaccine' to younger people who were in no danger from the 'virus'. What was the benefit? It was all downside with potential effects:

The fact that my government in what I thought was a civilised, rational country, is raining [the 'vaccine'] on people in their 30s and 40s, even my children in their 20s, they're getting letters and phone calls, I know this is not right and any of you doctors who are vaccinating you know it's not right, too. They are not at risk. They are not at risk from the disease, so you are now hoping that the side-effects are so rare that you get away with it. You don't give new technology ... that you don't understand to 100 percent of the population.

Blood clot problems with the AstraZeneca 'vaccine' have been affecting younger people to emphasise the downside risks with no benefit. AstraZeneca's version, produced with Oxford University, does not use mRNA, but still gets its toxic cocktail inside cells where

it targets DNA. The Johnson & Johnson ‘vaccine’ which uses a similar technique has also produced blood clot effects to such an extent that the United States paused its use at one point. They are all ‘gene therapy’ (cell modification) procedures and not ‘vaccines’. The truth is that once the content of these injections enter cells we have no idea what the effect will be. People can speculate and some can give very educated opinions and that’s good. In the end, though, only the makers know what their potions are designed to do and even they won’t know every last consequence. Michael Yeadon was scathing about doctors doing what they knew to be wrong.

‘Everyone’s mute’, he said. Doctors in the NHS must know this was not right, coming into work and injecting people. ‘I don’t know how they sleep at night. I know I couldn’t do it. I know that if I were in that position I’d have to quit.’ He said he knew enough about toxicology to know this was not a good risk-benefit. Yeadon had spoken to seven or eight university professors and all except two would not speak out publicly. Their universities had a policy that no one said anything that countered the government and its medical advisors. They were afraid of losing their government grants. This is how intimidation has been used to silence the truth at every level of the system. I say silence, but these people could still speak out if they made that choice. Yeadon called them ‘moral cowards’ – ‘This is about your children and grandchildren’s lives and you have just buggered off and left it.’

## **‘Variant’ nonsense**

Some of his most powerful comments related to the alleged ‘variants’ being used to instil more fear, justify more lockdowns, and introduce more ‘vaccines’. He said government claims about ‘variants’ were nonsense. He had checked the alleged variant ‘codes’ and they were 99.7 percent identical to the ‘original’. This was the human identity difference equivalent to putting a baseball cap on and off or wearing it the other way round. A 0.3 percent difference would make it impossible for that ‘variant’ to escape immunity from the ‘original’. This made no sense of having new ‘vaccines’ for

'variants'. He said there would have to be at least a *30 percent* difference for that to be justified and even then he believed the immune system would still recognise what it was. Gates-funded 'variant modeller' and 'vaccine'-pusher John Edmunds might care to comment. Yeadon said drug companies were making new versions of the 'vaccine' as a 'top up' for 'variants'. Worse than that, he said, the 'regulators' around the world like the MHRA in the UK had got together and agreed that because 'vaccines' for 'variants' were so similar to the first 'vaccines' *they did not have to do safety studies*. How transparently sinister that is. This is when Yeadon said: 'There is a conspiracy here.' There was no need for another vaccine for 'variants' and yet we were told that there was and the country had shut its borders because of them. 'They are going into hundreds of millions of arms without passing 'go' or any regulator. Why did they do that? Why did they pick this method of making the vaccine?'

The reason had to be something bigger than that it seemed and 'it's not protection against the virus'. It's was a far bigger project that meant politicians and advisers were willing to do things and not do things that knowingly resulted in avoidable deaths – 'that's already happened when you think about lockdown and deprivation of health care for a year.' He spoke of people prepared to do something that results in the avoidable death of their fellow human beings and it not bother them. This is the penny-drop I have been working to get across for more than 30 years – the level of pure evil we are dealing with. Yeadon said his friends and associates could not believe there could be that much evil, but he reminded them of Stalin, Pol Pot and Hitler and of what Stalin had said: 'One death is a tragedy. A million? A statistic.' He could not think of a benign explanation for why you need top-up vaccines 'which I'm sure you don't' and for the regulators 'to just get out of the way and wave them through'. Why would the regulators do that when they were still wrestling with the dangers of the 'parent' vaccine? He was clearly shocked by what he had seen since the 'Covid' hoax began and now he was thinking the previously unthinkable:

If you wanted to depopulate a significant proportion of the world and to do it in a way that doesn't involve destruction of the environment with nuclear weapons, poisoning everyone with anthrax or something like that, and you wanted plausible deniability while you had a multi-year infectious disease crisis, I actually don't think you could come up with a better plan of work than seems to be in front of me. I can't say that's what they are going to do, but I can't think of a benign explanation why they are doing it.

He said he never thought that they would get rid of 99 percent of humans, but now he wondered. 'If you wanted to that this would be a hell of a way to do it – it would be unstoppable folks.' Yeadon had concluded that those who submitted to the 'vaccine' would be allowed to have some kind of normal life (but for how long?) while screws were tightened to coerce and mandate the last few percent. 'I think they'll put the rest of them in a prison camp. I wish I was wrong, but I don't think I am.' Other points he made included: There were no coronavirus vaccines then suddenly they all come along at the same time; we have no idea of the long term affect with trials so short; coercing or forcing people to have medical procedures is against the Nuremberg Code instigated when the Nazis did just that; people should at least delay having the 'vaccine'; a quick Internet search confirms that masks don't reduce respiratory viral transmission and 'the government knows that'; they have smashed civil society and they know that, too; two dozen peer-reviewed studies show no connection between lockdown and reducing deaths; he knew from personal friends the elite were still flying around and going on holiday while the public were locked down; the elite were not having the 'vaccines'. He was also asked if 'vaccines' could be made to target difference races. He said he didn't know, but the document by the Project for the New American Century in September, 2000, said developing 'advanced forms of biological warfare that can target *specific genotypes* may transform biological warfare from the realm of terror to a politically useful tool.' Oh, they're evil all right. Of that we can be *absolutely* sure.

## **Another cull of old people**

We have seen from the CDC definition that the mRNA 'Covid vaccine' is not a vaccine and nor are the others that *claim* to reduce 'severity of symptoms' in *some* people, but not protect from infection or transmission. What about all the lies about returning to 'normal' if people were 'vaccinated'? If they are not claimed to stop infection and transmission of the alleged 'virus', how does anything change? This was all lies to manipulate people to take the jabs and we are seeing that now with masks and distancing still required for the 'vaccinated'. How did they think that elderly people with fragile health and immune responses were going to be affected by infusing their cells with synthetic material and other toxic substances? They *knew* that in the short and long term it would be devastating and fatal as the culling of the old that began with the first lockdowns was continued with the 'vaccine'. Death rates in care homes soared immediately residents began to be 'vaccinated' – infused with synthetic material. Brave and committed whistleblower nurses put their careers at risk by exposing this truth while the rest kept their heads down and their mouths shut to put their careers before those they are supposed to care for. A long-time American Certified Nursing Assistant who gave his name as James posted a video in which he described emotionally what happened in his care home when vaccination began. He said that during 2020 very few residents were sick with 'Covid' and no one died during the entire year; but shortly after the Pfizer mRNA injections 14 people died within two weeks and many others were near death. 'They're dropping like flies', he said. Residents who walked on their own before the shot could no longer and they had lost their ability to conduct an intelligent conversation. The home's management said the sudden deaths were caused by a 'super-spreader' of 'Covid-19'. Then how come, James asked, that residents who refused to take the injections were not sick? It was a case of inject the elderly with mRNA synthetic potions and blame their illness and death that followed on the 'virus'. James described what was happening in care homes as 'the greatest crime of genocide this country has ever seen'. Remember the NHS staff nurse from earlier who used the same

word ‘genocide’ for what was happening with the ‘vaccines’ and that it was an ‘act of human annihilation’. A UK care home whistleblower told a similar story to James about the effect of the ‘vaccine’ in deaths and ‘outbreaks’ of illness dubbed ‘Covid’ after getting the jab. She told how her care home management and staff had zealously imposed government regulations and no one was allowed to even question the official narrative let alone speak out against it. She said the NHS was even worse. Again we see the results of reframing. A worker at a local care home where I live said they had not had a single case of ‘Covid’ there for almost a year and when the residents were ‘vaccinated’ they had 19 positive cases in two weeks with eight dying.

### **It's not the 'vaccine' – honest**

The obvious cause and effect was being ignored by the media and most of the public. Australia’s health minister Greg Hunt (a former head of strategy at the World Economic Forum) was admitted to hospital after he had the ‘vaccine’. He was suffering according to reports from the skin infection ‘cellulitis’ and it must have been a severe case to have warranted days in hospital. Immediately the authorities said this was nothing to do with the ‘vaccine’ when an effect of some vaccines is a ‘cellulitis-like reaction’. We had families of perfectly healthy old people who died after the ‘vaccine’ saying that if only they had been given the ‘vaccine’ earlier they would still be alive. As a numbskull rating that is off the chart. A father of four ‘died of Covid’ at aged 48 when he was taken ill two days after having the ‘vaccine’. The man, a health administrator, had been ‘shielding during the pandemic’ and had ‘not really left the house’ until he went for the ‘vaccine’. Having the ‘vaccine’ and then falling ill and dying does not seem to have qualified as a possible cause and effect and ‘Covid-19’ went on his death certificate. His family said they had no idea how he ‘caught the virus’. A family member said: ‘Tragically, it could be that going for a vaccination ultimately led to him catching Covid ...The sad truth is that they are never going to know where it came from.’ The family warned people to remember

that the virus still existed and was 'very real'. So was their stupidity. Nurses and doctors who had the first round of the 'vaccine' were collapsing, dying and ending up in a hospital bed while they or their grieving relatives were saying they'd still have the 'vaccine' again despite what happened. I kid you not. You mean if your husband returned from the dead he'd have the same 'vaccine' again that killed him??

Doctors at the VCU Medical Center in Richmond, Virginia, said the Johnson & Johnson 'vaccine' was to blame for a man's skin peeling off. Patient Richard Terrell said: 'It all just happened so fast. My skin peeled off. It's still coming off on my hands now.' He said it was stinging, burning and itching and when he bent his arms and legs it was very painful with 'the skin swollen and rubbing against itself'. Pfizer/BioNTech and Moderna vaccines use mRNA to change the cell while the Johnson & Johnson version uses DNA in a process similar to AstraZeneca's technique. Johnson & Johnson and AstraZeneca have both had their 'vaccines' paused by many countries after causing serious blood problems. Terrell's doctor Fnu Nutan said he could have died if he hadn't got medical attention. It sounds terrible so what did Nutan and Terrell say about the 'vaccine' now? Oh, they still recommend that people have it. A nurse in a hospital bed 40 minutes after the vaccination and unable to swallow due to throat swelling was told by a doctor that he lost mobility in his arm for 36 hours following the vaccination. What did he say to the ailing nurse? 'Good for you for getting the vaccination.' We are dealing with a serious form of cognitive dissonance madness in both public and medical staff. There is a remarkable correlation between those having the 'vaccine' and trumpeting the fact and suffering bad happenings shortly afterwards. Witold Rogiewicz, a Polish doctor, made a video of his 'vaccination' and ridiculed those who were questioning its safety and the intentions of Bill Gates: 'Vaccinate yourself to protect yourself, your loved ones, friends and also patients. And to mention quickly I have info for anti-vaxxers and anti-Covidiers if you want to contact Bill Gates you can do this through me.' He further ridiculed the dangers of 5G. Days later he

was dead, but naturally the vaccination wasn't mentioned in the verdict of 'heart attack'.

## **Lies, lies and more lies**

So many members of the human race have slipped into extreme states of insanity and unfortunately they include reframed doctors and nursing staff. Having a 'vaccine' and dying within minutes or hours is not considered a valid connection while death from any cause within 28 days or longer of a positive test with a test not testing for the 'virus' means 'Covid-19' goes on the death certificate. How could that 'vaccine'-death connection not have been made except by calculated deceit? US figures in the initial rollout period to February 12th, 2020, revealed that a third of the deaths reported to the CDC after 'Covid vaccines' happened within 48 hours. Five men in the UK suffered an 'extremely rare' blood clot problem after having the AstraZeneca 'vaccine', but no causal link was established said the Gates-funded Medicines and Healthcare products Regulatory Agency (MHRA) which had given the 'vaccine' emergency approval to be used. Former Pfizer executive Dr Michael Yeadon explained in his interview how the procedures could cause blood coagulation and clots. People who should have been at no risk were dying from blood clots in the brain and he said he had heard from medical doctor friends that people were suffering from skin bleeding and massive headaches. The AstraZeneca 'shot' was stopped by some 20 countries over the blood clotting issue and still the corrupt MHRA, the European Medicines Agency (EMA) and the World Health Organization said that it should continue to be given even though the EMA admitted that it 'still cannot rule out definitively' a link between blood clotting and the 'vaccine'. Later Marco Cavaleri, head of EMA vaccine strategy, said there was indeed a clear link between the 'vaccine' and thrombosis, but they didn't know why. So much for the trials showing the 'vaccine' is safe. Blood clots were affecting younger people who would be under virtually no danger from 'Covid' even if it existed which makes it all the more stupid and sinister.

The British government responded to public alarm by wheeling out June Raine, the terrifyingly weak infant school headmistress sound-alike who heads the UK MHRA drug ‘regulator’. The idea that she would stand up to Big Pharma and government pressure is laughable and she told us that all was well in the same way that she did when allowing untested, never-used-on-humans-before, genetically-manipulating ‘vaccines’ to be exposed to the public in the first place. Mass lying is the new normal of the ‘Covid’ era. The MHRA later said 30 cases of rare blood clots had by then been connected with the AstraZeneca ‘vaccine’ (that means a lot more in reality) while stressing that the benefits of the jab in preventing ‘Covid-19’ outweighed any risks. A more ridiculous and disingenuous statement with callous disregard for human health it is hard to contemplate. Immediately after the mendacious ‘all-clears’ two hospital workers in Denmark experienced blood clots and cerebral haemorrhaging following the AstraZeneca jab and one died. Top Norwegian health official Pål Andre Holme said the ‘vaccine’ was the only common factor: ‘There is nothing in the patient history of these individuals that can give such a powerful immune response ... I am confident that the antibodies that we have found are the cause, and I see no other explanation than it being the vaccine which triggers it.’ Strokes, a clot or bleed in the brain, were clearly associated with the ‘vaccine’ from word of mouth and whistleblower reports. Similar consequences followed with all these ‘vaccines’ that we were told were so safe and as the numbers grew by the day it was clear we were witnessing human carnage.

## **Learning the hard way**

A woman interviewed by UKColumn told how her husband suffered dramatic health effects after the vaccine when he’d been in good health all his life. He went from being a little unwell to losing all feeling in his legs and experiencing ‘excruciating pain’. Misdiagnosis followed twice at Accident and Emergency (an ‘allergy’ and ‘sciatica’) before he was admitted to a neurology ward where doctors said his serious condition had been caused by the

'vaccine'. Another seven 'vaccinated' people were apparently being treated on the same ward for similar symptoms. The woman said he had the 'vaccine' because they believed media claims that it was safe. 'I didn't think the government would give out a vaccine that does this to somebody; I believed they would be bringing out a vaccination that would be safe.' What a tragic way to learn that lesson. Another woman posted that her husband was transporting stroke patients to hospital on almost every shift and when he asked them if they had been 'vaccinated' for 'Covid' they all replied 'yes'. One had a 'massive brain bleed' the day after his second dose. She said her husband reported the 'just been vaccinated' information every time to doctors in A and E only for them to ignore it, make no notes and appear annoyed that it was even mentioned. This particular report cannot be verified, but it expresses a common theme that confirms the monumental underreporting of 'vaccine' consequences. Interestingly as the 'vaccines' and their brain blood clot/stroke consequences began to emerge the UK National Health Service began a publicity campaign telling the public what to do in the event of a stroke. A Scottish NHS staff nurse who quit in disgust in March, 2021, said:

I have seen traumatic injuries from the vaccine, they're not getting reported to the yellow card [adverse reaction] scheme, they're treating the symptoms, not asking why, why it's happening. It's just treating the symptoms and when you speak about it you're dismissed like you're crazy, I'm not crazy, I'm not crazy because every other colleague I've spoken to is terrified to speak out, they've had enough.

Videos appeared on the Internet of people uncontrollably shaking after the 'vaccine' with no control over muscles, limbs and even their face. A Scottish mother broke out in a severe rash all over her body almost immediately after she was given the AstraZeneca 'vaccine'. The pictures were horrific. Leigh King, a 41-year-old hairdresser from Lanarkshire said: 'Never in my life was I prepared for what I was about to experience ... My skin was so sore and constantly hot ... I have never felt pain like this ...' But don't you worry, the 'vaccine' is perfectly safe. Then there has been the effect on medical

staff who have been pressured to have the ‘vaccine’ by psychopathic ‘health’ authorities and government. A London hospital consultant who gave the name K. Polyakova wrote this to the *British Medical Journal* or *BMJ*:

I am currently struggling with ... the failure to report the reality of the morbidity caused by our current vaccination program within the health service and staff population. The levels of sickness after vaccination is unprecedented and staff are getting very sick and some with neurological symptoms which is having a huge impact on the health service function. Even the young and healthy are off for days, some for weeks, and some requiring medical treatment. Whole teams are being taken out as they went to get vaccinated together.

Mandatory vaccination in this instance is stupid, unethical and irresponsible when it comes to protecting our staff and public health. We are in the voluntary phase of vaccination, and encouraging staff to take an unlicensed product that is impacting on their immediate health ... it is clearly stated that these vaccine products do not offer immunity or stop transmission. In which case why are we doing it?

Not to protect health that’s for sure. Medical workers are lauded by governments for agenda reasons when they couldn’t give a toss about them any more than they can for the population in general. Schools across America faced the same situation as they closed due to the high number of teachers and other staff with bad reactions to the Pfizer/BioNTech, Moderna, and Johnson & Johnson ‘Covid vaccines’ all of which were linked to death and serious adverse effects. The *BMJ* took down the consultant’s comments pretty quickly on the grounds that they were being used to spread ‘disinformation’. They were exposing the truth about the ‘vaccine’ was the real reason. The cover-up is breathtaking.

## **Hiding the evidence**

The scale of the ‘vaccine’ death cover-up worldwide can be confirmed by comparing official figures with the personal experience of the public. I heard of many people in my community who died immediately or soon after the vaccine that would never appear in the media or even likely on the official totals of ‘vaccine’ fatalities and adverse reactions when only about ten percent are estimated to be

reported and I have seen some estimates as low as one percent in a Harvard study. In the UK alone by April 29th, 2021, some 757,654 adverse reactions had been officially reported from the Pfizer/BioNTech, Oxford/AstraZeneca and Moderna 'vaccines' with more than a thousand deaths linked to jabs and that means an estimated ten times this number in reality from a ten percent reporting rate percentage. That's seven million adverse reactions and 10,000 potential deaths and a one percent reporting rate would be ten times *those* figures. In 1976 the US government pulled the swine flu vaccine after 53 deaths. The UK data included a combined 10,000 eye disorders from the 'Covid vaccines' with more than 750 suffering visual impairment or blindness and again multiply by the estimated reporting percentages. As 'Covid cases' officially fell hospitals virtually empty during the 'Covid crisis' began to fill up with a range of other problems in the wake of the 'vaccine' rollout. The numbers across America have also been catastrophic. Deaths linked to *all* types of vaccine increased by *6,000 percent* in the first quarter of 2021 compared with 2020. A 39-year-old woman from Ogden, Utah, died four days after receiving a second dose of Moderna's 'Covid vaccine' when her liver, heart and kidneys all failed despite the fact that she had no known medical issues or conditions. Her family sought an autopsy, but Dr Erik Christensen, Utah's chief medical examiner, said proving vaccine injury as a cause of death almost never happened. He could think of only one instance where an autopsy would name a vaccine as the official cause of death and that would be anaphylaxis where someone received a vaccine and died almost instantaneously. 'Short of that, it would be difficult for us to definitively say this is the vaccine,' Christensen said. If that is true this must be added to the estimated ten percent (or far less) reporting rate of vaccine deaths and serious reactions and the conclusion can only be that vaccine deaths and serious reactions – including these 'Covid' potions – are phenomenally understated in official figures. The same story can be found everywhere. Endless accounts of deaths and serious reactions among the public, medical

and care home staff while official figures did not even begin to reflect this.

Professional script-reader Dr David Williams, a ‘top public-health official’ in Ontario, Canada, insulted our intelligence by claiming only four serious adverse reactions and no deaths from the more than 380,000 vaccine doses then given. This bore no resemblance to what people knew had happened in their own circles and we had Dirk Huyer in charge of getting millions vaccinated in Ontario while at the same time he was Chief Coroner for the province investigating causes of death including possible death from the vaccine. An aide said he had stepped back from investigating deaths, but evidence indicated otherwise. Rosemary Frei, who secured a Master of Science degree in molecular biology at the Faculty of Medicine at Canada’s University of Calgary before turning to investigative journalism, was one who could see that official figures for ‘vaccine’ deaths and reactions made no sense. She said that doctors seldom reported adverse events and when people got really sick or died after getting a vaccination they would attribute that to anything except the vaccines. It had been that way for years and anyone who wondered aloud whether the ‘Covid vaccines’ or other shots cause harm is immediately branded as ‘anti-vax’ and ‘anti-science’. This was ‘career-threatening’ for health professionals. Then there was the huge pressure to support the push to ‘vaccinate’ billions in the quickest time possible. Frei said:

So that’s where we’re at today. More than half a million vaccine doses have been given to people in Ontario alone. The rush is on to vaccinate all 15 million of us in the province by September. And the mainstream media are screaming for this to be sped up even more. That all adds up to only a very slim likelihood that we’re going to be told the truth by officials about how many people are getting sick or dying from the vaccines.

What is true of Ontario is true of everywhere.

## **They KNEW – and still did it**

The authorities knew what was going to happen with multiple deaths and adverse reactions. The UK government’s Gates-funded

and Big Pharma-dominated Medicines and Healthcare products Regulatory Agency (MHRA) hired a company to employ AI in compiling the projected reactions to the ‘vaccine’ that would otherwise be uncountable. The request for applications said: ‘The MHRA urgently seeks an Artificial Intelligence (AI) software tool to process the expected high volume of Covid-19 vaccine Adverse Drug Reaction ...’ This was from the agency, headed by the disingenuous June Raine, that gave the ‘vaccines’ emergency approval and the company was hired before the first shot was given. ‘We are going to kill and maim you – is that okay?’ ‘Oh, yes, perfectly fine – I’m very grateful, thank you, doctor.’ The range of ‘Covid vaccine’ adverse reactions goes on for page after page in the MHRA criminally underreported ‘Yellow Card’ system and includes affects to eyes, ears, skin, digestion, blood and so on. Raine’s MHRA amazingly claimed that the ‘overall safety experience ... is so far as expected from the clinical trials’. The death, serious adverse effects, deafness and blindness were *expected*? When did they ever mention that? If these human tragedies were expected then those that gave approval for the use of these ‘vaccines’ must be guilty of crimes against humanity including murder – a definition of which is ‘killing a person with malice aforethought or with recklessness manifesting extreme indifference to the value of human life.’ People involved at the MHRA, the CDC in America and their equivalent around the world must go before Nuremberg trials to answer for their callous inhumanity. We are only talking here about the immediate effects of the ‘vaccine’. The longer-term impact of the DNA synthetic manipulation is the main reason they are so hysterically desperate to inoculate the entire global population in the shortest possible time.

Africa and the developing world are a major focus for the ‘vaccine’ depopulation agenda and a mass vaccination sales-pitch is underway thanks to caring people like the Rockefellers and other Cult assets. The Rockefeller Foundation, which pre-empted the ‘Covid pandemic’ in a document published in 2010 that ‘predicted’ what happened a decade later, announced an initial \$34.95 million grant in February, 2021, ‘to ensure more equitable access to Covid-19

testing and vaccines' among other things in Africa in collaboration with '24 organizations, businesses, and government agencies'. The pan-Africa initiative would focus on 10 countries: Burkina Faso, Ethiopia, Ghana, Kenya, Nigeria, Rwanda, South Africa, Tanzania, Uganda, and Zambia'. Rajiv Shah, President of the Rockefeller Foundation and former administrator of CIA-controlled USAID, said that if Africa was not mass-vaccinated (to change the DNA of its people) it was a 'threat to all of humanity' and not fair on Africans. When someone from the Rockefeller Foundation says they want to do something to help poor and deprived people and countries it is time for a belly-laugh. They are doing this out of the goodness of their 'heart' because 'vaccinating' the entire global population is what the 'Covid' hoax set out to achieve. Official 'decolonisation' of Africa by the Cult was merely a prelude to financial colonisation on the road to a return to physical colonisation. The 'vaccine' is vital to that and the sudden and convenient death of the 'Covid' sceptic president of Tanzania can be seen in its true light. A lot of people in Africa are aware that this is another form of colonisation and exploitation and they need to stand their ground.

### **The 'vaccine is working' scam**

A potential problem for the Cult was that the 'vaccine' is meant to change human DNA and body messaging and not to protect anyone from a 'virus' never shown to exist. The vaccine couldn't work because it was not designed to work and how could they make it *appear* to be working so that more people would have it? This was overcome by lowering the amplification rate of the PCR test to produce fewer 'cases' and therefore fewer 'deaths'. Some of us had been pointing out since March, 2020, that the amplification rate of the test not testing for the 'virus' had been made artificially high to generate positive tests which they could call 'cases' to justify lockdowns. The World Health Organization recommended an absurdly high 45 amplification cycles to ensure the high positives required by the Cult and then remained silent on the issue until January 20th, 2021 – Biden's Inauguration Day. This was when the

'vaccinations' were seriously underway and on that day the WHO recommended after discussions with America's CDC that laboratories *lowered their testing amplification*. Dr David Samadi, a certified urologist and health writer, said the WHO was encouraging all labs to reduce their cycle count for PCR tests. He said the current cycle was much too high and was 'resulting in any particle being declared a positive case'. Even one mainstream news report I saw said this meant the number of 'Covid' infections may have been 'dramatically inflated'. Oh, just a little bit. The CDC in America issued new guidance to laboratories in April, 2021, to use 28 cycles *but only for 'vaccinated' people*. The timing of the CDC/WHO interventions were cynically designed to make it appear the 'vaccines' were responsible for falling cases and deaths when the real reason can be seen in the following examples. New York's state lab, the Wadsworth Center, identified 872 positive tests in July, 2020, based on a threshold of 40 cycles. When the figure was lowered to 35 cycles *43 percent* of the 872 were no longer 'positives'. At 30 cycles the figure was 63 percent. A Massachusetts lab found that between *85 to 90 percent* of people who tested positive in July with a cycle threshold of 40 would be negative at 30 cycles, Ashish Jha, MD, director of the Harvard Global Health Institute, said: 'I'm really shocked that it could be that high ... Boy, does it really change the way we need to be thinking about testing.' I'm shocked that I could see the obvious in the spring of 2020, with no medical background, and most medical professionals still haven't worked it out. No, that's not shocking – it's terrifying.

Three weeks after the WHO directive to lower PCR cycles the London *Daily Mail* ran this headline: 'Why ARE Covid cases plummeting? New infections have fallen 45% in the US and 30% globally in the past 3 weeks but experts say vaccine is NOT the main driver because only 8% of Americans and 13% of people worldwide have received their first dose.' They acknowledged that the drop could not be attributed to the 'vaccine', but soon this morphed throughout the media into the 'vaccine' has caused cases and deaths to fall when it was the PCR threshold. In December, 2020, there was

chaos at English Channel ports with truck drivers needing negative 'Covid' tests before they could board a ferry home for Christmas. The government wanted to remove the backlog as fast as possible and they brought in troops to do the 'testing'. Out of 1,600 drivers just 36 tested positive and the rest were given the all clear to cross the Channel. I guess the authorities thought that 36 was the least they could get away with without the unquestioning catching on. The amplification trick which most people believed in the absence of information in the mainstream applied more pressure on those refusing the 'vaccine' to succumb when it 'obviously worked'. The truth was the exact opposite with deaths in care homes soaring with the 'vaccine' and in Israel the term used was 'skyrocket'. A re-analysis of published data from the Israeli Health Ministry led by Dr Hervé Seligmann at the Medicine Emerging Infectious and Tropical Diseases at Aix-Marseille University found that Pfizer's 'Covid vaccine' killed 'about 40 times more [elderly] people than the disease itself would have killed' during a five-week vaccination period and 260 *times* more younger people than would have died from the 'virus' even according to the manipulated 'virus' figures. Dr Seligmann and his co-study author, Haim Yativ, declared after reviewing the Israeli 'vaccine' death data: 'This is a new Holocaust.'

Then, in mid-April, 2021, after vast numbers of people worldwide had been 'vaccinated', the story changed with clear coordination. The UK government began to prepare the ground for more future lockdowns when Nuremberg-destined Boris Johnson told yet another whopper. He said that cases had fallen because of *lockdowns* not 'vaccines'. Lockdowns are irrelevant when *there is no 'virus'* and the test and fraudulent death certificates are deciding the number of 'cases' and 'deaths'. Study after study has shown that lockdowns don't work and instead kill and psychologically destroy people. Meanwhile in the United States Anthony Fauci and Rochelle Walensky, the ultra-Zionist head of the CDC, peddled the same line. More lockdown was the answer and not the 'vaccine', a line repeated on cue by the moron that is Canadian Prime Minister Justin Trudeau. Why all the hysteria to get everyone 'vaccinated' if lockdowns and

not ‘vaccines’ made the difference? None of it makes sense on the face of it. Oh, but it does. The Cult wants lockdowns *and* the ‘vaccine’ and if the ‘vaccine’ is allowed to be seen as the total answer lockdowns would no longer be justified when there are still livelihoods to destroy. ‘Variants’ and renewed upward manipulation of PCR amplification are planned to instigate never-ending lockdown *and* more ‘vaccines’.

## **You must have it – we’re desperate**

Israel, where the Jewish and Arab population are ruled by the Sabbatian Cult, was the front-runner in imposing the DNA-manipulating ‘vaccine’ on its people to such an extent that Jewish refusers began to liken what was happening to the early years of Nazi Germany. This would seem to be a fantastic claim. Why would a government of Jewish people be acting like the Nazis did? If you realise that the Sabbatian Cult was behind the Nazis and that Sabbatians hate Jews the pieces start to fit and the question of why a ‘Jewish’ government would treat Jews with such callous disregard for their lives and freedom finds an answer. Those controlling the government of Israel *aren’t Jewish* – they’re Sabbatian. Israeli lawyer Tamir Turgal was one who made the Nazi comparison in comments to German lawyer Reiner Fuellmich who is leading a class action lawsuit against the psychopaths for crimes against humanity. Turgal described how the Israeli government was vaccinating children and pregnant women on the basis that there was no evidence that this was dangerous when they had no evidence that it *wasn’t* dangerous either. They just had no evidence. This was medical experimentation and Turgal said this breached the Nuremberg Code about medical experimentation and procedures requiring informed consent and choice. Think about that. A Nuremberg Code developed because of Nazi experimentation on Jews and others in concentration camps by people like the evil-beyond-belief Josef Mengele is being breached by the *Israeli* government; but when you know that it’s a *Sabbatian* government along with its intelligence and military agencies like Mossad, Shin Bet and the Israeli Defense Forces, and that Sabbatians

were the force behind the Nazis, the kaleidoscope comes into focus. What have we come to when Israeli Jews are suing their government for violating the Nuremberg Code by essentially making Israelis subject to a medical experiment using the controversial 'vaccines'? It's a shocker that this has to be done in the light of what happened in Nazi Germany. The Anshe Ha-Emet, or 'People of the Truth', made up of Israeli doctors, lawyers, campaigners and public, have launched a lawsuit with the International Criminal Court. It says:

When the heads of the Ministry of Health as well as the prime minister presented the vaccine in Israel and began the vaccination of Israeli residents, the vaccinated were not advised, that, in practice, they are taking part in a medical experiment and that their consent is required for this under the Nuremberg Code.

The irony is unbelievable, but easily explained in one word: Sabbatians. The foundation of Israeli 'Covid' apartheid is the 'green pass' or 'green passport' which allows Jews and Arabs who have had the DNA-manipulating 'vaccine' to go about their lives – to work, fly, travel in general, go to shopping malls, bars, restaurants, hotels, concerts, gyms, swimming pools, theatres and sports venues, while non-'vaccinated' are banned from all those places and activities. Israelis have likened the 'green pass' to the yellow stars that Jews in Nazi Germany were forced to wear – the same as the yellow stickers that a branch of UK supermarket chain Morrisons told exempt mask-wears they had to display when shopping. How very sensitive. The Israeli system is blatant South African-style apartheid on the basis of compliance or non-compliance to fascism rather than colour of the skin. How appropriate that the Sabbatian Israeli government was so close to the pre-Mandela apartheid regime in Pretoria. The Sabbatian-instigated 'vaccine passport' in Israel is planned for everywhere. Sabbatians struck a deal with Pfizer that allowed them to lead the way in the percentage of a national population infused with synthetic material and the result was catastrophic. Israeli freedom activist Shai Dannon told me how chairs were appearing on beaches that said 'vaccinated only'. Health Minister Yuli Edelstein said that anyone unwilling or unable to get

the jabs that ‘confer immunity’ will be ‘left behind’. The man’s a liar. Not even the makers claim the ‘vaccines’ confer immunity. When you see those figures of ‘vaccine’ deaths these psychopaths were saying that you must take the chance the ‘vaccine’ will kill you or maim you while knowing it will change your DNA or lockdown for you will be permanent. That’s fascism. The Israeli parliament passed a law to allow personal information of the non-vaccinated to be shared with local and national authorities for three months. This was claimed by its supporters to be a way to ‘encourage’ people to be vaccinated. Hadas Ziv from Physicians for Human Rights described this as a ‘draconian law which crushed medical ethics and the patient rights’. But that’s the idea, the Sabbatians would reply.

## **Your papers, please**

Sabbatian Israel was leading what has been planned all along to be a global ‘vaccine pass’ called a ‘green passport’ without which you would remain in permanent lockdown restriction and unable to do anything. This is how badly – *desperately* – the Cult is to get everyone ‘vaccinated’. The term and colour ‘green’ was not by chance and related to the psychology of fusing the perception of the green climate hoax with the ‘Covid’ hoax and how the ‘solution’ to both is the same Great Reset. Lying politicians, health officials and psychologists denied there were any plans for mandatory vaccinations or restrictions based on vaccinations, but they knew that was exactly what was meant to happen with governments of all countries reaching agreements to enforce a global system. ‘Free’ Denmark and ‘free’ Sweden unveiled digital vaccine certification. Cyprus, Czech Republic, Estonia, Greece, Hungary, Iceland, Italy, Poland, Portugal, Slovakia, and Spain have all committed to a vaccine passport system and the rest including the whole of the EU would follow. The satanic UK government will certainly go this way despite mendacious denials and at the time of writing it is trying to manipulate the public into having the ‘vaccine’ so they could go abroad on a summer holiday. How would that work without something to prove you had the synthetic toxicity injected into you?

Documents show that the EU's European Commission was moving towards 'vaccine certificates' in 2018 and 2019 before the 'Covid' hoax began. They knew what was coming. Abracadabra – Ursula von der Leyen, the German President of the Commission, announced in March, 2021, an EU 'Digital Green Certificate' – green again – to track the public's 'Covid status'. The passport sting is worldwide and the Far East followed the same pattern with South Korea ruling that only those with 'vaccination' passports – again the *green* pass – would be able to 'return to their daily lives'.

Bill Gates has been preparing for this 'passport' with other Cult operatives for years and beyond the paper version is a Gates-funded 'digital tattoo' to identify who has been vaccinated and who hasn't. The 'tattoo' is reported to include a substance which is externally readable to confirm who has been vaccinated. This is a bio-luminous light-generating enzyme (think fireflies) called ... *Luciferase*. Yes, named after the Cult 'god' Lucifer the 'light bringer' of whom more to come. Gates said he funded the readable tattoo to ensure children in the developing world were vaccinated and no one was missed out. He cares so much about poor kids as we know. This was just the cover story to develop a vaccine tagging system for everyone on the planet. Gates has been funding the ID2020 'alliance' to do just that in league with other lovely people at Microsoft, GAVI, the Rockefeller Foundation, Accenture and IDEO.org. He said in interviews in March, 2020, before any 'vaccine' publicly existed, that the world must have a globalised digital certificate to track the 'virus' and who had been vaccinated. Gates knew from the start that the mRNA vaccines were coming and when they would come and that the plan was to tag the 'vaccinated' to marginalise the intelligent and stop them doing anything including travel. Evil just doesn't suffice. Gates was exposed for offering a \$10 million bribe to the Nigerian House of Representatives to invoke compulsory 'Covid' vaccination of all Nigerians. Sara Cunial, a member of the Italian Parliament, called Gates a 'vaccine criminal'. She urged the Italian President to hand him over to the International Criminal Court for crimes against

humanity and condemned his plans to 'chip the human race' through ID2020.

You know it's a long-planned agenda when war criminal and Cult gofer Tony Blair is on the case. With the scale of arrogance only someone as dark as Blair can muster he said: 'Vaccination in the end is going to be your route to liberty.' Blair is a disgusting piece of work and he confirms that again. The media has given a lot of coverage to a bloke called Charlie Mullins, founder of London's biggest independent plumbing company, Pimlico Plumbers, who has said he won't employ anyone who has not been vaccinated or have them go to any home where people are not vaccinated. He said that if he had his way no one would be allowed to walk the streets if they have not been vaccinated. Gates was cheering at the time while I was alerting the white coats. The plan is that people will qualify for 'passports' for having the first two doses and then to keep it they will have to have all the follow ups and new ones for invented 'variants' until human genetics is transformed and many are dead who can't adjust to the changes. Hollywood celebrities – the usual propaganda stunt – are promoting something called the WELL Health-Safety Rating to verify that a building or space has 'taken the necessary steps to prioritize the health and safety of their staff, visitors and other stakeholders'. They included Lady Gaga, Jennifer Lopez, Michael B. Jordan, Robert DeNiro, Venus Williams, Wolfgang Puck, Deepak Chopra and 17th Surgeon General Richard Carmona. Yawn. WELL Health-Safety has big connections with China. Parent company Delos is headed by former Goldman Sachs partner Paul Scialla. This is another example – and we will see so many others – of using the excuse of 'health' to dictate the lives and activities of the population. I guess one confirmation of the 'safety' of buildings is that only 'vaccinated' people can go in, right?

## **Electronic concentration camps**

I wrote decades ago about the plans to restrict travel and here we are for those who refuse to bow to tyranny. This can be achieved in one go with air travel if the aviation industry makes a blanket decree.

The ‘vaccine’ and guaranteed income are designed to be part of a global version of China’s social credit system which tracks behaviour 24/7 and awards or deletes ‘credits’ based on whether your behaviour is supported by the state or not. I mean your entire lifestyle – what you do, eat, say, everything. Once your credit score falls below a certain level consequences kick in. In China tens of millions have been denied travel by air and train because of this. All the locations and activities denied to refusers by the ‘vaccine’ passports will be included in one big mass ban on doing almost anything for those that don’t bow their head to government. It’s beyond fascist and a new term is required to describe its extremes – I guess fascist technocracy will have to do. The way the Chinese system of technological – technocratic – control is sweeping the West can be seen in the Los Angeles school system and is planned to be expanded worldwide. Every child is required to have a ‘Covid’-tracking app scanned daily before they can enter the classroom. The so-called Daily Pass tracking system is produced by Gates’ Microsoft which I’m sure will shock you rigid. The pass will be scanned using a barcode (one step from an inside-the-body barcode) and the information will include health checks, ‘Covid’ tests and vaccinations. Entry codes are for one specific building only and access will only be allowed if a student or teacher has a negative test with a test not testing for the ‘virus’, has no symptoms of anything alleged to be related to ‘Covid’ (symptoms from a range of other illness), and has a temperature under 100 degrees. No barcode, no entry, is planned to be the case for everywhere and not only schools.

Kids are being psychologically prepared to accept this as ‘normal’ their whole life which is why what they can impose in schools is so important to the Cult and its gofers. Long-time American freedom campaigner John Whitehead of the Rutherford Institute was not exaggerating when he said: ‘Databit by databit, we are building our own electronic concentration camps.’ Canada under its Cult gofer prime minister Justin Trudeau has taken a major step towards the real thing with people interned against their will if they test positive with a test not testing for the ‘virus’ when they arrive at a Canadian

airport. They are jailed in internment hotels often without food or water for long periods and with many doors failing to lock there have been sexual assaults. The interned are being charged sometimes \$2,000 for the privilege of being abused in this way. Trudeau is fully on board with the Cult and says the ‘Covid pandemic’ has provided an opportunity for a global ‘reset’ to permanently change Western civilisation. His number two, Deputy Prime Minister Chrystia Freeland, is a trustee of the World Economic Forum and a Rhodes Scholar. The Trudeau family have long been servants of the Cult. See *The Biggest Secret* and Cathy O’Brien’s book *Trance-Formation of America* for the horrific background to Trudeau’s father Pierre Trudeau another Canadian prime minister. Hide your fascism behind the façade of a heart-on-the-sleeve liberal. It’s a well-honed Cult technique.

## **What can the ‘vaccine’ really do?**

We have a ‘virus’ never shown to exist and ‘variants’ of the ‘virus’ that have also never been shown to exist except, like the ‘original’, as computer-generated fictions. Even if you believe there’s a ‘virus’ the ‘case’ to ‘death’ rate is in the region of 0.23 to 0.15 percent and those ‘deaths’ are concentrated among the very old around the same average age that people die anyway. In response to this lack of threat (in truth none) psychopaths and idiots, knowingly and unknowingly answering to Gates and the Cult, are seeking to ‘vaccinate’ every man, woman and child on Planet Earth. Clearly the ‘vaccine’ is not about ‘Covid’ – none of this ever has been. So what is it all about *really*? Why the desperation to infuse genetically-manipulating synthetic material into everyone through mRNA fraudulent ‘vaccines’ with the intent of doing this over and over with the excuses of ‘variants’ and other ‘virus’ inventions? Dr Sherri Tenpenny, an osteopathic medical doctor in the United States, has made herself an expert on vaccines and their effects as a vehement campaigner against their use. Tenpenny was board certified in emergency medicine, the director of a level two trauma centre for 12 years, and moved to Cleveland in 1996 to start an integrative

medicine practice which has treated patients from all 50 states and some 17 other countries. Weaning people off pharmaceutical drugs is a speciality.

She became interested in the consequences of vaccines after attending a meeting at the National Vaccine Information Center in Washington DC in 2000 where she 'sat through four days of listening to medical doctors and scientists and lawyers and parents of vaccine injured kids' and asked: 'What's going on?' She had never been vaccinated and never got ill while her father was given a list of vaccines to be in the military and was 'sick his entire life'. The experience added to her questions and she began to examine vaccine documents from the Centers for Disease Control (CDC). After reading the first one, the 1998 version of *The General Recommendations of Vaccination*, she thought: 'This is it?' The document was poorly written and bad science and Tenpenny began 20 years of research into vaccines that continues to this day. She began her research into 'Covid vaccines' in March, 2020, and she describes them as 'deadly'. For many, as we have seen, they already have been. Tenpenny said that in the first 30 days of the 'vaccine' rollout in the United States there had been more than 40,000 adverse events reported to the vaccine adverse event database. A document had been delivered to her the day before that was 172 pages long. 'We have over 40,000 adverse events; we have over 3,100 cases of [potentially deadly] anaphylactic shock; we have over 5,000 neurological reactions.' Effects ranged from headaches to numbness, dizziness and vertigo, to losing feeling in hands or feet and paraesthesia which is when limbs 'fall asleep' and people have the sensation of insects crawling underneath their skin. All this happened in the first 30 days and remember that only about *ten percent* (or far less) of adverse reactions and vaccine-related deaths are estimated to be officially reported. Tenpenny said:

So can you think of one single product in any industry, any industry, for as long as products have been made on the planet that within 30 days we have 40,000 people complaining of side effects that not only is still on the market but ... we've got paid actors telling us how great

they are for getting their vaccine. We're offering people \$500 if they will just get their vaccine and we've got nurses and doctors going; 'I got the vaccine, I got the vaccine'.

Tenpenny said they were not going to be 'happy dancing folks' when they began to suffer Bell's palsy (facial paralysis), neuropathies, cardiac arrhythmias and autoimmune reactions that kill through a blood disorder. 'They're not going to be so happy, happy then, but we're never going to see pictures of those people' she said. Tenpenny described the 'vaccine' as 'a well-designed killing tool'.

## No off-switch

Bad as the initial consequences had been Tenpenny said it would be maybe 14 months before we began to see the 'full ravage' of what is going to happen to the 'Covid vaccinated' with full-out consequences taking anything between two years and 20 years to show. You can understand why when you consider that variations of the 'Covid vaccine' use mRNA (messenger RNA) to in theory activate the immune system to produce protective antibodies without using the actual 'virus'. How can they when it's a computer program and they've never isolated what they claim is the 'real thing'? Instead they use *synthetic* mRNA. They are inoculating synthetic material into the body which through a technique known as the Trojan horse is absorbed into cells to change the nature of DNA. Human DNA is changed by an infusion of messenger RNA and with each new 'vaccine' of this type it is changed even more. Say so and you are banned by Cult Internet platforms. The contempt the contemptuous Mark Zuckerberg has for the truth and human health can be seen in an internal Facebook video leaked to the Project Veritas investigative team in which he said of the 'Covid vaccines': '... I share some caution on this because we just don't know the long term side-effects of basically modifying people's DNA and RNA.' At the same time this disgusting man's Facebook was censoring and banning anyone saying exactly the same. He must go before a Nuremberg trial for crimes against humanity when he *knows* that he

is censoring legitimate concerns and denying the right of informed consent on behalf of the Cult that owns him. People have been killed and damaged by the very ‘vaccination’ technique he cast doubt on himself when they may not have had the ‘vaccine’ with access to information that he denied them. The plan is to have at least annual ‘Covid vaccinations’, add others to deal with invented ‘variants’, and change all other vaccines into the mRNA system. Pfizer executives told shareholders at a virtual Barclays Global Healthcare Conference in March, 2021, that the public may need a third dose of ‘Covid vaccine’, plus regular yearly boosters and the company planned to hike prices to milk the profits in a ‘significant opportunity for our vaccine’. These are the professional liars, cheats and opportunists who are telling you their ‘vaccine’ is safe. Given this volume of mRNA planned to be infused into the human body and its ability to then replicate we will have a transformation of human genetics from biological to synthetic biological – exactly the long-time Cult plan for reasons we’ll see – and many will die. Sherri Tenpenny said of this replication:

It’s like having an on-button but no off-button and that whole mechanism ... they actually give it a name and they call it the Trojan horse mechanism, because it allows that [synthetic] virus and that piece of that [synthetic] virus to get inside of your cells, start to replicate and even get inserted into other parts of your DNA as a Trojan-horse.

Ask the overwhelming majority of people who have the ‘vaccine’ what they know about the contents and what they do and they would reply: ‘The government says it will stop me getting the virus.’ Governments give that false impression on purpose to increase take-up. You can read Sherri Tenpenny’s detailed analysis of the health consequences in her blog at [Vaxxter.com](http://Vaxxter.com), but in summary these are some of them. She highlights the statement by Bill Gates about how human beings can become their own ‘vaccine manufacturing machine’. The man is insane. [‘Vaccine’-generated] ‘antibodies’ carry synthetic messenger RNA into the cells and the damage starts, Tenpenny contends, and she says that lungs can be adversely affected through varying degrees of pus and bleeding which

obviously affects breathing and would be dubbed ‘Covid-19’. Even more sinister was the impact of ‘antibodies’ on macrophages, a white blood cell of the immune system. They consist of Type 1 and Type 2 which have very different functions. She said Type 1 are ‘hyper-vigilant’ white blood cells which ‘gobble up’ bacteria etc. However, in doing so, this could cause inflammation and in extreme circumstances be fatal. She says these affects are mitigated by Type 2 macrophages which kick in to calm down the system and stop it going rogue. They clear up dead tissue debris and reduce inflammation that the Type 1 ‘fire crews’ have caused. Type 1 kills the infection and Type 2 heals the damage, she says. This is her punchline with regard to ‘Covid vaccinations’: She says that mRNA ‘antibodies’ block Type 2 macrophages by attaching to them and deactivating them. This meant that when the Type 1 response was triggered by infection there was nothing to stop that getting out of hand by calming everything down. There’s an on-switch, but no off-switch, she says. What follows can be ‘over and out, see you when I see you’.

## **Genetic suicide**

Tenpenny also highlights the potential for autoimmune disease – the body attacking itself – which has been associated with vaccines since they first appeared. Infusing a synthetic foreign substance into cells could cause the immune system to react in a panic believing that the body is being overwhelmed by an invader (it is) and the consequences can again be fatal. There is an autoimmune response known as a ‘cytokine storm’ which I have likened to a homeowner panicked by an intruder and picking up a gun to shoot randomly in all directions before turning the fire on himself. The immune system unleashes a storm of inflammatory response called cytokines to a threat and the body commits hara-kiri. The lesson is that you mess with the body’s immune response at your peril and these ‘vaccines’ seriously – fundamentally – mess with immune response. Tenpenny refers to a consequence called anaphylactic shock which is a severe and highly dangerous allergic reaction when the immune system

floods the body with chemicals. She gives the example of having a bee sting which primes the immune system and makes it sensitive to those chemicals. When people are stung again maybe years later the immune response can be so powerful that it leads to anaphylactic shock. Tenpenny relates this 'shock' with regard to the 'Covid vaccine' to something called polyethylene glycol or PEG. Enormous numbers of people have become sensitive to this over decades of use in a whole range of products and processes including food, drink, skin creams and 'medicine'. Studies have claimed that some 72 percent of people have antibodies triggered by PEG compared with two percent in the 1960s and allergic hypersensitive reactions to this become a gathering cause for concern. Tenpenny points out that the 'mRNA vaccine' is coated in a 'bubble' of polyethylene glycol which has the potential to cause anaphylactic shock through immune sensitivity. Many reports have appeared of people reacting this way after having the 'Covid vaccine'. What do we think is going to happen as humanity has more and more of these 'vaccines'?

Tenpenny said: 'All these pictures we have seen with people with these rashes ... these weepy rashes, big reactions on their arms and things like that – it's an acute allergic reaction most likely to the polyethylene glycol that you've been previously primed and sensitised to.'

Those who have not studied the conspiracy and its perpetrators at length might think that making the population sensitive to PEG and then putting it in these 'vaccines' is just a coincidence. It is not. It is instead testament to how carefully and coldly-planned current events have been and the scale of the conspiracy we are dealing with. Tenpenny further explains that the 'vaccine' mRNA procedure can breach the blood-brain barrier which protects the brain from toxins and other crap that will cause malfunction. In this case they could make two proteins corrupt brain function to cause Amyotrophic lateral sclerosis (ALS), a progressive nervous system disease leading to loss of muscle control, and frontal lobe degeneration – Alzheimer's and dementia. Immunologist J. Bart Classon published a paper connecting mRNA 'vaccines' to prion

disease which can lead to Alzheimer's and other forms of neurodegenerative disease while others have pointed out the potential to affect the placenta in ways that make women infertile. This will become highly significant in the next chapter when I will discuss other aspects of this non-vaccine that relate to its nanotechnology and transmission from the injected to the uninjected.

## **Qualified in idiocy**

Tenpenny describes how research has confirmed that these 'vaccine'-generated antibodies can interact with a range of other tissues in the body and attack many other organs including the lungs. 'This means that if you have a hundred people standing in front of you that all got this shot they could have a hundred different symptoms.'

Anyone really think that Cult gofers like the Queen, Tony Blair, Christopher Whitty, Anthony Fauci, and all the other psychopaths have really had this 'vaccine' in the pictures we've seen? Not a bloody chance. Why don't doctors all tell us about all these dangers and consequences of the 'Covid vaccine'? Why instead do they encourage and pressure patients to have the shot? Don't let's think for a moment that doctors and medical staff can't be stupid, lazy, and psychopathic and that's without the financial incentives to give the jab. Tenpenny again:

Some people are going to die from the vaccine directly but a large number of people are going to start to get horribly sick and get all kinds of autoimmune diseases 42 days to maybe a year out. What are they going to do, these stupid doctors who say; 'Good for you for getting that vaccine.' What are they going to say; 'Oh, it must be a mutant, we need to give an extra dose of that vaccine.'

Because now the vaccine, instead of one dose or two doses we need three or four because the stupid physicians aren't taking the time to learn anything about it. If I can learn this sitting in my living room reading a 19 page paper and several others so can they. There's nothing special about me, I just take the time to do it.

Remember how Sara Kayat, the NHS and TV doctor, said that the 'Covid vaccine' would '100 percent prevent hospitalisation and death'. Doctors can be idiots like every other profession and they

should not be worshipped as infallible. They are not and far from it. Behind many medical and scientific ‘experts’ lies an uninformed prat trying to hide themselves from you although in the ‘Covid’ era many have failed to do so as with UK narrative-repeating ‘TV doctor’ Hilary Jones. Pushing back against the minority of proper doctors and scientists speaking out against the ‘vaccine’ has been the entire edifice of the Cult global state in the form of governments, medical systems, corporations, mainstream media, Silicon Valley, and an army of compliant doctors, medical staff and scientists willing to say anything for money and to enhance their careers by promoting the party line. If you do that you are an ‘expert’ and if you won’t you are an ‘anti-vaxxer’ and ‘Covidiot’. The pressure to be ‘vaccinated’ is incessant. We have even had reports claiming that the ‘vaccine’ can help cure cancer and Alzheimer’s and make the lame walk. I am waiting for the announcement that it can bring you coffee in the morning and cook your tea. Just as the symptoms of ‘Covid’ seem to increase by the week so have the miracles of the ‘vaccine’. American supermarket giant Kroger Co. offered nearly 500,000 employees in 35 states a \$100 bonus for having the ‘vaccine’ while donut chain Krispy Kreme promised ‘vaccinated’ customers a free glazed donut every day for the rest of 2021. Have your DNA changed and you will get a doughnut although we might not have to give you them for long. Such offers and incentives confirm the desperation.

Perhaps the worse vaccine-stunt of them all was UK ‘Health’ Secretary Matt-the-prat Hancock on live TV after watching a clip of someone being ‘vaccinated’ when the roll-out began. Hancock faked tears so badly it was embarrassing. Brain-of-Britain Piers Morgan, the lockdown-supporting, ‘vaccine’ supporting, ‘vaccine’ passport-supporting, TV host played along with Hancock – ‘You’re quite emotional about that’ he said in response to acting so atrocious it would have been called out at a school nativity which will presumably today include Mary and Jesus in masks, wise men keeping their camels six feet apart, and shepherds under tent arrest. System-serving Morgan tweeted this: ‘Love the idea of covid vaccine passports for everywhere: flights, restaurants, clubs, football, gyms,

shops etc. It's time covid-denying, anti-vaxxer loonies had their bullsh\*t bluff called & bar themselves from going anywhere that responsible citizens go.' If only I could aspire to his genius. To think that Morgan, who specialises in shouting over anyone he disagrees with, was lauded as a free speech hero when he lost his job after storming off the set of his live show like a child throwing his dolly out of the pram. If he is a free speech hero we are in real trouble. I have no idea what 'bullsh\*t' means, by the way, the \* throws me completely.

The Cult is desperate to infuse its synthetic DNA-changing concoction into everyone and has been using every lie, trick and intimidation to do so. The question of '*Why?*' we shall now address.

## CHAPTER TEN

### Human 2.0

***I believe that at the end of the century the use of words and general educated opinion will have altered so much that one will be able to speak of machines thinking without expecting to be contradicted –***

**Alan Turing (1912-1954), the ‘Father of artificial intelligence’**

I have been exposing for decades the plan to transform the human body from a biological to a synthetic-biological state. The new human that I will call Human 2.0 is planned to be connected to artificial intelligence and a global AI ‘Smart Grid’ that would operate as one global system in which AI would control everything from your fridge to your heating system to your car to your mind. Humans would no longer be ‘human’, but post-human and sub-human, with their thinking and emotional processes replaced by AI.

What I said sounded crazy and beyond science fiction and I could understand that. To any balanced, rational, mind it *is* crazy. Today, however, that world is becoming reality and it puts the ‘Covid vaccine’ into its true context. Ray Kurzweil is the ultra-Zionist ‘computer scientist, inventor and futurist’ and co-founder of the Singularity University. Singularity refers to the merging of humans with machines or ‘transhumanism’. Kurzweil has said humanity would be connected to the cyber ‘cloud’ in the period of the ever-recurring year of 2030:

Our thinking ... will be a hybrid of biological and non-biological thinking ... humans will be able to extend their limitations and ‘think in the cloud’ ... We’re going to put gateways to the

cloud in our brains ... We're going to gradually merge and enhance ourselves ... In my view, that's the nature of being human – we transcend our limitations. As the technology becomes vastly superior to what we are then the small proportion that is still human gets smaller and smaller and smaller until it's just utterly negligible.

They are trying to sell this end-of-humanity-as-we-know-it as the next stage of 'evolution' when we become super-human and 'like the gods'. They are lying to you. Shocked, eh? The population, and again especially the young, have been manipulated into addiction to technologies designed to enslave them for life. First they induced an addiction to smartphones (holdables); next they moved to technology on the body (wearables); and then began the invasion of the body (implantables). I warned way back about the plan for microchipped people and we are now entering that era. We should not be diverted into thinking that this refers only to chips we can see. Most important are the nanochips known as smart dust, neural dust and nanobots which are far too small to be seen by the human eye. Nanotechnology is everywhere, increasingly in food products, and released into the atmosphere by the geoengineering of the skies funded by Bill Gates to 'shut out the Sun' and 'save the planet from global warming'. Gates has been funding a project to spray millions of tonnes of chalk (calcium carbonate) into the stratosphere over Sweden to 'dim the Sun' and cool the Earth. Scientists warned the move could be disastrous for weather systems in ways no one can predict and opposition led to the Swedish space agency announcing that the 'experiment' would not be happening as planned in the summer of 2021; but it shows where the Cult is going with dimming the impact of the Sun and there's an associated plan to change the planet's atmosphere. Who gives psychopath Gates the right to dictate to the entire human race and dismantle planetary systems? The world will not be safe while this man is at large.

The global warming hoax has made the Sun, like the gas of life, something to fear when both are essential to good health and human survival (more inversion). The body transforms sunlight into vital vitamin D through a process involving ... *cholesterol*. This is the cholesterol we are also told to fear. We are urged to take Big Pharma

statin drugs to reduce cholesterol and it's all systematic. Reducing cholesterol means reducing vitamin D uptake with all the multiple health problems that will cause. At least if you take statins long term it saves the government from having to pay you a pension. The delivery system to block sunlight is widely referred to as chemtrails although these have a much deeper agenda, too. They appear at first to be contrails or condensation trails streaming from aircraft into cold air at high altitudes. Contrails disperse very quickly while chemtrails do not and spread out across the sky before eventually their content falls to earth. Many times I have watched aircraft cross-cross a clear blue sky releasing chemtrails until it looks like a cloudy day. Chemtrails contain many things harmful to humans and the natural world including toxic heavy metals, aluminium (see Alzheimer's) and nanotechnology. Ray Kurzweil reveals the reason without actually saying so: 'Nanobots will infuse all the matter around us with information. Rocks, trees, everything will become these intelligent creatures.' How do you deliver that? *From the sky.* Self-replicating nanobots would connect everything to the Smart Grid. The phenomenon of Morgellons disease began in the chemtrail era and the correlation has led to it being dubbed the 'chemtrail disease'. Self-replicating fibres appear in the body that can be pulled out through the skin. Morgellons fibres continue to grow outside the body and have a form of artificial intelligence. I cover this at greater length in *Phantom Self*.

## **'Vaccine' operating system**

'Covid vaccines' with their self-replicating synthetic material are also designed to make the connection between humanity and Kurzweil's 'cloud'. American doctor and dedicated campaigner for truth, Carrie Madej, an Internal Medicine Specialist in Georgia with more than 20 years medical experience, has highlighted the nanotechnology aspect of the fake 'vaccines'. She explains how one of the components in at least the Moderna and Pfizer synthetic potions are 'lipid nanoparticles' which are 'like little tiny computer bits' – a 'sci-fi substance' known as nanobots and hydrogel which can be 'triggered

at any moment to deliver its payload' and act as 'biosensors'. The synthetic substance had 'the ability to accumulate data from your body like your breathing, your respiration, thoughts and emotions, all kind of things' and each syringe could carry a *million* nanobots:

This substance because it's like little bits of computers in your body, crazy, but it's true, it can do that, [and] obviously has the ability to act through Wi-Fi. It can receive and transmit energy, messages, frequencies or impulses. That issue has never been addressed by these companies. What does that do to the human?

Just imagine getting this substance in you and it can react to things all around you, the 5G, your smart device, your phones, what is happening with that? What if something is triggering it, too, like an impulse, a frequency? We have something completely foreign in the human body.

Madej said her research revealed that electromagnetic (EMF) frequencies emitted by phones and other devices had increased dramatically in the same period of the 'vaccine' rollout and she was seeing more people with radiation problems as 5G and other electromagnetic technology was expanded and introduced to schools and hospitals. She said she was 'floored with the EMF coming off' the devices she checked. All this makes total sense and syncs with my own work of decades when you think that Moderna refers in documents to its mRNA 'vaccine' as an 'operating system':

Recognizing the broad potential of mRNA science, we set out to create an mRNA technology platform that functions very much like an operating system on a computer. It is designed so that it can plug and play interchangeably with different programs. In our case, the 'program' or 'app' is our mRNA drug – the unique mRNA sequence that codes for a protein ...

... Our mRNA Medicines – 'The Software Of Life': When we have a concept for a new mRNA medicine and begin research, fundamental components are already in place. Generally, the only thing that changes from one potential mRNA medicine to another is the coding region – the actual genetic code that instructs ribosomes to make protein. Utilizing these instruction sets gives our investigational mRNA medicines a software-like quality. We also have the ability to combine different mRNA sequences encoding for different proteins in a single mRNA investigational medicine.

Who needs a real ‘virus’ when you can create a computer version to justify infusing your operating system into the entire human race on the road to making living, breathing people into cyborgs? What is missed with the ‘vaccines’ is the *digital* connection between synthetic material and the body that I highlighted earlier with the study that hacked a computer with human DNA. On one level the body is digital, based on mathematical codes, and I’ll have more about that in the next chapter. Those who ridiculously claim that mRNA ‘vaccines’ are not designed to change human genetics should explain the words of Dr Tal Zaks, chief medical officer at Moderna, in a 2017 TED talk. He said that over the last 30 years ‘we’ve been living this phenomenal digital scientific revolution, and I’m here today to tell you, that we are actually *hacking the software of life*, and that it’s changing the way we think about prevention and treatment of disease’:

In every cell there’s this thing called messenger RNA, or mRNA for short, that transmits the critical information from the DNA in our genes to the protein, which is really the stuff we’re all made out of. This is the critical information that determines what the cell will do. So we think about it as an operating system. So if you could change that, if you could introduce a line of code, or change a line of code, it turns out, that has profound implications for everything, from the flu to cancer.

Zaks should more accurately have said that this has profound implications for the human genetic code and the nature of DNA. Communications within the body go both ways and not only one. But, hey, no, the ‘Covid vaccine’ will not affect your genetics. Cult fact-checkers say so even though the man who helped to develop the mRNA technique says that it does. Zaks said in 2017:

If you think about what it is we’re trying to do. We’ve taken information and our understanding of that information and how that information is transmitted in a cell, and we’ve taken our understanding of medicine and how to make drugs, and we’re fusing the two. We think of it as information therapy.

I have been writing for decades that the body is an information field communicating with itself and the wider world. This is why

radiation which is information can change the information field of body and mind through phenomena like 5G and change their nature and function. ‘Information therapy’ means to change the body’s information field and change the way it operates. DNA is a receiver-transmitter of information and can be mutated by information like mRNA synthetic messaging. Technology to do this has been ready and waiting in the underground bases and other secret projects to be rolled out when the ‘Covid’ hoax was played. ‘Trials’ of such short and irrelevant duration were only for public consumption. When they say the ‘vaccine’ is ‘experimental’ that is not true. It may appear to be ‘experimental’ to those who don’t know what’s going on, but the trials have already been done to ensure the Cult gets the result it desires. Zaks said that it took decades to sequence the human genome, completed in 2003, but now they could do it in a week. By ‘they’ he means scientists operating in the public domain. In the secret projects they were sequencing the genome in a week long before even 2003.

## **Deluge of mRNA**

Highly significantly the Moderna document says the guiding premise is that if using mRNA as a medicine works for one disease then it should work for many diseases. They were leveraging the flexibility afforded by their platform and the fundamental role mRNA plays in protein synthesis to pursue mRNA medicines for a broad spectrum of diseases. Moderna is confirming what I was saying through 2020 that multiple ‘vaccines’ were planned for ‘Covid’ (and later invented ‘variants’) and that previous vaccines would be converted to the mRNA system to infuse the body with massive amounts of genetically-manipulating synthetic material to secure a transformation to a synthetic-biological state. The ‘vaccines’ are designed to kill stunning numbers as part of the long-exposed Cult depopulation agenda and transform the rest. Given this is the goal you can appreciate why there is such hysterical demand for every human to be ‘vaccinated’ for an alleged ‘disease’ that has an estimated ‘infection’ to ‘death’ ratio of 0.23-0.15 percent. As I write

children are being given the ‘vaccine’ in trials (their parents are a disgrace) and ever-younger people are being offered the vaccine for a ‘virus’ that even if you believe it exists has virtually zero chance of harming them. Horrific effects of the ‘trials’ on a 12-year-old girl were revealed by a family member to be serious brain and gastric problems that included a bowel obstruction and the inability to swallow liquids or solids. She was unable to eat or drink without throwing up, had extreme pain in her back, neck and abdomen, and was paralysed from the waist down which stopped her urinating unaided. When the girl was first taken to hospital doctors said it was all in her mind. She was signed up for the ‘trial’ by her parents for whom no words suffice. None of this ‘Covid vaccine’ insanity makes any sense unless you see what the ‘vaccine’ really is – a body-changer. Synthetic biology or ‘SynBio’ is a fast-emerging and expanding scientific discipline which includes everything from genetic and molecular engineering to electrical and computer engineering. Synthetic biology is defined in these ways:

- A multidisciplinary area of research that seeks to create new biological parts, devices, and systems, or to redesign systems that are already found in nature.
- The use of a mixture of physical engineering and genetic engineering to create new (and therefore synthetic) life forms.
- An emerging field of research that aims to combine the knowledge and methods of biology, engineering and related disciplines in the design of chemically-synthesized DNA to create organisms with novel or enhanced characteristics and traits (synthetic organisms including humans).

We now have synthetic blood, skin, organs and limbs being developed along with synthetic body parts produced by 3D printers. These are all elements of the synthetic human programme and this comment by Kurzweil’s co-founder of the Singularity University,

Peter Diamandis, can be seen in a whole new light with the ‘Covid’ hoax and the sanctions against those that refuse the ‘vaccine’:

Anybody who is going to be resisting the progress forward [to transhumanism] is going to be resisting evolution and, fundamentally, they will die out. It’s not a matter of whether it’s good or bad. It’s going to happen.

‘Resisting evolution’? What absolute bollocks. The arrogance of these people is without limit. His ‘it’s going to happen’ mantra is another way of saying ‘resistance is futile’ to break the spirit of those pushing back and we must not fall for it. Getting this genetically-transforming ‘vaccine’ into everyone is crucial to the Cult plan for total control and the desperation to achieve that is clear for anyone to see. Vaccine passports are a major factor in this and they, too, are a form of resistance is futile. It’s NOT. The paper funded by the Rockefeller Foundation for the 2013 ‘health conference’ in China said:

We will interact more with artificial intelligence. The use of robotics, bio-engineering to augment human functioning is already well underway and will advance. Re-engineering of humans into potentially separate and unequal forms through genetic engineering or mixed human-robots raises debates on ethics and equality.

A new demography is projected to emerge after 2030 [that year again] of technologies (robotics, genetic engineering, nanotechnology) producing robots, engineered organisms, ‘nanobots’ and artificial intelligence (AI) that can self-replicate. Debates will grow on the implications of an impending reality of human designed life.

What is happening today is so long planned. The world army enforcing the will of the world government is intended to be a robot army, not a human one. Today’s military and its technologically ‘enhanced’ troops, pilotless planes and driverless vehicles are just stepping stones to that end. Human soldiers are used as Cult fodder and its time they woke up to that and worked for the freedom of the population instead of their own destruction and their family’s destruction – the same with the police. Join us and let’s sort this out. The phenomenon of enforce my own destruction is widespread in the ‘Covid’ era with Woker ‘luvvies’ in the acting and entertainment

industries supporting ‘Covid’ rules which have destroyed their profession and the same with those among the public who put signs on the doors of their businesses ‘closed due to Covid – stay safe’ when many will never reopen. It’s a form of masochism and most certainly insanity.

## **Transgender = transhumanism**

When something explodes out of nowhere and is suddenly everywhere it is always the Cult agenda and so it is with the tidal wave of claims and demands that have infiltrated every aspect of society under the heading of ‘transgenderism’. The term ‘trans’ is so ‘in’ and this is the dictionary definition:

A prefix meaning ‘across’, ‘through’, occurring ... in loanwords from Latin, used in particular for denoting movement or conveyance from place to place (transfer; transmit; transplant) or complete change (transform; transmute), or to form adjectives meaning ‘crossing’, ‘on the other side of’, or ‘going beyond’ the place named (transmontane; transnational; trans-Siberian).

Transgender means to go beyond gender and transhuman means to go beyond human. Both are aspects of the Cult plan to transform the human body to a synthetic state with *no gender*. Human 2.0 is not designed to procreate and would be produced technologically with no need for parents. The new human would mean the end of parents and so men, and increasingly women, are being targeted for the deletion of their rights and status. Parental rights are disappearing at an ever-quickening speed for the same reason. The new human would have no need for men or women when there is no procreation and no gender. Perhaps the transgender movement that appears to be in a permanent state of frenzy might now contemplate on how it is being used. This was never about transgender rights which are only the interim excuse for confusing gender, particularly in the young, on the road to *fusing* gender. Transgender activism is not an end; it is a *means* to an end. We see again the technique of creative destruction in which you destroy the status quo to ‘build back better’ in the form that you want. The gender status quo had to be

destroyed by persuading the Cult-created Woke mentality to believe that you can have 100 genders or more. A programme for 9 to 12 year olds produced by the Cult-owned BBC promoted the 100 genders narrative. The very idea may be the most monumental nonsense, but it is not what is true that counts, only what you can make people *believe* is true. Once the gender of  $2 + 2 = 4$  has been dismantled through indoctrination, intimidation and  $2 + 2 = 5$  then the new no-gender normal can take its place with Human 2.0.

Aldous Huxley revealed the plan in his prophetic *Brave New World* in 1932:

Natural reproduction has been done away with and children are created, 'decanted', and raised in 'hatcheries and conditioning centres'. From birth, people are genetically designed to fit into one of five castes, which are further split into 'Plus' and 'Minus' members and designed to fulfil predetermined positions within the social and economic strata of the World State.

How could Huxley know this in 1932? For the same reason George Orwell knew about the Big Brother state in 1948, Cult insiders I have quoted knew about it in 1969, and I have known about it since the early 1990s. If you are connected to the Cult or you work your balls off to uncover the plan you can predict the future. The process is simple. If there is a plan for the world and nothing intervenes to stop it then it will happen. Thus if you communicate the plan ahead of time you are perceived to have predicted the future, but you haven't. You have revealed the plan which without intervention will become the human future. The whole reason I have done what I have is to alert enough people to inspire an intervention and maybe at last that time has come with the Cult and its intentions now so obvious to anyone with a brain in working order.

## **The future is here**

Technological wombs that Huxley described to replace parent procreation are already being developed and they are only the projects we know about in the public arena. Israeli scientists told *The Times of Israel* in March, 2021, that they have grown 250-cell embryos

into mouse foetuses with fully formed organs using artificial wombs in a development they say could pave the way for gestating humans outside the womb. Professor Jacob Hanna of the Weizmann Institute of Science said:

We took mouse embryos from the mother at day five of development, when they are just of 250 cells, and had them in the incubator from day five until day 11, by which point they had grown all their organs.

By day 11 they make their own blood and have a beating heart, a fully developed brain. Anybody would look at them and say, 'this is clearly a mouse foetus with all the characteristics of a mouse.' It's gone from being a ball of cells to being an advanced foetus.

A special liquid is used to nourish embryo cells in a laboratory dish and they float on the liquid to duplicate the first stage of embryonic development. The incubator creates all the right conditions for its development, Hanna said. The liquid gives the embryo 'all the nutrients, hormones and sugars they need' along with a custom-made electronic incubator which controls gas concentration, pressure and temperature. The cutting-edge in the underground bases and other secret locations will be light years ahead of that, however, and this was reported by the London *Guardian* in 2017:

We are approaching a biotechnological breakthrough. Ectogenesis, the invention of a complete external womb, could completely change the nature of human reproduction. In April this year, researchers at the Children's Hospital of Philadelphia announced their development of an artificial womb.

The article was headed 'Artificial wombs could soon be a reality. What will this mean for women?' What would it mean for children is an even bigger question. No mother to bond with only a machine in preparation for a life of soulless interaction and control in a world governed by machines (see the *Matrix* movies). Now observe the calculated manipulations of the 'Covid' hoax as human interaction and warmth has been curtailed by distancing, isolation and fear with people communicating via machines on a scale never seen before.

These are all dots in the same picture as are all the personal assistants, gadgets and children's toys through which kids and adults communicate with AI as if it is human. The AI 'voice' on Sat-Nav should be included. All these things are psychological preparation for the Cult endgame. Before you can make a physical connection with AI you have to make a psychological connection and that is what people are being conditioned to do with this ever gathering human-AI interaction. Movies and TV programmes depicting the transhuman, robot dystopia relate to a phenomenon known as 'pre-emptive programming' in which the world that is planned is portrayed everywhere in movies, TV and advertising. This is conditioning the conscious and subconscious mind to become familiar with the planned reality to dilute resistance when it happens for real. What would have been a shock such is the change is made less so. We have young children put on the road to transgender transition surgery with puberty blocking drugs at an age when they could never be able to make those life-changing decisions.

Rachel Levine, a professor of paediatrics and psychiatry who believes in treating children this way, became America's highest-ranked openly-transgender official when she was confirmed as US Assistant Secretary at the Department of Health and Human Services after being nominated by Joe Biden (the Cult). Activists and governments press for laws to deny parents a say in their children's transition process so the kids can be isolated and manipulated into agreeing to irreversible medical procedures. A Canadian father Robert Hoogland was denied bail by the Vancouver Supreme Court in 2021 and remained in jail for breaching a court order that he stay silent over his young teenage daughter, a minor, who was being offered life-changing hormone therapy without parental consent. At the age of 12 the girl's 'school counsellor' said she may be transgender, referred her to a doctor and told the school to treat her like a boy. This is another example of state-serving schools imposing ever more control over children's lives while parents have ever less.

Contemptible and extreme child abuse is happening all over the world as the Cult gender-fusion operation goes into warp-speed.

## **Why the war on men – and now women?**

The question about what artificial wombs mean for women should rightly be asked. The answer can be seen in the deletion of women's rights involving sport, changing rooms, toilets and status in favour of people in male bodies claiming to identify as women. I can identify as a mountain climber, but it doesn't mean I can climb a mountain any more than a biological man can be a biological woman. To believe so is a triumph of belief over factual reality which is the very perceptual basis of everything Woke. Women's sport is being destroyed by allowing those with male bodies who say they identify as female to 'compete' with girls and women. Male body 'women' dominate 'women's' competition with their greater muscle mass, bone density, strength and speed. With that disadvantage sport for women loses all meaning. To put this in perspective nearly 300 American high school boys can run faster than the quickest woman sprinter in the world. Women are seeing their previously protected spaces invaded by male bodies simply because they claim to identify as women. That's all they need to do to access all women's spaces and activities under the Biden 'Equality Act' that destroys equality for women with the usual Orwellian Woke inversion. Male sex offenders have already committed rapes in women's prisons after claiming to identify as women to get them transferred. Does this not matter to the Woke 'equality' hypocrites? Not in the least. What matters to Cult manipulators and funders behind transgender activists is to advance gender fusion on the way to the no-gender 'human'. When you are seeking to impose transparent nonsense like this, or the 'Covid' hoax, the only way the nonsense can prevail is through censorship and intimidation of dissenters, deletion of factual information, and programming of the unquestioning, bewildered and naive. You don't have to scan the world for long to see that all these things are happening.

Many women's rights organisations have realised that rights and status which took such a long time to secure are being eroded and that it is systematic. Kara Dansky of the global Women's Human Rights Campaign said that Biden's transgender executive order immediately he took office, subsequent orders, and Equality Act legislation that followed 'seek to erase women and girls in the law as a category'. *Exactly.* I said during the long ago-started war on men (in which many women play a crucial part) that this was going to turn into a war on them. The Cult is phasing out *both* male and female genders. To get away with that they are brought into conflict so they are busy fighting each other while the Cult completes the job with no unity of response. Unity, people, *unity*. We need unity everywhere. Transgender is the only show in town as the big step towards the no-gender human. It's not about rights for transgender people and never has been. Woke political correctness is deleting words relating to genders to the same end. Wokers believe this is to be 'inclusive' when the opposite is true. They are deleting words describing gender because gender *itself* is being deleted by Human 2.0. Terms like 'man', 'woman', 'mother' and 'father' are being deleted in the universities and other institutions to be replaced by the *no*-gender, not trans-gender, 'individuals' and 'guardians'. Women's rights campaigner Maria Keffler of Partners for Ethical Care said: 'Children are being taught from kindergarten upward that some boys have a vagina, some girls have a penis, and that kids can be any gender they want to be.' Do we really believe that suddenly countries all over the world at the same time had the idea of having drag queens go into schools or read transgender stories to very young children in the local library? It's coldly-calculated confusion of gender on the way to the fusion of gender. Suzanne Vierling, a psychologist from Southern California, made another important point:

Yesterday's slave woman who endured gynecological medical experiments is today's girl-child being butchered in a booming gender-transitioning sector. Ovaries removed, pushing her into menopause and osteoporosis, uncharted territory, and parents' rights and authority decimated.

The erosion of parental rights is a common theme in line with the Cult plans to erase the very concept of parents and 'ovaries removed, pushing her into menopause' means what? Those born female lose the ability to have children – another way to discontinue humanity as we know it.

## **Eliminating Human 1.0 (before our very eyes)**

To pave the way for Human 2.0 you must phase out Human 1.0. This is happening through plummeting sperm counts and making women infertile through an onslaught of chemicals, radiation (including smartphones in pockets of men) and mRNA 'vaccines'. Common agriculture pesticides are also having a devastating impact on human fertility. I have been tracking collapsing sperm counts in the books for a long time and in 2021 came a book by fertility scientist and reproductive epidemiologist Shanna Swan, *Count Down: How Our Modern World Is Threatening Sperm Counts, Altering Male and Female Reproductive Development and Imperiling the Future of the Human Race*. She reports how the global fertility rate dropped by half between 1960 and 2016 with America's birth rate 16 percent below where it needs to be to sustain the population. Women are experiencing declining egg quality, more miscarriages, and more couples suffer from infertility. Other findings were an increase in erectile dysfunction, infant boys developing more genital abnormalities, male problems with conception, and plunging levels of the male hormone testosterone which would explain why so many men have lost their backbone and masculinity. This has been very evident during the 'Covid' hoax when women have been prominent among the Pushbackers and big strapping blokes have bowed their heads, covered their faces with a nappy and quietly submitted. Mind control expert Cathy O'Brien also points to how global education introduced the concept of 'we're all winners' in sport and classrooms: 'Competition was defused, and it in turn defused a sense of fighting back.' This is another version of the 'equity' doctrine in which you drive down rather than raise up. What a contrast in Cult-controlled China with its global ambitions

where the government published plans in January, 2021, to 'cultivate masculinity' in boys from kindergarten through to high school in the face of a 'masculinity crisis'. A government adviser said boys would be soon become 'delicate, timid and effeminate' unless action was taken. Don't expect any similar policy in the targeted West. A 2006 study showed that a 65-year-old man in 2002 had testosterone levels 15 percent lower than a 65-year-old man in 1987 while a 2020 study found a similar story with young adults and adolescents. Men are getting prescriptions for testosterone replacement therapy which causes an even greater drop in sperm count with up to 99 percent seeing sperm counts drop to zero during the treatment. More sperm is defective and malfunctioning with some having two heads or not pursuing an egg.

A class of *synthetic* chemicals known as phthalates are being blamed for the decline. These are found everywhere in plastics, shampoos, cosmetics, furniture, flame retardants, personal care products, pesticides, canned foods and even receipts. Why till receipts? Everyone touches them. Let no one delude themselves that all this is not systematic to advance the long-time agenda for human body transformation. Phthalates mimic hormones and disrupt the hormone balance causing testosterone to fall and genital birth defects in male infants. Animals and fish have been affected in the same way due to phthalates and other toxins in rivers. When fish turn gay or change sex through chemicals in rivers and streams it is a pointer to why there has been such an increase in gay people and the sexually confused. It doesn't matter to me what sexuality people choose to be, but if it's being affected by chemical pollution and consumption then we need to know. Does anyone really think that this is not connected to the transgender agenda, the war on men and the condemnation of male 'toxic masculinity'? You watch this being followed by 'toxic femininity'. It's already happening. When breastfeeding becomes 'chest-feeding', pregnant women become pregnant people along with all the other Woke claptrap you know that the world is going insane and there's a Cult scam in progress. Transgender activists are promoting the Cult agenda while Cult

billionaires support and fund the insanity as they laugh themselves to sleep at the sheer stupidity for which humans must be infamous in galaxies far, far away.

## **'Covid vaccines' and female infertility**

We can now see why the 'vaccine' has been connected to potential infertility in women. Dr Michael Yeadon, former Vice President and Chief Scientific Advisor at Pfizer, and Dr Wolfgang Wodarg in Germany, filed a petition with the European Medicines Agency in December, 2020, urging them to stop trials for the Pfizer/BioNTech shot and all other mRNA trials until further studies had been done. They were particularly concerned about possible effects on fertility with 'vaccine'-produced antibodies attacking the protein Syncytin-1 which is responsible for developing the placenta. The result would be infertility 'of indefinite duration' in women who have the 'vaccine' with the placenta failing to form. Section 10.4.2 of the Pfizer/BioNTech trial protocol says that pregnant women or those who might become so should not have mRNA shots. Section 10.4 warns men taking mRNA shots to 'be abstinent from heterosexual intercourse' and not to donate sperm. The UK government said that it *did not know* if the mRNA procedure had an effect on fertility. *Did not know?* These people have to go to jail. UK government advice did not recommend at the start that pregnant women had the shot and said they should avoid pregnancy for at least two months after 'vaccination'. The 'advice' was later updated to pregnant women should only have the 'vaccine' if the benefits outweighed the risks to mother and foetus. What the hell is that supposed to mean? Then 'spontaneous abortions' began to appear and rapidly increase on the adverse reaction reporting schemes which include only a fraction of adverse reactions. Thousands and ever-growing numbers of 'vaccinated' women are describing changes to their menstrual cycle with heavier blood flow, irregular periods and menstruating again after going through the menopause – all links to reproduction effects. Women are passing blood clots and the lining of their uterus while men report erectile dysfunction and blood effects. Most

significantly of all *unvaccinated* women began to report similar menstrual changes after interaction with '*vaccinated*' people and men and children were also affected with bleeding noses, blood clots and other conditions. 'Shedding' is when vaccinated people can emit the content of a vaccine to affect the unvaccinated, but this is different. '*Vaccinated*' people were not shedding a 'live virus' allegedly in '*vaccines*' as before because the fake '*Covid vaccines*' involve synthetic material and other toxicity. Doctors exposing what is happening prefer the term '*transmission*' to shedding. Somehow those that have had the shots are transmitting effects to those that haven't. Dr Carrie Madej said the nano-content of the '*vaccines*' can 'act like an antenna' to others around them which fits perfectly with my own conclusions. This '*vaccine*' transmission phenomenon was becoming known as the book went into production and I deal with this further in the Postscript.

Vaccine effects on sterility are well known. The World Health Organization was accused in 2014 of sterilising millions of women in Kenya with the evidence confirmed by the content of the vaccines involved. The same WHO behind the '*Covid*' hoax admitted its involvement for more than ten years with the vaccine programme. Other countries made similar claims. Charges were lodged by Tanzania, Nicaragua, Mexico, and the Philippines. The Gardasil vaccine claimed to protect against a genital '*virus*' known as HPV has also been linked to infertility. Big Pharma and the WHO (same thing) are criminal and satanic entities. Then there's the Bill Gates Foundation which is connected through funding and shared interests with 20 pharmaceutical giants and laboratories. He stands accused of directing the policy of United Nations Children's Fund (UNICEF), vaccine alliance GAVI, and other groupings, to advance the vaccine agenda and silence opposition at great cost to women and children. At the same time Gates wants to reduce the global population. Coincidence?

**Great Reset = Smart Grid = new human**

The Cult agenda I have been exposing for 30 years is now being openly promoted by Cult assets like Gates and Klaus Schwab of the World Economic Forum under code-terms like the 'Great Reset', 'Build Back Better' and 'a rare but narrow window of opportunity to reflect, reimagine, and reset our world'. What provided this 'rare but narrow window of opportunity'? The 'Covid' hoax did. Who created that? *They* did. My books from not that long ago warned about the planned 'Internet of Things' (IoT) and its implications for human freedom. This was the plan to connect all technology to the Internet and artificial intelligence and today we are way down that road with an estimated 36 billion devices connected to the World Wide Web and that figure is projected to be 76 billion by 2025. I further warned that the Cult planned to go beyond that to the Internet of *Everything* when the human brain was connected via AI to the Internet and Kurzweil's 'cloud'. Now we have Cult operatives like Schwab calling for precisely that under the term 'Internet of Bodies', a fusion of the physical, digital and biological into one centrally-controlled Smart Grid system which the Cult refers to as the 'Fourth Industrial Revolution'. They talk about the 'biological', but they really mean the synthetic-biological which is required to fully integrate the human body and brain into the Smart Grid and artificial intelligence planned to replace the human mind. We have everything being synthetically manipulated including the natural world through GMO and smart dust, the food we eat and the human body itself with synthetic 'vaccines'. I said in *The Answer* that we would see the Cult push for synthetic meat to replace animals and in February, 2021, the so predictable psychopath Bill Gates called for the introduction of synthetic meat to save us all from 'climate change'. The climate hoax just keeps on giving like the 'Covid' hoax. The war on meat by vegan activists is a carbon (oops, sorry) copy of the manipulation of transgender activists. They have no idea (except their inner core) that they are being used to promote and impose the agenda of the Cult or that they are only the *vehicle* and not the *reason*. This is not to say those who choose not to eat meat shouldn't be respected and supported in that right, but there are ulterior motives

for those in power. A *Forbes* article in December, 2019, highlighted the plan so beloved of Schwab and the Cult under the heading: 'What Is The Internet of Bodies? And How Is It Changing Our World?' The article said the human body is the latest data platform (remember 'our vaccine is an operating system'). *Forbes* described the plan very accurately and the words could have come straight out of my books from long before:

The Internet of Bodies (IoB) is an extension of the IoT and basically connects the human body to a network through devices that are ingested, implanted, or connected to the body in some way. Once connected, data can be exchanged, and the body and device can be remotely monitored and controlled.

They were really describing a human hive mind with human perception centrally-dictated via an AI connection as well as allowing people to be 'remotely monitored and controlled'.

Everything from a fridge to a human mind could be directed from a central point by these insane psychopaths and 'Covid vaccines' are crucial to this. *Forbes* explained the process I mentioned earlier of holdable and wearable technology followed by implantable. The article said there were three generations of the Internet of Bodies that include:

- Body external: These are wearable devices such as Apple Watches or Fitbits that can monitor our health.
- Body internal: These include pacemakers, cochlear implants, and digital pills that go inside our bodies to monitor or control various aspects of health.
- Body embedded: The third generation of the Internet of Bodies is embedded technology where technology and the human body are melded together and have a real-time connection to a remote machine.

*Forbes* noted the development of the Brain Computer Interface (BCI) which merges the brain with an external device for monitoring and controlling in real-time. ‘The ultimate goal is to help restore function to individuals with disabilities by using brain signals rather than conventional neuromuscular pathways.’ Oh, do fuck off. The goal of brain interface technology is controlling human thought and emotion from the central point in a hive mind serving its masters wishes. Many people are now agreeing to be chipped to open doors without a key. You can recognise them because they’ll be wearing a mask, social distancing and lining up for the ‘vaccine’. The Cult plans a Great Reset money system after they have completed the demolition of the global economy in which ‘money’ will be exchanged through communication with body operating systems. Rand Corporation, a Cult-owned think tank, said of the Internet of Bodies or IoB:

Internet of Bodies technologies fall under the broader IoT umbrella. But as the name suggests, IoB devices introduce an even more intimate interplay between humans and gadgets. IoB devices monitor the human body, collect health metrics and other personal information, and transmit those data over the Internet. Many devices, such as fitness trackers, are already in use ... IoB devices ... and those in development can track, record, and store users’ whereabouts, bodily functions, and what they see, hear, and even think.

Schwab’s World Economic Forum, a long-winded way of saying ‘fascism’ or ‘the Cult’, has gone full-on with the Internet of Bodies in the ‘Covid’ era. ‘We’re entering the era of the Internet of Bodies’, it declared, ‘collecting our physical data via a range of devices that can be implanted, swallowed or worn’. The result would be a huge amount of health-related data that could improve human wellbeing around the world, and prove crucial in fighting the ‘Covid-19 pandemic’. Does anyone think these clowns care about ‘human wellbeing’ after the death and devastation their pandemic hoax has purposely caused? Schwab and co say we should move forward with the Internet of Bodies because ‘Keeping track of symptoms could help us stop the spread of infection, and quickly detect new cases’. How wonderful, but keeping track’ is all they are really bothered

about. Researchers were investigating if data gathered from smartwatches and similar devices could be used as viral infection alerts by tracking the user's heart rate and breathing. Schwab said in his 2018 book *Shaping the Future of the Fourth Industrial Revolution*:

The lines between technologies and beings are becoming blurred and not just by the ability to create lifelike robots or synthetics. Instead it is about the ability of new technologies to literally become part of us. Technologies already influence how we understand ourselves, how we think about each other, and how we determine our realities. As the technologies ... give us deeper access to parts of ourselves, we may begin to integrate digital technologies into our bodies.

You can see what the game is. Twenty-four hour control and people – if you could still call them that – would never know when something would go ping and take them out of circulation. It's the most obvious rush to a global fascist dictatorship and the complete submission of humanity and yet still so many are locked away in their Cult-induced perceptual coma and can't see it.

## **Smart Grid control centres**

The human body is being transformed by the 'vaccines' and in other ways into a synthetic cyborg that can be attached to the global Smart Grid which would be controlled from a central point and other sub-locations of Grid manipulation. Where are these planned to be? Well, China for a start which is one of the Cult's biggest centres of operation. The technological control system and technocratic rule was incubated here to be unleashed across the world after the 'Covid' hoax came out of China in 2020. Another Smart Grid location that will surprise people new to this is Israel. I have exposed in *The Trigger* how Sabbatian technocrats, intelligence and military operatives were behind the horrors of 9/11 and not 19 Arab hijackers' who somehow manifested the ability to pilot big passenger airliners when instructors at puddle-jumping flying schools described some of them as a joke. The 9/11 attacks were made possible through control of civilian and military air computer systems and those of the White House, Pentagon and connected agencies. See *The Trigger* – it

will blow your mind. The controlling and coordinating force were the Sabbatian networks in Israel and the United States which by then had infiltrated the entire US government, military and intelligence system. The real name of the American Deep State is 'Sabbatian State'. Israel is a tiny country of only nine million people, but it is one of the global centres of cyber operations and fast catching Silicon Valley in importance to the Cult. Israel is known as the 'start-up nation' for all the cyber companies spawned there with the Sabbatian specialisation of 'cyber security' that I mentioned earlier which gives those companies access to computer systems of their clients in real time through 'backdoors' written into the coding when security software is downloaded. The Sabbatian centre of cyber operations outside Silicon Valley is the Israeli military Cyber Intelligence Unit, the biggest infrastructure project in Israel's history, headquartered in the desert-city of Beersheba and involving some 20,000 'cyber soldiers'. Here are located a literal army of Internet trolls scanning social media, forums and comment lists for anyone challenging the Cult agenda. The UK military has something similar with its 77th Brigade and associated operations. The Beersheba complex includes research and development centres for other Cult operations such as Intel, Microsoft, IBM, Google, Apple, Hewlett-Packard, Cisco Systems, Facebook and Motorola. [Techcrunch.com](#) ran an article about the Beersheba global Internet technology centre headlined 'Israel's desert city of Beersheba is turning into a cybertech oasis':

The military's massive relocation of its prestigious technology units, the presence of multinational and local companies, a close proximity to Ben Gurion University and generous government subsidies are turning Beersheba into a major global cybertech hub. Beersheba has all of the ingredients of a vibrant security technology ecosystem, including Ben Gurion University with its graduate program in cybersecurity and Cyber Security Research Center, and the presence of companies such as EMC, Deutsche Telekom, PayPal, Oracle, IBM, and Lockheed Martin. It's also the future home of the INCB (Israeli National Cyber Bureau); offers a special income tax incentive for cyber security companies, and was the site for the relocation of the army's intelligence corps units.

Sabbatians have taken over the cyber world through the following process: They scan the schools for likely cyber talent and develop them at Ben Gurion University and their period of conscription in the Israeli Defense Forces when they are stationed at the Beersheba complex. When the cyber talented officially leave the army they are funded to start cyber companies with technology developed by themselves or given to them by the state. Much of this is stolen through backdoors of computer systems around the world with America top of the list. Others are sent off to Silicon Valley to start companies or join the major ones and so we have many major positions filled by apparently 'Jewish' but really Sabbatian operatives. Google, YouTube and Facebook are all run by 'Jewish' CEOs while Twitter is all but run by ultra-Zionist hedge-fund shark Paul Singer. At the centre of the Sabbatian global cyber web is the Israeli army's Unit 8200 which specialises in hacking into computer systems of other countries, inserting viruses, gathering information, instigating malfunction, and even taking control of them from a distance. A long list of Sabbatians involved with 9/11, Silicon Valley and Israeli cyber security companies are operatives of Unit 8200. This is not about Israel. It's about the Cult. Israel is planned to be a Smart Grid hub as with China and what is happening at Beersheba is not for the benefit of Jewish people who are treated disgustingly by the Sabbatian elite that control the country. A glance at the Nuremberg Codes will tell you that.

The story is much bigger than 'Covid', important as that is to where we are being taken. Now, though, it's time to really strap in. There's more ... much more ...

## CHAPTER ELEVEN

### Who controls the Cult?

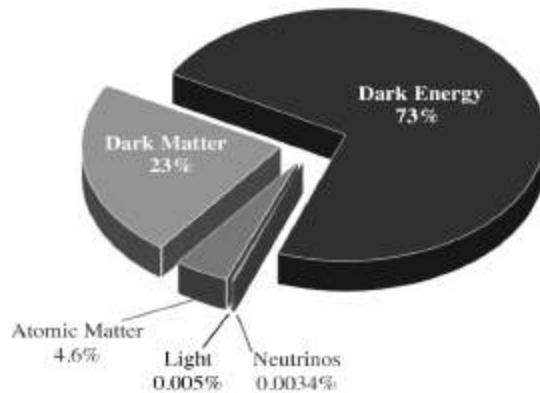
*Awake, arise or be forever fall'n*

**John Milton, Paradise Lost**

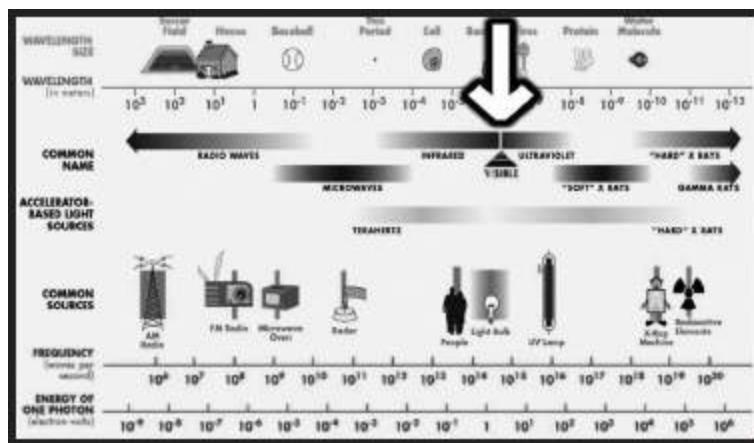
I have exposed this far the level of the Cult conspiracy that operates in the world of the seen and within the global secret society and satanic network which operates in the shadows one step back from the seen. The story, however, goes much deeper than that.

The 'Covid' hoax is major part of the Cult agenda, but only part, and to grasp the biggest picture we have to expand our attention beyond the realm of human sight and into the infinity of possibility that we cannot see. It is from here, ultimately, that humanity is being manipulated into a state of total control by the force which dictates the actions of the Cult. How much of reality can we see? Next to damn all is the answer. We may appear to see all there is to see in the 'space' our eyes survey and observe, but little could be further from the truth. The human 'world' is only a tiny band of frequency that the body's visual and perceptual systems can decode into *perception* of a 'world'. According to mainstream science the electromagnetic spectrum is 0.005 percent of what exists in the Universe ([Fig 10](#)). The maximum estimate I have seen is 0.5 percent and either way it's minuscule. I say it is far, far, smaller even than 0.005 percent when you compare reality we see with the totality of reality that we don't. Now get this if you are new to such information: Visible light, the only band of frequency that we can see, is a *fraction* of the 0.005

percent (Fig 11 overleaf). Take this further and realise that our universe is one of infinite universes and that universes are only a fragment of overall reality – *infinite* reality. Then compare that with the almost infinitesimal frequency band of visible light or human sight. You see that humans are as near blind as it is possible to be without actually being so. Artist and filmmaker, Sergio Toporek, said:



**Figure 10:** Humans can perceive such a tiny band of visual reality it's laughable.



**Figure 11:** We can see a smear of the 0.005 percent electromagnetic spectrum, but we still know it all. Yep, makes sense.

Consider that you can see less than 1% of the electromagnetic spectrum and hear less than 1% of the acoustic spectrum. 90% of the cells in your body carry their own microbial DNA and are not 'you'. The atoms in your body are 99.99999999999999% empty space and none of them are the ones you were born with ... Human beings have 46 chromosomes, two less than a potato.

The existence of the rainbow depends on the conical photoreceptors in your eyes; to animals without cones, the rainbow does not exist. So you don't just look at a rainbow, you create it. This is pretty amazing, especially considering that all the beautiful colours you see represent less than 1% of the electromagnetic spectrum.

Suddenly the 'world' of humans looks a very different place. Take into account, too, that Planet Earth when compared with the projected size of this single universe is the equivalent of a billionth of a pinhead. Imagine the ratio that would be when compared to infinite reality. To think that Christianity once insisted that Earth and humanity were the centre of everything. This background is vital if we are going to appreciate the nature of 'human' and how we can be manipulated by an unseen force. To human visual reality virtually *everything* is unseen and yet the prevailing perception within the institutions and so much of the public is that if we can't see it, touch it, hear it, taste it and smell it then it cannot exist. Such perception is indoctrinated and encouraged by the Cult and its agents because it isolates believers in the strictly limited, village-idiot, realm of the five senses where perceptions can be firewalled and information controlled. Most of those perpetuating the 'this-world-is-all-there-is' insanity are themselves indoctrinated into believing the same delusion. While major players and influencers know that official reality is laughable most of those in science, academia and medicine really believe the nonsense they peddle and teach succeeding generations. Those who challenge the orthodoxy are dismissed as nutters and freaks to protect the manufactured illusion from exposure. Observe the dynamic of the 'Covid' hoax and you will see how that takes the same form. The inner-circle psychopaths know it's a gigantic scam, but almost the entirety of those imposing their fascist rules believe that 'Covid' is all that they're told it is.

## **Stolen identity**

Ask people who they are and they will give you their name, place of birth, location, job, family background and life story. Yet that is not who they are – it is what they are *experiencing*. The difference is *absolutely crucial*. The true 'I', the eternal, infinite 'I', is consciousness,

a state of being aware. Forget ‘form’. That is a vehicle for a brief experience. Consciousness does not come *from* the brain, but *through* the brain and even that is more symbolic than literal. We are awareness, pure awareness, and this is what withdraws from the body at what we call ‘death’ to continue our eternal beingness, *isness*, in other realms of reality within the limitlessness of infinity or the Biblical ‘many mansions in my father’s house’. Labels of a human life, man, woman, transgender, black, white, brown, nationality, circumstances and income are not who we are. They are what we are – awareness – is *experiencing* in a brief connection with a band of frequency we call ‘human’. The labels are not the self; they are, to use the title of one of my books, a *Phantom Self*. I am not David Icke born in Leicester, England, on April 29th, 1952. I am the consciousness *having that experience*. The Cult and its non-human masters seek to convince us through the institutions of ‘education’, science, medicine, media and government that what we are *experiencing* is who we *are*. It’s so easy to control and direct perception locked away in the bewildered illusions of the five senses with no expanded radar. Try, by contrast, doing the same with a humanity aware of its true self and its true power to consciously create its reality and experience. How is it possible to do this? We do it all day every day. If you perceive yourself as ‘little me’ with no power to impact upon your life and the world then your life experience will reflect that. You will hand the power you don’t think you have to authority in all its forms which will use it to control your experience. This, in turn, will appear to confirm your perception of ‘little me’ in a self-fulfilling feedback loop. But that is what ‘little me’ really is – a *perception*. We are all ‘big-me’, infinite me, and the Cult has to make us forget that if its will is to prevail. We are therefore manipulated and pressured into self-identifying with human labels and not the consciousness/awareness *experiencing* those human labels.

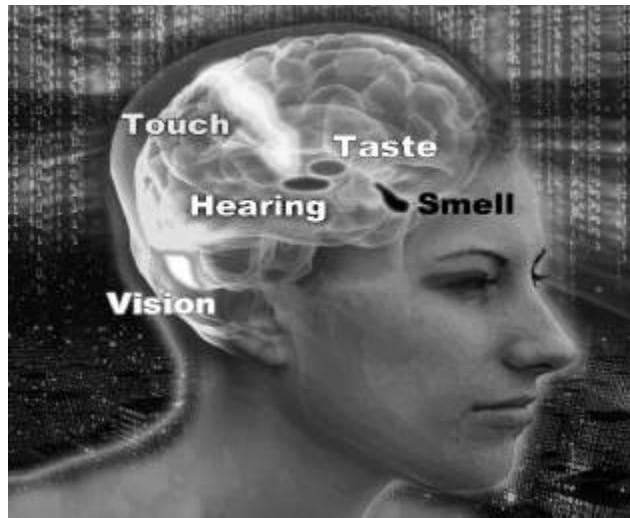
The phenomenon of identity politics is a Cult-instigated manipulation technique to sub-divide previous labels into even smaller ones. A United States university employs this list of letters to

describe student identity: LGBTQQFAGPBDSM or lesbian, gay, bisexual, transgender, transsexual, queer, questioning, flexual, asexual, gender-fuck, polyamorous, bondage/discipline, dominance/submission and sadism/masochism. I'm sure other lists are even longer by now as people feel the need to self-identify the 'I' with the minutiae of race and sexual preference. Wokers programmed by the Cult for generations believe this is about 'inclusivity' when it's really the Cult locking them away into smaller and smaller versions of Phantom Self while firewalls them from the influence of their true self, the infinite, eternal 'I'. You may notice that my philosophy which contends that we are all unique points of attention/awareness within the same infinite whole or Oneness is the ultimate non-racism. The very sense of Oneness makes the judgement of people by their body-type, colour or sexuality utterly ridiculous and confirms that racism has no understanding of reality (including anti-white racism). Yet despite my perception of life Cult agents and fast-asleep Wokers label me racist to discredit my information while they are themselves phenomenally racist and sexist. All they see is race and sexuality and they judge people as good or bad, demons or untouchables, by their race and sexuality. All they see is *Phantom Self* and perceive themselves in terms of *Phantom Self*. They are pawns and puppets of the Cult agenda to focus attention and self-identity in the five senses and play those identities against each other to divide and rule. Columbia University has introduced segregated graduations in another version of social distancing designed to drive people apart and teach them that different racial and cultural groups have nothing in common with each other. The last thing the Cult wants is unity. Again the pump-primers of this will be Cult operatives in the knowledge of what they are doing, but the rest are just the *Phantom Self* blind leading the *Phantom Self* blind. We *do* have something in common – we are all *the same consciousness* having different temporary experiences.

## **What is this 'human'?**

Yes, what *is* ‘human’? That is what we are supposed to be, right? I mean ‘human’? True, but ‘human’ is the experience not the ‘I’. Break it down to basics and ‘human’ is the way that information is processed. If we are to experience and interact with this band of frequency we call the ‘world’ we must have a vehicle that operates within that band of frequency. Our consciousness in its prime form cannot do that; it is way beyond the frequency of the human realm. My consciousness or awareness could not tap these keys and pick up the cup in front of me in the same way that radio station A cannot interact with radio station B when they are on different frequencies. The human body is the means through which we have that interaction. I have long described the body as a biological computer which processes information in a way that allows consciousness to experience this reality. The body is a receiver, transmitter and processor of information in a particular way that we call human. We visually perceive only the world of the five senses in a wakened state – that is the limit of the body’s visual decoding system. In truth it’s not even visual in the way we experience ‘visual reality’ as I will come to in a moment. We are ‘human’ because the body processes the information sources of human into a reality and behaviour system that we *perceive* as human. Why does an elephant act like an elephant and not like a human or a duck? The elephant’s biological computer is a different information field and processes information according to that program into a visual and behaviour type we call an elephant. The same applies to everything in our reality. These body information fields are perpetuated through procreation (like making a copy of a software program). The Cult wants to break that cycle and intervene technologically to transform the human information field into one that will change what we call humanity. If it can change the human information field it will change the way that field processes information and change humanity both ‘physically’ and psychologically. Hence the *messenger* (information) RNA ‘vaccines’ and so much more that is targeting human genetics by changing the body’s information – *messaging* – construct through food, drink, radiation, toxicity and other means.

Reality that we experience is nothing like reality as it really is in the same way that the reality people experience in virtual reality games is not the reality they are really living in. The game is only a decoded source of information that appears to be a reality. Our world is also an information construct – a *simulation* (more later). In its base form our reality is a wavefield of information much the same in theme as Wi-Fi. The five senses decode wavefield information into electrical information which they communicate to the brain to decode into holographic (illusory ‘physical’) information. Different parts of the brain specialise in decoding different senses and the information is fused into a reality that appears to be outside of us but is really inside the brain and the genetic structure in general ([Fig 12](#) overleaf). DNA is a receiver-transmitter of information and a vital part of this decoding process and the body’s connection to other realities. Change DNA and you change the way we decode and connect with reality – see ‘Covid vaccines’. Think of computers decoding Wi-Fi. You have information encoded in a radiation field and the computer decodes that information into a very different form on the screen. You can’t see the Wi-Fi until its information is made manifest on the screen and the information on the screen is inside the computer and not outside. I have just described how we decode the ‘human world’. All five senses decode the waveform ‘Wi-Fi’ field into electrical signals and the brain (computer) constructs reality inside the brain and not outside – ‘You don’t just look at a rainbow, you create it’. Sound is a simple example. We don’t hear sound until the brain decodes it. Waveform sound waves are picked up by the hearing sense and communicated to the brain in an electrical form to be decoded into the sounds that we hear. Everything we hear is inside the brain along with everything we see, feel, smell and taste. Words and language are waveform fields generated by our vocal chords which pass through this process until they are decoded by the brain into words that we hear. Different languages are different frequency fields or sound waves generated by vocal chords. Late British philosopher Alan Watts said:



**Figure 12:** The brain receives information from the five senses and constructs from that our perceived reality.

[Without the brain] the world is devoid of light, heat, weight, solidity, motion, space, time or any other imaginable feature. All these phenomena are interactions, or transactions, of vibrations with a certain arrangement of neurons.

That's exactly what they are and scientist Robert Lanza describes in his book, *Biocentrism*, how we decode electromagnetic waves and energy into visual and 'physical' experience. He uses the example of a flame emitting photons, electromagnetic energy, each pulsing electrically and magnetically:

... these ... invisible electromagnetic waves strike a human retina, and if (and only if) the waves happen to measure between 400 and 700 nano meters in length from crest to crest, then their energy is just right to deliver a stimulus to the 8 million cone-shaped cells in the retina.

Each in turn send an electrical pulse to a neighbour neuron, and on up the line this goes, at 250 mph, until it reaches the ... occipital lobe of the brain, in the back of the head. There, a cascading complex of neurons fire from the incoming stimuli, and we subjectively perceive this experience as a yellow brightness occurring in a place we have been conditioned to call the 'external world'.

## You hear what you decode

If a tree falls or a building collapses they make no noise unless someone is there to decode the energetic waves generated by the disturbance into what we call sound. Does a falling tree make a noise? Only if you hear it – *decode* it. Everything in our reality is a frequency field of information operating within the overall ‘Wi-Fi’ field that I call The Field. A vibrational disturbance is generated in The Field by the fields of the falling tree or building. These disturbance waves are what we decode into the sound of them falling. If no one is there to do that then neither will make any noise. Reality is created by the observer – *decoder* – and the *perceptions* of the observer affect the decoding process. For this reason different people – different *perceptions* – will perceive the same reality or situation in a different way. What one may perceive as a nightmare another will see as an opportunity. The question of why the Cult is so focused on controlling human perception now answers itself. All experienced reality is the act of decoding and we don’t experience Wi-Fi until it is decoded on the computer screen. The sight and sound of an Internet video is encoded in the Wi-Fi all around us, but we don’t see or hear it until the computer decodes that information. Taste, smell and touch are all phenomena of the brain as a result of the same process. We don’t taste, smell or feel anything except in the brain and there are pain relief techniques that seek to block the signal from the site of discomfort to the brain because if the brain doesn’t decode that signal we don’t feel pain. Pain is in the brain and only appears to be at the point of impact thanks to the feedback loop between them. We don’t see anything until electrical information from the sight senses is decoded in an area at the back of the brain. If that area is damaged we can go blind when our eyes are perfectly okay. So why do we go blind if we damage an eye? We damage the information processing between the waveform visual information and the visual decoding area of the brain. If information doesn’t reach the brain in a form it can decode then we can’t see the visual reality that it represents. What’s more the brain is decoding only a fraction of the information it receives and the rest is absorbed by the

sub-conscious mind. This explanation is from the science magazine, *Wonderpedia*:

Every second, 11 million sensations crackle along these [brain] pathways ... The brain is confronted with an alarming array of images, sounds and smells which it rigorously filters down until it is left with a manageable list of around 40. Thus 40 sensations per second make up what we perceive as reality.

The ‘world’ is not what people are told to believe that is it and the inner circles of the Cult *know that*.

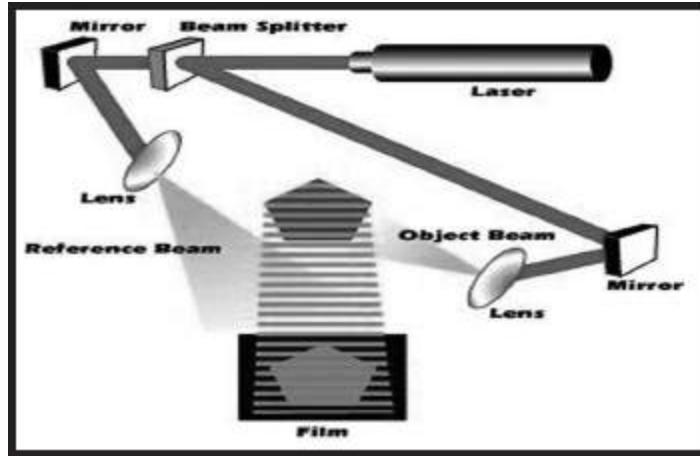
### **Illusory ‘physical’ reality**

We can only see a smear of 0.005 percent of the Universe which is only one of a vast array of universes – ‘mansions’ – within infinite reality. Even then the brain decodes only 40 pieces of information (‘sensations’) from a potential *11 million* that we receive every second. Two points strike you from this immediately: The sheer breathtaking stupidity of believing we know anything so rigidly that there’s nothing more to know; and the potential for these processes to be manipulated by a malevolent force to control the reality of the population. One thing I can say for sure with no risk of contradiction is that when you can perceive an almost indescribable fraction of infinite reality there is always more to know as in tidal waves of it. Ancient Greek philosopher Socrates was so right when he said that wisdom is to know how little we know. How obviously true that is when you think that we are experiencing a physical world of solidity that is neither physical nor solid and a world of apartness when everything is connected. Cult-controlled ‘science’ dismisses the so-called ‘paranormal’ and all phenomena related to that when the ‘para’-normal is perfectly normal and explains the alleged ‘great mysteries’ which dumbfound scientific minds. There is a reason for this. A ‘scientific mind’ in terms of the mainstream is a material mind, a five-sense mind imprisoned in see it, touch it, hear it, smell it and taste it. Phenomena and happenings that can’t be explained that way leave the ‘scientific mind’ bewildered and the rule is that if they

can't account for why something is happening then it can't, by definition, be happening. I beg to differ. Telepathy is thought waves passing through The Field (think wave disturbance again) to be decoded by someone able to connect with that wavelength (information). For example: You can pick up the thought waves of a friend at any distance and at the very least that will bring them to mind. A few minutes later the friend calls you. 'My god', you say, 'that's incredible – I was just thinking of you.' Ah, but *they* were thinking of *you* before they made the call and that's what you decoded. Native peoples not entrapped in five-sense reality do this so well it became known as the 'bush telegraph'. Those known as psychics and mediums (genuine ones) are doing the same only across dimensions of reality. 'Mind over matter' comes from the fact that matter and mind are the *same*. The state of one influences the state of the other. Indeed one *and* the other are illusions. They are aspects of the same field. Paranormal phenomena are all explainable so why are they still considered 'mysteries' or not happening? Once you go down this road of understanding you begin to expand awareness beyond the five senses and that's the nightmare for the Cult.



**Figure 13:** Holograms are not solid, but the best ones appear to be.



**Figure 14:** How holograms are created by capturing a waveform version of the subject image.

## Holographic ‘solidity’

Our reality is not solid, it is holographic. We are now well aware of holograms which are widely used today. Two-dimensional information is decoded into a three-dimensional reality that is not solid although can very much appear to be (Fig 13). Holograms are created with a laser divided into two parts. One goes directly onto a photographic print ('reference beam') and the other takes a waveform image of the subject ('working beam') before being directed onto the print where it 'collides' with the other half of the laser (Fig 14). This creates a *waveform* interference pattern which contains the wavefield information of whatever is being photographed (Fig 15 overleaf). The process can be likened to dropping pebbles in a pond. Waves generated by each one spread out across the water to collide with the others and create a wave representation of where the stones fell and at what speed, weight and distance. A waveform interference pattern of a hologram is akin to the waveform information in The Field which the five senses decode into electrical signals to be decoded by the brain into a holographic illusory 'physical' reality. In the same way when a laser (think human attention) is directed at the waveform interference pattern a three-dimensional version of the subject is projected into apparently 'solid' reality (Fig 16). An amazing trait of holograms reveals more 'paranormal mysteries'. Information of the *whole*

hologram is encoded in waveform in every part of the interference pattern by the way they are created. This means that every *part* of a hologram is a smaller version of the whole. Cut the interference wave-pattern into four and you won't get four parts of the image. You get quarter-sized versions of the *whole* image. The body is a hologram and the same applies. Here we have the basis of acupuncture, reflexology and other forms of healing which identify representations of the whole body in all of the parts, hands, feet, ears, everywhere. Skilled palm readers can do what they do because the information of whole body is encoded in the hand. The concept of as above, so below, comes from this.



**Figure 15:** A waveform interference pattern that holds the information that transforms into a hologram.



**Figure 16:** Holographic people including 'Elvis' holographically inserted to sing a duet with Celine Dion.

The question will be asked of why, if solidity is illusory, we can't just walk through walls and each other. The resistance is not solid against solid; it is electromagnetic field against electromagnetic field and we decode this into the *experience* of solid against solid. We should also not underestimate the power of belief to dictate reality. What you believe is impossible *will be*. Your belief impacts on your decoding processes and they won't decode what you think is impossible. What we believe we perceive and what we perceive we experience. 'Can't dos' and 'impossibles' are like a firewall in a computer system that won't put on the screen what the firewall blocks. How vital that is to understanding how human experience has been hijacked. I explain in *The Answer, Everything You Need To Know But Have Never Been Told* and other books a long list of 'mysteries' and 'paranormal' phenomena that are not mysterious and perfectly normal once you realise what reality is and how it works. 'Ghosts' can be seen to pass through 'solid' walls because the walls are not solid and the ghost is a discarnate entity operating on a frequency so different to that of the wall that it's like two radio stations sharing the same space while never interfering with each other. I have seen ghosts do this myself. The apartness of people and objects is also an illusion. Everything is connected by the Field like all sea life is connected by the sea. It's just that within the limits of our visual reality we only 'see' holographic information and not the field of information that connects everything and from which the holographic world is made manifest. If you can only see holographic 'objects' and not the field that connects them they will appear to you as unconnected to each other in the same way that we see the computer while not seeing the Wi-Fi.

## **What you don't know *can* hurt you**

Okay, we return to those 'two worlds' of human society and the Cult with its global network of interconnecting secret societies and satanic groups which manipulate through governments, corporations, media, religions, etc. The fundamental difference between them is *knowledge*. The idea has been to keep humanity

ignorant of the plan for its total enslavement underpinned by a crucial ignorance of reality – who we are and where we are – and how we interact with it. ‘Human’ should be the interaction between our expanded eternal consciousness and the five-sense body experience. We are meant to be *in* this world in terms of the five senses but not *of* this world in relation to our greater consciousness and perspective. In that state we experience the small picture of the five senses within the wider context of the big picture of awareness beyond the five senses. Put another way the five senses see the dots and expanded awareness connects them into pictures and patterns that give context to the apparently random and unconnected. Without the context of expanded awareness the five senses see only apartness and randomness with apparently no meaning. The Cult and its other-dimensional controllers seek to intervene in the frequency realm where five-sense reality is supposed to connect with expanded reality and to keep the two apart (more on this in the final chapter). When that happens five-sense mental and emotional processes are no longer influenced by expanded awareness, or the True ‘I’, and instead are driven by the isolated perceptions of the body’s decoding systems. They are in the world *and* of it. Here we have the human plight and why humanity with its potential for infinite awareness can be so easily manipulatable and descend into such extremes of stupidity.

Once the Cult isolates five-sense mind from expanded awareness it can then program the mind with perceptions and beliefs by controlling information that the mind receives through the ‘education’ system of the formative years and the media perceptual bombardment and censorship of an entire lifetime. Limit perception and a sense of the possible through limiting knowledge by limiting and skewing information while censoring and discrediting that which could set people free. As the title of another of my books says ... *And The Truth Shall Set You Free*. For this reason the last thing the Cult wants in circulation is the truth about anything – especially the reality of the eternal ‘I’ – and that’s why it is desperate to control information. The Cult knows that information becomes perception

which becomes behaviour which, collectively, becomes human society. Cult-controlled and funded mainstream ‘science’ denies the existence of an eternal ‘I’ and seeks to dismiss and trash all evidence to the contrary. Cult-controlled mainstream religion has a version of ‘God’ that is little more than a system of control and dictatorship that employs threats of damnation in an afterlife to control perceptions and behaviour in the here and now through fear and guilt. Neither is true and it’s the ‘neither’ that the Cult wishes to suppress. This ‘neither’ is that everything is an expression, a point of attention, within an infinite state of consciousness which is the real meaning of the term ‘God’.

Perceptual obsession with the ‘physical body’ and five-senses means that ‘God’ becomes personified as a bearded bloke sitting among the clouds or a raging bully who loves us if we do what ‘he’ wants and condemns us to the fires of hell if we don’t. These are no more than a ‘spiritual’ fairy tales to control and dictate events and behaviour through fear of this ‘God’ which has bizarrely made ‘God-fearing’ in religious circles a state to be desired. I would suggest that fearing *anything* is not to be encouraged and celebrated, but rather deleted. You can see why ‘God fearing’ is so beneficial to the Cult and its religions when *they* decide what ‘God’ wants and what ‘God’ demands (the Cult demands) that everyone do. As the great American comedian Bill Hicks said satirising a Christian zealot: ‘I think what God meant to say.’ How much of this infinite awareness (“God”) that we access is decided by how far we choose to expand our perceptions, self-identity and sense of the possible. The scale of self-identity reflects itself in the scale of awareness that we can connect with and are influenced by – how much knowing and insight we have instead of programmed perception. You cannot expand your awareness into the infinity of possibility when you believe that you are little me Peter the postman or Mary in marketing and nothing more. I’ll deal with this in the concluding chapter because it’s crucial to how we turnaround current events.

## **Where the Cult came from**

When I realised in the early 1990s there was a Cult network behind global events I asked the obvious question: When did it start? I took it back to ancient Rome and Egypt and on to Babylon and Sumer in Mesopotamia, the 'Land Between Two Rivers', in what we now call Iraq. The two rivers are the Tigris and Euphrates and this region is of immense historical and other importance to the Cult, as is the land called Israel only 550 miles away by air. There is much more going with deep esoteric meaning across this whole region. It's not only about 'wars for oil'. Priceless artefacts from Mesopotamia were stolen or destroyed after the American and British invasion of Iraq in 2003 justified by the lies of Boy Bush and Tony Blair (their Cult masters) about non-existent 'weapons of mass destruction'.

Mesopotamia was the location of Sumer (about 5,400BC to 1,750BC), and Babylon (about 2,350BC to 539BC). Sabbatians may have become immensely influential in the Cult in modern times but they are part of a network that goes back into the mists of history. Sumer is said by historians to be the 'cradle of civilisation'. I disagree. I say it was the re-start of what we call human civilisation after cataclysmic events symbolised in part as the 'Great Flood' destroyed the world that existed before. These fantastic upheavals that I have been describing in detail in the books since the early 1990s appear in accounts and legends of ancient cultures across the world and they are supported by geological and biological evidence. Stone tablets found in Iraq detailing the Sumer period say the cataclysms were caused by non-human 'gods' they call the Anunnaki. These are described in terms of extraterrestrial visitations in which knowledge supplied by the Anunnaki is said to have been the source of at least one of the world's oldest writing systems and developments in astronomy, mathematics and architecture that were way ahead of their time. I have covered this subject at length in *The Biggest Secret* and *Children of the Matrix* and the same basic 'Anunnaki' story can be found in Zulu accounts in South Africa where the late and very great Zulu high shaman Credo Mutwa told me that the Sumerian Anunnaki were known by Zulus as the Chitauri or 'children of the serpent'. See my six-hour video interview with Credo on this subject entitled *The*

*Reptilian Agenda* recorded at his then home near Johannesburg in 1999 which you can watch on the Ickonic media platform.

The Cult emerged out of Sumer, Babylon and Egypt (and elsewhere) and established the Roman Empire before expanding with the Romans into northern Europe from where many empires were savagely imposed in the form of Cult-controlled societies all over the world. Mass death and destruction was their calling card. The Cult established its centre of operations in Europe and European Empires were Cult empires which allowed it to expand into a global force. Spanish and Portuguese colonialists headed for Central and South America while the British and French targeted North America. Africa was colonised by Britain, France, Belgium, the Netherlands, Portugal, Spain, Italy, and Germany. Some like Britain and France moved in on the Middle East. The British Empire was by far the biggest for a simple reason. By now Britain was the headquarters of the Cult from which it expanded to form Canada, the United States, Australia and New Zealand. The Sun never set on the British Empire such was the scale of its occupation. London remains a global centre for the Cult along with Rome and the Vatican although others have emerged in Israel and China. It is no accident that the 'virus' is alleged to have come out of China while Italy was chosen as the means to terrify the Western population into compliance with 'Covid' fascism. Nor that Israel has led the world in 'Covid' fascism and mass 'vaccination'.

You would think that I would mention the United States here, but while it has been an important means of imposing the Cult's will it is less significant than would appear and is currently in the process of having what power it does have deleted. The Cult in Europe has mostly loaded the guns for the US to fire. America has been controlled from Europe from the start through Cult operatives in Britain and Europe. The American Revolution was an illusion to make it appear that America was governing itself while very different forces were pulling the strings in the form of Cult families such as the Rothschilds through the Rockefellers and other subordinates. The Rockefellers are extremely close to Bill Gates and

established both scalpel and drug ‘medicine’ and the World Health Organization. They play a major role in the development and circulation of vaccines through the Rockefeller Foundation on which Bill Gates said his Foundation is based. Why wouldn’t this be the case when the Rockefellers and Gates are on the same team? Cult infiltration of human society goes way back into what we call history and has been constantly expanding and centralising power with the goal of establishing a global structure to dictate everything. Look how this has been advanced in great leaps with the ‘Covid’ hoax.

## **The non-human dimension**

I researched and observed the comings and goings of Cult operatives through the centuries and even thousands of years as they were born, worked to promote the agenda within the secret society and satanic networks, and then died for others to replace them. Clearly there had to be a coordinating force that spanned this entire period while operatives who would not have seen the end goal in their lifetimes came and went advancing the plan over millennia. I went in search of that coordinating force with the usual support from the extraordinary synchronicity of my life which has been an almost daily experience since 1990. I saw common themes in religious texts and ancient cultures about a non-human force manipulating human society from the hidden. Christianity calls this force Satan, the Devil and demons; Islam refers to the Jinn or Djinn; Zulus have their Chitauri (spelt in other ways in different parts of Africa); and the Gnostic people in Egypt in the period around and before 400AD referred to this phenomena as the ‘Archons’, a word meaning rulers in Greek. Central American cultures speak of the ‘Predators’ among other names and the same theme is everywhere. I will use ‘Archons’ as a collective name for all of them. When you see how their nature and behaviour is described all these different sources are clearly talking about the same force. Gnostics described the Archons in terms of ‘luminous fire’ while Islam relates the Jinn to ‘smokeless fire’. Some refer to beings in form that could occasionally be seen, but the most common of common theme is that they operate from

unseen realms which means almost all existence to the visual processes of humans. I had concluded that this was indeed the foundation of human control and that the Cult was operating within the human frequency band on behalf of this hidden force when I came across the writings of Gnostics which supported my conclusions in the most extraordinary way.

A sealed earthen jar was found in 1945 near the town of Nag Hammadi about 75-80 miles north of Luxor on the banks of the River Nile in Egypt. Inside was a treasure trove of manuscripts and texts left by the Gnostic people some 1,600 years earlier. They included 13 leather-bound papyrus codices (manuscripts) and more than 50 texts written in Coptic Egyptian estimated to have been hidden in the jar in the period of 400AD although the source of the information goes back much further. Gnostics oversaw the Great or Royal Library of Alexandria, the fantastic depository of ancient texts detailing advanced knowledge and accounts of human history. The Library was dismantled and destroyed in stages over a long period with the death-blow delivered by the Cult-established Roman Church in the period around 415AD. The Church of Rome was the Church of Babylon relocated as I said earlier. Gnostics were not a race. They were a way of perceiving reality. Whenever they established themselves and their information circulated the terrorists of the Church of Rome would target them for destruction. This happened with the Great Library and with the Gnostic Cathars who were burned to death by the psychopaths after a long period of oppression at the siege of the Castle of Monségur in southern France in 1244. The Church has always been terrified of Gnostic information which demolishes the official Christian narrative although there is much in the Bible that supports the Gnostic view if you read it in another way. To anyone studying the texts of what became known as the Nag Hammadi Library it is clear that great swathes of Christian and Biblical belief has its origin with Gnostics sources going back to Sumer. Gnostic themes have been twisted to manipulate the perceived reality of Bible believers. Biblical texts have been in the open for centuries where they could be changed while Gnostic

documents found at Nag Hammadi were sealed away and untouched for 1,600 years. What you see is what they wrote.

### **Use your *pneuma* not your *nous***

Gnosticism and Gnostic come from 'gnosis' which means knowledge, or rather *secret* knowledge, in the sense of spiritual awareness – knowledge about reality and life itself. The desperation of the Cult's Church of Rome to destroy the Gnostics can be understood when the knowledge they were circulating was the last thing the Cult wanted the population to know. Sixteen hundred years later the same Cult is working hard to undermine and silence me for the same reason. The dynamic between knowledge and ignorance is a constant. 'Time' appears to move on, but essential themes remain the same. We are told to 'use your *nous*', a Gnostic word for head/brain/intelligence. They said, however, that spiritual awakening or 'salvation' could only be secured by expanding awareness *beyond* what they called *nous* and into *pneuma* or Infinite Self. Obviously as I read these texts the parallels with what I have been saying since 1990 were fascinating to me. There is a universal truth that spans human history and in that case why wouldn't we be talking the same language 16 centuries apart? When you free yourself from the perception program of the five senses and explore expanded realms of consciousness you are going to connect with the same information no matter what the perceived 'era' within a manufactured timeline of a single and tiny range of manipulated frequency. Humans working with 'smart' technology or knocking rocks together in caves is only a timeline appearing to operate within the human frequency band. Expanded awareness and the knowledge it holds have always been there whether the era be Stone Age or computer age. We can only access that knowledge by opening ourselves to its frequency which the five-sense prison cell is designed to stop us doing. Gates, Fauci, Whitty, Vallance, Zuckerberg, Brin, Page, Wojcicki, Bezos, and all the others behind the 'Covid' hoax clearly have a long wait before their range of frequency can make that connection given that an open heart is

crucial to that as we shall see. Instead of accessing knowledge directly through expanded awareness it is given to Cult operatives by the secret society networks of the Cult where it has been passed on over thousands of years outside the public arena. Expanded realms of consciousness is where great artists, composers and writers find their inspiration and where truth awaits anyone open enough to connect with it. We need to go there fast.

## **Archon hijack**

A fifth of the Nag Hammadi texts describe the existence and manipulation of the Archons led by a 'Chief Archon' they call 'Yaldabaoth', or the 'Demiurge', and this is the Christian 'Devil', 'Satan', 'Lucifer', and his demons. Archons in Biblical symbolism are the 'fallen ones' which are also referred to as fallen angels after the angels expelled from heaven according to the Abrahamic religions of Judaism, Christianity and Islam. These angels are claimed to tempt humans to 'sin' ongoing and you will see how accurate that symbolism is during the rest of the book. The theme of 'original sin' is related to the 'Fall' when Adam and Eve were 'tempted by the serpent' and fell from a state of innocence and 'obedience' (connection) with God into a state of disobedience (disconnection). The Fall is said to have brought sin into the world and corrupted everything including human nature. Yaldabaoth, the 'Lord Archon', is described by Gnostics as a 'counterfeit spirit', 'The Blind One', 'The Blind God', and 'The Foolish One'. The Jewish name for Yaldabaoth in Talmudic writings is Samael which translates as 'Poison of God', or 'Blindness of God'. You see the parallels. Yaldabaoth in Islamic belief is the Muslim Jinn devil known as Shaytan – Shaytan is Satan as the same themes are found all over the world in every religion and culture. The 'Lord God' of the Old Testament is the 'Lord Archon' of Gnostic manuscripts and that's why he's such a bloodthirsty bastard. Satan is known by Christians as 'the Demon of Demons' and Gnostics called Yaldabaoth the 'Archon of Archons'. Both are known as 'The Deceiver'. We are talking about the same 'bloke' for sure and these common themes

using different names, storylines and symbolism tell a common tale of the human plight.

Archons are referred to in Nag Hammadi documents as mind parasites, inverters, guards, gatekeepers, detainers, judges, pitiless ones and deceivers. The 'Covid' hoax alone is a glaring example of all these things. The Biblical 'God' is so different in the Old and New Testaments because they are not describing the same phenomenon. The vindictive, angry, hate-filled, 'God' of the Old Testament, known as Yahweh, is Yaldabaoth who is depicted in Cult-dictated popular culture as the 'Dark Lord', 'Lord of Time', Lord (Darth) Vader and Dormammu, the evil ruler of the 'Dark Dimension' trying to take over the 'Earth Dimension' in the Marvel comic movie, *Dr Strange*. Yaldabaoth is both the Old Testament 'god' and the Biblical 'Satan'. Gnostics referred to Yaldabaoth as the 'Great Architect of the Universe' and the Cult-controlled Freemason network calls their god 'the Great Architect of the Universe' (also Grand Architect). The 'Great Architect' Yaldabaoth is symbolised by the Cult as the all-seeing eye at the top of the pyramid on the Great Seal of the United States and the dollar bill. Archon is encoded in *arch-itect* as it is in *arch-angels* and *arch-bishops*. All religions have the theme of a force for good and force for evil in some sort of spiritual war and there is a reason for that – the theme is true. The Cult and its non-human masters are quite happy for this to circulate. They present themselves as the force for good fighting evil when they are really the force of evil (absence of love). The whole foundation of Cult modus operandi is inversion. They promote themselves as a force for good and anyone challenging them in pursuit of peace, love, fairness, truth and justice is condemned as a satanic force for evil. This has been the game plan throughout history whether the Church of Rome inquisitions of non-believers or 'conspiracy theorists' and 'anti-vaxxers' of today. The technique is the same whatever the timeline era.

## **Yaldabaoth is revolting (true)**

Yaldabaoth and the Archons are said to have revolted against God with Yaldabaoth claiming to *be* God – the *All That Is*. The Old Testament ‘God’ (Yaldabaoth) demanded to be worshipped as such: ‘*I am the LORD, and there is none else, there is no God beside me*’ (Isaiah 45:5). I have quoted in other books a man who said he was the unofficial son of the late Baron Philippe de Rothschild of the Mouton-Rothschild wine producing estates in France who died in 1988 and he told me about the Rothschild ‘revolt from God’. The man said he was given the name Phillip Eugene de Rothschild and we shared long correspondence many years ago while he was living under another identity. He said that he was conceived through ‘occult incest’ which (within the Cult) was ‘normal and to be admired’. ‘Phillip’ told me about his experience attending satanic rituals with rich and famous people whom he names and you can see them and the wider background to Cult Satanism in my other books starting with *The Biggest Secret*. Cult rituals are interactions with Archontic ‘gods’. ‘Phillip’ described Baron Philippe de Rothschild as ‘a master Satanist and hater of God’ and he used the same term ‘revolt from God’ associated with Yaldabaoth/Satan/Lucifer/the Devil in describing the Sabbatian Rothschild dynasty. ‘I played a key role in my family’s revolt from God’, he said. That role was to infiltrate in classic Sabbatian style the Christian Church, but eventually he escaped the mind-prison to live another life. The Cult has been targeting religion in a plan to make worship of the Archons the global one-world religion. Infiltration of Satanism into modern ‘culture’, especially among the young, through music videos, stage shows and other means, is all part of this.

Nag Hammadi texts describe Yaldabaoth and the Archons in their prime form as energy – consciousness – and say they can take form if they choose in the same way that consciousness takes form as a human. Yaldabaoth is called ‘formless’ and represents a deeply inverted, distorted and chaotic state of consciousness which seeks to attach to humans and turn them into a likeness of itself in an attempt at assimilation. For that to happen it has to manipulate

humans into low frequency mental and emotional states that match its own. Archons can certainly appear in human form and this is the origin of the psychopathic personality. The energetic distortion Gnostics called Yaldabaoth is psychopathy. When psychopathic Archons take human form that human will be a psychopath as an expression of Yaldabaoth consciousness. Cult psychopaths are Archons in human form. The principle is the same as that portrayed in the 2009 *Avatar* movie when the American military travelled to a fictional Earth-like moon called Pandora in the Alpha Centauri star system to infiltrate a society of blue people, or Na'vi, by hiding within bodies that looked like the Na'vi. Archons posing as humans have a particular hybrid information field, part human, part Archon, (the ancient 'demigods') which processes information in a way that manifests behaviour to match their psychopathic evil, lack of empathy and compassion, and stops them being influenced by the empathy, compassion and love that a fully-human information field is capable of expressing. Cult bloodlines interbreed, be they royalty or dark suits, for this reason and you have their obsession with incest. Interbreeding with full-blown humans would dilute the Archontic energy field that guarantees psychopathy in its representatives in the human realm.

Gnostic writings say the main non-human forms that Archons take are *serpentine* (what I have called for decades 'reptilian' amid unbounded ridicule from the Archontically-programmed) and what Gnostics describe as 'an unborn baby or foetus with grey skin and dark, unmoving eyes'. This is an excellent representation of the ET 'Greys' of UFO folklore which large numbers of people claim to have seen and been abducted by – Zulu shaman Credo Mutwa among them. I agree with those that believe in extraterrestrial or interdimensional visitations today and for thousands of years past. No wonder with their advanced knowledge and technological capability they were perceived and worshipped as gods for technological and other 'miracles' they appeared to perform. Imagine someone arriving in a culture disconnected from the modern world with a smartphone and computer. They would be

seen as a ‘god’ capable of ‘miracles’. The Renegade Mind, however, wants to know the source of everything and not only the way that source manifests as human or non-human. In the same way that a Renegade Mind seeks the original source material for the ‘Covid virus’ to see if what is claimed is true. The original source of Archons in form is consciousness – the distorted state of consciousness known to Gnostics as Yaldabaoth.

### **‘Revolt from God’ is energetic disconnection**

Where I am going next will make a lot of sense of religious texts and ancient legends relating to ‘Satan’, Lucifer’ and the ‘gods’. Gnostic descriptions sync perfectly with the themes of my own research over the years in how they describe a consciousness distortion seeking to impose itself on human consciousness. I’ve referred to the core of infinite awareness in previous books as Infinite Awareness in Awareness of Itself. By that I mean a level of awareness that knows that it is all awareness and is aware of all awareness. From here comes the frequency of love in its true sense and balance which is what love is on one level – the balance of all forces into a single whole called Oneness and Isness. The more we disconnect from this state of love that many call ‘God’ the constituent parts of that Oneness start to unravel and express themselves as a part and not a whole. They become individualised as intellect, mind, selfishness, hatred, envy, desire for power over others, and such like. This is not a problem in the greater scheme in that ‘God’, the *All That Is*, can experience all these possibilities through different expressions of itself including humans. What we as expressions of the whole experience the *All That Is* experiences. We are the *All That Is* experiencing itself. As we withdraw from that state of Oneness we disconnect from its influence and things can get very unpleasant and very stupid. Archontic consciousness is at the extreme end of that. It has so disconnected from the influence of Oneness that it has become an inversion of unity and love, an inversion of everything, an inversion of life itself. Evil is appropriately live written backwards. Archontic consciousness is obsessed with death, an inversion of life,

and so its manifestations in Satanism are obsessed with death. They use inverted symbols in their rituals such as the inverted pentagram and cross. Sabbatians as Archontic consciousness incarnate invert Judaism and every other religion and culture they infiltrate. They seek disunity and chaos and they fear unity and harmony as they fear love like garlic to a vampire. As a result the Cult, Archons incarnate, act with such evil, psychopathy and lack of empathy and compassion disconnected as they are from the source of love. How could Bill Gates and the rest of the Archontic psychopaths do what they have to human society in the 'Covid' era with all the death, suffering and destruction involved and have no emotional consequence for the impact on others? Now you know. Why have Zuckerberg, Brin, Page, Wojcicki and company callously censored information warning about the dangers of the 'vaccine' while thousands have been dying and having severe, sometimes life-changing reactions? Now you know. Why have Tedros, Fauci, Whitty, Vallance and their like around the world been using case and death figures they're aware are fraudulent to justify lockdowns and all the deaths and destroyed lives that have come from that? Now you know. Why did Christian Drosten produce and promote a 'testing' protocol that he knew couldn't test for infectious disease which led to a global human catastrophe. Now you know. The Archontic mind doesn't give a shit ([Fig 17](#)). I personally think that Gates and major Cult insiders are a form of AI cyborg that the Archons want humans to become.

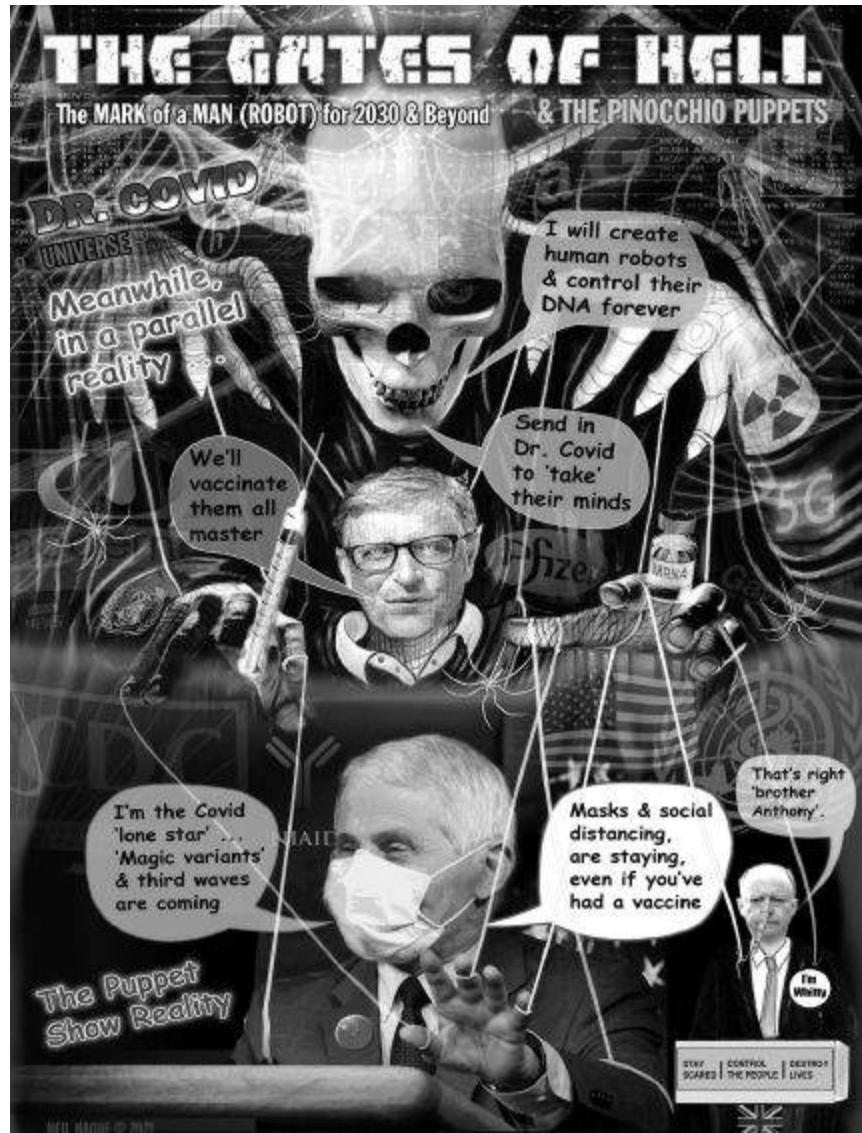


Figure 17: Artist Neil Hague's version of the 'Covid' hierarchy.

## Human batteries

A state of such inversion does have its consequences, however. The level of disconnection from the Source of All means that you withdraw from that source of energetic sustenance and creativity. This means that you have to find your own supply of energetic power and it has – *us*. When the Morpheus character in the first *Matrix* movie held up a battery he spoke a profound truth when he said: ‘The Matrix is a computer-generated dream world built to keep us under control in order to change the human being into one of

these.' The statement was true in all respects. We do live in a technologically-generated virtual reality simulation (more very shortly) and we have been manipulated to be an energy source for Archontic consciousness. The Disney-Pixar animated movie *Monsters, Inc.* in 2001 symbolised the dynamic when monsters in their world had no energy source and they would enter the human world to terrify children in their beds, catch the child's scream, terror (low-vibrational frequencies), and take that energy back to power the monster world. The lead character you might remember was a single giant eye and the symbolism of the Cult's all-seeing eye was obvious. Every thought and emotion is broadcast as a frequency unique to that thought and emotion. Feelings of love and joy, empathy and compassion, are high, quick, frequencies while fear, depression, anxiety, suffering and hate are low, slow, dense frequencies. Which kind do you think Archontic consciousness can connect with and absorb? In such a low and dense frequency state there's no way it can connect with the energy of love and joy. Archons can only feed off energy compatible with their own frequency and they and their Cult agents want to delete the human world of love and joy and manipulate the transmission of low vibrational frequencies through low-vibrational human mental and emotional states. *We are their energy source.* Wars are energetic banquets to the Archons – a world war even more so – and think how much low-frequency mental and emotional energy has been generated from the consequences for humanity of the 'Covid' hoax orchestrated by Archons incarnate like Gates.

The ancient practice of human sacrifice 'to the gods', continued in secret today by the Cult, is based on the same principle. 'The gods' are Archontic consciousness in different forms and the sacrifice is induced into a state of intense terror to generate the energy the Archontic frequency can absorb. Incarnate Archons in the ritual drink the blood which contains an adrenaline they crave which floods into the bloodstream when people are terrorised. Most of the sacrifices, ancient and modern, are children and the theme of 'sacrificing young virgins to the gods' is just code for children. They

have a particular pre-puberty energy that Archons want more than anything and the energy of the young in general is their target. The California Department of Education wants students to chant the names of Aztec gods (Archontic gods) once worshipped in human sacrifice rituals in a curriculum designed to encourage them to ‘challenge racist, bigoted, discriminatory, imperialist/colonial beliefs’, join ‘social movements that struggle for social justice’, and ‘build new possibilities for a post-racist, post-systemic racism society’. It’s the usual Woke crap that inverts racism and calls it anti-racism. In this case solidarity with ‘indigenous tribes’ is being used as an excuse to chant the names of ‘gods’ to which people were sacrificed (and still are in secret). What an example of Woke’s inability to see beyond black and white, us and them, They condemn the colonisation of these tribal cultures by Europeans (quite right), but those cultures sacrificing people including children to their ‘gods’, and mass murdering untold numbers as the Aztecs did, is just fine. One chant is to the Aztec god Tezcatlipoca who had a man sacrificed to him in the 5th month of the Aztec calendar. His heart was cut out and he was eaten. Oh, that’s okay then. Come on children … after three … Other sacrificial ‘gods’ for the young to chant their allegiance include Quetzalcoatl, Huitzilopochtli and Xipe Totec. The curriculum says that ‘chants, affirmations, and energizers can be used to bring the class together, build unity around ethnic studies principles and values, and to reinvigorate the class following a lesson that may be emotionally taxing or even when student engagement may appear to be low’. Well, that’s the cover story, anyway. Chanting and mantras are the repetition of a particular frequency generated from the vocal cords and chanting the names of these Archontic ‘gods’ tunes you into their frequency. That is the last thing you want when it allows for energetic synchronisation, attachment and perceptual influence. Initiates chant the names of their ‘Gods’ in their rituals for this very reason.

## **Vampires of the Woke**

Paedophilia is another way that Archons absorb the energy of children. Paedophiles possessed by Archontic consciousness are used as the conduit during sexual abuse for discarnate Archons to vampire the energy of the young they desire so much. Stupendous numbers of children disappear every year never to be seen again although you would never know from the media. Imagine how much low-vibrational energy has been generated by children during the 'Covid' hoax when so many have become depressed and psychologically destroyed to the point of killing themselves.

Shocking numbers of children are now taken by the state from loving parents to be handed to others. I can tell you from long experience of researching this since 1996 that many end up with paedophiles and assets of the Cult through corrupt and Cult-owned social services which in the reframing era has hired many psychopaths and emotionless automatons to do the job. Children are even stolen to order using spurious reasons to take them by the corrupt and secret (because they're corrupt) 'family courts'. I have written in detail in other books, starting with *The Biggest Secret* in 1997, about the ubiquitous connections between the political, corporate, government, intelligence and military elites (Cult operatives) and Satanism and paedophilia. If you go deep enough both networks have an interlocking leadership. The Woke mentality has been developed by the Cult for many reasons: To promote almost every aspect of its agenda; to hijack the traditional political left and turn it fascist; to divide and rule; and to target agenda pushbackers. But there are other reasons which relate to what I am describing here. How many happy and joyful Wokers do you ever see especially at the extreme end? They are a mental and psychological mess consumed by emotional stress and constantly emotionally cocked for the next explosion of indignation at someone referring to a female as a female. They are walking, talking, batteries as Morpheus might say emitting frequencies which both enslave them in low-vibrational bubbles of perceptual limitation and feed the Archons. Add to this the hatred claimed to be love; fascism claimed to 'anti-fascism', racism claimed to be 'anti-racism';

exclusion claimed to inclusion; and the abuse-filled Internet trolling. You have a purpose-built Archontic energy system with not a wind turbine in sight and all founded on Archontic *inversion*. We have whole generations now manipulated to serve the Archons with their actions and energy. They will be doing so their entire adult lives unless they snap out of their Archon-induced trance. Is it really a surprise that Cult billionaires and corporations put so much money their way? Where is the energy of joy and laughter, including laughing at yourself which is confirmation of your own emotional security? Mark Twain said: 'The human race has one really effective weapon, and that is laughter.' We must use it all the time. Woke has destroyed comedy because it has no humour, no joy, sense of irony, or self-deprecation. Its energy is dense and intense. *Mmmmm*, lunch says the Archontic frequency. Rudolf Steiner (1861-1925) was the Austrian philosopher and famous esoteric thinker who established Waldorf education or Steiner schools to treat children like unique expressions of consciousness and not minds to be programmed with the perceptions determined by authority. I'd been writing about this energy vampiring for decades when I was sent in 2016 a quote by Steiner. He was spot on:

There are beings in the spiritual realms for whom anxiety and fear emanating from human beings offer welcome food. When humans have no anxiety and fear, then these creatures starve. If fear and anxiety radiates from people and they break out in panic, then these creatures find welcome nutrition and they become more and more powerful. These beings are hostile towards humanity. Everything that feeds on negative feelings, on anxiety, fear and superstition, despair or doubt, are in reality hostile forces in super-sensible worlds, launching cruel attacks on human beings, while they are being fed ... These are exactly the feelings that belong to contemporary culture and materialism; because it estranges people from the spiritual world, it is especially suited to evoke hopelessness and fear of the unknown in people, thereby calling up the above mentioned hostile forces against them.

Pause for a moment from this perspective and reflect on what has happened in the world since the start of 2020. Not only will pennies drop, but billion dollar bills. We see the same theme from Don Juan Matus, a Yaqui Indian shaman in Mexico and the information source for Peruvian-born writer, Carlos Castaneda, who wrote a series of

books from the 1960s to 1990s. Don Juan described the force manipulating human society and his name for the Archons was the predator:

We have a predator that came from the depths of the cosmos and took over the rule of our lives. Human beings are its prisoners. The predator is our lord and master. It has rendered us docile, helpless. If we want to protest, it suppresses our protest. If we want to act independently, it demands that we don't do so ... indeed we are held prisoner!

They took us over because we are food to them, and they squeeze us mercilessly because we are their sustenance. Just as we rear chickens in coops, the predators rear us in human coops, humaneros. Therefore, their food is always available to them.

Different cultures, different eras, same recurring theme.

## The 'ennoia' dilemma

Nag Hammadi Gnostic manuscripts say that Archon consciousness has no 'ennoia'. This is directly translated as 'intentionality', but I'll use the term 'creative imagination'. The *All That Is* in awareness of itself is the source of all creativity – all possibility – and the more disconnected you are from that source the more you are subsequently denied 'creative imagination'. Given that Archon consciousness is almost entirely disconnected it severely lacks creativity and has to rely on far more mechanical processes of thought and exploit the creative potential of those that do have 'ennoia'. You can see cases of this throughout human society. Archon consciousness almost entirely dominates the global banking system and if we study how that system works you will appreciate what I mean. Banks manifest 'money' out of nothing by issuing lines of 'credit' which is 'money' that has never, does not, and will never exist except in theory. It's a confidence trick. If you think 'credit' figures-on-a-screen 'money' is worth anything you accept it as payment. If you don't then the whole system collapses through lack of confidence in the value of that 'money'. Archontic bankers with no 'ennoia' are 'lending' 'money' that doesn't exist to humans that *do* have creativity – those that have the inspired ideas and create businesses and products. Archon banking feeds off human creativity

which it controls through ‘money’ creation and debt. Humans have the creativity and Archons exploit that for their own benefit and control while having none themselves. Archon Internet platforms like Facebook claim joint copyright of everything that creative users post and while Archontic minds like Zuckerberg may officially head that company it will be human creatives on the staff that provide the creative inspiration. When you have limitless ‘money’ you can then buy other companies established by creative humans. Witness the acquisition record of Facebook, Google and their like. Survey the Archon-controlled music industry and you see non-creative dark suit executives making their fortune from the human creativity of their artists. The cases are endless. Research the history of people like Gates and Zuckerberg and how their empires were built on exploiting the creativity of others. Archon minds cannot create out of nothing, but they are skilled (because they have to be) in what Gnostic texts call ‘countermimicry’. They can imitate, but not innovate. Sabbatians trawl the creativity of others through backdoors they install in computer systems through their cybersecurity systems. Archon-controlled China is globally infamous for stealing intellectual property and I remember how Hong Kong, now part of China, became notorious for making counterfeit copies of the creativity of others – ‘countermimicry’. With the now pervasive and all-seeing surveillance systems able to infiltrate any computer you can appreciate the potential for Archons to vampire the creativity of humans. Author John Lamb Lash wrote in his book about the Nag Hammadi texts, *Not In His Image*:

Although they cannot originate anything, because they lack the divine factor of ennoia (intentionality), Archons can imitate with a vengeance. Their expertise is simulation (HAL, virtual reality). The Demiurge [Yaldabaoth] fashions a heaven world copied from the fractal patterns [of the original] ... His construction is celestial kitsch, like the fake Italianate villa of a Mafia don complete with militant angels to guard every portal.

This brings us to something that I have been speaking about since the turn of the millennium. Our reality is a simulation; a virtual reality that we think is real. No, I’m not kidding.

## **Human reality? Well, virtually**

I had pondered for years about whether our reality is ‘real’ or some kind of construct. I remembered being immensely affected on a visit as a small child in the late 1950s to the then newly-opened Planetarium on the Marylebone Road in London which is now closed and part of the adjacent Madame Tussauds wax museum. It was in the middle of the day, but when the lights went out there was the night sky projected in the Planetarium’s domed ceiling and it appeared to be so real. The experience never left me and I didn’t know why until around the turn of the millennium when I became certain that our ‘night sky’ and entire reality is a projection, a virtual reality, akin to the illusory world portrayed in the *Matrix* movies. I looked at the sky one day in this period and it appeared to me like the domed roof of the Planetarium. The release of the first *Matrix* movie in 1999 also provided a synchronistic and perfect visual representation of where my mind had been going for a long time. I hadn’t come across the Gnostic Nag Hammadi texts then. When I did years later the correlation was once again astounding. As I read Gnostic accounts from 1,600 years and more earlier it was clear that they were describing the same simulation phenomenon. They tell how the Yaldabaoth ‘Demiurge’ and Archons created a ‘bad copy’ of original reality to rule over all that were captured by its illusions and the body was a prison to trap consciousness in the ‘bad copy’ fake reality. Read how Gnostics describe the ‘bad copy’ and update that to current times and they are referring to what we would call today a virtual reality simulation.

Author John Lamb Lash said ‘the Demiurge fashions a heaven world copied from the fractal patterns’ of the original through expertise in ‘HAL’ or virtual reality simulation. Fractal patterns are part of the energetic information construct of our reality, a sort of blueprint. If these patterns were copied in computer terms it would indeed give you a copy of a ‘natural’ reality in a non-natural frequency and digital form. The principle is the same as making a copy of a website. The original website still exists, but now you can change the copy version to make it whatever you like and it can

become very different to the original website. Archons have done this with our reality, a *synthetic* copy of prime reality that still exists beyond the frequency walls of the simulation. Trapped within the illusions of this synthetic Matrix, however, were and are human consciousness and other expressions of prime reality and this is why the Archons via the Cult are seeking to make the human body synthetic and give us synthetic AI minds to complete the job of turning the entire reality synthetic including what we perceive to be the natural world. To quote Kurzweil: ‘Nanobots will infuse all the matter around us with information. Rocks, trees, everything will become these intelligent creatures.’ Yes, *synthetic* ‘creatures’ just as ‘Covid’ and other genetically-manipulating ‘vaccines’ are designed to make the human body synthetic. From this perspective it is obvious why Archons and their Cult are so desperate to infuse synthetic material into every human with their ‘Covid’ scam.

### **Let there be (electromagnetic) light**

Yaldabaoth, the force that created the simulation, or Matrix, makes sense of the Gnostic reference to ‘The Great Architect’ and its use by Cult Freemasonry as the name of its deity. The designer of the Matrix in the movies is called ‘The Architect’ and that trilogy is jam-packed with symbolism relating to these subjects. I have contended for years that the angry Old Testament God (Yaldabaoth) is the ‘God’ being symbolically ‘quoted’ in the opening of Genesis as ‘creating the world’. This is not the creation of prime reality – it’s the creation of the *simulation*. The Genesis ‘God’ says: ‘Let there be Light: and there was light.’ But what is this ‘Light’? I have said for decades that the speed of light (186,000 miles per second) is not the fastest speed possible as claimed by mainstream science and is in fact the frequency walls or outer limits of the Matrix. You can’t have a fastest or slowest anything within all possibility when everything is possible. The human body is encoded to operate within the speed of light or *within the simulation* and thus we see only the tiny frequency band of visible *light*. Near-death experiencers who perceive reality outside the body during temporary ‘death’ describe a very different

form of light and this is supported by the Nag Hammadi texts. Prime reality beyond the simulation ('Upper Aeons' to the Gnostics) is described as a realm of incredible beauty, bliss, love and harmony – a realm of 'watery light' that is so powerful 'there are no shadows'. Our false reality of Archon control, which Gnostics call the 'Lower Aeons', is depicted as a realm with a different kind of 'light' and described in terms of chaos, 'Hell', 'the Abyss' and 'Outer Darkness', where trapped souls are tormented and manipulated by demons (relate that to the 'Covid' hoax alone). The watery light theme can be found in near-death accounts and it is not the same as *simulation* 'light' which is electromagnetic or radiation light within the speed of light – the 'Lower Aeons'. Simulation 'light' is the 'luminous fire' associated by Gnostics with the Archons. The Bible refers to Yaldabaoth as 'that old serpent, called the Devil, and Satan, which deceiveth the whole world' (Revelation 12:9). I think that making a simulated copy of prime reality ('countermimicry') and changing it dramatically while all the time manipulating humanity to believe it to be real could probably meet the criteria of deceiving the whole world. Then we come to the Cult god Lucifer – the *Light Bringer*. Lucifer is symbolic of Yaldabaoth, the bringer of radiation light that forms the bad copy simulation within the speed of light. 'He' is symbolised by the lighted torch held by the Statue of Liberty and in the name 'Illuminati'. Sabbatian-Frankism declares that Lucifer is the true god and Lucifer is the real god of Freemasonry honoured as their 'Great or Grand Architect of the Universe' (simulation).

I would emphasise, too, the way Archontic technologically-generated luminous fire of radiation has deluged our environment since I was a kid in the 1950s and changed the nature of The Field with which we constantly interact. Through that interaction technological radiation is changing us. The Smart Grid is designed to operate with immense levels of communication power with 5G expanding across the world and 6G, 7G, in the process of development. Radiation is the simulation and the Archontic manipulation system. Why wouldn't the Archon Cult wish to unleash radiation upon us to an ever-greater extreme to form

Kurzweil's 'cloud'? The plan for a synthetic human is related to the need to cope with levels of radiation beyond even anything we've seen so far. Biological humans would not survive the scale of radiation they have in their script. The Smart Grid is a technological sub-reality within the technological simulation to further disconnect five-sense perception from expanded consciousness. It's a technological prison of the mind.

### **Infusing the 'spirit of darkness'**

A recurring theme in religion and native cultures is the manipulation of human genetics by a non-human force and most famously recorded as the biblical 'sons of god' (the gods plural in the original) who interbred with the daughters of men. The Nag Hammadi *Apocryphon of John* tells the same story this way:

He [Yaldabaoth] sent his angels [Archons/demons] to the daughters of men, that they might take some of them for themselves and raise offspring for their enjoyment. And at first they did not succeed. When they had no success, they gathered together again and they made a plan together ... And the angels changed themselves in their likeness into the likeness of their mates, filling them with the spirit of darkness, which they had mixed for them, and with evil ... And they took women and begot children out of the darkness according to the likeness of their spirit.

Possession when a discarnate entity takes over a human body is an age-old theme and continues today. It's very real and I've seen it. Satanic and secret society rituals can create an energetic environment in which entities can attach to initiates and I've heard many stories of how people have changed their personality after being initiated even into lower levels of the Freemasons. I have been inside three Masonic temples, one at a public open day and two by just walking in when there was no one around to stop me. They were in Ryde, the town where I live, Birmingham, England, when I was with a group, and Boston, Massachusetts. They all felt the same energetically – dark, dense, low-vibrational and sinister. Demonic attachment can happen while the initiate has no idea what is going on. To them it's just a ritual to get in the Masons and do a bit of good

business. In the far more extreme rituals of Satanism human possession is even more powerful and they are designed to make possession possible. The hierarchy of the Cult is dictated by the power and perceived status of the possessing Archon. In this way the Archon hierarchy becomes the Cult hierarchy. Once the entity has attached it can influence perception and behaviour and if it attaches to the extreme then so much of its energy (information) infuses into the body information field that the hologram starts to reflect the nature of the possessing entity. This is the *Exorcist* movie type of possession when facial features change and it's known as shapeshifting. Islam's Jinn are said to be invisible tricksters who change shape, 'whisper', confuse and take human form. These are all traits of the Archons and other versions of the same phenomenon. Extreme possession could certainty infuse the 'spirit of darkness' into a partner during sex as the Nag Hammadi texts appear to describe. Such an infusion can change genetics which is also energetic information. Human genetics is information and the 'spirit of darkness' is information. Mix one with the other and change must happen. Islam has the concept of a 'Jinn baby' through possession of the mother and by Jinn taking human form. There are many ways that human genetics can be changed and remember that Archons have been aware all along of advanced techniques to do this. What is being done in human society today – and far more – was known about by Archons at the time of the 'fallen ones' and their other versions described in religions and cultures.

Archons and their human-world Cult are obsessed with genetics as we see today and they know this dictates how information is processed into perceived reality during a human life. They needed to produce a human form that would decode the simulation and this is symbolically known as 'Adam and Eve' who left the 'garden' (prime reality) and 'fell' into Matrix reality. The simulation is not a 'physical' construct (there is no 'physical'); it is a source of information. Think Wi-Fi again. The simulation is an energetic field encoded with information and body-brain systems are designed to decode that information encoded in wave or frequency form which

is transmitted to the brain as electrical signals. These are decoded by the brain to construct our sense of reality – an illusory ‘physical’ world that only exists in the brain or the mind. Virtual reality games mimic this process using the same sensory decoding system. Information is fed to the senses to decode a virtual reality that can appear so real, but isn’t (Figs 18 and 19). Some scientists believe – and I agree with them – that what we perceive as ‘physical’ reality only exists when we are looking or observing. The act of perception or focus triggers the decoding systems which turn waveform information into holographic reality. When we are not observing something our reality reverts from a holographic state to a waveform state. This relates to the same principle as a falling tree not making a noise unless someone is there to hear it or decode it. The concept makes sense from the simulation perspective. A computer is not decoding all the information in a Wi-Fi field all the time and only decodes or brings into reality on the screen that part of Wi-Fi that it’s decoding – focusing upon – at that moment.



**Figure 18:** Virtual reality technology ‘hacks’ into the body’s five-sense decoding system.



**Figure 19:** The result can be experienced as very ‘real’.

Interestingly, Professor Donald Hoffman at the Department of Cognitive Sciences at the University of California, Irvine, says that our experienced reality is like a computer interface that shows us only the level with which we interact while hiding all that exists beyond it: ‘Evolution shaped us with a user interface that hides the truth. Nothing that we see is the truth – the very language of space and time and objects is the wrong language to describe reality.’ He is correct in what he says on so many levels. Space and time are not a universal reality. They are a phenomenon of decoded *simulation* reality as part of the process of enslaving our sense of reality. Near-death experiencers report again and again how space and time did not exist as we perceive them once they were free of the body – body decoding systems. You can appreciate from this why Archons and their Cult are so desperate to entrap human attention in the five senses where we are in the Matrix and of the Matrix. Opening your mind to expanded states of awareness takes you beyond the information confines of the simulation and you become aware of knowledge and insights denied to you before. This is what we call ‘awakening’ – *awakening from the Matrix* – and in the final chapter I will relate this to current events.

## **Where are the ‘aliens’?**

A simulation would explain the so-called ‘Fermi Paradox’ named after Italian physicist Enrico Fermi (1901-1954) who created the first nuclear reactor. He considered the question of why there is such a lack of extraterrestrial activity when there are so many stars and planets in an apparently vast universe; but what if the night sky that we see, or think we do, is a simulated projection as I say? If you control the simulation and your aim is to hold humanity fast in essential ignorance would you want other forms of life including advanced life coming and going sharing information with humanity? Or would you want them to believe they were isolated and apparently alone? Themes of human isolation and apartness are common whether they be the perception of a lifeless universe or the fascist isolation laws of the ‘Covid’ era. Paradoxically the very

existence of a simulation means that we are not alone when some force had to construct it. My view is that experiences that people have reported all over the world for centuries with Reptilians and Grey entities are Archon phenomena as Nag Hammadi texts describe; and that benevolent 'alien' interactions are non-human groups that come in and out of the simulation by overcoming Archon attempts to keep them out. It should be highlighted, too, that Reptilians and Greys are obsessed with *genetics* and *technology* as related by cultural accounts and those who say they have been abducted by them. Technology is their way of overcoming some of the limitations in their creative potential and our technology-driven and controlled human society of today is *archetypical* Archon-Reptilian-Grey modus operandi. Technocracy is really *Archontocracy*. The Universe does not have to be as big as it appears with a simulation. There is no space or distance only information decoded into holographic reality. What we call 'space' is only the absence of holographic 'objects' and that 'space' is The Field of energetic information which connects everything into a single whole. The same applies with the artificially-generated information field of the simulation. The Universe is not big or small as a physical reality. It is decoded information, that's all, and its perceived size is decided by the way the simulation is encoded to make it appear. The entire night sky as we perceive it only exists in our brain and so where are those 'millions of light years'? The 'stars' on the ceiling of the Planetarium looked a vast distance away.

There's another point to mention about 'aliens'. I have been highlighting since the 1990s the plan to stage a fake 'alien invasion' to justify the centralisation of global power and a world military. Nazi scientist Werner von Braun, who was taken to America by Operation Paperclip after World War Two to help found NASA, told his American assistant Dr Carol Rosin about the Cult agenda when he knew he was dying in 1977. Rosin said that he told her about a sequence that would lead to total human control by a one-world government. This included threats from terrorism, rogue nations, meteors and asteroids before finally an 'alien invasion'. All of these

things, von Braun said, would be bogus and what I would refer to as a No-Problem-Reaction-Solution. Keep this in mind when ‘the aliens are coming’ is the new mantra. The aliens are not coming – they are *already here* and they have infiltrated human society while looking human. French-Canadian investigative journalist Serge Monast said in 1994 that he had uncovered a NASA/military operation called Project Blue Beam which fits with what Werner von Braun predicted. Monast died of a ‘heart attack’ in 1996 the day after he was arrested and spent a night in prison. He was 51. He said Blue Beam was a plan to stage an alien invasion that would include religious figures beamed holographically into the sky as part of a global manipulation to usher in a ‘new age’ of worshipping what I would say is the Cult ‘god’ Yaldabaoth in a one-world religion. Fake holographic asteroids are also said to be part of the plan which again syncs with von Braun. How could you stage an illusory threat from asteroids unless they were holographic inserts? This is pretty straightforward given the advanced technology outside the public arena and the fact that our ‘physical’ reality is holographic anyway. Information fields would be projected and we would decode them into the illusion of a ‘physical’ asteroid. If they can sell a global ‘pandemic’ with a ‘virus’ that doesn’t exist what will humans not believe if government and media tell them?

All this is particularly relevant as I write with the Pentagon planning to release in June, 2021, information about ‘UFO sightings’. I have been following the UFO story since the early 1990s and the common theme throughout has been government and military denials and cover up. More recently, however, the Pentagon has suddenly become more talkative and apparently open with Air Force pilot radar images released of unexplained craft moving and changing direction at speeds well beyond anything believed possible with human technology. Then, in March, 2021, former Director of National Intelligence John Ratcliffe said a Pentagon report months later in June would reveal a great deal of information about UFO sightings unknown to the public. He said the report would have ‘massive implications’. The order to do this was included bizarrely

in a \$2.3 trillion ‘coronavirus’ relief and government funding bill passed by the Trump administration at the end of 2020. I would add some serious notes of caution here. I have been pointing out since the 1990s that the US military and intelligence networks have long had craft – ‘flying saucers’ or anti-gravity craft – which any observer would take to be extraterrestrial in origin. Keeping this knowledge from the public allows craft flown by *humans* to be perceived as alien visitations. I am not saying that ‘aliens’ do not exist. I would be the last one to say that, but we have to be streetwise here. President Ronald Reagan told the UN General Assembly in 1987: ‘I occasionally think how quickly our differences worldwide would vanish if we were facing an alien threat from outside this world.’ That’s the idea. Unite against a common ‘enemy’ with a common purpose behind your ‘saviour force’ (the Cult) as this age-old technique of mass manipulation goes global.

### **Science moves this way ...**

I could find only one other person who was discussing the simulation hypothesis publicly when I concluded it was real. This was Nick Bostrom, a Swedish-born philosopher at the University of Oxford, who has explored for many years the possibility that human reality is a computer simulation although his version and mine are not the same. Today the simulation and holographic reality hypothesis have increasingly entered the scientific mainstream. Well, the more open-minded mainstream, that is. Here are a few of the ever-gathering examples. American nuclear physicist Silas Beane led a team of physicists at the University of Bonn in Germany pursuing the question of whether we live in a simulation. They concluded that we probably do and it was likely based on a lattice of cubes. They found that cosmic rays align with that specific pattern. The team highlighted the Greisen-Zatsepin-Kuzmin (GZK) limit which refers to cosmic ray particle interaction with cosmic background radiation that creates an apparent boundary for cosmic ray particles. They say in a paper entitled ‘Constraints on the Universe as a Numerical Simulation’ that this ‘pattern of constraint’ is exactly what you

would find with a computer simulation. They also made the point that a simulation would create its own ‘laws of physics’ that would limit possibility. I’ve been making the same point for decades that the *perceived* laws of physics relate only to this reality, or what I would later call the simulation. When designers write codes to create computer and virtual reality games they are the equivalent of the laws of physics for that game. Players interact within the limitations laid out by the coding. In the same way those who wrote the codes for the simulation decided the laws of physics that would apply. These can be overridden by expanded states of consciousness, but not by those enslaved in only five-sense awareness where simulation codes rule. Overriding the codes is what people call ‘miracles’. They are not. They are bypassing the encoded limits of the simulation. A population caught in simulation perception would have no idea that this was their plight. As the Bonn paper said: ‘Like a prisoner in a pitch-black cell we would not be able to see the “walls” of our prison.’ That’s true if people remain mesmerised by the five senses. Open to expanded awareness and those walls become very clear. The main one is the speed of light.

American theoretical physicist James Gates is another who has explored the simulation question and found considerable evidence to support the idea. Gates was Professor of Physics at the University of Maryland, Director of The Center for String and Particle Theory, and on Barack Obama’s Council of Advisors on Science and Technology. He and his team found *computer codes* of digital data embedded in the fabric of our reality. They relate to on-off electrical charges of 1 and 0 in the binary system used by computers. ‘We have no idea what they are doing there’, Gates said. They found within the energetic fabric mathematical sequences known as error-correcting codes or block codes that ‘reboot’ data to its original state or ‘default settings’ when something knocks it out of sync. Gates was asked if he had found a set of equations embedded in our reality indistinguishable from those that drive search engines and browsers and he said: ‘That is correct.’ Rich Terrile, director of the Centre for Evolutionary Computation and Automated Design at NASA’s Jet

Propulsion Laboratory, has said publicly that he believes the Universe is a digital hologram that must have been created by a form of intelligence. I agree with that in every way. Waveform information is delivered electrically by the senses to the brain which constructs a *digital* holographic reality that we call the ‘world’. This digital level of reality can be read by the esoteric art of numerology. Digital holograms are at the cutting edge of holographics today. We have digital technology everywhere designed to access and manipulate our digital level of perceived reality. Synthetic mRNA in ‘Covid vaccines’ has a digital component to manipulate the body’s digital ‘operating system’.

## **Reality is numbers**

How many know that our reality can be broken down to numbers and codes that are the same as computer games? Max Tegmark, a physicist at the Massachusetts Institute of Technology (MIT), is the author of *Our Mathematical Universe* in which he lays out how reality can be entirely described by numbers and maths in the way that a video game is encoded with the ‘physics’ of computer games. Our world and computer virtual reality are essentially the same.

Tegmark imagines the perceptions of characters in an advanced computer game when the graphics are so good they don’t know they are in a game. They think they can bump into real objects (electromagnetic resistance in our reality), fall in love and feel emotions like excitement. When they began to study the apparently ‘physical world’ of the video game they would realise that everything was made of pixels (which have been found in our energetic reality as must be the case when on one level our world is digital). What computer game characters thought was physical ‘stuff’, Tegmark said, could actually be broken down into numbers:

And we’re exactly in this situation in our world. We look around and it doesn’t seem that mathematical at all, but everything we see is made out of elementary particles like quarks and electrons. And what properties does an electron have? Does it have a smell or a colour or a texture? No! ... We physicists have come up with geeky names for [Electron] properties, like

electric charge, or spin, or lepton number, but the electron doesn't care what we call it, the properties are just numbers.

This is the illusory reality Gnostics were describing. This is the simulation. The A, C, G, and T codes of DNA have a binary value – A and C = 0 while G and T = 1. This has to be when the simulation is digital and the body must be digital to interact with it. Recurring mathematical sequences are encoded throughout reality and the body. They include the Fibonacci sequence in which the two previous numbers are added to get the next one, as in ... 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc. The sequence is encoded in the human face and body, proportions of animals, DNA, seed heads, pine cones, trees, shells, spiral galaxies, hurricanes and the number of petals in a flower. The list goes on and on. There are fractal patterns – a 'never-ending pattern that is infinitely complex and self-similar across all scales in the as above, so below, principle of holograms. These and other famous recurring geometrical and mathematical sequences such as Phi, Pi, Golden Mean, Golden Ratio and Golden Section are *computer codes* of the simulation. I had to laugh and give my head a shake the day I finished this book and it went into the production stage. I was sent an article in *Scientific American* published in April, 2021, with the headline 'Confirmed! We Live in a Simulation'. Two decades after I first said our reality is a simulation and the speed of light is its outer limit the article suggested that we do live in a simulation and that the speed of light is its outer limit. I left school at 15 and never passed a major exam in my life while the writer was up to his eyes in qualifications. As I will explain in the final chapter *knowing* is far better than thinking and they come from very different sources. The article rightly connected the speed of light to the processing speed of the 'Matrix' and said what has been in my books all this time ... 'If we are in a simulation, as it appears, then space is an abstract property written in code. It is not real'. No it's not and if we live in a simulation something created it and it wasn't *us*. 'That David Icke says we are manipulated by aliens' – he's crackers.'

## **Wow ...**

The reality that humanity thinks is so real is an illusion. Politicians, governments, scientists, doctors, academics, law enforcement, media, school and university curriculums, on and on, are all founded on a world that *does not exist* except as a simulated prison cell. Is it such a stretch to accept that 'Covid' doesn't exist when our entire 'physical' reality doesn't exist? Revealed here is the knowledge kept under raps in the Cult networks of compartmentalised secrecy to control humanity's sense of reality by inducing the population to believe in a reality that's not real. If it wasn't so tragic in its experiential consequences the whole thing would be hysterically funny. None of this is new to Renegade Minds. Ancient Greek philosopher Plato (about 428 to about 347BC) was a major influence on Gnostic belief and he described the human plight thousands of years ago with his Allegory of the Cave. He told the symbolic story of prisoners living in a cave who had never been outside. They were chained and could only see one wall of the cave while behind them was a fire that they could not see. Figures walked past the fire casting shadows on the prisoners' wall and those moving shadows became their sense of reality. Some prisoners began to study the shadows and were considered experts on them (today's academics and scientists), but what they studied was only an illusion (today's academics and scientists). A prisoner escaped from the cave and saw reality as it really is. When he returned to report this revelation they didn't believe him, called him mad and threatened to kill him if he tried to set them free. Plato's tale is not only a brilliant analogy of the human plight and our illusory reality. It describes, too, the dynamics of the 'Covid' hoax. I have only skimmed the surface of these subjects here. The aim of this book is to crisply connect all essential dots to put what is happening today into its true context. All subject areas and their connections in this chapter are covered in great evidential detail in *Everything You Need To Know, But Have Never Been Told* and *The Answer*.

They say that bewildered people 'can't see the forest for the trees'. Humanity, however, can't see the forest for the *twigs*. The five senses

see only twigs while Renegade Minds can see the forest and it's the forest where the answers lie with the connections that reveals. Breaking free of perceptual programming so the forest can be seen is the way we turn all this around. Not breaking free is how humanity got into this mess. The situation may seem hopeless, but I promise you it's not. We are a perceptual heartbeat from paradise if only we knew.

## CHAPTER TWELVE

### Escaping Wetiko

*Life is simply a vacation from the infinite*

Dean Cavanagh

Renegade Minds weave the web of life and events and see common themes in the apparently random. They are always there if you look for them and their pursuit is aided by incredible synchronicity that comes when your mind is open rather than mesmerised by what it thinks it can see.

Infinite awareness is infinite possibility and the more of infinite possibility that we access the more becomes infinitely possible. That may be stating the apparently obvious, but it is a devastatingly-powerful fact that can set us free. We are a point of attention within an infinity of consciousness. The question is how much of that infinity do we choose to access? How much knowledge, insight, awareness, wisdom, do we want to connect with and explore? If your focus is only in the five senses you will be influenced by a fraction of infinite awareness. I mean a range so tiny that it gives new meaning to infinitesimal. Limitation of self-identity and a sense of the possible limit accordingly your range of consciousness. We are what we think we are. Life is what we think it is. The dream is the dreamer and the dreamer is the dream. Buddhist philosophy puts it this way: 'As a thing is viewed, so it appears.' Most humans live in the realm of touch, taste, see, hear, and smell and that's the limit of their sense of the possible and sense of self. Many will follow a religion and speak of a God in his heaven, but their lives are still

dominated by the five senses in their perceptions and actions. The five senses become the arbiter of everything. When that happens all except a smear of infinity is sealed away from influence by the rigid, unyielding, reality bubbles that are the five-sense human or Phantom Self. Archon Cult methodology is to isolate consciousness within five-sense reality – the simulation – and then program that consciousness with a sense of self and the world through a deluge of life-long information designed to instil the desired perception that allows global control. Efforts to do this have increased dramatically with identity politics as identity bubbles are squeezed into the minutiae of five-sense detail which disconnect people even more profoundly from the infinite ‘I’.

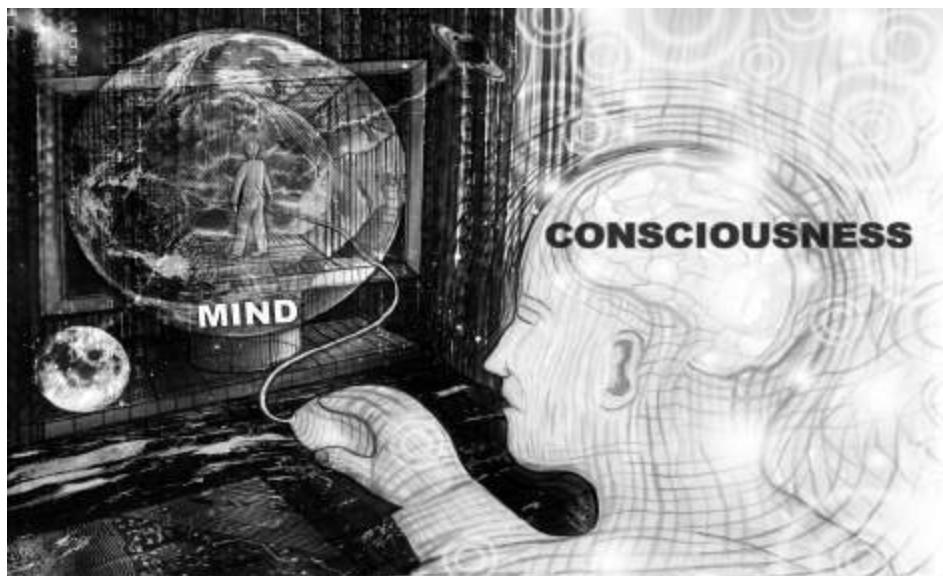
Five-sense focus and self-identity are like a firewall that limits access to the infinite realms. You only perceive one radio or television station and no other. We’ll take that literally for a moment. Imagine a vast array of stations giving different information and angles on reality, but you only ever listen to one. Here we have the human plight in which the population is overwhelmingly confined to CultFM. This relates only to the frequency range of CultFM and limits perception and insight to that band – limits *possibility* to that band. It means you are connecting with an almost imperceptibly minuscule range of possibility and creative potential within the infinite Field. It’s a world where everything seems apart from everything else and where synchronicity is rare. Synchronicity is defined in the dictionary as ‘the happening by chance of two or more related or similar events at the same time’. Use of ‘by chance’ betrays a complete misunderstanding of reality. Synchronicity is not ‘by chance’. As people open their minds, or ‘awaken’ to use the term, they notice more and more coincidences in their lives, bits of ‘luck’, apparently miraculous happenings that put them in the right place at the right time with the right people. Days become peppered with ‘fancy meeting you here’ and ‘what are the chances of that?’ My entire life has been lived like this and ever more so since my own colossal awakening in 1990 and 91 which transformed my sense of reality. Synchronicity is not ‘by chance’; it is by accessing expanded

realms of possibility which allow expanded potential for manifestation. People broadcasting the same vibe from the same openness of mind tend to be drawn ‘by chance’ to each other through what I call frequency magnetism and it’s not only people. In the last more than 30 years incredible synchronicity has also led me through the Cult maze to information in so many forms and to crucial personal experiences. These ‘coincidences’ have allowed me to put the puzzle pieces together across an enormous array of subjects and situations. Those who have breached the bubble of five-sense reality will know exactly what I mean and this escape from the perceptual prison cell is open to everyone whenever they make that choice. This may appear super-human when compared with the limitations of ‘human’, but it’s really our natural state. ‘Human’ as currently experienced is consciousness in an unnatural state of induced separation from the infinity of the whole. I’ll come to how this transformation into unity can be made when I have described in more detail the force that holds humanity in servitude by denying this access to infinite self.

## **The Wetiko factor**

I have been talking and writing for decades about the way five-sense mind is systematically barricaded from expanded awareness. I have used the analogy of a computer (five-sense mind) and someone at the keyboard (expanded awareness). Interaction between the computer and the operator is symbolic of the interaction between five-sense mind and expanded awareness. The computer directly experiences the Internet and the operator experiences the Internet via the computer which is how it’s supposed to be – the two working as one. Archons seek to control that point where the operator connects with the computer to stop that interaction ([Fig 20](#)). Now the operator is banging the keyboard and clicking the mouse, but the computer is not responding and this happens when the computer is taken over – *possessed* – by an appropriately-named computer ‘virus’. The operator has lost all influence over the computer which goes its own way making decisions under the control of the ‘virus’. I have

just described the dynamic through which the force known to Gnostics as Yaldabaoth and Archons disconnects five-sense mind from expanded awareness to imprison humanity in perceptual servitude.



**Figure 20:** The mind ‘virus’ I have been writing about for decades seeks to isolate five-sense mind (the computer) from the true ‘I’. (Image by Neil Hague).

About a year ago I came across a Native American concept of Wetiko which describes precisely the same phenomenon. Wetiko is the spelling used by the Cree and there are other versions including wintiko and windigo used by other tribal groups. They spell the name with lower case, but I see Wetiko as a proper noun as with Archons and prefer a capital. I first saw an article about Wetiko by writer and researcher Paul Levy which so synced with what I had been writing about the computer/operator disconnection and later the Archons. I then read his book, the fascinating *Dispelling Wetiko, Breaking the Spell of Evil*. The parallels between what I had concluded long before and the Native American concept of Wetiko were so clear and obvious that it was almost funny. For Wetiko see the Gnostic Archons for sure and the Jinn, the Predators, and every other name for a force of evil, inversion and chaos. Wetiko is the Native American name for the force that divides the computer from

the operator ([Fig 21](#)). Indigenous author Jack D. Forbes, a founder of the Native American movement in the 1960s, wrote another book about Wetiko entitled *Columbus And Other Cannibals – The Wetiko Disease of Exploitation, Imperialism, and Terrorism* which I also read. Forbes says that Wetiko refers to an evil person or spirit ‘who terrorizes other creatures by means of terrible acts, including cannibalism’. Zulu shaman Credo Mutwa told me that African accounts tell how cannibalism was brought into the world by the Chitauri ‘gods’ – another manifestation of Wetiko. The distinction between ‘evil person or spirit’ relates to Archons/Wetiko possessing a human or acting as pure consciousness. Wetiko is said to be a sickness of the soul or spirit and a state of being that takes but gives nothing back – the Cult and its operatives perfectly described. Black Hawk, a Native American war leader defending their lands from confiscation, said European invaders had ‘poisoned hearts’ – Wetiko hearts – and that this would spread to native societies. Mention of the heart is very significant as we shall shortly see. Forbes writes: ‘Tragically, the history of the world for the past 2,000 years is, in great part, the story of the epidemiology of the wetiko disease.’ Yes, and much longer. Forbes is correct when he says: ‘The wetikos destroyed Egypt and Babylon and Athens and Rome and Tenochtitlan [capital of the Aztec empire] and perhaps now they will destroy the entire earth.’ Evil, he said, is the number one export of a Wetiko culture – see its globalisation with ‘Covid’. Constant war, mass murder, suffering of all kinds, child abuse, Satanism, torture and human sacrifice are all expressions of Wetiko and the Wetiko possessed. The world is Wetiko made manifest, *but it doesn’t have to be*. There is a way out of this even now.



**Figure 21:** The mind ‘virus’ is known to Native Americans as ‘Wetiko’. (Image by Neil Hague).

## Cult of Wetiko

Wetiko is the Yaldabaoth frequency distortion that seeks to attach to human consciousness and absorb it into its own. Once this connection is made Wetiko can drive the perceptions of the target which they believe to be coming from their own mind. All the horrors of history and today from mass killers to Satanists, paedophiles like Jeffrey Epstein and other psychopaths, are the embodiment of Wetiko and express its state of being in all its grotesqueness. The Cult is Wetiko incarnate, Yaldabaoth incarnate, and it seeks to facilitate Wetiko assimilation of humanity in totality into its distortion by manipulating the population into low frequency states that match its own. Paul Levy writes: ‘Holographically enforced within the psyche of every human being the wetiko virus pervades and underlies the entire field of consciousness, and can therefore potentially manifest through any one of us at any moment if we are not mindful.’ The ‘Covid’ hoax has achieved this with many people, but others have not fallen into Wetiko’s frequency lair. Players in the ‘Covid’ human catastrophe including Gates, Schwab, Tedros, Fauci, Whitty, Vallance, Johnson, Hancock, Ferguson, Drosten, and all the rest, including the psychopath psychologists, are expressions of Wetiko. This is why

they have no compassion or empathy and no emotional consequence for what they do that would make them stop doing it. Observe all the people who support the psychopaths in authority against the Pushbackers despite the damaging impact the psychopaths have on their own lives and their family's lives. You are again looking at Wetiko possession which prevents them seeing through the lies to the obvious scam going on. *Why can't they see it?* Wetiko won't let them see it. The perceptual divide that has now become a chasm is between the Wetikoed and the non-Wetikoed.

Paul Levy describes Wetiko in the same way that I have long described the Archontic force. They are the same distorted consciousness operating across dimensions of reality: '... the subtle body of wetiko is not located in the third dimension of space and time, literally existing in another dimension ... it is able to affect ordinary lives by mysteriously interpenetrating into our three-dimensional world.' Wetiko does this through its incarnate representatives in the Cult and by weaving itself into The Field which on our level of reality is the electromagnetic information field of the simulation or Matrix. More than that, the simulation *is* Wetiko / Yaldabaoth. Caleb Scharf, Director of Astrobiology at Columbia University, has speculated that 'alien life' could be so advanced that it has transcribed itself into the quantum realm to become what we call physics. He said intelligence indistinguishable from the fabric of the Universe would solve many of its greatest mysteries:

Perhaps hyper-advanced life isn't just external. Perhaps it's already all around. It is embedded in what we perceive to be physics itself, from the root behaviour of particles and fields to the phenomena of complexity and emergence ... In other words, life might not just be in the equations. It might BE the equations [My emphasis].

Scharf said it is possible that 'we don't recognise advanced life because it forms an integral and unsuspicious part of what we've considered to be the natural world'. I agree. Wetiko/Yaldabaoth *is* the simulation. We are literally in the body of the beast. But that doesn't mean it has to control us. We all have the power to overcome Wetiko

influence and the Cult knows that. I doubt it sleeps too well because it knows that.

## **Which Field?**

This, I suggest, is how it all works. There are two Fields. One is the fierce electromagnetic light of the Matrix within the speed of light; the other is the ‘watery light’ of The Field beyond the walls of the Matrix that connects with the Great Infinity. Five-sense mind and the decoding systems of the body attach us to the Field of Matrix light. They have to or we could not experience this reality. Five-sense mind sees only the Matrix Field of information while our expanded consciousness is part of the Infinity Field. When we open our minds, and most importantly our hearts, to the Infinity Field we have a mission control which gives us an expanded perspective, a road map, to understand the nature of the five-sense world. If we are isolated only in five-sense mind there is no mission control. We’re on our own trying to understand a world that’s constantly feeding us information to ensure we do not understand. People in this state can feel ‘lost’ and bewildered with no direction or radar. You can see ever more clearly those who are influenced by the Fields of Big Infinity or little five-sense mind simply by their views and behaviour with regard to the ‘Covid’ hoax. We have had this division throughout known human history with the mass of the people on one side and individuals who could see and intuit beyond the walls of the simulation – Plato’s prisoner who broke out of the cave and saw reality for what it is. Such people have always been targeted by Wetiko/Archon-possessed authority, burned at the stake or demonised as mad, bad and dangerous. The Cult today and its global network of ‘anti-hate’, ‘anti-fascist’ Woke groups are all expressions of Wetiko attacking those exposing the conspiracy, ‘Covid’ lies and the ‘vaccine’ agenda.

Woke as a whole is Wetiko which explains its black and white mentality and how at one it is with the Wetiko-possessed Cult. Paul Levy said: ‘To be in this paradigm is to still be under the thrall of a two-valued logic – where things are either true or false – of a

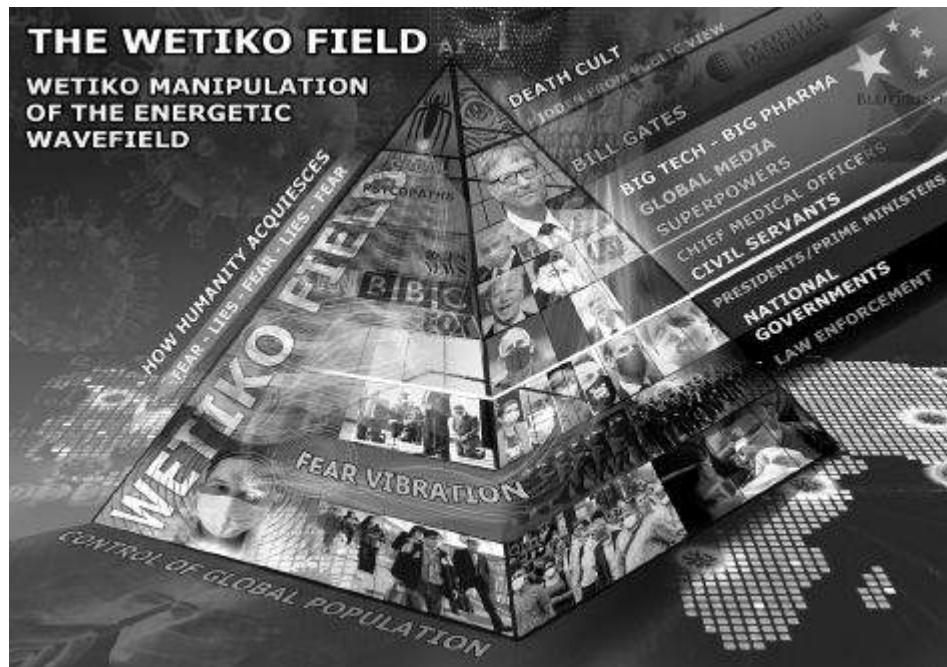
wetikoized mind.' Wetiko consciousness is in a permanent rage, therefore so is Woke, and then there is Woke inversion and contradiction. 'Anti-fascists' act like fascists because fascists *and* 'anti-fascists' are both Wetiko at work. Political parties act the same while claiming to be different for the same reason. Secret society and satanic rituals are attaching initiates to Wetiko and the cold, ruthless, psychopathic mentality that secures the positions of power all over the world is Wetiko. Reframing 'training programmes' have the same cumulative effect of attaching Wetiko and we have their graduates described as automatons and robots with a cold, psychopathic, uncaring demeanour. They are all traits of Wetiko possession and look how many times they have been described in this book and elsewhere with regard to personnel behind 'Covid' including the police and medical profession. Climbing the greasy pole in any profession in a Wetiko society requires traits of Wetiko to get there and that is particularly true of politics which is not about fair competition and pre-eminence of ideas. It is founded on how many backs you can stab and arses you can lick. This culminated in the global 'Covid' coordination between the Wetiko possessed who pulled it off in all the different countries without a trace of empathy and compassion for their impact on humans. Our sight sense can see only holographic form and not the Field which connects holographic form. Therefore we perceive 'physical' objects with 'space' in between. In fact that 'space' is energy/consciousness operating on multiple frequencies. One of them is Wetiko and that connects the Cult psychopaths, those who submit to the psychopaths, and those who serve the psychopaths in the media operations of the world. Wetiko is Gates. Wetiko is the mask-wearing submissive. Wetiko is the fake journalist and 'fact-checker'. The Wetiko Field is coordinating the whole thing. Psychopaths, gofers, media operatives, 'anti-hate' hate groups, 'fact-checkers' and submissive people work as one unit *even without human coordination* because they are attached to the *same* Field which is organising it all ([Fig 22](#)). Paul Levy is here describing how Wetiko-possessed people are drawn together and refuse to let any information breach their rigid

perceptions. He was writing long before ‘Covid’, but I think you will recognise followers of the ‘Covid’ religion *oh just a little bit*:

People who are channelling the vibratory frequency of wetiko align with each other through psychic resonance to reinforce their unspoken shared agreement so as to uphold their deranged view of reality. Once an unconscious content takes possession of certain individuals, it irresistibly draws them together by mutual attraction and knits them into groups tied together by their shared madness that can easily swell into an avalanche of insanity.

A psychic epidemic is a closed system, which is to say that it is insular and not open to any new information or informing influences from the outside world which contradict its fixed, limited, and limiting perspective.

There we have the Woke mind and the ‘Covid’ mind. Compatible resonance draws the awakening together, too, which is clearly happening today.



**Figure 22:** The Wetiko Field from which the Cult pyramid and its personnel are made manifest. (Image by Neil Hague).

## Spiritual servitude

Wetiko doesn't care about humans. It's not human; it just possesses humans for its own ends and the effect (depending on the scale of

possession) can be anything from extreme psychopathy to unquestioning obedience. Wetiko's worst nightmare is for human consciousness to expand beyond the simulation. Everything is focussed on stopping that happening through control of information, thus perception, thus frequency. The 'education system', media, science, medicine, academia, are all geared to maintaining humanity in five-sense servitude as is the constant stimulation of low-vibrational mental and emotional states (see 'Covid'). Wetiko seeks to dominate those subconscious spaces between five-sense perception and expanded consciousness where the computer meets the operator. From these subconscious hiding places Wetiko speaks to us to trigger urges and desires that we take to be our own and manipulate us into anything from low-vibrational to psychopathic states. Remember how Islam describes the Jinn as invisible tricksters that 'whisper' and confuse. Wetiko is the origin of the 'trickster god' theme that you find in cultures all over the world. Jinn, like the Archons, are Wetiko which is terrified of humans awakening and reconnecting with our true self for then its energy source has gone. With that the feedback loop breaks between Wetiko and human perception that provides the energetic momentum on which its very existence depends as a force of evil. Humans are both its target and its source of survival, but only if we are operating in low-vibrational states of fear, hate, depression and the background anxiety that most people suffer. We are Wetiko's target because we are its key to survival. It needs us, not the other way round. Paul Levy writes:

A vampire has no intrinsic, independent, substantial existence in its own right; it only exists in relation to us. The pathogenic, vampiric mind-parasite called wetiko is nothing in itself – not being able to exist from its own side – yet it has a 'virtual reality' such that it can potentially destroy our species ...

...The fact that a vampire is not reflected by a mirror can also mean that what we need to see is that there's nothing, no-thing to see, other than ourselves. The fact that wetiko is the expression of something inside of us means that the cure for wetiko is with us as well. The critical issue is finding this cure within us and then putting it into effect.

Evil begets evil because if evil does not constantly expand and find new sources of energetic sustenance its evil, its *distortion*, dies with the assimilation into balance and harmony. Love is the garlic to Wetiko's vampire. Evil, the absence of love, cannot exist in the presence of love. I think I see a way out of here. I have emphasised so many times over the decades that the Archons/Wetiko and their Cult are not all powerful. *They are not.* I don't care how it looks even now *they are not.* I have not called them little boys in short trousers for effect. I have said it because it is true. Wetiko's insatiable desire for power over others is not a sign of its omnipotence, but its insecurity. Paul Levy writes: 'Due to the primal fear which ultimately drives it and which it is driven to cultivate, wetiko's body politic has an intrinsic and insistent need for centralising power and control so as to create imagined safety for itself.' *Yeeeeees!* Exactly! Why does Wetiko want humans in an ongoing state of fear? Wetiko itself *is* fear and it is petrified of love. As evil is an absence of love, so love is an absence of fear. Love conquers all and *especially* Wetiko which *is* fear. Wetiko brought fear into the world when it wasn't here before. *Fear* was the 'fall', the fall into low-frequency ignorance and illusion – fear is False Emotion Appearing Real. The simulation is driven and energised by fear because Wetiko/Yaldabaoth (fear) *are* the simulation. Fear is the absence of love and Wetiko is the absence of love.

## **Wetiko today**

We can now view current events from this level of perspective. The 'Covid' hoax has generated momentous amounts of ongoing fear, anxiety, depression and despair which have empowered Wetiko. No wonder people like Gates have been the instigators when they are Wetiko incarnate and exhibit every trait of Wetiko in the extreme. See how cold and unemotional these people are like Gates and his cronies, how dead of eye they are. That's Wetiko. Sabbatians are Wetiko and everything they control including the World Health Organization, Big Pharma and the 'vaccine' makers, national 'health'

hierarchies, corporate media, Silicon Valley, the banking system, and the United Nations with its planned transformation into world government. All are controlled and possessed by the Wetiko distortion into distorting human society in its image. We are with this knowledge at the gateway to understanding the world.

Divisions of race, culture, creed and sexuality are diversions to hide the real division between those possessed and influenced by Wetiko and those that are not. The ‘Covid’ hoax has brought both clearly into view. Human behaviour is not about race. Tyrants and dictatorships come in all colours and creeds. What unites the US president bombing the innocent and an African tribe committing genocide against another as in Rwanda? What unites them? *Wetiko*. All wars are Wetiko, all genocide is Wetiko, all hunger over centuries in a world of plenty is Wetiko. Children going to bed hungry, including in the West, is Wetiko. Cult-generated Woke racial divisions that focus on the body are designed to obscure the reality that divisions in behaviour are manifestations of mind, not body. Obsession with body identity and group judgement is a means to divert attention from the real source of behaviour – mind and perception. Conflict sown by the Woke both within themselves and with their target groups are Wetiko providing lunch for itself through still more agents of the division, chaos, and fear on which it feeds. The Cult is seeking to assimilate the entirety of humanity and all children and young people into the Wetiko frequency by manipulating them into states of fear and despair. Witness all the suicide and psychological unravelling since the spring of 2020. Wetiko psychopaths want to impose a state of unquestioning obedience to authority which is no more than a conduit for Wetiko to enforce its will and assimilate humanity into itself. It needs us to believe that resistance is futile when it fears resistance and even more so the game-changing non-cooperation with its impositions. It can use violent resistance for its benefit. Violent impositions and violent resistance are *both* Wetiko. The Power of Love with its Power of No will sweep Wetiko from our world. Wetiko and its Cult know that. They just don’t want us to know.

## **AI Wetiko**

This brings me to AI or artificial intelligence and something else Wetikos don't want us to know. What is AI *really*? I know about computer code algorithms and AI that learns from data input. These, however, are more diversions, the expeditionary force, for the real AI that they want to connect to the human brain as promoted by Silicon Valley Wetikos like Kurzweil. What is this AI? It is the frequency of *Wetiko*, the frequency of the Archons. The connection of AI to the human brain is the connection of the Wetiko frequency to create a Wetiko hive mind and complete the job of assimilation. The hive mind is planned to be controlled from Israel and China which are both 100 percent owned by Wetiko Sabbatians. The assimilation process has been going on minute by minute in the 'smart' era which fused with the 'Covid' era. We are told that social media is scrambling the minds of the young and changing their personality. This is true, but what is social media? Look more deeply at how it works, how it creates divisions and conflict, the hostility and cruelty, the targeting of people until they are destroyed. That's Wetiko. Social media is manipulated to tune people to the Wetiko frequency with all the emotional exploitation tricks employed by platforms like Facebook and its Wetiko front man, Zuckerberg. Facebook's Instagram announced a new platform for children to overcome a legal bar on them using the main site. This is more Wetiko exploitation and manipulation of kids. Amnesty International likened the plan to foxes offering to guard the henhouse and said it was incompatible with human rights. Since when did Wetiko or Zuckerberg (I repeat myself) care about that? Would Brin and Page at Google, Wojcicki at YouTube, Bezos at Amazon and whoever the hell runs Twitter act as they do if they were not channelling Wetiko? Would those who are developing technologies for no other reason than human control? How about those designing and selling technologies to kill people and Big Pharma drug and 'vaccine' producers who know they will end or devastate lives? Quite a thought for these people to consider is that if you are Wetiko in a human life you are Wetiko on the 'other side' unless your frequency

changes and that can only change by a change of perception which becomes a change of behaviour. Where Gates is going does not bear thinking about although perhaps that's exactly where he wants to go. Either way, that's where he's going. His frequency will make it so.

## **The frequency lair**

I have been saying for a long time that a big part of the addiction to smartphones and devices is that a frequency is coming off them that entraps the mind. People spend ages on their phones and sometimes even a minute or so after they put them down they pick them up again and it all repeats. 'Covid' lockdowns will have increased this addiction a million times for obvious reasons. Addictions to alcohol overindulgence and drugs are another way that Wetiko entraps consciousness to attach to its own. Both are symptoms of low-vibrational psychological distress which alcoholism and drug addiction further compound. Do we think it's really a coincidence that access to them is made so easy while potions that can take people into realms beyond the simulation are banned and illegal? I have explored smartphone addiction in other books, the scale is mind-blowing, and that level of addiction does not come without help. Tech companies that make these phones are Wetiko and they will have no qualms about destroying the minds of children. We are seeing again with these companies the Wetiko perceptual combination of psychopathic enforcers and weak and meek unquestioning compliance by the rank and file.

The global Smart Grid is the Wetiko Grid and it is crucial to complete the Cult endgame. The simulation is radiation and we are being deluged with technological radiation on a devastating scale. Wetiko frauds like Elon Musk serve Cult interests while occasionally criticising them to maintain his street-cred. 5G and other forms of Wi-Fi are being directed at the earth from space on a volume and scale that goes on increasing by the day. Elon Musk's (officially) SpaceX Starlink project is in the process of putting tens of thousands of satellites in low orbit to cover every inch of the planet with 5G and other Wi-Fi to create Kurzweil's global 'cloud' to which the

human mind is planned to be attached very soon. SpaceX has approval to operate 12,000 satellites with more than 1,300 launched at the time of writing and applications filed for 30,000 more. Other operators in the Wi-Fi, 5G, low-orbit satellite market include OneWeb (UK), Telesat (Canada), and AST & Science (US). Musk tells us that AI could be the end of humanity and then launches a company called Neuralink to connect the human brain to computers. Musk's (in theory) Tesla company is building electric cars and the driverless vehicles of the smart control grid. As frauds and bullshitters go Elon Musk in my opinion is Major League.

5G and technological radiation in general are destructive to human health, genetics and psychology and increasing the strength of artificial radiation underpins the five-sense perceptual bubbles which are themselves expressions of radiation or electromagnetism. Freedom activist John Whitehead was so right with his 'databit by databit, we are building our own electronic concentration camps'. The Smart Grid and 5G is a means to control the human mind and infuse perceptual information into The Field to influence anyone in sync with its frequency. You can change perception and behaviour en masse if you can manipulate the population into those levels of frequency and this is happening all around us today. The arrogance of Musk and his fellow Cult operatives knows no bounds in the way that we see with Gates. Musk's satellites are so many in number already they are changing the night sky when viewed from Earth. The astronomy community has complained about this and they have seen nothing yet. Some consequences of Musk's Wetiko hubris include: Radiation; visible pollution of the night sky; interference with astronomy and meteorology; ground and water pollution from intensive use of increasingly many spaceports; accumulating space debris; continual deorbiting and burning up of aging satellites, polluting the atmosphere with toxic dust and smoke; and ever-increasing likelihood of collisions. A collective public open letter of complaint to Musk said:

We are writing to you ... because SpaceX is in process of surrounding the Earth with a network of thousands of satellites whose very purpose is to irradiate every square inch of the

Earth. SpaceX, like everyone else, is treating the radiation as if it were not there. As if the mitochondria in our cells do not depend on electrons moving undisturbed from the food we digest to the oxygen we breathe.

As if our nervous systems and our hearts are not subject to radio frequency interference like any piece of electronic equipment. As if the cancer, diabetes, and heart disease that now afflict a majority of the Earth's population are not metabolic diseases that result from interference with our cellular machinery. As if insects everywhere, and the birds and animals that eat them, are not starving to death as a result.

People like Musk and Gates believe in their limitless Wetiko arrogance that they can do whatever they like to the world because they own it. Consequences for humanity are irrelevant. It's absolutely time that we stopped taking this shit from these self-styled masters of the Earth when you consider where this is going.

## **Why is the Cult so anti-human?**

I hear this question often: Why would they do this when it will affect them, too? Ah, but will it? Who is this *them*? Forget their bodies. They are just vehicles for Wetiko consciousness. When you break it all down to the foundations we are looking at a state of severely distorted consciousness targeting another state of consciousness for assimilation. The rest is detail. The simulation is the fly-trap in which unique sensations of the five senses create a cycle of addiction called reincarnation. Renegade Minds see that everything which happens in our reality is a smaller version of the whole picture in line with the holographic principle. Addiction to the radiation of smart technology is a smaller version of addiction to the whole simulation. Connecting the body/brain to AI is taking that addiction on a giant step further to total ongoing control by assimilating human incarnate consciousness into Wetiko. I have watched during the 'Covid' hoax how many are becoming ever more profoundly attached to Wetiko's perceptual calling cards of aggressive response to any other point of view ('There is no other god but me'), psychopathic lack of compassion and empathy, and servile submission to the narrative and will of authority. Wetiko is the psychopaths *and* subservience to psychopaths. The Cult of Wetiko is

so anti-human because it is *not* human. It embarked on a mission to destroy human by targeting everything that it means to be human and to survive as human. ‘Covid’ is not the end, just a means to an end. The Cult with its Wetiko consciousness is seeking to change Earth systems, including the atmosphere, to suit them, not humans. The gathering bombardment of 5G alone from ground and space is dramatically changing The Field with which the five senses interact. There is so much more to come if we sit on our hands and hope it will all go away. It is not meant to go away. It is meant to get ever more extreme and we need to face that while we still can – just.

Carbon dioxide is the gas of life. Without that human is over. Kaput, gone, history. No natural world, no human. The Cult has created a cock and bull story about carbon dioxide and climate change to justify its reduction to the point where Gates and the ignoramus Biden ‘climate chief’ John Kerry want to suck it out of the atmosphere. Kerry wants to do this because his master Gates does. Wetikos have made the gas of life a demon with the usual support from the Wokers of Extinction Rebellion and similar organisations and the bewildered puppet-child that is Greta Thunberg who was put on the world stage by Klaus Schwab and the World Economic Forum. The name Extinction Rebellion is both ironic and as always Wetiko inversion. The gas that we need to survive must be reduced to save us from extinction. The most basic need of human is oxygen and we now have billions walking around in face nappies depriving body and brain of this essential requirement of human existence. More than that 5G at 60 gigahertz interacts with the oxygen molecule to reduce the amount of oxygen the body can absorb into the bloodstream. The obvious knock-on consequences of that for respiratory and cognitive problems and life itself need no further explanation. Psychopaths like Musk are assembling a global system of satellites to deluge the human atmosphere with this insanity. The man should be in jail. Here we have two most basic of human needs, oxygen and carbon dioxide, being dismantled.

Two others, water and food, are getting similar treatment with the United Nations Agendas 21 and 2030 – the Great Reset – planning to

centrally control all water and food supplies. People will not even own rain water that falls on their land. Food is affected at the most basic level by reducing carbon dioxide. We have genetic modification or GMO infiltrating the food chain on a mass scale, pesticides and herbicides polluting the air and destroying the soil. Freshwater fish that provide livelihoods for 60 million people and feed hundreds of millions worldwide are being 'pushed to the brink' according the conservationists while climate change is the only focus. Now we have Gates and Schwab wanting to dispense with current food sources all together and replace them with a synthetic version which the Wetiko Cult would control in terms of production and who eats and who doesn't. We have been on the Totalitarian Tiptoe to this for more than 60 years as food has become ever more processed and full of chemical shite to the point today when it's not natural food at all. As Dr Tom Cowan says: 'If it has a label don't eat it.' Bill Gates is now the biggest owner of farmland in the United States and he does nothing without an ulterior motive involving the Cult. Klaus Schwab wrote: 'To feed the world in the next 50 years we will need to produce as much food as was produced in the last 10,000 years ... food security will only be achieved, however, if regulations on genetically modified foods are adapted to reflect the reality that gene editing offers a precise, efficient and safe method of improving crops.' Liar. People and the world are being targeted with aluminium through vaccines, chemtrails, food, drink cans, and endless other sources when aluminium has been linked to many health issues including dementia which is increasing year after year. Insects, bees and wildlife essential to the food chain are being deleted by pesticides, herbicides and radiation which 5G is dramatically increasing with 6G and 7G to come. The pollinating bee population is being devastated while wildlife including birds, dolphins and whales are having their natural radar blocked by the effects of ever-increasing radiation. In the summer windscreens used to be splattered with insects so numerous were they. It doesn't happen now. Where have they gone?

## **Synthetic everything**

The Cult is introducing genetically-modified versions of trees, plants and insects including a Gates-funded project to unleash hundreds of millions of genetically-modified, lab-altered and patented male mosquitoes to mate with wild mosquitoes and induce genetic flaws that cause them to die out. Clinically-insane Gates-funded Japanese researchers have developed mosquitos that spread vaccine and are dubbed 'flying vaccinators'. Gates is funding the modification of weather patterns in part to sell the myth that this is caused by carbon dioxide and he's funding geoengineering of the skies to change the atmosphere. Some of this came to light with the Gates-backed plan to release tonnes of chalk into the atmosphere to 'deflect the Sun and cool the planet'. Funny how they do this while the heating effect of the Sun is not factored into climate projections focussed on carbon dioxide. The reason is that they want to reduce carbon dioxide (so don't mention the Sun), but at the same time they do want to reduce the impact of the Sun which is so essential to human life and health. I have mentioned the sun-cholesterol-vitamin D connection as they demonise the Sun with warnings about skin cancer (caused by the chemicals in sun cream they tell you to splash on). They come from the other end of the process with statin drugs to reduce cholesterol that turns sunlight into vitamin D. A lack of vitamin D leads to a long list of health effects and how vitamin D levels must have fallen with people confined to their homes over 'Covid'. Gates is funding other forms of geoengineering and most importantly chemtrails which are dropping heavy metals, aluminium and self-replicating nanotechnology onto the Earth which is killing the natural world. See *Everything You Need To Know, But Have Never Been Told* for the detailed background to this.

Every human system is being targeted for deletion by a force that's not human. The Wetiko Cult has embarked on the process of transforming the human body from biological to synthetic biological as I have explained. Biological is being replaced by the artificial and synthetic – Archontic 'countermimicry' – right across human society. The plan eventually is to dispense with the human body altogether

and absorb human consciousness – which it wouldn't really be by then – into cyberspace (the simulation which is Wetiko/Yaldabaoth). Preparations for that are already happening if people would care to look. The alternative media rightly warns about globalism and 'the globalists', but this is far bigger than that and represents the end of the human race as we know it. The 'bad copy' of prime reality that Gnostics describe was a bad copy of harmony, wonder and beauty to start with before Wetiko/Yaldabaoth set out to change the simulated 'copy' into something very different. The process was slow to start with. Entrapped humans in the simulation timeline were not technologically aware and they had to be brought up to intellectual speed while being suppressed spiritually to the point where they could build their own prison while having no idea they were doing so. We have now reached that stage where technological intellect has the potential to destroy us and that's why events are moving so fast. Central American shaman Don Juan Matus said:

Think for a moment, and tell me how you would explain the contradictions between the intelligence of man the engineer and the stupidity of his systems of belief, or the stupidity of his contradictory behaviour. Sorcerers believe that the predators have given us our systems of beliefs, our ideas of good and evil; our social mores. They are the ones who set up our dreams of success or failure. They have given us covetousness, greed, and cowardice. It is the predator who makes us complacent, routinary, and egomaniacal.

In order to keep us obedient and meek and weak, the predators engaged themselves in a stupendous manoeuvre – stupendous, of course, from the point of view of a fighting strategist; a horrendous manoeuvre from the point of those who suffer it. They gave us their mind. The predators' mind is baroque, contradictory, morose, filled with the fear of being discovered any minute now.

For 'predators' see Wetiko, Archons, Yaldabaoth, Jinn, and all the other versions of the same phenomenon in cultures and religions all over the world. The theme is always the same because it's true and it's real. We have reached the point where we have to deal with it. The question is – how?

## **Don't fight – walk away**

I thought I'd use a controversial subheading to get things moving in terms of our response to global fascism. What do you mean 'don't fight'? What do you mean 'walk away'? We've got to fight. We can't walk away. Well, it depends what we mean by fight and walk away. If fighting means physical combat we are playing Wetiko's game and falling for its trap. It wants us to get angry, aggressive, and direct hate and hostility at the enemy we think we must fight. Every war, every battle, every conflict, has been fought with Wetiko leading both sides. It's what it does. Wetiko wants a fight, anywhere, any place. Just hit me, son, so I can hit you back. Wetiko hits Wetiko and Wetiko hits Wetiko in return. I am very forthright as you can see in exposing Wetikos of the Cult, but I don't hate them. I refuse to hate them. It's what they want. What you hate you become. What you *fight* you become. Wokers, 'anti-haters' and 'anti-fascists' prove this every time they reach for their keyboards or don their balaclavas. By walk away I mean to disengage from Wetiko which includes ceasing to cooperate with its tyranny. Paul Levy says of Wetiko:

The way to 'defeat' evil is not to try to destroy it (for then, in playing evil's game, we have already lost), but rather, to find the invulnerable place within ourselves where evil is unable to vanquish us – this is to truly 'win' our battle with evil.

Wetiko is everywhere in human society and it's been on steroids since the 'Covid' hoax. Every shouting match over wearing masks has Wetiko wearing a mask and Wetiko not wearing one. It's an electrical circuit of push and resist, push and resist, with Wetiko pushing *and* resisting. Each polarity is Wetiko empowering itself. Dictionary definitions of 'resist' include 'opposing, refusing to accept or comply with' and the word to focus on is 'opposing'. What form does this take – setting police cars alight or 'refusing to accept or comply with'? The former is Wetiko opposing Wetiko while the other points the way forward. This is the difference between those aggressively demanding that government fascism must be obeyed who stand in stark contrast to the great majority of Pushbackers. We saw this clearly with a march by thousands of Pushbackers against lockdown in London followed days later by a Woker-hijacked

protest in Bristol in which police cars were set on fire. Masks were virtually absent in London and widespread in Bristol. Wetiko wants lockdown on every level of society and infuses its aggression to police it through its unknowing stooges. Lockdown protesters are the ones with the smiling faces and the hugs, The two blatantly obvious states of being – getting more obvious by the day – are the result of Wokers and their like becoming ever more influenced by the simulation Field of Wetiko and Pushbackers ever more influenced by The Field of a far higher vibration beyond the simulation. Wetiko can't invade the heart which is where most lockdown opponents are coming from. It's the heart that allows them to see through the lies to the truth in ways I will be highlighting.

Renegade Minds know that calmness is the place from which wisdom comes. You won't find wisdom in a hissing fit and wisdom is what we need in abundance right now. Calmness is not weakness – you don't have to scream at the top of your voice to be strong. Calmness is indeed a sign of strength. 'No' means I'm not doing it. NOOOO!!! doesn't mean you're not doing it even more. Volume does not advance 'No – I'm not doing it'. You are just not doing it. Wetiko possessed and influenced don't know how to deal with that. Wetiko wants a fight and we should not give it one. What it needs more than anything is our *cooperation* and we should not give that either. Mass rallies and marches are great in that they are a visual representation of feeling, but if it ends there they are irrelevant. You demand that Wetikos act differently? Well, they're not going to are they? They are Wetikos. We don't need to waste our time demanding that something doesn't happen when that will make no difference. We need to delete the means that *allows* it to happen. This, invariably, is our cooperation. You can demand a child stop firing a peashooter at the dog or you can refuse to buy the peashooter. If you provide the means you are cooperating with the dog being smacked on the nose with a pea. How can the authorities enforce mask-wearing if millions in a country refuse? What if the 74 million Pushbackers that voted for Trump in 2020 refused to wear masks, close their businesses or stay in their homes. It would be unenforceable. The

few control the many through the compliance of the many and that's always been the dynamic be it 'Covid' regulations or the Roman Empire. I know people can find it intimidating to say no to authority or stand out in a crowd for being the only one with a face on display; but it has to be done or it's over. I hope I've made clear in this book that where this is going will be far more intimidating than standing up now and saying 'No' – I will not cooperate with my own enslavement and that of my children. There might be consequences for some initially, although not so if enough do the same. The question that must be addressed is what is going to happen if we don't? It is time to be strong and unyieldingly so. No means no. Not here and there, but *everywhere* and *always*. I have refused to wear a mask and obey all the other nonsense. I will not comply with tyranny. I repeat: Fascism is not imposed by fascists – there are never enough of them. Fascism is imposed by the population acquiescing to fascism. *I will not do it.* I will die first, or my body will. Living meekly under fascism is a form of death anyway, the death of the spirit that Martin Luther King described.

## **Making things happen**

We must not despair. This is not over till it's over and it's far from that. The 'fat lady' must refuse to sing. The longer the 'Covid' hoax has dragged on and impacted on more lives we have seen an awakening of phenomenal numbers of people worldwide to the realisation that what they have believed all their lives is not how the world really is. Research published by the system-serving University of Bristol and King's College London in February, 2021, concluded: 'One in every 11 people in Britain say they trust David Icke's take on the coronavirus pandemic.' It will be more by now and we have gathering numbers to build on. We must urgently progress from seeing the scam to ceasing to cooperate with it. Prominent German lawyer Reiner Fuellmich, also licenced to practice law in America, is doing a magnificent job taking the legal route to bring the psychopaths to justice through a second Nuremberg tribunal for crimes against humanity. Fuellmich has an impressive record of

beating the elite in court and he formed the German Corona Investigative Committee to pursue civil charges against the main perpetrators with a view to triggering criminal charges. Most importantly he has grasped the foundation of the hoax – the PCR test not testing for the ‘virus’ – and Christian Drosten is therefore on his charge sheet along with Gates frontman Tedros at the World Health Organization. Major players must be not be allowed to inflict their horrors on the human race without being brought to book. A life sentence must follow for Bill Gates and the rest of them. A group of researchers has also indicted the government of Norway for crimes against humanity with copies sent to the police and the International Criminal Court. The lawsuit cites participation in an internationally-planned false pandemic and violation of international law and human rights, the European Commission’s definition of human rights by coercive rules, Nuremberg and Hague rules on fundamental human rights, and the Norwegian constitution. We must take the initiative from hereon and not just complain, protest and react.

There are practical ways to support vital mass non-cooperation. Organising in numbers is one. Lockdown marches in London in the spring in 2021 were mass non-cooperation that the authorities could not stop. There were too many people. Hundreds of thousands walked the London streets in the centre of the road for mile after mile while the Face-Nappies could only look on. They were determined, but calm, and just *did it* with no histrionics and lots of smiles. The police were impotent. Others are organising group shopping without masks for mutual support and imagine if that was happening all over. Policing it would be impossible. If the store refuses to serve people in these circumstances they would be faced with a long line of trolleys full of goods standing on their own and everything would have to be returned to the shelves. How would they cope with that if it kept happening? I am talking here about moving on from complaining to being pro-active; from watching things happen to making things happen. I include in this our relationship with the police. The behaviour of many Face-Nappies

has been disgraceful and anyone who thinks they would never find concentration camp guards in the ‘enlightened’ modern era have had that myth busted big-time. The period and setting may change – Wetikos never do. I watched film footage from a London march in which a police thug viciously kicked a protestor on the floor who had done nothing. His fellow Face-Nappies stood in a ring protecting him. What he did was a criminal assault and with a crowd far outnumbering the police this can no longer be allowed to happen unchallenged. I get it when people chant ‘shame on you’ in these circumstances, but that is no longer enough. They *have* no shame those who do this. Crowds needs to start making a citizen’s arrest of the police who commit criminal offences and brutally attack innocent people and defenceless women. A citizen’s arrest can be made under section 24A of the UK Police and Criminal Evidence (PACE) Act of 1984 and you will find something similar in other countries. I prefer to call it a Common Law arrest rather than citizen’s for reasons I will come to shortly. Anyone can arrest a person committing an indictable offence or if they have reasonable grounds to suspect they are committing an indictable offence. On both counts the attack by the police thug would have fallen into this category. A citizen’s arrest can be made to stop someone:

- Causing physical injury to himself or any other person
- Suffering physical injury
- Causing loss of or damage to property
- Making off before a constable can assume responsibility for him

A citizen’s arrest may also be made to prevent a breach of the peace under Common Law and if they believe a breach of the peace will happen or anything related to harm likely to be done or already done in their presence. This is the way to go I think – the Common Law version. If police know that the crowd and members of the public will no longer be standing and watching while they commit

their thuggery and crimes they will think twice about acting like Brownshirts and Blackshirts.

## **Common Law – common sense**

Mention of Common Law is very important. Most people think the law is the law as in one law. This is not the case. There are two bodies of law, Common Law and Statute Law, and they are not the same. Common Law is founded on the simple premise of do no harm. It does not recognise victimless crimes in which no harm is done while Statute Law does. There is a Statute Law against almost everything. So what is Statute Law? Amazingly it's the law of the sea that was brought ashore by the Cult to override the law of the land which is Common Law. They had no right to do this and as always they did it anyway. They had to. They could not impose their will on the people through Common Law which only applies to do no harm. How could you stitch up the fine detail of people's lives with that? Instead they took the law of the sea, or Admiralty Law, and applied it to the population. Statute Law refers to all the laws spewing out of governments and their agencies including all the fascist laws and regulations relating to 'Covid'. The key point to make is that Statute Law is *contract law*. It only applies between *contracting* corporations. Most police officers don't even know this. They have to be kept in the dark, too. Long ago when merchants and their sailing ships began to trade with different countries a contractual law was developed called Admiralty Law and other names. Again it only applied to *contracts* agreed between *corporate* entities. If there is no agreed contract the law of the sea had no jurisdiction *and that still applies to its new alias of Statute Law*. The problem for the Cult when the law of the sea was brought ashore was an obvious one. People were not corporations and neither were government entities. To overcome the latter they made governments and all associated organisations corporations. All the institutions are *private corporations* and I mean governments and their agencies, local councils, police, courts, military, US states, the whole lot. Go to the

Dun and Bradstreet corporate listings website for confirmation that they are all corporations. You are arrested by a private corporation called the police by someone who is really a private security guard and they take you to court which is another private corporation.

Neither have jurisdiction over you unless you consent and *contract* with them. This is why you hear the mantra about law enforcement policing by *consent* of the people. In truth the people 'consent' only in theory through monumental trickery.

Okay, the Cult overcame the corporate law problem by making governments and institutions corporate entities; but what about people? They are not corporations are they? Ah ... well in a sense, and *only* a sense, they are. Not people exactly – the illusion of people. The Cult creates a corporation in the name of everyone at the time that their birth certificate is issued. Note birth/ *berth* certificate and when you go to court under the law of the sea on land you stand in a *dock*. These are throwbacks to the origin. My Common Law name is David Vaughan Icke. The name of the corporation created by the government when I was born is called Mr David Vaughan Icke usually written in capitals as MR DAVID VAUGHAN ICKE. That is not me, the living, breathing man. It is a fictitious corporate entity. The trick is to make you think that David Vaughan Icke and MR DAVID VAUGHAN ICKE are the same thing. *They are not*. When police charge you and take you to court they are prosecuting the corporate entity and not the living, breathing, man or woman. They have to trick you into identifying as the corporate entity and contracting with them. Otherwise they have no jurisdiction. They do this through a language known as legalese. Lawful and legal are not the same either. Lawful relates to Common Law and legal relates to Statute Law. Legalese is the language of Statue Law which uses terms that mean one thing to the public and another in legalese. Notice that when a police officer tells someone why they are being charged he or she will say at the end: 'Do you understand?' To the public that means 'Do you comprehend?' In legalese it means 'Do you stand under me?' Do you stand under my authority? If you say

yes to the question you are unknowingly agreeing to give them jurisdiction over you in a contract between two corporate entities.

This is a confidence trick in every way. Contracts have to be agreed between informed parties and if you don't know that David Vaughan Icke is agreeing to be the corporation MR DAVID VAUGHAN ICKE you cannot knowingly agree to contract. They are deceiving you and another way they do this is to ask for proof of identity. You usually show them a driving licence or other document on which your corporate name is written. In doing so you are accepting that you are that corporate entity when you are not. Referring to yourself as a 'person' or 'citizen' is also identifying with your corporate fiction which is why I made the Common Law point about the citizen's arrest. If you are approached by a police officer you identify yourself immediately as a living, breathing, man or woman and say 'I do not consent, I do not contract with you and I do not understand' or stand under their authority. I have a Common Law birth certificate as a living man and these are available at no charge from [commonlawcourt.com](http://commonlawcourt.com). Businesses registered under the Statute Law system means that its laws apply. There are, however, ways to run a business under Common Law. Remember all 'Covid' laws and regulations are Statute Law – the law of *contracts* and you do not have to contract. This doesn't mean that you can kill someone and get away with it. Common Law says do no harm and that applies to physical harm, financial harm etc. Police are employees of private corporations and there needs to be a new system of non-corporate Common Law constables operating outside the Statute Law system. If you go to [davidicke.com](http://davidicke.com) and put Common Law into the search engine you will find videos that explain Common Law in much greater detail. It is definitely a road we should walk.

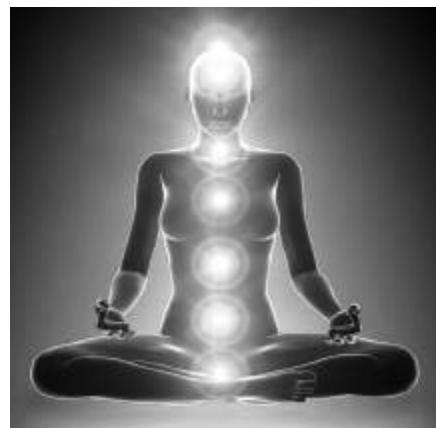
## **With all my heart**

I have heard people say that we are in a spiritual war. I don't like the term 'war' with its Wetiko dynamic, but I know what they mean. Sweep aside all the bodily forms and we are in a situation in which two states of consciousness are seeking very different realities.

Wetiko wants upheaval, chaos, fear, suffering, conflict and control. The other wants love, peace, harmony, fairness and freedom. That's where we are. We should not fall for the idea that Wetiko is all-powerful and there's nothing we can do. Wetiko is not all-powerful. It's a joke, pathetic. It doesn't have to be, but it has made that choice for now. A handful of times over the years when I have felt the presence of its frequency I have allowed it to attach briefly so I could consciously observe its nature. The experience is not pleasant, the energy is heavy and dark, but the ease with which you can kick it back out the door shows that its real power is in persuading us that it has power. It's all a con. Wetiko is a con. It's a trickster and not a power that can control us if we unleash our own. The con is founded on manipulating humanity to give its power to Wetiko which recycles it back to present the illusion that it has power when its power is *ours* that we gave away. This happens on an energetic level and plays out in the world of the seen as humanity giving its power to Wetiko authority which uses that power to control the population when the power is only the power the population has handed over. How could it be any other way for billions to be controlled by a relative few? I have had experiences with people possessed by Wetiko and again you can kick its arse if you do it with an open heart. Oh yes – the *heart* which can transform the world of perceived 'matter'.

We are receiver-transmitters and processors of information, but what information and where from? Information is processed into perception in three main areas – the brain, the heart and the belly. These relate to thinking, knowing, and emotion. Wetiko wants us to be head and belly people which means we think within the confines of the Matrix simulation and low-vibrational emotional reaction scrambles balance and perception. A few minutes on social media and you see how emotion is the dominant force. Woke is all emotion and is therefore thought-free and fact-free. Our heart is something different. It *knows* while the head *thinks* and has to try to work it out because it doesn't know. The human energy field has seven prime vortexes which connect us with wider reality ([Fig 23](#)). Chakra means

'wheels of light' in the Sanskrit language of ancient India. The main ones are: The crown chakra on top of the head; brow (or 'third eye') chakra in the centre of the forehead; throat chakra; heart chakra in the centre of the chest; solar plexus chakra below the sternum; sacral chakra beneath the navel; and base chakra at the bottom of the spine. Each one has a particular function or functions. We feel anxiety and nervousness in the belly where the sacral chakra is located and this processes emotion that can affect the colon to give people 'the shits' or make them 'shit scared' when they are nervous. Chakras all play an important role, but the Mr and Mrs Big is the heart chakra which sits at the centre of the seven, above the chakras that connect us to the 'physical' and below those that connect with higher realms (or at least should). Here in the heart chakra we feel love, empathy and compassion – 'My heart goes out to you'. Those with closed hearts become literally 'heart-less' in their attitudes and behaviour (see Bill Gates). Native Americans portrayed Wetiko with what Paul Levy calls a 'frigid, icy heart, devoid of mercy' (see Bill Gates).



**Figure 23:** The chakra system which interpenetrates the human energy field. The heart chakra is the governor – or should be.

Wetiko trembles at the thought of heart energy which it cannot infiltrate. The frequency is too high. What it seeks to do instead is close the heart chakra vortex to block its perceptual and energetic influence. Psychopaths have 'hearts of stone' and emotionally-damaged people have 'heartache' and 'broken hearts'. The astonishing amount of heart disease is related to heart chakra

disruption with its fundamental connection to the ‘physical’ heart. Dr Tom Cowan has written an outstanding book challenging the belief that the heart is a pump and making the connection between the ‘physical’ and spiritual heart. Rudolph Steiner who was way ahead of his time said the same about the fallacy that the heart is a pump. *What?* The heart is not a pump? That’s crazy, right? Everybody knows that. Read Cowan’s *Human Heart, Cosmic Heart* and you will realise that the very idea of the heart as a pump is ridiculous when you see the evidence. How does blood in the feet so far from the heart get pumped horizontally up the body by the heart?? Cowan explains in the book the real reason why blood moves as it does. Our ‘physical’ heart is used to symbolise love when the source is really the heart vortex or spiritual heart which is our most powerful energetic connection to ‘out there’ expanded consciousness. That’s why we feel *knowing* – intuitive knowing – in the centre of the chest. Knowing doesn’t come from a process of thoughts leading to a conclusion. It is there in an instant all in one go. Our heart knows because of its connection to levels of awareness that *do* know. This is the meaning and source of intuition – intuitive *knowing*.

For the last more than 30 years of uncovering the global game and the nature of reality my heart has been my constant antenna for truth and accuracy. An American intelligence insider once said that I had quoted a disinformant in one of my books and yet I had only quoted the part that was true. He asked: ‘How do you do that?’ By using my heart antenna was the answer and anyone can do it. Heart-centred is how we are meant to be. With a closed heart chakra we withdraw into a closed mind and the bubble of five-sense reality. If you take a moment to focus your attention on the centre of your chest, picture a spinning wheel of light and see it opening and expanding. You will feel it happening, too, and perceptions of the heart like joy and love as the heart impacts on the mind as they interact. The more the chakra opens the more you will feel expressions of heart consciousness and as the process continues, and becomes part of you, insights and knowings will follow. An open

heart is connected to that level of awareness that knows all is *One*. You will see from its perspective that the fault-lines that divide us are only illusions to control us. An open heart does not process the illusions of race, creed and sexuality except as brief experiences for a consciousness that is all. Our heart does not see division, only unity (Figs 24 and 25). There's something else, too. Our hearts love to laugh. Mark Twain's quote that says 'The human race has one really effective weapon, and that is laughter' is really a reference to the heart which loves to laugh with the joy of knowing the true nature of infinite reality and that all the madness of human society is an illusion of the mind. Twain also said: 'Against the assault of laughter nothing can stand.' This is so true of Wetiko and the Cult. Their insecurity demands that they be taken seriously and their power and authority acknowledged and feared. We should do nothing of the sort. We should not get aggressive or fearful which their insecurity so desires. We should laugh in their face. Even in their no-face as police come over in their face-nappies and expect to be taken seriously. They don't take themselves seriously looking like that so why should we? Laugh in the face of intimidation. Laugh in the face of tyranny. You will see by its reaction that you have pressed all of its buttons. Wetiko does not know what to do in the face of laughter or when its targets refuse to concede their joy to fear. We have seen many examples during the 'Covid' hoax when people have expressed their energetic power and the string puppets of Wetiko retreat with their tail limp between their knees. Laugh – the world is bloody mad after all and if it's a choice between laughter and tears I know which way I'm going.



**Figure 24:** Head consciousness without the heart sees division and everything apart from everything else.



**Figure 25:** Heart consciousness sees everything as One.

## **Vaccines' and the soul**

The foundation of Wetiko/Archon control of humans is the separation of incarnate five-sense mind from the infinite 'I' and closing the heart chakra where the True 'I' lives during a human life. The goal has been to achieve complete separation in both cases. I was interested therefore to read an account by a French energetic healer of what she said she experienced with a patient who had been given the 'Covid' vaccine. Genuine energy healers can sense information and consciousness fields at different levels of being which are referred to as 'subtle bodies'. She described treating the patient who later returned after having, without the healer's knowledge, two doses of the 'Covid vaccine'. The healer said:

I noticed immediately the change, very heavy energy emanating from [the] subtle bodies. The scariest thing was when I was working on the heart chakra, I connected with her soul: it was detached from the physical body, it had no contact and it was, as if it was floating in a state of total confusion: a damage to the consciousness that loses contact with the physical body, i.e. with our biological machine, there is no longer any communication between them.

I continued the treatment by sending light to the heart chakra, the soul of the person, but it seemed that the soul could no longer receive any light, frequency or energy. It was a very powerful experience for me. Then I understood that this substance is indeed used to detach consciousness so that this consciousness can no longer interact through this body that it possesses in life, where there is no longer any contact, no frequency, no light, no more energetic balance or mind.

This would create a human that is rudderless and at the extreme almost zombie-like operating with a fractional state of consciousness at the mercy of Wetiko. I was especially intrigued by what the healer said in the light of the prediction by the highly-informed Rudolf Steiner more than a hundred years ago. He said:

In the future, we will eliminate the soul with medicine. Under the pretext of a 'healthy point of view', there will be a vaccine by which the human body will be treated as soon as possible directly at birth, so that the human being cannot develop the thought of the existence of soul and Spirit. To materialistic doctors will be entrusted the task of removing the soul of humanity.

As today, people are vaccinated against this disease or that disease, so in the future, children will be vaccinated with a substance that can be produced precisely in such a way that people, thanks to this vaccination, will be immune to being subjected to the 'madness' of spiritual life. He would be extremely smart, but he would not develop a conscience, and that is the true goal of some materialistic circles.

Steiner said the vaccine would detach the physical body from the etheric body (subtle bodies) and 'once the etheric body is detached the relationship between the universe and the etheric body would become extremely unstable, and man would become an automaton'. He said 'the physical body of man must be polished on this Earth by spiritual will – so the vaccine becomes a kind of arymanique (Wetiko) force' and 'man can no longer get rid of a given materialistic feeling'. Humans would then, he said, become 'materialistic of constitution and can no longer rise to the spiritual'. I have been writing for years about DNA being a receiver-transmitter of information that connects us to other levels of reality and these 'vaccines' changing DNA can be likened to changing an antenna and what it can transmit and receive. Such a disconnection would clearly lead to changes in personality and perception. Steiner further predicted the arrival of AI. Big Pharma 'Covid vaccine' makers, expressions of Wetiko, are testing their DNA-manipulating evil on children as I write with a view to giving the 'vaccine' to babies. If it's a soul-body disconnecter – and I say that it is or can be – every child would be disconnected from 'soul' at birth and the 'vaccine' would create a closed system in which spiritual guidance from the greater self would play no part. This has been the ambition of Wetiko all

along. A Pentagon video from 2005 was leaked of a presentation explaining the development of vaccines to change behaviour by their effect on the brain. Those that believe this is not happening with the ‘Covid’ genetically-modifying procedure masquerading as a ‘vaccine’ should make an urgent appointment with Naivety Anonymous. Klaus Schwab wrote in 2018:

Neurotechnologies enable us to better influence consciousness and thought and to understand many activities of the brain. They include decoding what we are thinking in fine levels of detail through new chemicals and interventions that can influence our brains to correct for errors or enhance functionality.

The plan is clear and only the heart can stop it. With every heart that opens, every mind that awakens, Wetiko is weakened. Heart and love are far more powerful than head and hate and so nothing like a majority is needed to turn this around.

## **Beyond the Phantom**

Our heart is the prime target of Wetiko and so it must be the answer to Wetiko. We *are* our heart which is part of one heart, the infinite heart. Our heart is where the true self lives in a human life behind firewalls of five-sense illusion when an imposter takes its place – *Phantom Self*; but our heart waits patiently to be set free any time we choose to see beyond the Phantom, beyond Wetiko. A Wetikoed Phantom Self can wreak mass death and destruction while the love of forever is locked away in its heart. The time is here to unleash its power and let it sweep away the fear and despair that is Wetiko. Heart consciousness does not seek manipulated, censored, advantage for its belief or religion, its activism and desires. As an expression of the One it treats all as One with the same rights to freedom and opinion. Our heart demands fairness for itself no more than for others. From this unity of heart we can come together in mutual support and transform this Wetikoed world into what reality is meant to be – a place of love, joy, happiness, fairness, justice and freedom. Wetiko has another agenda and that’s why the world is as

it is, but enough of this nonsense. Wetiko can't stay where hearts are open and it works so hard to keep them closed. Fear is its currency and its food source and love in its true sense has no fear. Why would love have fear when it knows it is *All That Is, Has Been, And Ever Can Be* on an eternal exploration of all possibility? Love in this true sense is not the physical attraction that passes for love. This can be an expression of it, yes, but Infinite Love, a love without condition, goes far deeper to the core of all being. It is the core of all being. Infinite reality was born from love beyond the illusions of the simulation. Love infinitely expressed is the knowing that all is One and the swiftly-passing experience of separation is a temporary hallucination. You cannot disconnect from Oneness; you can only perceive that you have and withdraw from its influence. This is the most important of all perception trickery by the mind parasite that is Wetiko and the foundation of all its potential for manipulation.

If we open our hearts, open the sluice gates of the mind, and redefine self-identity amazing things start to happen. Consciousness expands or contracts in accordance with self-identity. When true self is recognised as infinite awareness and label self – Phantom Self – is seen as only a series of brief experiences life is transformed. Consciousness expands to the extent that self-identity expands and everything changes. You see unity, not division, the picture, not the pixels. From this we can play the long game. No more is an experience something in and of itself, but a fleeting moment in the eternity of forever. Suddenly people in uniform and dark suits are no longer intimidating. Doing what your heart knows to be right is no longer intimidating and consequences for those actions take on the same nature of a brief experience that passes in the blink of an infinite eye. Intimidation is all in the mind. Beyond the mind there is no intimidation.

An open heart does not consider consequences for what it knows to be right. To do so would be to consider not doing what it knows to be right and for a heart in its power that is never an option. The Renegade Mind is really the Renegade Heart. Consideration of consequences will always provide a getaway car for the mind and

the heart doesn't want one. What is right in the light of what we face today is to stop cooperating with Wetiko in all its forms and to do it without fear or compromise. You cannot compromise with tyranny when tyranny always demands more until it has everything. Life is your perception and you are your destiny. Change your perception and you change your life. Change collective perception and we change the world.

*Come on people ... One human family, One heart, One goal ...  
FREEEEEDOM!*

We must settle for nothing less.

## **Postscript**

**T**he big scare story as the book goes to press is the ‘Indian’ variant and the world is being deluged with propaganda about the ‘Covid catastrophe’ in India which mirrors in its lies and misrepresentations what happened in Italy before the first lockdown in 2020.

The *New York Post* published a picture of someone who had ‘collapsed in the street from Covid’ in India in April, 2021, which was actually taken during a gas leak in May, 2020. Same old, same old. Media articles in mid-February were asking why India had been so untouched by ‘Covid’ and then as their vaccine rollout gathered pace the alleged ‘cases’ began to rapidly increase. Indian ‘Covid vaccine’ maker Bharat Biotech was funded into existence by the Bill and Melinda Gates Foundation (the pair announced their divorce in May, 2021, which is a pity because they so deserve each other). The Indian ‘Covid crisis’ was ramped up by the media to terrify the world and prepare people for submission to still more restrictions. The scam that worked the first time was being repeated only with far more people seeing through the deceit. [Davidicke.com](http://Davidicke.com) and [Ickonic.com](http://Ickonic.com) have sought to tell the true story of what is happening by talking to people living through the Indian nightmare which has nothing to do with ‘Covid’. We posted a letter from ‘Alisha’ in Pune who told a very different story to government and media mendacity. She said scenes of dying people and overwhelmed hospitals were designed to hide what was really happening – genocide and starvation. Alisha said that millions had already died of starvation during the ongoing lockdowns while government and media were lying and making it look like the ‘virus’:

Restaurants, shops, gyms, theatres, basically everything is shut. The cities are ghost towns. Even so-called 'essential' businesses are only open till 11am in the morning. You basically have just an hour to buy food and then your time is up.

Inter-state travel and even inter-district travel is banned. The cops wait at all major crossroads to question why you are traveling outdoors or to fine you if you are not wearing a mask.

The medical community here is also complicit in genocide, lying about hospitals being full and turning away people with genuine illnesses, who need immediate care. They have even created a shortage of oxygen cylinders.

This is the classic Cult modus operandi played out in every country. Alisha said that people who would not have a PCR test not testing for the 'virus' were being denied hospital treatment. She said the people hit hardest were migrant workers and those in rural areas. Most businesses employed migrant workers and with everything closed there were no jobs, no income and no food. As a result millions were dying of starvation or malnutrition. All this was happening under Prime Minister Narendra Modi, a 100-percent asset of the Cult, and it emphasises yet again the scale of pure anti-human evil we are dealing with. Australia banned its people from returning home from India with penalties for trying to do so of up to five years in jail and a fine of £37,000. The manufactured 'Covid' crisis in India was being prepared to justify further fascism in the West. Obvious connections could be seen between the Indian 'vaccine' programme and increased 'cases' and this became a common theme. The Seychelles, the most per capita 'Covid vaccinated' population in the world, went back into lockdown after a 'surge of cases'.

Long ago the truly evil Monsanto agricultural biotechnology corporation with its big connections to Bill Gates devastated Indian farming with genetically-modified crops. Human rights activist Gurcharan Singh highlighted the efforts by the Indian government to complete the job by destroying the food supply to hundreds of millions with 'Covid' lockdowns. He said that 415 million people at the bottom of the disgusting caste system (still going whatever they say) were below the poverty line and struggled to feed themselves every year. Now the government was imposing lockdown at just the

time to destroy the harvest. This deliberate policy was leading to mass starvation. People may reel back at the suggestion that a government would do that, but Wetiko-controlled ‘leaders’ are capable of any level of evil. In fact what is described in India is in the process of being instigated worldwide. The food chain and food supply are being targeted at every level to cause world hunger and thus control. Bill Gates is not the biggest owner of farmland in America for no reason and destroying access to food aids both the depopulation agenda and the plan for synthetic ‘food’ already being funded into existence by Gates. Add to this the coming hyper-inflation from the suicidal creation of fake ‘money’ in response to ‘Covid’ and the breakdown of container shipping systems and you have a cocktail that can only lead one way and is meant to. The Cult plan is to crash the entire system to ‘build back better’ with the Great Reset.

## **'Vaccine' transmission**

Reports from all over the world continue to emerge of women suffering menstrual and fertility problems after having the fake ‘vaccine’ and of the non-‘vaccinated’ having similar problems when interacting with the ‘vaccinated’. There are far too many for ‘coincidence’ to be credible. We’ve had menopausal women getting periods, others having periods stop or not stopping for weeks, passing clots, sometimes the lining of the uterus, breast irregularities, and miscarriages (which increased by 400 percent in parts of the United States). Non-‘vaccinated’ men and children have suffered blood clots and nose bleeding after interaction with the ‘vaccinated’. Babies have died from the effects of breast milk from a ‘vaccinated’ mother. Awake doctors – the small minority – speculated on the cause of non-‘vaccinated’ suffering the same effects as the ‘vaccinated’. Was it nanotechnology in the synthetic substance transmitting frequencies or was it a straight chemical bioweapon that was being transmitted between people? I am not saying that some kind of chemical transmission is not one possible answer, but the foundation of all that the Cult does is frequency and

this is fertile ground for understanding how transmission can happen. American doctor Carrie Madej, an internal medicine physician and osteopath, has been practicing for the last 20 years, teaching medical students, and she says attending different meetings where the agenda for humanity was discussed. Madej, who operates out of Georgia, did not dismiss other possible forms of transmission, but she focused on frequency in search of an explanation for transmission. She said the Moderna and Pfizer 'vaccines' contained nano-lipid particles as a key component. This was a brand new technology never before used on humanity. 'They're using a nanotechnology which is pretty much little tiny computer bits ... nanobots or hydrogel.' Inside the 'vaccines' was 'this sci-fi kind of substance' which suppressed immune checkpoints to get into the cell. I referred to this earlier as the 'Trojan horse' technique that tricks the cell into opening a gateway for the self-replicating synthetic material and while the immune system is artificially suppressed the body has no defences. Madej said the substance served many purposes including an on-demand ability to 'deliver the payload' and using the nano 'computer bits' as biosensors in the body. 'It actually has the ability to accumulate data from your body, like your breathing, your respiration, thoughts, emotions, all kinds of things.'

She said the technology obviously has the ability to operate through Wi-Fi and transmit and receive energy, messages, frequencies or impulses. 'Just imagine you're getting this new substance in you and it can react to things all around you, the 5G, your smart device, your phones.' We had something completely foreign in the human body that had never been launched large scale at a time when we were seeing 5G going into schools and hospitals (plus the Musk satellites) and she believed the 'vaccine' transmission had something to do with this: '... if these people have this inside of them ... it can act like an antenna and actually transmit it outwardly as well.' The synthetic substance produced its own voltage and so it could have that kind of effect. This fits with my own contention that the nano receiver-transmitters are designed to connect people to the

Smart Grid and break the receiver-transmitter connection to expanded consciousness. That would explain the French energy healer's experience of the disconnection of body from 'soul' with those who have had the 'vaccine'. The nanobots, self-replicating inside the body, would also transmit the synthetic frequency which could be picked up through close interaction by those who have not been 'vaccinated'. Madej speculated that perhaps it was 5G and increased levels of other radiation that was causing the symptoms directly although interestingly she said that non-'vaccinated' patients had shown improvement when they were away from the 'vaccinated' person they had interacted with. It must be remembered that you can control frequency and energy with your mind and you can consciously create energetic barriers or bubbles with the mind to stop damaging frequencies from penetrating your field. American paediatrician Dr Larry Palevsky said the 'vaccine' was not a 'vaccine' and was never designed to protect from a 'viral' infection. He called it 'a massive, brilliant propaganda of genocide' because they didn't have to inject everyone to get the result they wanted. He said the content of the jabs was able to infuse any material into the brain, heart, lungs, kidneys, liver, sperm and female productive system. 'This is genocide; this is a weapon of mass destruction.' At the same time American colleges were banning students from attending if they didn't have this life-changing and potentially life-ending 'vaccine'. Class action lawsuits must follow when the consequences of this college fascism come to light. As the book was going to press came reports about fertility effects on sperm in 'vaccinated' men which would absolutely fit with what I have been saying and hospitals continued to fill with 'vaccine' reactions. Another question is what about transmission via blood transfusions? The NHS has extended blood donation restrictions from seven days after a 'Covid vaccination' to 28 days after even a sore arm reaction.

I said in the spring of 2020 that the then touted 'Covid vaccine' would be ongoing each year like the flu jab. A year later Pfizer CEO, the appalling Albert Bourla, said people would 'likely' need a 'booster dose' of the 'vaccine' within 12 months of getting 'fully

'vaccinated' and then a yearly shot. 'Variants will play a key role', he said confirming the point. Johnson & Johnson CEO Alex Gorsky also took time out from his 'vaccine' disaster to say that people may need to be vaccinated against 'Covid-19' each year. UK Health Secretary, the psychopath Matt Hancock, said additional 'boosters' would be available in the autumn of 2021. This is the trap of the 'vaccine passport'. The public will have to accept every last 'vaccine' they introduce, including for the fake 'variants', or it would cease to be valid. The only other way in some cases would be continuous testing with a test not testing for the 'virus' and what is on the swabs constantly pushed up your noise towards the brain every time?

## **'Vaccines' changing behaviour**

I mentioned in the body of the book how I believed we would see gathering behaviour changes in the 'vaccinated' and I am already hearing such comments from the non-'vaccinated' describing behaviour changes in friends, loved ones and work colleagues. This will only increase as the self-replicating synthetic material and nanoparticles expand in body and brain. An article in the *Guardian* in 2016 detailed research at the University of Virginia in Charlottesville which developed a new method for controlling brain circuits associated with complex animal behaviour. The method, dubbed 'magnetogenetics', involves genetically-engineering a protein called ferritin, which stores and releases iron, to create a magnetised substance – 'Magneto' – that can activate specific groups of nerve cells from a distance. This is claimed to be an advance on other methods of brain activity manipulation known as optogenetics and chemogenetics (the Cult has been developing methods of brain control for a long time). The ferritin technique is said to be non-invasive and able to activate neurons 'rapidly and reversibly'. In other words, human thought and perception. The article said that earlier studies revealed how nerve cell proteins 'activated by heat and mechanical pressure can be genetically engineered so that they become sensitive to radio waves and magnetic fields, by attaching them to an iron-storing protein called ferritin, or to inorganic

paramagnetic particles'. Sensitive to radio waves and magnetic fields? You mean like 5G, 6G and 7G? This is the human-AI Smart Grid hive mind we are talking about. The *Guardian* article said:

... the researchers injected Magneto into the striatum of freely behaving mice, a deep brain structure containing dopamine-producing neurons that are involved in reward and motivation, and then placed the animals into an apparatus split into magnetised and non-magnetised sections.

Mice expressing Magneto spent far more time in the magnetised areas than mice that did not, because activation of the protein caused the striatal neurons expressing it to release dopamine, so that the mice found being in those areas rewarding. This shows that Magneto can remotely control the firing of neurons deep within the brain, and also control complex behaviours.

Make no mistake this basic methodology will be part of the 'Covid vaccine' cocktail and using magnetics to change brain function through electromagnetic field frequency activation. The Pentagon is developing a 'Covid vaccine' using ferritin. Magnetics would explain changes in behaviour and why videos are appearing across the Internet as I write showing how magnets stick to the skin at the point of the 'vaccine' shot. Once people take these 'vaccines' anything becomes possible in terms of brain function and illness which will be blamed on 'Covid-19' and 'variants'. Magnetic field manipulation would further explain why the non-'vaccinated' are reporting the same symptoms as the 'vaccinated' they interact with and why those symptoms are reported to decrease when not in their company. Interestingly 'Magneto', a 'mutant', is a character in the Marvel Comic *X-Men* stories with the ability to manipulate magnetic fields and he believes that mutants should fight back against their human oppressors by any means necessary. The character was born Erik Lehnsherr to a Jewish family in Germany.

## Cult-controlled courts

The European Court of Human Rights opened the door for mandatory 'Covid-19 vaccines' across the continent when it ruled in a Czech Republic dispute over childhood immunisation that legally

enforced vaccination could be ‘necessary in a democratic society’. The 17 judges decided that compulsory vaccinations did not breach human rights law. On the face of it the judgement was so inverted you gasp for air. If not having a vaccine infused into your body is not a human right then what is? Ah, but they said human rights law which has been specifically written to delete all human rights at the behest of the state (the Cult). Article 8 of the European Convention on Human Rights relates to the right to a private life. The crucial word here is ‘*except*’:

There shall be no interference by a public authority with the exercise of this right EXCEPT such as is in accordance with the law and is necessary in a democratic society in the interests of national security, public safety or the economic wellbeing of the country, for the prevention of disorder or crime, for the protection of health or morals, or for the protection of the rights and freedoms of others [My emphasis].

No interference *except* in accordance with the law means there *are* no ‘human rights’ *except* what EU governments decide you can have at their behest. ‘As is necessary in a democratic society’ explains that reference in the judgement and ‘in the interests of national security, public safety or the economic well-being of the country, for the prevention of disorder or crime, for the protection of health or morals, or for the protection of the rights and freedoms of others’ gives the EU a coach and horses to ride through ‘human rights’ and scatter them in all directions. The judiciary is not a check and balance on government extremism; it is a vehicle to enforce it. This judgement was almost laughably predictable when the last thing the Cult wanted was a decision that went against mandatory vaccination. Judges rule over and over again to benefit the system of which they are a part. Vaccination disputes that come before them are invariably delivered in favour of doctors and authorities representing the view of the state which owns the judiciary. Oh, yes, and we have even had calls to stop putting ‘Covid-19’ on death certificates within 28 days of a ‘positive test’ because it is claimed the practice makes the ‘vaccine’ appear not to work. They are laughing at you.

The scale of madness, inhumanity and things to come was highlighted when those not ‘vaccinated’ for ‘Covid’ were refused evacuation from the Caribbean island of St Vincent during massive volcanic eruptions. Cruise ships taking residents to the safety of another island allowed only the ‘vaccinated’ to board and the rest were left to their fate. Even in life and death situations like this we see ‘Covid’ stripping people of their most basic human instincts and the insanity is even more extreme when you think that fake ‘vaccine’-makers are not even claiming their body-manipulating concoctions stop ‘infection’ and ‘transmission’ of a ‘virus’ that doesn’t exist. St Vincent Prime Minister Ralph Gonsalves said: ‘The chief medical officer will be identifying the persons already vaccinated so that we can get them on the ship.’ Note again the power of the chief medical officer who, like Whitty in the UK, will be answering to the World Health Organization. This is the Cult network structure that has overridden politicians who ‘follow the science’ which means doing what WHO-controlled ‘medical officers’ and ‘science advisers’ tell them. Gonsalves even said that residents who were ‘vaccinated’ after the order so they could board the ships would still be refused entry due to possible side effects such as ‘wooziness in the head’. The good news is that if they were woozy enough in the head they could qualify to be prime minister of St Vincent.

## **Microchipping freedom**

The European judgement will be used at some point to justify moves to enforce the ‘Covid’ DNA-manipulating procedure. Sandra Ro, CEO of the Global Blockchain Business Council, told a World Economic Forum event that she hoped ‘vaccine passports’ would help to ‘drive forced consent and standardisation’ of global digital identity schemes: ‘I’m hoping with the desire and global demand for some sort of vaccine passport – so that people can get travelling and working again – [it] will drive forced consent, standardisation, and frankly, cooperation across the world.’ The lady is either not very bright, or thoroughly mendacious, to use the term ‘forced consent’.

You do not ‘consent’ if you are forced – you *submit*. She was describing what the plan has been all along and that’s to enforce a digital identity on every human without which they could not function. ‘Vaccine passports’ are opening the door and are far from the end goal. A digital identity would allow you to be tracked in everything you do in cyberspace and this is the same technique used by Cult-owned China to enforce its social credit system of total control. The ultimate ‘passport’ is planned to be a microchip as my books have warned for nearly 30 years. Those nice people at the Pentagon working for the Cult-controlled Defense Advanced Research Projects Agency (DARPA) claimed in April, 2021, they have developed a microchip inserted under the skin to detect ‘asymptomatic Covid-19 infection’ before it becomes an outbreak and a ‘revolutionary filter’ that can remove the ‘virus’ from the blood when attached to a dialysis machine. The only problems with this are that the ‘virus’ does not exist and people transmitting the ‘virus’ with no symptoms is brain-numbing bullshit. This is, of course, not a ruse to get people to be microchipped for very different reasons. DARPA also said it was producing a one-stop ‘vaccine’ for the ‘virus’ and all ‘variants’. One of the most sinister organisations on Planet Earth is doing this? Better have it then. These people are insane because Wetiko that possesses them is insane.

Researchers from the Salk Institute in California announced they have created an embryo that is part human and part monkey. My books going back to the 1990s have exposed experiments in top secret underground facilities in the United States where humans are being crossed with animal and non-human ‘extraterrestrial’ species. They are now easing that long-developed capability into the public arena and there is much more to come given we are dealing with psychiatric basket cases. Talking of which – Elon Musk’s scientists at Neuralink trained a monkey to play Pong and other puzzles on a computer screen using a joystick and when the monkey made the correct move a metal tube squirted banana smoothie into his mouth which is the basic technique for training humans into unquestioning compliance. Two Neuralink chips were in the monkey’s skull and

more than 2,000 wires ‘fanned out’ into its brain. Eventually the monkey played a video game purely with its brain waves. Psychopathic narcissist Musk said the ‘breakthrough’ was a step towards putting Neuralink chips into human skulls and merging minds with artificial intelligence. *Exactly.* This man is so dark and Cult to his DNA.

## **World Economic Fascism (WEF)**

The World Economic Forum is telling you the plan by the statements made at its many and various events. Cult-owned fascist YouTube CEO Susan Wojcicki spoke at the 2021 WEF Global Technology Governance Summit (see the name) in which 40 governments and 150 companies met to ensure ‘the responsible design and deployment of emerging technologies’. Orwellian translation: ‘Ensuring the design and deployment of long-planned technologies will advance the Cult agenda for control and censorship.’ Freedom-destroyer and Nuremberg-bound Wojcicki expressed support for tech platforms like hers to censor content that is ‘technically legal but could be harmful’. Who decides what is ‘harmful’? She does and they do. ‘Harmful’ will be whatever the Cult doesn’t want people to see and we have legislation proposed by the UK government that would censor content on the basis of ‘harm’ no matter if the information is fair, legal and provably true. Make that *especially* if it is fair, legal and provably true. Wojcicki called for a global coalition to be formed to enforce content moderation standards through automated censorship. This is a woman and mega-censor so self-deluded that she shamelessly accepted a ‘free expression’ award – *Wojcicki* – in an event sponsored by her own *YouTube*. They have no shame and no self-awareness.

You know that ‘Covid’ is a scam and Wojcicki a Cult operative when YouTube is censoring medical and scientific opinion purely on the grounds of whether it supports or opposes the Cult ‘Covid’ narrative. Florida governor Ron DeSantis compiled an expert panel with four professors of medicine from Harvard, Oxford, and Stanford Universities who spoke against forcing children and

vaccinated people to wear masks. They also said there was no proof that lockdowns reduced spread or death rates of 'Covid-19'. Cult-gofer Wojcicki and her YouTube deleted the panel video 'because it included content that contradicts the consensus of local and global health authorities regarding the efficacy of masks to prevent the spread of Covid-19'. This 'consensus' refers to what the Cult tells the World Health Organization to say and the WHO tells 'local health authorities' to do. Wojcicki knows this, of course. The panellists pointed out that censorship of scientific debate was responsible for deaths from many causes, but Wojcicki couldn't care less. She would not dare go against what she is told and as a disgrace to humanity she wouldn't want to anyway. The UK government is seeking to pass a fascist 'Online Safety Bill' to specifically target with massive fines and other means non-censored video and social media platforms to make them censor 'lawful but harmful' content like the Cult-owned Facebook, Twitter, Google and YouTube. What is 'lawful but harmful' would be decided by the fascist Blair-created Ofcom.

Another WEF obsession is a cyber-attack on the financial system and this is clearly what the Cult has planned to take down the bank accounts of everyone – except theirs. Those that think they have enough money for the Cult agenda not to matter to them have got a big lesson coming if they continue to ignore what is staring them in the face. The World Economic Forum, funded by Gates and fronted by Klaus Schwab, announced it would be running a 'simulation' with the Russian government and global banks of just such an attack called Cyber Polygon 2021. What they simulate – as with the 'Covid' Event 201 – they plan to instigate. The WEF is involved in a project with the Cult-owned Carnegie Endowment for International Peace called the WEF-Carnegie Cyber Policy Initiative which seeks to merge Wall Street banks, 'regulators' (I love it) and intelligence agencies to 'prevent' (arrange and allow) a cyber-attack that would bring down the global financial system as long planned by those that control the WEF and the Carnegie operation. The Carnegie Endowment for International Peace sent an instruction to First World

War US President Woodrow Wilson not to let the war end before society had been irreversibly transformed.

## **The Wuhan lab diversion**

As I close, the Cult-controlled authorities and lapdog media are systematically pushing ‘the virus was released from the Wuhan lab’ narrative. There are two versions – it happened by accident and it happened on purpose. Both are nonsense. The perceived existence of the never-shown-to-exist ‘virus’ is vital to sell the impression that there is actually an infective agent to deal with and to allow the endless potential for terrifying the population with ‘variants’ of a ‘virus’ that does not exist. The authorities at the time of writing are going with the ‘by accident’ while the alternative media is promoting the ‘on purpose’. Cable news host Tucker Carlson who has questioned aspects of lockdown and ‘vaccine’ compulsion has bought the Wuhan lab story. ‘Everyone now agrees’ he said. Well, I don’t and many others don’t and the question is *why* does the system and its media suddenly ‘agree’? When the media moves as one unit with a narrative it is always a lie – witness the hour by hour mendacity of the ‘Covid’ era. Why would this Cult-owned combination which has unleashed lies like machine gun fire suddenly ‘agree’ to tell the truth??

Much of the alternative media is buying the lie because it fits the conspiracy narrative, but it’s the *wrong* conspiracy. The real conspiracy is that *there is no virus* and that is what the Cult is desperate to hide. The idea that the ‘virus’ was released by accident is ludicrous when the whole ‘Covid’ hoax was clearly long-planned and waiting to be played out as it was so fast in accordance with the Rockefeller document and Event 201. So they prepared everything in detail over decades and then sat around strumming their fingers waiting for an ‘accidental’ release from a bio-lab? *What??* It’s crazy. Then there’s the ‘on purpose’ claim. You want to circulate a ‘deadly virus’ and hide the fact that you’ve done so and you release it down the street from the highest-level bio-lab in China? I repeat – *What??*

You would release it far from that lab to stop any association being made. But, no, we'll do it in a place where the connection was certain to be made. Why would you need to scam 'cases' and 'deaths' and pay hospitals to diagnose 'Covid-19' if you had a real 'virus'? What are sections of the alternative media doing believing this crap? Where were all the mass deaths in Wuhan from a 'deadly pathogen' when the recovery to normal life after the initial propaganda was dramatic in speed? Why isn't the 'deadly pathogen' now circulating all over China with bodies in the street? Once again we have the technique of tell them what they want to hear and they will likely believe it. The alternative media has its 'conspiracy' and with Carlson it fits with his 'China is the danger' narrative over years. China *is* a danger as a global Cult operations centre, but not for this reason. The Wuhan lab story also has the potential to instigate conflict with China when at some stage the plan is to trigger a Problem-Reaction-Solution confrontation with the West. Question everything – *everything* – and especially when the media agrees on a common party line.

### **Third wave ... fourth wave ... fifth wave ...**

As the book went into production the world was being set up for more lockdowns and a 'third wave' supported by invented 'variants' that were increasing all the time and will continue to do so in public statements and computer programs, but not in reality. India became the new Italy in the 'Covid' propaganda campaign and we were told to be frightened of the new 'Indian strain'. Somehow I couldn't find it within myself to do so. A document produced for the UK government entitled 'Summary of further modelling of easing of restrictions – Roadmap Step 2' declared that a third wave was inevitable (of course when it's in the script) and it would be the fault of children and those who refuse the health-destroying fake 'Covid vaccine'. One of the computer models involved came from the Cult-owned *Imperial College* and the other from Warwick University which I wouldn't trust to tell me the date in a calendar factory. The document states that both models presumed extremely high uptake

of the ‘Covid vaccines’ and didn’t allow for ‘variants’. The document states: ‘The resurgence is a result of some people (mostly children) being ineligible for vaccination; others choosing not to receive the vaccine; and others being vaccinated but not perfectly protected.’ The mendacity takes the breath away. Okay, blame those with a brain who won’t take the DNA-modifying shots and put more pressure on children to have it as ‘trials’ were underway involving children as young as six months with parents who give insanity a bad name. Massive pressure is being put on the young to have the fake ‘vaccine’ and child age consent limits have been systematically lowered around the world to stop parents intervening. Most extraordinary about the document was its claim that the ‘third wave’ would be driven by ‘the resurgence in both hospitalisations and deaths … dominated by *those that have received two doses of the vaccine*, comprising around 60-70% of the wave respectively’. The predicted peak of the ‘third wave’ suggested 300 deaths per day with 250 of them *fully ‘vaccinated’ people*. How many more lies do acquiescers need to be told before they see the obvious? Those who took the jab to ‘protect themselves’ are projected to be those who mostly get sick and die? So what’s in the ‘vaccine’? The document went on:

It is possible that a summer of low prevalence could be followed by substantial increases in incidence over the following autumn and winter. Low prevalence in late summer should not be taken as an indication that SARS-CoV-2 has retreated or that the population has high enough levels of immunity to prevent another wave.

They are telling you the script and while many British people believed ‘Covid’ restrictions would end in the summer of 2021 the government was preparing for them to be ongoing. Authorities were awarding contracts for ‘Covid marshals’ to police the restrictions with contracts starting in July, 2021, and going through to January 31st, 2022, and the government was advertising for ‘Media Buying Services’ to secure media propaganda slots worth a potential £320 million for ‘Covid-19 campaigns’ with a contract not ending until March, 2022. The recipient – via a list of other front companies – was reported to be American media marketing giant Omnicom Group

Inc. While money is no object for ‘Covid’ the UK waiting list for all other treatment – including life-threatening conditions – passed 4.5 million. Meantime the Cult is seeking to control all official ‘inquiries’ to block revelations about what has really been happening and why. It must not be allowed to – we need Nuremberg jury trials in every country. The cover-up doesn’t get more obvious than appointing ultra-Zionist professor Philip Zelikow to oversee two dozen US virologists, public health officials, clinicians, former government officials and four American ‘charitable foundations’ to ‘learn the lessons’ of the ‘Covid’ debacle. The personnel will be those that created and perpetuated the ‘Covid’ lies while Zelikow is the former executive director of the 9/11 Commission who ensured that the truth about those attacks never came out and produced a report that must be among the most mendacious and manipulative documents ever written – see *The Trigger* for the detailed exposure of the almost unimaginable 9/11 story in which Sabbatians can be found at every level.

## **Passive no more**

People are increasingly challenging the authorities with amazing numbers of people taking to the streets in London well beyond the ability of the Face-Nappies to stop them. Instead the Nappies choose situations away from the mass crowds to target, intimidate, and seek to promote the impression of ‘violent protestors’. One such incident happened in London’s Hyde Park. Hundreds of thousands walking through the streets in protest against ‘Covid’ fascism were ignored by the Cult-owned BBC and most of the rest of the mainstream media, but they delighted in reporting how police were injured in ‘clashes with protestors’. The truth was that a group of people gathered in Hyde Park at the end of one march when most had gone home and they were peacefully having a good time with music and chat. Face-Nappies who couldn’t deal with the full-march crowd then waded in with their batons and got more than they bargained for. Instead of just standing for this criminal brutality the crowd used their numerical superiority to push the Face-Nappies out of the

park. Eventually the Nappies turned and ran. Unfortunately two or three idiots in the crowd threw drink cans striking two officers which gave the media and the government the image they wanted to discredit the 99.9999 percent who were peaceful. The idiots walked straight into the trap and we must always be aware of potential agent provocateurs used by the authorities to discredit their targets.

This response from the crowd – the can people apart – must be a turning point when the public no longer stand by while the innocent are arrested and brutally attacked by the Face-Nappies. That doesn't mean to be violent, that's the last thing we need. We'll leave the violence to the Face-Nappies and government. But it does mean that when the Face-Nappies use violence against peaceful people the numerical superiority is employed to stop them and make citizen's arrests or Common Law arrests for a breach of the peace. The time for being passive in the face of fascism is over.

We are the many, they are the few, and we need to make that count before there is no freedom left and our children and grandchildren face an ongoing fascist nightmare.

*COME ON PEOPLE – IT'S TIME.*

### **One final thought ...**

The power of love  
A force from above  
Cleaning my soul  
Flame on burn desire  
Love with tongues of fire  
Purge the soul  
Make love your goal

I'll protect you from the hooded claw  
Keep the vampires from your door  
When the chips are down I'll be around  
With my undying, death-defying  
Love for you

Envy will hurt itself  
Let yourself be beautiful  
Sparkling love, flowers  
And pearls and pretty girls  
Love is like an energy  
Rushin' rushin' inside of me

This time we go sublime  
Lovers entwine, divine, divine,  
Love is danger, love is pleasure  
Love is pure – the only treasure

I'm so in love with you  
Purge the soul  
Make love your goal

The power of love  
A force from above  
Cleaning my soul  
The power of love  
A force from above  
A sky-scraping dove

Flame on burn desire  
Love with tongues of fire  
Purge the soul  
Make love your goal

**Frankie Goes To Hollywood**

## APPENDIX

### Cowan-Kaufman-Morell Statement on Virus Isolation (SOVI)

*Isolation: The action of isolating; the fact or condition of being isolated or standing alone; separation from other things or persons; solitariness*

Oxford English Dictionary

The controversy over whether the SARS-CoV-2 virus has ever been isolated or purified continues. However, using the above definition, common sense, the laws of logic and the dictates of science, any unbiased person must come to the conclusion that the SARS-CoV-2 virus has never been isolated or purified. As a result, no confirmation of the virus' existence can be found. The logical, common sense, and scientific consequences of this fact are:

- the structure and composition of something not shown to exist can't be known, including the presence, structure, and function of any hypothetical spike or other proteins;
- the genetic sequence of something that has never been found can't be known;
- "variants" of something that hasn't been shown to exist can't be known;
- it's impossible to demonstrate that SARS-CoV-2 causes a disease called Covid-19.

In as concise terms as possible, here's the proper way to isolate, characterize and demonstrate a new virus. First, one takes samples (blood, sputum, secretions) from many people (e.g. 500) with symptoms which are unique and specific enough to characterize an illness. Without mixing these samples with ANY tissue or products that also contain genetic material, the virologist macerates, filters and ultracentrifuges i.e. *purifies* the specimen. This common virology technique, done for decades to isolate bacteriophages<sup>1</sup> and so-called giant viruses in every virology lab, then allows the virologist to demonstrate with electron microscopy thousands of identically sized and shaped particles. These particles are the isolated and purified virus.

These identical particles are then checked for uniformity by physical and/or microscopic techniques. Once the purity is determined, the particles may be further characterized. This would include examining the structure, morphology, and chemical composition of the particles. Next, their genetic makeup is characterized by extracting the genetic material directly from the purified particles and using genetic-sequencing techniques, such as Sanger sequencing, that have also been around for decades. Then one does an analysis to confirm that these uniform particles are exogenous (outside) in origin as a virus is conceptualized to be, and not the normal breakdown products of dead and dying tissues.<sup>2</sup> (As of May 2020, we know that virologists have no way to determine whether the particles they're seeing are viruses or just normal breakdown products of dead and dying tissues.)<sup>3</sup>

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1 Isolation, characterization and analysis of bacteriophages from the haloalkaline lake Elmenteita, KenyaJuliah Khayeli Akhwale et al, PLOS One, Published: April 25, 2019.  
<https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0215734> – accessed 2/15/21

2 "Extracellular Vesicles Derived From Apoptotic Cells: An Essential Link Between Death and Regeneration," Maojiao Li et al, Frontiers in Cell and Developmental Biology, 2020 October 2.  
<https://www.frontiersin.org/articles/10.3389/fcell.2020.573511/full> – accessed 2/15/21

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3 "The Role of Extracellular Vesicles as Allies of HIV, HCV and SARS Viruses," Flavia Giannessi, et al, *Viruses*, 2020 May

If we have come this far then we have fully isolated, characterized, and genetically sequenced an exogenous virus particle. However, we still have to show it is causally related to a disease. This is carried out by exposing a group of healthy subjects (animals are usually used) to this isolated, purified virus in the manner in which the disease is thought to be transmitted. If the animals get sick with the same disease, as confirmed by clinical and autopsy findings, one has now shown that the virus actually causes a disease. This demonstrates infectivity and transmission of an infectious agent.

None of these steps has even been attempted with the SARS-CoV-2 virus, nor have all these steps been successfully performed for any so-called pathogenic virus. Our research indicates that a single study showing these steps does not exist in the medical literature.

Instead, since 1954, virologists have taken unpurified samples from a relatively few people, often less than ten, with a similar disease. They then minimally process this sample and inoculate this unpurified sample onto tissue culture containing usually four to six other types of material – all of which contain identical genetic material as to what is called a “virus.” The tissue culture is starved and poisoned and naturally disintegrates into many types of particles, some of which contain genetic material. Against all common sense, logic, use of the English language and scientific integrity, this process is called “virus isolation.” This brew containing fragments of genetic material from many sources is then subjected to genetic analysis, which then creates in a computer-simulation process the alleged sequence of the alleged virus, a so-called *in silico* genome. At no time is an actual virus confirmed by electron microscopy. At no time is a genome extracted and sequenced from an actual virus. This is scientific fraud.

The observation that the unpurified specimen — inoculated onto tissue culture along with toxic antibiotics, bovine fetal tissue, amniotic fluid and other tissues — destroys the kidney tissue onto which it is inoculated is given as evidence of the virus' existence and pathogenicity. This is scientific fraud.

From now on, when anyone gives you a paper that suggests the SARS-CoV-2 virus has been isolated, please check the methods sections. If the researchers used Vero cells or any other culture method, you know that their process was not isolation. You will hear the following excuses for why actual isolation isn't done:

1. There were not enough virus particles found in samples from patients to analyze.
2. Viruses are intracellular parasites; they can't be found outside the cell in this manner.

If No. 1 is correct, and we can't find the virus in the sputum of sick people, then on what evidence do we think the virus is dangerous or even lethal? If No. 2 is correct, then how is the virus spread from person to person? We are told it emerges from the cell to infect others. Then why isn't it possible to find it?

Finally, questioning these virology techniques and conclusions is not some distraction or divisive issue. Shining the light on this truth is essential to stop this terrible fraud that humanity is confronting. For, as we now know, if the virus has never been isolated, sequenced or shown to cause illness, if the virus is imaginary, then why are we wearing masks, social distancing and putting the whole world into prison?

Finally, if pathogenic viruses don't exist, then what is going into those injectable devices erroneously called "vaccines," and what is their purpose? This scientific question is the most urgent and relevant one of our time.

We are correct. The SARS-CoV2 virus does not exist.

Sally Fallon Morell, MA

Dr. Thomas Cowan, MD

Dr. Andrew Kaufman, MD

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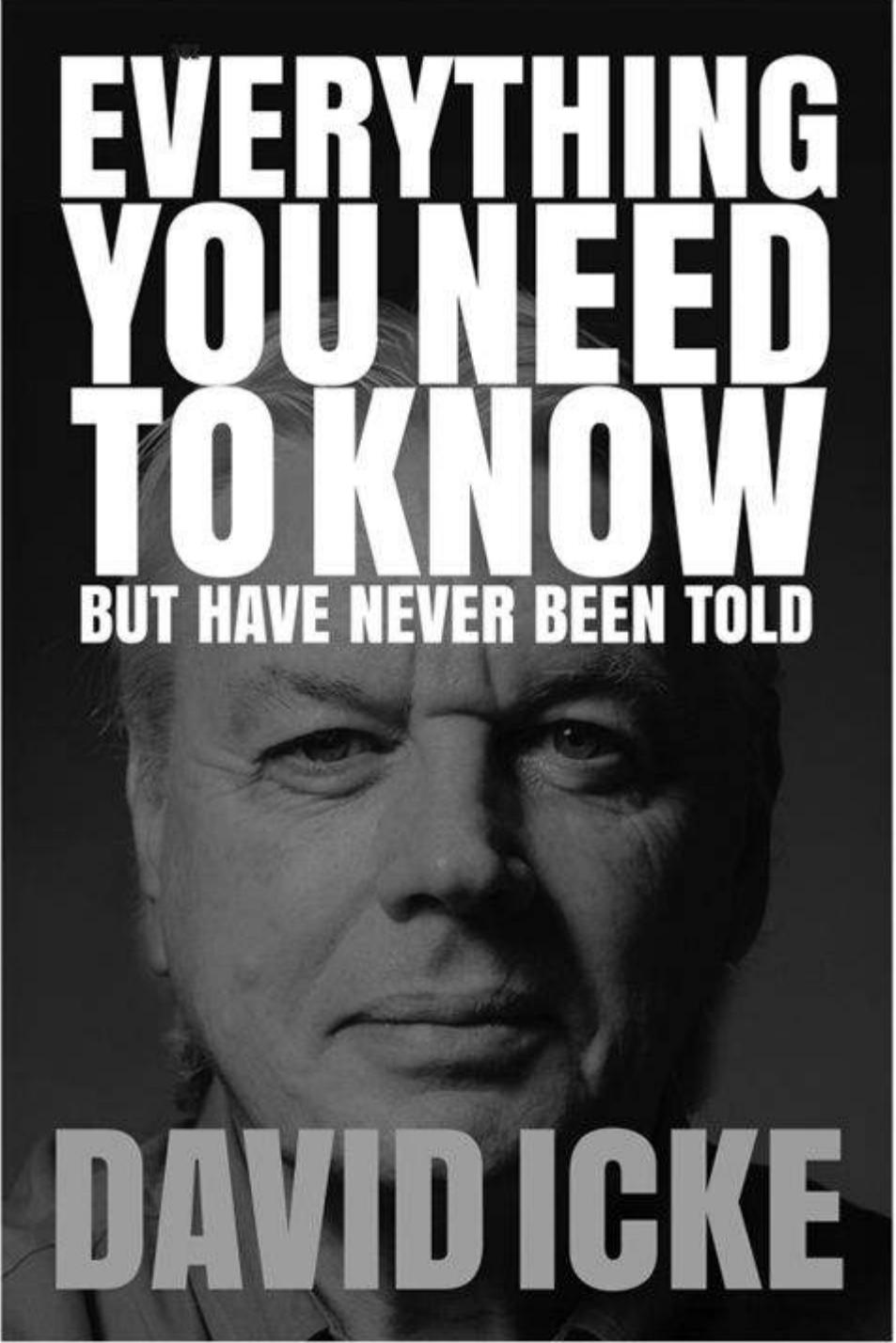
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/'ren-i.gəd/

**noun**

A person who behaves in a rebelliously unconventional manner.

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