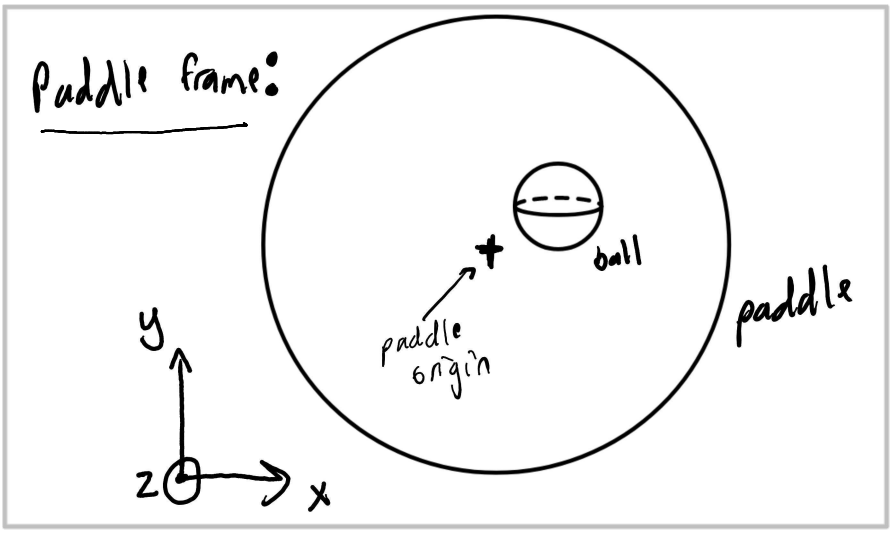
Setup:

world frame F_0

paddle frame F_p

origin at center of paddle



let $p_0(t)$ be the paddle origin.

let $R_s(t)$ be a rotation matrix that defines orientation, such that $R_s(t) \in SO(3)$

describes paddle surface

normal vector:
$n_p = R_s [0 \ 0 \ 1]^T$

Basic idea: only track x and y coordinates of the ping pong ball relative to the surface of the paddle, then translate that to 3D via rotation matrix $R_s(t)$

Assumptions:

Assume no slip, no bouncing, ping-pong ball is constantly touching the surface of the paddle.

Minimum coordinates:

2 degrees of freedom, x and y (cartesian)

$$q = \begin{bmatrix} x \\ y \end{bmatrix}$$

radius of ping-pong ball: r

Position: (equations describing the position of the center of the ball)

2D: $p(t) \in \mathbb{R}^2$

arbitrary scalars for vector

$$p(t) - p_0(t) = [m_1 \; m_2] \begin{bmatrix} x \\ y \end{bmatrix}$$

$$p(t) = [m_1 \; m_2] \begin{bmatrix} x \\ y \end{bmatrix} + p_0(t)$$

3D: $p(t) \in \mathbb{R}^3$

same thing, but adding a z-component

$$p(t) = R_s(t) \begin{bmatrix} x \\ y \\ r \end{bmatrix} + p_0(t)$$

the center of the ball should be offset by radius r

☆ note: $p_0(t)$ tracks the world position of the paddle origin

Position:

$$p(t) = R_s(t)\begin{bmatrix} x \\ y \\ r \end{bmatrix} + p_0(t)$$

Velocity:

$$\dot{p}(t) = R_s(t)\begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} + \dot{R}_s(t)\begin{bmatrix} x \\ y \\ r \end{bmatrix} + \dot{p}_0(t)$$

Need to define $R_s(t)$ $\quad [R_s(t) \in SO(3)]$

Can either define using Euler angles or a quaternion

Euler

(θ, ϕ, ψ)

x y z

"roll" "pitch" "yaw"

Extrinsic

$$R_s(t) = R_z(\psi) R_y(\phi) R_x(\theta)$$

where

$$R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Quaternion

$$q(t) = (w, x, y, z)$$

scalar

$$R_s(t) = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-wz) & 2(xz+wy) \\ 2(xy+wz) & 1-2(x^2+z^2) & 2(yz-wx) \\ 2(xz-wy) & 2(yz+wx) & 1-2(x^2+y^2) \end{bmatrix}$$

Since $R_s(t) \in SO(3)$

$$\dot{R}_s(t) = R_s(t)\,\hat{\omega}_s$$

angular velocity skew symmetric matrix

(time derivative of Rodriguez' Rotation Formula)

[see HW7 for proof of this property]

Velocity:

$$\dot{p}(t) = R_s(t)\begin{bmatrix}\dot{x}\\ \dot{y}\\ 0\end{bmatrix} + \dot{R}_s(t)\begin{bmatrix}x\\ y\\ r\end{bmatrix} + \dot{p}_0(t)$$

$$\boxed{\dot{p}(t) = R_s(t)\begin{bmatrix}\dot{x}\\ \dot{y}\\ 0\end{bmatrix} + R_s(t)\,\hat{\omega}_s\begin{bmatrix}x\\ y\\ r\end{bmatrix} + \dot{p}_0(t)}$$

$\hookrightarrow$ plug into Lagrangian...

Kinetic Energy

$$T_{total} = T_{translational} + T_{rotational}$$

$$T_{translational} = \frac{1}{2} m \|v\|^2 = \frac{1}{2} m_p \|\underbrace{\dot{p}(t)}_{\text{plug in velocity equation here}}\|^2$$

mass of ping pong ball

$$T_{rotational} = \frac{1}{2} I_p \|\omega_p\|^2$$

$$I_p = \frac{2}{3} m_p r^2 \quad [\text{hollow sphere}]$$

☆ here, I'm arguing that the angular velocity can be represented as a function of the linear velocity and radius of the ping-pong ball as it rolls across the paddle's surface

$$v_r = r \times \omega_p$$

$$\omega_p = \frac{v_r}{r}$$, where v_r is the tangential component of $\dot{p}(t)$ as it rolls

$$\omega_p = \frac{v_r}{r} = \frac{1}{r}\left[R_s(t)\begin{bmatrix}\dot{x}\\ \dot{y}\\ 0\end{bmatrix}\right]$$

$$T = \frac{1}{2} m_p \|\dot{p}(t)\|^2 + \frac{1}{2}\cdot\frac{2}{3} m_p r^2 \left(\frac{1}{r}\left[R_s(t)\begin{bmatrix}\dot{x}\\ \dot{y}\\ 0\end{bmatrix}\right]\right)^2$$

$$\boxed{T = \frac{1}{2} m_p \|\dot{p}(t)\|^2 + \underbrace{\frac{1}{3} m_p \left(\left[R_s(t)\begin{bmatrix}\dot{x}\\ \dot{y}\\ 0\end{bmatrix}\right]\right)^2}_{\text{or can just write as } \frac{1}{2} I_p \frac{1}{R^2} \|\omega_p\|^2}}$$

Potential Energy

$$V = mgh$$

$$= mg\left[p_o(t)_z + R_s(t)_z \begin{bmatrix} x \\ y \\ r \end{bmatrix} \right]_z$$

take z-components only

$$L = T - V$$

$$L(x, y, \dot{x}, \dot{y}, t) = T(x, y, \dot{x}, \dot{y}, t) - V(x, y, \dot{x}, \dot{y}, t)$$

$$L = \frac{1}{2} m_p \|\dot{p}(t)\|^2 + \frac{1}{2} I_p \|\omega_p\|^2 - mg\left[p_o(t)_z + R_s(t)_z \begin{bmatrix} x \\ y \\ r \end{bmatrix} \right]_z$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad q = \begin{bmatrix} x \\ y \end{bmatrix}$$

Next steps: Find $\frac{\partial L}{\partial \dot{q}_i}$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$, $\frac{\partial L}{\partial q_i}$, arrange into formula: $M\ddot{q} + C\dot{q} + G = F_{paddle}(t)$

[see code for this]

In summary...

Position of ball

$$\boxed{p(t) = \begin{bmatrix} p_{o_x}(t) + x(t) + r\phi(t) \\ p_{o_y}(t) + y(t) - r\theta(t) \\ p_{o_z}(t) + r - \phi(t)x(t) + \theta(t)y(t) \end{bmatrix}}$$

Velocity of ball

$$\dot{p}_o(t) = \begin{bmatrix} \dot{p}_{o_x}(t) \\ \dot{p}_{o_y}(t) \\ \dot{p}_{o_z}(t) \end{bmatrix}, \quad w(t) = \begin{bmatrix} \dot{\theta}(t) \\ \dot{\phi}(t) \\ 0 \end{bmatrix}, \quad \hat{w} = \begin{bmatrix} 0 & 0 & \dot{\phi} \\ 0 & 0 & -\dot{\theta} \\ -\dot{\phi} & \dot{\theta} & 0 \end{bmatrix}$$

velocity of paddle center

$$, \quad R_s = \begin{bmatrix} 1 & 0 & \phi \\ 0 & 1 & -\theta \\ -\phi & \theta & 1 \end{bmatrix}$$

← See linearization doc for explanation (Euler angles)

$$\dot{p}(t) = \dot{p}_o + R_s \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} + R_s \hat{w} \begin{bmatrix} x \\ y \\ r \end{bmatrix}$$

$$\boxed{\dot{p}(t) = \begin{bmatrix} \dot{p}_{o_x} + \dot{x} + r\dot{\phi} - x\phi\dot{\phi} + y\phi\dot{\theta} \\ \dot{p}_{o_y} + \dot{y} - r\dot{\theta} + x\theta\dot{\phi} - y\theta\dot{\theta} \\ \dot{p}_{o_z} + r(-\phi\dot{\phi} - \theta\dot{\theta}) - \phi\dot{x} + \theta\dot{y} - x\dot{\phi} + y\dot{\theta} \end{bmatrix}}$$

Translational Kinetic Energy

$$T_{trans} = \frac{1}{2} m \dot{p}^T \dot{p}$$

$$= \frac{1}{2} m \left(\dot{p}_x^{\,2} + \dot{p}_y^{\,2} + \dot{p}_z^{\,2} \right)$$

$$\boxed{\begin{aligned} T_{trans} = \frac{1}{2} m \Big[& \left(\dot{p}_{0x} + \dot{x} + r\dot{\phi} - x\phi\dot{\phi} + y\phi\dot{\theta} \right)^2 + \\ & \left(\dot{p}_{0y} + \dot{y} - r\dot{\theta} + x\theta\dot{\phi} - y\theta\dot{\theta} \right)^2 + \\ & \left(\dot{p}_{0z} + r(-\phi\dot{\phi} - \theta\dot{\theta}) - \phi\dot{x} + \theta\dot{y} - x\dot{\phi} + y\dot{\theta} \right)^2 \Big] \end{aligned}}$$

(1 x 1 expression)

Rotational Kinetic Energy

$$\omega_b = \frac{1}{r} R_s \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix}$$

$$T_{rot} = \frac{1}{2} I_p \omega_p^T \omega_p = \frac{I_p}{2r^2} \left[(\dot{x} - \phi\dot{y})^2 + (\dot{y} + \theta\dot{x})^2 + 0^2 \right]$$

$$\boxed{T_{rot} = \frac{I_p}{2r^2} \left(\dot{x}^2 + \dot{y}^2 + (\dot{x}\phi - \dot{y}\theta)^2 \right)}$$

$$\boxed{T_{total} = T_{trans} + T_{rot}}$$

Potential Energy

$$\boxed{V = gm\left(r + p_{0z}(t) - x\phi + y\theta \right)}$$

$$M\ddot{q} + C\dot{q} + G = 0$$ ← no external forces acting on ball (that isn't already covered by G)

M matrix

$$M(q) = \begin{bmatrix} \dfrac{I_p\phi^2 + I_p + mr^2(\phi^2+1)}{r^2} & \dfrac{(-I_p - mr^2)\phi\theta}{r^2} \\ \dfrac{(-I_p - mr^2)\phi\theta}{r^2} & \dfrac{I_p\theta^2 + I_p + mr^2(\theta^2+1)}{r^2} \end{bmatrix}$$

C matrix

$$C(q,\dot{q}) = \frac{1}{r^2}\begin{bmatrix} 2(I_p + mr^2)\phi\dot{\phi} & -(I_p + mr^2)(\phi\dot{\theta} + \theta\dot{\phi}) \\ -(I_p + mr^2)(\phi\dot{\theta} + \theta\dot{\phi}) & 2(I_p + mr^2)\theta\dot{\theta} \end{bmatrix}$$

G Matrix

$$G = \begin{bmatrix} G_1(x, y, \theta, \phi) \\ G_2(x, y, \theta, \phi) \end{bmatrix}$$

$$\begin{aligned} G_1 = m(&-g\phi + r\phi^2\ddot{\phi} + r\phi\theta\ddot{\theta} + 2r\phi\dot{\phi}^2 + r\phi\dot{\theta}^2 + r\theta\dot{\phi}\dot{\theta} \\ &+ r\ddot{\phi} - \phi^2 x(t)\dot{\phi} + \phi^2 y(t)\dot{\phi}\dot{\theta} + \phi\dot{p}_{ox}(t)\dot{\phi} \\ &- \phi\ddot{p}_{oz}(t) - \theta^2 x(t)\dot{\phi}^2 + \theta^2 y(t)\dot{\phi}\dot{\theta} - \theta\dot{p}_{oy}(t)\dot{\phi} \\ &- x(t)\dot{\phi}^2 + y(t)\dot{\phi}\dot{\theta} + \ddot{p}_{ox}(t)) \end{aligned}$$

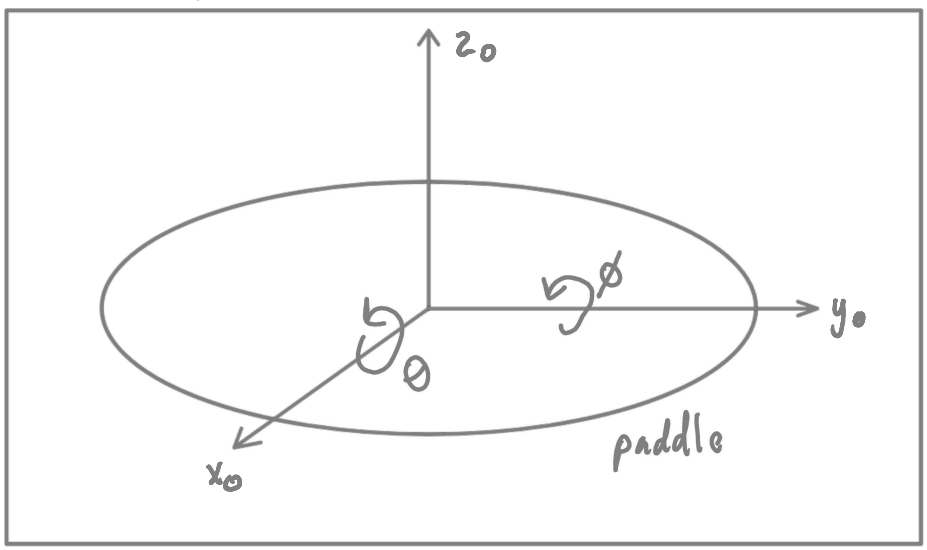
$$\begin{aligned} G_2 = m\Big(&g\theta - r\phi\theta\ddot{\phi} - r\phi\dot{\phi}\dot{\theta} - r\theta^2\ddot{\theta} - r\theta\dot{\phi}^2 - 2r\theta\dot{\theta}^2 \\ &- r\ddot{\theta} + \phi^2 x(t)\dot{\phi}\dot{\theta} - \phi^2 y(t)\dot{\theta}^2 - \phi\dot{p}_{ox}(t)\dot{\theta} + \theta^2 x(t)\dot{\phi}\dot{\theta} \\ &- \theta^2 y(t)\dot{\theta}^2 + \theta\dot{p}_{oy}(t)\dot{\theta} + \theta\ddot{p}_{oz}(t) + x(t)\dot{\phi}\dot{\theta} - y(t)\dot{\theta}^2 \\ &+ \ddot{p}_{oy}(t)\Big) \end{aligned}$$

Linearization

First, let's go over inputs and outputs for this function.

Inputs

We want to control the ping-pong ball using roll and pitch tilt on the paddle in order to redirect it back to its equilibrium state (center of the paddle)

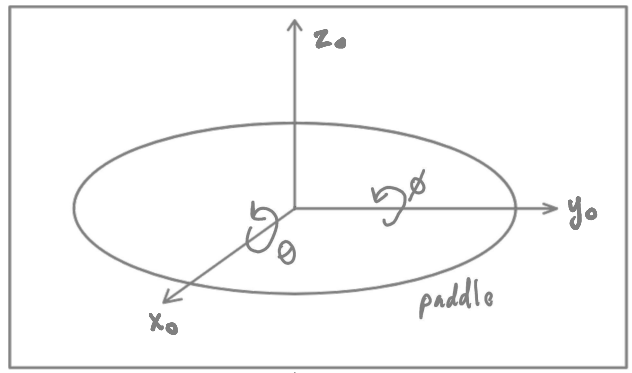


Paddle Frame

Here, roll and pitch are the only tilts we need to worry about. Yaw is unnecessary, as we shouldn't have to worry about the spin of the ball too much.

In our Lagrangian Dynamics Model, we defined the origin of the paddle frame as the center of the paddle surface. We shall do the same here too for linearization.

Inputs (continued):



Paddle Frame

Thus, are final inputs are:

- θ (roll tilt)
- ϕ (pitch tilt)

$$U(t) = \begin{bmatrix} \theta(t) \\ \phi(t) \end{bmatrix}$$

- $\dot{\theta}$ angular velocity about x-axis
- $\dot{\phi}$ angular velocity about y-axis

$$\hat{\omega}_s = \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

☆ we assume the paddle to be a rigid body, so each point on the paddle undergoes the same rotation (doesn't matter if we rotate with respect to handle or center)

So...

$$R_s(t) = R_y(\phi) R_x(\theta)$$ [defined extrinsically]

where

$$R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_s(t) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\phi) & \sin(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) \\ 0 & \cos(\theta) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi)\sin(\theta) & \cos(\phi)\cos(\theta) \end{bmatrix}$$

$\theta\phi$ is a product of small angles, so ≈ 0

☆ note that the paddle stays close to level the entire time → applying small angle approximation...

$$\begin{pmatrix} \cos(\theta) = 1 & \sin(\theta) = \theta \\ \cos(\phi) = 1 & \sin(\phi) = \phi \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \phi \\ 0 & 1 & -\theta \\ -\phi & \theta & 1 \end{bmatrix}$$

State variables (outputs)

$$\left.\begin{matrix} \cdot\, x \\ \cdot\, y \end{matrix}\right] q = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \left.\begin{matrix} \cdot\, \dot{x} \\ \cdot\, \dot{y} \end{matrix}\right] \dot{q}$$

$$s = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}, \qquad \dot{s} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \ddot{x} \\ \ddot{y} \end{bmatrix}$$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

Since there are no actuators on the ping-pong ball, and all external forces are already accounted for via $G(q)$, $\tau = 0$

Thus for our purposes...

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q,t) = 0$$

solve for $\ddot{q}$...

$$\ddot{q} = M^{-1}(q)\left[C(q,\dot{q})\dot{q} + G(q,t)\right]$$

$$\ddot{q} = f(q,\dot{q},t)$$

$$\dot{s} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ M^{-1}(q)\left[C(q,\dot{q})\dot{q} + G(q,t)\right] \end{bmatrix}$$

Equilibrium point (necessary for setup)

at equilibrium:

$$\boxed{\begin{gathered} \theta = 0, \quad \phi = 0, \quad \dot{\theta} = 0, \quad \dot{\phi} = 0 \\ x = 0, \quad y = 0, \quad \dot{x} = 0, \quad \dot{y} = 0 \\ p_0(t) \text{ is constant} \rightarrow \dot{p}_0 = 0, \quad \ddot{p}_0 = 0 \end{gathered}}$$

Small angle approximation (otherwise math is cooked 💀)

- Again, since the paddle will be mostly flat during its run time. we can approximate the terms in the M, C, and G matrix to follow the small angle approximation.

where:

$$\boxed{\begin{gathered} \theta^2 \approx 0, \quad \phi^2 \approx 0, \quad \phi\theta \approx 0, \\ \dot{\phi}\dot{\theta} \approx 0, \quad \dot{\theta}^2 \approx 0, \quad \dot{\phi}^2 \approx 0, \quad \theta\dot{\theta} \approx 0, \quad \phi\dot{\phi} \approx 0, \\ \theta\dot{\phi} \approx 0, \quad \dot{\theta}\phi \approx 0 \end{gathered}}$$

[continues on next page] →

Small angle approximation (continued)

So $M(q)$ becomes:

$$M(q) = \begin{bmatrix} \frac{I_p\phi^2 + I_p + mr^2(\phi^2+1)}{r^2} & \frac{(-I_p - mr^2)\phi\theta}{r^2} \\ \frac{(-I_p - mr^2)\phi\theta}{r^2} & \frac{I_p\theta^2 + I_p + mr^2(\theta^2+1)}{r^2} \end{bmatrix}$$

$$\downarrow$$

$$M(q) = \begin{bmatrix} \frac{I_p + mr^2}{r^2} & 0 \\ 0 & \frac{I_p + mr^2}{r^2} \end{bmatrix}$$

$$\boxed{M(q) = \left(\frac{I_p}{r^2} + m\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

and $C(q,\dot{q})$ becomes:

$$C(q,\dot{q}) = \frac{1}{r^2}\begin{bmatrix} 2(I_p + mr^2)\phi\dot{\phi} & -(I_p + mr^2)(\phi\dot{\theta} + \theta\dot{\phi}) \\ -(I_p + mr^2)(\phi\dot{\theta} + \theta\dot{\phi}) & 2(I_p + mr^2)\theta\dot{\theta} \end{bmatrix}$$

$$\downarrow$$

$$\boxed{C(q,\dot{q}) = \frac{1}{r^2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

Small angle approximation (continued)

And lastly, the G matrix becomes:

$$G_1 = m\Big(-g\phi + r\phi^2\ddot{\phi} + r\phi\theta\ddot{\theta} + 2r\phi\dot{\phi}^2 + r\phi\dot{\theta}^2 + r\theta\dot{\phi}\dot{\theta} + r\ddot{\phi} - \phi^2 x(t)\dot{\phi} + \phi^2 y(t)\dot{\phi}\dot{\theta} + \phi\dot{p}_{ox}(t)\dot{\phi} - \phi\ddot{p}_{oz}(t) - \theta^2 x(t)\dot{\phi}^2 + \theta^2 y(t)\dot{\phi}\dot{\theta} - \theta\dot{p}_{oy}(t)\dot{\phi} - x(t)\dot{\phi}^2 + y(t)\dot{\phi}\dot{\theta} + \ddot{p}_{ox}(t)\Big)$$

$$G_2 = m\Big(g\theta - r\phi\theta\ddot{\phi} - r\phi\dot{\phi}\dot{\theta} - r\theta^2\ddot{\theta} - r\theta\dot{\phi}^2 - 2r\theta\dot{\theta}^2 - r\ddot{\theta} + \phi^2 x(t)\dot{\phi}\dot{\theta} - \phi^2 y(t)\dot{\theta}^2 - \phi\dot{p}_{ox}(t)\dot{\theta} + \theta^2 x(t)\dot{\phi}\dot{\theta} - \theta^2 y(t)\dot{\theta}^2 + \theta\dot{p}_{oy}(t)\dot{\theta} + \theta\ddot{p}_{oz}(t) + x(t)\dot{\phi}\dot{\theta} - y(t)\dot{\theta}^2 + \ddot{p}_{oy}(t)\Big)$$

cut small angle terms (angle × angle)

$$G_1 = m\Big(-g\phi + \cancel{r\phi^2\ddot{\phi}} + \cancel{r\phi\theta\ddot{\theta}} + \cancel{2r\phi\dot{\phi}^2} + \cancel{r\phi\dot{\theta}^2} + \cancel{r\theta\dot{\phi}\dot{\theta}} + r\ddot{\phi} - \cancel{\phi^2 x(t)\dot{\phi}} + \cancel{\phi^2 y(t)\dot{\phi}\dot{\theta}} + \cancel{\phi\dot{p}_{ox}(t)\dot{\phi}} - \phi\ddot{p}_{oz}(t) - \cancel{\theta^2 x(t)\dot{\phi}^2} + \cancel{\theta^2 y(t)\dot{\phi}\dot{\theta}} - \cancel{\theta\dot{p}_{oy}(t)\dot{\phi}} - \cancel{x(t)\dot{\phi}^2} + \cancel{y(t)\dot{\phi}\dot{\theta}} + \ddot{p}_{ox}(t)\Big)$$

$$G_2 = m\Big(g\theta - \cancel{r\phi\theta\ddot{\phi}} - \cancel{r\phi\dot{\phi}\dot{\theta}} - \cancel{r\theta^2\ddot{\theta}} - \cancel{r\theta\dot{\phi}^2} - \cancel{2r\theta\dot{\theta}^2} - r\ddot{\theta} + \cancel{\phi^2 x(t)\dot{\phi}\dot{\theta}} - \cancel{\phi^2 y(t)\dot{\theta}^2} - \cancel{\phi\dot{p}_{ox}(t)\dot{\theta}} + \cancel{\theta^2 x(t)\dot{\phi}\dot{\theta}} - \cancel{\theta^2 y(t)\dot{\theta}^2} + \cancel{\theta\dot{p}_{oy}(t)\dot{\theta}} + \theta\ddot{p}_{oz}(t) + \cancel{x(t)\dot{\phi}\dot{\theta}} - \cancel{y(t)\dot{\theta}^2} + \ddot{p}_{oy}(t)\Big)$$

simplify

$$G_1 = m\left(-g\phi + r\ddot{\phi} - \phi\ddot{p}_{oz}(t) + \ddot{p}_{ox}(t)\right)$$

$$G_2 = m\left(g\theta - r\ddot{\theta} + \theta\ddot{p}_{oz}(t) + \ddot{p}_{oy}(t)\right)$$

drop 2nd order terms

$$G_1 = -mg\phi$$

$$G_2 = mg\theta$$

$$G = \begin{bmatrix} -mg\phi \\ mg\theta \end{bmatrix}$$

So $M\ddot{q} + C\dot{q} + G = 0$ becomes

$$\left(\frac{I_p}{r^2} + m\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{q} + 0 + \begin{bmatrix} -mg\phi \\ mg\theta \end{bmatrix} = 0$$

For state space ...

$$\dot{s} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ M^{-1}(q)\left[C(q,\dot{q})\dot{q} - G(q,t)\right] \end{bmatrix}$$

$$\dot{s} = f(s,u) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \left[\begin{bmatrix} \left(\frac{I_p}{r^2}+m\right) & 0 \\ 0 & \left(\frac{I_p}{r^2}+m\right) \end{bmatrix}^{-1} \cdot (-1)\begin{bmatrix} -mg\phi \\ mg\theta \end{bmatrix}\right] \end{bmatrix}$$

A matrix

$$A = \frac{\partial f}{\partial s} = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial \dot{x}} & \frac{\partial \dot{x}}{\partial \dot{y}} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial \dot{x}} & \frac{\partial \dot{y}}{\partial \dot{y}} \\ \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial y} & \frac{\partial \ddot{x}}{\partial \dot{x}} & \frac{\partial \ddot{x}}{\partial \dot{y}} \\ \frac{\partial \ddot{y}}{\partial x} & \frac{\partial \ddot{y}}{\partial y} & \frac{\partial \ddot{y}}{\partial \dot{x}} & \frac{\partial \ddot{y}}{\partial \dot{y}} \end{bmatrix}$$

Equilibrium terms:

$\theta = 0$, $\phi = 0$, $\dot{\theta} = 0$, $\dot{\phi} = 0$

$x = 0$, $y = 0$, $\dot{x} = 0$, $\dot{y} = 0$

$p_0(t)$ is constant $\rightarrow$ $\dot{p}_0 = 0$, $\ddot{p}_0 = 0$

$$A = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial \dot{x}} & \frac{\partial \dot{x}}{\partial \dot{y}} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial \dot{x}} & \frac{\partial \dot{y}}{\partial \dot{y}} \\ \multicolumn{4}{c}{\frac{\partial \ddot{q}}{\partial s}} \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$B \text{ matrix} = \frac{\partial f}{\partial u}, \qquad u = \begin{bmatrix} \theta \\ \phi \end{bmatrix}$$

input variables

$$B = \begin{bmatrix} \frac{\partial \dot{x}}{\partial \theta} & \frac{\partial \dot{x}}{\partial \phi} \\ \frac{\partial \dot{y}}{\partial \theta} & \frac{\partial \dot{y}}{\partial \phi} \\ \frac{\partial \ddot{x}}{\partial \theta} & \frac{\partial \ddot{x}}{\partial \phi} \\ \frac{\partial \ddot{y}}{\partial \theta} & \frac{\partial \ddot{y}}{\partial \phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\partial \ddot{q}}{\partial u} \end{bmatrix}$$

$$\ddot{q} = \left[\begin{bmatrix} \left(\frac{I_p}{r^2} + m\right) & 0 \\ 0 & \left(\frac{I_p}{r^2} + m\right) \end{bmatrix}^{-1} \cdot (-1) \begin{bmatrix} -mg\phi \\ mg\theta \end{bmatrix} \right]$$

$$\frac{\partial \ddot{q}}{\partial u} = \frac{\partial}{\partial u} \left(-\frac{r^2}{I_p + mr^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -mg\phi \\ mg\theta \end{bmatrix} \right)$$

$$= \frac{r^2}{I_p + mr^2} \frac{\partial}{\partial u} \left(-mg \begin{bmatrix} -\phi \\ \theta \end{bmatrix} \right) = \begin{bmatrix} 0 & \frac{mgr^2}{I_p + mr^2} \\ -\frac{mgr^2}{I_p + mr^2} & 0 \end{bmatrix}$$

$$\boxed{B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \alpha \\ -\alpha & 0 \end{bmatrix} \quad \text{where} \quad \alpha = \frac{mgr^2}{I_p + mr^2}}$$

since we can neglect disturbances...

$$\boxed{\begin{aligned} C &= \text{Identity matrix} \\ D &= \text{zero matrix} \end{aligned}}$$