LINEAR KALMAN FILTERS

SUBTEAM 2

General Framework

- State space model
- Discrete time
- Two sets of data: **latent states** and **observables**
- Goal: estimate state at time (k + 1) based on time (k)
- Prior versus Posterior predictions
- Follow 2 steps
  - Propagation Step
  - Update Step

Model Set Up

Transition Matrix Process Noise

Process Equation

$$x_{k+1} = F_{k+1,k}x_k + w_k \quad (1.1)$$

Measurement Equation

$$y_k = H_k x_k + v_k \quad (1.3)$$

Measurement Matrix Measurement Noise

Where F and H both known

Process and Measurement Noise

- White, additive, Gaussian with mean 0
- Process noise covariance:

$$E[w_n w_k^T] = \begin{cases} Q_k \text{ for n = k} \\ 0 \text{ for } \ n \neq k \end{cases} \quad (1.2)$$

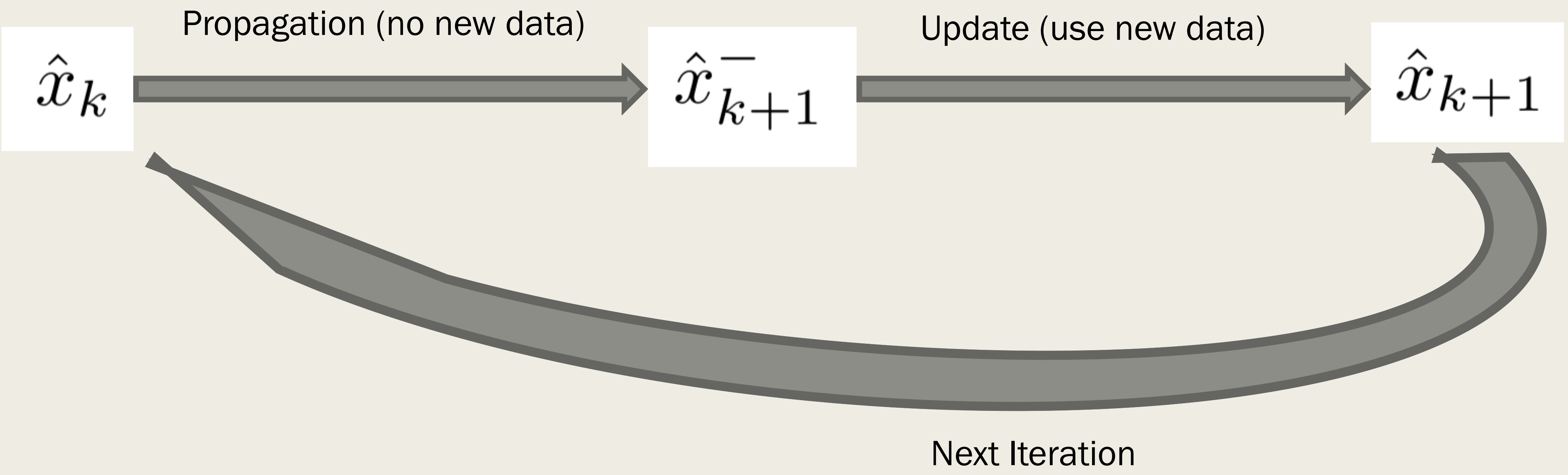
- Measurement noise covariance:

$$E[v_n v_k^T] = \begin{cases} R_k \text{ for n = k} \\ 0 \text{ for } \ n \neq k \end{cases} \quad (1.4)$$

- Noises independent of one another

Notation

- $\hat{x_k}$ - posterior prediction
- $\hat{x_k}^-$ - prior prediction
- $\tilde{x_k}$ -posterior prediction error
- $\tilde{x}_k^-$ - prior prediction error
- $\tilde{x_k} = x_k - \hat{x_k}$
- $\tilde{x}_k^- = x_k - \hat{x}_k^-$
- $P_k = E[\tilde{x_k} * \tilde{x_k}^T]$ - posterior covariance
- $P_k^- = E[\tilde{x_k}^- * \tilde{x_k}^{-T}]$ - prior covariance


$\hat{x}_k$
Propagation (no new data)
$\hat{x}^-_{k+1}$
Update (use new data)
$\hat{x}_{k+1}$
Next Iteration

Approaching the Problem

- Minimizing mean squared error of state prediction
- I.e. find x to minimize the following:

$$\mathrm{E}[(x_k - \hat{x_k})^2]$$

Principle of Orthogonality

Thm 1.2 (Principle of Orthogonality): Let the stochastic processes $\{x_k\}$ and $\{y_k\}$ be of zero mean, that is,

$$E[x_k] = E[y_k] = 0 \ for \ all \ k$$

Then:
(i) the stochastic processes x_k and y_k are jointly Gaussian; or
(ii) if the optimal estimate $\hat{x}_k$ is restricted to be a linear function of the observables and the cost function is the mean square error,
(iii) then the optimum estimate $\hat{x}_k$ given the observables $y_1, y_2, ..., y_k$ is the orthogonal projection of x_k on the space spanned by those observables.

What does this buy us?

$$E[\tilde{x}_k y_i^T] = 0 \ for \ all \ i = 1, ..., k-1 \ (1.7)$$

Relate Prior and Posterior Predictions

- Before able to predict (k + 1), need posterior prediction for (k)
- Assume the following linear relationship:

Observable Data

$$\hat{x}_k = G_k^{(1)} \hat{x}_k^{-} + G_k y_k \quad (1.5)$$

Prior Prediction

Kalman Gain

Current Problems

- G_k unknown
- Write $G_k^{(1)}$ in terms of G_k

Solving for $G_k^{(1)}$ in terms of G_k

Begin with:

$$E[\tilde{x}_k y_i^T] = 0 \ for \ all \ i = 1, ..., k-1 \ (1.7)$$

Substitute in for x and y to yield:

$$E[(x_k - G_k{}^{(1)} \hat{x_k}{}^{-} - G_k H_k x_k - G_k w_k) * y_i{}^T] = 0$$

From which it follows from the principle of orthoganlity that:

$$(I - G_k H_k - G_k^{(1)}) E[x_k y_i^T] = 0$$

Which is only satisfied if:

$$G_k{}^{(1)} = I - G_k H_k$$

The Result: State Estimate Update

$$\hat{x}_k = \hat{x}_k^- + G_k(y_k - H_k * \hat{x}_k^-)$$

Not Yet Defined

The Prior Prediction

$\hat{x_k}^-$ represents the prior prediction:
Can apply transition matrix to previous posterior prediction to obtain as follows:

$$\hat{x_k}^- = F_{k,k-1}\hat{x}_{k-1}$$

This update known as **State Estimate Propagation**

Finding the Kalman Gain

From Thm 1.2:

$$E[(x_k - \hat{x_k})\hat{y_k}^T] = 0$$

After bringing in $\tilde{y_k}$ have that:

$$E[(x_k - \hat{x_k})\tilde{y_k}^T] = 0$$

After substitions for first and second terms and some algebra:

$$(I - G_k H_k)E[\tilde{x_k}^- \tilde{x_k}^{-T}]H_k^T - G_k E[v_k v_k^T] = 0$$

P_k^- R_k

Finally, Solve for G_k

$$G_k = P_k{}^- * H_k{}^T * [H_k * P_k{}^- * H_k{}^T + R_k]^{-1}$$

Error Covariance Propagation

- Need closed form way to calculate P_k and P_k^-
- Done in 2 states
  - Given P_k^- , find P_k
  - Given P_{k-1} , find P_k^-

Using P_k^- to Find P_k

Consider definition of P_k as:

$$P_k = E[(x_k - \hat{x_k})(x_k - \hat{x_k})^T]$$

After some substitutions and using independence of v_k and $\tilde{x_k}^-$:

$$P_k = (I - G_k H_k) P_k^- (I - G_k H_k)^T + G_k R_k G_k^T$$

Finally after some more expansion:

$$P_k = (I - G_k H_k) P_k^-$$

Given P_{k-1} Find P_k^-

Use definition of prior estimate $\hat{x_k}^-$ to write:

$$\tilde{x_k}^- = x_k - \hat{x_k}^- as$$

$$\tilde{x_k}^- = F_{k,k-1} x\tilde{_{k-1}} + w_{k-1}$$

Inserting this expression into $P_k^- = E[\tilde{x_k}^- * \tilde{x_k}^{-T}]$ results in:

$$P_k^- = F_{k,k-1} P_{k-1} F^T_{k,k-1} + Q_{k-1}$$

Initialization (what to do for k = 0?)

$$\hat{x_0} = E[x_0]$$

$$P_0 = E[(x_0 - E[x_0])(x_0 - E[x_0])^T]$$

Summary

1. Model
   - Describe latent variable x through $x_{k+1} = F_{k+1,k} * x_k + w_k$
   - Describe observable data though $y_k = H_k * x_k + v_k$
2. Initialization
   - For k = 0, set $\hat{x_0} = E[x_0]$
   - Also set $P_0 = E[(x_0 - E[x_0])(x_0 - E[x_0])^T]$
3. Computation
   - Propagation Step
     - State estimate Propagation: $\hat{x_k}^- = F_{k,k-1} * \hat{x}_{k-1}$ (1.26)
     - Error Covariance Propagation: $P_k{}^- = F_{k,k-1} * P_{k-1} * F_{k,k-1}^T + Q_{k-1}$ (1.28)
   - Update Step
     - Kalman Gain Matrix: $G_k = P_k{}^- * H_k{}^T * [H_k * P_k{}^- * H_k{}^T + R_k]^{-1}$ (1.22)
     - State Estimate Update: $\hat{x}_k = \hat{x}_k^- + G_k(y_k - H_k * \hat{x}_k^-)$ (1.12)
     - Error Covariance Update: $P_k = (I - G_k H_k) P_k{}^-$ (1.25)

Example

- Assume system of 2 ODE's representing latent states (x's)
- Assume system of 3 variables representing observable data (y's)
- Assume transition and measurement matrices independent of time

Latent State ODE's:

$$\frac{dx^{(1)}}{dt} = \ldots$$

$$\frac{dx^{(2)}}{dt} = \ldots$$

Linearized System:

$$x_{k+1}^{(1)} = F^{(1)} * x_k^{(1)} + w^{(1)}$$

$$x_{k+1}^{(2)} = F^{(2)} * x_k^{(2)} + w^{(2)}$$

Where $x^{(z)}$ represents single element and $F^{(z)}$ represents single row

Initialization

Since x is a random variable, can initialize with:

$$\hat{x_0} = E[x_0] = \bar{X} \qquad x_0 = \begin{bmatrix} 2\text{x}1 \end{bmatrix}$$

Which means P_0 intialized to:

$$P_0 = E[(x_0 - \bar{X})(x_0 - \bar{X})^T] \quad P_0 = \begin{bmatrix} 2\text{x}2 \end{bmatrix}$$

Model Set Up (Latent States)

$$x_1 = Fx_0 + w_0 \ (1.1)$$

$$\begin{bmatrix} \\ 2\text{x}1 \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & 2\text{x}2 & \\ & & \end{bmatrix} \begin{bmatrix} \\ 2\text{x}1 \\ \\ \end{bmatrix} + \begin{bmatrix} \\ 2\text{x}1 \\ \\ \end{bmatrix}$$

Model Set Up (Observables)

$$y_0 = Hx_0 + v_0 \quad (1.3)$$

$$\begin{bmatrix} \\ 3\text{x}1 \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & 3\text{x}2 & \\ & & \end{bmatrix} \begin{bmatrix} \\ 2\text{x}1 \\ \\ \end{bmatrix} + \begin{bmatrix} \\ 3\text{x}1 \\ \\ \end{bmatrix}$$

State Estimate Propagation

$$\hat{x}_1^- = F\bar{X} \ (1.26)$$

$$\begin{bmatrix} \\ 2\text{x}1 \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & 2\text{x}2 & \\ & & \end{bmatrix} \begin{bmatrix} \\ 2\text{x}1 \\ \\ \end{bmatrix} ($$

Error Covariance Propagation

$$P_1^- = FP_0F^T + Q_0 \ (1.28)$$

$$\begin{bmatrix} & 2\text{x}2 & \end{bmatrix} = \begin{bmatrix} & 2\text{x}2 & \end{bmatrix}\begin{bmatrix} & 2\text{x}2 & \end{bmatrix}\begin{bmatrix} & 2\text{x}2 & \end{bmatrix} + \begin{bmatrix} & 2\text{x}2 & \end{bmatrix}$$

Kalman Gain Matrix

$$G_1 = P_1{}^{-} * H_1{}^{T} * [H_1 * P_1{}^{-} * H_1{}^{T} + R_1]^{-1} \; (1.22)$$

$$\begin{bmatrix} & & \\ & \text{2x3} & \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ & \text{2x2} & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & \text{2x3} & \\ & & \end{bmatrix}\left(\begin{bmatrix} & & \\ & \text{3x2} & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & \text{2x2} & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & \text{2x3} & \\ & & \end{bmatrix} + \begin{bmatrix} & & \\ & \text{3x3} & \\ & & \end{bmatrix}\right)^{-1}$$

State Estimate Update

$$\hat{x}_1 = \hat{x}_1^- + G_1(y_1 - H_1\hat{x}_1^-) \ (1.12)$$

$$\begin{bmatrix} \text{2x1} \end{bmatrix} = \begin{bmatrix} \text{2x1} \end{bmatrix} + \begin{bmatrix} \quad \text{2x3} \quad \end{bmatrix} \left(\begin{bmatrix} \text{3x1} \end{bmatrix} - \begin{bmatrix} \quad \text{3x2} \quad \end{bmatrix} \begin{bmatrix} \text{2x1} \end{bmatrix} \right)$$

Error Covariance Update

$$P_1 = (I - G_1 H_1) P_1^{-} \quad (1.25)$$

$$\begin{bmatrix} \quad 2\text{x}2 \quad \end{bmatrix} = \left(\begin{bmatrix} \quad 2\text{x}2 \quad \end{bmatrix} - \begin{bmatrix} \quad 2\text{x}3 \quad \end{bmatrix} \begin{bmatrix} \quad 3\text{x}2 \quad \end{bmatrix} \right) \begin{bmatrix} \quad 2\text{x}2 \quad \end{bmatrix}$$

CAN NOW PROCEED TO ITERATION K = 2 FOLLOWING THE SAME PROCESS