Lecture Notes for ECON 23620 Autumn 2017

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Disclaimer

These lecture notes are not intended to be self-contained. They are provided to accompany a live lecture. They may contain typos and errors. Please correct as necessary.

1 Discrete time consumption-savings problems

1.1 Deterministic consumption-savings problems

Sequence formulation The sequence formulation of the basic infinite-horizon consumption-savings problem in discrete time is

$$\max \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

$$\text{subject to}$$

$$\begin{aligned}
c_t + s_t &\leq y_t + (1+r)\, a_t \\
a_{t+1} &= s_t \\
s_t &\geq \underline{a} \\
a_0 &\text{ given} \\
\lim_{t\to\infty} a_t &\geq 0
\end{aligned}$$

The last condition is known as a No Ponzi Schemes condition. The equivalent finite horizon problem is the same as the infinite horizon problem, with only two changes:

1. The objective function is $\max \sum_{t=0}^{T} \beta^t u\left(c_t\right)$.
2. The No Ponzi Schemes condition is replaced with a terminal condition $a_{T+1} \geq 0$.

Euler equation without borrowing constraints Start by ignoring the borrowing constraint $s_t \geq \underline{a}$ and assume only that the No Ponzi condition must hold. This means that a household can borrow as much as they like as long as they can ultimately pay it back. Substituting the evolution equation into the borrowing constraint gives

$$c_t + a_{t+1} \leq y_t + (1+r)\, a_t$$

We attach a sequence of non-negative Lagrange multipliers λ_t to the time t budget constraints. The Lagrangian for the problem is

$$\max_{c_t, a_{t+1}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) - \sum_{t=0}^{\infty} \lambda_t \left[c_t + a_{t+1} - y_t - (1+r)\, a_t\right]$$

The FOC for c_t and a_{t+1} are

$$\beta^t u'(c_t) = \lambda_t$$
$$\lambda_t = (1+r)\lambda_{t+1}$$

The latter equation is known as the Euler equation. It relates the marginal value of wealth at time t (the multiplier on the budget constraint, which has the interpretation of the marginal value of relaxing the associated constraint) to the marginal value of wealth at time $t+1$. The equation says that the value of an extra dollar at time t is equal to $(1+r)$ times the value of a dollar at time $t+1$. The intuition is that you can always take that dollar and invest it for one period, after which it would be worth $1+r$. If this equation did not hold, then the household could increase the value of their wealth by moving wealth around.

The former equation says that the marginal value of wealth at time t must be equal to the marginal value of consumption at time t. If we substitute the two equations we get

$$\beta^t u'(c_t) = (1+r)\beta^{t+1} u'(c_{t+1})$$
$$u'(c_t) = (1+r)\beta u'(c_{t+1})$$

which is the Euler equation.

Implications of Euler equation If the utility function is strictly concave then $u''(c_t) < 0$. In the absence of borrowing constraints the Euler equation then implies that

$$c_{t+1} = c_t \text{ if } \beta R = 1$$
$$c_{t+1} > c_t \text{ if } \beta R > 1$$
$$c_{t+1} < c_t \text{ if } \beta R > 1$$

Note that these slopes for the consumption path are derived without knowing anything about either initial assets a_0 or the sequence of income $\{y_t\}$.

Smoothing motive for savings The optimal consumption path thus features income smoothing. For example if $\beta R = 1$, the household chooses a constant path for consumption even if income is moving around. Since income is moving around but consumption is not, the household budget constraint implies that when income is temporarily high, the household saves; and when income is temporarily low, the household dissaves. This motive for savings is known as the smoothing motive.

Transversality condition The set of Euler equations do not on their own fully characterize and optimal solution to the consumption savings problem. Intuitively, these equations only pin down the slope of the consumption profile. To pin down the level of the consumption profile we need to make use of the household budget constraint and the No Ponzi condition. We can ensure that these are satisfied by also imposing a transversality condition:

$$\begin{aligned}\lim_{t\to\infty}\lambda_t &= 0\\ \lim_{t\to\infty}\beta^t u'(c_t) &= 0\end{aligned}$$

CRRA Example Assume that

$$u(c)=\begin{cases}\frac{c^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma\in(0,1),\gamma>1\\ \log c & \text{if } \gamma=1\end{cases}$$

which implies

$$u'(c)=c^{-\gamma}.$$

In the absence of borrowing constraints The Euler equation implies

$$\begin{aligned}c_t^{-\gamma} &= \beta R c_{t+1}^{-\gamma}\\ c_t &= (\beta R)^{-\frac{1}{\gamma}} c_{t+1}\\ c_t &= (\beta R)^{\frac{t}{\gamma}} c_0\end{aligned}$$

Collapsing the budget constraints at equality gives

$$\sum_{t=0}^{\infty} R^{-t}c_t = Ra_0 + \sum_{t=0}^{\infty} R^{-t}y_t + \lim_{t\to\infty} R^{-t}a_{t+1}$$

We will impose a stronger version of the No Ponzi schemes condition that $\lim_{t\to\infty} R^{-t}a_{t+1}$. Substituting in the optimal decision rule for c_t gives

$$\sum_{t=0}^{\infty}\left(R^{\frac{1-\gamma}{\gamma}}\beta^{\frac{1}{\gamma}}\right)^{-t}c_0 = Ra_0 + \sum_{t=0}^{\infty} R^{-t}y_t$$

which can be rearranged as

$$c_0 = \mathfrak{m}\left(\beta, R, \gamma\right)\left(Ra_0 + \sum_{t=0}^{\infty} R^{-t}y_t\right)$$

where

$$\mathfrak{m}\left(\beta, R, \gamma\right) = 1 - R^{\frac{1-\gamma}{\gamma}}\beta^{\frac{1}{\gamma}}$$

is known as the marginal propensity to consume.

In the special case where interest rates is such that $\beta R = 1$ the marginal propensity to consume is

$$\mathfrak{m}\left(\beta, \beta, \gamma\right) = 1 - \beta$$

regardless of γ. Note also that with log utility ($\gamma = 1$) the marginal propensity to consume is also

$$\mathfrak{m}\left(\beta, R, 1\right) = 1 - \beta$$

regardless of R.

Only PDV of income matters We used the absence of borrowing constraints twice: once when we derived the Euler equation and once when we collapsed the intertemporal budget constraint. Without borrowing constraints, this formula reveals that consumption depends only on the present discounted value of income and not the timing of income. Later we will see that a version of this result holds in much more general environments and is the basis for Ricardian equivalence. Also, the smoothing motive implicit in this formula for optimal consumption reveals one of the most basic drivers of wealth inequality – even if two individual face the same present value of income, if they have different values of (β_i, R_i), they will choose to consume that income at different times, and hence will accumulate different amounts of wealth.

Euler equation with borrowing constraints We now attach an additional sequence of non-negative Lagrange multipliers μ_t to the time t borrowing constraints. The Lagrangian becomes

$$\max_{c_t, a_{t+1}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) - \sum_{t=0}^{\infty} \lambda_t \left[c_t + a_{t+1} - y_t - (1+r)\, a_t\right] - \sum_{t=0}^{\infty} \mu_t \left(\underline{a} - a_{t+1}\right)$$

The FOC for c_t is unchanged

$$\beta^t u'\left(c_t\right) = \lambda_t$$

The FOC for a_{t+1} becomes

$$\lambda_t = (1+r)\, \lambda_{t+1} + \mu_t$$

Substituting we get

$$u'\left(c_t\right) = (1+r)\, \beta u'\left(c_{t+1}\right) + \beta^{-t} \mu_t$$

Since $\beta^{-t}\mu_t \geq 0$, this equation is typically written as

$$u'\left(c_t\right) \geq (1+r)\, \beta u'\left(c_{t+1}\right)$$

Interpretation of Euler equation In periods where the borrowing constraint does not bind, we will have $a_{t+1} > \underline{a}$ and $\mu_t = 0$ and the Euler equation will hold at equality. In periods where the borrowing constraint binds, we will have $a_{t+1} = \underline{a}$ and $\mu_t > 0$ and the Euler equation will hold at equality. In these periods, the household has a higher marginal utility of consumption today than tomorrow. Thus they would like to bring resources from the future to the present by borrowing, but they are restricted in doing so by the borrowing constraint.

1.2 Deterministic dynamic programming

Optimal consumption and asset holdings The solution to the consumption-savings problem we just encountered consists of an optimal path of consumption $\{c_t\}$ that depends on the sequence of income $\{y_t\}$ and initial assets a_0. Associated with the optimal path of consumption is an optimal path of asset holdings $\{a_{t+1}\}$ that can be derived from the sequence of budget constraints. Different levels of initial assets a_0 will be associated with different paths of consumption and assets.

Value function We can substitute the optimal sequence of consumption into the objective function and find the maximized value of the household's present discounted value of utility, which of course depends on initial assets. We call this present discounted value the *value function*:

$$V\left(a_0\right) = \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

where the maximization is subject to the sequence of budget constraints from $t = 0$ onwards

Dynamic Programing The idea behind dynamic programing is to take this very big optimization problem and breaking it into a sequence of small problems. Rather than deciding on a full entire sequence of consumption choices $\{c_t\}_{t=0}^{\infty}$ at time 0 (and an associated path of asset holdings $\{a_{t+1}\}_{t=0}^{\infty}$), the household decides only on two things - how much to consume today c_t, and how much to save a_{t+1}.

Re-write the value function as

$$\begin{aligned}
V\left(a_0\right) &= \max_{c_0, a_1, \{c_t, a_{t+1}\}_{t=1}^{\infty}} \left[u\left(c_0\right) + \sum_{t=1}^{\infty} \beta^t u\left(c_t\right)\right] \\
&= \max_{c_0, a_1, \{c_s, a_{s+1}\}_{s=0}^{\infty}} \left[u\left(c_0\right) + \beta \sum_{s=0}^{\infty} \beta^s u\left(c_s\right)\right] \\
&= \max_{c_0, a_1} \left[u\left(c_0\right) + \beta \max_{\{c_s, a_{s+1}\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s u\left(c_s\right)\right]
\end{aligned}$$

where we have defined $s = t-1$. Each maximization over the pair $\{c_t, a_{t+1}\}$ is subject to the time t budget constraint. So the outer maximization is subject to only $c_0 + a_1 \leq Ra_0 + y_1$. The inner maximization is subject to the sequence of budget constraints from $t = 1$ (i.e. $s = 0$) onwards

Recursive formulation Note that the second term in the brackets looks exactly like the definition of the value function. The only difference is that when the household gets around to solving the second maximization problem, it will now longer have assets a_0 at its disposal but instead will have assets a_1, defined by the $t = 0$ budget constraint. An alternative way to see this is to note that the term in the brackets is exactly equal to $V\left(a_0\right)$ but with time subscripts indexed by s rather than t. Since $s = t - 1$, when time is indexed by t this term is $V\left(a_1\right)$

Bellman equation Using this property we get the Bellman equation

$$V(a_0) = \max_{c_0, a_1} u(c_0) + \beta V(a_1)$$
$$\text{subject to}$$
$$c_0 + a_1 \leq y + (1+r) a_0$$

The beauty of an infinite horizon, is that once the household has solved this problem, tomorrow looks like today. So we can drop the time subscripts and just write

$$V(a) = \max_{c, a'} u(c) + \beta V(a')$$
$$\text{subject to}$$
$$c + a' \leq y + (1+r) a$$

where we have used ' to denote next period values of variables. This equation is known as the Bellman equation and from now on we will write all of our economic decision problems in this from.

Functional equation In one sense this is a much easier problem than the sequence problem because we are solving for just 2 objects, rather than for an infinite sequence of objects. But this comes at a cost - there is an unknown function in the objective function! In other words the Bellman equation is a *functional* equation. It is an equation that defines a function, known as the value function.

Policy functions The solution to a Bellman equation consists of two types of objects:

1. A value function $V(a)$, which describes the present discounted value the households problem at any point in time for any possible level of assets.
2. A set of policy functions, $a'(a)$, and $c(a)$ which define the household's optimal choice of consumption c and next period assets a' as for any current level of assets.

State variables The arguments of the value function and policy functions are known as *state variables.* These are are the minimum set of variables that a household needs to know at any point in time in order to make an optimal decision. Finding the state variables is not always easy. Tom Sargent used to say that "finding the state is an art". In this simple deterministic consumption savings problem, it is simple to find the state variables.

Principle of optimality Note that, relative to the sequence formulation of the consumption-savings problem, one of the important constraints disappeared: the No Ponzi scheme condition. When we characterized the solution to the household problem in terms of the Euler equation, we had to impose an additional condition to be sure that the solution to the set of Euler equations corresponded to the solution to the underlying sequence problem and satisfied the the No Ponzi schemes condition. That condition was called a Transversality condition. In dynamic programming, there is a strong result called the Principle of Optimality that says that if a value function and policy functions solve the Bellman equation *and* they satisfy an additional boundedness condition, then they also solve the sequence problem. That additional boundedness condition is what ensures that the No Ponzi schemes condition is satisfied.

This is difficult to prove, so you will just have to take my word for it. The idea is that, starting from any given a_0, a set of policy functions defines a path of consumption and assets by iterating on the equations

$$\begin{aligned} c_t &= c\left(a_t\right) \\ a_{t+1} &= a'\left(a_t\right) \end{aligned}$$

We can ask whether the implied sequence of consumption is the solution to the sequence problem.

It is much easier to prove the converse: if a consumption plan solves the sequence problem, then the implied policy functions defined by

$$\begin{aligned} c\left(a_t\right) &= c_t \\ a'\left(a_t\right) &= y + (1+r)\,a_t - c_t \end{aligned}$$

and the implied value function defined by

$$V\left(a_t\right) = \max_{\{c_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u\left(c_s\right)$$

solve the Bellman equation.

Together the above two results are known as the Principle of Optimality.

Euler equation again We can also derive the Euler equation directly from the Bellman equation. Start by forming the Lagrangian

$$V(a) = \max_{c,a'} u(c) + \beta V(a) - \lambda [c + a' - y - (1+r)a] - \mu [\underline{a} - a']$$

where λ and μ are non-negative multipliers on the budget constraint and borrowing constraint, respectively. The FOC are

$$u'(c) = \lambda$$
$$\beta V_a(a') = \lambda - \mu$$

where we have used the convention that $f_x \equiv \frac{\partial f}{\partial x}$. We also get an envelope condition that comes from differentiating the value function with respect to a at the optimum. This means that we differentiate, assuming that we have already maximized with respect to c and a':

$$V_a(a) = \lambda(1+r)$$

We can move this equation forward one period in time to get

$$V_a(a') = \lambda'(1+r)$$

and substituting into the second FOC we get

$$\lambda - \mu = \beta(1+r)\lambda'$$

Then using the first order condition we get

$$u'(c) = \beta(1+r)u'(c') + \mu$$

Since $\mu \geq 0$ this is typically written as

$$u'(c) \geq \beta(1+r)u'(c')$$

When the borrowing constraint does not bind, the Euler equation holds at equality. When it binds, the inequality will be strict.

Value Function Iteration (VFI) Value function iteration is a very easy method for solving a Bellman equation on a computer. It is relatively slow and inaccurate, and later in the class we will study other more advanced methods for solving Bellman equations. The basic idea is to assume that assets can only take a finite number of values and then to iterate to convergence on the value function. The max operator boils down to choosing the savings and consumption option that generates the highest feasible value. We take the following steps:

1. Discretize the asset space by constructing a finite grid $\mathcal{A} \equiv \{a_1, a_2, \ldots a_N\}$ where N is the size of the grid. Set the lowest possible asset holdings to the borrowing limit, i.e. $a_1 = \underline{a}$.

2. Guess an initial value function $V_0(a)$ by assigning a value to each point on the asset grid and income grid. A reasonable first guess is

$$\begin{aligned} V_0(a) &= \sum_{t=0}^{\infty} \beta^t u(ra + y) \\ &= \frac{u(ra+y)}{1-\beta} \end{aligned}$$

   which assumes that the household consumes the annuity value of its financial wealth plus its labor income every period. Set the iteration counter, $l = 0$.

3. Loop over each point in $\mathcal{A}$, i.e. for $i = 1 \ldots N$. At each point a_i solve

$$a'_{l+1}(a_i) = \arg\max_{a' \in \mathcal{A}} u(y + (1+r)a_i - a') + \beta V_l(a')$$

   which generates the following update rule for the value function:

$$\begin{aligned} V_{l+1}(a_i) &= \max_{a' \in \mathcal{A}} u(y + (1+r)a_i - a') + \beta V_l(a') \\ &= u\left(y + (1+r)a_i - a'_{l+1}(a_i)\right) + \beta V_l\left(a'_{l+1}(a_i)\right) \end{aligned}$$

   Note the following:

   - The value function on the RHS is known – it is the one from the previous iteration. As we loop over the state space, we are filling in the values for the value function on the LHS. When we are done, we will have a new value function $V_{l+1}(a_k)$.

- We have substituted the budget constraint into the objective function.
- We can only choose asset amounts on the pre-defined grid for assets $\mathcal{A}$.
- By restricting our choice of savings to lie in the grid $\mathcal{A}$, we automatically impose the borrowing limit.

4. Check for convergence. Compute the difference between the old and new value functions as

$$\epsilon_l = \max_i |V_{l+1}(a_i) - V_l(a_i)|$$

If $\epsilon_l < \bar{\epsilon}$, where $\bar{\epsilon}$ is a pre-specified tolerance level, then go to Step 5. If not, increase the iteration counter $l \to l+1$ and go back to Step 3.

5. Extract policy functions. The optimal savings function and value functions are given by $a'(a) = a_{l+1}(a)$ and $V(a) = V_{l+1}(a)$. The optimal consumption function is then given by

$$c(a) = y + (1+r)a - a'(a)$$

Note that the consumption function is restricted to lie on an implicit grid. Hence, Euler equations will likely not hold very accurately.

Finite horizon dynamic programming For problems with a finite horizon, the value function depends on time t:

$$\begin{aligned} V_t(a) &= \max_{c,a'} u(c) + \beta V_{t+1}(a') \\ &\text{subject to} \\ c + a' &\leq y_t + (1+r)a \\ a' &\geq \underline{a} \end{aligned}$$

The solution to a finite horizon Bellman equation consists of a sequence of value functions $\{V_t(a)\}_{t=0}^T$ and an associated sequence of policy functions $\{c_t(a), a'_t(a)\}_{t=0}^T$

To solve finite horizon problems, we use backward iteration starting at the horizon $t = T$. For consumption savings problems, the condition that $a_{T+1} \geq 0$ means that maximization problem in the final period is trivial

$$\begin{aligned} V_T(a) &= \max_{c,a'} u(c) \\ &\text{subject to} \\ c + a' &\leq y_T + (1+r)a \\ a' &\geq \underline{a} \end{aligned}$$

Since the utility function u is strictly increasing, the solution is to consume all available cash.

$$\begin{aligned} a'_T(a) &= 0 \\ c_T(a) &= y_T + (1+r)a \\ V_T(a) &= u(y_T + (1+r)a) \end{aligned}$$

We then can solve backwards using the same steps as in VFI for infinite horizon problems.

1.3 Stochastic consumption-savings problems

Sequence formulation The sequence formulation of the basic infinite-horizon consumption-savings problem in discrete time is

$$\begin{aligned} \max E_t \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{subject to} \\ c_t + s_t &\leq y_t + (1+r)a_t \\ a_{t+1} &= s_t \\ s_t &\geq \underline{a} \\ a_0 &\text{ given} \\ \lim_{t\to\infty} a_t &\geq 0 \end{aligned}$$

In this problem, the exogenous random variable is the income level y_t so the expectation is over sequences of income $y^t \equiv \{y_0, y_1, \ldots y_t\}$. The endogenous choices are functions for consumption $c_t(y^t)$ and savings $s_t(y^t)$ that take as input histories of realizations of income.

The optimal functions for consumption c_t and savings s_t induce a joint stochastic process for the evolution of the household's assets and income (a_t, y_t).

Markov process for income We will assume that y_t evolves according to a known Markov process. A Markov process y_t is a stochastic process in which the distribution of y_t conditional on the whole history of past realizations of y^{t-1}, $F\left(y_t|y^{t-1}\right)$ is a function of only the most recent realization y_{t-1}. The conditional distribution thus satisfies $F\left(y_t|y^{t-1}\right) = F\left(y_t|y_{t-1}\right)$.

Recursive formulation: Bellman equation The recursive formulation of the basic consumption-savings problem in discrete time consists of the following Bellman equation (BE):

$$\begin{aligned} V\left(a, y\right) &= \max_{c,a'} u\left(c\right) + \beta E\left[V\left(a', y'\right)|y\right] \\ &\text{subject to} \\ c + a' &\leq y + \left(1+r\right)a \\ a' &\geq \underline{a} \end{aligned}$$

The solution to a Bellman equation consists of a value function $V\left(a, y\right)$ and policy functions $c\left(a, y\right)$ and $a'\left(a, y\right)$.

Cash-on-hand state variable When income is distributed independently over time we can reduce the state space to a single variable x, which we label as cash-on-hand, defined as

$$x = y + \left(1+r\right)a$$

In this case the Bellman equation becomes

$$\begin{aligned} V\left(x\right) &= \max_{c,s} u\left(c\right) + \beta E\left[V\left(\left(1+r\right)s + y'\right)\right] \\ &\text{subject to} \\ c + s &\leq x \\ s &\geq \underline{a} \end{aligned}$$

and the solution consists of a policy rule for consumption $c\left(x\right)$.

Stochastic Euler equation We can derive the Euler equation directly from the Bellman equation. Start by forming the Lagrangian

$$V\left(a,y\right)=\max_{c,a'}u\left(c\right)+\beta E\left[V\left(a',y'\right)|y\right]-\lambda\left[c+a'-y-\left(1+r\right)a\right]-\mu\left[\underline{a}-a'\right]$$

where λ and μ are non-negative multipliers on the budget constraint and borrowing constraint, respectively. The FOC are

$$u'\left(c\right)=\lambda$$
$$\beta E\left[V_a\left(a',y'\right)|y\right]=\lambda-\mu$$

where we have assumed the necessary requirements in order to exchange the order of the expectation operator and partial derivative. We will use the convention that $f_x\equiv\frac{\partial f}{\partial x}$. We also get an envelope condition that comes from differentiating the value function with respect to a at the optimum. This means that we differentiate, assuming that we have already maximized with respect to c and a':

$$V_a\left(a,y\right)=\lambda\left(1+r\right)$$

We can move this equation forward one period in time to get

$$V_a\left(a',y'\right)=\lambda'\left(1+r\right)$$

and substituting into the second FOC we get

$$\lambda-\mu=\beta\left(1+r\right)E\left[\lambda'|y\right]$$

Then using the first order condition we get

$$u'\left(c\right)=\beta\left(1+r\right)E\left[u'\left(c'\right)|y\right]+\mu$$

Since $\mu\geq0$ this is typically written as

$$u'\left(c\right)\geq\beta\left(1+r\right)E\left[u'\left(c'\right)\right]$$

where it is implicit that the expectation operator is conditional on all information available at time t. When the borrowing constraint does not bind, the Euler equation holds at equality. When it binds, the inequality will be strict.

Discrete state Markov process for income A simple type of first-order Markov process that we will use repeatedly in our analysis is a discrete state process in which income, y, can take 1 of only J possible realizations, $\{y_1, \cdots, y_J\}$. The probability of income changing from value $j \in \{1, \ldots, J\}$ to $j' \in \{1, \ldots, J\}$ across two periods is given by the $(j, j')\, th$ element of the Markov transition matrix P, which we denote as $p_{jj'}$. The only restrictions that we place on the matrix P are:

1. $p_{jj'} \in [0, 1]$ for all j, j'
2. $\sum_{j'=1}^{J} p_{jj'} = 1$ for all j

We denote the ergodic distribution implied by the Markov chain as the vector π with elements π_j.

Bellman equation with discrete state Markov process When income follows a first-order discrete state Markov chain, we can write the Bellman equation as

$$
\begin{aligned}
V(a, y_j) = \max_{c,a'} u(c) + \beta \sum_{j'=1}^{J} V(a', y_{j'})\, p_{jj'} \\
\text{subject to} \\
c + a' \leq y_j + (1 + r)\, a \\
a' \geq \underline{a}
\end{aligned}
$$

The corresponding Euler equation is

$$
u'(c(a, y_j)) = \beta (1 + r) \sum_{j'=1}^{J} u'(c(a, y_{j'}))\, p_{jj'}
$$

which is a set of J functional equations for $c(a, y_j)$.

Value function iteration We can use value function iteration to solve the stochastic Bellemna equation. We take the following steps:

1. Discretize the asset space by constructing a finite grid $\mathcal{A} \equiv \{a_1, a_2, \ldots a_N\}$ where N is the size of the grid. Set the lowest possible asset holdings to the borrowing limit, i.e. $a_1 = \underline{a}$.

2. Guess an initial value function $V_0(a, y_j)$ by assigning a value to each point on the asset grid and income grid. A reasonable first guess is

$$\begin{aligned} V_0(a, y) &= \sum_{t=0}^{\infty} \beta^t u(ra + y) \\ &= \frac{u(ra + y)}{1 - \beta} \end{aligned}$$

which assumes that the household consumes the annuity value of its financial wealth plus its labor income every period. A slightly better first guess may take stochastic nature of income into account and solve

$$V_0(a, y_j) = u(ra + y_j) + \beta \sum_{j'=1}^{J} V(a', y_{j'}) p_{jj'}$$

for each value of a. This can be solved very easily by noting that this is a system of J linear equations in J unknowns for each value of a. Set the iteration counter, $l = 0$.

3. Loop over each point in $\mathcal{A}$, i.e. for $i = 1 \ldots N$ and each point in the income grid. For each point (a_i, y_j) solve

$$a'_{l+1}(a_i, y_j) = \arg\max_{a' \in \mathcal{A}} u(y_j + (1 + r) a_i - a') + \beta \sum_{j'=1}^{J} V_l(a', y_{j'}) p_{jj'}$$

which generates the following update rule for the value function:

$$\begin{aligned} V_{l+1}(a_i, y_j) &= \max_{a' \in \mathcal{A}} u(y_j + (1 + r) a_i - a') + \beta \sum_{j'=1}^{J} V_l(a', y_{j'}) p_{jj'} \\ &= u\left(y_j + (1 + r) a_i - a'_{l+1}(a_i, y_j)\right) + \beta \sum_{j'=1}^{J} V_l\left(a'_{l+1}(a_i, y_j), y_{j'}\right) p_{jj'} \end{aligned}$$

Note the following:

- The value function on the RHS is known – it is the one from the previous iteration. As we loop over the state space, we are filling in the values for the value function on the LHS. When we are done, we will have a new value function $V_{l+1}(a_k, y_i)$.

- We have substituted the budget constraint into the objective function.
- We can only choose asset amounts on the pre-defined grid for assets $\mathcal{A}$.
- By restricting our choice of savings to lie in the grid $\mathcal{A}$, we automatically impose the borrowing limit.

4. Check for convergence. Compute the difference between the old and new value functions as

$$\epsilon_l = \max_{i,j} |V_{l+1}(a_i, y_j) - V_l(a_i, y_j)|$$

If $\epsilon_l < \bar{\epsilon}$, where $\bar{\epsilon}$ is a pre-specified tolerance level, then go to Step 5. If not, increase the iteration counter $l \to l+1$ and go back to Step 3.

5. Extract policy functions. The optimal savings function and value functions are given by $a'(a, y) = a_{l+1}(a, y)$ and $V(a, y) = V_{l+1}(a, y)$. The optimal consumption function is then given by

$$c(a, y) = y_i + (1+r)a - a'(a, y)$$

Note that the consumption function is restricted to lie on an implicit grid. Hence, Euler equations will likely not hold very accurately.

1.4 Certainty equivalence and precautionary savings

1.4.1 Permanent Income Hypothesis (PIH)

Assumptions The strict version of the PIH makes three important assumptions.

1. Quadratic utility $u(c) = \alpha_1 c - \frac{1}{2}\alpha_2 c^2$.

2. The interest rate on the one-period bond equals the inverse of the discount factor, or $\beta(1+r) = 1$.

3. We replace the borrowing constraints with a single No Ponzi Schemes condition

$$E_t\left[\lim_{\tau\to\infty} R^{-\tau} a_{t+1+\tau}\right] \geq 0 \text{ for all } t.$$

Consumption is a Martingale To solve for the optimal consumption plan, we use the Euler Equation at equality and then impose the No Ponzi Scheme condition.

$$\alpha_1 - \alpha_2 c_t = E_t\left(\alpha_1 - \alpha_2 c_{t+1}\right) \tag{1}$$

$$c_t = E_t c_{t+1} \tag{2}$$

from which we recover the well known result that consumption is a martingale. It is useful to note that from the law of iterated expectations implies that :

$$E_t c_{t+j} = c_t, \text{ for all } j \geq 0. \tag{3}$$

Consumption equals permanent income We multiply the budget constraint in period $t+j$ by R^{-j} and sum over j in order to solve for consumption allocations explicitly.

$$\begin{aligned} c_t &= y_t + Ra_t - a_{t+1} \\ R^{-1}c_{t+1} &= R^{-1}y_t + a_{t+1} - R^{-1}a_{t+2} \\ &\vdots \end{aligned}$$

Summing gives

$$\sum_{j=0}^{\infty} R^{-j} c_{t+j} = Ra_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j} + \lim_{j\to\infty} R^{-j} a_{t+j+1}$$

Taking expectations at time t and imposing the No Ponzi Schemes condition gives

$$\begin{aligned} \sum_{j=0}^{\infty} R^{-j} E_t c_{t+j} &= Ra_t + \sum_{j=0}^{\infty} R^{-j} E_t y_{t+j} \\ \frac{R}{r} c_t &= Ra_t + \sum_{j=0}^{\infty} R^{-j} E_t y_{t+j} \\ c_t &= r\left(a_t + H_t\right) \end{aligned} \tag{4}$$

where the left side of the second row uses the martingale property established in (3). We have denoted human wealth as the expected discounted value of future earnings:

$$H_t \equiv R^{-1} \sum_{j=0}^{\infty} R^{-j} E_t y_{t+j}$$

Recall that financial wealth is a_t. Hence total wealth is given by $W_t \equiv (a_t + H_t)$. We define permanent income to be the annuity value of total wealth rW_t and note that optimal consumption is equal to permanent income.

Certainty equivalence Note that the optimal level of consumption is not a function of the stochastic process for endowments, y_t, except through the expected present value of earnings, H_t. In other words, any two stochastic processes for y_t that generate the same H_t will yield the same consumption allocation. In particular, take any stochastic process $\{y_{t+\tau}\}$ and consider the deterministic process process defined by

$$\overline{y}_{t+\tau} = E_t\left[y_{t+\tau}\right].$$

Since these two processes generate the same expected present value, H_t, the resulting consumption paths are identical. Thus second and higher moments of endowments do not affect optimal consumption decisions. This property descends directly from the linear-quadratic objective function. Below we will modify the environment to break certainty equivalence, by relaxing some of the assumptions of the strict PIH.

Consumption distribution Consider an economy populated by a large number of PIH agents, indexed by i. How does the cross-sectional distribution of consumption evolve in such an economy?

Since $c_t^i = E_t\left[c_{t+1}^i\right]$ we can always write

$$c_{t+1}^i = c_t^i + \varepsilon_{t+1}^i \tag{5}$$

with $E_t\left[\varepsilon_{t+1}^i \mid c_t^i\right] = 0$. If endowments are independent across agents (i.e. no aggregate risk) then ε_{t+1}^i is also independent across agents. Moreover if $V\left[y_t\right] > 0$ for all t, then $V\left[\varepsilon_t\right] > 0$ for all t. We can then infer two important properties of the cross-sectional distribution of consumption from (5):

1. $E\left[c_{t+1}\right] = E\left[c_t\right]$. Mean consumption in the population is constant.
2. $\lim_{\tau\to\infty} V\left[c_{t+\tau}\right] = \infty$. The cross-sectional variance of consumption explodes. Note that this is true, even if the cross-sectional variance of endowments is constant.

Consumption dynamics The change in consumption at time t equals

$$\begin{aligned}
\Delta c_t &= c_t - c_{t-1} \\
&= c_t - E_{t-1} c_t \\
&= r\left[W_t - E_{t-1} W_t\right] \\
&= r\left[a_t - E_{t-1} a_t + R^{-1} \sum_{j=0}^{\infty} R^{-j}\left(E_t - E_{t-1}\right) y_{t+j}\right] \\
&= \frac{r}{1+r} \sum_{j=0}^{\infty} R^{-j}\left(E_t - E_{t-1}\right) y_{t+j} \qquad (6)
\end{aligned}$$

where we have used the random walk property, the law of iterated expectations and the fact that $a_t = E_{t-1} a_t$ since a_t is known at time $t-1$. Hence the change in consumption between $t-1$ and t is proportional to the revision in expected earnings due to the new information that arrives period t.

Wealth dynamics We now derive an expression for the evolution of assets and ask whether the solution that we derived above would continue to hold if we were to impose tighter borrowing limits than the No Ponzi Scheme condition. It turns out that whether this is the case depends crucially on the stochastic process for earnings.

From the budget constraint we have that

$$\Delta a_{t+1} = r a_t + y_t - c_t$$

Substituting the expression for the optimal consumption

$$c_t = r\left[a_t + R^{-1} \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right] \qquad (7)$$

we obtain

$$
\begin{aligned}
\Delta a_{t+1} &= ra_t + y_t - c_t \\
&= y_t - \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j} \\
(1+r)\,\Delta a_{t+1} &= (1+r)\,y_t - ry_t - r\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j} \\
&= y_t - \sum_{j=1}^{\infty}\left[\left(\frac{1}{1+r}\right)^{j-1} - \left(\frac{1}{1+r}\right)^j\right] E_t y_{t+j}
\end{aligned}
$$

where the last line uses the simple algebraic relationship

$$
\frac{r}{(1+r)^j} = \frac{1}{(1+r)^{j-1}} - \frac{1}{(1+r)^j}.
$$

Unfolding the expression in the sum listing first all the positive terms and then all the negative ones:

$$
\begin{aligned}
(1+r)\,\Delta a_{t+1} &= y_t - \left[E_t y_{t+1} + \left(\frac{1}{1+r}\right) E_t y_{t+2} + ... - \left(\frac{1}{1+r}\right) E_t y_{t+1} - \left(\frac{1}{1+r}\right)^2 E_t y_{t+2} - ...\right] \\
\Delta a_{t+1} &= -\sum_{j=1}^{\infty} R^{-j} E_t \Delta y_{t+j}
\end{aligned}
$$

To illustrate the importance of the stochastic process for y_t, we consider two alternative specifications:

1. Random Walk. Assume that the process for y_t is

$$
y_t = y_{t-1} + v_t
$$

where v_t is a white noise innovation. Then it is easy to see that $\Delta y_{t+j} = v_{t+j}$ and $\Delta a_{t+1} = 0$. Thus the initial wealth endowment perpetuates itself forever. Note that this means that there is no self-insurance in such a model, $\Delta c_t = \Delta y_t = v_t$. Imagine that we were to change the environment by replacing the No Ponzi Scheme condition with the stronger condition that

$$
a_{t+1} \geq b.
$$

Since $\Delta a_t = 0$, if an agents starts above the borrowing constraint, then the borrowing constraint will never bind. Thus the solution would not be affected. The intuition is that with random walk shocks, any changes in income also change permanent income by the same amount. Since consumption is equal to permanent income, the result is effectively autarky. Constantinides and Duffie (1996) and Heathcote, Storesletten and Violante (2009) gain a lot of milage by exploiting this result.

2. IID earnings. If the endowment process is iid, we have that $\Delta y_{t+1} = u_{t+1} - u_t$, where u_t is a white noise innovation. Hence:

$$\begin{aligned}
\Delta a_{t+1} &= -\sum_{j=1}^{\infty} R^{-j} E_t \Delta y_{t+j} \\
&= -\sum_{j=1}^{\infty} R^{-j} E_t \left[u_{t+j} - u_{t+j-1} \right] \\
&= u_t
\end{aligned}$$

since all other terms are zero. Hence a_t follows a random walk and, as a result, any constraint on asset holdings will eventually bind with probability one.

To summarize, we note that whether imposing borrowing constraints that are tighter than the No Ponzi Schemes condition affects consumption allocations, depends very much on the stochastic process for earnings. In general, borrowing constraints cannot be ignored.

Precautionary Savings In the next sections we study conditions under which savings decisions do react to changes in income uncertainty. Recall that in the PIH, when borrowing constraints do not bind, certainty equivalence implies that mean-preserving spreads to the income distribution do not impact savings decisions. We now relax the two key PIH assumptions: quadratic utility and the absence of borrowing constraints, and introduce the concept of precautionary savings.

1.4.2 Precautionary savings through prudence

A Two period model Consider a simple two-period consumption-saving problem

$$\begin{aligned} & \max_{\{c_0,c_1,a_1\}} u(c_0) + \beta E[u(c_1)] \\ & s.t. \\ c_0 + a_1 &= y_0 \\ c_1 &= Ra_1 + y_1 \end{aligned}$$

where y_0 is given, and income next period y_1 is also exogenous but stochastic. If we retain the assumption $\beta R = 1$ to simplify the algebra, the Euler equation gives

$$u'(y_0 - a_1) = E[u'(Ra_1 + y_1)],$$

which is one equation in one unknown, a_1. The LHS is increasing in a_1 since $u'' < 0$, and the RHS is decreasing since the sum (integral) of decreasing functions is decreasing., Hence a_1^* is uniquely determined. Note that current consumption c_0 is determined by the period-zero budget constraint

$$c_0^* = y_0 - a_1^*,$$

hence a rise in savings leads to a fall in current consumption.

Mean-preserving spread We start by recalling a simple result from Rothschild and Stiglitz (1970) about mean preserving spreads. Let f be a convex function and let X, Y, Z be three random variables satisfying $Y = X + Z$ with $E[Z|X] = 0$ for all X. By Jenson's inequality we have

$$\begin{aligned} E_z[f(Y)] &\geq f(E_z[Y]) \\ &= f(E_z[X+Z]) \\ &= f(X) \end{aligned}$$

Now take expectations over X to get

$$\begin{aligned} E_x E_z[f(Y)] &\geq E_x[f(X)] \\ E[f(Y)] &\geq E[f(X)] \end{aligned}$$

So if Y is a mean-preserving spread in X then $E[f(Y)] \geq E[f(X)]$ for f convex.

Now, what happens to optimal consumption at $t = 0$ if uncertainty over income next period y_1 rises, i.e. as future income becomes more risky? Consider a mean-preserving spread to y_1. Let:

$$y_2 = y_1 + \varepsilon,$$

where $E[\varepsilon] = 0$ and $var(\varepsilon_t) = \sigma^2 > 0$. The Euler equation becomes

$$u'(y_0 - a_1) = E[u'(Ra_1 + y_1 + \varepsilon)].$$

If u' *is convex*, i.e. $u''' > 0$ then the result above implies that the RHS is larger than $E[u'(Ra_1 + y_1)]$, and a rise in a_1^* is required to maintain equality. Since y_0 is unchanged, c_0^* must fall. The additional savings that results from this channel is known as *precautionary savings.*

Prudence The convexity of marginal utility (or $u''' > 0$) is called *prudence*. Prudence is a property of preferences, like risk aversion: whereas risk-aversion refers to the curvature of the utility function, prudence refers to the curvature of the marginal utility function. Kimball (1990) defines the index of absolute prudence as the ratio $-u'''(c)/u''(c)$, analogously to the Arrow-Pratt index of absolute risk-aversion $-u''(c)/u'(c)$. Hence we have seen that if an individual is prudent ($u''' > 0$), a rise in future income uncertainty leads to a rise in current savings and a decline in current consumption.

Which types of preferences display prudence? We can show that any monotonic increasing utility function with decreasing absolute risk aversion, i.e. in the DARA class (which includes CRRA) displays positive third derivative. Let $\alpha(c)$ be the coefficient of absolute risk aversion. Then:

$$\alpha(c) = \frac{-u''(c)}{u'(c)}$$
$$\alpha'(c) = \frac{-u'''(c)\,u'(c) + [u''(c)]^2}{[u'(c)]^2}.$$

DARA preferences imply that $\alpha'(c) < 0$. So we have that

$$\begin{aligned} u'''(c)\,u'(c) &> [u''(c)]^2 \\ u'''(c) &> \frac{[u''(c)]^2}{u'(c)} \\ &\geq 0. \end{aligned}$$

Intuitively, a rise in uncertainty reduces the certainty-equivalent income next period. With DARA preferences, this effectively increases the degree of risk-aversion, inducing the agent to save more.

Prudence is a motive for additional savings to protect against possible negative realizations of future income shocks. Savings that are induced by prudence are referred to as *precautionary savings* or *self-insurance*. In this simple two-period decision problem, one can define precautionary wealth due to income uncertainty σ^2 as the difference between the optimal savings decision under uncertainty $a_1^*(\sigma^2)$ and the optimal asset choice under certainty over next period income, i.e. $a_1^*(0)$ (which is the same as the PIH level of savings).

Saving motives This is a good time to make a short remark about saving motives. We distinguish the following reasons why agents save in consumption-savings problems:

1. *Inter-temporal Motive.* The savings motive associated with $\beta R > 1$. The agent desires to postpone consumption because of patience and/or returns to savings.

2. *Smoothing Motive.* The savings motive implicit in the pure PIH with quadratic utility and $\beta R = 1$. A risk-averse agent desires to smooth consumption when income is changing over time, whether deterministic or stochastic.

3. *Precautionary Motive.* The saving motive associated to future income uncertainty over and above the savings desired by a certainty-equivalent consumer.

4. We add that in a life-cycle model where an individual faces a retirement period, during the working stage of the life-cycle the individual would have a *life-cycle motive* for saving associated to the desire to smooth consumption between the working life and retirement. In the presence of altruism, there may be an additional *bequest motive* for savings, induced by the desire to pass on assets to the next generation.

A Multi Period Model We now generalize the two-period model to a multi-period model with *iid* income shocks and a finite horizon, $t = 0, 1 \ldots T$. In the multi-period case, the problem of the household can be written, in recursive form, as

$$\begin{aligned} V_t(a_t, y_t) &= \max_{\{c_t, a_{t+1}\}} u(c_t) + \beta E[V_{t+1}(a_{t+1}, y_{t+1})] \\ & s.t. \\ c_t + a_{t+1} &= Ra_t + y_t \end{aligned}$$

Note that when the income shocks $\{y_t\}$ are *iid*, we can define a unique state variable which is a sufficient statistic for the household choice, that we label "cash-on-hand" $x_t \equiv Ra_t + y_t$ since (a_t, y_t) always enter additively and current levels of y_t do not provide any information about the future realizations of income shocks. This leads to the simpler formulation

$$\begin{aligned} V_t(x_t) &= \max_{\{c_t, x_{t+1}\}} u(c_t) + \beta E[V_{t+1}(x_{t+1})] \\ & s.t. \\ x_{t+1} &= R(x_t - c_t) + y_{t+1} \end{aligned}$$

where the last constraint follows from the definition of cash in hand and the budget constraint:

$$x_{t+1} = Ra_{t+1} + y_{t+1} = R(x_t - c_t) + y_{t+1}.$$

From the FOC's with respect to c_t and the constraints, we obtain (with $\beta R = 1$)

$$u'(c_t) = E\left[V'_{t+1}(R(x_t - c_t) + y_{t+1})\right]. \tag{8}$$

Inspection of the Euler equation reveals that the precautionary saving result of the two-period model goes through as long as the derivative of the value function $\left(V'_{t+1}\right)$ is convex, i.e. $V'''_{t+1} > 0$.

Sibley (1975) showed that when the time-horizon T is finite, it can be proved that *if* $u''' > 0$, *then* $V'''_t > 0$ *for all* $t = 1, ..., T$. The proof is based on backward induction: in the last period $V'_T = u'$ which is convex by assumption. From this result, one can show that V'_{T-1} is also convex, and so on. Is a positive third derivative sufficient for the presence of precautionary savings? There are a number of technicalities that need to be taken care of and it turns out that in general they are not. See Huggett and Ospina (2001) and Huggett (2004) for details. In the next section we show that a positive third derivative is not necessary either.

1.4.3 Precautionary Savings Through Borrowing Constraints

In this section we show that in the presence of an occasionally binding borrowing constraint, it is possible to generate precautionary savings even in the absence of prudence. To see this, we will focus on the quadratic utility case and replace the No Ponzi Schemes condition with a *no-borrowing constraint* $a_{t+1} \geq 0$.

The Euler equation now needs to be modified to account for this constraint

$$c_t = \begin{cases} E_t c_{t+1} & \text{if } a_{t+1} > 0 \\ y_t + a_t & \text{if } a_{t+1} = 0 \end{cases}$$

where the first line is just the FOC of the agent when the constraint is not binding, while the second line comes directly from the budget constraint $a_{t+1} = R\left(y_t + a_t - c_t\right)$ when the constraint is binding ($a_{t+1} = 0$). The constrained household would like to borrow to finance consumption, but is not allowed to do so, so consumes all its resources.

The above pair of conditions can be written in compound form as

$$\begin{aligned} c_t &= \min\left\{y_t + a_t, E_t c_{t+1}\right\} \\ &= \min\left\{y_t + a_t, E_t\left[\min\left\{y_{t+1} + a_{t+1}, E_{t+1} c_{t+2}\right\}\right]\right\} \\ &= \ldots \end{aligned}$$

Suppose that uncertainty about income y_{t+1} increases. If the borrowing constraint is not already binding, i.e. $a_t > 0$, then low realizations of income y_{t+1} become more likely, which makes the borrowing constraint more likely to bind at $t+1$ and reduces the value of $E_t\left[\min\left\{y_{t+1} + a_{t+1}, E_{t+1} c_{t+2}\right\}\right]$. This, in turn, reduces the value of $E_t c_{t+1}$ and hence c_t. Thus, even if the borrowing constraint is not already binding at time t, the possibility that it may bind in future periods causes the agent to consume less today.

Recall what happened when we asked when a borrowing constraint would bind in the PIH model. We concluded that it depends crucially on the stochastic process for earnings.

We argued that if income was IID, then any borrowing constraint would eventually bind with probability one.

Intuitively, when agents face borrowing constraints, they fear getting several consecutive bad income realizations which would push them towards the constraint and force them to consume their income *without the ability to smooth consumption.* To prevent this situation, they increase their savings for self-insurance (precautionary savings). Thus, we have shown that prudence is not strictly necessary for precautionary saving behavior: an occasionally binding borrowing constraint is enough. Even though we showed this result for quadratic utility, it is a general results that holds for concave utility.

1.4.4 Natural Borrowing Limits

The previous section showed that if there is a non-zero probability that a borrowing constraint will bind at some time, then agents will have a precautionary motive to save. This begs the question of how tight an exogenous borrowing limit must be, in order for it to bind with positive probability. In other words, when will an exogenous borrowing constraint be occasionally binding? To answer this question we introduce the concept of a *natural borrowing limit.*

A Natural Borrowing Limit Suppose the income process $\{y_t\}_{t=0}^{\infty}$ is deterministic. Consider imposing non-negativity of consumption throughout the life of the household, i.e. $c_t \geq 0$ for all t. Iterating forward on the budget constraint yields

$$\begin{aligned}
c_t &\geq 0 \\
Ra_t + y_t - a_{t+1} &\geq 0 \\
a_t &\geq -R^{-1}y_t + R^{-1}a_{t+1} \\
a_t &\geq -R^{-1}y_t + R^{-1}\left[-R^{-1}y_{t+1} + R^{-1}a_{t+2}\right] \\
&\vdots \\
a_t &\geq -\sum_{j=0}^{\infty} R^{-j-1}y_{t+j} + \lim_{j\to\infty} R^{-j}a_{t+j} \\
&\geq -\sum_{j=0}^{\infty} R^{-j-1}y_{t+j}.
\end{aligned}$$

This constraint says that the household is not allowed to accumulate more debt than she will ever be able to repay by consuming just zero every period. Borrowing any more than this amount would imply negative consumption in at least one period, which would violate the non-negativity of consumption.

If the income process is stochastic, then we need to ensure that the household is able to repay her debt *almost surely (i.e. with probability 1).* A necessary and sufficient condition can be obtained by calculating the maximum amount that the agent could repay if she were to receive the lowest possible realization of the income shock, $y_{\min}$, in every future period. This generates the *natural borrowing limit*

$$a_t \geq -\frac{y_{\min}}{r}. \tag{9}$$

Inada conditions When the utility function satisfies the Inada condition $u(0) = -\infty$, the NBL is endogenous in the sense that the consumer will never want to borrow up to the NBL. If she did, there would be positive probability on a history with zero consumption which implies expected utility is equal to $-\infty$, which can never be optimal. Thus when preferences satisfy an Inada condition, the preferences alone ensure that the NBL will never bind. In other words, in solving for the optimal consumption you can safely assume interior solutions for the Euler equation.

However, for preferences that do not exhibit an Inada condition, this will not be true, since the NBL is derived from an ad-hoc non-negativity constraint on consumption, $c_t \geq 0$. Similarly, if the borrowing constraint is specified to be tighter than the NBL, it may bind with positive probability. Lastly, it is useful to note that for income distributions whose support is the positive real line, the NBL is zero. For example, if y_t is drawn from a log-normal distribution, the Inada condition alone ensures that the agent never chooses to borrow.

1.5 Numerical dynamic programming

We describe two new methods for solving consumption-savings problems that are faster and more accurate than VFI. Rather than starting with a guess of the value function, these methods start with a guess of the policy functions and use the Euler equations to update the guesses. As with VFI, both methods can be adapted to finite-horizon (or time-dependent) problems by iterating backwards from a terminal condition.

1.5.1 Euler equation iteration

EEI overview Euler equation iteration is faster than value function iteration, but is still pretty slow. However it is much more accurate. As with VFI, we assume a discrete state space for assets, but we make three main changes:

- Instead of iterating on the value function, we iterate on the policy functions for consumption, $c(a_i, y_j)$ and savings $a'(a_i, y_j)$.
- We update using the Euler equation, not the Bellman equations.
- We do not require optimal savings to lie on the asset grid $\mathcal{A}$; the policy functions for both savings can take any real value.

EEI steps Recall that the Euler equation is

$$u'(c) \geq \beta(1+r) \sum_{j'=1}^{J} u'(c')\, p_{jj'}$$

with equality whenever the borrowing constraint does not bind (i.e. when $a' > \underline{a}$), and with strict inequality whenever the borrowing constraint does bind (i.e. when $a' = \underline{a}$).

We take the following steps:

1. Discretize the asset space by constructing a finite grid $\mathcal{A} \equiv \{a_1, a_2, \ldots a_N\}$ where N is the size of the grid. Set the lowest possible asset holdings to the borrowing limit, i.e. $a_1 = \underline{a}$.
2. Guess an initial policy function for consumption $c_0(a_i, y_j)$ by assigning a consumption decision to each point on the asset grid and income grid. A reasonable first guess is

   $$c_0(a_i, y_j) = ra + y$$

   which assumes that the household consumes the annuity value of its financial wealth plus its labor income every period. Set the iteration counter, $l = 0$.
3. Loop over each point in $\mathcal{A}$, i.e. for $i = 1 \ldots N$ and each point in the income grid. For each point (a_i, y_j) solve the Euler equation for optimal savings c. Note that the

Euler equation can be written in terms of the consumption choice as

$$u'(c) \geq \beta (1+r) \sum_{j'=1}^{J} u'(c_l [y_j + (1+r) a_i - c, y_{j'}]) p_{jj'}$$

This is a single equation in a single unknown c. It is extremely well behaved. The LHS is decreasing in c (consuming more lowers marginal utility). The RHS is increasing in c (consuming more lowers assets tomorrow which lowers consumption tomorrow which raises marginal utility). We solve this in two steps. First we check whether the borrowing constraint binds. Second, if it doesn't, we solve the non-linear equation numerically.

(a) Check if the borrowing constraint binds. Assume it does. In that case, the optimal savings decision is $a' = \underline{a}$ so the optimal consumption decision is $c = (1+r) a_i + y_j - \underline{a}$. We can compute the LHS and the RHS of the EE as

$$\text{LHS} = u'((1+r) a_i + y_j - \underline{a})$$
$$\text{RHS} = \beta (1+r) \sum_{j'=1}^{J} u'(c_l [\underline{a}, y_{j'}]) p_{jj'}$$

If LHS $\leq$ RHS then this means the borrowing constraint binds. Even at the lowest possible savings level (highest possible consumption), the marginal of utility of consumption is higher today than tomorrow. So the household would like to consume more, but can't because of the borrowing constraint. Set $c_{l+1}(a_i, y_j) = (1+r) a_i + y_j - \underline{a}$. Go to Step 4

(b) If LHS $>$ RHS, then there is a solution to the non-linear equation. We can solve it by evaluating candidate values for c until LHS = RHS. Evaluating a candidate c for the LHS is easy. For the RHS, there are two difficulties. First we have to sum over all the possible income realizations. Second the candidate c will generically imply a value for next period assets a' that does not lie on the grid $\mathcal{A}$. Hence we have to interpolate the function $c_l(a', y_{j'})$.

If the grids are well chosen, linear interpolation is a fast and efficient choice. Even so, this can be time consuming because as we loop over the state space we will probably be repeating this same interpolation many times. Moreover, inside the sum we have to interpolate J times (once for each income point) for every candidate c.

It is therefore useful to construct an interpolating function, which I call $EMUC$ for expected marginal utility of consumption, to avoid doing the interpolation inside the sum. This function is defined as

$$EMUC\left(a', y_j\right) = \sum_{j'=1}^{J} u'\left(c\left(a', y_{j'}\right)\right) p_{ij}$$

Note that it depends on y_j (today's income), not $y_{j'}$ (tomorrow's income). The non-linear equation at the point $\left(a_i, y_j\right)$ then becomes

$$u'\left(c\right) = \beta\left(1+r\right) EMUC\left(\left(1+r\right) a_i + y_j - c, y_j\right)$$

There are many ways to solve a non-linear equation. In Matlab you can use fzero or fsolve. In Python you can use scipy.optimize.root or scipy.optimize.fsolve. This is a very well behaved equation so you can even just use a bisection method, secant method or quasi-Newton method if the built-in routine fails. Set $c_{l+1}\left(a_i, y_j\right) = c$

4. Check for convergence. Compute the difference between the old and new policy functions as

$$\epsilon_l = \max_{i,j} \left|c_{l+1}\left(a_i, y_j\right) - c_l\left(a_i, y_j\right)\right|$$

If $\epsilon_l < \bar{\epsilon}$, we are done. If not, increase the iteration counter $l \to l+1$ and go back to Step 3.

1.5.2 Endogenous grid points

EGP overview This is the method of choice for discrete-time consumption-savings problems. It is similar to the Euler equation iteration method but avoids the need to solve non-linear equations.

EGP steps We take the following steps:

1. Discretize the asset space by constructing a finite grid $\mathcal{A} \equiv \{a_1, a_2, \ldots a_N\}$ where N is the size of the grid. Set the lowest possible asset holdings to the borrowing limit, i.e. $a_1 = \underline{a}$.

2. Guess an initial policy function for consumption $c_0(a_i, y_j)$ by assigning a consumption decision to each point on the asset grid and income grid. A reasonable first guess is

$$c_0(a_i, y_j) = ra + y$$

which assumes that the household consumes the annuity value of its financial wealth plus its labor income every period. Set the iteration counter, $l = 0$.

3. The first thing we will do is to reinterpret the policy function $c_l(a_i, y_j)$ as a function of tomorrow's state variables $c_l(a_i', y_{j'})$ rather than as a function of today's state variables. It answers the question: what would consumption be tomorrow if we have assets a_i' and income $y_{j'}$ tomorrow? Using this we can construct expected marginal utility of consumption, similarly to how we constructed an interpolating function with the previous method:

$$\mathrm{EMUC}_l(a_i', y_j) = \sum_{j'=1}^{J} u'(c_l(a_i', y_{j'}))\, p_{jj'}$$

The next step is the heart of the EGP method. We use the Euler equation at equality to get the marginal utility of consumption today.

$$\mathrm{MUC}_l(a_i', y_j) = \beta(1+r)\,\mathrm{EMUC}_l(a_i', y_j)$$

Note that this gives the marginal utility consumption *today* as a function of income today and assets *tomorrow* (or, more precisely, the choice of savings today, which in this simple model is equal to assets tomorrow). From this it is straight forward to get consumption today and assets today as a function of income today and assets tomorrow:

$$\begin{aligned} c_l(a_i', y_j) &= u'^{-1}(\mathrm{MUC}_l(a_i', y_j)) \\ a_l(a_i', y_j) &= \frac{c_l(a_i', y_j) + a_i' - y_j}{1+r} \end{aligned}$$

This function answers the question: if I currently have income y_j and I am going to save a_i', how much assets a must I currently have today? This is the opposite of the questions we actually want to answer, i.e if I have a assets today, how much a' am I going to save ? We thus need to invert the function $a_l\left(a_i', y_j\right)$, i.e. switch the output with the first input. This would give a function $a_{l+1}'\left(a, y_j\right)$. We could then evaluate that function at the points $a_i \in \mathcal{A}$ and use the budget constraint to get our updated consumption function $c_{l+1}\left(a_i, y_j\right)$.

In practice, for every y_j the function $a_l\left(a_i', y_j\right)$ is just a list of possible savings amounts a_i' that all lie on the grid $\mathcal{A}$, and an associated list of values of current assets a not on the grid. So one way to invert the function is just to switch the order of these columns. The only problem is that then we would have a function $a'\left(a, y_j\right)$ but it would be on the wrong grid (hence the name of the method). So instead we interpolate this function at each point on the grid $\mathcal{A}$, giving us the function we are after, i.e. $a_{l+1}'\left(a_i, y_j\right)$. We then get the new consumption function from

$$c_{l+1}\left(a_i, y_j\right) = (1+r)\, a_i + y_j - a_{l+1}'\left(a_i, y_j\right)$$

4. Deal with the borrowing constraint. This is simpler than it seems. Given the function $a_l\left(a_i', y_j\right)$, which tells us what assets must be today if the agent optimally choose a_i', we can find the level of a such that the borrowing constraint binds exactly, which is just $a^*\left(y_j\right) = a_l\left(\underline{a}, y_j\right)$. Then when we do the interpolation in the previous step, we only do it for points on the grid $a_i \in \mathcal{A}$ such that $a_i > a^*\left(y_j\right)$. For points such that $a_i > a^*\left(y_j\right)$ where the borrowing constraint binds, we set:

$$\begin{aligned} a_{l+1}\left(a_i, y_j\right) &= \underline{a} \\ c_{j+1}\left(a_i, y_j\right) &= (1+r)\, a_i + y_j - \underline{a} \end{aligned}$$

EGP with cash-on-hand as state variable When income risk is IID we can economize on state variables by using cash-on-hand as single state variable. In this case, the individual chooses consumption c and savings s subject to

$$\begin{aligned} c + s &\leq x \\ s &\geq \underline{a} \end{aligned}$$

and then cash-on-hand evolves as

$$x' = (1+r)\,s + y'$$

With this formulation the choice today (s) is not identically equal to the state tomorrow (x') as it was in the previous formulation (where the choice if a' is both the current choice and the future state). In cases like this, the trick with EGP is to construct a grid over the current choice variable s. We take the following steps:

1. Discretize the cash-on-hand space by constructing a finite grid $\mathcal{X} \equiv \{x_1, x_2, \ldots x_N\}$ where N is the size of the grid. Set the lowest possible asset holdings to the borrowing limit plus the lowest income realization, i.e. $x_1 = R\underline{a} + y_{\min}$. Discretize the savings space by constructing a finite grid $\mathcal{S} \equiv \{s_1, s_2, \ldots s_N\}$. Set the lowest possible asset holdings to the borrowing limit, i.e. $s_1 = \underline{a}$.

2. Guess an initial policy function for consumption $c_0\,(x_i)$ on the grid $\mathcal{X}$. A reasonable first guess might be

$$c_0\,(x_i) = r x_i$$

   which assumes that the household consumes the annuity value of its cash-on-hand every period. Set the iteration counter, $l = 0$.

3. Compute expected marginal utility of consumption $\text{EMUC}\,(s_i)$ on the grid for savings $\mathcal{S}$

$$\text{EMUC}_l\,(s_i) = \sum_{j'=1}^{J} u'\left(c_l\left((1+r)\,s_i + y_j\right)\right) p_{j'}$$

   This steps requires interpolating either $c\,(x)$ or alternatively $\text{MUC}\,(x) \equiv u'\,(c\,(x))$.

4. Use the Euler equation at equality to get the marginal utility of consumption today on the grid for savings

$$\text{MUC}_l\,(s_i) = \beta\,(1+r)\,\text{EMUC}_l\,(s_i)$$

   Then invert the utility function and use the budget constraint to get current consumption and current cash on hands as a function of savings

$$
\begin{aligned}
c_l(s_i) &= u'^{-1}(\mathrm{MUC}_l(s_i)) \\
x_l(s_i) &= s_i + c_l(s_i)
\end{aligned}
$$

5. Invert the function $x_l(s_i)$ by interpolating on the grid $\mathcal{X}$, taking care to deal with the borrow straight just as in the previous case. This gives a function $s_{l+1}(x_i)$, from which we can use the budget constraint to get $c_{l+1}(x_i) = x_i + s_{l+1}(x_i)$.

EGP with endogenous labor supply In models with endogenous labor supply, there is a an additional first order condition. Given marginal utility of consumption $\mathrm{MUC}_l(a'_i, y_j)$, we need to solve for the optimal hours function $h_l(a'_i, y_j)$, in addition to the optimal consumption function $c_l(a'_i, y_j)$ and current assets $a_l(a'_i, y_j)$. This requires solving the following system of three equations in the three unknowns (c_l, h_l, a_l) for each value of (a'_i, y_j)

$$
\begin{aligned}
u_c(c_l, h_l) &= \mathrm{MUC}_l(a'_i, y_j) \\
-u_h(c_l, h_l) &= u_c(c_l, h_l)\, y_j \\
c_l + a'_i &= (1+r)\, a_l(a'_i, y_j) + h_l y_j
\end{aligned}
$$

For some utility functions these three equations can be solved in closed form. For example, with separable preferences $u(c,h) = v(c) - v(h)$, the solution has a simple recursive structure.

$$
\begin{aligned}
c_l &= v'^{-1}\left[\mathrm{MUC}_l(a'_i, y_j)\right] \\
h_l &= v'^{-1}\left[v'(c_l)\, y_j\right] \\
a_i &= \frac{c_l + a'_i - h_l y_j}{1+r}
\end{aligned}
$$

As always, we must take care to appropriately deal with the borrowing constraint and any constraints on the choice of hours.

1.5.3 Stationary distribution

Stationary distribution through simulation Once we have solved the Bellman equation to get the policy functions, we can use the policy functions to compute the stationary distribution that is associated with the solution to the Bellman equation. The easiest method is to simulate.

1. Set the seed of the random number generator so that you can reproduce your results.

2. Initialize arrays to hold consumption c_{it} and assets a_{it} for a large number I of agents for time periods $t = 1 \dots T$.

3. Loop over all individuals and for each one, draw a random draw of income y_{i0} from the stationary distribution associated with the Markov chain for y_{it}. Initialize each individual with $a_{i0} = 0$.

4. Loop over time periods. For each time period, use the policy function $c(a, y)$ to compute consumption for each agent. If a_{it} is not on the grid, use linear (or some other form of) interpolation. Then use the budget constraint to get next period asset holdings a_{it+1}. Note: If you have used value function iteration, it makes more sense to use the policy function for assets directly. In this case, you can get consumption from the budget constraint.

5. Compute mean asset holdings in the population as

$$A_t = \frac{1}{I} \sum_{i=1}^{I} a_{it}$$

   Continue until A_t converges.

For reasons that we will explain in a later section, for mean asset holdings A_t to converge and not explode to infinity, it is crucial that we chose the discount rate and interest rate so that $\beta(1 + r) < 1$.

1.5.4 Lifecycle models

Backward induction rather than iteration Lifecycle models with overlapping generations are even easier to solve than infinite horizon models since we can solve backward from a terminal condition rather than iterating to convergence. If there is a retirement period, then we first solve for optimal decisions in retirement and then for optimal decisions in the working life. Any of the three methods can be used.

Example lifecycle problem For example, consider the following model where individuals work until age T^w and then live to a maximum age of $T = T^w + T^r$. During the T^r years of retirement individuals survive with probabilities δ_t, with $\delta_T = 0$ so that all individuals are dead at the end of age T. While retired, individuals receive a pension benefit $b\left(y_{T^w}\right)$ that is a function of their income in the last period of the working phase. While working, individual i receive labor income given by $e^{\kappa_t} y_{it}$ where κ_t is an exogenous deterministic age component of earnings that is common to all individuals and y_{it} is a idiosyncratic stochastic component of earnings.

The problem in the last period of life, $t = T$, is:

$$V_T\left(a_T, b\right) = u\left(b + \left(1+r\right)a\right)$$

The problem during the retirement phase, $t = T^w + 1, \ldots, T-1$, is

$$\begin{gathered} V_t\left(a_t, b\right) = \max_{c_t, a_{t+1}} u\left(c\right) + \beta\delta_t V_{t+1}\left(a_{t+1}, b\right) \\ \text{subject to} \\ c_t + a_{t+1} \leq b + \left(1+r\right)a_t \end{gathered}$$

The problem in the final year of the working phase, $t = T^w$, is

$$\begin{gathered} V_t\left(a_t, y_t\right) = \max_{c_t, a_{t+1}} u\left(c_t\right) + \beta V_{T^w+1}\left(a_{t+1}, b\left(y_t\right)\right) \\ \text{subject to} \\ c_t + a_{t+1} \leq e^{\kappa_t} y_t + \left(1+r\right)a_t \end{gathered}$$

The problem during the remaining years of the working phase, $t = 1, \ldots, T^w - 1$ is

$$\begin{gathered} V_t\left(a_t, y_t\right) = \max_{c_t, a_{t+1}} u\left(c_t\right) + \beta E_t\left[V_{t+1}\left(a_{t+1}, y_{t+1}\right)|y_t\right] \\ \text{subject to} \\ c_t + a_{t+1} \leq e^{\kappa_t} y_t + \left(1+r\right)a_t \end{gathered}$$

Stationary distribution in lifecycle models To compute the stationary distribution, we can just simulate one large cohort and then re-weight average quantities at each age by the weight of that age group in the ergodic distribution that is implied by the assumed survival probabilities. For the above example, the relative size of age groups $t = 1, \ldots, T^w$ are each equal to $s_t = 1$, while the relative size of age groups $t = T^w + 1, \ldots, T$ are each given by s_t where

$$s_t = \prod_{\tau=T^w+1}^{t} \delta_\tau.$$

Mean consumption in the population can then be calculated as

$$E[c] = \frac{\sum_{t=1}^{T} s_t E[c|t]}{\sum_{t=1}^{T} s_t}$$

Other aggregate quantities can be computed similarly.

1.5.5 Choosing the grid for assets

Consumption and savings policy functions are typically very non-linear close to the borrowing constraint, and very close to linear away from the constraint. Since we use linear interpolation to evaluate policy functions off the grid, we can obtain more accurate solutions by putting more points close to the borrowing constraint and fewer at high asset levels.

A simple way to do this is to use a power-spaced grid. The goal is to construct a grid a on $[\underline{a}, \overline{a}]$ with more grid points closer to $\underline{a}$. Let z be an equi-spaced grid on $[0, 1]$. Define $x = z^\alpha$ for $\alpha \in (0, 1]$. Note that when $\alpha = 1$ the grid is unchanged, but as $\alpha \to 0$, the grid for x puts more and more points closer to 0. Thus x is a non-linear spaced grid on $[0, 1]$ with more points closer to 0. We then create a by re-scaling x as

$$a = \underline{a} + z(\overline{a} - \underline{a}).$$

1.5.6 Income process discretization

Review on how to discretize a distribution Assume that $x \sim F$ with domain $(-\infty, \infty)$. Consider approximating the distribution F with a discrete distribution that takes values $\{x_1, \ldots, x_n\}$. The goal is simply to assign probabilities p_i to each discrete value

x_i, such that the discrete distribution provides a good approximation to the underlying continuous distribution. When the points x_i are given, the simplest way to do this is to set

$$p_i = \begin{cases} F\left(\left(x_1 + \frac{1}{2}\left(x_2 - x_1\right)\right)\right) & \text{for } i = 1 \\ F\left(x_i + \frac{1}{2}\left(x_{i+1} - x_i\right)\right) - F\left(x_i - \frac{1}{2}\left(x_i - x_{i-1}\right)\right) & \text{for } i = 2, \ldots n-1 \\ 1 - F\left(x_n - \frac{1}{2}\left(x_n - x_{n-1}\right)\right) & \text{for } i = n \end{cases}$$

This will ensure that this is a valid probability distribution, i.e.

$$\sum_{i=1}^{n} p_i = 1, \;\; p_i > 0$$

but it will not necessarily ensure that the mean of the discrete distribution is the same as the mean of the continuos distribution. So often you will then want to re-scale to achieve this. You can do this by using probabilities q_i that satisfy

$$q_i = p_i \frac{E\left[X\right]}{\sum_{i=1}^{n} p_i x_i}$$

provided that none of the resulting $q_i > 1$. If it turns out that this is the case, then the x_i grid was not a good one to use, and you would be better off choosing it differently. This leads us to the next section.

Simple method to match moments A simple way to match the mean and the variance of a distribution is to set the grid as equally spaced on $[\hat{x} - \kappa\sigma, \hat{x} + \kappa\sigma]$. For a symmetric F you can set $\hat{x} = \mu = E\left[X\right]$, and this will guarantee that the the mean of the discrete distribution is μ. If the standard deviation is σ, you can write a little optimization routine to choose the value of κ so that the standard deviation of the discrete distribution equals σ. We can use a variant of this idea to come up with a simple way to discretize an $AR\left(1\right)$ or any other continuous Markov process.

Rouwenhurst method This description follows Kopecky and Suen (2009). Let $y_t = \rho y_{t-1} + \varepsilon_t$

1. When $N = 2$ define Θ_2 as

$$\Theta_2 = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$$

2. For $N > 2$ construct the $N \times N$ matrix of

$$p \begin{bmatrix} \Theta_{N-1} & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix} + (1-p) \begin{bmatrix} \mathbf{0} & \Theta_{N-1} \\ 0 & \mathbf{0}' \end{bmatrix} + (1-q) \begin{bmatrix} \mathbf{0}' & 0 \\ \Theta_{N-1} & \mathbf{0} \end{bmatrix} + q \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & \Theta_{N-1} \end{bmatrix}$$

   where $\mathbf{0}$ is an $N-1 \times 1$ vector of zeros.

3. Divide all but the top and bottom rows by 2 so that the elements in each row sum to 1.

4. The grid for y is evenly spaced between $-\psi$ and ψ. One can match the unconditional mean, unconditional variance, and first-order autocorrelation by setting

$$p = q = \frac{1+\rho}{2}$$
$$\psi = \sqrt{N-1}\sigma_{\varepsilon}^{2}$$

1.6 Solving non-linear equations

1.6.1 Basics of non-linear equations

Setup We consider 1D equations of the form $f(x) = 0$ subject to $x \in [a, b]$ where f is a continuous function.

Check boundaries Before attempting to solve a non-linear equation, it is important to check the boundaries to that you know whether a solution is guaranteed to exist. For example, if

$$f(a) > 0,\ f(b) < 0, \text{ or}$$
$$f(a) < 0,\ f(b) > 0$$

then continuity of f guarantees that a solution exists. Moreover, if we know that f is monotonic, which is often the case in economic problems, then checking the boundaries will immediately tell us that either there is a unique interior solution, or that there is a corner solution at one of the boundaries.

Bracketed roots In general it is best to bracket a root before trying to find one. Some basic solution methods will generate updates that push you outside your best bracket. You should never let this happen.

Tolerances Because of floating point arithmetic, computed value of $f(x)$ will never be exactly zero. For $x \approx 1$ then finding x to within $\pm 1.0e^{-6}$ is probably okay for most purposes. But for $x \approx 1.0e^{20}$ then finding x to within $\pm 1.0e^{-6}$ is far too accurate to be expected. This suggests using a relative tolerance rather than an absolute tolerance. However, using a relative tolerance is a bad idea when $x \approx 0$! The problem is that we don't know in advance whether the solution x is very large or close to zero. So in practice, we typically specify two tolerance levels for x(one relative and one absolute) and one tolerance level for f. If either of these are satisfied then we terminate the algorithm.

Bisection Bisection is the simplest method for finding a root. It is very robust and will find a root to any desired level of accuracy. The downside is that you need to supply a bracketing guess of of the root and it may take many iterations. Intuitively, with bisection the only information about f that you use is its sign. More advanced methods also make use of the shape of f.

1.6.2 Quasi-Newton methods

Newton's method Starting with a guess x^k, Newton's method uses a first-order Taylor expansion of $f(x)$ around $x = x_k$ to find an updated guess x_{k+1}:

$$f(x) \approx f(x_k) + f'(x_k)(x - x_k) = 0$$
$$x_{k+1} = x_k - f'(x_k)^{-1} f(x_k)$$

The main problems with Newton's method are (i) it requires being able to evaluate the derivative $f'(x)$ as well as the value $f(x)$; and (ii) it is not reliable when $f'(x_k)$ is large, i.e. when the function is very steep. On the other hand, Newton has an extremely fast order of convergence of 2. Quasi-Newton methods replace $f'(x_k)$ with an easier to compute approximation to $f'(x_k)$ or with something that is better conditioned. Both Newton and quasi-Newton methods are often "polished" by taking a couple of Newton steps once the algorithm is in a neighborhood of the root.

Secant method The secant method replaces $f'(x_k)$ with

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

so that the the updating rule becomes

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$$

The secant method requires you to provide two distinct guesses as initial conditions. These guesses do not have to bracket the root but the algorithm tends to perform better when they do. In a neighborhood of the root, the secant method has a fast order of convergence of 1.618 (the golden ratio). However, since the root does not necessarily remain bracketed, for functions that are not sufficiently continuous, the algorithm is not guaranteed to converge.

Broyden's method Broyden's method is basically a multi-dimensional version of the secant method. It starts with a guess x_0 and an initialization A_0 of the Jacobian matrix at the root. Two example choices of A^0 are the numerical Jacobian at x^0 and a re-scaled identity matrix. The Taylor expansion then gives:

$$\begin{aligned} f(x) &\approx f(x_k) + A_k (x - x_k) = 0 \\ x_{k+1} &= x_k - [A_k]^{-1} f(x_k) \end{aligned}$$

We update A_k using the relationship

$$f(x_{k+1}) - f(x_k) = A_{k+1}(x_{k+1} - x_k)$$

Note that A_k need not (and does not) converge to the Jacobian at the root.

False position method This is the same as the secant method except you use only points that bracket (straddle) the root. In other words, whereas the secant method uses the two most recent function values to compute an approximation to the derivative of f, false position uses the two most recent guesses with opposite signs. False position is more robust than the secant method but can be slightly slower. In practice, people typically use hybrid algorithms that have elements of both methods.

Backstepping The idea behind backstepping is that taking a full Newton step may not improve the function, but perhaps a smaller step will. This is the basic idea behind line-search methods. Let the norm of the function be defined as

$$\|f(x)\| = \frac{1}{2} f(x)' f(x)$$

or in the case of a 1D function

$$\|f(x)\| = \frac{1}{2} f(x)^2$$

Then compute the full Newton (or quasi-Newton) update as

$$\tilde{x}_{k+1} = x_k - f'(x_k)^{-1} f(x_k)$$

If $\|f(\tilde{x}_{k+1})\| < \|f(x_k)\|$ then set $x_{k+1} = \tilde{x}_{k+1}$. If not then set

$$x_{k+1} = x_k + \delta(\tilde{x}_{k+1} - x_k)$$

where $\delta \in (0, 1)$.

1.6.3 Hybrid methods

In general, these methods are better than either the secant or false position method.

Ridder's method Given a bracket $[x_1, x_2]$, let $x_3 = \frac{1}{2}(x_1 + x_2)$. This gives three points (x_1, x_2, x_3). The idea is to fit a curve through these three points:

$$p(x) = a + be^{cx}$$

and then set x_4 such that $p(x_4) = 0$. It can be shown that convergence is quadratic. In practice this method is also very robust. It is competitive with Brent's method (see below), but is much simpler.

Brent's method Brent's method combines bracketing, bisection and inverse quadratic interpolation. The key trick is to keep track of whether convergence is super-linear. If not, it sticks in a bisection step since we know that bisection converges linearly. This is the method of choices inside many commercial routines, including fzero in Matlab.