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The method of endogenous gridpoints with occasionally binding constraints among endogenous variables

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ABSTRACT

We show how the method of endogenous gridpoints can be extended to solve models with occasionally binding constraints among endogenous variables very efficiently. We present the method for a consumer problem with occasionally binding collateral constraints and non-separable utility in durable and non-durable consumption. This problem allows for a joint analysis of durable and non-durable consumption in models with uninsurable income risk which is important to understand patterns of consumption, saving and collateralized debt. We illustrate the algorithm and its efficiency by calibrating the model to US data.

1. Introduction

Many interesting problems in economics have to be analyzed with dynamic and stochastic models of equilibrium. Unless one is willing to make very restrictive assumptions about the underlying environment, equilibria in these models need to be approximated numerically. Maybe the most popular method for finding such approximations of solutions to dynamic stochastic optimization problems is value function iteration. This standard approach, however, is known to be particularly prone to a "curse of dimensionality" in problems whose recursive formulation needs to include multiple state variables. Therefore, the analysis of many interesting research questions has been hampered by the large amounts of required computing time.

In this paper we develop a much less time-consuming and very accurate solution algorithm, based on the endogenous gridpoints method (EGM) of Carroll (2006), to solve problems with occasionally binding constraints among multiple endogenous variables. This type of problem is frequently encountered in economics and has received much interest recently in the context of consumer models with durables and occasionally binding collateral constraints. We thus illustrate our method for such a model.

The EGM has been previously extended by Barillas and Fernández-Villaverde (2007) to problems with more than one *control* variable. There are two important differences of our paper compared with Barillas and Fernández-Villaverde (2007). Firstly, in our problem we have both an additional control variable and an additional *state* variable. Secondly, our algorithm

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handles constraints among endogenous variables which are occasionally binding. This is achieved much more simply than in the parameterized expectations approach described in Christiano and Fisher (2000) since we can fully characterize the policy functions and the multipliers on the constraints in terms of the next-period combinations of state variables that we condition on.

As a result we compute precise solutions with average Euler equation errors below 10^{-4} very efficiently: our algorithm needs one second per (period of time) iteration using Matlab on one of the 2.4 GHz dual-core processors for a PC with 2 GB RAM. Solving the problem with the same accuracy using standard value function iteration would be prohibitively slow.

The computational advantage of conditioning on the grid of *future* rather than *current* state variables (or post-decision rather than pre-decision state variables) has been explored in the engineering literature (Powell, 2007). There it has been used to avoid numerical complications in the computation of expectations in high-dimensional dynamic programming problems. In our algorithm we exploit at least two additional advantages. Firstly, occasionally binding constraints are dealt with very efficiently, since constraints among variables are naturally expressed in terms of future endogenous states. Thus, our algorithm derives the binding patterns of the constraints and the corresponding multipliers as functions of future endogenous states. Secondly, the economic structure of a problem may allow for a very efficient mapping between future endogenous states.

Importantly, our proposed algorithm can provide substantial efficiency gains even if the model of interest in a certain application can only be solved by value function iteration. The EGM may be applicable to a suitably defined auxiliary model, which allows to efficiently compute an accurate initial guess for the value function, reducing the number of iterations necessary for convergence. For example, such an auxiliary model could hold some dimensions of the choice problem constant while exploiting the advantages of the EGM for a subset of choice variables.

We illustrate our method for a consumer problem with durables and occasionally binding collateral constraints. This is an important problem that has received substantial attention in recent research since a joint analysis of durables and non-durables helps to understand patterns of consumption, saving and collateralized household debt. Introducing durables into classic models with uninsurable income risk (Aiyagari, 1994; Carroll, 1997; Deaton, 1991) has allowed to quantify the importance of durables for the precautionary savings motive and wealth inequality (Gruber and Martin, 2003; Díaz and Luengo-Prado, 2010) or to highlight the importance of durables for the hump of non-durable consumption over the life-cycle (Fernández-Villaverde and Krueger, forthcoming). Since durables serve as collateral in these models and most of consumer debt is collateralized in reality, the models' predictions also help to interpret observed patterns of consumer borrowing.

The numerical solution of models with durables and collateralized debt has been considered to be particularly challenging and computationally expensive for two reasons: First, durables enter as an additional state variable in the value function along with total wealth and persistent stochastic labor income (at least if non-separable utility from durable and non-durable consumption or adjustment costs are allowed for). Second, durables increase the dimensionality of portfolio choice beyond a simple savings decision. Hence, there is a substantial payoff for a method which allows to solve these models efficiently.

The rest of this paper is structured as follows. In Section 2 we present the consumer problem. We then discuss our algorithm for a rather general model structure before explaining details of the implementation in the framework of the consumer problem. In Section 3 we illustrate our method by calibrating the model to US data. We conclude in Section 4.

2. The EGM with collateral constraints and non-separable utility

In this section we first present the consumer problem before we discuss the algorithm for its efficient solution. The basic idea of the algorithm is to exploit the structure of the first-order conditions to avoid time-consuming and potentially unstable standard procedures such as root-finding or numeric optimization.

2.1. The model

Risk averse consumers with a (finite or infinite) time horizon T derive utility from a durable good d and a non-durable good c where the instantaneous utility $U(c,d)$ is assumed to be strictly concave, and is allowed to be non-separable in c and d.

Consumers are exposed to idiosyncratic labor-income shocks and the timing of these shocks is as follows. After the draw of uncertain exogenous labor income y_t, agents choose consumption c_t and the endogenous assets in the next period, i.e., the holdings of the durable good d_{t+1} and the financial risk-free asset a_{t+1}. They then derive utility from consumption before the returns of the assets accrue. Financial assets earn interest r and durables depreciate at rate δ. Markets are incomplete since the assets do not allow consumers to fully diversify their risk. As a consequence, consumers are heterogeneous ex post although they are identical ex ante: different histories of shocks imply different portfolios of the endogenous state variables.

Durables generate utility but the durability of these goods also allows consumers to use them as collateral against which they can borrow. Since most household debt in the US is mortgage debt or other credit that is secured by collateral (about 85% in the Survey of Consumer Finances 2004), we make the simplifying assumption that all credit needs to be collateralized. Denote $\underline{y}$ as the minimum labor income realization and $\mu \in [0,1)$ and $\gamma \in [0,1)$ as the collateralizable fractions of the durable stock and of minimum labor income, respectively. The collateral constraint takes the following form:

$$\underbrace{\mu(1-\delta)d_{t+1}+\gamma\underline{y}}_{\text{collateral}} \geq -(1+r)a_{t+1}.$$

Given the timing assumptions made above, this constraint guarantees full repayment by consumers. The assumption is that the lender, who lends at the risk-free rate, knows the financial portfolio choice (a_{t+1},d_{t+1}) and the minimum of the support of the income distribution $\underline{y}$ but he does not observe future individual income draws y_{t+1}. Note that whether and by what margin the collateral constraint binds in $t+1$ is entirely determined by the choices in period t.

Defining the available total wealth as

$$x_t \equiv (1+r)a_t+(1-\delta)d_t, \tag{1}$$

it is useful to rewrite the collateral constraint in terms of x_{t+1} and d_{t+1}:

$$x_{t+1} \geq -\gamma\underline{y}+(1-\mu)(1-\delta)d_{t+1}. \tag{2}$$

Written this way, we see that total wealth needs to be larger than the negative value of $\gamma\underline{y}$ and the fraction of durable wealth which cannot be used as collateral. If the constraint binds, d_{t+1} is determined for a given x_{t+1}, under our assumption that $\mu<1$.

We also consider the case in which durables can only be adjusted at a cost Ψ where

$$\Psi(d_{t+1},d_t)=\frac{\alpha}{2}\left(\frac{d_{t+1}-(1-\delta)d_t}{d_t}\right)^2 d_t.$$

This specification of quadratic adjustment costs is borrowed from the investment literature (see, for example, Adda and Cooper, 2003, p. 192) where the costs are differentiable in d_{t+1} and d_t and consumers can avoid adjustment costs if they let the durable stock depreciate.[1] The definition of total wealth above then implies that we can write the budget constraint as

$$a_{t+1}+d_{t+1}+c_t+\Psi(d_{t+1},d_t)=x_t+y_t. \tag{3}$$

2.1.1. The recursive formulation of the household problem

The recursive formulation of the household problem is

$$v_t(x_t,d_t,y_t)=\max_{a_{t+1},d_{t+1}}\left[U(\underbrace{x_t+y_t-a_{t+1}-d_{t+1}-\Psi(d_{t+1},d_t)}_{c_t},d_t)+\hat{v}_t(x_{t+1},d_{t+1},y_t)\right] \tag{4}$$

subject to the constraints

$$a_{t+1}+d_{t+1}+c_t+\Psi(d_{t+1},d_t)=x_t+y_t,$$

$$x_{t+1}=(1+r)a_{t+1}+(1-\delta)d_{t+1},$$

$$x_{t+1}\geq -\gamma\underline{y}+(1-\mu)(1-\delta)d_{t+1},$$

$$d_{t+1}\geq d_{\min},$$

where y_t enters as a state variable due to its persistence and the expected value of next period's value function, multiplied by the discount factor β, is denoted by

$$\hat{v}_t(x_{t+1},d_{t+1},y_t)\equiv \beta E_t v_{t+1}(x_{t+1},d_{t+1},y_{t+1}).$$

The last constraint restricts the portfolio position in durables where $d_{\min}$=0 in the model specification without adjustment costs and $d_{\min}>0$ in the specification with adjustment costs so that the adjustment cost function introduced above is well defined.

In the following we drop time indices on variables, on functions, and on the expectations operator, if these are at date t. Furthermore, we let primes "′" denote a one-period lead on variables.

2.1.2. Further assumptions and useful analytic properties

We now summarize further assumptions and useful analytic properties that apply in the model and which turn out beneficial to rely on in the construction of the algorithm. We consider the class of preferences which are characterized by

$$U(c,d)=\frac{\psi(c,d)^{1-\sigma}-1}{1-\sigma} \quad \text{and} \quad \psi(c,d)=c^{\theta}(d+\varepsilon_d)^{1-\theta},$$

[1] For simplicity, adjustment costs do not reduce the amount of collateral in (2). This is the case if adjustment costs do not affect sales of collateral seized by lenders. We show in Appendix A how this assumption may be relaxed. We prefer the simpler case in the main text for presentation purposes.

where $\varepsilon_d \geq 0$ can be interpreted as autonomous durable consumption. For $\varepsilon_d > 0$ marginal utility for durables is finite at $d=0$. Introducing the parameter ε_d adds to the flexibility of the class of preferences by allowing for solutions where agents choose not to hold durables for certain combinations of the state variables. This flexibility also helps to apply the framework of the model to situations that differ in the empirical interpretation of the durable good in terms of its necessity. Allowing for the potentially optimal choice $d' = d_{\min}$ means that the constraint for minimal durable holdings needs to be taken into account in the recursive problem as another occasionally binding constraint in addition to the collateral constraint.

Although our approach may be valid for a more general class of preferences, we focus on the parametric case introduced above since: (i) it encompasses many of the previous numerical applications which we are aware of, (ii) the Cobb–Douglas specification for the consumption index is roughly in line with empirical estimates on the substitutability between durables and non-durables (see Fernández-Villaverde and Krueger, forthcoming, for further discussion and references) and (iii) the Cobb–Douglas specification allows to invert the financial-asset Euler equation to retrieve c as a function of future choices in a straightforward manner.

Moreover, we use in the algorithm that the value function at iteration step n is differentiable in our problem with inequality constraints (Rendahl, 2007, Proposition 1). Furthermore, the value function is strictly concave at each iteration step (Stokey and Lucas, 1989, Chapter 9).

Finally, time iteration on the policy function converges to the true policy function (Rendahl, 2007, Proposition 2). At each iteration for given state variables, the unique maximizers are continuous policy functions (Stokey and Lucas, 1989, Chapter 9).

2.2. The algorithm

We pursue a numerical approach that relies on a substantial extension of the endogenous gridpoints method (EGM) which has been proposed by Carroll (2006) for a much simpler problem. Extending the EGM allows for an efficient and accurate solution of our model. Carroll's method avoids root-finding and forward maximization to solve for the optimal endogenous state next period x' given this period's state x. By specifying an exogenous grid for the state variable x' in the *next period* instead, the first-order conditions are used to determine the endogenous grid of the state variable x in this period implied by the optimal choices.

The challenge of extending the EGM to the problem in our paper is twofold. Firstly, we have to handle the mapping from *multiple* next-period state variables (x',d') to multiple current-period states. Secondly, we need to consider this mapping from tomorrow's endogenous states to today's states in a setting with constraints among endogenous choice variables. These constraints may be occasionally binding. The main contribution of our algorithm is thus to handle these challenges. Importantly, the algorithm is derived from a recursive formulation such that it can be applied to both life-cycle and infinite-horizon problems. We now proceed by discussing our algorithm for a rather general setting before applying it to our specific case of interest.

2.2.1. The general framework

Suppose the optimal solution of a general choice problem in its recursive formulation is characterized by a set of first-order conditions with equality constraints and occasionally binding constraints. Given the vector of endogenous state variables $\mathbf{s}$ and the vector of exogenous state variables $\mathbf{y}$ we thus have a system of equations and inequalities:

$$\mathcal{F}(\mathbf{s}',\mathbf{c},\lambda;\mathbf{s},\mathbf{y}) = \mathbf{0},$$

$$\mathcal{E}(\mathbf{s}',\mathbf{c};\mathbf{s},\mathbf{y}) = \mathbf{0},$$

$$\mathcal{O}(\mathbf{s}',\mathbf{c};\mathbf{s},\mathbf{y}) \geq \mathbf{0},$$

$$\lambda(\mathbf{s}',\mathbf{c};\mathbf{s},\mathbf{y}) \geq \mathbf{0},$$

$$\lambda(\mathbf{s}',\mathbf{c};\mathbf{s},\mathbf{y})\mathcal{O}(\mathbf{s}',\mathbf{c};\mathbf{s},\mathbf{y}) = \mathbf{0}$$

in the unknowns $\mathbf{s}'$, the vector of future endogenous states, $\mathbf{c}$, the vector of controls, and $\boldsymbol{\lambda}$, the vector of multipliers on the occasionally binding constraints.

$\mathcal{F}(\cdot) = \mathbf{0}$ collects the first-order derivative conditions, $\mathcal{E}(\cdot) = \mathbf{0}$ contains the equality constraints, and $\mathcal{O}(\cdot) \geq \mathbf{0}$ gathers the occasionally binding inequality constraints. The system also includes the complementary slackness conditions $\lambda(\cdot)\mathcal{O}(\cdot) = \mathbf{0}$.

The algorithm uses the system above to compute policy functions, and it proceeds in two steps, which we label "Step 1" and "Step 2" in the following. In Step 1 of the algorithm, we condition on a subset of the future endogenous state variables $\mathbf{s}'$, a subset of the current endogenous state variables $\mathbf{s}$ and all the exogenous state variables $\mathbf{y}$. We then exploit the first-order derivative conditions $\mathcal{F}(\cdot) = \mathbf{0}$ to determine the combinations of future endogenous state variables that can be attained through an optimal choice. Since the occasionally binding constraints are naturally expressed in terms of future endogenous state variables, we simultaneously find out where these constraints are binding. We are then able to compute those multipliers that only depend on the future endogenous state variables and the subset of current state variables which we conditioned on.

In Step 2 of the algorithm, we use the combinations of future endogenous state variables and the multipliers, which we computed in Step 1, and the first-order derivative conditions $\mathcal{F}(\cdot)=\mathbf{0}$ to determine the remaining controls $\mathbf{c}$. In doing so, we condition on a subset of the current endogenous state variables $\mathbf{s}$ and all the exogenous state variables $\mathbf{y}$. Finally, equality constraints from $\mathcal{E}(\cdot)=\mathbf{0}$ let us determine those endogenous current state variables that we did *not* condition on in the previous stages of the algorithm.

Given this general structure of the problem, it becomes evident that the efficiency gains of the endogenous gridpoint method decrease in the number of states that need to be conditioned on in Steps 1 and 2. This is particularly relevant in Step 1 since the characterization of the optimal combinations of future endogenous state variables will involve derivatives of the value function. In typical applications such characterization cannot be done in closed form. In contrast Step 2, which inverts first-order derivative conditions in closed form, copes well with a larger state space.

We now impose the structure of our application on this framework in order to elaborate in-depth on the explanation of the steps of the algorithm presented above. In terms of our model the key components of the general choice problem described above are the equality and inequality constraints of the recursive optimization problem in (4), and the first-order derivative conditions (5) and (6) discussed in the following.

2.2.2. The first-order conditions for our application

Assigning the multiplier κ to the collateral constraint and the multiplier η to the constraint on durables, the two first-order conditions for a' and d' in problem (4) are

$$-\kappa(1+r)+\frac{\partial U(c,d)}{\partial c}=(1+r)\frac{\partial \hat{v}}{\partial x'} \tag{5}$$

and

$$-\eta-\kappa\mu(1-\delta)+\frac{\partial U(c,d)}{\partial c}\left(1+\frac{\partial \Psi(d',d)}{\partial d'}\right)=(1-\delta)\frac{\partial \hat{v}}{\partial x'}+\frac{\partial \hat{v}}{\partial d'}, \tag{6}$$

where for our parametric specification of adjustment costs

$$\frac{\partial \Psi(d',d)}{\partial d'}=\alpha\left(\frac{d'}{d}-(1-\delta)\right).$$

For later reference note that the envelope conditions are

$$\frac{\partial v}{\partial x}=\frac{\partial U(c,d)}{\partial c} \quad \text{and} \quad \frac{\partial v}{\partial d}=\frac{\partial U(c,d)}{\partial d}-\frac{\partial U(c,d)}{\partial c}\frac{\partial \Psi(d',d)}{\partial d}, \tag{7}$$

where

$$\frac{\partial \Psi(d',d)}{\partial d}=\frac{\alpha}{2}\left((1-\delta)^2-\left(\frac{d'}{d}\right)^2\right).$$

Using the first optimality condition (5) to substitute for $\partial U(c,d)/\partial c$ in the second condition (6), we obtain

$$\kappa\left((1+r)\left(1+\frac{\partial \Psi(d',d)}{\partial d'}\right)-\mu(1-\delta)\right)-\eta=-\left(\frac{\partial \Psi(d',d)}{\partial d'}(1+r)+r+\delta\right)\frac{\partial \hat{v}}{\partial x'}+\frac{\partial \hat{v}}{\partial d'}. \tag{8}$$

Eq. (8) is one of the key equations for our solution method as it characterizes the implicit relationship between the two endogenous state variables in the next period x' and d'. Notice that Eq. (8) also relates the multipliers κ and η to the endogenous state variables in the next period.

Using that

$$\frac{\partial U(c,d)}{\partial c}=\frac{\partial \psi(c,d)}{\partial c}\psi(c,d)^{-\sigma}=\theta c^{\theta-1}(d+\varepsilon_d)^{1-\theta}[c^{\theta}(d+\varepsilon_d)^{1-\theta}]^{-\sigma},$$

we derive the other key equation, solving the first optimality condition (5) for c:

$$c=\left[(1+r)\left(\frac{\partial \hat{v}(x',d',y)}{\partial x'}+\kappa\right)\frac{(d+\varepsilon_d)^{(\theta-1)(1-\sigma)}}{\theta}\right]^{1/(\theta(1-\sigma)-1)}. \tag{9}$$

Conditional on one of the endogenous state variables in the current period, d, Eq. (9) relates the endogenous state variables in the next period, d' and x', to the current choice c. The presence of the occasionally binding collateral constraint, which is captured by the multiplier κ in Eq. (9), does not destroy the structure of the relationship. As mentioned above, κ can be characterized in terms of next-period states d' and x'.

Given the next-period states x' and d', the current choice c and conditional on the current state d, we describe in more detail below how Eqs. (1) and (3) allow us to determine the remaining endogenous state x in the current period.

2.2.3. The extension of the EGM in our application

For the numerical implementation of the EGM, we start with an exogenous grid for the future state x', $G_{x'} \equiv \{x'_1, x'_2, \ldots, x'_I\}$, the current state d, $G_d \equiv \{d_1, d_2, \ldots, d_J\}$, and use Tauchen's (1986) method to discretize the stochastic process for y on the grid $G_y \equiv \{y_1, y_2, \ldots, y_K\}$.

Step 1: *Mapping x' into d'*. The first step of the algorithm uses the optimal relationship in Eq. (8)

$$\kappa_{ijk}\left((1+r)\left(1+\frac{\partial\Psi(d'_{ijk},d_j)}{\partial d'}\right)-\mu(1-\delta)\right)-\eta_{ijk}=-\left(\frac{\partial\Psi(d'_{ijk},d_j)}{\partial d'}(1+r)+r+\delta\right)\frac{\partial\hat{v}(x'_i,d'_{ijk},y_k)}{\partial x'}+\frac{\partial\hat{v}(x'_i,d'_{ijk},y_k)}{\partial d'} \tag{10}$$

to determine the values of d'_{ijk} which correspond to each of the exogenous gridpoints x'_i for a given y_k and current durable stock d_j. Conditioning on d_j would not be necessary without adjustment costs, further improving the speed of our algorithm. The partial derivatives $\partial\hat{v}(x'_i,d'_{ijk},y_k)/\partial x'$ and $\partial\hat{v}(x'_i,d'_{ijk},y_k)/\partial d'$ are obtained by exploiting the envelope conditions (7). With adjustment costs this implies a term $\partial\Psi(d'',d')/\partial d'$ on the right-hand side of Eq. (8) which we handle in the second iteration (the "second-to-last" period) by using the terminal condition $d''=0$ (we then update d'' in each further iteration by the previously computed optimal policy).[2]

The collateral constraint and the lower bound on positions of d' imply that an optimal solution d'_{ijk} lies in the interval $[d_{\min}, \overline{d'_i}]$, where

$$\overline{d'_i} \equiv \frac{x'_i+\gamma\underline{y}}{(1-\mu)(1-\delta)}. \tag{11}$$

In order to compute the optimal d'_{ijk} we need to distinguish three possible cases for each income state y_k and current durable stock d_j. In the first case neither the collateral constraint nor the constraint on durables are binding so that $\kappa_{ijk}=\eta_{ijk}=0$. In this case there exists a d'_{ijk} in the feasible interval for which the right-hand side of Eq. (10) intersects zero. If there is no such intersection, the right-hand side of Eq. (10) is either positive or negative over the entire feasible interval of d', so that two more possible cases need to be considered.

In one case the right-hand side is positive, such that Eq. (10) can only be satisfied for $\kappa_{ijk}>0$ and $d'_{ijk}=\overline{d'_i}$ where

$$\kappa_{ijk}=\frac{\dfrac{\partial\hat{v}(x'_i,d'_{ijk},y_k)}{\partial d'}-\left(\dfrac{\partial\Psi(d'_{ijk},d_j)}{\partial d'}(1+r)+r+\delta\right)\dfrac{\partial\hat{v}(x'_i,d'_{ijk},y_k)}{\partial x'}}{(1+r)\left(1+\dfrac{\partial\Psi(d'_{ijk},d_j)}{\partial d'}\right)-\mu(1-\delta)},$$

and we interpolate the derivatives of $\hat{v}$ to determine their values at d'_{ijk}. The condition

$$\mu<(1+r)\left[\frac{1}{1-\delta}-\alpha\right] \tag{12}$$

ensures that in this case for all x'_i and d_j there exists a $d'_{ijk}=\overline{d'_i}$ that satisfies Eq. (10) with $\kappa_{ijk}>0$. Condition (12) is not very restrictive for plausible parameter values, as we will see below. It guarantees that the term in brackets which multiplies κ_{ijk} on the left-hand side of (10) is positive. This also implies that in the last of the three cases, in which the right-hand side of (10) is negative over the entire feasible interval, Eq. (10) can only be satisfied for $\eta_{ijk}>0$ and $d'_{ijk}=d_{\min}$. Specifying an exogenous grid for x' rather than x thus allows immediate characterization of the multipliers of the occasionally binding constraints which is a major advantage compared with standard methods.

Fig. 1 illustrates the solution $d'_{ik}(x'_i,y_k)$ without adjustment costs ($\alpha=0$) for each of the five income states and for the parameter values which we consider in our application below.[3] Since μ is very close to one in our calibration (i.e., most of the durable can be collateralized), the collateral constraint is very steep in the (x',d') space and this constraint is binding for small values of x'. The constraint intersects with the horizontal axis at $x'_i=-\gamma\underline{y}$ and, conditional on being collateral constrained, durable wealth in the consumer's portfolio is independent of the income state. Not surprisingly, the *share* of the consumer's portfolio in durable wealth decreases in the amount of total wealth x' if the collateral constraint is slack and the motive to accumulate durables rather than financial wealth becomes weaker. In this case, however, higher income increases the amount of durables in the consumer's portfolio since income shocks are persistent and durables are also a normal consumption good.

Step 2: *From optimal future combinations (x',d') to current choice c and state x*. The second step of the algorithm applies Eq. (9) at all gridpoints

$$c_{ijk}=\left[(1+r)\left(\frac{\partial\hat{v}(x'_i,d'_{ijk},y_k)}{\partial x'}+\kappa_{ijk}\right)\frac{(d_j+\varepsilon_d)^{(\theta-1)(1-\sigma)}}{\theta}\right]^{1/(\theta(1-\sigma)-1)} \tag{13}$$

[2] With adjustment costs, full decumulation is feasible for all x, d and y if $\mu<1-(1-\delta)\alpha/2$ which is satisfied for the chosen parameter values below.

[3] Recall that we do not need to condition on d_j in this case so that there is no index j.

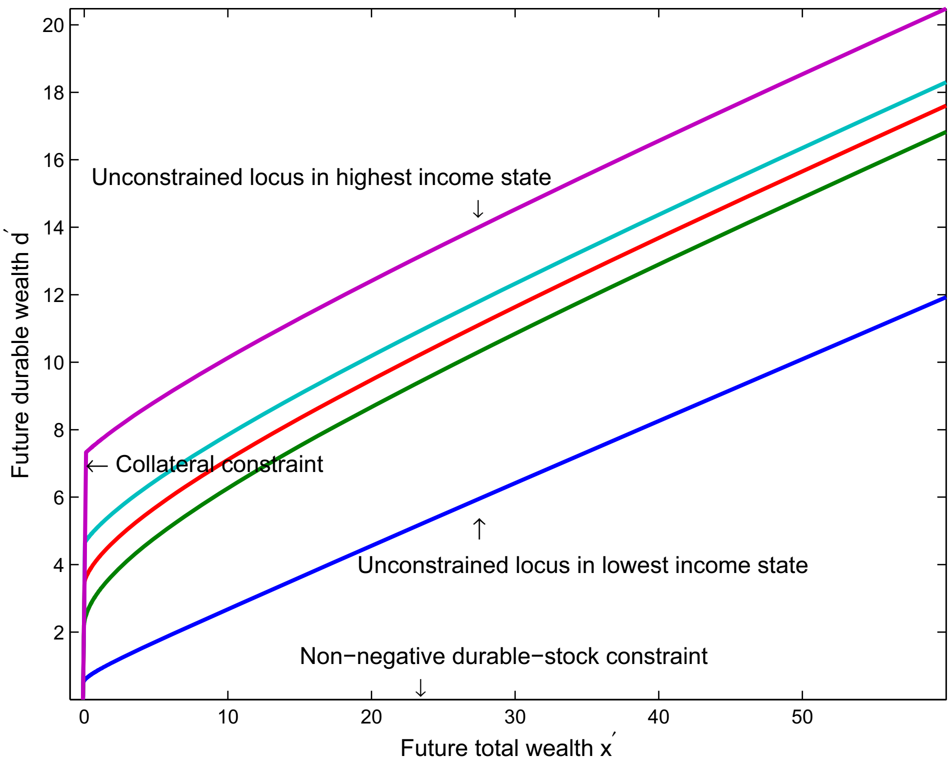

Fig. 1. The relationship between future total wealth and future durable wealth for five different income states without adjustment costs.

to determine consumption c_{ijk} where we need to condition on d_j also because of the non-separable utility function. Eq. (1) implies

$$a'_{ijk} = \frac{x'_i - (1-\delta)d'_{ijk}}{1+r},$$

so that we can use the budget constraint (3) to calculate the current state x as

$$x_{ijk} = a'_{ijk} + d'_{ijk} + c_{ijk} + \Psi(d'_{ijk}, d_j) - y_k. \tag{14}$$

These are the endogenous gridpoints in our problem. We then interpolate to retrieve the policies $c(x,d_j,y_k)$, $d'(x,d_j,y_k)$ on the exogenous grid of x for given d_j and y_k, where we use $G_{x'}$ as the exogenous grid for x. To simplify notation, we do not index this exogenous grid.

The structure of our application implies that low values of x are not attained when we apply (13) and (14) to map backwards from the combinations (x'_i, d'_{ijk}) computed in Step 1. The reason is that the optimal policies (x', d') for all these low values of x lead to the point $(-\gamma \underline{y}, d_{\min})$ where both constraints are binding. We now explain how we can determine the optimal consumption and portfolio choices in this case.

Define the minimum endogenous value of x_{ijk} for each d_j and y_k as

$$\tilde{x}_{jk} \equiv \min_i \{x_{ijk}\} \quad \text{for all } j \text{ and } k.$$

For $x < \tilde{x}_{jk}$, $x'(x,d_j,y_k) = -\gamma \underline{y}$ and $d'(x,d_j,y_k) = d_{\min}$, and the budget constraint implies

$$c(x,d_j,y_k) = x + \gamma \frac{\underline{y}}{1+r} + y_k - \Psi(d_{\min}, d_j). \tag{15}$$

Hence, the consumption propensity out of total wealth x is equal to one in this case so that $\Delta c = \Delta x$ in the range $x < \tilde{x}_{jk}$ for given d_j and y_k. Eq. (15) and the consumption policy $c(x,d_j,y_k)$, for $x \geq \tilde{x}_{jk}$ and conditional on all d_j and y_k, complete the update of the relevant policy functions $c(x,d,y)$ and $d'(x,d,y)$ that enter the next iteration. The budget constraint and definition for total wealth x' then allow us to calculate $a'(x,d,y)$ and $x'(x,d,y)$.

Further algorithmic issues. We now discuss some remaining issues concerning the algorithm. The constraint set in this problem is characterized by two inequality constraints, i.e., the collateral constraint and the constraint on the minimal holdings of durables. The smallest value of x', for which optimal combinations (x', d') exist, is determined by the intersection of the two constraints under condition (12). This value $-\gamma \underline{y} + (1-\mu)(1-\delta)d_{\min}$ is the lower bound of the grid for x'. The lower bound of the grid for d is $d_{\min}$. Our choices of the upper bounds guarantee that, for every x and d and for every realization of income y, the equilibrium policy will imply a value for x' and d' that remains within the implied interval. We choose a finer grid at the origin where the policy functions tend to have more curvature.

We initialize the choice of durables as $d'=0$ and the consumption policy function as $c(x,d_j,y_k)=x+y_k-\Psi(0,d_j)$. Therefore, we can interpret the solutions in each iteration step n as policy functions in a life-cycle problem n periods before the end of life. We obtain the stationary solution of the policy functions for the infinite horizon by iterating on Steps 1 and 2 above until convergence of the policy functions c and d'.

3. Illustration of the algorithm

We now provide an illustration of our algorithm for an infinite horizon model, with and without adjustment costs. We first briefly discuss the data and calibration of the model before discussing the efficiency of the algorithm.

3.1. The data

We use data on households with a head between 20 and 55 years of age from the Survey of Consumer Finances (SCF) 2004 to calibrate our model. The focus on this age group is motivated by our choice of an infinite horizon model in the illustration which abstracts from retirement. The SCF data has been widely used as it provides the most accurate information on consumer finances in the US. The data collectors of the Federal Reserve System pay special attention in their sampling procedures to accurately capture the right tail of the very right-skewed wealth distribution (see Kennickell, 2003, and the references therein). The data thus allow us to compute precise statistics for the consumers' wealth portfolio.

We largely follow Budría Rodríguez et al. (2002) and Díaz-Giménez et al. (1997) when computing statistics for net worth and labor earnings in the US. Net worth consists of net-financial wealth and durables where durables are defined as the sum of the value of homes, residential and non-residential property and vehicles. These are the most important durable items which can be used as collateral in real-world debt contracts.

We account for differences in household size using the equivalence scale reported in Fernández-Villaverde and Krueger (2007), Table 1, last column. To make the empirical data comparable with the data generated by the model, we normalize all variables by average net labor earnings. More precisely, we use SCF data on gross labor earnings and the NBER tax simulator described in Feenberg and Coutts (1993) to construct a measure of disposable labor earnings after taxes and transfers for each household in 2004.[4] Arguably, after-tax rather than pre-tax earnings matter for household consumption decisions since some of the uninsurable labor earnings risk may be eliminated by redistributive taxes and transfers. More detailed information is contained in Appendix B.

3.2. Calibration

For our numerical application we use a triple-exponential grid for x' and d with I=225 gridpoints for x' and J=100 gridpoints for d, and we use K=5 gridpoints for y. We set risk aversion $\sigma=2$, a commonly used value in the literature. We calibrate the remaining preference parameters θ and β to precisely match (up to precision 10^{-2}) the amount of total wealth in the data, which is 6.33 in terms average population labor earnings (after taxes and transfers), and the part of wealth accounted for by durables, 4.45.[5] The parameters are recalibrated for each model version we consider so that these targets are always matched in the calibrations below.

The annual real interest rate is set to 3% and the depreciation rate is 2% (see, for example, Caporale and Grier, 2000; Lustig and van Nieuwerburgh, 2005). The low depreciation rate is motivated by the dominating role of housing for consumer durables.

We have to specify three parameters for the collateral constraint: the minimum labor income $\underline{y}$ and the parameters γ and μ which determine the fractions of minimum income and durables that can be used as collateral. We choose $\mu=0.97$, consistent with data on the legal maximum of the loan-to-value ratio reported in Green and Wachter (2005), Table 2. We set $\gamma=0.95$, a value smaller than unity to ensure positive consumption at the smallest gridpoint for x'. The discrete approximation of the labor income process below implies $\underline{y}\simeq 0.09$. This value is positive since transfer income is included in labor earnings and US consumers in our sample, who receive transfers in the SCF 2004, receive about 9% of average net labor earnings through unemployment insurance, food stamps or child support.

In the calibration with adjustment costs, we set $\alpha=0.05$ to capture the order of magnitude of these costs observed in market transactions, such as the typical 5% fee charged by real-estate brokers in the US (Díaz and Luengo-Prado, 2010). Note that condition (12) holds for the chosen parameter values which imply that μ has to be smaller than 0.9995. This is barely restrictive since we require that $\mu<1$ anyway. Finally, the functional form for adjustment costs is well defined only if $d\geq d_{\min}>0$. We choose $d_{\min}=0.01$ to keep the grid very similar to the one without adjustment cost.

[4] We use the programs provided by Kevin Moore on http://www.nber.org/~taxsim/ for constructing the SCF data in 2004 which are fed into the tax simulator on the NBER website.

[5] When computing the statistics in the data, we use the sampling weights provided in the SCF. The normalization by net labor earnings and the use of equivalence scales implies that normalized (aggregate) wealth is larger than the wealth to *output* ratio.

3.2.1. Income process

We approximate the income process by a 5-state Markov chain.[6] We purge net labor earnings of life-cycle effects focusing on households with a head between 20 and 55 years of age. For this sample we regress net labor earnings on an age polynomial and compute the quintile means of the residual distribution around the normalized mean income of 1 in the SCF 2004. This results in

$$y = [0.09,\ 0.39,\ 0.74,\ 1.22,\ 2.57].$$

We approximate the income distribution in the SCF as

$$\log y \sim \mathcal{N}(-0.3663, 0.7325)$$

and assume an AR(1) process with first-order correlation of 0.95 which implies a variance of the innovations in the AR(1) process of $0.7325(1-0.95^2)$=0.07. This is larger than the variance of 0.02 for persistent innovations estimated by Storesletten et al. (2004) using PSID data but we consider our parametrization of the Markov chain reasonable for the following reasons. Firstly, as discussed for example in Heathcote et al. (2010), the earnings dispersion is larger in the SCF sample than in the PSID sample, which does not represent very wealthy consumers well. Secondly, the estimates of Storesletten et al. (2004) based on the PSID panel are not directly applicable to the SCF data cross-section since they are able to control for permanent effects and allow also for purely transitory innovations. As a robustness check we will also report results for a first-order correlation of 0.97 which implies a smaller variance of 0.04 for the innovations in the AR(1) process.

We then use Tauchen's (1986) method to compute the transition matrix which, for our benchmark parameters, is given by

$$\Gamma = \begin{bmatrix} 0.9854 & 0.0146 & 0 & 0 & 0 \\ 0.0045 & 0.8451 & 0.1491 & 0.0013 & 0 \\ 0 & 0.1359 & 0.6787 & 0.1843 & 0.0011 \\ 0 & 0.0029 & 0.2208 & 0.6963 & 0.0800 \\ 0 & 0 & 0.0006 & 0.1455 & 0.8539 \end{bmatrix}.$$

The stationary income distribution is

$$\pi = [0.0817,\ 0.2627,\ 0.2856,\ 0.2377,\ 0.1323].$$

Although the Markov chain with five states approximates the log-normally distributed AR(1) process very well, we implement a bias correction which ensures that the discrete Markov chain implies *exactly* the same mean and variance.[7]

3.2.2. Results

We calibrate the preference parameters θ and β to match the average durable stock and total wealth in the data. The steady-state statistics for the model are based on 110,000 simulated observations where we discard the first 10,000 observations to avoid that initial conditions influence our results.

Table 1 displays the calibrated preference parameters for the two different values of the autocorrelation discussed above, 0.95 and 0.97, for the case with and without adjustment costs. The calibrated preference parameters, θ and β, are similar to previous studies in all cases and align the statistics for durable and total wealth of the model with the data. Since the results with and without adjustment costs are very similar, we focus on the case without adjustment costs in our discussion below and only highlight important differences for the case with adjustment costs.

In terms of wealth dispersion, our benchmark case with an autocorrelation of 0.95 matches the Gini of net financial wealth in the data 0.97 quite closely with a Gini of 0.88.[8] The Ginis for durable wealth and total wealth are a bit smaller in the model than in the data: 0.54 in the model compared with 0.81 in the data for total wealth and 0.30 compared with 0.67 for durable wealth.

Quantitatively, the model generates more dispersion if shocks are more persistent (when the autocorrelation is 0.97). For example, the model then generates a Gini of 0.37 for durables which nonetheless is still smaller than in the data. This is not surprising since additional non-testable assumptions about the income processes are necessary to match the observed wealth dispersion (see for example Díaz and Luengo-Prado, 2010). The model correctly predicts the ranking of the Ginis: financial wealth is most unequally distributed, followed by total wealth and durable wealth. The ranking is intuitive since durables are also a consumption good which is smoothed over time. Finally, we find that the collateral constraint is binding for a non-negligible part of the population (between 5% and 15% in the four cases presented above) so that the very non-linear part of the policy functions (which are plotted in Fig. 2 for the case without adjustment costs) is relevant.

Fig. 2 displays the policy functions for financial assets, durables and non-durable consumption as function of total wealth for each of the five income states, in the benchmark case with an autocorrelation of 0.95 and no adjustment costs. Since the collateral constraint is more relevant for a small durable stock d and low values of total wealth x, which makes

[6] We make use of existing Matlab routines accompanying Miranda and Fackler (2002) and Ljungqvist and Sargent (2004).

[7] The idea is to choose the standard deviation which we use to compute the transition matrix so that the implied standard deviation of the Markov chain is exactly equal to the one in the data.

[8] Since net-financial wealth can be negative, we renormalize the Gini statistic so that it remains between 0 and 1 (see Chen et al., 1982).

Table 1
Calibrated parameters for first-order autocorrelation 0.95 and 0.97.

	Calibrated parameters	
	No adjustment costs: $\alpha=0$	Adjustment costs: $\alpha=0.05$
Autocorrelation 0.95		
	$\theta=0.8092$ $\beta=0.9391$	$\theta=0.807$ $\beta=0.93885$
Autocorrelation 0.97		
	$\theta=0.8098$ $\beta=0.94552$	$\theta=0.8083$ $\beta=0.94536$

Notes: The target statistics are the average total wealth, 6.33, and average durable stock, 4.45. These statistics are matched at precision 10^{-2} for all cases. Wealth is measured in units of average net-labor earnings for consumers between age 20 and 55.

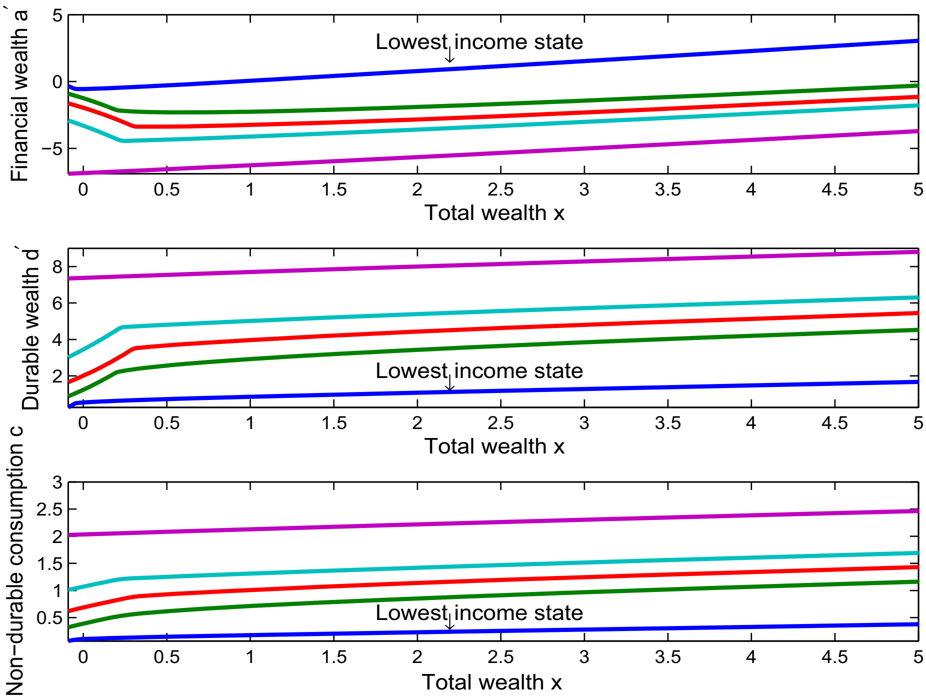

Fig. 2. Policy functions without adjustment costs ($\alpha=0$) for the five income states, conditional on d=0.05, plotted for total wealth $x\leq 5$.

the plots more interesting, the graphs in the figure are plotted for a small current durable stock d=0.05 and for values of total wealth $x\leq 5$. Not surprisingly, higher labor income shifts up the durable and non-durable consumption policies. The policies for financial wealth shift down for higher income levels instead, since income states are quite persistent and consumers can afford to consume more out of current income if that income is high.

We now discuss how the collateral constraint affects the shape of the policy functions. If consumers are not at the collateral constraint, financial assets and consumption increase in total wealth. At the constraint instead, the propensity to consume out of total wealth is higher and financial assets decrease in total wealth. Consumers borrow as much as possible against their additional durable collateral d' in this case. As total wealth increases, the collateral constraint eventually stops to bind. Note that the policies are non-linear when consumers are constrained where the shape also depends on the degree of complementarity between durables and non-durables.

The main difference for the policy functions with adjustment costs is that the policies for durables become flatter in total wealth, especially for small values of d. This is since quadratic adjustment costs make it optimal to slowly adjust durables from their current level.

3.3. *Computational accuracy and efficiency*

We iterate until the policy functions have converged at a precision of 10^{-6}. As has become standard in the literature (see, e.g., Judd, 1992; Aruoba et al., 2006; Barillas and Fernández-Villaverde, 2007), we evaluate the accuracy of our

Table 2
Maximum and average Euler equation errors for autocorrelation 0.95.

Number of gridpoints for (x',d)	Time to converge (seconds)	Max. Euler error for (a',d')	Average Euler error for (a',d')
No adj. costs: $\alpha=0$			
(225,100)	169	$(-3.94,-3.77)$	$(-4.69,-4.49)$
(225,300)	381	$(-4.44,-4.43)$	$(-5.22,-5.13)$
Adj. costs: $\alpha=0.05$			
(225,100)	218	$(-3.54,-2.79)$	$(-4.67,-4.08)$
(225,300)	462	$(-3.46,-3.51)$	$(-5.19,-4.88)$

Note: Absolute errors reported in units of base-10 logarithms.

solutions by the normalized Euler equation errors implied by the policy functions. The normalized Euler errors ρ_a and ρ_d in our problem are obtained by using the envelope conditions (7) in Eqs. (5) and (6) for $\kappa=\eta=0$,

$$\rho_a \equiv 1-\frac{\left[\beta(1+r)E\dfrac{\partial U(c',d')}{\partial c'}\dfrac{(d+\varepsilon_d)^{(\theta-1)(1-\sigma)}}{\theta}\right]^{1/(\theta(1-\sigma)-1)}}{c}$$

and

$$\rho_d \equiv 1-\frac{\left(\dfrac{\beta\left((1-\delta)E\dfrac{\partial U(c',d')}{\partial c'}+E\left(\dfrac{\partial U(c',d')}{\partial d'}-\dfrac{\partial U(c',d')}{\partial c'}\dfrac{\partial \Psi(d'',d')}{\partial d'}\right)\right)(d+\varepsilon_d)^{(\theta-1)(1-\sigma)}}{\theta\left(1+\dfrac{\partial \Psi(d',d)}{\partial d'}\right)}\right)^{1/(\theta(1-\sigma)-1)}}{c}.$$

These errors are expressed in units of non-durable consumption and thus have a straightforward interpretation. An error of 10^{-3}, or -3 in units of the base-10 logarithm, means a mistake of \$1 for \$1000 spent. Table 2 displays statistics for these errors in units of the base-10 logarithm for an autocorrelation of 0.95, both for the case with and without adjustment costs. In both cases, we present statistics for solutions with 100 and 300 gridpoints for durables to illustrate the trade-off between computing time and accuracy. The statistics in Table 2 show that the solution is fast and already quite accurate for relatively few gridpoints. With 100 gridpoints for durables and 225 gridpoints for total wealth, an iteration on the policy functions takes about a second, using Matlab on one of the 2.4 GHz dual-core processors for a PC with 2 GB RAM. Convergence of the policies for the infinite horizon case is achieved in about 4 minutes. For this setting, the average Euler errors are smaller than 10^{-4}, applying the weights of the stationary distribution, where the largest errors on the range where the Euler equations apply with equality are of order 10^{-3}. Computation time and accuracy are better in the case without adjustment costs but the speed and accuracy in the case with adjustment costs remain remarkable. Even the very accurate solutions with 300 gridpoints for durables and 225 gridpoints for total wealth are computed at a rate of about 2 seconds per iteration.

Standard approaches for solving our type of portfolio choice problem with occasionally binding constraints either discretize choices and states or rely on numerical root-finding or constrained optimization. Both approaches suffer from the additional dimension due to durables, in both the space of states and choices. To obtain reasonable accuracy in a fully discretized approach requires a large number of gridpoints, say m, for *each* dimension of states and choices. Having m gridpoints in 2×2=4 dimensions, would entail solving m^2 maximization problems each of which would need to be evaluated at m^2 candidate choices, an enormous computational burden.

Fig. 3 illustrates this point, comparing the computation time for different grid sizes for the model without adjustment costs.[9] The computation time for the solution obtained with EGM is plotted as the solid graph and the time for the solution obtained with discretization and value function iteration is shown as the dashed graph. The grid size at factor 1 in the figure is 8 gridpoints for x' and 4 gridpoints for d. At factor 10 it is 80 gridpoints for x' and 40 gridpoints for d and so forth. For small grid-size factors, the overhead-computing time dominates and both methods are approximately equally fast. The accuracy of the solutions is very different, however. Whereas the accuracy of the EGM is remarkable for so few gridpoints (the average Euler errors are of order -2 in base-10 logarithms), the average Euler errors with discretization and value function iteration are of order -1.

At the grid-size factor 10 in Fig. 3, the solution with the EGM is 20 times faster than with value function iteration, with average Euler errors of order -2.7 in base-10 logarithms that are half the size of the average errors for the solution

[9] We use a single (instead of triple) exponential grid for the comparison which improves the accuracy in terms of the normalized Euler error criteria for standard value function iteration.

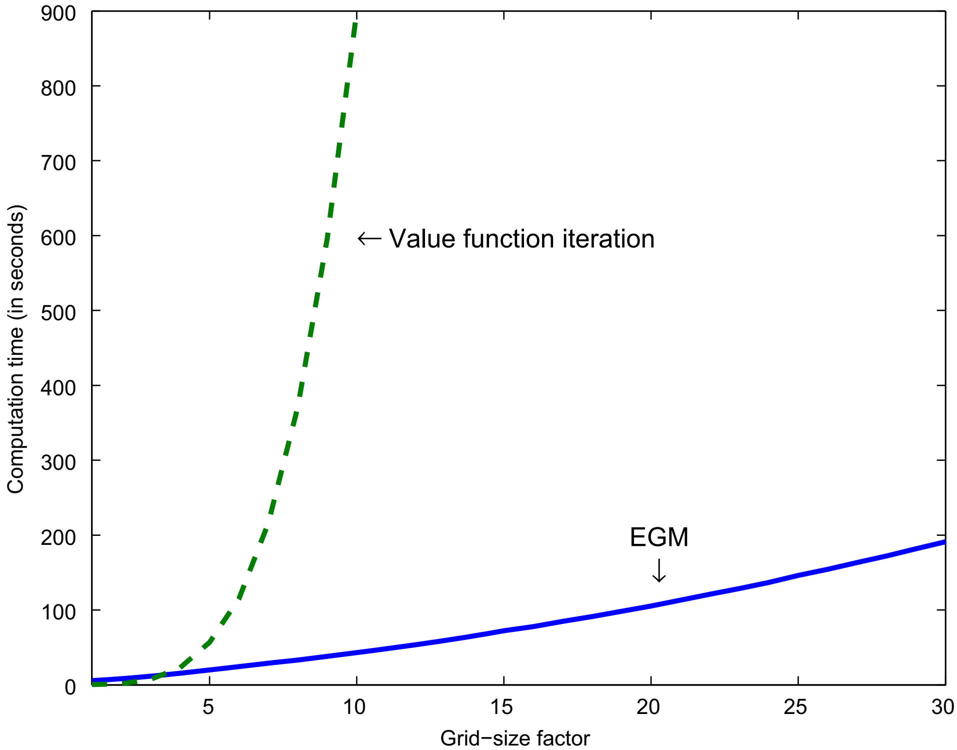

Fig. 3. Computation time of the model without adjustment costs, as a function of the grid size. Notes: Solid graph: computation time for the EGM and dashed graph: computation time for discretization with value function iteration. A grid-size factor of 10 means twice as many gridpoints for x' and twice as many points for d than for a grid-size factor of 5.

obtained with standard value function iteration. As illustrated in the figure, standard value function iteration is subject to a severe "curse of dimensionality" while the EGM is not. In fact, the EGM allows the computation of solutions at a level of accuracy that would be prohibitively costly to achieve with standard value function iteration. Moreover, the "curse of dimensionality" limits the scope of gradually increasing the number of gridpoints in the value function iteration. For the numbers of gridpoints that we use for the very accurate solution of the model with the EGM in Table 2, standard value function iteration would reach memory requirements of more than 30 gigabytes for one Markov state $((225\times300)^2\times8=36.5\times10^9$, where 8 bytes are allocated by Matlab to a double-precision floating point).

We now briefly discuss the advantages of the EGM in the context of two further alternatives which might be considered for the type of problem we have presented above. One such alternative is standard time iteration on the policy functions, which would need to solve a system of first-order conditions that includes the Kuhn–Tucker inequalities for the occasionally binding constraints. An example of this approach is the paper by Kubler and Schmedders (2003) who replace these inequalities by equalities via a change of variables. Massive application of root-finding routines then allows to solve for equilibrium policies and multipliers at each point in the state space. In the context of our application with three state variables, two choice variables and up to two multipliers, such an approach would encounter the difficulty of having to solve multidimensional root-finding problems on a very large scale.

Compared with such an approach our extension of the EGM has the advantage that the combination of future state variables computed in Step 1 of our algorithm immediately reflects the binding patterns of the constraints. We can then express the multipliers on the occasionally binding constraints in terms of these combinations of future state variables, and solve for the control variables in closed form.

Another alternative would be to stick to value function iteration and to allow for continuous optimization by relying upon numerical optimization routines. In our application this amounts to solving constrained optimization problems in two dimensions, one call of a numerical solver for each point in a three-dimensional state space. This presupposes that the stability of the solver employed is ascertained. Experimenting with various standard routines that are available in Matlab we have encountered numerical instability problems. The appropriate rectification of this numerical issue is in general quite cumbersome, it will depend on the particular problem at hand, and it asks for a fair degree of computational literacy. The advantage of the EGM is that it avoids such numerical challenges since it does not rely on any kind of multidimensional optimization or root-finding. We therefore hope that the availability of such a stable method will encourage research on economic questions in fields such as household portfolio choice, which involve solutions of multidimensional problems subject to occasionally binding constraints.

4. Conclusion

We have proposed an efficient method to solve dynamic stochastic models with occasionally binding constraints among endogenous variables. We have illustrated the method for an important consumer problem where the method is well founded in the analytical properties of the problem. In terms of performance, we find that the algorithm needs one second per iteration for a very accurate solution which would otherwise be prohibitively time consuming to compute. The efficient and accurate way of describing optimal individual behavior with our method will hopefully become an important building block for future research which studies DSGE models or attempts structural estimation.

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Appendix A. Adjustment costs and the collateral constraint

We sketch how adjustment costs may alter the collateral constraint in the application we study. If the bank seizes the collateralizable part of labor income $\gamma\underline{y}$ and incurs adjustment costs $\alpha/2((1+r)a'+\gamma\underline{y}/d)^2 d$ if it sells $-((1+r)a'+\gamma\underline{y})$ units of the durable to recover the debt, the collateral constraint becomes

$$\underbrace{\min\left\{\mu, 1-\frac{\frac{\alpha}{2}\left((1+r)a'+\gamma\frac{\underline{y}}{d}\right)^2 d}{(1-\delta)d'}\right\}(1-\delta)d'+\gamma\underline{y}}_{\text{collateral}}+(1+r)a'\geq 0$$

for $a'<0$. If

$$\mu \leq 1-\frac{\frac{\alpha}{2}\left((1+r)a'+\gamma\frac{\underline{y}}{d}\right)^2 d}{(1-\delta)d'}$$

then the collateral constraint is

$$\underbrace{\mu(1-\delta)d'+\gamma\underline{y}}_{\text{collateral}}\geq -(1+r)a'$$

as in the text.

If instead

$$\mu > 1-\frac{\frac{\alpha}{2}\left((1+r)a'+\gamma\frac{\underline{y}}{d}\right)^2 d}{(1-\delta)d'}$$

the constraint becomes

$$\underbrace{(1-\delta)d'-\frac{\alpha}{2}\left((1+r)a'+\gamma\frac{\underline{y}}{d}\right)^2 d+\gamma\underline{y}}_{\text{collateral}}\geq -(1+r)a'.$$

Expressing this constraint in terms of x' and d', we have to solve the quadratic equation

$$(1-\delta)d'-\frac{\alpha}{2}\left(x'-(1-\delta)d'+\gamma\frac{\underline{y}}{d}\right)^2 d+\gamma\underline{y}\geq -x'+(1-\delta)d',$$

which has the solution

$$x'\geq (1-\delta)d'-\gamma\underline{y}+\frac{d}{\alpha}-\sqrt{\frac{d^2}{\alpha^2}+2\frac{d}{\alpha}(1-\delta)d'}.$$

The negative root is the relevant one and the constraint simplifies to $x'\geq -\gamma\underline{y}$ if $d'=0$. Importantly, the constraint is differentiable in x' and d' so that the EGM still can be applied.

Appendix B. Data

This data appendix describes how we construct data counterparts for the wealth portfolio as well as labor earnings in the model, using data from the Survey of Consumer Finances (SCF) 2004. We construct all variables for the full SCF sample and then apply the sample-selection criteria mentioned below.

Gross labor income is the sum of wage and salary income. As in Budría Rodríguez et al. (2002) we add a fraction of the business income where this fraction is the average share of labor income in total income in the SCF. *Disposable labor income* is computed using the NBER tax simulator. We use the programs by Kevin Moore provided on http://www.nber.org/~taxsim/ to construct disposable labor earnings for each household in the SCF 2004. Following the standardized instructions on the NBER website, we feed the following required SCF data into the NBER tax simulator: the US state (where available, otherwise we use the average of the state tax payments across states), marital status, number of dependents, taxpayers above age 65 and dependent children in the household, wage income, dividend income, interest and other property income, pensions and gross social security benefits, non-taxable transfer income, rents paid, property tax, other itemized deductions, unemployment benefits, mortgage interest paid, short and long-term capital gains or losses. We then divide the resulting federal and state income tax payments as well as federal insurance contributions of each household by the household's gross total income in the SCF. This yields the implicit average tax rate for each household in 2004. The mean of that average tax rate for consumers between ages 26-65 in the SCF 2004 is 22%, for example. Finally, we use the average tax rate of each household in 2004 to compute household disposable labor income as (1 − household average tax rate)∗household gross labor income (including taxable transfers) and then add non-taxable transfers.

Durables are defined as the sum of the value of homes, residential and non-residential property and vehicles. These are the most important durable items which can be used as collateral in real-world debt contracts.

Net financial assets are defined as the sum of assets (besides the durables defined above) net of household debt. The assets are the sum of money in checking accounts, savings accounts, money-market accounts, money-market mutual funds, call accounts in brokerages, certificates of deposit, bonds, account-type pension plans, thrift accounts, the current value of life insurance, savings bonds, other managed funds, other financial assets, stocks and mutual funds, owned non-financial business assets, jewelry, antiques or other small durable items not included in the durable definition above. We then subtract the sum of mortgage and housing debt, other lines of credit and debt written against residential and non-residential property or vehicles, credit-card debt, non-auto consumer loans and other financial debt.

Net worth is then defined as the sum of durables and net-financial wealth.

Sample selection criteria: In order to contain the effect of outliers on the means, we drop observations if gross labor income is negative (11 observations are deleted) and net worth is smaller than −1.2 in terms of the population average of disposable labor income (additional 19 observations are deleted). We further restrict our attention to households with a household head between age 20 and 55 when matching the model to the data, for reasons discussed further in the main text.

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