Asset Allocation with Factor Models

Carlo A. Favero,[1]

[1]Bocconi University & CEPR

MAFINRISK

The Econometrics of Financial Returns

- Predicting the distribution of returns of financial assets is a task of primary importance for identifying desirable investments, performing optimal asset allocation within a portfolio, as well as measuring and managing portfolio risk.

- Horizon at which returns are defined matters for financial and statistical reasons

- Portfolio allocation, i.e. the choice of optimal weights to be attributed to the different financial assets in a portfolio, is typically based on a long horizon perspective,

- the measurement of risk of a given portfolio takes typically a very short-horizon perspective.

The Econometric Modelling Process

- Econometrics uses the "past available data" to predict the future distribution of returns. This is a process the provide some information only if past data are capable of giving us some information on the future and involves several steps:
    - Data collection and transformation
    - Graphical and descriptive data analysis
    - Model Specification
    - Model Estimation
    - Model Validation
    - Model Simulation

The Traditional Approach

To illustrate the econometric modelling process and its application for asset allocation and risk measurement let us consider the most basic model: the Constant Expected Return Model

Model Specification

The traditional approach is based on the constant expected returns model, that we can describe as follows considering the vector of one-period returns on assets used to build a portfolio:

$$\begin{aligned} \mathbf{r}_{t,t+1} &= \mu + \mathbf{H}\epsilon_{t+1} \\ \Sigma &= \mathbf{H}\mathbf{H}'. \\ \epsilon_{t+k} &\sim \mathcal{D}\left(\mathbf{0}, \mathbf{I}\right) \end{aligned}$$

where $\mathbf{r}_{t,t+k}$ is the vector of returs between time t and time t+k in which we are interested, μ is the vector of mean returns and the matrix $\mathbf{H}$ determines the time invarying variance-covariance matrix of returns.

Model Estimation

Model estimation allows to find values for $\mu, \mathbf{\Sigma}$. In the case of CER this step is easily solved by n OLS regressions of the n returns on a constant. Notice that once one-step ahead returns are known also n-step ahead returns are known:

$$E_t(\mathbf{r}_{t,t+n}) = n\mu$$

$$Var(\mathbf{r}_{t,t+n}) = n\Sigma$$

Monte-Carlo methods

- given some estimates of the unknown parameters in the model (μ, Σ in our case).
- an assumption is made on the distribution of ϵ_t.
- The an artificial sample for ϵ_t of the length matching that of the available can be computer simulated.
- The simulated residuals are then mapped into simulated returns via μ, Σ.
- This exercise can be replicated N times to construct the distribution of model predicted returns.

Bootstrap methods

do exactly like in Monte-Carlo but rather than using a theoretical distribution for ϵ_t use their empirical distribution (the distribution of the OLS residuals from the n regressions in the CER model) and resample from it with reimmission.

Model Validation

- Backtesting can be used to provide statistical evidence of the capability of the model to replicate the data.
- Statistical testing can be used to evaluate restrictions on coefficients predicted by the theory used to specify the empirical model.

An Application in R

The Rmarkdown code $prog20211.rmd$ illustrates the Econometric Modelling approach with an application to the CER model.

A Static Asset Allocation Problem with Constant Expected Returns

Let's denote with $\mathbf{r}$ the random vector of linear total returns from time t to time T from a given menu of N risky assets for interval $[t, T]$, $\mathbf{r} \sim \mathcal{D}(\mu, \Sigma)$

Given a degree of risk aversion λ, a standard *mean-variance* description of this allocation problem is the following:

$$\max_{\mathbf{w}} \left(1 - \mathbf{w}'\mathbf{e}\right) r^f + \mathbf{w}'\mu - \frac{1}{2}\lambda(\mathbf{w}'\Sigma\mathbf{w})$$

where $E[\mathbf{r}] = (1 - \mathbf{w}'\mathbf{e})\, r^f + \mathbf{w}'\mu = r^f + \mathbf{w}'(\mu - r^f\mathbf{e})$ and $Var[\mathbf{r}] = \mathbf{w}'\Sigma\mathbf{w}$

A Static Asset Allocation Problem with Constant Expected Returns

first-order conditions (FOCs) are necessary and sufficient and define the following system of N linear equations in N unknowns, the portfolio weights $\mathbf{w} \in \mathcal{R}^N$:

$$(\mu - r^f \mathbf{e}) - \lambda \Sigma \mathbf{w} = \mathbf{0}.$$

Solving the FOCs yields:

$$\hat{\mathbf{w}} = \frac{1}{\lambda} \Sigma^{-1} \left(\mu - r^f \mathbf{e} \right),$$

A Static Asset Allocation Problem with Constant Expected Returns

Consider now the special case in which $\hat{\mathbf{w}}'\mathbf{e} = 1$, that is no investment in the riskfree bond is allowed. The optimal portfolio in this case is the famous *tangency portfolio*:

$$\mathbf{e}'\hat{\mathbf{w}} = \frac{1}{\lambda}\mathbf{e}'\Sigma^{-1}\left(\mu - r^f\mathbf{e}\right) = 1 \Longrightarrow \lambda = \mathbf{e}'\Sigma^{-1}\left(\mu - r^f\mathbf{e}\right)$$

$$\hat{\mathbf{w}}^T = \frac{\Sigma^{-1}\left(\mu - r^f\mathbf{e}\right)}{\mathbf{e}'\Sigma^{-1}\left(\mu - r^f\mathbf{e}\right)},$$

The role of Econometric Modelling

- The role of econometric modelling is to provide empirical estimates of Σ, μ, that allows empirical implementation of the optimal asset allocation.
- After the estimation of μ,Σ and their mapping into $\hat{\mathbf{w}}$, it is possible to reconstruct via simulation the distribution of portfolio returns and determine the relevant measures of risk, such as the VaR.

Factor Models and Reduction in Dimensionality

- The problem with the traditional approach is dimensionality.
- The implementation of asset allocation and risk measurement requires the estimation of a very large number of parameters: $\frac{n(n+1)}{2} + n$.
- Factor models allow to simplify the structure of the model and to reduce the number of parameters to be estimated

Factor Models:Time-Series Representation

The statistical distribution of N assets (i=1...n) is conditioned on a vector of K factors **f** (where N is large and K is small)

$$
\begin{aligned}
r^i_{t,t+k} &= \gamma^i_0 + \gamma^{i\prime}_1 \mathbf{f}_{t,t+k} + v^i_{t,t+k} \\
\mathbf{f}_{t,t+k} &= \mu^f + \mathbf{H}^f \epsilon_{t,t+k} \\
\Sigma^f &= \mathbf{H}^f \mathbf{H}^{f\prime}. \\
\mathbf{E}\left(v^i_{t,t+k}, v^j_{t,t+k}\right) &= 0 \\
\mathbf{E}\left(v^i_{t,t+k}, \epsilon^j_{t,t+k}\right) &= 0 \\
\epsilon_{t+k} &\sim \mathcal{D}\left(\mathbf{0}, \mathbf{I}\right)
\end{aligned}
$$

Factor Models: Cross-Sectional representation

The multifactor model has the following cross-sectional representation for the (Nx1) vector of returns at time t

$$
\begin{aligned}
\underset{(Nx1)}{\mathbf{r}_{t,t+k}} &= \underset{(Nx1)}{\alpha} + \underset{(NxK)}{B}\underset{(Kx1)}{\mathbf{f}_{t,t+k}} + \underset{(Nx1)}{\mathbf{v}_t} \\
\underset{(Kx1)}{\mathbf{f}_{t,t+k}} &= \underset{(Kx1)}{\mu^f} + \underset{(KxK)}{\mathbf{H}^f}\underset{(Kx1)}{\epsilon^f} \\
\Sigma^v &= \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ .. & .. & .. & .. \\ 0 & 0 & 0 & \sigma_n \end{bmatrix} \\
\Sigma^f &= \mathbf{H}^f \mathbf{H}^{f\prime}.
\end{aligned}
$$

Factor Models as Parsimoniuos Representation

- Using the joint distribution of returns to estimate the variance covariance matrix (the CER model) requires the estimation of n+n(n+1)/2 parameters;
- using a structure of k factors requires the estimation of (2n+nk)+ (k+k(k+1)/2) parameters.
- Think for example of an asset allocation problem with 30 assets and 4 factors. The CER would require the estimation of 505 parameters, the factor model would reduce that number to 194.

A single factor model:The CAPM

The time series representation

$$
\begin{aligned}
\left(r_t^i - r_t^{rf}\right) &= \beta_{0,i} + \beta_{1,i}\left(r_t^m - r_t^{rf}\right) + u_{i,t} \\
\left(r_t^m - r_t^{rf}\right) &= \mu_m + u_{m,t} \\
u_{i,t} &\sim n.i.d.\left(0, \sigma_i^2\right) \\
\begin{pmatrix} u_{i,t} \\ u_{m,t} \end{pmatrix} &\sim n.i.d.\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{ii} & 0 \\ 0 & \sigma_{mm} \end{pmatrix}\right]
\end{aligned}
$$

The CAPM: cross-section representation

Considering the cross-section representation for the returns :

$$\begin{aligned}
\mathbf{r}_t &= \beta_0 + \beta_1 r_t^m + \mathbf{u}_t \\
r_t^m &= E\left(r^m\right) + \sigma_m \mathbf{u}_{m,t} \\
\Sigma &= \beta_1 \beta_1' \sigma_m^2 + \Sigma_u \\
\mu &= \beta_0 + \beta_1 E\left(r^m\right)
\end{aligned}$$

and μ, Σ can be obtained with the estimation of 3n+2 parameters only.

Asset Allocation with the CER and the CAPM in R

The R programme $factormodels2021.r$ allows you to

- upload a data set of US stock market returns
- perform descriptive and graphical analysis
- implement optimal portfolio allocation with the CER model
- implement optimal portfolio allocation with the CAPM model
- compare the results

Asset Allocation with the CER and the CAPM in R

In the application portfolio optimization is aimed at computing the minimum variance portfolio:

$$\min_{\mathbf{w}}(\mathbf{w}'\Sigma\mathbf{w})$$

subject to

$$\mathbf{w}'\mathbf{e} = 1$$

with solution:

$$\mathbf{w} = \frac{\Sigma^{-1}\mathbf{e}}{\mathbf{e}'\Sigma^{-1}\mathbf{e}}$$

Validating Factor Models

- The diagonality of the variance-covariance matrix of the residuals coming from projecting asset returns on factors is a necessary–and testable–requirement for the validity of any factor model.
- Further validation is based on testing restrictions on their coefficients

Validation by testing restrictions

Consider once again the time-series representation of a factor model

$$r^i_{t+1} = \alpha_1 + \beta_i^{f^1} f^1_{t+1} + \beta_i^{f^2} f^2_{t+1} + \cdots + \beta_i^{f^k} f^k_{t+1} + v_{t+1} \qquad (1)$$

After having estimated N equations for the N assets you have available the following k vectors of coefficients, each of length N: $\boldsymbol{\beta}^{f1}$, $\boldsymbol{\beta}^{f2}$,..., $\boldsymbol{\beta}^{fk}$. Using the sample of t observations on the returns of the N assets you can compute the vector of length N of average sample returns for the assets: $E(\mathbf{r})$.

Validation by testing restrictions

You can now run the affine expected return-beta cross-sectional regression is:

$$E(\mathbf{r}) = \gamma_0 + \gamma_1 \boldsymbol{\beta}_{f1} + \gamma_2 \boldsymbol{\beta}_{f2} + \cdots + \gamma_k \boldsymbol{\beta}_{fk} + \mathbf{u}$$

A two-step test (FamaMacBeth) for the validity of any factor model can be run by considering the following null hypothesis:

$$\hat{\gamma}_0 = \bar{r}^f, \; \hat{\gamma}_i = E\left(f^i\right)$$

Fama-MacBeth

- care must be exercise in the test as the variance-covariance matrix of the residuals in the cross-sectional regression will not be diagonal and corrections for heteroscedasticity should be implemented.

- note also that, if both test assets and factors are excess returns, the validity of the model can be simply tested by evaluating the null that all intercepts in the time-series model for excess returns are zero.

- this null is inevitably rejected. Two industries have emerged (i) the factors "zoo", thath looks for omitted factors (ii) the performance evaluation industry that classifies fund manager performance according to their

Which Factors ?

Many different set of factors have been considered in the literature :

- Fundamental Factors
  - Fama-French five factors with observable characteristics and estimated betas (MKT, SMB, HML, RMW, CMA and momentum MOM
  - BARRA factors with know time-invariant betas and unobservable factor realizations estimated by cross-sectional regressions (see the program factormodels.R for a practical illustration)
- Macroeconomic Factors (inflation, growth and uncertainty)
- Statistical Factors (for example principal components)

Factor Exposures

Exposure to portfolios to factors can be assessed by computing the share of the total portfolio variance attributable to each factor.

$$r^i_{t+1} = \alpha_1 + \beta_i^{f^1} f^1_{t+1} + \beta_i^{f^2} f^2_{t+1} + \cdots + \beta_i^{f^k} f^k_{t+1} + v_{t+1}$$

$$\begin{aligned} Var\left(r^i_{t+1}\right) &= Cov\left(r^i_{t+1}, r^i_{t+1}\right) \\ &= \beta_i^{f^1} Cov\left(f^1_{t+1}, r^i_{t+1}\right) + \ldots \beta_i^{f^k} Cov\left(f^k_{t+1}, r^i_{t+1}\right) \\ &+ Cov\left(v_{t+1}, r^i_{t+1}\right) \end{aligned}$$

Smart Beta Strategies

Smart beta aims to outperform the capitalization-weighted market through alternative weighting methods that emphasize factors such as size, value, momentum, or low volatility.

From returns to prices

- Factor models have the general form:

$$r_{i,t+1} = \alpha_i + \beta_i' \mathbf{f}_{t+1} + v_{i,t+1}. \tag{2}$$

  Construct

  - Prices of any test asset as cumulative returns: $\ln P_{i,t} = \ln P_{i,t-1} + \mathbf{r}_{i,t}$.
  - Price-level risk drivers as cumulative returns of the factors: $\ln \mathbf{F}_t = \ln \mathbf{F}_{t-1} + \mathbf{f}_t$.

- Prices and risk-drivers follow stochastic trends that are not related under (2).

The problem(s) with standard factor models

- Standard Factor models that do no relate the stochastic trends in prices and risk drivers generate two potential problems.
  - using wrongly a "good" factor model
  - using correctly a "bad" factor model

A new approach: From prices to returns

- Start with a model that describes the exposure of a given portfolio $P_{i,t}$ to (price-level) risk drivers $\mathbf{F}_t$:

$$\ln P_{i,t} = \alpha_{0,i} + \alpha_{1,i} t + \underbrace{\beta_i' \ln \mathbf{F}_t}_{\text{intrinsic value}} + u_{i,t} \tag{3}$$

- The residuals $u_{i,t}$ are stationary if the risk drivers capture the stochastic trend in the long-run dynamics of prices.

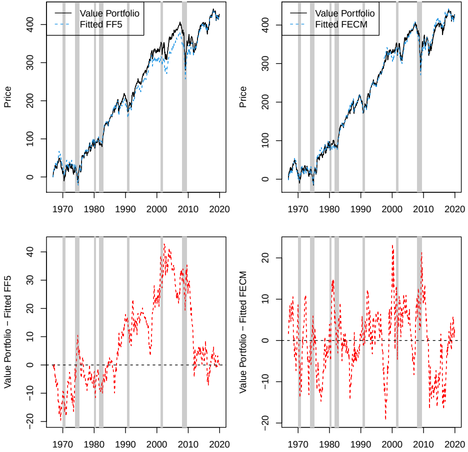
- For ease of exposition assume:

$$u_{i,t+1} = \rho_i u_{i,t} + v_{i,t+1}$$

- Taking first differences of our model in (3) we obtain

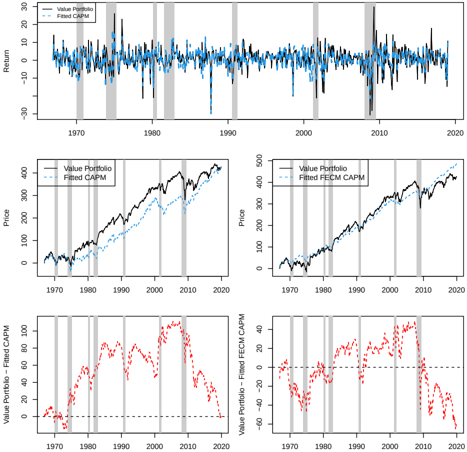
$$r_{i,t+1} = \alpha_{1,i} + \beta_i' \mathbf{f}_{t+1} + \underbrace{(\rho_i - 1)}_{\delta_i} \underbrace{u_{i,t}}_{\equiv ECT_{i,t}} + v_{i,t+1} \tag{4}$$

Price Dynamics in FF5 and its FECM Specification



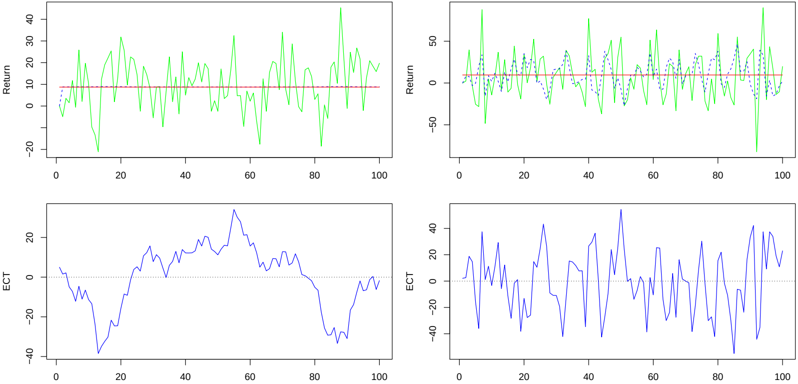
▸ MC simulation ▸ CAPM

Price Dynamics in CAPM and its FECM Specification



Back

Monte-Carlo Simulation



(a) DGP without cointegration. (b) DGP with cointegration.

▸ Back

A co-integrated approach to factor modelling

We model the joint distribution of portfolio prices, factors, and risk drivers as follows:

$$
\begin{aligned}
\ln P_{t+1}^{i} &= \alpha_0^i + \alpha_1^i t + \beta_i' \ln \mathbf{F}_{t+1} + u_{t+1}^i \\
u_{t+1}^i &= \rho_i u_t^i + v_{t+1}^i \\
\mathbf{f}_{t+1} &= E\left(\mathbf{f}_{t+1} \mid I_t\right) + \epsilon_{t+1} \\
\ln P_t^i &= \ln P_{t-1}^i + r_t^i \\
\ln \mathbf{F}_t &= \ln \mathbf{F}_{t-1} + \mathbf{f}_t \\
\epsilon_{t+1} &\sim \mathcal{D}\left(\mathbf{0}, \mathbf{\Sigma}\right) \\
Cov\left(v_{t+1}^i, v_{t+1}^j\right) &= 0
\end{aligned}
$$

If u_t^i is stationary, then prices and risk drivers are cointegrated.

What's next?

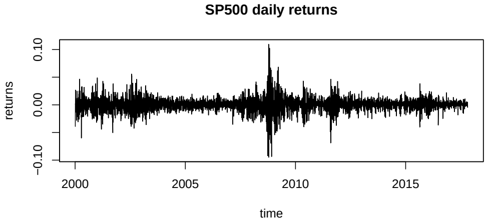
- ECT, Risk management and Portfolio timing
  - Modeling the relation between risk drivers and asset prices contributes to the description of the dynamics of returns
  - The predictive distribution of returns at time $t+1$ is centered on the ECT observed at time t.
- ECT and time-varying alphas
  - Recall

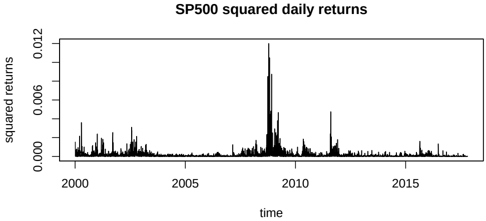
$$r_{i,t+1} = \alpha_{1,i} + \beta_i' \mathbf{f}_{t+1} + \delta_i ECT_{i,t} + \ v_{i,t+1}.$$

  - The test that the intercept is zero becomes a test that $\widehat{\alpha}_{1,i} + \widehat{\delta}_i ECT_{i,t} = 0$.

The Evidence form high frequency data

- Data at high-frequency (monthly or higher) show:
  - very little or no persistence in first moments
  - persistence in the variance
  - non-normality
- These features of the data can be used to measure VaR, using appropriate models for heteroscedasticity and non-normality





- Visibly, volatility "clusters" in time: high (low) volatility tends to be followed by high (low) volatility

GARCH

A parsimonious model capable of capturing all the features of high-frequency returns:

$$
\begin{aligned}
R_{t+1} &= \mu_t + \sigma_{t+1} z_{t+1} \qquad z_{t+1} \sim \text{IID } \mathcal{N}(0,1), \\
\sigma_{t+1}^2 &= \omega + \alpha \left(R_t - \mu_t\right)^2 + \beta \sigma_t^2 \\
\alpha + \beta &< 1
\end{aligned}
$$

where returns have a constant mean (that is usually zero) and a time varying GARCH(1,1) structure.
In a model like this the innovation $\epsilon_t \equiv \sigma_t z_t$ has zero mean and is serially uncorrelated at all lags $j \geqslant 1$.

GARCH Properties

R_{t+1} has a finite unconditional long-run variance of $\frac{\omega}{1-\alpha-\beta}$

$$\begin{aligned}\sigma^2 &= E\left(\sigma_{t+1}^2\right) = \omega + \alpha E\left(R_t - \mu\right)^2 + \beta\sigma^2 \\ &= \omega + \alpha\sigma^2 + \beta\sigma^2 \\ &= \frac{\omega}{1-\alpha-\beta}\end{aligned}$$

Substituting ω out of the GARCH expression:

$$\begin{aligned}\sigma_{t+1}^2 &= \left(1-\alpha-\beta\right)\sigma^2 + \alpha R_t^2 + \beta\sigma_t^2 \\ &= \sigma^2 + \alpha\left(\left(R_t-\mu\right)^2 - \sigma^2\right) + \beta\left(\sigma_t^2 - \sigma^2\right)\end{aligned}$$

which illustrates the relation between predicted variance and long-run variance in a GARCH model.

GARCH Forecasting

$$
\begin{aligned}
\sigma^2_{t+1|t} &= \bar{\sigma}^2 + \alpha \left[(R_t - \mu_t)^2 - \bar{\sigma}^2 \right] + \beta \left(\sigma^2_t - \bar{\sigma}^2 \right), \\
\sigma^2_{t+2|t} &= \bar{\sigma}^2 + (\alpha + \beta)\, \sigma^2_{t+1|t} \\
\sigma^2_{t+n+1|t} &= \bar{\sigma}^2 + (\alpha + \beta)^n\, \sigma^2_{t+1|t}
\end{aligned}
$$

From GARCH to VaR

After estimation a GARCH model can be simulated using bootstrap or Monte-Carlo to derive the distribution of returns and the relevant VaR

$$
\begin{aligned}
R_{t+1} &= \mu + \sigma_{t+1} z_{t+1} \qquad z_{t+1} \sim \text{IID } \mathcal{N}(0,1), \\
\sigma_{t+1}^2 &= \omega + \alpha \left(R_t - \mu_t \right)^2 + \beta \sigma_t^2 \\
\alpha + \beta &< 1
\end{aligned}
$$

Given estimation, derive $\hat{z}_t = \frac{R_t}{\hat{\sigma}_t}$. At time t you can now predict σ_{t+1}^2 and the distribution of R_{t+1} can now be simulated via the preferred method.
Recursion can then be applied to derive the distribution of R_{t+n} with n > 1.

GARCH with factors

think of modelling the returns of many assets at a high frequency with a single factor model

$$
\begin{aligned}
R_{t+1}^{i} &= \gamma_0 + \gamma_1 f_{t+1} + \sigma^{i} v_{i,t+1} \\
f_{t+1} &= \mu_t + \sigma_{t+1} z_{t+1} \\
\sigma_{t+1}^{2} &= \omega + \alpha \left(R_t - \mu_t \right)^2 + \beta \sigma_t^2 \\
v_{i,t+1} &\sim \text{IID } \mathcal{N}(0,1), \\
z_{t+1} &\sim \text{IID } \mathcal{N}(0,1), \\
\alpha + \beta &< 1
\end{aligned}
$$

one GARCH estimation will allow to model many returns distribution. Again factor models allow parsimonious representation