线性静态系统最小二乘辨识

1. 建模

假设一个静态线性系统的参数为：

$$\theta = \begin{bmatrix} \theta_0 & \theta_1 & \dots & \theta_n \end{bmatrix}^T \tag{1.1}$$

系统在k时刻(或编号为k的采样)的输入为：

$$\phi_k = \begin{bmatrix} 1 & u_1(k) & \dots & \theta_n(k) \end{bmatrix}^T \tag{1.2}$$

系统根据物理定律，和系统输入相关的系统输出为：

$$y_u(k) = \phi_k^T(k)\theta \tag{1.3}$$

系统输出的观测是系统真实输出与扰动的叠加：

$$y_P(k) = y_u(k) + n(k) \tag{1.4}$$

其中$n(k)$是离散时间平稳随机信号(discrete-time stationary random signal)[1],且$E\{n(k)\} = 0$[2]，$D\{n(k)\} = \sigma^2$.

对真实物理系统的参数辨识问题，是用模型参数θ_M来估计物理系统的参数θ，则模型输出为：

$$y_M(k) = \phi_k^T(k)\theta_M \tag{1.5}$$

则模型的观测误差(observation error)[3]（在更多的材料中，a.k.a. residual，残差）

$$e(k) = y_P(k) - y_M(k) \tag{1.6}$$

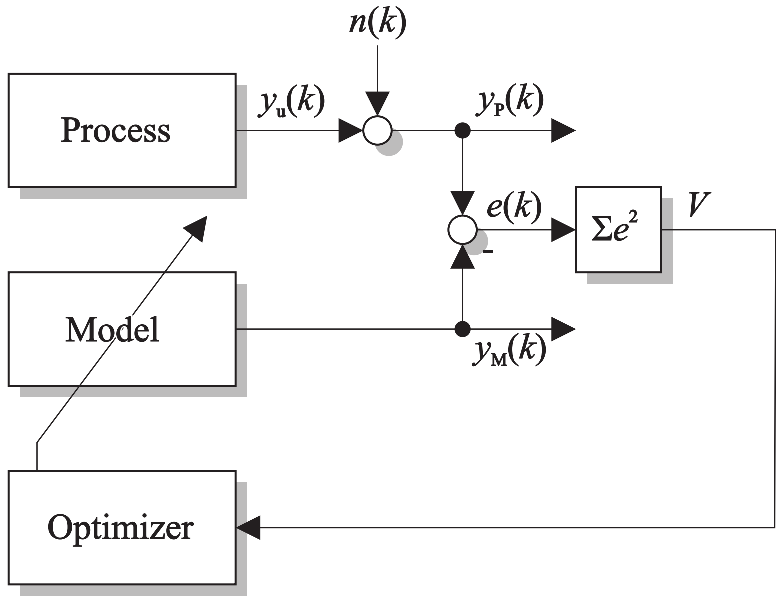

Fig. 8.1. Schematic diagram of the general arrangement for the method of least squares

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2. 最小二乘辨识

2.1. 推导

设采样的数据对$(\phi_k, y_P(k)), k = 1, 2, \ldots, N, N > n$，于是：

$$y_N = \begin{bmatrix} y_P(1) & y_P(2) & \cdots & y_P(N) \end{bmatrix}^T$$

$$n_N = \begin{bmatrix} n(1) & n(2) & \cdots & n(N) \end{bmatrix}^T$$

$$e_N = \begin{bmatrix} e(1) & e(2) & \cdots & e(N) \end{bmatrix}^T$$

$$\Phi_N = \begin{bmatrix} \phi_1^T \\ \phi_2^T \\ \vdots \\ \phi_N^T \end{bmatrix} = \begin{bmatrix} 1 & u_1(1) & \cdots & u_n(1) \\ 1 & u_1(2) & \cdots & u_n(2) \\ \vdots & \vdots & \cdots & \vdots \\ 1 & u_1(N) & \cdots & u_n(N) \end{bmatrix}$$

于是(1.6)可以扩展为

$$\underset{\text{Residual}}{e_N} = \underset{\text{Observation}}{y_N} - \underset{\text{Model Prediction}}{\Phi_N \theta_M}$$

最小二乘法的损失函数为

$$J(\theta_M) = \|e_N\|_2^2 = e_N^T e_N = (y_N - \Phi_N \theta_M)^T (y_N - \Phi_N \theta_M)$$

于是参数估计问题转化为优化问题

$$\hat{\theta} = \arg\min_{\theta_M} J$$

J是一个下凸函数，极值点就是最小值点，对参数θ_M求导并令其等于零

$$\frac{\partial J}{\mathrm{d}\theta_M} = -2\Phi_N^T (y_N - \Phi_N \theta_M) = 0$$

得到如下被称为**正规方程**的方程组

$$\Phi_N^T \Phi_N \theta_M = \Phi_N^T y_N$$

假设上面的矩阵Φ_N满足如下性质

$$\det(\Phi_N^T \Phi_N) \neq 0$$

解上面正规方程可以得到系统参数θ的估计值$\hat{\theta}$

$$\hat{\theta} = (\Phi_N^T \Phi_N)^{-1} \Phi_N^T y_N$$

2.2. 性质

在下面的假设下，讨论静态线性系统最小二乘估计的性质

- 假设噪声信号$n(k)$是零均值，不相关，方差相等(假设为σ^2).
- 假设Φ_N是列满秩的，即$\operatorname{rank}(\Phi_N) = n + 1$.

2.2.1. 参数估计性质

性质 . 估计误差的线性性

参数估计误差是扰动$n(k)$的线性函数

$$\tilde{\theta} = \hat{\theta} - \theta = (\Phi_N^T\Phi)^{-1}\Phi_N^T(\Phi_N\theta + n_N) - \theta = (\Phi_N^T\Phi)^{-1}\Phi_N^T n_N$$

性质 . 无偏性

$$\begin{aligned}
\hat{\theta} &= (\Phi_N^T\Phi_N)^{-1}\Phi_N^T y_N \\
&= (\Phi_N^T\Phi_N)^{-1}\Phi_N^T(\Phi_N\theta + n_N) \\
&= \theta + (\Phi_N^T\Phi_N)^{-1}\Phi_N^T n_N \\
&= \theta + \left(\frac{1}{N}\Phi_N^T\Phi_N\right)^{-1}\left(\frac{1}{N}\Phi_N^T n_N\right) \\
&= \theta + \left(\overline{\phi_k\phi_k^T}\right) + \left(\overline{\phi_k n(k)}\right)
\end{aligned}$$

$$\begin{aligned}
E\{\hat{\theta}\} &= E\{(\Phi_N^T\Phi_N)^{-1}\Phi_N^T y_N\} \\
&= E\{(\Phi_N^T\Phi_N)^{-1}\Phi_N^T(\Phi_N\theta + n_N)\} \\
&= \theta
\end{aligned}$$

性质 . 协方差

$$\begin{aligned}
D\{\hat{\theta}\} &= E\{(\hat{\theta} - E\{\hat{\theta}\})(\hat{\theta} - E\{\hat{\theta}\})^T\} \\
&= E\{(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T\} \\
&= E\{\tilde{\theta}\tilde{\theta}^T\} \\
&= E\{(\Phi_N^T\Phi_N)^{-1}\Phi_N^T n_N n_N^T\Phi_N(\Phi_N^T\Phi_N)^{-1}\} \\
&= (\Phi_N^T\Phi_N)^{-1}\Phi_N^T E\{n_N n_N^T\}\Phi_N(\Phi_N^T\Phi_N)^{-1} \\
&= \sigma^2(\Phi_N^T\Phi_N)^{-1}
\end{aligned}$$

性质 . 均方误差(Mean Squared Error, MSE)

由于Φ_N是列满秩的，因此$\Phi_N^T\Phi_N$是正定对称矩阵，所以存在正交矩阵P使得$P^T\Phi_N^T\Phi_N P = \mathrm{diag}\left(\lambda_i\right)$，其中$\lambda_i(i=0,1,\ldots,n)$是矩阵$\Phi_N^T\Phi_N$的特征根。所以均方误差

$$\begin{aligned}
\mathrm{MSE} &= \frac{1}{n+1}\sum_{i=0}^{n} E\{(\theta_i - \hat{\theta}_i)^2\} \\
&= E\{\tilde{\theta}^T\tilde{\theta}\} \\
&= \mathrm{tr}(E\{\tilde{\theta}\tilde{\theta}^T\}) \\
&= \sigma^2 \mathrm{tr}((\Phi_N^T\Phi_N)^{-1}) \\
&= \sigma^2 \mathrm{tr}((\Phi_N^T\Phi_N)^{-1} P P^T) \\
&= \sigma^2 \mathrm{tr}((P^T\Phi_N^T\Phi_N P)^{-1}) \\
&= \sigma^2 \sum_{i=0}^{n} \lambda_i^{-1}
\end{aligned}$$

性质 . 方差最小性

$\hat{\theta}$是θ的方差最小的线性无偏估计

性质 .噪声方差的估计

噪声的方差$\hat{\sigma^2}$的一个无偏估计是

$$\hat{\sigma^2} = \frac{1}{N-n-1} J(\hat{\theta})$$

2.2.1. 预测性质

性质 . 残差

$$e_N = -\tilde{y_N} = y_N - \Phi_N\hat{\theta} = [I - \Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T]n_N$$

性质 . 无偏性

$$E\{\hat{y_N}\} = y_N$$

2.3. 几何解释

2.4. 讨论

2.4.1 一般情况

3. 病态方程

Appendix

A.1. 矩阵求导

A.2. 矩阵特征根的性质

A.3. 矩阵对角化

A.4. 协方差与均方误差

1. 分布律或分布参数不随时间变化的随机信号. ↩
2. 这里的零均值假设并不是特别强，因为总能通过调整参数的常数项θ_0的大小来满足. ↩
3. Isermann, Rolf, and Marco Münchhof. Identification of dynamic systems: an introduction with applications. Springer Science & Business Media, 2010:pp.204. ↩