Modelling Dependence with Copulas

An Introduction

Giovanni Della Lunga

WORKSHOP IN QUANTITATIVE FINANCE

Bologna - May 3-4, 2018

Outline

1. Introduction
2. Joint Distributions
3. Copulas
   - Survival Copula
   - Density
4. Dependence Concepts
   - Linear Correlation Coefficient
   - Concordance
   - Kendall's tau and Spearman's rho
5. Copula Families
   - Gaussian copulas
   - t-copulas
   - Archimedean copulas
6. Tail Dependence

Introduction

Introduction

- Many real-life situations can be modelled by a large number of random variables which play a significant role, and such variates are generally not independent.
- Therefore, it is often of fundamental importance to be able to link the marginal distributions of different variables in order to give a flexible and accurate descrip- tion of the joint law of the variables of interest.
- Copulas were introduced in 1959 in the context of probabilistic metric spaces and later exploited as a tool for understanding relationships among multivariate outcomes.
- A copula is a function that links univariate marginals to their joint multivariate distribution in such a way that it captures the entire dependence structure in the multivariate distribution.

Introduction

- The main advantage provided by a copula-approach in dependence modelling is that the selection of an appropriate model for the dependence between variables X and Y , represented by the copula, can proceed independently from the choice of the marginal distributions.
- The seminal result in the history of copulas is due to Sklar that introduced in 1959 the notion, and the name, of copula, and proved the theo- rem that now bears his name (Sklar, 1959).
- The latter states that any multivariate distribution can be expressed as its copula function evaluated at its marginal distribution functions.
- Moreover, any copula function when evaluated at any marginal distributions is a multivariate distribution.

Introduction

- This presentation is so organized: in section 2 we recall some basic concepts from multivariate distributions theory, after Section 3 in which we define the concept of copula in full generality, we turn in Section 4 to an overview of the most important notions of dependence used in IRM. Section 5 introduces the most important families of copulas, their properties both methodological as well as with respect to simulation. Finally in Section 6 we discuss the concept of tail dependence.
- I would like to stress that this presentation only gives a first introduction, topics not included are statistical estimation of copulas and the modelling of dependence, through copulas, in a dynamic environment.

The Problem (Embrechts, 2009)

- A problem well known by all those who deal with Risk Management is the following: "Here we are given a multi-dimensional (i.e. several risk factors) problem for which we have marginal information together with an idea of dependence".
- Now the question is: **When is this problem well-posed?**

The Problem (Embrechts, 2009)

- One concrete question could be this: given two marginal, one-period risk factors X_1 ed X_2 with lognormal distribution functions (dfs) $F_1 = LN(\mu = 0, \sigma = 1)$ and $F_1 = LN(\mu = 0, \sigma = 4)$. How can we simulate from such a model if X_1 and X_2 have linear correlation $\rho = 0.5$ say?
- First of all, the correlation information say something (but what?) about the bivariate df of the random vectors $(X_1, X_2)^T$, i.e. about $F(x_1, x_2) = Prob[x_1 \leq X_1, x_2 \leq X_2]$;
- Note however that, in the above, that information is **not** given; we **only** know F_1, F_2 e ρ;
- What else would one need?

The easy Copula argument

- First note that for random variables (rvs) X_i with continuous dfs F_i, le variabili $U_i = F_i(X_i)$ are uniformly distributed rvs on $[0, 1]$;
- Hence for some joint df F with marginal dfs F_1 e F_2 we have:

$$\begin{aligned} F(X_1, X_2) &= P(X_1 \leq x_1, X_2 \leq x_2) \\ &= P(F_1(X_1) \leq F_1(x_1), F_1(X_2) \leq F_1(x_2)) \\ &= P(U_1 \leq F_1(x_1), U_2 \leq F_2(x_2)) \\ &:= C(F_1(x_1), F_2(x_2)) \end{aligned} \tag{1}$$

- The function C above is exactly *a* (careful here!) **copula**, a df on $I^2 = [0, 1] \times [0, 1]$ with standard uniform marginals, C is the df of the random vector $(U_1, U_2)^T$.

The easy Copula argument

- If we return to our lognormal example, we see no immediate reason how the number ρ should determine the function C, it is not even clear whether the problem has none, one or infinitely many solutions;
- In this case, it turns out that the problem posed has **no** solution.
- The time has come to move on to some definition ...

Joint Distributions

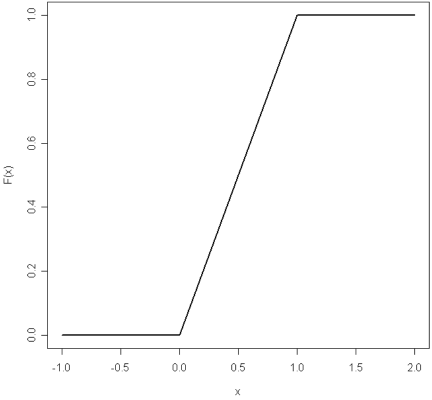
Distribution Function

- Lets take a random real value array $X = (X_1, \dots, X_d)$ we can define a multivariate distribution function F:

$$F(\mathbf{x}) = Prob[X_1 \leq x_1, \dots, X_d \leq x_d]$$

- In general a distribution function F defined from $\mathbb{R}$ in I, must meet the following conditions:
  - $F(x) \geq 0 \quad \forall x$
  - $\lim_{x_i \to +\infty} F(x_1, \dots, x_d) = 1 \quad \forall x_i$
  - $\lim_{x_i \to -\infty} F(x_1, \dots, x_d) = 0 \quad \forall x_i$

Distribution Function



- Let **X** be a uniform random variable $[0, 1]$, the distribution function is:

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

Joint Cumulative Distribution Function

Remember that the **joint cumulative function** of two random variables X and Y is defined as

$$F_{XY}(x, y) = \mathbb{P}(X \leq x, Y \leq y) \tag{2}$$

The joint CDF satisfies the following properties:

1. $F_X(x) = F_{XY}(x, \infty)$, for any x (marginal CDF of X);
2. $F_Y(y) = F_{XY}(\infty, y)$, for any y (marginal CDF of Y);
3. $F_{XY}(\infty, \infty) = 1$;
4. $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0$;
5. $\mathbb{P}(x_1 < X \leq x_2, \;\; y_1 < Y \leq y_2) =$ $F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$;
6. if X and Y are independent, then $F_{XY}(x, y) = F_X(x)F_Y(y)$.

Joint Cumulative Distribution Function

In particular from property 5, putting $x_2 \to +\infty$ and $y_2 \to +\infty$ we have

$$\begin{aligned}
\mathbb{P}(x_1 < X \leq +\infty,\ \ y_1 < Y \leq +\infty) &= F_{XY}(+\infty, +\infty) - F_{XY}(x_1, +\infty) \\
&\quad - F_{XY}(+\infty, y_1) + F_{XY}(x_1, y_1) \\
&= 1 - F_X(x) - F_Y(y) + F_{XY}(x_1, y_1)
\end{aligned} \tag{3}$$

If we denote with $\bar{F}_{XY}(x, y) = \mathbb{P}[X > x, Y > y]$, we finally obtain

$$\bar{F}_{XY}(x, y) = 1 - F_X(x) - F_Y(y) + F_{XY}(x, y) \tag{4}$$

Copulas

Copulas

- The concept of **copula** arises from the idea of breaking down a multivariate distribution F into components allow one to easily model and estimate the distribution of random vectors by estimating marginals and dependency structure separately;
- The importance of the copula functions derives entirely from a noteworthy result known in the literature as **Sklar's theorem** that states that any multivariate joint distribution can be written in terms of univariate marginal distribution functions and a copula which describes the dependence structure between the variables;
- Before discussing the theorem we are gong to describe some of the fundamental properties of copula functions limiting (for simplicity) to the bivariate case and recalling our *definition* of copula function as a bi-variate distribution of two marginally uniform variables ...

Copulas

- According to our previous definition

$$C(x,y) = P(U_1 \leq x, U_2 \leq y) \tag{5}$$

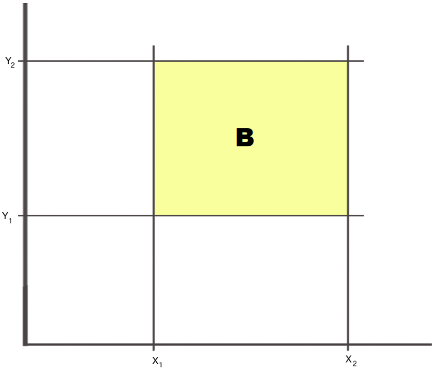
- From the above equation we can write:

$$\begin{aligned} C(x,0) &= P(U_1 \leq x, U_2 \leq 0) = 0 \\ C(0,y) &= P(U_1 \leq 0, U_2 \leq y) = 0 \\ C(x,1) &= P(U_1 \leq x, U_2 \leq 1) = P(U_1 \leq x) = x \\ C(1,y) &= P(U_1 \leq 1, U_2 \leq y) = P(U_2 \leq y) = y \end{aligned} \tag{6}$$

- in particular from the last two relations of (6) we obtain

$$C(x,y) \leq C(x,1) = x \quad C(x,y) \leq C(1,y) = y \Rightarrow C(x,y) \leq \min(x,y) \tag{7}$$

Copulas



- The volume of the square in the picture is clearly equal to

$$V([x_1, x_2] \times [y_1, y_2]) = (x_2 - x_1) \cdot (y_2 - y_1) = x_2 y_2 - x_2 y_1 - x_1 y_2 + x_1 y_1 \quad (8)$$

- Generaly speaking, for each function H we can define a generalized-volume or *H-Volume* by:

$$V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1) \quad (9)$$

Copulas

- For the C-Volume to be considered as a probability, it must be always positive definite

$$C(x_2, y_2) - C(x_2, y_1) - C(x_1, y_2) + C(x_1, y_1) \geq 0 \qquad (10)$$

- This must be true however the four values x_i, y_i are chosen, in particular if we choose $(x_2, y_2) = (x, y)$ e $(x_1, y_1) = (1, 1)$ we have

$$C(x, y) - C(x, 1) - C(1, y) + C(1, 1) \geq 0 \qquad (11)$$

- Remember that $C(x, 1) = x$ e $C(1, y) = y$, so we have $C(x, y) \geq x + y - 1$ and $C(x, y) \geq 0$ then:

$$C(x, y) \geq \max(x + y - 1, 0) \qquad (12)$$

Fréchet Limits

- Putting together (12) and (7) we have

$$\max(x+y-1,0) \leq C(x,y) \leq \min(x,y) \tag{13}$$

- This is a special case of one of the most important results of multivariate statistics the**Fréchet- Hoeffding Theorem**.

Theorem (Fréchet-Hoeffding Bounds)

Suppose $F_1,\dots,F_d$ are marginal dfs and F any joint df with those given marginals, then $\forall \mathbf{x} \in \mathbb{R}^d$,

$$\left(\sum_{k=1}^{d} F_k(x_k) + 1 - d\right)^+ \leq F(\mathbf{x}) \leq \min(F_1(x_1),\dots,F_d(x_d)) \tag{14}$$

Fréchet Limits

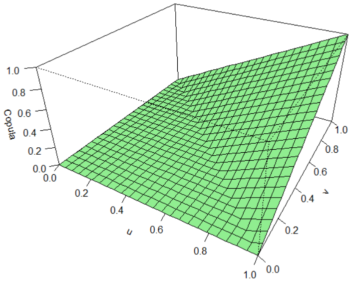


Figure: a) Minimum Copula

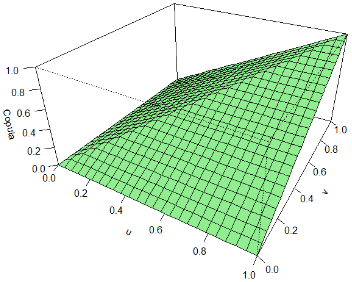


Figure: b) Maximum Copula

Sklar's Theorem

- The key to link copula functions and distribution functions is represented by the **Sklar Theorem**:

Theorem (A. Sklar, 1959)

Suppose $X_1, \ldots, X_d$ are rvs with continuous dfs $F_1, \ldots, F_d$ and joint df F, then there exists a unique copula C (a df on $[0,1]^d$ with uniform marginals) such that for all $\mathbf{x} \in \mathbb{R}^d$:

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \tag{15}$$

Conversely given any dfs $F_1, \ldots, F_d$ and copula C, F defined through (15) is a d-variate df with marginal dfs $F_1, \ldots, F_d$.

Sklar's Theorem

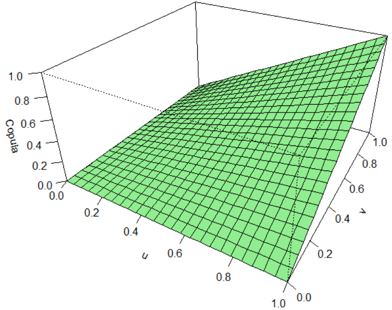
- The theorem guarantees that for continuous random variables, the univariate margins and the multivariate dependence structure can be univocally separated and that the copula fully describes the dependence structure;
- The theorem can be reversed, so that the copula can be described in terms of a joint distribution function and quantile functions of the marginal ones.

Corollary

Let be $F, F_1, \ldots, F_d$ and C the same as in Sklar Theorem and $F_1^{-1}, \ldots, F_d^{-1}$ the quantile functions of $F_1, \ldots, F_d$ then we can write:

$$C(\mathbf{u}) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)) \tag{16}$$

Independence



- Also the concept of independence between random variables can be represented in terms of copulas;
- In terms of copulas, if the random variables are independent, the functional that describes the link between the marginal distribution functions and the joint one is the product copula, conversely if some variables are associated with the product copula then they are independent

Survival Copula

- The survival copula associated with the copula C is

$$\bar{C}(u,v) = u + v - 1 + C(1-u, 1-v) \tag{17}$$

- It is easy to verify that $\bar{C}$ ha the copula properties. Once computed in $(1-u, 1-v)$ is equivalent to the complementary distribution function of a bivariate uniform distribution, since

$$\begin{aligned}\bar{C}(1-u,1-v) &= 1 - u + 1 - v - 1 + C(u,v) \\ &= 1 - \mathbb{P}(U_1 \le u) + 1 - \mathbb{P}(U_2 \le v) - 1 \\ &+ \mathbb{P}(U_1 \le u, U_2 \le v) \\ &= 1 - \mathbb{P}(U_1 \le u) - \mathbb{P}(U_2 \le v) + \mathbb{P}(U_1 \le u, U_2 \le v) \\ &= \mathbb{P}(U_1 > u, U_2 > v)\end{aligned} \tag{18}$$

Density

- If C is absolutely continuous then it can be written in the form

$$C(\mathbf{u}) = \int_{[0,\mathbf{u}]^d} c(\mathbf{s})\, d\mathbf{s} \tag{19}$$

where c is a suitable function called *density* of C.

- In particular, for almost all $\mathbf{u} \in \mathbb{I}^d$ one has

$$c(\mathbf{u}) = \frac{\partial^d C(\mathbf{u})}{\partial u_1 \ldots \partial u_d} \tag{20}$$

- As stressed by many authors, this equation is far from obvious. In fact, there are some facts that are implicitly used: first, the mixed partial derivatives of order d of C exist and are equal almost everywhere on $\mathbb{I}^d$; second each mixed partial derivative is actually almost everywhere equal to the density c.

Density

- We will use this result to formally define a measure induced on $\mathbb{I}^d$ by C.

$$dC(\mathbf{u}) = c(\mathbf{u})\, d\mathbf{u} \tag{21}$$

- In particular for a bivariate copula we have

$$dC(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v} dudv \tag{22}$$

Copula and Measure: Example

- Assume that C is absolutely continuous. The following identity is true

$$\iint_{[0,1]^2} uv dC(u,v) = \iint_{[0,1]^2} C(u,v)\, dudv \tag{23}$$

- If C is absolutely continuous then we can write

$$dC(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v} dudv \tag{24}$$

Sostitution of this definition in the left hand member and evaluating the inner integrals by parts give us...

Copula and Measure: Example

$$
\begin{aligned}
\int_0^1\int_0^1 uv\,dC(u,v) &= \int_0^1\int_0^1 uv\frac{\partial^2 C(u,v)}{\partial u\partial v}dudv = \int_0^1 du\; u\int_0^1 dv\; v\frac{\partial^2 C(u,v)}{\partial u\partial v}\\
&= \int_0^1 du\; u\left(\left[v\frac{\partial C(u,v)}{\partial u}\right]_{v=0}^{v=1} - \int_0^1 \frac{\partial C(u,v)}{\partial u}\right) = \int_0^1 du\; u\left(1-\int_0^1\frac{\partial C(u,v)}{\partial u}\right)\\
&= \int_0^1 du\; u - \int_0^1 dv\int_0^1 du\; u\frac{\partial C(u,v)}{\partial u} = \int_0^1 du\; u - \int_0^1 dv\left(\Big[uC(u,v)\Big]_{u=0}^{u=1} + \int_0^1 C(u,v)\,du\right)\\
&= \int_0^1 du\; u - \int_0^1 dv\; v + \int_0^1\int_0^1 C(u,v)\,dudv = \int_0^1\int_0^1 C(u,v)\,dudv
\end{aligned}
\tag{25}
$$

Copula and Measure: Example

Exercise Compute the following expression

$$\int_0^1 \int_0^1 C(u, v) \frac{\partial^2 C(u, v)}{\partial u \partial v} \, du \, dv \tag{26}$$

Copula and Measure: Example

Answer Evaluate the inner integral by parts

$$\begin{aligned} &\int_0^1 C(u,v)\frac{\partial^2}{\partial u\partial v}C(u,v)\,du \\ &= C(u,v)\frac{\partial}{\partial v}C(u,v)\Big|_{u=0}^{u=1} - \int_0^1 \frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v)\,du \\ &= v - \int_0^1 \frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v)\,du \end{aligned} \tag{27}$$

performing the second integral we have

$$\begin{aligned} &\int_0^1\int_0^1 C(u,v)\frac{\partial^2}{\partial u\partial v}C(u,v)\,dudv \\ &= \frac{1}{2} - \int_0^1\int_0^1 \frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v)\,dudv. \end{aligned} \tag{28}$$

Dependence Concepts

Dependence Measures

- Copulas provide a natural way to study and measure dependence between random variables.
- Copula properties are invariant under strictly increasing transformations of the underlying random variables.
- Linear correlation (or Pearson's correlation) is most frequently used in practice as a measure of dependence. However, since linear correlation is not a copula-based measure of dependence, it can often be quite misleading and should not be taken as the canonical dependence measure. In the next slides we recall the basic properties of linear correlation, and then continue with some copula based measures of dependence.

Linear Correlation Coefficient

Definition. Let $(X, Y)^T$ be a vector of random variables with nonzero finite variances. The linear correlation coefficient for $(X, Y)^T$ is

$$\rho(X, Y) = \frac{Cov(X, Y}{\sqrt{Var(X)}\sqrt{Var(Y)}} \tag{29}$$

where $Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ is the covariance of $(X, Y)^T$, and $Var(X)$ and $Var(Y)$ are the variances of X and Y.

Linear Correlation Coefficient

- Linear correlation is a popular but also often misunderstood measure of dependence.
- The popularity of linear correlation stems from the ease with which it can be calculated and it is a natural scalar measure of dependence in elliptical distributions (with well known members such as the multivariate normal and the multivariate t- distribution).
- However most random variables are not jointly elliptically distributed, and using linear correlation as a measure of dependence in such situations might prove very misleading.
- Even for jointly elliptically distributed random variables there are situations where using linear correlation does not make sense. We might choose to model some scenario using heavy-tailed distributions such as t2-distributions. In such cases the linear correlation coefficient is not even defined because of infinite second moments.

Linear Correlation Coefficient

Property. $\rho(X, Y)$ is bounded:

$$\rho_l \leq \rho \leq \rho_u$$

where the bounds ρ_l and ρ_u are defined as

$$\begin{aligned}
\rho_l &= \frac{\iint_D \left[C^-(F_1(x), F_2(y)) - F_1(x)F_2(y)\right] dx\, dy}{\sqrt{\left[\int_{Dom(F_1)}(x - \mathbb{E}(x))^2 dF_1(x)\right]\left[\int_{Dom(F_2)}(y - \mathbb{E}(y))^2 dF_2(y)\right]}} \\
\rho_u &= \frac{\iint_D \left[C^+(F_1(x), F_2(y)) - F_1(x)F_2(y)\right] dx\, dy}{\sqrt{\left[\int_{Dom(F_1)}(x - \mathbb{E}(x))^2 dF_1(x)\right]\left[\int_{Dom(F_2)}(y - \mathbb{E}(y))^2 dF_2(y)\right]}}
\end{aligned} \tag{30}$$

and are attained respectively when X and Y are countermonotonic and comonotonic.

Linear Correlation Coefficient

Proof. The bound for $\rho(X, Y)$ can be obtained directly from Hoeffding's expression for covariance together with the Fréchet inequality.

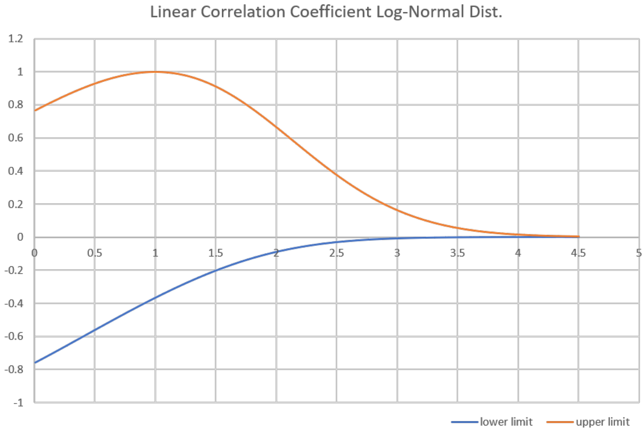
Example. Let $X \sim LN(0, \sigma_1^2)$ and $Y \sim LN(0, \sigma_2^2)$. Then $\rho_{min} = \rho(e^{\sigma_1 Z}, e^{-\sigma_2 Z})$ and $\rho_{max} = \rho(e^{\sigma_1 Z}, e^{\sigma_2 Z})$, where $Z \sim N(0, 1)$. ρ_{min} and ρ_{max} can be computed yielding:

$$\begin{aligned} \rho_l &= \frac{exp(-\sigma_1\sigma_2) - 1}{\sqrt{(exp(\sigma_1^2) - 1)}\sqrt{(exp(\sigma_2^2) - 1)}} \leq 0 \\ \rho_u &= \frac{exp(\sigma_1\sigma_2) - 1}{\sqrt{(exp(\sigma_1^2) - 1)}\sqrt{(exp(\sigma_2^2) - 1)}} \geq 0 \end{aligned} \tag{31}$$

Linear Correlation Coefficient

Note that

$$\lim_{\sigma\to\infty}\rho_{min} = \lim_{\sigma\to\infty}\rho_{max} = 0$$



Concordance

- Concordance concepts, loosely speaking, aim at capturing the fact that the probability of having "large" (or "small") values of both X and Y is high, while the probability of having "large" values of X together with "small" values of "Y" - or viceversa - is low.
- Let $(x, y)^T$ and $(\tilde{x}, \tilde{y})^T$ be two observations from a vector $(X, Y)^T$ of continuous random variables.
- Then $(x, y)^T$ and $(\tilde{x}, \tilde{y})^T$ are said to be concordant if $(x - \tilde{x})(y - \tilde{y}) > 0$, and discordant if $(x - \tilde{x})(y - \tilde{y}) < 0$.
- The following theorem can be found in Nelsen (1999) p. 127. Many of the results in this section are direct consequences of this theorem.

Concordance

Theorem

Let $(X, Y)^T$ and $(\tilde{X}, \tilde{Y})^T$ be independent vectors of continuous random variables with joint distribution functions H and $\tilde{H}$, respectively, with common margins F (of X and $\tilde{X}$) and G (of Y and $\tilde{Y}$). Let C and $\tilde{C}$ denote the copulas of $(X, Y)^T$ and $(\tilde{X}, \tilde{Y})^T$ respectively,so that $H(x, y) = C(F(x), G(y))$ and $\tilde{H}(x, y) = \tilde{C}(F(x), G(y))$. Let Q denote the difference between the probability of concordance and discordance of $(X, Y)^T$ and $(\tilde{X}, \tilde{Y})^T$, i.e. let

$$Q = \mathbb{P}[(X - \tilde{X})(Y - \tilde{Y}) > 0] - \mathbb{P}[(X - \tilde{X})(Y - \tilde{Y}) < 0] \tag{32}$$

Then

$$Q = Q(C, \tilde{C}) = 4 \iint_{[0,1]^2} \tilde{C}(u, v) dC(u, v) - 1 \tag{33}$$

Concordance

Proof. Since the random variables are all continuous,

$\mathbb{P}[(X-\tilde{X})(Y-\tilde{Y})<0]=1-\mathbb{P}[(X-\tilde{X})(Y-\tilde{Y})>0] \Rightarrow Q=2\mathbb{P}[(X-\tilde{X})(Y-\tilde{Y})>0]-1$

But

$$\mathbb{P}[(X-\tilde{X})(Y-\tilde{Y})>0]=\mathbb{P}[X>\tilde{X}, Y>\tilde{Y}]+\mathbb{P}[X<\tilde{X}, Y<\tilde{Y}]$$

and these probabilities can be evaluated by integrating over the distribution of one of the vectors $(X, Y)^T$ or $(\tilde{X}, \tilde{Y})^T$. Hence

$$\begin{aligned}
\mathbb{P}[X>\tilde{X}, Y>\tilde{Y}] &= \mathbb{P}[\tilde{X}<X, \tilde{Y}<Y] \\
&= \iint_{\mathbb{R}^2} \mathbb{P}[\tilde{X}<x, \tilde{Y}<y]\, dC[F(x), G(y)] \\
&= \iint_{\mathbb{R}^2} \tilde{C}[F(x), G(y)]\, dC[F(x), G(y)]
\end{aligned} \tag{34}$$

Concordance

Employing the probability-integral transform $u = F(x)$ and $v = G(y)$ then yields

$$\mathbb{P}[X > \tilde{X}, Y > \tilde{Y}] = \mathbb{P}[\tilde{X} < X, \tilde{Y} < Y] = \iint_{[0,1]^2} \tilde{C}(u, v)\, dC(u, v) \tag{35}$$

Similarly,

$$\begin{aligned}
\mathbb{P}[X < \tilde{X}, Y < \tilde{Y}] &= \iint_{\mathbb{R}^2} \mathbb{P}[\tilde{X} > x, \tilde{Y} > y]\, dC[F(x), G(y)] \\
&= \iint_{\mathbb{R}^2} \{1 - F(x) - G(y) + \tilde{C}[F(x), G(y)]\}\, dC[F(x), G(y)] \\
&= \iint_{[0,1]^2} \{1 - u - v + \tilde{C}(u, v)\}\, dC(u, v)
\end{aligned} \tag{36}$$

Concordance

But since C is the joint distribution function of a vector $(U, V)^T$ of $U(0,1)$ random variables, $\mathbb{E}(U) = \mathbb{E}(V) = 1/2$, and hence

$$\mathbb{P}[X < \tilde{X}, Y < \tilde{Y}] = 1 - \frac{1}{2} - \frac{1}{2} + \iint_{[0,1]^2} \tilde{C}(u,v)\, dC(u,v) = \iint_{[0,1]^2} \tilde{C}(u,v)\, dC(u,v) \quad (37)$$

Thus

$$\mathbb{P}[(X - \tilde{X})(Y - \tilde{Y}) > 0] = 2 \iint_{[0,1]^2} \tilde{C}(u,v)\, dC(u,v) \quad (38)$$

and the conclusion follows

$$Q = 2\mathbb{P}[(X - \tilde{X})(Y - \tilde{Y}) > 0] - 1 = 4 \iint_{[0,1]^2} \tilde{C}(u,v)\, dC(u,v) - 1 \quad (39)$$

Kendall's tau and Spearman's rho

Definition. Kendall's tau for the random vector $(X, Y)^T$ is defined as

$$\tau(X, Y) = \mathbb{P}[(X - \tilde{X})(Y - \tilde{Y}) > 0] - \mathbb{P}[(X - \tilde{X})(Y - \tilde{Y}) < 0] \quad (40)$$

where $(\tilde{X}, \tilde{Y})^T$ is an independent copy of $(X, Y)^T$. Hence Kendall's tau for $(X, Y)^T$ is simply the probability of concordance minus the probability of discordance and since the copula of $(\tilde{X}, \tilde{Y})^T$ is the same of $(X, Y)^T$ is also simply equal to $Q(C, C)$:

Kendall's tau and Spearman's rho

Theorem. Let $(X, Y)^T$ be a vector of continuous random variables with copula C. Then Kendall's tau for $(X, Y)^T$ is given by

$$\tau(X, Y) = Q(C, C) = 4 \iint_{[0,1]^2} C(u, v)\, dC(u, v) - 1 \tag{41}$$

Note that the integral above is the expected value of the random variable $C(U, V)$, where $U, V \sim U(0, 1)$ with joint distribution function C, i.e. $\tau = 4\mathbb{E}(C(U, V)) - 1$.

Kendall's tau and Spearman's rho

- **Definition**. Spearman's rho for the random vector $(X, Y)^T$ is defined as

$$\rho_S(X, Y) = 3(\mathbb{P}[(X - \tilde{X})(Y - Y') > 0] - \mathbb{P}[(X - \tilde{X})(Y - Y') < 0]) \tag{42}$$

where $(X, Y)^T$, $(\tilde{X}, \tilde{Y})^T$ and $(X', Y')^T$ are **independent** copies.

- **Theorem**. Let $(X, Y)^T$ be a vector of continuous random variables with copula C. Then Spearman's rho for $(X, Y)^T$ is given by (note that $\tilde{X}$ and Y' are independent so their copula is the product copula)

$$\begin{aligned}\rho_S(X, Y) &= 3Q(C, \Pi) = 12 \iint_{[0,1]^2} uv\, dC(u, v) - 3 = 12 \iint_{[0,1]^2} C(u, v)\, du\, dv - 3 \\ &= \frac{\mathbb{E}(UV) - 1/4}{1/12} = \frac{Cov(U, V)}{\sqrt{Var(U)}\sqrt{Var(V)}} \\ &= \rho[F(X), G(Y)]\end{aligned} \tag{43}$$

Copula Families

Elliptical Copulas

- The class of elliptical distributions provides a rich source of multivariate distributions which share many of the tractable properties of the multivariate normal distribution and enables modelling of multivariate extremes and other forms of nonnormal dependences.
- Elliptical copulas are simply the copulas of elliptical distributions.
- Simulation from elliptical distributions is easy, and as a consequence of Sklar's Theorem so is simulation from elliptical copulas.
- Furthermore, we will show that rank correlation and tail dependence coefficients can be easily calculated

Gaussian Copula

- The copula of the *n-variate* normal distribution with linear correlation matrix ρ is

$$C_\rho^{Ga}(u_1, \ldots, u_2) = \mathbf{\Phi}_\rho^d(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_d))$$

  where $\mathbf{\Phi}_\rho^d$ denotes the joint distribution function of the n-variate standard normal distribution function with linear correlation matrix ρ, and ϕ^{-1} denotes the inverse of the distribution function of the univariate standard normal distribution.
- Copulas of the above form are called Gaussian copulas.

Gaussian Copula

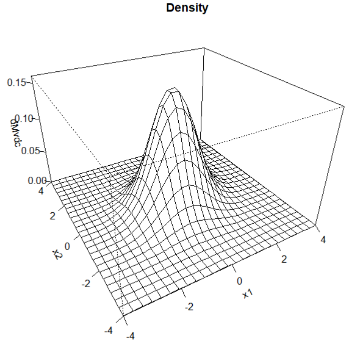
- We now address the question of random variate generation from the Gaussian copula C_ρ^{Ga}.
- For our purpose, it is sufficient to consider only strictly positive definite matrices Σ. Write $\Sigma = AA^T$ for some $n \times n$ matrix A, and if $Z_1, \ldots, Z_n \sim N(0,1)$ are independent, then $\mu + AZ \sim N_n(\mu, \Sigma)$.
- One natural choice of A is the Cholesky decomposition of Σ. The Cholesky decomposition of Σ is the unique lower-triangular matrix L with $LL^T = \Sigma$. Furthermore Cholesky decomposition routines are implemented in most mathematical software.
- This provides an easy algorithm for random variate generation from the Gaussian n-copula C_ρ^{Ga}.

Gaussian Copula: Simulation Algorithm

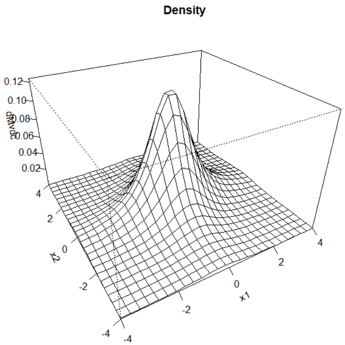
- Find the Cholesky decomposition A of Σ.
- Simulate n independent random variates $z_1, \ldots, z_n$ from $N(0, 1)$.
- Set $x = Az$.
- Set $u_i = \Phi(x_i), \quad i = 1, \ldots, n$.
- $(u_1, \ldots, u_n)^T \sim C_\rho^{Ga}$

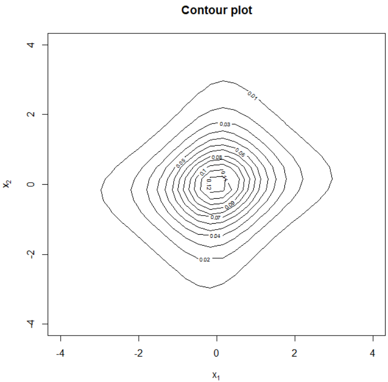
As usual Φ denotes the univariate standard normal distribution function.

Gaussian Copula with Gaussian Marginals ($\rho = 0.2$)

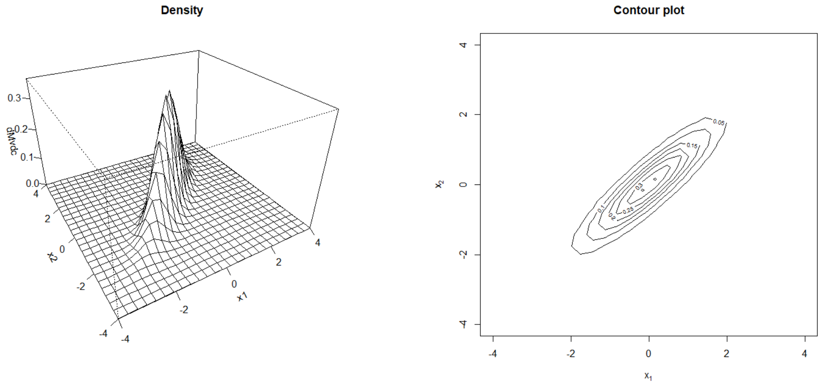


Gaussian Copula with t_ν Marginals ($\rho = 0.2$)

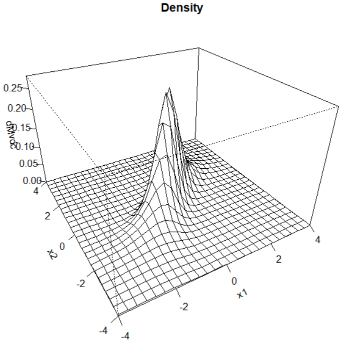


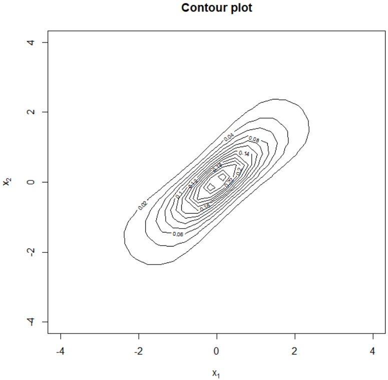


Gaussian Copula with Gaussian Marginals ($\rho = 0.9$)



Gaussian Copula with t_ν Marginals ($\rho = 0.9$)





t copula

- Similarly to the case of a normal distribution we can consider the distribution t_ν; the corresponding copula with correlation ρ is:

$$C^t_{\nu,\rho}(u_1, \dots, u_d) = \Theta^d_{\nu,\rho}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d)) \tag{44}$$

where $\Theta^d_{\nu,\rho}$ is the t multivariate distribution with dimension d, ν degree of freedom and linear correlation ρ while t_ν^{-1} is the inverse univariate standard t-distribution t_ν.

- Also for this functional you have a simple simulation algorithm based on the following result: If X has the stochastic representation

$$X = \frac{\sqrt{\nu}}{\sqrt{Z}} Y \quad \text{with} \quad Y \sim \mathcal{N}_d(0, \rho), Z \sim \chi^2_\nu$$

then X has an $n - variate$ t_ν-distribution with mean μ (for $\nu > 1$) and covariance matrix $\nu\Sigma/(\nu - 2)$ (for $\nu > 2$). If $\nu \leq 2$ then $Cov(X)$ is not defined. In this case we just interpret Σ as being the shape parameter of the distribution of X.

t copula: Simulation Algorithm

- Find the Cholesky decomposition A of Σ.
- Simulate n independent random variates $z_1, ..., z_n$ from $N(0, 1)$.
- Simulate a random variate s from χ^2_ν independent of $z_1, ..., z_n$.
- Set $x = Az$
- Set $y = \sqrt{\frac{n}{s}}x$
- Set $u_i = T_\nu(y_i)$ for $i = 1, \dots, n$
- $(u_1, ..., u_n)^T \sim C^t_{\nu,\Sigma}$.

Archimedean copulas

- In this section we discuss an important class of copulas called Archimedean copulas.
- They have proved to be useful in several applications since they are capable of capturing wide ranges of dependence structures.
- Furthermore, in contrast to elliptical copulas, all commonly encountered Archimedean copulas have closed form expressions.
- We divide the discussion of Archimedean copulas in two subsections: the first introduces them and their main properties while the second presents different one-parameter families of Archimedean copulas.

Archimedean copulas

- Archimedean copulas may be constructed using a function $\phi : \mathbb{I} \to [0, \infty]$, continuous, decreasing, convex and such that $\phi(1) = 0$.
- Such a function ϕ is called a generator. It is called a strict generator whenever $\phi(0) = +\infty$.
- The pseudo-inverse of ϕ is defined as follows:

$$\varphi^{[-1]}(v) = \begin{cases} \varphi^{-1}(v) & 0 \leq v \leq \varphi(0) \\ 0 & \varphi(0) \leq v \leq +\infty \end{cases}$$

Archimedean copulas

The pseudo-inverse is such that, by composition with the generator, it gives the identity, and it coincides with the usual inverse if ϕ is a strict generator.

Definition Given a generator and its pseudo-inverse, an Archimedean 2-copula takes the form

$$C(u,v) = \phi^{[-1]}\Big(\phi(u) + \phi(v)\Big) \tag{45}$$

If the generator is strict, the copula is said to be a strict Archimedean 2-copula.

Archimedean copulas

Selected Archimedean 2-copulas and their generators.

	$C(u,v)$	$\varphi_\alpha(t)$	range for α
Gumbel	$\exp\{-[(-\log u)^\alpha + (-\log v)^\alpha]\}$	$(-\log t)^\alpha$	$[1, +\infty)$
Clayton	$(u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}$	$\alpha^{-1}(t^{-\alpha} - 1)$	$(0, +\infty)$
Clayton*	$\max[(u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}, 0]$	$\alpha^{-1}(t^{-\alpha} - 1)$	$(-1, +\infty)\backslash\{0\}$
Frank	$-\frac{1}{\alpha}\log\left(1 + \frac{(\exp(-\alpha u)-1)(\exp(-\alpha v)-1)}{\exp(-\alpha)-1}\right)$	$-\log\frac{\exp(-\alpha t)-1}{\exp(-\alpha)-1}$	$(-\infty, \infty)\backslash\{0\}$

*For Clayton, the two cases correspond to strict and nonstrict generator, respectively.

Multivariate extensions can be obtained if restrictions are placed on the generator (see, *e.g.*, Durante & Sempi, 2010; Cherubini et al., 2004).

Archimedean copulas: Gumbel d-copula

The Gumbel family has been introduced by Gumbel (1960). Since it has been discussed in Hougaard (1986), it is also known as the Gumbel–Hougaard family. The standard expression for members of this family of d-copulas is

$$C(u_1, \ldots, u_d) = \exp\left(-\left(\sum_{i=1}^{d}(-\log(u_i))^{\alpha}\right)^{\frac{1}{\alpha}}\right), \alpha \geq 1.$$

The case $\alpha = 1$ gives the product copula as a special case, and the limit for $\alpha \to +\infty$ is the comonotonicity copula. It follows that the Gumbel family can represent independence and positive dependence only. The generator is given by

$$\phi_\alpha(u) = (-\log\ u)^{\alpha}, \quad \alpha \geq 1 \tag{46}$$

Archimedean copulas: Clayton d-copula

The Clayton family was first proposed by Clayton (1978), and studied by Oakes (1982). The standard expression for members of this family of d-copulas is

$$C(u_1, \ldots, u_d) = \left(\sum_{i=1}^{d} u_i^{-\alpha} - (d-1) \right)^{-\frac{1}{\alpha}} \alpha > 0.$$

The limiting case $\alpha = 0$ corresponds to the independence copula. The generator has the form

$$\phi_\alpha(u) = u^{-\alpha} - 1, \quad \alpha > 0 \tag{47}$$

Archimedean copulas: Frank d-copula

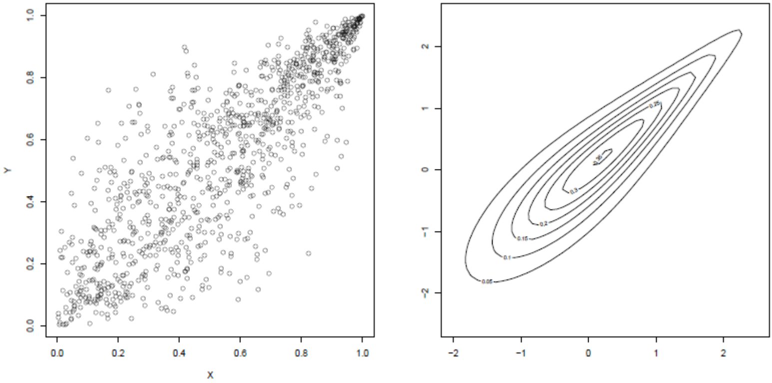
Copulas of this family have been introduced by Frank (1979), and have the expression:

$$C(u_1, \ldots, u_d) = -\frac{1}{\alpha} \log \left\{ 1 + \frac{\prod_{i=1}^{d} (e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{d-1}} \right\}, \alpha > 0.$$

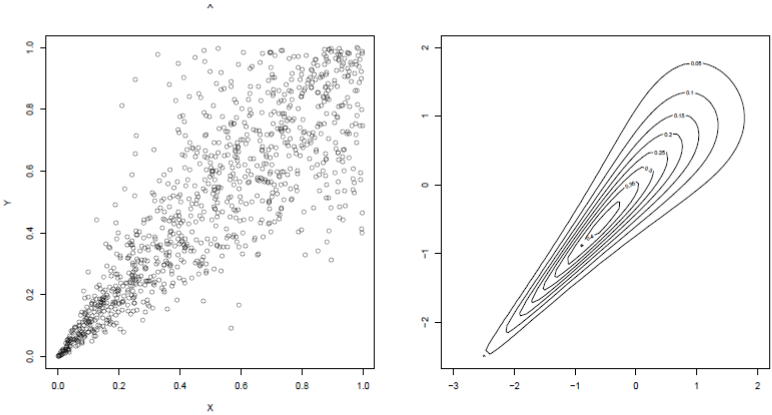
It reduces to the product copula if $\alpha = 0$. For the case $d = 2$, the parameter α can be extended also to the case $\alpha < 0$. The generator is given by

$$\varphi_\alpha(u) = -\log \left(\frac{e^{-\alpha u} - 1}{e^{-\alpha} - 1} \right), \alpha > 0.$$

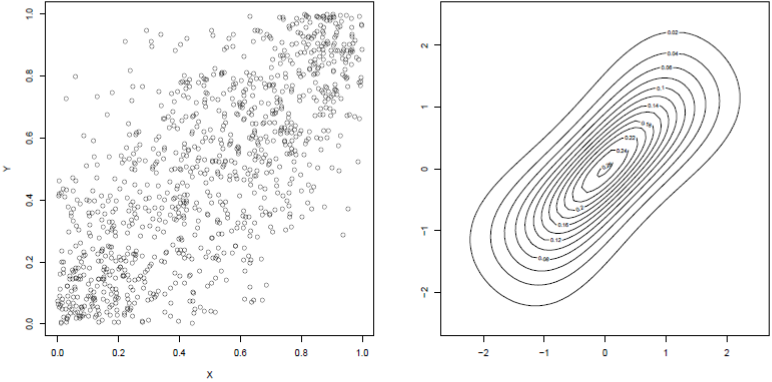
Bivariate sample from the Gumbel copula



Bivariate sample from the Clayton copula



Bivariate sample from the Frank copula



Tail Dependence

Tail Dependence

- The concept of tail dependence relates to the amount of dependence in the upper-right- quadrant tail or lower-left- quadrant tail of a bivariate distribution.
- It is a concept that is relevant for the study of dependence between extreme values.
- It turns out that tail dependence between two continuous random variables X and Y is a copula property and hence the amount of tail dependence is invariant under strictly increasing transformations of X and Y .

Tail Dependence

Definition 3.6. Let $(X,Y)^T$ be a vector of continuous random variables with marginal distribution functions F and G. The coefficient of upper tail dependence of $(X,Y)^T$ is

$$\lim_{u \nearrow 1} \mathbb{P}\{Y > G^{-1}(u) | X > F^{-1}(u)\} = \lambda_U$$

provided that the limit $\lambda_U \in [0,1]$ exists. If $\lambda_U \in (0,1]$, X and Y are said to be asymptotically dependent in the upper tail; if $\lambda_U = 0$, X and Y are said to be asymptotically independent in the upper tail. $\square$

Since $\mathbb{P}\{Y > G^{-1}(u) \mid X > F^{-1}(u)\}$ can be written as

$$\frac{1 - \mathbb{P}\{X \leq F^{-1}(u)\} - \mathbb{P}\{Y \leq G^{-1}(u)\} + \mathbb{P}\{X \leq F^{-1}(u), Y \leq G^{-1}(u)\}}{1 - \mathbb{P}\{X \leq F^{-1}(u)\}},$$

Tail Dependence

Definition 3.7. If a bivariate copula C is such that

$$\lim_{u \nearrow 1}(1 - 2u + C(u, u))/(1 - u) = \lambda_U$$

exists, then C has upper tail dependence if $\lambda_U \in (0, 1]$, and upper tail independence if $\lambda_U = 0$. □

Example 3.3. Consider the bivariate Gumbel family of copulas given by

$$C_\theta(u, v) = \exp(-[(-\ln u)^\theta + (-\ln v)^\theta]^{1/\theta}),$$

for $\theta \geq 1$. Then

$$\frac{1 - 2u + C(u, u)}{1 - u} = \frac{1 - 2u + \exp(2^{1/\theta} \ln u)}{1 - u} = \frac{1 - 2u + u^{2^{1/\theta}}}{1 - u},$$

and hence

$$\lim_{u \nearrow 1}(1 - 2u + C(u, u))/(1 - u) = 2 - \lim_{u \nearrow 1} 2^{1/\theta} u^{2^{1/\theta} - 1} = 2 - 2^{1/\theta}.$$

Tail Dependence

- item 1

Tail Dependence

- item 1

Tail Dependence

- item 1

Titolo

- item 1
- item 2
- item 3