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基於總體成本與駕駛與乘客間公平性之最佳化共乘演
算法

An Optimization Based Carpool Algorithm in
Consideration of the Total Cost and Fairness among the
Driver and All Passengers

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摘要

共享經濟是隨著網路以及行動裝置普及下逐漸興起的熱潮，傳統的計程車與共乘服務在共享經濟的風潮下產生出新的商業模式，不同於以往使用者必須到目的地才能知道價錢，或是只能針對特定路線如通勤、通學等才容易有共乘機會；透過線上叫車服務，使用者除了可以事先知道價錢，還可以透過共乘配對服務，自動配對有相近路線的乘客，共同分攤車資，提供使用者更便宜經濟的選擇。

在自動配對的共乘服務中，往往需要多繞路以同時滿足不同乘客間的載運需求，如何讓使用者之間的车資分配符合公平性，便是重要的挑戰。本研究考慮共享經濟中共乘服務的多乘客路線規劃問題，將乘客可接受的抵達時間、繞多少路的可接受程度，車輛人數限制，以及載客的優先順序納入考量，以最大化最小的乘客共乘節費比例，並透過拉格朗日鬆弛法以取得最佳解。

關鍵字：共乘、共享經濟、車輛路徑問題、公平性、斯坦納樹問題、拉格朗日鬆弛法

Abstract

The sharing economy is a boom with the rise of the Internet and mobile devices. Traditional taxis and ride-sharing services have produced new business models under the sharing economy. Unlike in the past, users can know the price only if they go to the destination. It is not easy to have shared ride opportunities if it is not the commutation route for work and school. Instead, in the online ride-hailing service, users can know the price in advance. The service can also automatically match passengers with similar routes in real-time. Sharing the fare provides users with a cheaper and more economical choice.

In the ride-sharing service with automatic matching, detours often happen when there are multiple riders with different needs at the same time. How to make it fair to charge the riders is a significant challenge. This research considers the problem of multi-riders route planning for ride-sharing services. Considering the acceptable arrival time, the acceptable degree of detours, vehicles' capacity, and passengers' priority order constraints. Maximize the minimal rider's savings ratio compared to self ride-hailing with carpooling. Use the

Lagrangian relaxation method to obtain the best solution.

Keywords: Carpooling, Sharing economy, Vehicle routing problem, Fairness, Steiner Tree Problem, Lagrangian relaxation

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Chapter 1 Introduction

1.1 Background

In recent years, with the rising of the Internet and mobile devices, information sharing has become a daily thing. Some people began to put idle resources on the Internet to share and exchange with people. However, sharing with strangers means User risk. With the invention of user rating systems, sharing platforms can begin to know whether users are good or bad through the rating, which reduces the risk in online sharing[1]. As a trend, from food delivery, accommodation to transportation, the sharing platforms seem to be a part of daily life. People call this new concept a "sharing economy."

There is currently no clear definition of the sharing economy so far. Generally speaking, activities in the sharing economy contain four categories: recycling of goods, increasing the use rate of durable goods, exchange of services, and sharing of operating assets[1]. All of the activities are about the re-allocation of "idle assets" [2]. No matter the exchange of resources or earning from the sharing platforms, low transaction costs make the idle assets more valuable. The demander can use the asset at a lower cost, increase the overall utilization rate of social assets. "Renting rather than buying" changes the concept of assets.

The spirit of re-allocation of idle assets in the sharing economy has extended the taxis and ride-sharing services. Compared with traditional public transportation services, the new type of transportation services under the sharing economy have the following features:

1. Efficient use of resources: Hailing by the roadside, phone appointments are the traditional taxis method. Take Taiwan as an example. According to data from the Ministry of Transport, street hailing, taxi stands, and scheduling [3] are more than 50%, which can increase the time vacancy rate of taxis and cause waste of resource. Through the ride-hailing platform, a ride request can be matched on the platform at any time. Riders do not need to wait on the roadside. They only need to use their mobile phones to know the arrival time and journey time, reducing uncertainty and improving car usage rate. 2. Dynamic resources: Drivers on the ride-hailing service can decide their working hours with flexibility by log in and logout. Therefore, compared with traditional taxis with fixed resources, the ride-hailing platform's resources are highly dynamic. 3. Dynamic ride-sharing: Some ride-hailing apps also provide ride-sharing services, such as UberPool and Lyft Line. As long as drivers turned on the ride-sharing option, riders with similar routes can be matched in time. Not only can the fare be shared, but it also can reduce the waste of vacant seats in the journey.

From the above features, we can understand that both the supplier and the demander have a high degree of dynamics and uncertainty on the new type of ride-sharing service. Finding a suitable route from many dynamic constraints will be a big challenge.

1.2 Motivation

In the early, ride-sharing was necessary to go carpooling by matching at a taxi station, a fixed commuter route or self-organized among people who knew each other [4]. With the maturity of mobile devices, they are widely used in every corner of life, such as food, clothing, housing, and transportation. Carpooling is a classic example. In the sharing economy era, when one takes out his/her mobile phone, the ride-sharing service can make better use of the "idle assets" such as seats and vehicles. Drivers can now earn more by detours for ride-sharing. Riders can share the fare with others at a cost-effective price.

However, a trip may consist of many riders with different origins and destinations; charging riders with fairness is a big problem. There are detours and waiting times among the riders, which may be varied. Some people likely detour a lot in a shared ride, and some people hardly detour but have to pay a similar fare. It is very unfair for the riders in the same carpooling. Therefore, making it fair between passengers in the ride-sharing problem is the object of this article.

1.3 Objective

This research aims to maximize the minimal percentage of cost-saving between a rider choose to hail a ride on his/her own and go carpooling. In order to ensure there is no significant deviation from what a rider expects. Constraints are applied with a ride request of origin and destination, car capacity, and detour limitation.

1.4 Thesis Organization

The rest of the paper is organized as follows. We will go through the related work of the carpooling problem in Chapter 2. Chapter 3 will describe the carpooling problem in detail and formulate it into a mathematical model.

Chapter 2 Literature Review

2.1 Related Work on Carpooling

This section will go through the related work and discuss the approaches for the carpooling problem. Before this section, we will make a brief throughout the history of carpooling. Carpooling has risen since World War II, as all the resources are for the war. At that time, carpooling is a tactic for rationing gasoline through car-sharing clubs, which match riders and drivers via a bulletin board. As technology progresses, the carpooling service starts to integrate the Internet and mobile phones in recent years. Inbuilt location-based service (LBS) on smartphones makes it easy to get a user's location in real-time. Making it possible to match drivers and riders on-demand at any time [4].

Coja-Oghlan et al. [5] see the carpooling problem as a dial-a-ride problem (DARP) that allocates all resources through a centralized server. This paper treats the graph of the road network as a caterpillar tree, which is an undirected graph $G = (V, E)$ and all the vertices are on a central backbone. The self-developed Minimal Spanning Tree (MST) heuristic method can find an optimal solution in polynomial time with probability. By applying the bipartite Steiner tree, we can also find the optimal solution in polynomial time at random, which indicates that $NP = RP$. Overall, NP-hard is the worst cast, while on average is solvable efficiently.

Farin and Williams [6] propose a fair carpooling scheduling algorithm in different approaches focused on driving rotation on any given day when a given number of members form a carpooling club. There are the following factors to concern: who drives today and how many riders are on the carpool. This research also concerns the "cost" of the trip as the least common multiple of $1, 2, \dots, n$, where n is the largest number of people participating in the carpool. The driver earns the "cost" while the riders share it. Who has the least "cost" is the driver of the next day.

Wolfson and Lin [7] indicate that ridesharing fairness is a new a problem in studies. The concept in this paper is related to the Nash Equilibrium. This paper proposes a payment scheme called Guaranteed-Ridesharing-Fairness (GRF) that guarantees each passenger/trip saves in an optimal ridesharing plan (RSP), which has no other RSP has saving greater than any RSP. In a GRF method, it combines both fair and optimal ridesharing queries.

2.1.1 Dial-a-ride Problem

When it comes to carpooling, the dial-a-ride problem could be one of the best discussions about this kind of scenario. Dial-a-rider problem (DARP) is a problem for finding vehicle routes to serve users with requests with specific origins and destinations, with objective to minimize cost or maximize satisfied demand [8]. DARP has been widely researched for about 50 years [9] and is considered as an NP-hard problem [10]. There are many variations in the discussion. Cordeau and Laporte [8] indicate that DARP mainly consists of the following factors:

- Vehicles: There might be 1 or n vehicles in the system. Most DARP studies assume

that vehicles are with homogeneous capacity; some studies may use a heterogeneous fleet with different capacity.

- Requests: A request usually consists of these parts: passengers, origins, destinations, and time constraints. Static or dynamic is also a significant concern. For a static DARP, all the ride requests will be known in advance, and the central scenario is to allocate all the resources once. A dynamic DARP usually discusses ride request planning in a specific time window. For better understanding, we will use ride requests instead of requests.
- Routes: Routes of the road network usually consist of length and duration.
- Detour: Because a route of DARP is aggregate from many must-pass nodes proposed by ride requests. Rather than the shortest path, every passenger may have a detour.

A DARP is a kind of problem that consists of n must-pass nodes with the lowest cost.

We will use a Steiner tree problem for finding routes.

2.1.2 Steiner Tree Problem

A Steiner tree problem is to find a minimum cost tree in given subset vertices of a given graph. When it comes to the Steiner tree, a minimum spanning tree problem can be mentioned together. While a minimum spanning tree problem is to find a minimum total weight that the links have to go through all the vertices in a given graph, a Steiner tree problem treats the vertices as terminal and non-terminal vertices. All terminals must go through, and the non-terminal vertices are allowed to contain to reduce cost [11]. Figure

2.1 as an example; this figure is an undirected graph $G = (V, E, w)$ with nonnegative weights. A set of vertices $L \subset V$ is the terminals. The minimal Steiner tree is to find a tree that passes all the terminals that can include other vertices in the graph. We could find a Steiner minimal tree shown in Figure 2.2 with cost = 5 [12]. According to Foulds and Graham [13], finding a Steiner minimal tree (SMT) is an NP-Complete problem, which has no deterministic algorithms to solve in the polynomial time. As a result, we tend to use heuristic approaches to this problem.



Figure 2.1: Example graph of a Steiner tree problem

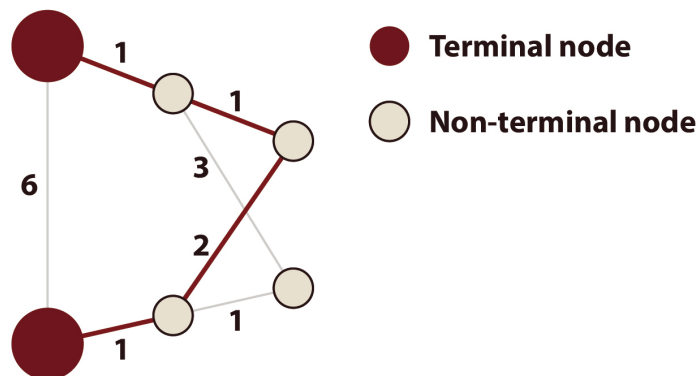


Figure 2.2: Steiner minimal tree with cost = 5

2.2 Fairness

Fairness is the central issue in this research. According to the Cambridge dictionary definition, fairness is "the quality of treating people equally or in a way that is right or reasonable" [14]. There are many factors to be considered when it comes to carpooling. One of them is the mechanism of dispatching ride requests to drivers. Suppose a driver in the carpooling system is not treated equally, such as not receiving any ride request in the rush hour while others receive a lot. In that case, the driver might choose not to use the carpooling system. However, getting rid of unequal ride requests dispatching is not easy. There are many possibilities for drivers to think they are treated equally or unequally. For instance, one may think it is equal to receive the same ride requests as others. In contrast, another may think to be treated unequally for participating in carpooling for a long time. To get rid of the issue in such a resource allocation scenario, we define fairness in a mathematical approach to measure fairness. There are many indices to measure the fairness of resource allocation [15]. In this paper, we use Raj Jain's approach, Jain's fairness index [16], to know the status of ride request allocation by calculation.

2.2.1 Jain's Fairness Index

Jain's fairness index is a metric to measure the fairness of the resource allocation in a system, which is used in the networking engineering field. By definition, there are n users share resource in a system, a user $i \in \{1, 2, 3, \dots, n\}$ allocates x_i resource [16]. The following is the formula:

$$Fairness\ Index = f_A(x) = \frac{\left(\sum_{i=1}^n x_i\right)^2}{n \sum_{i=1}^n x_i^2}$$

The index ranges from $\frac{1}{n}$ to 1. When there is only one user monopoly to all resource, which means the others get no resource, the fairness index will be $\frac{1}{n}$. It is the most unfair scenario. When all the users get the same resource, the fairness index will be 1, which is the fairest scenario.

Chapter 3 Problem Formulation

In this chapter, we will introduce the scenario of the carpooling fairness problem. Then formulate the problem into a mathematical model with given parameters and decision variables.

3.1 Problem Description

The object of this research is to minimize the maximum percentage of cost saving when a passenger chooses to carpool, with consideration of the number of drivers, car capacity and routing limit constraints.

Take Figure 3-1 as an example scenario; Passenger 1 and Passenger 2 are requesting a trip. When Passenger 1 chooses to take a ride by himself/herself, called "exclusive" riding shown as Figure 3-2, it would cost \$5. We can describe the scenario as a shortest path problem from S to D_1 , and P_1 is a must-pass node. We use Steiner tree to solve this kind of shortest path problem with must-pass nodes. Passenger 2's "exclusive" riding would also cost \$5 in Figure 3-3. In this case, it would be a shortest path problem from S to D_2 with P_2 as a must-pass node.

When the passengers choose to take a carpool, which is called "sharing" riding. We

can describe the scenario as a shortest problem from S to D_1 with P_1 , P_2 and D_2 as must-pass nodes or S to D_2 with P_1 , P_2 and D_1 as must-pass nodes. In Figure 3-4 is one of the best case, which would cost \$8 for all the passengers, in Figure 3-5 would cost the same \$8 and both of the fares they could share are \$5 ($\overline{P_2 D_1}$ and $\overline{P_2 D_2}$ respectively); however, in Figure 3-4 would cost $\frac{\overline{P_1 P_2}}{1} + \frac{1}{2} \times \frac{\overline{P_2 D_1}}{5} = \3.5 for Passenger 1 and $\frac{1}{2} \times \frac{\overline{P_2 D_1}}{5} + \frac{\overline{D_1 D_2}}{1} = \3.5 for Passenger 2, in Figure 3-5 would cost $\frac{\overline{P_1 P_2}}{1} + \frac{1}{2} \times \frac{\overline{P_2 D_2}}{5} + \frac{\overline{D_2 D_1}}{1} = \4.5 for Passenger 1 and $\frac{1}{2} \times \frac{\overline{P_2 D_2}}{5} = \2.5 for Passenger 2.



Figure 3.1: Example road network with driving fare and points of passengers and their destinations

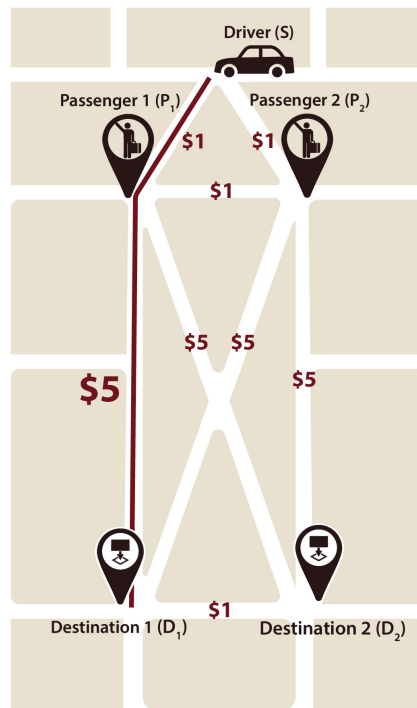


Figure 3.2: Best routing path when the driver only serving Passenger 1



Figure 3.3: Best routing path when the driver only serving Passenger 2

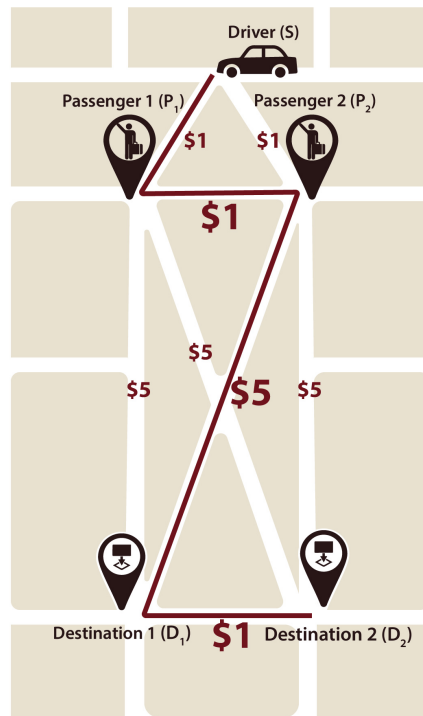


Figure 3.4: One of the best routing when both Passenger 1 and Passenger 2 carpool

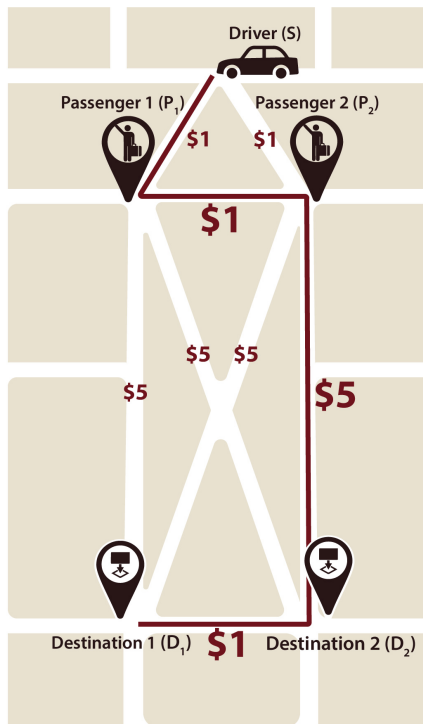


Figure 3.5: Another best routing when both Passenger 1 and Passenger 2 carpool

3.1.1 Assumptions

There are three assumption in our system:

1. Every driving request only includes one passenger. This driving request will be split into multiple one-person driving request for a driving request with multiple people.
2. System calculation time is not considered, that is, assume that drivers' location will still keep unchanged after the time of system calculation.
3. The carpooling fare is shared equally by the time passengers in the vehicle. For a specific link in the route with n passengers, the fare of the link for a passenger will be divided by n .

3.1.2 Fairness of Vehicle Dispatching

In order to consider the fairness when dispatching driver, we introduce Jain's fairness index to avoid unfair situation:

$$Fairness\ Index = f_A(x) = \frac{\left(\sum_{i=1}^n x_i\right)^2}{n \sum_{i=1}^n x_i^2}$$

We take accumulate driving requests that a driver has received as the metrics for fairness. That is, when a driver receives more driving requests in the past, he/she should give his/her chance of receiving a new driving request to the one who takes fewer driving requests.

As a new driving request comes, we need to ensure our system must get more or at least keep the same fairness. Hence, we form a constraint that the fairness index can not decrease compared to the previous fairness index.

However, when the driving resource changes, such as a new driver joins our system, it will violate our constraint for fairness. Because the newcomer has not taken any driving request before, the fairness will decrease. Even if a driver leaves our system, the fairness will increase or decrease due to the driver's situation. As a result, we need to reset the fairness index to the initial state and reset all the drivers' accumulated driving requests to 0. Nevertheless, it will make the fairness index's denominator to 0, which can not be divided. To avoid this situation, we assign a virtual driver with one accumulated driving request, which will make the fairness index to $\frac{1}{(n+1)}$, where n is the number of drivers in our system. Moreover, for any situation that the fairness index will decrease when dispatching a new driving request, the fairness will reset.

3.2 Mathematical Model

We will go through the detailed given parameters and decision variables in the mathematical model before describing our mathematical model.

Table 3.1: Notations of given parameters

Notation	Description
C	Set of passengers in the system (take j as index)
D	Set of destinations in the system (take k as index)
R	Set of drivers in the system (take i as index)
W	Set of passengers or destinations in the system, where $W = P \cup D$
B_r	Set of passengers that driver r has picked up, where $r \in R$
L	Set of links on the road network
L_c	Set of links on the road network that pass by the passenger $c \in C$
L_d	Set of links on the road network that pass by the destination $d \in D$
P_{pw}	Set of all paths for driver r from initial location to a passenger or a destination, where $r \in R$ and $w \in W$
P_{rc_i}	Set of all paths for driver r from initial location to passenger c_i , where $r \in R$ and $c_i \in C$
P_{rd_i}	Set of all paths for driver r from initial location to destination d_i , where $r \in R$ and $d_i \in D$
Q_r	Maximum load capacity for driver r , where $r \in R$
f_l	The fare for the link l , where $l \in L$
g_r	The capacity that the driver $r \in R$ has loaded in the previous decision round.
α	The fairness index in the previous decision round.
δ_{pl}	Indicator function which is 1 if link $l \in L$ on the route $p \in P_r$ when driver $r \in R$ goes carpooling; 0 otherwise
ζ_{pl}	Indicator function which is 1 if link $l \in L$ on the route $p \in P_{rd_i}$, $i \in \{1, 2, 3, \dots, n\}$ is selected when passenger $c_i \in C$ with index i ; 0 otherwise

Table 3.2: Notations of decision variables

Notation	Description
P_r	Set of all paths for driver r , where $r \in R$
E_r	Set of customers for driver r , where $r \in R$ to pick up when the driver goes carpooling
z_p	Binary variable, 1 if route $p \in P_{rw}$ is chosen; 0 otherwise.
u_l	Binary variable, 1 if link $l \in L$ is on the shortest path; 0 otherwise.
x_{pr}	Binary variable, 1 if route $p \in P_r$ is chosen when driver $r \in R$ goes carpooling; 0 otherwise.
y_{pr}	Binary variable, 1 if route $p \in P_{rd_i}, i \in \{1, 2, 3, \dots, n\}$ is chosen when passenger $c_i \in C$ with index i does not take a carpooling; 0 otherwise.
v_{cr}	Binary variable, 1 if passenger $c \in C$ is assigned to driver $r \in R$
a_r	Total fare that the driver $r \in R$ has earned before current decision round.
b_r	Total fare that the driver $r \in R$ has earned after current decision round.

Objective Function

The objective function is to maximize the minimum discount percentage when applying carpooling among passengers in one trip.

$$\max_r \min_{i \in E_r} \frac{Z_i - \text{carpool cost}_i}{Z_i} \quad (\text{IP1})$$

$$\text{where } Z_i = \sum_{p \in P_{rd_i}} \sum_{l \in L} y_p \zeta_{pl} a_l$$

Constraints

Constraint 3.1 ensures there is only one path selected for a driver.

$$\sum_{p \in P_r} x_p \leq 1 \quad (3.1)$$

Constraint 3.2 ensures the selected link on the route we chose is on the shortest path.

$$\sum_{p \in P_r} x_p \delta_{pl} \leq u_l \quad \forall l \in L \quad (3.2)$$

Constraint 3.3 ensures all the links pass by the node of passenger or destination we chose is the shortest path.

$$u_l \leq 1 \quad \forall l \in L_c \cup L_d \quad (3.3)$$

Constraint 3.4 ensures that all the routes to passengers or destinations we chose is on the shortest path and overlap the route we chose.

$$\sum_{p \in P_w} z_p \delta_{pl} \leq u_l \quad \forall w \in W \quad (3.4)$$

Constraint 3.5 ensures the route we chose will only pass by the node of passenger or destination once.

$$\sum_{p \in P_w} z_p \leq 1 \quad \forall w \in W \quad (3.5)$$

Constraint 3.6 ensures that we must pick up the passenger before we drop off the passenger.

$$\sum_{l \in L} \sum_{p \in P_{rc_i}} z_p \delta_{pl} f_l \leq \sum_{l \in L} \sum_{p \in P_{rd_i}} z_p \delta_{pl} f_l \quad \forall i \in \{1, 2, \dots, n\}, r \in D \quad (3.6)$$

Constraint 3.7 ensures the fairness index after current decision round will not less than the previous decision round.

$$\frac{\left(\sum_{r \in R} b_r\right)^2}{\sum_{r \in R} 1 \sum_{r \in R} b_r^2} \geq \alpha \quad (3.7)$$

Constraint 3.8 is the fairness index in the previous decision round.

$$\frac{\left(\sum_{r \in R} a_r\right)^2}{\sum_{r \in R} 1 \sum_{r \in R} a_r^2} \leq \alpha \quad (3.8)$$

Constraint 3.9 ensures there is only one driver selected for a passenger.

$$\sum_{r \in R} v_{cr} \leq 1 \quad (3.9)$$

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