Homework #01: Solutions to chapter 2 exercises

Isaac Ayala Lozano

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Presented below are the answers to exercises 2.1 through 2.12 from Chapter 2 in [1].

1. System (a)'s output is described with a function of the form $y(u) = mu$, where m is a constant. Testing the system to verify the propery of superposition, the system is shown to be *linear*.

$$
\begin{aligned}
\alpha y(u) = \alpha m u \quad y(\alpha u) &= m \alpha u \\
\alpha y(u) &= y(\alpha u) \\
y(u_1) &= m u_1 \\
y(u_1 + u_2) &= m(u_1 + u_2) \\
y(u_1) + y(u_2) &= m u_1 + m u_2 \\
y(u_1 + u_2) &= y(u_1) + y(u_2)
\end{aligned}
$$

The output of system (b) is described by a function $y(u) = mu + y_0$. It is sufficient to test additivity to prove that the system is *nonlinear*.

$$
\begin{aligned}
y(u_1) = m u_1 + y_0 \quad y(u_2) &= m u_2 + y_0 \\
y(u_1) + y(u_2) &= m(u_1 + u_2) + 2y_0 \\
y(u_1 + u_2) &= m(u_1 + u_2) + y_0 \\
y(u_1) + y(u_2) &\neq y(u_1 + u_2)
\end{aligned}
$$

System (c) is described by a function $y(u) = f(u)$. It is observed that $y \in (-k, k), u \in (-\infty, \infty)$. Consider a case where $y(u) = k$ and test for the property of superposition. The system is *nonlinear*.

$$\alpha y(u) = \alpha k$$
$$y(\alpha u) = k$$
$$\alpha y(u) \neq y(\alpha u)$$

System (b) could be linearized by defining a new operating point/defining a new output $\bar{y}$, where $\bar{y} = y - y_0$.

2. The ideal low pass filter is not a causal system. Given $t < t_0$, the system would be presenting an output to a time in the future instead of the present and past values. It is impossible to construct a non-causal system in the real world.

3. The system is linear. Consider $u_1 \neq u_2$ and verify that the system posseses the propery of superposition.

$$\text{Assuming} \quad t < \alpha$$
$$y_1 = y(u_1) = u_1(t) \quad y_2 = y(u_2) = u_2(t)$$
$$y_1 + y_2 = u_1(t) + u_2(t)$$
$$y(u_1 + u_2) = u_1(t) + u_2(t)$$
$$y_1 + y_2 = y(u_1 + u_2)$$
$$\alpha y(u) = \alpha u(t)$$
$$y(\alpha u) = \alpha u(t)$$
$$\alpha y(u) = y(\alpha u)$$

For values of $t > \alpha$ the system is also linear, having its output be 0 for all cases of $t > \alpha$.

The system is not time-invariant. Consider the case where $t < \alpha$, $t + T > \alpha$ and $u(t) = u(t + T) > 0$.

$$y(t) = u(t)$$
$$y(t + T) = 0$$
$$y(t) \neq y(t + T)$$

The system's output is not the same, despite the input value being the same at t and $t + T$.

The system is causal. The system is memoryless, depending only on the present input value hence not reacting to an input value in the future.

4. If the operator H is linear, then $P_\alpha y = P_\alpha H u = P_\alpha H P_\alpha u$ is true.

$$P_\alpha H u = \begin{cases} Hu & t \leq \alpha \\ 0 & t > \alpha \end{cases}$$
$$P_\alpha u = \begin{cases} u & t \leq \alpha \\ 0 & t > \alpha \end{cases}$$
$$P_\alpha H P_\alpha u = \begin{cases} Hu & t \leq \alpha \\ 0 & t > \alpha \end{cases}$$

The systems $P_\alpha H u$ and $P_\alpha H P_\alpha u$ have the same output value for the same values of t, thus it is inferred that $P_\alpha H u = P_\alpha H P_\alpha u$.

This expression is false if H is a nonlinear operator, because the output of $H P_\alpha u$ would be different from $P_\alpha H u$ for $t > \alpha$.

$$\begin{aligned} &\text{Consider } t > \alpha \quad H := u + k \\ &H P_\alpha u = k \\ &P_\alpha H u = 0 \\ &H P_\alpha u \neq P_\alpha H u \end{aligned}$$

Consider an operator H that can be nonzero for $t > \alpha$, for example $H := u + k$. It is shown that for $t \leq \alpha$ the expression $(P_\alpha H u)(t) = (H P_\alpha u)(t)$ is true. The same is not true for values of $t > \alpha$.

$$\begin{aligned} &\text{Given } H := u + k \\ &(P_\alpha H u)(t) = \begin{cases} u + k & t \leq \alpha \\ 0 & t > \alpha \end{cases} \\ &(H P_\alpha u)(t) = \begin{cases} u + k & t \leq \alpha \\ k & t > \alpha \end{cases} \end{aligned}$$

The expression $(P_\alpha H u)(t) = (H P_\alpha u)(t)$ is thus false.

5. For $\mathbf{x}(0) \neq 0$ only statement 2 is true. Testing superposition for the three statements at t_0 it is shown that statements 1 and 3 are false.

$$
\begin{aligned}
&\text{Statement 1} \\
y(u(t_0)) &= \mathbf{x}(0) \\
y(u_1(t_0)) + y(u_2(t_0)) &= \mathbf{x}(0) + \mathbf{x}(0) = 2\mathbf{x}(0) \\
y(u_3(t_0)) &= \mathbf{x}(0) \\
y(u_3(t_0)) &\neq y(u_1(t_0)) + y(u_2(t_0)) \\
&\text{Statement 2} \\
0.5(y(u_1(t_0)) + y(u_2(t_0))) &= \mathbf{x}(0) \\
y(u_3(t_0)) &= \mathbf{x}(0) \\
y(u_3(t_0)) &= 0.5(y(u_1(t_0)) + y(u_2(t_0))) \\
&\text{Statement 3} \\
y(u_1(t_0)) - y(u_2(t_0)) &= \mathbf{x}(0) - \mathbf{x}(0) = 0 \\
y(u_3(t_0)) &= \mathbf{x}(0) \\
y(u_3(t_0)) &\neq y(u_1(t_0)) + y(u_2(t_0))
\end{aligned}
$$

For $\mathbf{x}(0) = 0$ all statements are true. Given that the system starts at rest, then all three statements are true because $y_1(t_0) = y_2(t_0) = y_3(t_0) = 0$.

6. Consider $u_1 \neq u_2$, the output for each input would be

$$
\begin{aligned}
y_1 = y(u_1) &= \frac{u_1^2(t)}{u_1(t-1)} \\
y_2 = y(u_2) &= \frac{u_2^2(t)}{u_2(t-1)}
\end{aligned}
$$

Testing the system for additivity shows that the system does not comply with it.

$$
\begin{aligned}
y_1 + y_2 &= \frac{u_1^2(t)}{u_1(t-1)} + \frac{u_2^2(t)}{u_2(t-1)} \\
y(u_1 + u_2) &= \frac{(u_1(t) + u_2(t))^2}{u_1(t-1) + u_2(t-1)} \\
y_1 + y_2 &\neq y(u_1 + u_2)
\end{aligned}
$$

Testing the system for homogeneity proves that it possesses that property.

$$\begin{aligned}
y(\alpha u) &= \frac{(\alpha u(t))^2}{\alpha u(t-1)} \\
&= \frac{\alpha u^2(t)}{u(t-1)} \\
\alpha y(u) &= \alpha \frac{u^2(t)}{u(t-1)} \\
y(\alpha u) &= \alpha y(u)
\end{aligned}$$

7. All rational numbers α can be expressed as the ratio of two numbers such that $\alpha = \frac{i}{j}, \quad i \neq j$. The product of two numbers a and b is defined as the sum of number a a total of b times $a * b = \sum_{i=1}^{n=b} a$. Recall that the output response of a system can be expressed as the sum of the zero-input response f_{zi} and the zero-state response f_{zs}.

$$\begin{aligned}
f(nu) &= f(u + u + \ldots u) \\
&= f(u) + f(u) + \cdots + f(u) = nf(u) \\
\text{Let } n &= \frac{i}{j} \\
f(nu) &= f(\frac{i}{j}u) \\
&= if(u/j) \\
&= \frac{i}{j}f(u) \\
&= nf(u)
\end{aligned}$$

Recall that $\alpha = i/j = n$, thus $\alpha y(u) = y(\alpha u)$.

8. Given $x = t + \tau$ and $y = t - \tau$.

$$\begin{aligned}
x + y &= 2t \\
t &= \frac{x+y}{2} \\
x - y &= 2\tau \\
\tau &= \frac{x-y}{2} \\
g(t,\tau) &= g(\frac{x+y}{2}, \frac{x-y}{2}) \\
\frac{\partial g(t,\tau)}{\partial x} &= \frac{\partial g(\frac{x+y}{2}, \frac{x-y}{2})}{\partial x}
\end{aligned}$$

Recall the definition of the derivative

$$\frac{\partial f(t)}{\partial t} \equiv \lim_{t\to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

We can rewrite the partial derivative of g in the following manner

$$\begin{aligned}
\frac{\partial g(t,\tau)}{\partial x} &= \lim_{x\to 0} \frac{g(\frac{x+\Delta x+y}{2}, \frac{x+\Delta x-y}{2}) - g(\frac{x+y}{2}, \frac{x-y}{2})}{\Delta x} \\
&= \lim_{x\to 0} \frac{g(\frac{x+y}{2} + \frac{\Delta x}{2}, \frac{x-y}{2} + \frac{\Delta x}{2}) - g(\frac{x+y}{2}, \frac{x-y}{2})}{\Delta x} \\
&= \lim_{x\to 0} \frac{g(\frac{x+y}{2}, \frac{x-y}{2}) + g(\frac{\Delta x}{2}, \frac{\Delta x}{2}) - g(\frac{x+y}{2}, \frac{x-y}{2})}{\Delta x} \\
&= \lim_{x\to 0} \frac{g(\frac{x+y}{2}, \frac{x-y}{2}) + 0 - g(\frac{x+y}{2}, \frac{x-y}{2})}{\Delta x} \\
&= 0
\end{aligned}$$

The function $g(t,\tau)$ does not depend on x, it depends only on $t - \tau$.

9. The impulse response of the system is

$$g(t) = \begin{cases} t & 0 \le t < 1 \\ 2 - t & 1 \le t < 2 \end{cases}$$

Similarly, input $u(t)$ is

$$u(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & 1 \le t < 2 \end{cases}$$

The convolution integral is then $y(t) = g(t) \circledast u(t)$.

$$\begin{aligned}
y(t) &= \int_0^t u(\tau)g(t-\tau)d\tau \\
&= \begin{cases} \int_0^t u(t)g(t-\tau)d\tau & 0 \le t < 1 \\ \int_0^{t-1} u(\tau)g(t-\tau)d\tau + \int_{t-1}^1 u(\tau)g(t-\tau)d\tau + \int_1^t u(\tau)g(t-\tau)d\tau & 1 \le t < 2 \end{cases} \\
&= \begin{cases} \int_0^t (1)(t-\tau)d\tau & 0 \le t < 1 \\ \int_0^{t-1}(1)(t-\tau)d\tau + \int_{t-1}^1 (1)(2-(t-\tau))d\tau + 0 & 1 \le t < 2 \end{cases} \\
y(t) &= \begin{cases} \frac{1}{2}t^2 & 0 \le t < 1 \\ -\frac{3}{2}t^2 + 4t - 2 & 1 \le t < 2 \\ 0 & \text{all other cases} \end{cases}
\end{aligned}$$

10. Assuming that the system is a relaxed system then the transfer function $\hat{g}(s)$ can be obtained using the Laplace transform. The impulse response $g(t)$ is the inverse Laplace transform of the transfer function.

$$\begin{aligned}
\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y &= \frac{du}{dt} - u \\
\hat{y} = \hat{y}(s) &= \mathfrak{L}\{y(t)\} \\
s^2\hat{y}(s) + 2s\hat{y}(s) - 3\hat{y}(s) &= s\hat{u}(s) - \hat{u}(s) \\
\hat{y}(s^2+2s-3) = \hat{y}(s-1)(s+3) &= \hat{u}(s-1) \\
\hat{g} = \frac{\hat{y}}{\hat{u}} &= \frac{1}{s+3} \\
g(t) = \mathfrak{L}^{-1}\{\frac{1}{s+3}\} &= exp(-3t)
\end{aligned}$$

11. Let $\bar{y}(t) = g(t) \circledast u(t)$, where $g(t)$ is the impulse response of the system and $u(t)$ is the unit-step function. Solve for $g(t)$.

$$\begin{aligned}
\mathfrak{L}\{\bar{y}(t)\} &= \hat{y} = \hat{g}\hat{u} \\
\hat{u} &= \mathfrak{L}\{u(t)\} = s^{-1} \\
\hat{y} &= s^{-1}\hat{g} \\
\hat{g} &= s\hat{y} \\
g(t) = \mathfrak{L}^{-1}\{\hat{g}\} &= \mathfrak{L}^{-}1\{s\hat{y}\} \\
g(t) &= \frac{d\bar{y}}{dt}
\end{aligned}$$

12. Apply the Laplace transform to the system of equations and rewrite the system as matrices. Because D are polynomials d/dt, their Laplace transform are polynomials of the form $\hat{D} = \hat{D}(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$. Solve for the transfer matrix $\mathbf{G}$.

$$\begin{aligned}
D_{11}(p)y_1(t) + D_{12}(p)y_2(t) &= N_{11}(p)u_1(t) + N_{12}(p)u_2(t) \\
D_{21}(p)y_1(t) + D_{22}(p)y_2(t) &= N_{21}(p)u_1(t) + N_{22}(p)u_2(t) \\
\hat{y} = \hat{y}(s) &= \mathfrak{L}\{y(t)\} \\
\hat{D}_{11}\hat{y}_1 + \hat{D}_{12}\hat{y}_2 &= \hat{N}_{11}\hat{u}_1 + \hat{N}_{12}\hat{u}_2 \\
\hat{D}_{21}\hat{y}_1 + \hat{D}_{22}\hat{y}_2 &= \hat{N}_{21}\hat{u}_1 + \hat{N}_{22}\hat{u}_2
\end{aligned}$$

The system of equations can now be expressed as a product of matrices.

$$\begin{gathered}
\begin{pmatrix} \hat{D}_{11} & \hat{D}_{12} \\ \hat{D}_{21} & \hat{D}_{22} \end{pmatrix} \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} = \begin{pmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} \\
\mathbf{D} = \begin{pmatrix} \hat{D}_{11} & \hat{D}_{12} \\ \hat{D}_{21} & \hat{D}_{22} \end{pmatrix} \quad \mathbf{N} = \begin{pmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{pmatrix} \\
\mathbf{Y} = \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} \quad \mathbf{U} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} \\
\mathbf{DY} = \mathbf{NU} \\
\mathbf{G} = \mathbf{YU}^{-1} = \mathbf{D}^{-1}\mathbf{N} \\
\mathbf{G} = \begin{pmatrix} \hat{D}_{11} & \hat{D}_{12} \\ \hat{D}_{21} & \hat{D}_{22} \end{pmatrix}^{-1} \begin{pmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{pmatrix}
\end{gathered}$$

References

[1] C.T. Chen. *Linear System Theory and Design*. Oxford series in electrical and computer engineering. Oxford University Press, 1999. ISBN: 9780195117776.