Homework 02

Isaac Ayala Lozano

2020-02-09

3.1 Consider Fig 1. What is the representation of the vector $\mathbf{x}$ with respect to the basis $[\mathbf{q_1}, \mathbf{i_2}]$? What is the representation of $\mathbf{q_1}$ with respect to the basis $[\mathbf{i_2}, \mathbf{q_2}]$

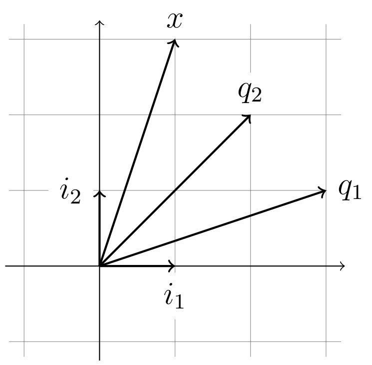

Figure 1: Different representations of vector x.

Given the vector $\mathbf{x} = [1\ 3]'$, we search for a linear combination of $[\mathbf{q_1}\ \mathbf{i_2}]$ to represent $\mathbf{x}$ in that basis.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = [\mathbf{q_1}\ \mathbf{i_2}] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

We solve this equation for $[\alpha_1\ \alpha_2]'$.

$$\begin{aligned}
\begin{bmatrix} 1 \\ 3 \end{bmatrix} &= \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\
\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} &= \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\
\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} &= \begin{bmatrix} 1/3 \\ 8/3 \end{bmatrix}
\end{aligned}$$

Given the vector $\mathbf{q_1} = [3\ 1]'$, we search for a linear combination of $[\mathbf{i_2}\ \mathbf{q_2}]$ to represent $\mathbf{q_1}$ in that basis.

$$\mathbf{q_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = [\mathbf{i_2}\ \mathbf{q_2}] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

We solve this equation for $[\beta_1\ \beta_2]'$.

$$\begin{aligned} \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \\ \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \\ \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} &= \begin{bmatrix} -2 \\ 3/2 \end{bmatrix} \end{aligned}$$

3.2 What are the 1-norm, 2-norm, and infinite-norm of the vectors

$$\mathbf{x_1} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \qquad \mathbf{x_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

From [1], we have the following definitions for each norm:

$$\begin{aligned} \|\mathbf{x}\|_1 &= \sum_{i=1}^{n} |x_i| \\ \|\mathbf{x}\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \\ \|\mathbf{x}\|_\infty &= \max{}_i\ |x_i| \end{aligned}$$

The norms for each vector are calculated accordingly.

- 1-norm

$$\begin{aligned} \|\mathbf{x_1}\|_1 &= 2 + 3 + 1 \\ &= 6 \\ \|\mathbf{x_2}\|_1 &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

– 2-norm

$$\begin{aligned}\|\mathbf{x_1}\|_2 &= \sqrt{2^2 + (-3)^2 + 1^2} \\ &= \sqrt{14} \\ \|\mathbf{x_2}\|_2 &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3}\end{aligned}$$

– inifinite-norm

$$\begin{aligned}\|\mathbf{x_1}\|_\infty &= \max\{|2|, |-3|, |1|\} \\ &= 3 \\ \|\mathbf{x_2}\|_\infty &= \max\{|1|, |1|, |1|\} \\ &= 1\end{aligned}$$

3.3 Find two orthonormal vectors that span the same space as the two vectors in Problem 3.2.

We obtain an orthonornal set of vector from $\{\mathbf{x}_1, \mathbf{x}_2\}$ employing the Schmidt normalization procedure.

$$\begin{aligned}\mathbf{u}_1 &:= \mathbf{x}_1 & \mathbf{q}_1 &:= \mathbf{u}_1/\|\mathbf{u}_1\| \\ \mathbf{u}_2 &:= \mathbf{x}_2 - (\mathbf{q}_1'\mathbf{x}_2)\mathbf{q}_1 & \mathbf{q}_2 &:= \mathbf{u}_2/\|\mathbf{u}_2\|\end{aligned}$$

We first obtain the vectors $\mathbf{u}_1, \mathbf{q}_1$ and $\mathbf{u}_2$.

$$
\begin{aligned}
\mathbf{u}_1 &\coloneqq \mathbf{x}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \\
\mathbf{q}_1 &\coloneqq \mathbf{u}_1/\|\mathbf{u}_1\| \\
&= \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \\
\mathbf{u}_2 &\coloneqq \mathbf{x}_2 - (\mathbf{q}_1'\mathbf{x}_2)\mathbf{q}_1 \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{14} \left(\begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{14}(0) \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

We notice that the original vectors were already orthogonal to each other. Thus, it is only necessary to normalize each of them. We continue with the Schmidt normalization procedure as the normalization step is the only one pending.

$$
\begin{aligned}
\mathbf{q}_2 &\coloneqq \mathbf{u}_2/\|\mathbf{u}_2\| \\
&= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

The set of orthonormal vectors $\{\mathbf{q}_1, \mathbf{q}_2\}$ has now been obtained, where

$$\mathbf{q}_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$\mathbf{q}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3.4 Consider an $n \times m$ matrix $\mathbf{A}$ with $n \geq m$. If all columns of $\mathbf{A}$ are orthonormal, then $\mathbf{A}'\mathbf{A} = \mathbf{I}_m$. What can you say about $\mathbf{A}\mathbf{A}'$?

The result of $\mathbf{A}\mathbf{A}'$ will be a $n \times n$ matrix with values different from the identity matrix of the same dimensions ($\mathbf{A}\mathbf{A}' \neq \mathbf{I}_n$). This is due to the fact that while the *columns* of A are orthogonal to one another, its *rows* are not orthogonal to one another. As such, the product of its rows will yield a result different to the identity matrix.

3.5 Find the ranks and nullities of the following matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We evaluate the determinant of each matrix to determine if the matrices are full rank. If not, then the determinants of its minors are evaluated until finding a minor $\mathbf{M}_{ij}$ of dimensions $m \times m$, where $m < n$, such that $\det(M) \neq 0$. Note that for $\mathbf{A}_3$, a similar method employing its minors must be used due to the fact that $\mathbf{A}_3$ is not a square matrix.

- $\mathbf{A}_1$

$$\det(\mathbf{A}_1) = \det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

We observe that its determinant is zero, thus not full rank. We now proceed to evaluate its minors to determine its rank.

$$\det(\mathbf{M}) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$$

Having verified that one of its 2×2 minors is invertible, we conclude that the rank and nullity of $\mathbf{A}_1$ are

$$\begin{aligned} \rho(\mathbf{A}_1) &= 2 \\ \text{nullity}(\mathbf{A}_1) &= \text{columns of } \mathbf{A}_1 - \rho(\mathbf{A}_1) = 3 - 2 = 1 \end{aligned}$$

- $\mathbf{A}_2$
  We obtain the determinant of the matrix to determine its rank.

$$\det(\mathbf{A}_2) = \det\left(\begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}\right) = -1$$

  The matrix is full rank, thus

$$\begin{aligned} \rho(\mathbf{A}_2) &= 3 \\ \text{nullity}(\mathbf{A}_2) &= 0 \end{aligned}$$

- $\mathbf{A}_3$
  The third column of matrix $\mathbf{A}_3$ is a linear combination of the first two. A 3×3 matrix $\mathbf{B}$ can be constructed with columns one, two and four of $\mathbf{A}_3$ and its rank can then be determined.

$$\begin{aligned} \mathbf{A}_3 &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ \det(\mathbf{B}) &= \det\left(\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\right) \\ &= -1 \end{aligned}$$

  We conclude that the rank and nullity of $\mathbf{A}_3$ is thus

$$\begin{aligned} \rho(\mathbf{A}_3) &= \rho(\mathbf{B}) = 3 \\ \text{nullity}(\mathbf{A}_3) &= 1 \end{aligned}$$

3.6 Find bases of the range spaces and null spaces of the matrices in Problem 3.5.

- $\mathbf{A}_1$

  Given that $\mathbf{A}_1 = [\mathbf{a}_1\, \mathbf{a}_2\, \mathbf{a}_3]$, we can construct a basis for the range space of the matrix with columns $\mathbf{a}_2$ and $\mathbf{a}_3$.

$$\text{range space}(\mathbf{A}_1) = \{\mathbf{a}_2, \mathbf{a}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

  The basis for the null space of $\mathbf{A}_1$ is determined by finding a vector $\mathbf{n}$ that satisfies the equation

$$\mathbf{A}_1\mathbf{n} = \mathbf{0}$$

  Let $\mathbf{n} = \begin{bmatrix} n_{11} & n_{21} & n_{31} \end{bmatrix}'$ and solve the previous system of equations for $\mathbf{n}$.

$$\begin{aligned} \mathbf{A}_1\mathbf{n} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \end{bmatrix} = \mathbf{0} \\ &= \begin{bmatrix} 0 + n_{21} + 0 \\ 0 + 0 + 0 \\ 0 + 0 + n_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ n_{21} = 0 \qquad & n_{31} = 0 \\ \mathbf{n} &= \begin{bmatrix} n_{11} \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

  We propose $n_{11} = 1$ as a particular solution.

$$\text{null space}(\mathbf{A}_1) = \{\mathbf{n}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

- $\mathbf{A}_2$

  Matrix $\mathbf{A}_2$ is full rank, thus its columns can form a basis for the range space. By consequence the only solution to $\mathbf{A}_2\mathbf{n} = \mathbf{0}$ is the trivial solution.

$$\text{range space}(\mathbf{A}_2) = \left\{ \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\text{null space}(\mathbf{A}_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- $\mathbf{A}_3$

  For matrix $\mathbf{A}_3$, three of its four columns can be used as a basis for the range space. It is only necessary for the set of vectors to be linearly independent. As such, the basis of $\mathbf{A}_3$ is constructed as follows

$$\mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$$

$$\text{range space}(\mathbf{A}_3) = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}$$

  As for the basis for the null space of $\mathbf{A}_3$, it is necessary to solve the equation $\mathbf{A}_3\mathbf{n} = \mathbf{0}$.

$$\begin{aligned}
\mathbf{A}_3\mathbf{n} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \\ n_{41} \end{bmatrix} &= \begin{bmatrix} n_{11} + 2n_{21} + 3n_{31} + 4n_{41} \\ 0 - n_{21} - 2n_{31} + 2n_{41} \\ 0 + 0 + 0 + n_{41} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
n_{41} &= 0 \\
-n_{21} - 2n_{31} + 2(0) = 0 \rightarrow n_{21} &= -2n_{31} \\
n_{11} + 2(-2n_{31}) + 3n_{31} + 4(0) = 0 \rightarrow n_{11} &= n_{31} \\
\mathbf{n} &= \begin{bmatrix} n_{11} \\ -2n_{11} \\ n_{11} \\ 0 \end{bmatrix}
\end{aligned}$$

We propose $n_{11} = 1$ as a particular solution.

$$\text{null space}(\mathbf{A}_3) = \mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

3.7 Consider the linear algebraic equation

$$\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{y}$$

It has three equations and two unkowns. Does a solution $\mathbf{x}$ exist in the equations? Is the solution unique? Does a solution exist if $\mathbf{y} = [111]'$?

Yes, there is a solution to the system of equations. It is an *unique* solution. This is due to the fact that the second row of the expanded matrix is a linear combination of the other two rows of the matrix. In other words, the rank of the matrix matches the number of unkowns ($\rho(\mathbf{A}) = 2$).

$$\begin{aligned} \begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2x_1 - x_2 \\ -3x_1 + 3x_2 \\ -x_1 + 2x_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ -3x_1 + 3x_2 &= 0 \rightarrow x_1 = x_2 \\ 2x_1 - x_2 &= 1 \rightarrow x_1 = 1 \rightarrow x_2 = 1 \\ -x_1 + 2x_2 &= 1 \rightarrow x_2 = 1 \end{aligned}$$

There is no contradiction in the system of equations.

For $\mathbf{y} = [1\ 1\ 1]'$ there is no solution to the system of equations. The second row is no longer a linear combination of the other two. When solving the system of equations we arrive at a contradiction.

$$\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2x_1 - x_2 \\ -3x_1 + 3x_2 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$-3x_1 + 3x_2 = 1 \to x_1 = x_2 - \frac{1}{3}$$
$$2x_1 - x_2 = 1 \to x_2 = \frac{5}{3} \to x_1 = \frac{4}{3}$$
But
$$-x_1 + 2x_2 = 1 = -\frac{4}{3} + 2(\frac{5}{3}) = 2 \neq 1$$

We have arrived at a contradiction, thus the system has no solution.

3.8 Find the general solution of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

How many parameters do you have?

We first find $\mathbf{x}$ for the system of equations.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 \\ -x_2 - 2x_3 + 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$
$$x_4 = 1$$
$$-x_2 - 2x_3 + 2(1) = 2 \to x_2 + 2x_3 = 0$$
$$x_1 + 0 + x_2 + x_3 + 4(1) = 3 \to x_1 + x_2 + x_3 = -1$$

Let $x_2, x_3 = 0$ such that

$$\mathbf{x}_p = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The general solution for the system of equations employs the basis of the null space (see Problem 3.6).

$$\mathbf{x} = \mathbf{x}_p + \alpha \mathbf{n}$$
$$\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

There is a single parameter α for the general solution.

3.9 Find the solution in Example 3.3 that has the smallest Euclidean norm.

Given

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 0 \\ -4 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 \\ -4 + \alpha_1 + 2\alpha_2 \\ -\alpha_1 \\ -\alpha_2 \end{bmatrix} \end{aligned}$$

We obtain an equation from the Euclidean norm of $\mathbf{x}$ and find its minimum.

$$
\begin{aligned}
\|\mathbf{x}\|_2 &= \sqrt{\alpha_1^2 + (-4+\alpha_1+2\alpha_2)^2 + \alpha_1^2 + \alpha_2^2} \\
\|\mathbf{x}\|_2^2 = f(\alpha_1,\alpha_2) &= 3\alpha_1^2 + 5\alpha_2^2 - 8\alpha_1 - 16\alpha_2 + 4\alpha_1\alpha_2 + 16 \\
\frac{\partial f}{\partial \alpha_1} &= 0 = 6\alpha_1 - 8 + 4\alpha_2 \\
\frac{\partial f}{\partial \alpha_2} &= 0 = 10\alpha_2 - 16 + 4\alpha_1 \\
\begin{bmatrix} 6 & 4 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} &= \begin{bmatrix} 8 \\ 16 \end{bmatrix} \\
\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} &= \begin{bmatrix} 4/11 \\ 16/11 \end{bmatrix}
\end{aligned}
$$

Evaluate $\mathbf{x}$ with the new values for α_1 and α_2.

$$
\mathbf{x} = \begin{bmatrix} 4/11 \\ -4 + (36/11) \\ -4/11 \\ -16/11 \end{bmatrix} = \begin{bmatrix} 4/11 \\ -8/11 \\ -4/11 \\ -16/11 \end{bmatrix}
$$

3.10 Find the solution in Problem 3.8 that has the smallest Euclidean norm.

The process is the same as Problem 3.9. Obtain an equation from the Euclidean norm and find its minimum.

$$
\begin{aligned}
\mathbf{x} &= \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1+\alpha \\ -2\alpha \\ \alpha \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\|\mathbf{x}\|_2 &= \sqrt{(-1+\alpha)^2 + 4\alpha^2 + \alpha^2 + 1} \\
\|\mathbf{x}\|_2^2 &= (-1+\alpha)^2 + 4\alpha^2 + \alpha^2 + 1 \\
&= \alpha^2 - 2\alpha + 1 + 5\alpha^2 + 1 \\
f(\alpha) &= 6\alpha^2 - 2\alpha + 2 \\
\frac{\partial f}{\partial \alpha} &= 0 = 12\alpha - 2 \\
\alpha &= 1/6
\end{aligned}
$$

Evaluate $\mathbf{x}$ with the new value for α.

$$
\mathbf{x} = \begin{bmatrix} -1+1/6 \\ -2(1/6) \\ 1/6 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/6 \\ -1/3 \\ 1/6 \\ 1 \end{bmatrix}
$$

3.11 Consider the equation

$$
\mathbf{x}[n] = \mathbf{A}^n x[0] + \mathbf{A}^{n-1}\mathbf{b}u[0] + \mathbf{A}^{n-2}\mathbf{b}u[1] + \ldots \mathbf{A}\mathbf{b}u[n-2] + \mathbf{b}u[n-1]
$$

where $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{b}$ is an $n \times 1$ column vector. Under what conditions on $\mathbf{A}$ and $\mathbf{b}$ will there exist $u[0], u[1], \ldots, u[n-1]$ to meet the equation for any $\mathbf{x}[n]$ and $\mathbf{x}[0]$? *Hint*: Write the equation in the form

$$
\mathbf{x}[n] - \mathbf{A}^n\mathbf{x}[0] = [\mathbf{b}\ \mathbf{A}\mathbf{b}\ \ldots \mathbf{A}^{n-1}\mathbf{b}] \begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[0] \end{bmatrix}
$$

Let $\mathbf{C} = [\mathbf{b}\ \mathbf{A}\mathbf{b}\ \ldots \mathbf{A}^{n-1}\mathbf{b}]$ and $\mathbf{u} = \begin{bmatrix} u[n-1] & u[n-2] & \cdots & u[0] \end{bmatrix}'$ such that the equation above can be rewritten as

$$
\mathbf{x}[n] - \mathbf{A}^n\mathbf{x}[0] = \mathbf{C}\mathbf{u}
$$

There will exist $\mathbf{u}$ for any $\mathbf{x}[n]$ and $\mathbf{x}[0]$ *if and only if* matrix $\mathbf{C}$ is invertible ($\mathbf{C}^{-1}$ must exist) . Such that

$$\mathbf{C}^{-1}(\mathbf{x}[n] - \mathbf{A}^n\mathbf{x}[0]) = \mathbf{u}$$

3.12 Given

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \bar{\mathbf{b}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

what are the representations of $\mathbf{A}$ with respect to the basis $[\mathbf{b},\ \mathbf{Ab},\ \mathbf{A}^2\mathbf{b},\ \mathbf{A}^3\mathbf{b}]$ and the basis $[\bar{\mathbf{b}},\ \mathbf{A}\bar{\mathbf{b}},\ \mathbf{A}^2\bar{\mathbf{b}},\ \mathbf{A}^3\bar{\mathbf{b}}]$, respectively? (Note that the representations are the same!)

– **b**

We obtain the vectors that form the basis.

$$\begin{aligned} \mathbf{Ab} &= \begin{bmatrix} 0 & 1 & 2 & 1 \end{bmatrix}' & \mathbf{A}^2\mathbf{b} &= \begin{bmatrix} 1 & 4 & 4 & 1 \end{bmatrix}' \\ \mathbf{A}^3\mathbf{b} &= \begin{bmatrix} 6 & 12 & 8 & 1 \end{bmatrix}' & \mathbf{A}^4\mathbf{b} &= \begin{bmatrix} 24 & 32 & 16 & 1 \end{bmatrix}' \end{aligned}$$

Verify that the set of vectors $\begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}$ is linearly independent.

$$\det(\begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}) = 1$$

The vectors are indeed linearly independent and can be used as a basis.

$$\begin{aligned} \mathbf{A}(\mathbf{b}) &= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}' \\ \mathbf{A}^2(\mathbf{b}) &= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}' \\ \mathbf{A}^3(\mathbf{b}) &= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}' \\ \mathbf{A}^4(\mathbf{b}) &= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} -8 & 20 & -18 & 7 \end{bmatrix}' \end{aligned}$$

The new representation of $\mathbf{A}$ is

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

– $\bar{\mathbf{b}}$

As stated in the problem, the representaion of $\mathbf{A}$ is the same as with $\begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}$

3.13 Find Jordan-form representations of the following matrices:

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$
$$\mathbf{A}_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \mathbf{A}_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

Note that all except A_4 can be diagonalized.

– $\mathbf{A}_1$

– $\mathbf{A}_2$

– $\mathbf{A}_3$

– $\mathbf{A}_4$

3.14 Consider the companion-form matrix

$$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Show that its characteristic polynomial is given by

$$\Delta(\lambda) = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4$$

Show also that if λ_i is an eigenvalue of A or a solution of $\Delta(\lambda) = 0$, then $[\lambda_i^3\lambda_i^2\lambda_i 1]'$ is an eigenvector of A associated with λ_i.

3.15 Show that the Vandermonde determinant

$$\begin{bmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

equals $\Pi_{1\leq i<j\leq 4}(\lambda_j - \lambda_i)$. Thus we conclude that the matrix is nonsingular or, equivalently, the eigenvectors are linearly independent if all eigenvalues are distinct.

3.16 Show that the companion-form matrix in Problem 3.14 is nonsingular if and only if $\alpha_4 \neq 0$. Under this assumption, show that its inverse equals

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\alpha_4 & -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{bmatrix}$$

3.17 Consider

$$A = \begin{bmatrix} \lambda & \lambda T & \lambda T^2/2 \\ 0 & \lambda & \lambda T \\ 0 & 0 & \lambda \end{bmatrix}$$

with $\lambda \neq 0$ and $T > 0$. Show that $[001]'$ is a generalized eigenvector of grade 3 and the three columns of

$$Q = \begin{bmatrix} \lambda^2 T^2 & \lambda T^2 & 0 \\ 0 & \lambda T & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

constitute a chain of generalized eigenvectors of length 3. Verify

$$Q^{-1}AQ = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

3.18 Find the characteristic polynomials and the minimal polynomials of the following matrices:

$$\begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \quad \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$
$$\begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} \quad \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

3.19 Show that if λ is an eigenvalue of A with eigenvector x, then $f(\lambda)$ is an eigenvalue of $f(A)$ with the same eigenvector x.

3.20 Show that an $n \times n$ matrix has the property $A^k = 0$ for $k \geq m$ if and only if A has eigenvalues 0 with multiplicity n and index m or less. Such a matrix is a $n$ilpotent matrix.

3.21 GIven

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

find A^{10}, A^{103}, and e^{At}.

3.22 Use two different methods to compute e^{At} for A_1 and A_4 in Problem 3.13.

3.23 Show that functions of the same matrix commute; that is,

$$f(A)g(A) = g(A)f(A)$$

Consequently we have $Ae^{At} = e^{At}A$.

3.24 Let

$$C = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Find a matrix B such that $e^B = C$. Show that if $\lambda_i = 0$ for some i, then B does not exist. Let

$$C = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Find a B such that $e^B = C$. Is it true that, for any nonsingular C, there exists a matrix B such that $e^B = C$?

- Obtain eigenvectors of the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Obtain a transformation matrix P such that $P^{-1}AP$ is diagonal. A is given by

$$\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$$

- Find a transformation matrix S such that

$$S^{-1}AS = J$$

where

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}, \qquad J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Find the Jordan-canonical-form representations of the following matrices:

$$A_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & -150 & -120 \\ 0 & 200 & 160 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 7/2 & 21/2 & 14 \\ -1/2 & -3/2 & -2 \\ -1/2 & -3/2 & -2 \end{bmatrix} \quad A_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Find the Jordan-canonical-form representations of the following matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & -4 & -3 & 4 \end{bmatrix}$$

References

[1] C.T. Chen. *Linear System Theory and Design*. Oxford series in electrical and computer engineering. Oxford University Press, 1999. ISBN: 9780195117776.