Homework 02

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3.1 Consider Fig 1. What is the representation of the vector x with respest to the basis $[q_1, i_2]$? What is the representation of q_1 with respect to the basis $[i_2, q_2]$

3.2 What are the 1-norm, 2-norm, and infinite-norm of the vectors

$$x_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3.3 Find two orthonormal vectors that span the same space as the two vectors in Problem 3.2.

3.4 Consider an $n \times m$ matrix A with $n \geq m$. If all columns of A are orthonormal, then $A'A = I_m$. What can you say about AA'?

3.5 Find the ranks and nullities of the following matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.6 Find bases of the range spaces and null spaces of the matrices in Problem 3.5.

3.7 Consider the linear algebraic equation

Figure 1: Different representations of vetor x.

$$\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = y$$

It has three equations and two unkowns. Does a solution x exist in the equations? Is the solution unique? Does a solution exist if $y = [111]'$?

3.8 Find the general solution of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

How many parameters do you have?

3.9 Find the solution in Example 3.3 that has the smallest Euclidean norm.

3.10 Find the solution in Problem 3.8 that has the smalles Euclidean norm.

3.11 Consider the equation

$$x[n] = A^n x[0] + A^{n-1} bu[0] + A^{n-2} bu[1] + \dots Abu[n-2] + bu[n-1]$$

where A is an $n \times n$ matrix and b is an $n \times 1$ column vector. Under what conditions on A and b will there exist $u[0], u[1], \dots, u[n-1]$ to meet the equation for any $x[n]$ and $x[0]$? *Hint*: Write the equation in the form

$$x[n] - A^n x[0] = [bAb \dots A^{n-1} b] \begin{bmatrix} u[n-1] \\ u[n-1] \\ \vdots \\ u[0] \end{bmatrix}$$

3.12 Given

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

what are the representations of A with respect to the basis $[b, Ab, A^2b, A^3b]$ and the basis $[\bar{b}, A\bar{b}, A^2\bar{b}, A^3\bar{b}]$, respectively? (Note that the representations are the same!)

3.13 Find Jordan-form representations of the following matrices:

$$A_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

Note that all except A_4 can be diagonalized.

3.14 Consider the companion-form matrix

$$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Show that its characteristic polynomial is given by

$$\Delta(\lambda) = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4$$

Show also that if λ_i is an eigenvalue of A or a solution of $\Delta(\lambda) = 0$, then $[\lambda_i^3 \lambda_i^2 \lambda_i 1]'$ is an eigenvector of A associated with λ_i.

3.15 Show that the Vandermonde determinant

$$\begin{bmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

equals $\Pi_{1\le i<j\le 4}(\lambda_j - \lambda_i)$. Thus we conclude that the matrix is nonsingular or, equivalently, the eigenvectors are linearly independent if all eigenvalues are distinct.

3.16 Show that the companion-form matrix in Problem 3.14 is nonsingular if and only if $\alpha_4 \neq 0$. Under this assumption, show that its inverse equals

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\alpha_4 & -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{bmatrix}$$

3.17 Consider

$$A = \begin{bmatrix} \lambda & \lambda T & \lambda T^2/2 \\ 0 & \lambda & \lambda T \\ 0 & 0 & \lambda \end{bmatrix}$$

with $\lambda \neq 0$ and $T > 0$. Show that $[001]'$ is a generalized eigenvector of grade 3 and the three columns of

$$Q = \begin{bmatrix} \lambda^2 T^2 & \lambda T^2 & 0 \\ 0 & \lambda T & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

constitute a chain of generalized eigenvectors of length 3. Verify

$$Q^{-1}AQ = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

3.18 Find the characteristic polynomials and the minimal polynomials of the following matrices:

$$\begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \quad \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} \quad \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

3.19 Show that if λ is an eigenvalue of A with eigenvector x, then $f(\lambda)$ is an eigenvalue of $f(A)$ with the same eigenvector x.

3.20 Show that an $n \times n$ matrix has the property $A^k = 0$ for $k \geq m$ if and only if A has eigenvalues 0 with multiplicity n and index m or less. Such a matrix is a $n$ilpotent matrix.

3.21 GIven

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

find A^{10}, A^{103}, and e^{At}.

3.22 Use two different methods to compute e^{At} for A_1 and A_4 in Problem 3.13.

3.23 Show that functions of the same matrix commute; that is,

$$f(A)g(A) = g(A)f(A)$$

Consequently we have $Ae^{At} = e^{At}A$.

3.24 Let

$$C = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Find a matrix B such that $e^B = C$. Show that if $\lambda_i = 0$ for some i, then B does not exist. Let

$$C = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Find a B such that $e^B = C$. Is it true that, for any nonsingular C, there exists a matrix B such that $e^B = C$?

- Obtain eigenvectors of the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Obtain a transformation matrix P such that $P^{-1}AP$ is diagonal. A is given by

$$\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$$

- Find a transformation matrix S such that

$$S^{-1}AS = J$$

where

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}, \qquad J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Find the Jordan-canonical-form representations of the following matrices:

$$A_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & -150 & -120 \\ 0 & 200 & 160 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 7/2 & 21/2 & 14 \\ -1/2 & -3/2 & -2 \\ -1/2 & -3/2 & -2 \end{bmatrix} \quad A_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Find the Jordan-canonical-form representations of the following matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & -4 & -3 & 4 \end{bmatrix}$$