Homework # 2

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3.1 Consider Fig 1. What is the representation of the vector $\mathbf{x}$ with respect to the basis $[\mathbf{q_1}, \mathbf{i_2}]$? What is the representation of $\mathbf{q_1}$ with respect to the basis $[\mathbf{i_2}, \mathbf{q_2}]$

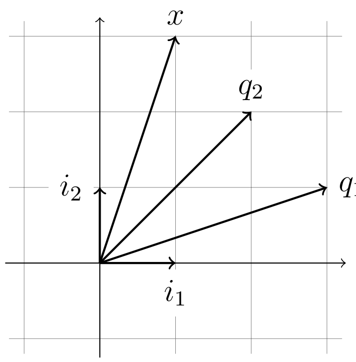

Figure 1: Different representations of vector x.

Given the vector $\mathbf{x} = [1\ 3]'$, we search for a linear combination of $[\mathbf{q_1}\ \mathbf{i_2}]$ to represent $\mathbf{x}$ in that basis.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = [\mathbf{q_1}\ \mathbf{i_2}] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

We solve this equation for $[\alpha_1\ \alpha_2]'$.

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$
$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 8/3 \end{bmatrix}$$

Given the vector $\mathbf{q_1} = [3\ 1]'$, we search for a linear combination of $[\mathbf{i_2}\ \mathbf{q_2}]$ to represent $\mathbf{q_1}$ in that basis.

$$\mathbf{q_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = [\mathbf{i_2}\ \mathbf{q_2}] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

We solve this equation for $[\beta_1\ \beta_2]'$.

$$\begin{aligned}
\begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \\
\begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \\
\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} &= \begin{bmatrix} -2 \\ 3/2 \end{bmatrix}
\end{aligned}$$

3.2 What are the 1-norm, 2-norm, and infinite-norm of the vectors

$$\mathbf{x_1} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \qquad \mathbf{x_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

From [1], we have the following definitions for each norm:

$$\begin{aligned}
\|\mathbf{x}\|_1 &= \sum_{i=1}^{n} |x_i| \\
\|\mathbf{x}\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \\
\|\mathbf{x}\|_\infty &= \max{}_i\ |x_i|
\end{aligned}$$

The norms for each vector are calculated accordingly.

- 1-norm

$$\begin{aligned}
\|\mathbf{x_1}\|_1 &= 2 + 3 + 1 \\
&= 6 \\
\|\mathbf{x_2}\|_1 &= 1 + 1 + 1 \\
&= 3
\end{aligned}$$

- 2-norm

$$\begin{aligned}
\|\mathbf{x_1}\|_2 &= \sqrt{2^2 + (-3)^2 + 1^2} \\
&= \sqrt{14} \\
\|\mathbf{x_2}\|_2 &= \sqrt{1^2 + 1^2 + 1^2} \\
&= \sqrt{3}
\end{aligned}$$

- inifinite-norm

$$\begin{aligned}
\|\mathbf{x_1}\|_\infty &= \max\{|2|, |-3|, |1|\} \\
&= 3 \\
\|\mathbf{x_2}\|_\infty &= \max\{|1|, |1|, |1|\} \\
&= 1
\end{aligned}$$

3.3 Find two orthonormal vectors that span the same space as the two vectors in Problem 3.2.

We obtain an orthonorнal set of vector from $\{\mathbf{x}_1, \mathbf{x}_2\}$ employing the Schmidt normalization procedure.

$$\begin{aligned}
\mathbf{u}_1 &:= \mathbf{x}_1 & \mathbf{q}_1 &:= \mathbf{u}_1/\|\mathbf{u}_1\| \\
\mathbf{u}_2 &:= \mathbf{x}_2 - (\mathbf{q}_1'\mathbf{x}_2)\mathbf{q}_1 & \mathbf{q}_2 &:= \mathbf{u}_2/\|\mathbf{u}_2\|
\end{aligned}$$

We first obtain the vectors $\mathbf{u}_1, \mathbf{q}_1$ and $\mathbf{u}_2$.

$$
\begin{aligned}
\mathbf{u}_1 &\coloneqq \mathbf{x}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \\
\mathbf{q}_1 &\coloneqq \mathbf{u}_1 / \|\mathbf{u}_1\| \\
&= \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \\
\mathbf{u}_2 &\coloneqq \mathbf{x}_2 - (\mathbf{q}_1' \mathbf{x}_2)\mathbf{q}_1 \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{14} \left(\begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{14}(0) \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

We notice that the original vectors were already orthogonal to each other. Thus, it is only necessary to normalize each of them. We continue with the Schmidt normalization procedure as the normalization step is the only one pending.

$$
\begin{aligned}
\mathbf{q}_2 &\coloneqq \mathbf{u}_2 / \|\mathbf{u}_2\| \\
&= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

The set of orthonormal vectors $\{\mathbf{q}_1, \mathbf{q}_2\}$ has now been obtained, where

$$\mathbf{q}_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$\mathbf{q}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3.4 Consider an $n \times m$ matrix $\mathbf{A}$ with $n \geq m$. If all columns of $\mathbf{A}$ are orthonormal, then $\mathbf{A}'\mathbf{A} = \mathbf{I}_m$. What can you say about $\mathbf{A}\mathbf{A}'$?

The result of $\mathbf{A}\mathbf{A}'$ will be a $n \times n$ matrix with values different from the identity matrix of the same dimensions ($\mathbf{A}\mathbf{A}' \neq \mathbf{I}_n$). This is due to the fact that while the *columns* of A are orthogonal to one another, its *rows* are not orthogonal to one another. As such, the product of its rows will yield a result different to the identity matrix.

3.5 Find the ranks and nullities of the following matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We evaluate the determinant of each matrix to determine if the matrices are full rank. If not, then the determinants of its minors are evaluated until finding a minor $\mathbf{M}_{ij}$ of dimensions $m \times m$, where $m < n$, such that $\det(M) \neq 0$. Note that for $\mathbf{A}_3$, a similar method employing its minors must be used due to the fact that $\mathbf{A}_3$ is not a square matrix.

– $\mathbf{A}_1$

$$\det(\mathbf{A}_1) = \det \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

We observe that its determinant is zero, thus not full rank. We now proceed to evaluate its minors to determine its rank.

$$\det(\mathbf{M}) = \det \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 1$$

Having verified that one of its 2×2 minors is invertible, we conclude that the rank and nullity of $\mathbf{A}_1$ are

$$\begin{aligned}\rho(\mathbf{A}_1) &= 2 \\ \text{nullity}(\mathbf{A}_1) &= \text{columns of } \mathbf{A}_1 - \rho(\mathbf{A}_1) = 3 - 2 = 1\end{aligned}$$

– $\mathbf{A}_2$
We obtain the determinant of the matrix to determine its rank.

$$\det(\mathbf{A}_2) = \det\left(\begin{bmatrix}4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0\end{bmatrix}\right) = -1$$

The matrix is full rank, thus

$$\begin{aligned}\rho(\mathbf{A}_2) &= 3 \\ \text{nullity}(\mathbf{A}_2) &= 0\end{aligned}$$

– $\mathbf{A}_3$
The third column of matrix $\mathbf{A}_3$ is a linear combination of the first two. A 3×3 matrix $\mathbf{B}$ can be constructed with columns one, two and four of $\mathbf{A}_3$ and its rank can then be determined.

$$\begin{aligned}\mathbf{A}_3 &= \begin{bmatrix}1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4\end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_4\end{bmatrix} = \begin{bmatrix}1 & 2 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 1\end{bmatrix} \\ \det(\mathbf{B}) &= \det\left(\begin{bmatrix}1 & 2 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 1\end{bmatrix}\right) \\ &= -1\end{aligned}$$

We conclude that the rank and nullity of $\mathbf{A}_3$ is thus

$$\begin{aligned}\rho(\mathbf{A}_3) &= \rho(\mathbf{B}) = 3 \\ \text{nullity}(\mathbf{A}_3) &= 1\end{aligned}$$

3.6 Find bases of the range spaces and null spaces of the matrices in Problem 3.5.

- $\mathbf{A}_1$

  Given that $\mathbf{A}_1 = [\mathbf{a}_1\, \mathbf{a}_2\, \mathbf{a}_3]$, we can construct a basis for the range space of the matrix with columns $\mathbf{a}_2$ and $\mathbf{a}_3$.

$$\text{range space}(\mathbf{A}_1) = \{\mathbf{a}_2, \mathbf{a}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

  The basis for the null space of $\mathbf{A}_1$ is determined by finding a vector $\mathbf{n}$ that satisfies the equation

$$\mathbf{A}_1\mathbf{n} = \mathbf{0}$$

  Let $\mathbf{n} = \begin{bmatrix} n_{11} & n_{21} & n_{31} \end{bmatrix}'$ and solve the previous system of equations for $\mathbf{n}$.

$$\begin{aligned} \mathbf{A}_1\mathbf{n} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \end{bmatrix} = \mathbf{0} \\ &= \begin{bmatrix} 0 + n_{21} + 0 \\ 0 + 0 + 0 \\ 0 + 0 + n_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ n_{21} = 0 \qquad & n_{31} = 0 \\ \mathbf{n} &= \begin{bmatrix} n_{11} \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

  We propose $n_{11} = 1$ as a particular solution.

$$\text{null space}(\mathbf{A}_1) = \{\mathbf{n}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

- $\mathbf{A}_2$

  Matrix $\mathbf{A}_2$ is full rank, thus its columns can form a basis for the range space. By consequence the only solution to $\mathbf{A}_2\mathbf{n} = \mathbf{0}$ is the trivial solution.

$$\text{range space}(\mathbf{A}_2) = \left\{ \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\text{null space}(\mathbf{A}_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

– $\mathbf{A}_3$

For matrix $\mathbf{A}_3$, three of its four columns can be used as a basis for the range space. It is only necessary for the set of vectors to be linearly independent. As such, the basis of $\mathbf{A}_3$ is constructed as follows

$$\mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$$

$$\text{range space}(\mathbf{A}_3) = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}$$

As for the basis for the null space of $\mathbf{A}_3$, it is necessary to solve the equation $\mathbf{A}_3\mathbf{n} = \mathbf{0}$.

$$\begin{aligned}
\mathbf{A}_3\mathbf{n} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \\ n_{41} \end{bmatrix} &= \begin{bmatrix} n_{11} + 2n_{21} + 3n_{31} + 4n_{41} \\ 0 - n_{21} - 2n_{31} + 2n_{41} \\ 0 + 0 + 0 + n_{41} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
n_{41} &= 0 \\
-n_{21} - 2n_{31} + 2(0) = 0 \rightarrow n_{21} &= -2n_{31} \\
n_{11} + 2(-2n_{31}) + 3n_{31} + 4(0) = 0 \rightarrow n_{11} &= n_{31} \\
\mathbf{n} &= \begin{bmatrix} n_{11} \\ -2n_{11} \\ n_{11} \\ 0 \end{bmatrix}
\end{aligned}$$

We propose $n_{11} = 1$ as a particular solution.

$$\text{null space}(\mathbf{A}_3) = \mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

3.7 Consider the linear algebraic equation

$$\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{y}$$

It has three equations and two unkowns. Does a solution $\mathbf{x}$ exist in the equations? Is the solution unique? Does a solution exist if $\mathbf{y} = [111]'$?

Yes, there is a solution to the system of equations. It is an *unique* solution. This is due to the fact that the second row of the expanded matrix is a linear combination of the other two rows of the matrix. In other words, the rank of the matrix matches the number of unkowns ($\rho(\mathbf{A}) = 2$).

$$\begin{aligned} \begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2x_1 - x_2 \\ -3x_1 + 3x_2 \\ -x_1 + 2x_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ -3x_1 + 3x_2 &= 0 \rightarrow x_1 = x_2 \\ 2x_1 - x_2 &= 1 \rightarrow x_1 = 1 \rightarrow x_2 = 1 \\ -x_1 + 2x_2 &= 1 \rightarrow x_2 = 1 \end{aligned}$$

There is no contradiction in the system of equations.

For $\mathbf{y} = [1\ 1\ 1]'$ there is no solution to the system of equations. The second row is no longer a linear combination of the other two. When solving the system of equations we arrive at a contradiction.

$$\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 - x_2 \\ -3x_1 + 3x_2 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$-3x_1 + 3x_2 = 1 \to x_1 = x_2 - \frac{1}{3}$$

$$2x_1 - x_2 = 1 \to x_2 = \frac{5}{3} \to x_1 = \frac{4}{3}$$

But

$$-x_1 + 2x_2 = 1 = -\frac{4}{3} + 2(\frac{5}{3}) = 2 \neq 1$$

We have arrived at a contradiction, thus the system has no solution.

3.8 Find the general solution of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

How many parameters do you have?

We first find $\mathbf{x}$ for the system of equations.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 \\ -x_2 - 2x_3 + 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$x_4 = 1$$

$$-x_2 - 2x_3 + 2(1) = 2 \to x_2 + 2x_3 = 0$$

$$x_1 + 0 + x_2 + x_3 + 4(1) = 3 \to x_1 + x_2 + x_3 = -1$$

Let $x_2, x_3 = 0$ such that

$$\mathbf{x}_p = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The general solution for the system of equations employs the basis of the null space (see Problem 3.6).

$$\mathbf{x} = \mathbf{x}_p + \alpha \mathbf{n}$$

$$\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

There is a single parameter α for the general solution.

3.9 Find the solution in Example 3.3 that has the smallest Euclidean norm.

Given

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 0 \\ -4 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 \\ -4 + \alpha_1 + 2\alpha_2 \\ -\alpha_1 \\ -\alpha_2 \end{bmatrix} \end{aligned}$$

We obtain an equation from the Euclidean norm of $\mathbf{x}$ and find its minimum.

$$
\begin{aligned}
\|\mathbf{x}\|_2 &= \sqrt{\alpha_1^2 + (-4 + \alpha_1 + 2\alpha_2)^2 + \alpha_1^2 + \alpha_2^2} \\
\|\mathbf{x}\|_2^2 = f(\alpha_1, \alpha_2) &= 3\alpha_1^2 + 5\alpha_2^2 - 8\alpha_1 - 16\alpha_2 + 4\alpha_1\alpha_2 + 16 \\
\frac{\partial f}{\partial \alpha_1} &= 0 = 6\alpha_1 - 8 + 4\alpha_2 \\
\frac{\partial f}{\partial \alpha_2} &= 0 = 10\alpha_2 - 16 + 4\alpha_1 \\
\begin{bmatrix} 6 & 4 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} &= \begin{bmatrix} 8 \\ 16 \end{bmatrix} \\
\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} &= \begin{bmatrix} 4/11 \\ 16/11 \end{bmatrix}
\end{aligned}
$$

Evaluate $\mathbf{x}$ with the new values for α_1 and α_2.

$$
\mathbf{x} = \begin{bmatrix} 4/11 \\ -4 + (36/11) \\ -4/11 \\ -16/11 \end{bmatrix} = \begin{bmatrix} 4/11 \\ -8/11 \\ -4/11 \\ -16/11 \end{bmatrix}
$$

3.10 Find the solution in Problem 3.8 that has the smallest Euclidean norm.

The process is the same as Problem 3.9. Obtain an equation from the Euclidean norm and find its minimum.

$$
\begin{aligned}
\mathbf{x} &= \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 + \alpha \\ -2\alpha \\ \alpha \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\|\mathbf{x}\|_2 &= \sqrt{(-1+\alpha)^2 + 4\alpha^2 + \alpha^2 + 1} \\
\|\mathbf{x}\|_2^2 &= (-1+\alpha)^2 + 4\alpha^2 + \alpha^2 + 1 \\
&= \alpha^2 - 2\alpha + 1 + 5\alpha^2 + 1 \\
f(\alpha) &= 6\alpha^2 - 2\alpha + 2 \\
\frac{\partial f}{\partial \alpha} &= 0 = 12\alpha - 2 \\
\alpha &= 1/6
\end{aligned}
$$

Evaluate $\mathbf{x}$ with the new value for α.

$$
\mathbf{x} = \begin{bmatrix} -1 + 1/6 \\ -2(1/6) \\ 1/6 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/6 \\ -1/3 \\ 1/6 \\ 1 \end{bmatrix}
$$

3.11 Consider the equation

$$
\mathbf{x}[n] = \mathbf{A}^n x[0] + \mathbf{A}^{n-1}\mathbf{b}u[0] + \mathbf{A}^{n-2}\mathbf{b}u[1] + \ldots \mathbf{Ab}u[n-2] + \mathbf{b}u[n-1]
$$

where $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{b}$ is an $n \times 1$ column vector. Under what conditions on $\mathbf{A}$ and $\mathbf{b}$ will there exist $u[0], u[1], \ldots, u[n-1]$ to meet the equation for any $\mathbf{x}[n]$ and $\mathbf{x}[0]$? *Hint*: Write the equation in the form

$$
\mathbf{x}[n] - \mathbf{A}^n\mathbf{x}[0] = [\mathbf{b}\ \mathbf{Ab}\ \ldots \mathbf{A}^{n-1}\mathbf{b}] \begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[0] \end{bmatrix}
$$

Let $\mathbf{C} = [\mathbf{b}\ \mathbf{Ab}\ \ldots \mathbf{A}^{n-1}\mathbf{b}]$ and $\mathbf{u} = \begin{bmatrix} u[n-1] & u[n-2] & \cdots & u[0] \end{bmatrix}'$ such that the equation above can be rewritten as

$$
\mathbf{x}[n] - \mathbf{A}^n\mathbf{x}[0] = \mathbf{Cu}
$$

There will exist $\mathbf{u}$ for any $\mathbf{x}[n]$ and $\mathbf{x}[0]$ *if and only if* matrix $\mathbf{C}$ is invertible ($\mathbf{C}^{-1}$ must exist) . Such that

$$\mathbf{C}^{-1}(\mathbf{x}[n] - \mathbf{A}^n\mathbf{x}[0]) = \mathbf{u}$$

3.12 Given

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \bar{\mathbf{b}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

what are the representations of $\mathbf{A}$ with respect to the basis $[\mathbf{b},\ \mathbf{Ab},\ \mathbf{A}^2\mathbf{b},\ \mathbf{A}^3\mathbf{b}]$ and the basis $[\bar{\mathbf{b}},\ \mathbf{A}\bar{\mathbf{b}},\ \mathbf{A}^2\bar{\mathbf{b}},\ \mathbf{A}^3\bar{\mathbf{b}}]$, respectively? (Note that the representations are the same!)

- **b**

  We obtain the vectors that form the basis.

$$\begin{aligned} \mathbf{Ab} &= \begin{bmatrix} 0 & 1 & 2 & 1 \end{bmatrix}' & \mathbf{A}^2\mathbf{b} &= \begin{bmatrix} 1 & 4 & 4 & 1 \end{bmatrix}' \\ \mathbf{A}^3\mathbf{b} &= \begin{bmatrix} 6 & 12 & 8 & 1 \end{bmatrix}' & \mathbf{A}^4\mathbf{b} &= \begin{bmatrix} 24 & 32 & 16 & 1 \end{bmatrix}' \end{aligned}$$

  Verify that the set of vectors $\begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}$ is linearly independent.

$$\det(\begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}) = 1$$

  The vectors are indeed linearly independent and can be used as a basis.

$$\begin{aligned} \mathbf{A}(\mathbf{b}) &= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}' \\ \mathbf{A}^2(\mathbf{b}) &= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}' \\ \mathbf{A}^3(\mathbf{b}) &= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}' \\ \mathbf{A}^4(\mathbf{b}) &= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} -8 & 20 & -18 & 7 \end{bmatrix}' \end{aligned}$$

  The new representation of $\mathbf{A}$ is

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

- $\bar{\mathbf{b}}$

  As stated in the problem, the representaion of $\mathbf{A}$ is the same as with $\begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}$

3.13 Find Jordan-form representations of the following matrices:

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$
$$\mathbf{A}_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \mathbf{A}_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

Note that all except A_4 can be diagonalized.

For all matrices, the Jordan form is obtained from the eigenvalues and eigenvectors of the matrix. This done by obtaining solutions to the following equations

$$\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = 0$$
$$(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{q}_i = 0$$

- $\mathbf{A}_1$

  The matrix is triangular, thus its eigenvalues are the values of the diagonal.

$$\hat{\mathbf{A}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

  We obtain $\mathbf{Q} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix}$.

$$(\mathbf{A} - \lambda_1)\mathbf{q}_1 = \mathbf{0}$$
$$\begin{bmatrix} 0 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{q}_1 = \mathbf{0}$$

$$\begin{bmatrix} 0 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

$$(\mathbf{A} - \lambda_2)\mathbf{q}_2 = \mathbf{0}$$
$$\begin{bmatrix} -1 & 4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{q}_2 = \mathbf{0}$$

$$\begin{bmatrix} -1 & 4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$$

$$(\mathbf{A} - \lambda_3)\mathbf{q}_3 = \mathbf{0}$$
$$\begin{bmatrix} -2 & 4 & 10 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{q}_3 = \mathbf{0}$$

$$\begin{bmatrix} -2 & 4 & 10 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

Thus the Jordan form is

$$\hat{\mathbf{A}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
$$\mathbf{Q} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– $\mathbf{A}_2$
  We obtain the eigenvalues of the matrix.

$$\det(\lambda\mathbf{I} - \mathbf{A}_2) = \det(\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2 & 4 & \lambda+3 \end{bmatrix}) = \mathbf{0}$$
$$= \lambda(\lambda^2 + 3\lambda + 4) - 2$$
$$= \lambda^3 + 3\lambda^2 + 4\lambda - 2$$
$$= (\lambda+1)(\lambda+1+j)(\lambda+1-j)$$

$$\hat{\mathbf{A}}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1-j & 0 \\ 0 & 0 & -1+j \end{bmatrix}$$

Determine the eigenvectors of matrix $\mathbf{A}_2$.

$$(\mathbf{A} - \lambda_1)\mathbf{q}_1 = \mathbf{0}$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix} \mathbf{q}_1 = \mathbf{0}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \mathbf{0}$$

$$(\mathbf{A} - \lambda_2)\mathbf{q}_2 = \mathbf{0}$$
$$\begin{bmatrix} 1+j & 1 & 0 \\ 0 & 1+j & 1 \\ -2 & -4 & -2+j \end{bmatrix} \mathbf{q}_2 = \mathbf{0}$$

$$\begin{bmatrix} 1+j & 1 & 0 \\ 0 & 1+j & 1 \\ -2 & -4 & -2+j \end{bmatrix} \begin{bmatrix} -1 \\ 1+j \\ -2j \end{bmatrix} = \mathbf{0}$$

$$(\mathbf{A} - \lambda_3)\mathbf{q}_3 = \mathbf{0}$$
$$\begin{bmatrix} 1-j & 1 & 0 \\ 0 & 1-j & 1 \\ -2 & -4 & -2-j \end{bmatrix} \mathbf{q}_3 = \mathbf{0}$$

$$\begin{bmatrix} 1-j & 1 & 0 \\ 0 & 1-j & 1 \\ -2 & -4 & -2-j \end{bmatrix} \begin{bmatrix} -1 \\ 1-j \\ 2j \end{bmatrix} = \mathbf{0}$$

Thus the Jordan form is

$$\hat{\mathbf{A}}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1-j & 0 \\ 0 & 0 & -1+j \end{bmatrix}$$
$$\mathbf{Q} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1+j & 1-j \\ 1 & -2j & 2j \end{bmatrix}$$

– $\mathbf{A}_3$

The matrix is a triangular matrix, just like $\mathbf{A}_1$.

$$\hat{\mathbf{A}}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We obtain $\mathbf{Q}$ by finding the eigenvectors of the matrix.

$$(\mathbf{A} - \lambda_1)\mathbf{q}_1 = \mathbf{0}$$
$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{q}_1 = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

$$(\mathbf{A} - \lambda_2)\mathbf{q}_2 = \mathbf{0}$$
$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{q}_2 = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$$

$$(\mathbf{A} - \lambda_3)\mathbf{q}_3 = \mathbf{0}$$
$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{q}_3 = \mathbf{0}$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

Thus the Jordan form is

$$\hat{\mathbf{A}}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– $\mathbf{A}_4$

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}_3) &= \det\left(\begin{bmatrix} \lambda & -4 & -3 \\ 0 & \lambda - 20 & -16 \\ 0 & 25 & \lambda + 20 \end{bmatrix}\right) \\ &= \lambda((\lambda - 20)(\lambda + 20) - 25(-16)) \\ &= \lambda(\lambda^2 - 400 + 400) \\ \lambda^3 = 0 &\to \lambda_{1,2,3} = 0 \end{aligned}$$

The new representation of A is then

$$\hat{\mathbf{A}}_4 = \mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Due to the repeated eigenvalues of the matrix, it is necessary to obtain generalized vectors for the matrix.
We obtain the basis.

$$(\mathbf{A}-\lambda\mathbf{I}) = \mathbf{A}$$
$$(\mathbf{A}-\lambda\mathbf{I})^2 = \begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$(\mathbf{A}-\lambda\mathbf{I})^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find a solution to

$$(\mathbf{A}-\lambda\mathbf{I})^3\mathbf{v} = \mathbf{0}$$
$$(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v} = \mathbf{v}_2$$
$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v} = \mathbf{v}_3$$

$$(\mathbf{A}-\lambda\mathbf{I})^3\mathbf{v} = \mathbf{0}$$
$$(\mathbf{A}-\lambda\mathbf{I})^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v} = \mathbf{v}_2$$
$$(\mathbf{A}-\lambda\mathbf{I})^2 = \begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v} = \mathbf{v}_3$$
$$(\mathbf{A}-\lambda\mathbf{I}) = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} \to \mathbf{v}_3 = \begin{bmatrix} 4 \\ 20 \\ -25 \end{bmatrix}$$

Thus the Jordan form is

$$\hat{\mathbf{A}}_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 5 & 4 & 0 \\ 0 & 20 & 1 \\ 0 & -25 & 0 \end{bmatrix}$$

3.14 Consider the companion-form matrix

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Show that its characteristic polynomial is given by

$$\Delta(\lambda) = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4$$

Show also that if λ_i is an eigenvalue of $\mathbf{A}$ or a solution of $\Delta(\lambda) = 0$, then $[\lambda_i^3 \ \lambda_i^2 \ \lambda_i \ 1]'$ is an eigenvector of $\mathbf{A}$ associated with λ_i.

We obtain the characteristic polynomial by solving $\det(\lambda\mathbf{I} - \mathbf{A}) = \mathbf{0}$.

$$\begin{aligned}
\det(\lambda\mathbf{I} - \mathbf{A}) &= \det\left(\begin{bmatrix} \lambda + \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{bmatrix}\right) \\
&= (\lambda + \alpha_1)\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{bmatrix}\right) \\
&- (-1)\det\left(\begin{bmatrix} \alpha_2 & \alpha_3 & \alpha_4 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{bmatrix}\right) \\
&= (\lambda + \alpha_1)(\lambda^3) + \alpha_2(\lambda^2) - \alpha_3(-\lambda) + \alpha_4(1) \\
&= \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 \\
&= \Delta(\lambda)
\end{aligned}$$

If $\Delta(\lambda_i) = 0$ then

$$
\begin{aligned}
\Delta(\lambda_i) &= 0 \\
&= \lambda_i^4 + \alpha_1\lambda_i^3 + \alpha_2\lambda_i^2 + \alpha_3\lambda_i + \alpha_4 \\
\lambda_i^4 &= -\alpha_1\lambda_i^3 - \alpha_2\lambda_i^2 - \alpha_3\lambda_i - \alpha_4
\end{aligned}
$$

Evaluate $\mathbf{A}\begin{bmatrix}\lambda_i^3 & \lambda_i^2 & \lambda_i & 1\end{bmatrix}'$.

$$
\begin{aligned}
\mathbf{A}\begin{bmatrix}\lambda_i^3\\ \lambda_i^2\\ \lambda_i\\ 1\end{bmatrix} &= \begin{bmatrix}-\alpha_1\lambda_i^3 - \alpha_2\lambda_i^2 - \alpha_3\lambda_i - \alpha_4\\ \lambda_i^3\\ \lambda_i^2\\ \lambda_i\end{bmatrix} \\
&= \begin{bmatrix}\lambda_i^4\\ \lambda_i^3\\ \lambda_i^2\\ \lambda_i\end{bmatrix} \\
\mathbf{A}\begin{bmatrix}\lambda_i^3\\ \lambda_i^2\\ \lambda_i\\ 1\end{bmatrix} &= \lambda_i\begin{bmatrix}\lambda_i^3\\ \lambda_i^2\\ \lambda_i\\ 1\end{bmatrix}
\end{aligned}
$$

The vector $\begin{bmatrix}\lambda_i^3 & \lambda_i^2 & \lambda_i & 1\end{bmatrix}'$ is an eigenvector.

3.15 Show that the Vandermonde determinant

$$
\begin{bmatrix}\lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3\\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2\\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\\ 1 & 1 & 1 & 1\end{bmatrix}
$$

equals $\Pi_{1\le i<j\le 4}(\lambda_j - \lambda_i)$. Thus we conclude that the matrix is nonsingular or, equivalently, the eigenvectors are linearly independent if all eigenvalues are distinct.

By properties of the determinant, we first subtract the fourth column from all other columns.

$$
\det\left(\begin{bmatrix}\lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3\\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2\\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\\ 1 & 1 & 1 & 1\end{bmatrix}\right) = \det\left(\begin{bmatrix}\lambda_1^3-\lambda_4^3 & \lambda_2^3-\lambda_4^3 & \lambda_3^3-\lambda_4^3 & \lambda_4^3\\ \lambda_1^2-\lambda_4^2 & \lambda_2^2\lambda_4^2 & \lambda_3^2-\lambda_4^2 & \lambda_4^2\\ \lambda_1-\lambda_4 & \lambda_2-\lambda_4 & \lambda_3-\lambda_4 & \lambda_4\\ 0 & 0 & 0 & 1\end{bmatrix}\right)
$$

The determinant of this 4×4 matrix can then be reduced to the determinant of a 3×3 matrix because of the zeros in the 4th row.

$$\det\left(\begin{bmatrix} \lambda_1^3 - \lambda_4^3 & \lambda_2^3 - \lambda_4^3 & \lambda_3^3 - \lambda_4^3 & \lambda_4^3 \\ \lambda_1^2 - \lambda_4^2 & \lambda_2^2\lambda_4^2 & \lambda_3^2 - \lambda_4^2 & \lambda_4^2 \\ \lambda_1 - \lambda_4 & \lambda_2 - \lambda_4 & \lambda_3 - \lambda_4 & \lambda_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda_1^3 - \lambda_4^3 & \lambda_2^3 - \lambda_4^3 & \lambda_3^3 - \lambda_4^3 \\ \lambda_1^2 - \lambda_4^2 & \lambda_2^2\lambda_4^2 & \lambda_3^2 - \lambda_4^2 \\ \lambda_1 - \lambda_4 & \lambda_2 - \lambda_4 & \lambda_3 - \lambda_4 \end{bmatrix}\right)$$

From the identity

$$x^m - y^m = (x-y)(x^{m-1} + x^{m-2}y + \cdots + y^{m-1})$$

we conclude that for any column i, it is divisible by $x_i - x_j$. This implies that $x_i - x_j$ can also divide the determinant.

$$= (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)\det\left(\begin{bmatrix} \lambda_1^2 - \lambda_1\lambda_4 + \lambda_4^2 & \lambda_2^2 - \lambda_2\lambda_4 + \lambda_4^2 & \lambda_3^2 - \lambda_3\lambda_4 + \lambda_4^2 \\ \lambda_1 + \lambda_4 & \lambda_2 + \lambda_4 & \lambda_3 + \lambda_4 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

We continue the use of the two properties presented to further reduce the matrix and simplify the result.

$$\begin{aligned}
&= \prod_{i<j\le 4}(\lambda_i - \lambda_j)\det\left(\begin{bmatrix} \lambda_1^2 - \lambda_1\lambda_4 + \lambda_4^2 & \lambda_2^2 - \lambda_2\lambda_4 + \lambda_4^2 & \lambda_3^2 - \lambda_3\lambda_4 + \lambda_4^2 \\ \lambda_1 + \lambda_4 & \lambda_2 + \lambda_4 & \lambda_3 + \lambda_4 \\ 1 & 1 & 1 \end{bmatrix}\right) \\
&= \prod_{i<j\le 4}(\lambda_i - \lambda_j)\times \\
&\quad \det\left(\begin{bmatrix} (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 + \lambda_4) & (\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 + \lambda_4) & \lambda_3^2 - \lambda_3\lambda_4 + \lambda_4^2 \\ \lambda_1 - \lambda_2 & \lambda_2 - \lambda_3 & \lambda_3 + \lambda_4 \\ 0 & 0 & 1 \end{bmatrix}\right) \\
&= \prod_{i<j\le 4}(\lambda_i - \lambda_j)\det\left(\begin{bmatrix} (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 + \lambda_4) & (\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 + \lambda_4) \\ \lambda_1 - \lambda_2 & \lambda_2 - \lambda_3 \end{bmatrix}\right) \\
&= \prod_{i<j\le 4}(\lambda_i - \lambda_j)(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)\det\left(\begin{bmatrix} \lambda_1 + \lambda_2 + \lambda_4 & \lambda_2 + \lambda_3 + \lambda_4 \\ 1 & 1 \end{bmatrix}\right) \\
&= \prod_{i<j\le 4}(\lambda_i - \lambda_j)(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3) \\
&= \prod_{1\le i<j\le 4}(\lambda_i - \lambda_j)
\end{aligned}$$

The determinant will be different from zero if all eigenvalues are distinct.

3.16 Show that the companion-form matrix in Problem 3.14 is nonsingular if and only if $\alpha_4 \neq 0$. Under this assumption, show that its inverse equals

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\alpha_4 & -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{bmatrix}$$

We test the assumption by multiplying $\mathbf{A}$ with the proposed inverse.

$$\begin{aligned} \mathbf{A}\mathbf{A}^{-1} &= \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\alpha_4 & -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{bmatrix} \\ &= \begin{bmatrix} -\alpha_4(-1/\alpha_4) & -\alpha_1 - \alpha_4(-\alpha_1/\alpha_4) & -\alpha_2 - \alpha_4(-\alpha_2/\alpha_4) & -\alpha_3 - \alpha_4(-\alpha_3/\alpha_4) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \mathbf{I} \end{aligned}$$

The proposed matrix is indeed the inverse of $\mathbf{A}$. For this to be true, α_4 must be different from zero. If not, then the matrix becomes singular.

3.17 Consider

$$\mathbf{A} = \begin{bmatrix} \lambda & \lambda T & \lambda T^2/2 \\ 0 & \lambda & \lambda T \\ 0 & 0 & \lambda \end{bmatrix}$$

with $\lambda \neq 0$ and $T > 0$. Show that $[0\ 0\ 1]'$ is a generalized eigenvector of grade 3 and the three columns of

$$\mathbf{Q} = \begin{bmatrix} \lambda^2 T^2 & \lambda T^2 & 0 \\ 0 & \lambda T & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

constitute a chain of generalized eigenvectors of length 3. Verify

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

Obtain the eigenvectors of $\mathbf{A}$.

$$(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 0 & \lambda T & \lambda T^2/2 \\ 0 & 0 & \lambda T \\ 0 & 0 & 0 \end{bmatrix}$$

Given $T > 0$ and $\lambda \neq 0$ then $\rho(\mathbf{A}) = 2$. Thus its nullity is one.

We construct generalized vectors for the matrix employing the same procedure as Problem 3.13.

We obtain the basis.

$$(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 0 & \lambda T & \lambda T^2/2 \\ 0 & 0 & \lambda T \\ 0 & 0 & 0 \end{bmatrix}$$
$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & \lambda^2 T^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$(\mathbf{A} - \lambda\mathbf{I})^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find a solution to

$$(\mathbf{A} - \lambda\mathbf{I})^3\mathbf{v} = \mathbf{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v} = \mathbf{v}_2$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{v}_3$$

$$(\mathbf{A} - \lambda\mathbf{I})^3\mathbf{v} = \mathbf{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v} = \mathbf{v}_2$$
$$(\mathbf{A}-\lambda\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & \lambda^2T^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \mathbf{v}_2 = \begin{bmatrix} \lambda^2T^2 \\ 0 \\ 0 \end{bmatrix}$$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v} = \mathbf{v}_3$$
$$(\mathbf{A}-\lambda\mathbf{I}) = \begin{bmatrix} 0 & \lambda T & \lambda T^2/2 \\ 0 & 0 & \lambda T \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \mathbf{v}_3 = \begin{bmatrix} \lambda T^2/2 \\ \lambda T \\ 0 \end{bmatrix}$$

Thus the Jordan form is

$$\hat{\mathbf{A}} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$
$$\mathbf{Q} = \begin{bmatrix} \lambda^2T^2 & \lambda T^2/2 & 0 \\ 0 & \lambda T & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We verify $\hat{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ by performing the multiplication of $\mathbf{Q}\hat{\mathbf{A}} = \mathbf{A}\mathbf{Q}$.

$$\begin{aligned} \mathbf{A}\mathbf{Q} &= \begin{bmatrix} \lambda & \lambda T & \lambda T^2/2 \\ 0 & \lambda & \lambda T \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda^2T^2 & \lambda T^2 & 0 \\ 0 & \lambda T & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda^3 & 3\lambda^2T^2/2 & \lambda T^2/2 \\ 0 & \lambda^2 T & 0 \\ 0 & 0 & \lambda \end{bmatrix} \\ \mathbf{Q}\hat{\mathbf{A}} &= \begin{bmatrix} \lambda^2T^2 & \lambda T^2 & 0 \\ 0 & \lambda T & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & \lambda T & \lambda T^2/2 \\ 0 & \lambda & \lambda T \\ 0 & 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} \lambda^3 & 3\lambda^2T^2/2 & \lambda T^2/2 \\ 0 & \lambda^2 T & 0 \\ 0 & 0 & \lambda \end{bmatrix} \end{aligned}$$

It is the same matrix, thus we have proofed the original statement.

Note The matrix $\mathbf{Q}$ presented in the intructions is wrong. The answer provided meets the criteria, whilst the original one does not.

3.18 Find the characteristic polynomials and the minimal polynomials of the following matrices:

$$m_1 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \quad m_2 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$
$$m_3 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} \quad m_4 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

We determine the polynomials employing the following equations

$$\Delta(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) = \prod_i (\lambda - \lambda_i)^{n_i} \quad \text{Characteristic polynomial}$$
$$\psi(\lambda) = \prod_i (\lambda - \lambda_i)^{\bar{n}_i} \quad \text{Minimal polynomial}$$

where $\bar{n}_i$ is the largest order of all Jordan blocks of the matrix.

– m_1

For the triangular matrix, the characteristic equation is obtained from the elements in its diagonal.

$$\Delta(m_1) = (\lambda - \lambda_1)^3(\lambda - \lambda_2)$$

The Jordan block for this matrix are sizes one and three, thus

$$\psi(m_1) = \Delta(m_1) = (\lambda - \lambda_1)^3(\lambda - \lambda_2)$$

– m_2

For the triangular matrix, the characteristic equation is obtained from the elements in its diagonal.

$$\Delta(m_2) = (\lambda - \lambda_1)^4$$

The Jordan block for this matrix ($\mathbf{J} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$) is size three, thus

$$\psi(m_2) = (\lambda - \lambda_1)^3$$

- m_3

  For the triangular matrix, the characteristic equation is obtained from the elements in its diagonal.

  $$\Delta(m_3) = (\lambda - \lambda_1)^4$$

  The Jordan block for this matrix ($\mathbf{J} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$) is size two, thus

  $$\psi(m_3) = (\lambda - \lambda_1)^2$$

- m_4

  For the triangular matrix, the characteristic equation is obtained from the elements in its diagonal.

  $$\Delta(m_4) = (\lambda - \lambda_1)^4$$

  The Jordan block for this matrix is size one, thus

  $$\psi(m_4) = (\lambda - \lambda_1)$$

3.19 Show that if λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{x}$, then $f(\lambda)$ is an eigenvalue of $f(\mathbf{A})$ with the same eigenvector $\mathbf{x}$.

We prove this statement by using the Cayley-Hamilton theorem. If λ is an eigenvalue of $\mathbf{A}$ then

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \rightarrow \mathbf{A}^n\mathbf{x} = \lambda^n\mathbf{x}$$

Then $f(\mathbf{A})$ can be expressed as a linear combination of linear matrices.

$$\begin{aligned} f(\mathbf{A}) &= \beta_0\mathbf{I} + \beta_1\mathbf{A} + \cdots + \beta_n\mathbf{A}^{n-1} \\ &= \beta_0 + \beta_1\lambda + \cdots + \beta_n\lambda^{n-1} \\ f(\mathbf{A})\mathbf{x} &= (\beta_0 + \beta_1\lambda + \cdots + \beta_n\lambda^{n-1})\mathbf{x} \\ &= f(\lambda)\mathbf{x} \end{aligned}$$

This proves that $f(\lambda)$ is an eigenvalue of $f(\mathbf{A})$.

3.20 Show that an $n \times n$ matrix has the property $\mathbf{A}^k = \mathbf{0}$ for $k \geq m$ if and only if $\mathbf{A}$ has eigenvalues 0 with multiplicity n and index m or less. Such a matrix is a *nilpotent* matrix.

Assume that $\mathbb{A}$ is in Jordan form, such that $\mathbf{A}^k = \mathbf{Q}\hat{\mathbf{A}}^k\mathbf{Q}^{-1}$.

Given the premise $\mathbf{A}^k = \mathbf{0}$ then $\hat{\mathbf{A}}^k = \mathbf{0}$. Recall that

$$\hat{\mathbf{A}} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

where λ_i are the eigenvalues of $\mathbf{A}$.

For $\hat{\mathbf{A}}^k = \mathbf{0}$ to be true, then the eigenvalues of $\mathbf{A}$ must be zero or there must be an eigenvector $\lambda_j = 0$ with multiplicity n.

Let A be a diagonal matrix with Jordan blocks $\mathbf{A}_i$ with eigenvalues zero. If $\mathbf{A}^k = \mathbf{0}$ then $\mathbf{A}^k = \operatorname{diag}\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n\} = \mathbf{0}$. For this to be valid for $k \geq m$, the order n_i of all Jordan blocks must be $n_i \leq m$. Thus, the index of $\mathbf{A}$ must also be m or less.

3.21 Given

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

find $\mathbf{A}^{10}$, $\mathbf{A}^{103}$, and $e^{\mathbf{A}t}$.

We first determine the eigenvalues of $\mathbf{A}$.

$$\lambda_1 = 1 \qquad \lambda_2 = 0 \qquad \lambda_3 = 1$$

Let $h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$. From Cayley-Hamilton's theorem, we know that $f(\lambda) = f(\mathbf{A})$. Using this property, we calculate the three functions given by finding linear combinations of the basis $\begin{bmatrix} \mathbf{I} & \mathbf{A} & \mathbf{A}^2 \end{bmatrix}$

- $\mathbf{A}^{10}$
  Let $f(\lambda) = \lambda^{10}$. Then $\frac{df(\lambda)}{d\lambda} = 10\lambda^9$.

$$
\begin{aligned}
\lambda = 0 \rightarrow f(0) = 0^{10} &= \beta_0 = 0 \\
\lambda = 1 \rightarrow f(1) = 1^{10} &= \beta_1 + \beta_2 = 1 \\
\rightarrow \frac{df(\lambda)}{d\lambda}\Big|_{\lambda=1} &= 10 = \beta_1 + 2\beta_2 \\
\beta_0 &= 0 \\
\beta_1 &= -8 \\
\beta_2 &= 9 \\
\mathbf{A}^2 &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathbf{A}^{10} &= \beta_0\mathbf{I} + \beta_1\mathbf{A} + \beta_2\mathbf{A}^2 \\
&= -8\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathbf{A}^{10} &= \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

- $\mathbf{A}^{103}$

  Let $f(\lambda) = \lambda^{103}$. Then $\frac{df(\lambda)}{d\lambda} = 103\lambda^{102}$.

$$
\begin{aligned}
\lambda = 0 \rightarrow f(0) = 0^{103} &= \beta_0 = 0 \\
\lambda = 1 \rightarrow f(1) = 1^{103} &= \beta_1 + \beta_2 = 1 \\
\rightarrow \frac{df(\lambda)}{d\lambda}\Big|_{\lambda=1} &= 103 = \beta_1 + 2\beta_2 \\
\beta_0 &= 0 \\
\beta_1 &= -101 \\
\beta_2 &= 102
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}^{103} &= \beta_0\mathbf{I} + \beta_1\mathbf{A} + \beta_2\mathbf{A}^2 \\
&= -101\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix} + 102\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix} \\
\mathbf{A}^{103} &= \begin{bmatrix}1 & 1 & 102\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}
\end{aligned}
$$

– $e^{\mathbf{A}t}$

Let $f(\lambda) = \exp(\lambda t)$. Then $\frac{df(\lambda)}{d\lambda} = t\exp(\lambda t)$.

$$
\begin{aligned}
\lambda = 0 &\to f(0) = \exp(0) = \beta_0 = 1 \\
\lambda = 1 &\to f(1) = \exp(t) = 1 + \beta_1 + \beta_2 \\
&\to \left.\frac{df(\lambda)}{d\lambda}\right|_{\lambda=1} = t\exp(t) = \beta_1 + 2\beta_2 \\
\beta_0 &= 1 \\
\beta_1 &= (2-t)\exp(t) - 2 \\
\beta_2 &= (t-1)\exp(t) + 1
\end{aligned}
$$

$$
\begin{aligned}
e^{\mathbf{A}t} &= \beta_0\mathbf{I} + \beta_1\mathbf{A} + \beta_2\mathbf{A}^2 \\
&= \mathbf{I} + ((2-t)e^t - 2)\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix} + ((t-1)e^t + 1)\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix} \\
e^{\mathbf{A}t} &= \begin{bmatrix}e^t & e^t - 1 & te^t - e^t + 1\\ 0 & 1 & e^t - 1\\ 0 & 0 & e^t\end{bmatrix}
\end{aligned}
$$

3.22 Use two different methods to compute $e^{\mathbf{A}t}$ for $\mathbf{A}_1$ and $\mathbf{A}_4$ in Problem 3.13.

– $\mathbf{A}_1$

The eigenvalues of the matrix are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. Let $f(\lambda) = \exp(\lambda t)$.

$$
\begin{aligned}
\lambda = 1 \rightarrow f(1) = \exp(t) &= \beta_0 + \beta_1 + \beta_2 \\
\lambda = 2 \rightarrow f(2) = \exp(2t) &= \beta_0 + 2\beta_1 + 4\beta_2 \\
\lambda = 3 \rightarrow f(3) = \exp(3t) &= \beta_0 + 3\beta_1 + 9\beta_2 \\
\beta_0 &= 3\exp(t) + 3\exp(2t) + \exp(3t) \\
\beta_1 &= 4\exp(2t) - 1.5\exp(3t) - 2.5\exp(t) \\
\beta_2 &= 0.5(\exp(3t) - 2\exp(2t) + \exp(t))
\end{aligned}
$$

$$
\begin{aligned}
e^{\mathbf{A}t} &= \beta_0\mathbf{I} + \beta_1\mathbf{A} + \beta_2\mathbf{A}^2 \\
&= \beta_0\mathbf{I} + \beta_1 \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 & 12 & 40 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \\
e^{\mathbf{A}t} &= \begin{bmatrix} e^t & 4(e^{2t} - e^t) & 5(e^{3t} - e^t) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}
\end{aligned}
$$

Employing the representation $\mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1}$.

$$
\begin{aligned}
\mathbf{A}_1 &= \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
e^{\mathbf{A}_1 t} &= \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} e^t & 4(e^{2t} - e^t) & 5(e^{3t} - e^t) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}
\end{aligned}
$$

– $\mathbf{A}_4$

There is one eigenvector $\lambda = 0$ with duplicity of 3. Let $f(\lambda) = \exp(\lambda t)$. Then $\frac{df(\lambda)}{d\lambda} = t\exp(\lambda t)$ and $\frac{d^2 f(\lambda)}{d\lambda^2} = t^2 \exp(\lambda t)$.

$$
\begin{aligned}
\lambda = 0 \rightarrow f(0) &= \exp(0) = \beta_0 = 1 \\
\rightarrow \left.\frac{df(\lambda)}{d\lambda}\right|_{\lambda=1} &= t\exp(0) = t = \beta_1 \\
\rightarrow \left.\frac{d^2 f(\lambda)}{d\lambda^2}\right|_{\lambda=1} &= t^2\exp(0) = t^2 = 2\beta_2 \\
\beta_0 &= 1 \\
\beta_1 &= t \\
\beta_2 &= \frac{t^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
e^{\mathbf{A}t} &= \beta_0 \mathbf{I} + \beta_1 \mathbf{A} + \beta_2 \mathbf{A}^2 \\
&= \mathbf{I} + t\begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
e^{\mathbf{A}t} &= \begin{bmatrix} 1 & 4t + 2.5t^2 & 3t + 2t^2 \\ 0 & 1 + 20t & 16t \\ 0 & -25t & 1 - 20t \end{bmatrix}
\end{aligned}
$$

Employing the representation $\mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1}$.

$$
\begin{aligned}
\mathbf{A}_4 &= \begin{bmatrix} 5 & 4 & 0 \\ 0 & 20 & 1 \\ 0 & -25 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 4 & 0 \\ 0 & 20 & 1 \\ 0 & -25 & 0 \end{bmatrix}^{-1} \\
e^{\mathbf{A}_4 t} &= \begin{bmatrix} 5 & 4 & 0 \\ 0 & 20 & 1 \\ 0 & -25 & 0 \end{bmatrix}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 4 & 0 \\ 0 & 20 & 1 \\ 0 & -25 & 0 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 4t + 2.5t^2 & 3t + 2t^2 \\ 0 & 1 + 20t & 16t \\ 0 & -25t & 1 - 20t \end{bmatrix}
\end{aligned}
$$

3.23 Show that functions of the same matrix commute; that is,

$$
f(\mathbf{A})g(\mathbf{A}) = g(\mathbf{A})f(\mathbf{A})
$$

Consequently we have $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$.

The functions $f(\mathbf{A})$ and $g(\mathbf{A})$ can be represented as linear combinations of $\mathbf{A}^i$.

$$\begin{aligned}
f(\mathbf{A}) &= \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \cdots + \alpha_n\mathbf{A}^{n-1} \\
g(\mathbf{A}) &= \beta_0\mathbf{I} + \beta_1\mathbf{A}^{n-1} + \cdots + \beta_n\mathbf{A}^{n-1} \\
f(\mathbf{A})g(\mathbf{A}) &= \alpha_0\mathbf{I}(\beta_0\mathbf{I} + \beta_1\mathbf{A}^{n-1} + \cdots + \beta_n\mathbf{A}^{n-1}) + \dots \\
&\quad + \alpha_n\mathbf{A}^{n-1}(\beta_0\mathbf{I} + \beta_1\mathbf{A}^{n-1} + \cdots + \beta_n\mathbf{A}^{n-1}) \\
&= \beta_0\mathbf{I}(\alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \cdots + \alpha_n\mathbf{A}^{n-1}) + \dots \\
&\quad + \beta_n\mathbf{A}^{n-1}(\alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \cdots + \alpha_n\mathbf{A}^{n-1}) \\
&= g(\mathbf{A})f(\mathbf{A})
\end{aligned}$$

From this proof we conclude that $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$.

3.24 Let

$$\mathbf{C} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Find a matrix $\mathbf{B}$ such that $e^{\mathbf{B}} = \mathbf{C}$. Show that if $\lambda_i = 0$ for some i, then $\mathbf{B}$ does not exist. Let

$$\mathbf{C} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Find a $\mathbf{B}$ such that $e^{\mathbf{B}} = \mathbf{C}$. Is it true that, for any nonsingular $\mathbf{C}$, there exists a matrix $\mathbf{B}$ such that $e^{\mathbf{B}} = \mathbf{C}$?

Using properties of logarithms we can find $\mathbf{B}$ as a function of $\mathbf{C}$.

$$\begin{aligned}
\exp(\mathbf{B}) &= \mathbf{C} \\
&= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \\
\ln(\exp(\mathbf{B})) &= \ln(\mathbf{C}) \\
\mathbf{B} &= \begin{bmatrix} \ln(\lambda_1) & 0 & 0 \\ 0 & \ln(\lambda_2) & 0 \\ 0 & 0 & \ln(\lambda_3) \end{bmatrix}
\end{aligned}$$

This holds true as long as $\lambda_i \neq 0$ because $\ln(0)$ is not defined.

For

$$\mathbf{C} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

we employ the representation $\mathbf{Q}\hat{\mathbf{C}}\mathbf{Q}^{-1}$.

$$\begin{aligned}
\exp(\mathbf{B}) &= \mathbf{C} \\
&= \mathbf{Q}\hat{\mathbf{C}}\mathbf{Q}^{-1} \\
&= \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \\
\ln(\exp(\mathbf{B})) &= \ln(\mathbf{C}) \\
\mathbf{B} &= \begin{bmatrix} \ln(\lambda) & 1/\lambda & 0 \\ 0 & \ln(\lambda) & 0 \\ 0 & 0 & \ln(\lambda) \end{bmatrix} \\
&= \mathbf{Q}\ln(\hat{\mathbf{C}})\mathbf{Q}^{-1}
\end{aligned}$$

For $\hat{\mathbf{C}}$ to exist, $\mathbf{C}$ must be nonsingular.

- Obtain eigenvectors of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

We solve the characteristic equation of the matrix.

$$\begin{aligned}
\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) &= 0 \\
&= \det\left(\begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -1 & 0 & 0 & \lambda \end{bmatrix}\right)
\end{aligned}$$

$$= \lambda^4 - 1 = (\lambda^2+1)(\lambda^2-1)$$
$$\lambda = \pm j, \pm 1$$

We find the eigenvectors of the matrix by solving $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{q}_i = 0$.

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{q}_i = \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} q_{i1} \\ q_{i2} \\ q_{i3} \\ q_{i4} \end{bmatrix}$$
$$\begin{bmatrix} -\lambda q_{i1} + q_{i2} \\ -\lambda q_{i2} + q_{i3} \\ -\lambda q_{i3} + q_{i4} \\ -\lambda q_{i4} + q_{i1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \mathbf{q}_i = \begin{bmatrix} \lambda q_{i4} \\ \lambda^2 q_{i4} \\ \lambda^3 q_{i4} \\ q_{i4} \end{bmatrix}$$

Consider $q_{i4} = 1$ such that

$$\lambda = j \rightarrow \mathbf{q}_i = \begin{bmatrix} j \\ -1 \\ -j \\ 1 \end{bmatrix}$$
$$\lambda = -j \rightarrow \mathbf{q}_i = \begin{bmatrix} -j \\ -1 \\ j \\ 1 \end{bmatrix}$$
$$\lambda = 1 \rightarrow \mathbf{q}_i = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$\lambda = -1 \rightarrow \mathbf{q}_i = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Solving for $\mathbf{q}_i$ we obtain the eigenvectors of the matrix.

$$
\begin{aligned}
\mathbf{Q} &= \begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4\end{bmatrix} \\
&= \begin{bmatrix} j & -j & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -j & j & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

- Obtain a transformation matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal. $\mathbf{A}$ is given by

$$
\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}
$$

$$
\begin{aligned}
\lambda\mathbf{I} - \mathbf{A} &= \begin{bmatrix} \lambda & -1 & 0 \\ 3 & \lambda & -2 \\ 12 & -7 & \lambda+6 \end{bmatrix} \\
\Delta(\lambda) &= \det\left(\begin{bmatrix} \lambda & -1 & 0 \\ 3 & \lambda & -2 \\ 12 & -7 & \lambda+6 \end{bmatrix}\right) \\
&= \lambda(\lambda^2 + 6\lambda - 14) + 3\lambda + 18 + 24 \\
&= \lambda \;\; 3 + 6\lambda^2 - 11\lambda + 24 \\
&= 0
\end{aligned}
$$

The eigenvectors are

$$
\lambda = -8.17053 \quad \lambda = 0.41934 \quad \lambda 1.75118
$$

We now solve $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{q}_i = \mathbf{0}$ to find the eigenvectors. These vectors are the columns of $\mathbf{P}$.

$$
(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{q}_i = \mathbf{0}
$$

$$
\begin{bmatrix} -\lambda & 1 & 0 \\ 3 & -\lambda & 2 \\ -12 & 7 & -6-\lambda \end{bmatrix}
\begin{bmatrix} q_{i1} \\ q_{i2} \\ q_{i3} \end{bmatrix}
\rightarrow
\begin{bmatrix} -\lambda q_{i1} + q_{i2} \\ 3q_{i1} - \lambda q_{i2} + 2q_{i3} \\ -12q_{i1} + 7q_{i2} + (-6-\lambda)q_{i3} \end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$\mathbf{P} = \begin{bmatrix} 0.030373 & -0.561673 & 0.495818 \\ -0.248161 & -0.235534 & 0.868269 \\ 0.968243 & 0.793125 & 0.016522 \end{bmatrix}$$

- Find a transformation matrix $\mathbf{S}$ such that

$$\mathbf{S}^{-1}\mathbf{AS} = \mathbf{J}$$

where

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We first obtain the eigenvalues of the matrix.

$$\begin{aligned} \Delta(\lambda) &= \det(\lambda\mathbf{I} - \mathbf{A}) = 0 \\ &= \det\left(\begin{bmatrix} \lambda - 4 & -1 & 2 \\ -1 & \lambda & -2 \\ -1 & 1 & \lambda - 3 \end{bmatrix}\right) \end{aligned}$$

$$\lambda = 3 \quad \lambda = 3 \quad \lambda = 1$$

Due to the repeated roots, it is necessary to construct generalized eignevectors to form $\mathbf{S}$.

We obtain the basis.

$$\begin{aligned} (\mathbf{A} - \lambda\mathbf{I}) &= \begin{bmatrix} 4 - \lambda & 1 & -2 \\ 1 & -\lambda & 2 \\ 1 & -1 & 3 - \lambda \end{bmatrix} \\ \lambda = 1 \rightarrow (\mathbf{A} - \lambda\mathbf{I}) &= \begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \\ \lambda = 3 \rightarrow (\mathbf{A} - \lambda\mathbf{I}) &= \begin{bmatrix} 1 & 1 & -2 \\ 1 & -3 & 2 \\ 1 & -1 & 0 \end{bmatrix} \\ \rightarrow (\mathbf{A} - \lambda\mathbf{I})^2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & -8 \\ 0 & 4 & -4 \end{bmatrix} \end{aligned}$$

Find a solution to

$$
\begin{aligned}
(\mathbf{A}-\mathbf{I})\mathbf{v}_3 &= \mathbf{0} \\
(\mathbf{A}-3\mathbf{I})^2\mathbf{v}_2 &= \mathbf{0} \\
(\mathbf{A}-3\mathbf{I})\mathbf{v}_2 &= \mathbf{v}_1
\end{aligned}
$$

$$
(\mathbf{A}-\mathbf{I})\mathbf{v}_3 = \mathbf{0}
$$

$$
(\mathbf{A}-\lambda\mathbf{I}) = \begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \rightarrow \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}
$$

$$
(\mathbf{A}-3\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}
$$

$$
(\mathbf{A}-\lambda\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & -8 \\ 0 & 4 & -4 \end{bmatrix} \rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

$$
(\mathbf{A}-3\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1
$$

$$
(\mathbf{A}-\lambda\mathbf{I}) = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -3 & 2 \\ 1 & -1 & 0 \end{bmatrix} \rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
$$

Thus the Jordan form is

$$
\mathbf{J} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathbf{S} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}
$$

- Find the Jordan-canonical-form representations of the following matrices:

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & -150 & -120 \\ 0 & 200 & 160 \end{bmatrix} \quad \mathbf{A}_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

$$\mathbf{A}_5 = \begin{bmatrix} 7/2 & 21/2 & 14 \\ -1/2 & -3/2 & -2 \\ -1/2 & -3/2 & -2 \end{bmatrix} \quad \mathbf{A}_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

– $\mathbf{A}_1$

From Problem 3.13 we have:

The matrix is triangular, thus its eigenvalues are the values of the diagonal.

$$\hat{\mathbf{A}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

We obtain $\mathbf{Q} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix}$.

$$(\mathbf{A} - \lambda_1)\mathbf{q}_1 = \mathbf{0}$$

$$\begin{bmatrix} 0 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{q}_1 = \mathbf{0}$$

$$\begin{bmatrix} 0 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

$$(\mathbf{A} - \lambda_2)\mathbf{q}_2 = \mathbf{0}$$

$$\begin{bmatrix} -1 & 4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{q}_2 = \mathbf{0}$$

$$\begin{bmatrix} -1 & 4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$$

$$(\mathbf{A} - \lambda_3)\mathbf{q}_3 = \mathbf{0}$$
$$\begin{bmatrix} -2 & 4 & 10 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{q}_3 = \mathbf{0}$$

$$\begin{bmatrix} -2 & 4 & 10 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

Thus the Jordan form is

$$\hat{\mathbf{A}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
$$\mathbf{Q} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– $\mathbf{A}_2$

We obtain the eigenvalues of the matrix.

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}_2) &= \det(\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2 & 4 & \lambda+3 \end{bmatrix}) = \mathbf{0} \\ &= \lambda(\lambda^2 + 3\lambda + 4) - 2 \\ &= \lambda^3 + 3\lambda^2 + 4\lambda - 2 \\ &= (\lambda+1)(\lambda+1+j)(\lambda+1-j) \end{aligned}$$

$$\hat{\mathbf{A}}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1-j & 0 \\ 0 & 0 & -1+j \end{bmatrix}$$

Determine the eigenvectors of matrix $\mathbf{A}_2$.

$$(\mathbf{A} - \lambda_1)\mathbf{q}_1 = \mathbf{0}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix} \mathbf{q}_1 = \mathbf{0}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \mathbf{0}$$

$$(\mathbf{A} - \lambda_2)\mathbf{q}_2 = \mathbf{0}$$

$$\begin{bmatrix} 1+j & 1 & 0 \\ 0 & 1+j & 1 \\ -2 & -4 & -2+j \end{bmatrix} \mathbf{q}_2 = \mathbf{0}$$

$$\begin{bmatrix} 1+j & 1 & 0 \\ 0 & 1+j & 1 \\ -2 & -4 & -2+j \end{bmatrix} \begin{bmatrix} -1 \\ 1+j \\ -2j \end{bmatrix} = \mathbf{0}$$

$$(\mathbf{A} - \lambda_3)\mathbf{q}_3 = \mathbf{0}$$

$$\begin{bmatrix} 1-j & 1 & 0 \\ 0 & 1-j & 1 \\ -2 & -4 & -2-j \end{bmatrix} \mathbf{q}_3 = \mathbf{0}$$

$$\begin{bmatrix} 1-j & 1 & 0 \\ 0 & 1-j & 1 \\ -2 & -4 & -2-j \end{bmatrix} \begin{bmatrix} -1 \\ 1-j \\ 2j \end{bmatrix} = \mathbf{0}$$

Thus the Jordan form is

$$\hat{\mathbf{A}}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1-j & 0 \\ 0 & 0 & -1+j \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1+j & 1-j \\ 1 & -2j & 2j \end{bmatrix}$$

- $\mathbf{A}_3$

$$\begin{aligned}
\det(\lambda\mathbf{I} - \mathbf{A}_3) &= \det\left(\begin{bmatrix} \lambda & -4 & -3 \\ 0 & \lambda + 150 & 120 \\ 0 & -200 & \lambda - 160 \end{bmatrix}\right) \\
&= \lambda((\lambda + 150)(\lambda - 160) + 200(120)) \\
&= \lambda^2(\lambda - 10) \\
\lambda^2(\lambda - 10) = 0 &\rightarrow \lambda_{1,2} = 0, \lambda_3 = 10
\end{aligned}$$

Due to the repeated roots, it is necessary to construct generalized eignevectors to form $\mathbf{S}$.
We obtain the basis.

$$\begin{aligned}
(\mathbf{A} - \lambda\mathbf{I}) &= \begin{bmatrix} -\lambda & 4 & 3 \\ 0 & -150 - \lambda & -120 \\ 0 & 200 & 160 - \lambda \end{bmatrix} \\
\lambda = 10 \rightarrow (\mathbf{A} - \lambda\mathbf{I}) &= \begin{bmatrix} -10 & 4 & 3 \\ 0 & -160 & -120 \\ 0 & 200 & 150 \end{bmatrix} \\
\lambda = 0 \rightarrow (\mathbf{A} - \lambda\mathbf{I}) &= \begin{bmatrix} 0 & 4 & 3 \\ 0 & -150 & -120 \\ 0 & 200 & 160 \end{bmatrix} \\
\rightarrow (\mathbf{A} - \lambda\mathbf{I})^2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1500 & -1200 \\ 0 & 2000 & 1600 \end{bmatrix}
\end{aligned}$$

Find a solution to

$$\begin{aligned}
(\mathbf{A} - 10\mathbf{I})\mathbf{v}_3 &= \mathbf{0} \\
(\mathbf{A} - 0\mathbf{I})^2\mathbf{v}_2 &= \mathbf{0} \\
(\mathbf{A} - 0\mathbf{I})\mathbf{v}_2 &= \mathbf{v}_1
\end{aligned}$$

$$(\mathbf{A} - 10\mathbf{I})\mathbf{v}_3 = \mathbf{0}$$

$$(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} -10 & 4 & 3 \\ 0 & -160 & -120 \\ 0 & 200 & 150 \end{bmatrix} \rightarrow \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix}$$

$$(\mathbf{A} - 0\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$$

$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1500 & -1200 \\ 0 & 2000 & 1600 \end{bmatrix} \rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}$$

$$(\mathbf{A} - 0\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$$

$$(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 0 & 4 & 3 \\ 0 & -150 & -120 \\ 0 & 200 & 160 \end{bmatrix} \rightarrow \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Thus the Jordan form is

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -4 & 3 \\ 0 & 5 & -4 \end{bmatrix}$$

- $\mathbf{A}_4$

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}_3) &= \det\left(\begin{bmatrix} \lambda & -4 & -3 \\ 0 & \lambda - 20 & -16 \\ 0 & 25 & \lambda + 20 \end{bmatrix}\right) \\ &= \lambda((\lambda - 20)(\lambda + 20) - 25(-16)) \\ &= \lambda(\lambda^2 - 400 + 400) \\ \lambda^3 = 0 \rightarrow \lambda_{1,2,3} = 0 \end{aligned}$$

The new representation of A is then

$$\hat{\mathbf{A}}_4 = \mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Due to the repeated eigenvalues of the matrix, it is necessary to obtain generalized vectors for the matrix.

We obtain the basis.

$$
\begin{aligned}
(\mathbf{A}-\lambda\mathbf{I}) &= \mathbf{A} \\
(\mathbf{A}-\lambda\mathbf{I})^2 &= \begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
(\mathbf{A}-\lambda\mathbf{I})^3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Find a solution to

$$
\begin{aligned}
(\mathbf{A}-\lambda\mathbf{I})^3\mathbf{v} &= \mathbf{0} \\
(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v} &= \mathbf{v}_2 \\
(\mathbf{A}-\lambda\mathbf{I})\mathbf{v} &= \mathbf{v}_3
\end{aligned}
$$

$$(\mathbf{A}-\lambda\mathbf{I})^3\mathbf{v} = \mathbf{0}$$

$$(\mathbf{A}-\lambda\mathbf{I})^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v} = \mathbf{v}_2$$

$$(\mathbf{A}-\lambda\mathbf{I})^2 = \begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v} = \mathbf{v}_3$$

$$(\mathbf{A}-\lambda\mathbf{I}) = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} \rightarrow \mathbf{v}_3 = \begin{bmatrix} 4 \\ 20 \\ -25 \end{bmatrix}$$

Thus the Jordan form is

$$
\begin{aligned}
\hat{\mathbf{A}}_4 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\
\mathbf{Q} &= \begin{bmatrix} 5 & 4 & 0 \\ 0 & 20 & 1 \\ 0 & -25 & 0 \end{bmatrix}
\end{aligned}
$$

– $\mathbf{A}_5$

We obtain the eigenvalues of the matrix.

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det\left(\begin{bmatrix} \lambda - 7/2 & -21/2 & -14 \\ 1/2 & \lambda + 3/2 & 2 \\ 1/2 & 3/2 & \lambda + 2 \end{bmatrix}\right) = \mathbf{0}$$

The eigenvalue is zero with multiplicity of three.

We construct generalized vectors for the matrix employing the same procedure as Problem 3.13.

We obtain the basis.

$$(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 7/2 & 21/2 & 14 \\ -1/2 & -3/2 & -2 \\ -1/2 & -3/2 & -2 \end{bmatrix}$$
$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$(\mathbf{A} - \lambda\mathbf{I})^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find a solution to

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_3 = \mathbf{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$$

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_3 = \mathbf{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3/4 \end{bmatrix}$$

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{v}_3$$

$$(\mathbf{A}-\lambda\mathbf{I})=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\rightarrow\mathbf{v}_3=\begin{bmatrix}14\\-2\\-2\end{bmatrix}$$

Thus the Jordan form is

$$\hat{\mathbf{A}}=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$\mathbf{Q}=\begin{bmatrix}14&0&0\\-2&0&1\\-2&1&-3/4\end{bmatrix}$$

– $\mathbf{A}_6$

Because the matrix is in triangular form, we know that the eigenvalues of the matrix are all zero.

We obtain the basis for the eigenvectors.

$$(\mathbf{A}-\lambda\mathbf{I})=\begin{bmatrix}0&1&1&1&1\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}(\mathbf{A}-\lambda\mathbf{I})^2=\begin{bmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

$$(\mathbf{A}-\lambda\mathbf{I})^3=\begin{bmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}(\mathbf{A}-\lambda\mathbf{I})^4=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

$$(\mathbf{A}-\lambda\mathbf{I})^5=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

Find a solution to

$$
\begin{aligned}
(\mathbf{A}-\lambda\mathbf{I})^5\mathbf{v}_5 &= \mathbf{0}\\
(\mathbf{A}-\lambda\mathbf{I})^4\mathbf{v}_5 &= \mathbf{v}_4\\
(\mathbf{A}-\lambda\mathbf{I})^3\mathbf{v}_4 &= \mathbf{v}_3\\
(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_3 &= \mathbf{v}_2\\
(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 &= \mathbf{v}_1
\end{aligned}
$$

Thus the Jordan form is

$$
\hat{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 1 & -1\\ 0 & 0 & 1 & -2 & 3\\ 0 & 0 & 0 & 1 & -3\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Find the Jordan-canonical-form representations of the following matrices:

$$
\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 1 & -2 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -8 & -12 & -6 \end{bmatrix} \quad \mathbf{A}_3 = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 4 & -4 & -3 & 4 \end{bmatrix}
$$

  - $\mathbf{A}_1$

    Determine the eigenvalues of the matrix first.

$$
\begin{aligned}
\det(\lambda\mathbf{I}-\mathbf{A}_1) = \det(\begin{bmatrix} \lambda & -1 & 0\\ 0 & \lambda & -1\\ -2 & -1 & \lambda+2 \end{bmatrix}) &= \mathbf{0}\\
&= \lambda(\lambda^2+2\lambda-1)-2\\
&= \lambda^3+2\lambda^2-\lambda-2\\
&= (\lambda-1)(\lambda+1)(\lambda+2)
\end{aligned}
$$

Obtain the eigenvectors of the matrix

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{q}_i = \mathbf{0}$$

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & 1 & -\lambda - 2 \end{bmatrix} \begin{bmatrix} q_{i1} \\ q_{i2} \\ q_{i3} \end{bmatrix} \rightarrow \begin{bmatrix} -\lambda q_{i1} + q_{i2} \\ -\lambda q_{i2} + q_{i3} \\ 2q_{i1} + q_{i2} - (\lambda + 2)q_{i3} \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} q_{i1} \\ \lambda q_{i1} \\ \lambda^2 q_{i1} \end{bmatrix}$$

Consider $q_{i1} = 1$.

$$\lambda = -2 \rightarrow \mathbf{q}_i = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$
$$\lambda = -1 \rightarrow \mathbf{q}_i = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$
$$\lambda = 1 \rightarrow \mathbf{q}_i = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The Jordan form is

$$\hat{\mathbf{A}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\mathbf{Q} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

– $\mathbf{A}_2$
  Determine the eigenvalues of the matrix first.

$$
\begin{aligned}
\det(\lambda\mathbf{I}-\mathbf{A}_2) &= \det(\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 8 & 12 & \lambda+6 \end{bmatrix}) = \mathbf{0} \\
&= \lambda(\lambda^2+6\lambda+12)+8 \\
&= \lambda^3+6\lambda^2+12\lambda+8 \\
&= (\lambda+2)^3
\end{aligned}
$$

Obtain the eigenvectors of the matrix

$$
\begin{aligned}
(\mathbf{A}-\lambda\mathbf{I})\mathbf{q}_i &= \mathbf{0} \\
&= \begin{bmatrix} -\lambda & 1-\lambda & 0 \\ 0 & -\lambda & 1 \\ -8 & -12 & -6-\lambda \end{bmatrix} \begin{bmatrix} q_{i1} \\ q_{i2} \\ q_{i3} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathbf{A}} &= \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \\
\mathbf{Q} &= \begin{bmatrix} 1 & 1 & 3/4 \\ -2 & -1 & -1/2 \\ 4 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

– $\mathbf{A}_3$

Determine the eigenvalues of the matrix first.

$$
\begin{aligned}
\det(\lambda\mathbf{I}-\mathbf{A}_1) = \det\left(\begin{bmatrix} \lambda & -1 & 6 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -4 & 4 & 3 & \lambda-4 \end{bmatrix}\right) = \mathbf{0} \\
= (\lambda-1)(\lambda+1)(\lambda-2)^2
\end{aligned}
$$

Obtain the eigenvectors of the matrix

$$
\begin{aligned}
(\mathbf{A}-\lambda\mathbf{I})\mathbf{q}_i &= \mathbf{0} \\
&= \begin{bmatrix} -\lambda & 1 & 6 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 4 & -4 & -3 & -\lambda+4 \end{bmatrix} \begin{bmatrix} q_{i1} \\ q_{i2} \\ q_{i3} \\ q_{i4} \end{bmatrix}
\end{aligned}
$$

The Jordan form is then

$$
\hat{\mathbf{A}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}
$$

$$
\mathbf{Q} = \begin{bmatrix} -1 & 1 & 1 & -3/2 \\ 1 & 1 & 2 & -2 \\ -1 & 1 & 4 & -2 \\ 1 & 1 & 8 & 0 \end{bmatrix}
$$

References

[1] C.T. Chen. *Linear System Theory and Design*. Oxford series in electrical and computer engineering. Oxford University Press, 1999. ISBN: 9780195117776.