Notes on Dynamic Systems

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Chapter 1

Signals & Systems

1.1 What is a system?

A system is defined as a grouping of elements to be analysed together. They can be categorised as linear or non-linear, depending on the equations used to describe them. Linear systems are considered to be idealised systems, whilst non-linear systems are those representing real-world conditions.

Systems may also be categorised based on the *order* of their differential equations. Some examples of categorised systems are shown in table 1.1.

	Linear	Non-Linear
1st order	RC circuit	Population growth
2nd order	Spring Mass Damper	Pendulum
3rd order	-	Chaotic Systems
...	...	...
Nth order	Wave Equation	General Relativity

Table 1.1: Examples of linear and non-linear systems.

1.2 Linear Systems

A system is considered linear if they meet the following two principles:

- Proportionality. Given an input t, the system will return an output $f(t)$, and if given an input αt the system will return an output $f(\alpha t) = \alpha f(t)$.

- Superposition. Given inputs t_1 and t_2, the system will return outputs $f(t_1)$ and $f(t_2)$ respectively. If given an input $t_1 + t_2$ the system will return an output $f(t_1 + t_2) = f(t_1) + f(t_2)$.

1.2.1 Exercise: Linear systems

Given the following equations, determine if they belong to linear systems or not.

Case 1: $f(s) = 3s$. Evaluating the function to test proportionality, we obtain the following:

$$\begin{aligned} f(2s) &= 3(2s) \\ &= 6s \\ 2f(s) &= 2(3s) \\ &= 6s \\ f(2s) &= 2f(s) \end{aligned}$$

For the superposition principle, we have:

$$\begin{aligned} f(s_1) &= 3s_1 \\ f(s_2) &= 3s_2 \\ f(s_1 + s_2) &= 3(s_1 + s_2) \\ &= 3s_1 + 3s_2 \\ f(s_1 + s_2) &= f(s_1) + f(s_2) \end{aligned}$$

We conclude that case 1 is linear.

An equation can be tested to meet both principles simultaneously. Let $s = \alpha(s_1 + s_2)$ and $f(s)$ be the output of the system. A system is linear if the following condition is met:

$$f(\alpha(s_1 + s_2)) = \alpha f(s_1) + \alpha f(s_2) \tag{1.1}$$

Evaluating 1.1 for case 1 as an example:

$$\begin{aligned} f(\alpha(s_1 + s_2)) &= 3(\alpha(s_1 + s_2)) \\ &= 3\alpha s_1 + 3\alpha s_2 \\ f(\alpha s) &= 3\alpha s \\ f(\alpha(s_1 + s_2)) &= \alpha f(s_1) + \alpha f(s_2) \end{aligned}$$

Case 2: $f(s) = 3s + 1$.

$$\begin{aligned}
f(\alpha s) &= 3\alpha s + 1 \\
\alpha f(s) &= 3\alpha s + \alpha \\
f(\alpha s) &\neq \alpha f(s) \\
f(s_1) + f(s_2) &= 3s_1 + 3s_2 + 2 \\
f(s_1 + s_2) &= 3(s_1 + s_2) + 1 \\
f(s_1 + s_2) &\neq f(s_1) + f(s_2)
\end{aligned}$$

Case 2 is non-linear.
Case 3: $f(s) = 0.5\cos(0.1s)$

$$\begin{aligned}
f(\alpha s) &= 0.5\cos(0.1\alpha s) \\
\alpha f(s) &= 0.5\alpha\cos(0.1s) \\
f(\alpha s) &\neq \alpha f(s) \\
f(s_1) + f(s_2) &= 0.5\cos(0.1s_1) + 0.5\cos(0.1s_2) \\
f(s_1 + s_2) &= 0.5\cos(0.1s_1 + 0.1s_2) \\
f(s_1 + s_2) &\neq f(s_1) + f(s_2)
\end{aligned}$$

Case 3 is non-linear.
Case 4: $f(s) = 1.2\mathrm{e}^{0.1s}$

$$\begin{aligned}
f(\alpha(s_1 + s_2)) &= 1.2\mathrm{e}^{0.1\alpha(s_1+s_2)} \\
\alpha f(s_1) &= 1.2\alpha\mathrm{e}^{0.1s_1} \\
\alpha f(s_1) + \alpha f(s_2) &= 1.2\alpha(\mathrm{e}^{0.1s_1} + \mathrm{e}^{0.1s_2}) \\
f(\alpha(s_1 + s_2)) &\neq \alpha f(s_1) + \alpha f(s_2)
\end{aligned}$$

Case 4 is non-linear.

Case 5: $f(s) = \int\limits_0^t s(t)dt$

$$\begin{aligned}
f(\alpha(s_1 + s_2)) &= \alpha \int\limits_0^t \Big(s_1(t) + s_2(t)\Big)dt \\
&= \alpha \int\limits_0^t s_1(t)dt + \alpha \int\limits_0^t s_2(t)dt \\
\alpha f(s_1) + \alpha f(s_2) &= \alpha \int\limits_0^t s_1(t)dt + \alpha \int\limits_0^t s_2(t)dt \\
f(\alpha(s_1 + s_2)) &= \alpha f(s_1) + \alpha f(s_2)
\end{aligned}$$

Case 5 is linear.

Case 6: $f(s) = \frac{ds(t)}{dt}$

$$\begin{aligned}
f(\alpha(s_1 + s_2)) &= \alpha \frac{d\Big(s_1(t) + s_2(t)\Big)}{dt} \\
&= \alpha \frac{ds_1(t)}{dt} + \alpha \frac{ds_2(t)}{dt} \\
\alpha f(s_1) + \alpha f(s_2) &= \alpha \frac{ds_1(t)}{dt} + \alpha \frac{ds_2(t)}{dt} \\
f(\alpha(s_1 + s_2)) &= \alpha f(s_1) + \alpha f(s_2)
\end{aligned}$$

Case 6 is linear.

Note that for all equations $\alpha \neq 1$ because using the identity is not valid for proving compliance.

1.2.2 Order of a system

The order of a system is dependent on the highest order of derivatives in the equations that describe it. A system with no derivative terms is a *zero order* system, a system with a differential equation of order 1 is a *first-order* system, and so on.

Revisiting the cases shown in Exercise 1, we can classify the systems depending on their order as well. Table 1.2.2 contains the classification of each case.

Equation	Type	Order
$3s$	Linear	0
$3s+1$	Non-linear	0
$0.5\cos(0.1s)$	Non-Linear	0
$1.2\mathrm{e}^{0.1s}$	Non-Linear	0
$\int_0^t s(t)dt$	Linear	0
$\frac{ds(t)}{dt}$	Linear	1

Table 1.2: Classification of systems.

1.3 Example: Spring-Mass-Damper System

The Spring-Mass-Damper System (abbreviated SMD) is the most commonly used abstraction for systems. Depending on the initial conditions of the systems, it may or may not be linear. An example is presented below.

Given the forces interacting in the Free Body Diagram, the system's equation is obtained as follows:

$$\begin{aligned} \sum F &= 0 \\ F_s(t) + F_d(t) - F(t) &= 0 \end{aligned} \tag{1.2}$$

Recall the equations for springs and dampers, substituting them in (1.2).

$$\begin{aligned} F_s &= b\dot{x} \\ F_d &= kx(t) \\ b\dot{x} + kx(t) &= F(t) \end{aligned}$$

Rewriting the equation into the form $\dot{x} + f(t)x(t) = g(t)$.

$$\dot{x} + \frac{k}{b}x(t) = \frac{1}{b}F(t) \tag{1.3}$$

1.3.1 Linear System Case

Solving the system via Laplace Transforms to obtain an equation for x(t). Consider $x(0) = 0$ as the initial condition.

$$\mathfrak{L}\Big\{\dot{x}+\frac{k}{b}x(t)=\frac{1}{b}F(t)\Big\}\rightarrow s\mathcal{X}-x(0)+\frac{k}{b}\mathcal{X}=\frac{1}{b}\mathcal{F} \tag{1.4}$$

Evaluating $x(0) = 0$ gives us the system equation in the frequency domain.

$$s\mathcal{X}+\frac{k}{b}\mathcal{X}=\frac{1}{b}\mathcal{F} \tag{1.5}$$

Solve 1.5 for $\mathcal{X}$ and then obtain the inverse Laplace transform of the equation.

$$\begin{aligned}\mathcal{X}(s+\frac{k}{b}) &= \frac{1}{b}\mathcal{F}\\ \mathcal{X} &= \frac{1}{b}\frac{1}{s+\frac{k}{b}}\mathcal{F}\end{aligned}$$

$$\mathfrak{L}^{-1}\Big\{\mathcal{X}=\frac{1}{b}\frac{1}{s+\frac{k}{b}}\mathcal{F}\Big\}\rightarrow x(F(t))=\frac{1}{b}\mathrm{e}^{-\frac{k}{b}t}F(t) \tag{1.6}$$

Testing the principles of proportionality and superposition on 1.6 we have

$$\begin{aligned}x(\alpha(F_1+F_2)) &= \frac{\alpha}{b}\mathrm{e}^{-\frac{k}{b}t}(F_1+F_2)\\ \alpha x(F_1) &= \frac{\alpha}{b}\mathrm{e}^{-\frac{k}{b}t}F_1\\ \alpha x(F_1)+\alpha x(F_2) &= \frac{\alpha}{b}\mathrm{e}^{-\frac{k}{b}t}(F_1+F_2)\\ x(\alpha(F_1+F_2)) &= \alpha x(F_1)+\alpha x(F_2)\end{aligned}$$

We confirm that the system is linear for an initial condition of $x(0) = 0$.

1.4 Homework

1.4.1 Non-Linear Case

Given the conditions $x(0) = 0.5$ and $F(t) = F$, we evaluate the Laplace transform of 1.3 and obtain

$$s\mathcal{X} - 0.5 + \frac{k}{b}\mathcal{X} = \frac{1}{b}\frac{F}{s} \tag{1.7}$$

Solving for $\mathcal{X}$.

$$\mathcal{X} = \frac{F}{b}\frac{1}{s(s+\frac{k}{b})} + \frac{0.5}{s+\frac{k}{b}} \tag{1.8}$$

Using partial fractions, find an equivalent expression for 1.8 .

$$\frac{1}{s(s+\frac{k}{b})} = \frac{A}{s} + \frac{B}{s+\frac{k}{b}}$$
$$A = \frac{b}{k}$$
$$B = -\frac{b}{k}$$

$$\frac{1}{s(s+\frac{k}{b})} = \frac{1}{s}\frac{b}{k} - \frac{b}{k}\frac{1}{s+\frac{k}{b}} \tag{1.9}$$

Substitute 1.9 in 1.8 and then obtain the inverse Laplace transform of the equation.

$$\begin{aligned}\mathcal{X} &= \frac{F}{b}\Big(\frac{1}{s}\frac{b}{k} - \frac{b}{k}\frac{1}{s+\frac{k}{b}}\Big) + \frac{0.5}{s+\frac{k}{b}} \\ &= \frac{F}{k}\frac{1}{s} - \frac{F}{k}\frac{1}{s+\frac{k}{b}} + \frac{0.5}{s+\frac{k}{b}}\end{aligned}$$

$$\mathfrak{L}^{-1}\Big\{\mathcal{X}\Big\} \to x(t) = \frac{F}{k} - \frac{F}{k}\mathrm{e}^{-\frac{k}{b}t} + 0.5\mathrm{e}^{-\frac{k}{b}t} \tag{1.10}$$

Test the principles of proportionality and superposition on 1.10.

$$
\begin{aligned}
x(\alpha t) &= \frac{F}{k} - \frac{F}{k}\mathrm{e}^{-\alpha\frac{k}{b}t} + 0.5\mathrm{e}^{-\alpha\frac{k}{b}t} \\
\alpha x(t) &= \alpha\frac{F}{k} - \alpha\frac{F}{k}\mathrm{e}^{-\frac{k}{b}t} + 0.5\alpha\mathrm{e}^{-\frac{k}{b}t} \\
x(\alpha t) &\neq \alpha x(t) \\
x(t_1+t_2) &= \frac{F}{k} - \frac{F}{k}\mathrm{e}^{-\frac{k}{b}(t_1+t_2)} + 0.5\mathrm{e}^{-\frac{k}{b}(t_1+t_2)} \\
x(t_1)+x(t_2) &= 2\frac{F}{k} - \frac{F}{k}\Big(\mathrm{e}^{-\frac{k}{b}t_1} - \mathrm{e}^{-\frac{k}{b}t_2}\Big) + 0.5\Big(\mathrm{e}^{-\frac{k}{b}t_1} + \mathrm{e}^{-\frac{k}{b}t_2}\Big) \\
x(t_1+t_2) &\neq x(t_1)+x(t_2)
\end{aligned}
$$

We conclude that the system in Non-linear for the given initial conditions.

Chapter 2

Discrete Systems

2.1 Homework

Chapter 3

System Dynamics

3.1 Homework

3.1.1 Introduction

Given the system shown in figure 3.1 develop the equations that describe the movement of the system.

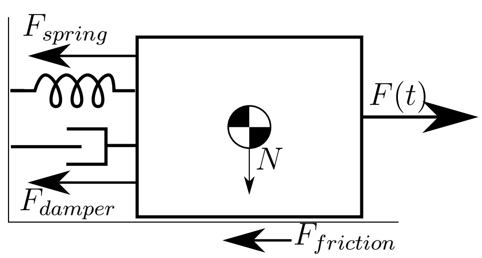

Figure 3.1: Spring Mass Damper System.

Consider the following parameters:

$$\begin{aligned} b &= 2 & m &= 0.5 & x(0) &= 0 \\ k &= 1 & \mu &= 0.01 & \dot{x}(0) &= -0.5 \end{aligned}$$

The force $F(t)$ is defined as the Heaviside equation $U(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$

3.1.2 Solution

From figure 3.1 we establish the following equation

$$\begin{aligned} \sum F &= m\ddot{x}(t) \\ &= F(t) - F_{friction} - F_{spring} - F_{damper} \end{aligned} \tag{3.1}$$

Each term in (3.1) can be substituted for its respective formula.

$$F_{friction} = \mu N \qquad F_{spring} = kx(t) \qquad F_{damper} = b\dot{x}(t) \qquad N = mg$$

We rewrite (3.1) and evaluate the known values of its coefficients.

$$\begin{gathered} U(t) - \mu N - kx(t) - b\dot{x}(t) = m\ddot{x}(t) \\ U(t) - 0.04905 - x(t) - 2\dot{x}(t) = 0.5\ddot{x}(t) \end{gathered}$$

Apply the Laplace transform to both sides of the equation and rearrange its terms

$$\frac{\mathrm{e}^{-s}}{s} - \frac{0.04905}{s} - \mathfrak{X} - 2\left(s\mathfrak{X} - \mathfrak{X}_\circ\right) = 0.5\left(s^2\mathfrak{X} - s\mathfrak{X} - \mathfrak{X}_\circ\right)$$

We factor $\mathfrak{X}$ in the left side of the equation and rewrite it as a function of s.

$$asd \tag{3.2}$$