

Homework #04: Random Variable and Discrete Distribution Functions

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1 Problem

1. Plot the behaviour of the Binomial, Geometric and Poisson distributions for the following values
 - Binomial distribution: $n \in [10, 20, 30]$ and $p \in [0.1, 0.3, 0.5]$
 - Geometric distribution: $p \in [0.1, 0.3, 0.5]$
 - Poisson distribution: $\lambda \in [1, 2, 3]$
2. Show that the Poisson distribution is a good approximation to the Binomial distribution for large values of n and small values of p .

2 Solution

2.1 Plots

Figure 1 shows the plots for the Binomial distribution.

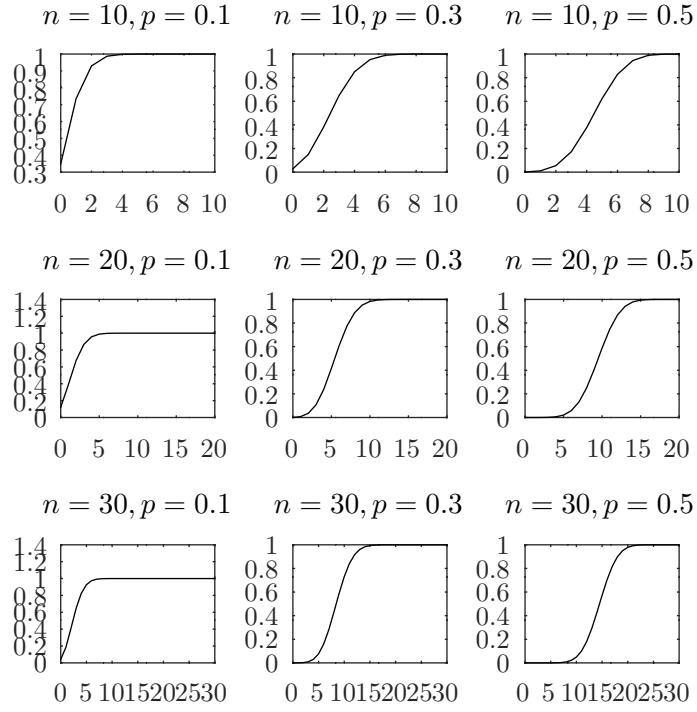


Figure 1: Binomial distribution.

Figure 2 shows the plots for the Geometric distribution.

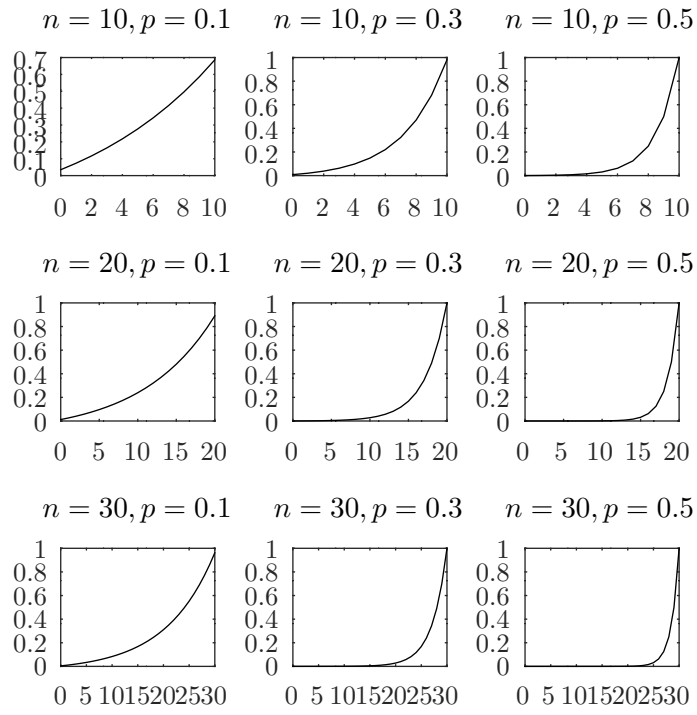


Figure 2: Geometric distribution

Figure 3 shows the plots for the Poisson distribution.

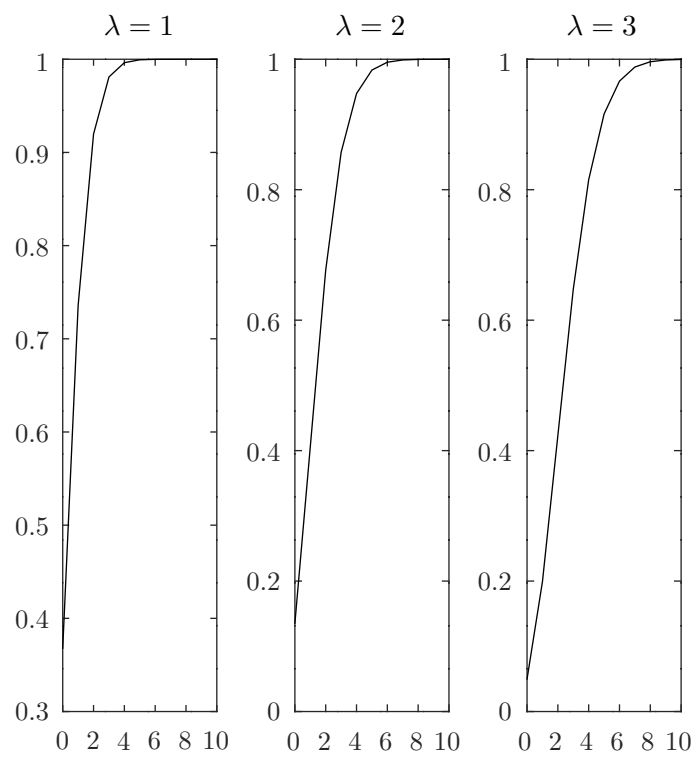


Figure 3: Poisson distribution

2.2 Proof

From [1] we obtain the equations for the Binomial and Poisson distributions, equations (1) and (2) respectively.

$$P(X) = \binom{n}{i} p^i (1-p)^{n-i} \quad (1)$$

$$P(X) = \exp(-\lambda) \frac{\lambda^i}{i!} \quad (2)$$

Let $\lambda = np$ such that (1) can be written as follows

$$\binom{n}{i} p^i (1-p)^{n-i} = \frac{n!}{(n-i)!i!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

Rewrite the last term of the equation.

$$\frac{n!}{(n-i)!i!} \left(\frac{\lambda}{n}\right)^i \frac{(1 - \lambda/n)^n}{(1 - \lambda/n)^i} \quad (3)$$

Recall that for large values of n the term $(1 - \lambda/n)^n$ is approximately equal to $\exp(-\lambda)$. Thus the equation above can be expressed as

$$\frac{n!}{(n-i)!i!} \left(\frac{\lambda}{n}\right)^i \frac{\exp(-\lambda)}{(1 - \lambda/n)^i} \quad (4)$$

The term $\frac{n!}{(n-i)!i!}$ can be expanded as follows

$$\frac{n!}{(n-i)!i!} = \frac{n * (n-1)!}{(n-i)!i!} = \frac{n * (n-1) * (n-2)!}{(n-i)!i!}$$

This expansion permits us to rewrite the term as

$$\frac{n!}{(n-i)!i!} = \frac{n * (n-1) * (n-2) \dots (n-i+1) * (n-i)!}{(n-i)!i!}$$

Notice how this new expression permits us to eliminate $(n-i)!$ from the equation.

$$\frac{n!}{(n-i)!i!} = \frac{n * (n-1) * (n-2) \dots (n-i+1)}{i!}$$

Equation (4) can now be rewritten as

$$\frac{n(n-1)(n-2) \dots (n-i+1)}{i!} \frac{\lambda^i}{n^i} \frac{\exp(-\lambda)}{(1 - \lambda/n)^i} \quad (5)$$

For large values of n, the term $(1 - \lambda/n)^i$ is approximately equal to one. This simplifies the equation once again.

$$\frac{n(n-1)(n-2)\dots(n-i+1)}{i!} \frac{\lambda^i}{n^i} \exp(-\lambda) \quad (6)$$

Exchanging the denominators in the fractions we obtain

$$\frac{n(n-1)(n-2)\dots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \exp(-\lambda) \quad (7)$$

Expanding the polynomial equation from the terms $n(n-1)\dots(n-i+1)$

$$\begin{cases} n(n-1) & = n^2 - n \\ n(n-1)(n-2) & = n^3 - 3n^2 + 2n \\ \vdots & \\ n(n-1)(n-2)\dots(n-i+1) & = n^i - in^{i-1} + (i-1)n^{i-2}\dots + n \end{cases}$$

Rewriting the first term of the equation and evaluating the limit as n goes to infinity we have

$$\lim_{n \rightarrow \infty} \frac{n^i - \beta_1 n^{i-1} + \beta_2 n^{i-2} \dots + \beta_n n}{n^i} = 1 \quad (8)$$

Equation (7) is now the equation for the Poisson distribution due to the result of the limit shown above. We conclude that for large values of n and small values of p, the Poisson distribution can approximate the result of the Binomial distribution.

$$P(X) = \binom{n}{i} (p)^i (1-p)^{n-i} \approx \frac{\lambda^i}{i!} \exp(-\lambda) \quad (9)$$

References

- [1] S.M. Ross. *Introduction to Probability Models*. Elsevier Science, 2006. ISBN: 9780123756879. URL: <https://books.google.com.mx/books?id=0yDAZf1TfJEC>.

A Código Octave

```
1  close all;
2  clear all;
3  clc;
4  clf;
5  fmt = {"horizontalalignment", "center", "
        verticalalignment", "middle"};
6
7  %% n in [10, 20, 30]
8  %% p in [0.1, 0.3, 0.5]
9  %% lambda in [1, 2, 3]
10
11 figure(1)
12 n01 = 10;
13 n02 = 20;
14 n03 = 30;
15
16 p01 = 0.1;
17 p02 = 0.3;
18 p03 = 0.5;
19
20 binomial11 = zeros(11,1);
21 binomial12 = zeros(11,1);
22 binomial13 = zeros(11,1);
23
24 binomial21 = zeros(21,1);
25 binomial22 = zeros(21,1);
26 binomial23 = zeros(21,1);
27
28 binomial31 = zeros(31,1);
29 binomial32 = zeros(31,1);
30 binomial33 = zeros(31,1);
31
32
33 for i = 0:30;
34     if(i<11)
35 binomial11(i+1) = nchoosek(n01,i)*(p01)^i * (1-p01)^(
        n01-i);
36 binomial12(i+1) = nchoosek(n01,i)*(p02)^i * (1-p02)^(
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        n01-i);
37 binomial13(i+1) = nchoosek(n01,i)*(p03)^i * (1-p03)^(
        n01-i);
38     endif
39
40     if(i<21)
41 binomial21(i+1) = nchoosek(n02,i)*(p01)^i * (1-p01)^(
        n02-i);
42 binomial22(i+1) = nchoosek(n02,i)*(p02)^i * (1-p02)^(
        n02-i);
43 binomial23(i+1) = nchoosek(n02,i)*(p03)^i * (1-p03)^(
        n02-i);
44     endif
45
46 binomial31(i+1) = nchoosek(n03,i)*(p01)^i * (1-p01)^(
        n03-i);
47 binomial32(i+1) = nchoosek(n03,i)*(p02)^i * (1-p02)^(
        n03-i);
48 binomial33(i+1) = nchoosek(n03,i)*(p03)^i * (1-p03)^(
        n03-i);
49
50 endfor
51
52 cumulativeBinomial11 = cumsum(binomial11);
53 cumulativeBinomial12 = cumsum(binomial12);
54 cumulativeBinomial13 = cumsum(binomial13);
55
56 cumulativeBinomial21 = cumsum(binomial21);
57 cumulativeBinomial22 = cumsum(binomial22);
58 cumulativeBinomial23 = cumsum(binomial23);
59
60 cumulativeBinomial31 = cumsum(binomial31);
61 cumulativeBinomial32 = cumsum(binomial32);
62 cumulativeBinomial33 = cumsum(binomial33);
63
64 subplot(3,3,1)
65 plot(0:10, cumulativeBinomial11, 'k')
66 title('n = 10, p = 0.1$', 'Interpreter', 'latex')
67 set(gcf, 'Color', [1 1 1])
68
69 subplot(3,3,2)

```



```

70 plot(0:10, cumulativeBinomial12, 'k')
71 title(' $n = 10, p = 0.3$ ', 'Interpreter', 'latex')
72
73 subplot(3,3,3)
74 plot(0:10, cumulativeBinomial13, 'k')
75 title(' $n = 10, p = 0.5$ ', 'Interpreter', 'latex')
76
77 subplot(3,3,4)
78 plot(0:20, cumulativeBinomial21, 'k')
79 title(' $n = 20, p = 0.1$ ', 'Interpreter', 'latex')
80
81 subplot(3,3,5)
82 plot(0:20, cumulativeBinomial22, 'k')
83 title(' $n = 20, p = 0.3$ ', 'Interpreter', 'latex')
84
85 subplot(3,3,6)
86 plot(0:20, cumulativeBinomial23, 'k')
87 title(' $n = 20, p = 0.5$ ', 'Interpreter', 'latex')
88
89 subplot(3,3,7)
90 plot(0:30, cumulativeBinomial31, 'k')
91 title(' $n = 30, p = 0.1$ ', 'Interpreter', 'latex')
92
93 subplot(3,3,8)
94 plot(0:30, cumulativeBinomial32, 'k')
95 title(' $n = 30, p = 0.3$ ', 'Interpreter', 'latex')
96
97 subplot(3,3,9)
98 plot(0:30, cumulativeBinomial33, 'k')
99 title(' $n = 30, p = 0.5$ ', 'Interpreter', 'latex')
100
101 print(' -dpdflatex ', './img/hw04_binomial.tex ', '-S300
    ,300 ');
102
103 figure(2)
104
105 geometric11 = zeros(11,1);
106 geometric12 = zeros(11,1);
107 geometric13 = zeros(11,1);
108
109 geometric21 = zeros(21,1);

```

```

110 geometric22 = zeros(21,1);
111 geometric23 = zeros(21,1);
112
113 geometric31 = zeros(31,1);
114 geometric32 = zeros(31,1);
115 geometric33 = zeros(31,1);
116
117
118 for i = 0:30;
119     if (i<11)
120 geometric11(i+1) = p01 * (1-p01)^(n01-i);
121 geometric12(i+1) = p02 * (1-p02)^(n01-i);
122 geometric13(i+1) = p03 * (1-p03)^(n01-i);
123     endif
124
125     if (i<21)
126 geometric21(i+1) = p01 * (1-p01)^(n02-i);
127 geometric22(i+1) = p02 * (1-p02)^(n02-i);
128 geometric23(i+1) = p03 * (1-p03)^(n02-i);
129 endif
130
131 geometric31(i+1) = p01 * (1-p01)^(n03-i);
132 geometric32(i+1) = p02 * (1-p02)^(n03-i);
133 geometric33(i+1) = p03 * (1-p03)^(n03-i);
134
135 endfor
136
137 cumulativeGeometric11 = cumsum(geometric11);
138 cumulativeGeometric12 = cumsum(geometric12);
139 cumulativeGeometric13 = cumsum(geometric13);
140
141 cumulativeGeometric21 = cumsum(geometric21);
142 cumulativeGeometric22 = cumsum(geometric22);
143 cumulativeGeometric23 = cumsum(geometric23);
144
145 cumulativeGeometric31 = cumsum(geometric31);
146 cumulativeGeometric32 = cumsum(geometric32);
147 cumulativeGeometric33 = cumsum(geometric33);
148
149 subplot(3,3,1)
150 plot(0:10, cumulativeGeometric11, 'k')

```

```

151 title('$n = 10, p = 0.1$', 'Interpreter', 'latex')
152 set(gcf, 'Color', [1 1 1])
153
154 subplot(3,3,2)
155 plot(0:10, cumulativeGeometric12, 'k')
156 title('$n = 10, p = 0.3$', 'Interpreter', 'latex')
157
158 subplot(3,3,3)
159 plot(0:10, cumulativeGeometric13, 'k')
160 title('$n = 10, p = 0.5$', 'Interpreter', 'latex')
161
162 subplot(3,3,4)
163 plot(0:20, cumulativeGeometric21, 'k')
164 title('$n = 20, p = 0.1$', 'Interpreter', 'latex')
165
166 subplot(3,3,5)
167 plot(0:20, cumulativeGeometric22, 'k')
168 title('$n = 20, p = 0.3$', 'Interpreter', 'latex')
169
170 subplot(3,3,6)
171 plot(0:20, cumulativeGeometric23, 'k')
172 title('$n = 20, p = 0.5$', 'Interpreter', 'latex')
173
174 subplot(3,3,7)
175 plot(0:30, cumulativeGeometric31, 'k')
176 title('$n = 30, p = 0.1$', 'Interpreter', 'latex')
177
178 subplot(3,3,8)
179 plot(0:30, cumulativeGeometric32, 'k')
180 title('$n = 30, p = 0.3$', 'Interpreter', 'latex')
181
182 subplot(3,3,9)
183 plot(0:30, cumulativeGeometric33, 'k')
184 title('$n = 30, p = 0.5$', 'Interpreter', 'latex')
185
186 print('-dpdflatex', './img/hw04_geometric.tex', '-S300
    ,300');
187
188 figure(3)
189 lambda01 = 1;
190 lambda02 = 2;

```

```

191 lambda03 = 3;
192 poisson01 = zeros(11,1);
193 poisson02 = zeros(11,1);
194 poisson03 = zeros(11,1);
195 for i = 0:10;
196     poisson01(i+1)= exp(- lambda01) * (lambda01^i)/
        factorial(i);
197     poisson02(i+1)= exp(- lambda02) * (lambda02^i)/
        factorial(i);
198     poisson03(i+1) = exp(- lambda03) * (lambda03^i)/
        factorial(i);
199 endfor
200
201 cumulativePoisson01 = cumsum(poisson01);
202 cumulativePoisson02 = cumsum(poisson02);
203 cumulativePoisson03 = cumsum(poisson03);
204
205 subplot(1,3,1)
206 plot(0:10, cumulativePoisson01, 'k')
207 title(' $\lambda = 1$', 'Interpreter', 'latex')
208 set(gcf, 'Color', [1 1 1])
209
210 subplot(1,3,2)
211 plot(0:10, cumulativePoisson02, 'k')
212 title(' $\lambda = 2$', 'Interpreter', 'latex')
213
214 subplot(1,3,3)
215 plot(0:10, cumulativePoisson03, 'k')
216 title(' $\lambda = 3$', 'Interpreter', 'latex')
217
218 print(' -dpdflatex ', './img/hw04_poisson.tex ', '-S300
        ,300 ');

```