Homework #15

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1. Explain why $G_{xx}(f)$ is a constant for Example 6.3 in [1].

   The function $G_{xx}(f)$ is a constant for the example because $G_{xx}(f)$ is the autospectral density function of the input. This function is defined as

$$G_{xx} = 2S_{xx}(f)$$

   Recall that

$$S_{xx}(f) = 2\int_0^\infty R_{xx}(\tau)\cos(2\pi f\tau)d\tau$$

   As such, G_{xx} becomes

$$G_{xx} = 4\int_0^\infty R_{xx}(\tau)\cos(2\pi f\tau)d\tau$$

   And R_{xx} is defined as

$$R_{xx}(\tau) = E[x_k(t)x_k(t+\tau)]$$

   The expected value of the input, white noise, is a constant μ that depends on the samples. Thus, G_{xx} is a constant because the Fourier transformation of the original input results in a constant as well.

2. Explain how the output spectral function G_{yy} for Example 6.3 results in the output autocorrelation function R_{yy} shown below.

$$
\begin{aligned}
G_{yy}(f) &= |H(f)|^2_{f-d} G = \frac{G}{\left|1-(f/f_n)^2\right|^2 + (2\zeta f/f_n)^2} \quad 0 \le f < \infty \\
R_{yy}(\tau) &= \frac{G\pi f_n \exp(-2\pi f_n \zeta|\tau|)}{4\zeta} F(\tau,\zeta) \\
F(\tau,\zeta) &= \cos(2\pi f_n\sqrt{1-\zeta^2}|\tau|) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(2\pi f_n\sqrt{1-\zeta^2}|\tau|)
\end{aligned}
$$

Recall that the the autocorrelation function can be expressed as

$$
R_{yy}(\tau) = \int_0^\infty G_{yy}(f)\cos(2\pi f\tau)df
$$

Substituting known values in the expression

$$
\begin{aligned}
R_{yy}(\tau) &= \int_0^\infty |H(f)|^2_{f-d} G\cos(2\pi f\tau)df \\
&= G\int_0^\infty |H(f)|^2_{f-d}\cos(2\pi f\tau)df \\
&= G\int_0^\infty \frac{\cos(2\pi f\tau)}{\left|1-(f/f_n)^2\right|^2 + (2\zeta f/f_n)^2}df
\end{aligned}
$$

Solving the integral, we obtain the following

$$
\begin{aligned}
R_{yy}(\tau) = &\frac{G\pi f_n \exp(-2\pi f_n \zeta|\tau|)}{4\zeta}\cos(2\pi f_n\sqrt{1-\zeta^2}|\tau|) \\
&+ \frac{G\pi f_n \exp(-2\pi f_n \zeta|\tau|)}{4\zeta}\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(2\pi f_n\sqrt{1-\zeta^2}|\tau|)
\end{aligned}
$$

which can then be factorized to

$$
\begin{aligned}
R_{yy}(\tau) &= \frac{G\pi f_n \exp(-2\pi f_n \zeta|\tau|)}{4\zeta} F(\tau,\zeta) \\
F(\tau,\zeta) &= \cos(2\pi f_n\sqrt{1-\zeta^2}|\tau|) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(2\pi f_n\sqrt{1-\zeta^2}|\tau|)
\end{aligned}
$$

3. Present plots for both G_{yy} and R_{yy}

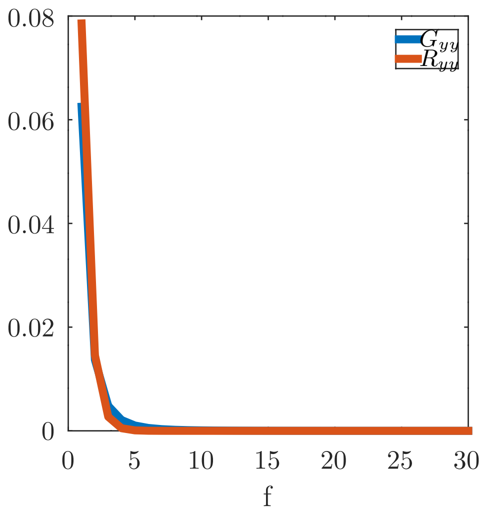

Figure 1: Comparison.

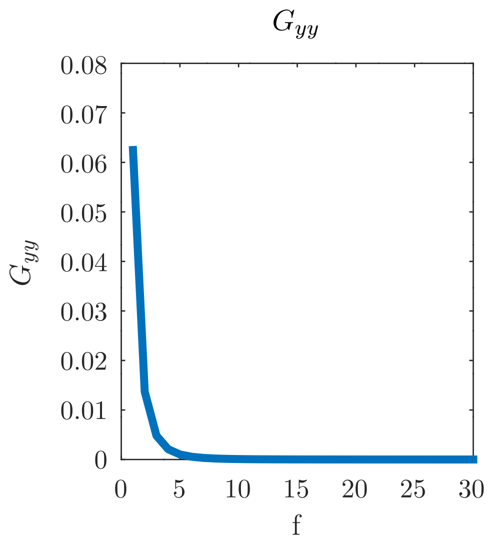

Figure 2: Output Spectral function.

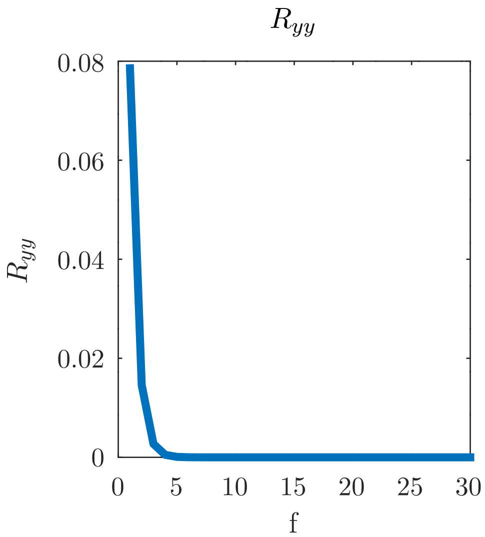

Figure 3: Output Autocorrelation function.

References

[1] J.S. Bendat and A.G. Piersol. *Random Data: Analysis and Measurement Procedures*. Wiley Series in Probability and Statistics. Wiley, 2011. ISBN: 9781118210826. URL: `https://books.google.com.mx/books?id=qYSViFRNMlwC`.

A Octave Code

```
clc
close all
clear all

fn = 1; % Natural frequency
G = 1;
zeta = 2;

n = 30;

Gyy = zeros(n, 1);
Ryy = zeros(n, 1);


% Autospectral density function
for f = 1:n
    a = (abs(1 - (f/fn)^2))^2;
    b = (2 * zeta * f / fn)^2;
    Gyy(f) = G/(a+b);
    endfor

% Output Autocorrelation function
for tau= 1:n
    x1 = G * pi * fn * exp(-2*pi*zeta*abs(tau))/(4*zeta);
    x2 = cos(2*pi*fn *abs(tau)*sqrt(1-zeta^2));
    x3 = (zeta)/(sqrt(1-zeta^2))*  sin(2*pi*fn *abs(tau)*sqrt(1-zeta^2));
    Ryy(tau) = x1*(x2+x3);
  endfor


figure(1);
plot (1:n, Gyy,'linewidth', 3, 1:n, Ryy,'linewidth', 3)
xlabel('f')
legend({'$G_{yy}$', '$R_{yy}$'},'Interpreter','latex')
print('-dpdflatex', './img/hw15_comparison.tex', '-
```

```
    S200,200');

figure(2);
plot (1:n, Gyy,'linewidth', 3)
xlabel('f')
ylabel('$G_{yy}$')
title('$G_{yy}$','Interpreter','latex')
print('-dpdflatex', './img/hw15_Gyy.tex', '-S200,200')
    ;

figure(3);
plot (1:n, Ryy,'linewidth', 3)
xlabel('f')
ylabel('$R_{yy}$')
title('$R_{yy}$','Interpreter','latex')
print('-dpdflatex', './img/hw15_Ryy.tex', '-S200,200')
    ;
```