Homework #07: Joint Probability Distributions of Functions of Random Variables

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2020-02-24

- Prove that $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|J(x_1, x_2)|^{-1}$, where $|J(x_1, x_2)|^{-1}$ is the inverse of the determinant of the Jacobian. Consider $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ for some functions g_1 and g_2.

  We begin the proof by considering the following

  1. The equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 such that $x_1 = h_1(y_1, y_2)$ and $x_2 = h_2(y_1, y_2)$.
  2. The functions g_1 and g_2 have continuous partial derivatives at all points (x_1, x_2) and are such that

     $$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} \equiv \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \neq 0$$

     at all points (x_1, x_2).

  Given the Probability Mass Function

  $$P\{Y_1 \leq y_1, Y_2 \leq y_2\} = \iint_R f_{X_1,X_2}(x_1, x_2) dx_1 dx_2$$

  the *joint density function* can be obtained by differentiating the previous equation.

  $$f_{Y_1,Y_2}(y_1, y_2) = \frac{\partial^2}{\partial y_1 \partial y_2} \iint_R f_{X_1,X_2}(x_1, x_2) dx_1 dx_2$$

Given that the original equation is a function of x_1 and x_2, it is necessary to apply a *one to one transformation* T on the region R that is being integrated, such that

$$T(g_1, g_2) = (x_1, x_2)$$

Let $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ be the equations that map the original coordinates to the new region S and $J(g_1, g_2)$ the determinant of the matrix of partial derivatives of g_1 and g_2 such that

$$J_g = J(g_1, g_2) = \begin{vmatrix} \frac{\partial x_1}{\partial g_1} & \frac{\partial x_1}{\partial g_2} \\ \frac{\partial x_2}{\partial g_1} & \frac{\partial x_2}{\partial g_2} \end{vmatrix} = (J(x_1, x_2))^{-1} = \frac{1}{\begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}}$$

Then we have the following solution to the equation, integrating by substitution yields

$$\begin{aligned} \iint_R f_{X_1,X_2}(x_1, x_2) dA &= \iint_S f_{X_1,X_2}(h_1(g_1, g_2), h_2(g_1, g_2)) J_g dg_1 dg_2 \\ &= \iint_S f_{X_1,X_2}(h_1(y_1, y_2), h_2(y_1, y_2)) J_g dy_1 dy_2 \end{aligned}$$

Differentiating this double integral will result in the joint density function

$$\begin{aligned} &\frac{\partial^2}{\partial y_1 \partial y_2} \iint_S f_{X_1,X_2}(h_1(y_1, y_2), h_2(y_1, y_2)) J_g dy_1 dy_2 \\ &= f_{X_1,X_2}(h_1(y_1, y_2), h_2(y_1, y_2)) J_g \\ &= f_{X_1,X_2}(h_1(g_1, g_2), h_2(g_1, g_2)) J_g \\ &= f_{X_1,X_2}(x_1, x_2) J_g \\ &= f_{X_1,X_2}(x_1, x_2) J(g_1, g_2) \\ &= f_{X_1,X_2}(x_1, x_2) |J(x_1, x_2)|^{-1} \end{aligned}$$

- Prove that $\dfrac{d\phi(t)}{dt} \equiv \dfrac{d}{dt} E[\exp(tX)] = E[\dfrac{d}{dt} \exp(tX)]$

From [1], the *moment generating function* is defined as

$$\phi(t) = E[\exp(tX)] = \begin{cases} \sum_x \exp(tx)p(x), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} \exp(tx)f(x)dx, & \text{if } X \text{ is continuous} \end{cases}$$

We obtain the derivative of the moment generating function.

$$\begin{aligned} \frac{d\phi(t)}{dt} &= \frac{d}{dt}\sum_x \exp(tx)p(x) \\ &= \sum_x \frac{d}{dt}\left(\exp(tx)p(x)\right) \\ &= \sum_x \left(\exp(tx)\frac{dp(x)}{dt} + \frac{d\exp(tx)}{dt}p(x)\right) \end{aligned}$$

Given that $p(x)$ is not a function of t ($\frac{dp(x)}{dt} = 0$), the expression becomes

$$\begin{aligned} \frac{d\phi(t)}{dt} &= \sum_x \left(0 + \frac{d\exp(tx)}{dt}p(x)\right) \\ &= \sum_x x\exp(tx)p(x) \\ &= E[X\exp(tX)] \\ &= E[\frac{d}{dt}\exp(tX)] \end{aligned}$$

As such, it is proven that

$$\frac{d\phi(t)}{dt} \equiv \frac{d}{dt}E[\exp(tX]) = E[\frac{d}{dt}\exp(tX)]$$

References

[1] S.M. Ross. *Introduction to Probability Models*. Elsevier Science, 2006. ISBN: 9780123756879. URL: `https://books.google.com.mx/books?id=0yDAZf1TfJEC`.