Homework #05: Continuous Random Variable

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1 Problem

1. Plot the probability distribution functions for the Exponential and Gamma Distributions. Consider $\lambda \in \{0.5, 1, 2\}$ for the Exponential Distribution and $\alpha \in \{2, 3, 7, 7\}, \lambda \in \{0.5, 2, 2, 1\}$ for the Gamma Distribution.

2. Provide proof that the Gamma distribution is indeed a probability density function.

3. Relate the Uniform, Exponenial and Gamma Distributions to physical phenomena or engineering problems.

2 Solution

Plots are presented for the Probability Density Function of the Exponential (figure 1) and Gamma (figure 2) Distributions.

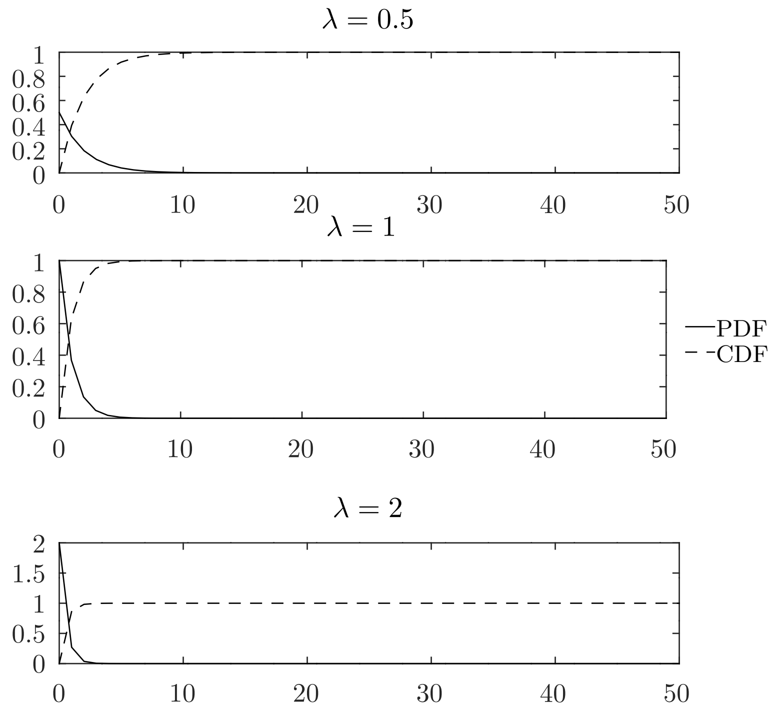

Figure 1: Exponential Distribution.

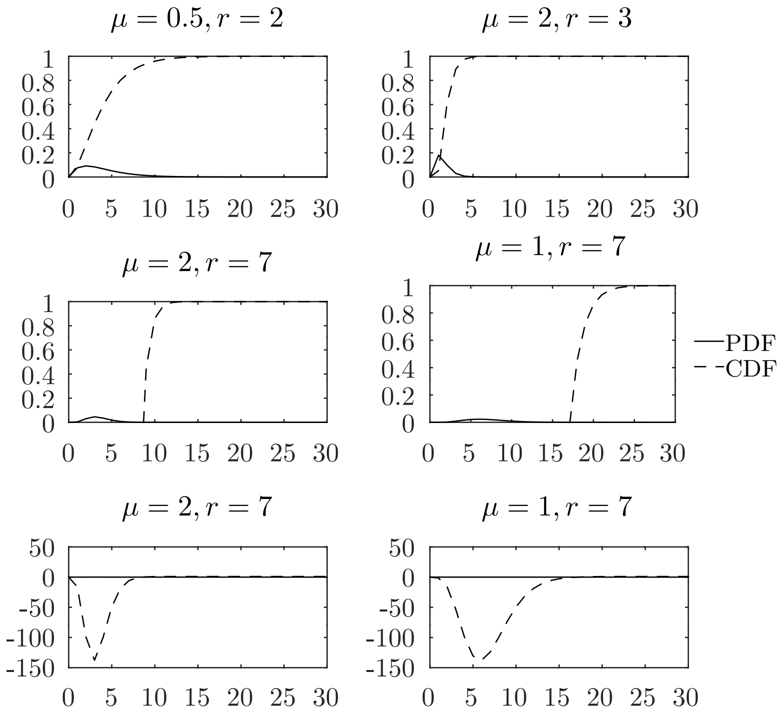

Figure 2: Gamma Distribution.

We provide proof that the Gamma Distribution probability function is indeed a probability density function. From [1], the Gamma Distribution is defined as

$$f(x) \equiv f(x; \beta, \alpha) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} \exp(-x/\beta), \quad x > 0 \tag{1}$$

By establishing $\alpha = r$ and $\beta = 1/\mu$, the function can be rewritten as

$$f(x) = \frac{\mu(\mu x)^{r-1} \exp(-\mu x)}{\Gamma(r)} \tag{2}$$

this form is also known as the *Erlang-r* distribution. Integrating this new expression we have the following

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{\mu(\mu x)^{r-1}\exp(-\mu x)}{\Gamma(r)}dx \tag{3}$$

$$u = \mu x \tag{4}$$

$$du = \mu dx \tag{5}$$

$$= \frac{1}{\Gamma(r)}\int_0^\infty (u)^{r-1}\exp(-u)du \tag{6}$$

Recall the definition of the Gamma function

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}exp(-y)dy \tag{7}$$

Thus the equation is simplified as

$$\int_0^\infty f(x)dx = \frac{\Gamma(r)}{\Gamma(r)} \tag{8}$$

$$= 1 \tag{9}$$

We present examples of the three distributions studied.

- Uniform Distribution. Events in which all results have the same probability of occurring such as the odds of any number of a six sided die being rolled or the result of a coin toss.
- Exponential Distribution. Events related to time and events, such as number of number of calls received during an hour, rainfall in a year, distribution of gas molecules at fixed temperature and pressure.
- Gamma Distribution. Also used for events related to time and rate, such as the amount of customers arriving to a restaurant in a specific time interval. Consider a restaurant where customers arrive at x customer per hour, what is the probability of k customers arriving in a given time interval?

References

[1] W.J. Stewart. *Probability, Markov Chains, Queues, and Simulation: The Mathematical Basis of Performance Modeling*. Princeton University Press, 2009. ISBN: 9781400832811. URL: https://books.google.com.mx/books?id=ZfRyBS1WbAQC.

A Octave Code

```
close all;
clear all;
clc;
clf;
%% fmt = {"horizontalalignment", "center", "verticalalignment", "middle"};

%% lambda [0.5, 1, 2]

expPDF01 = zeros(51,1);
expPDF02 = zeros(51,1);
expPDF03 = zeros(51,1);

expCDF01 = zeros(51,1);
expCDF02 = zeros(51,1);
expCDF03 = zeros(51,1);

lambda01 = 0.5;
lambda02 = 1;
lambda03 = 2;

for i = 0:50
  expPDF01(i+1) = lambda01*exp(-lambda01*i);
  expPDF02(i+1) = lambda02*exp(-lambda02*i);
  expPDF03(i+1) = lambda03*exp(-lambda03*i);

  expCDF01(i+1) = 1-exp(-lambda01*i);
  expCDF02(i+1) = 1-exp(-lambda02*i);
  expCDF03(i+1) = 1-exp(-lambda03*i);
endfor

%%expCDF01 = cumsum(expPDF01);
%%expCDF02 = cumsum(expPDF02);
%%expCDF03 = cumsum(expPDF03);

%% alpha [2, 3, 7, 7]
%% lambda [0.5, 2, 2, 1]

```

```
gammaPDF01 = zeros(31,1);
gammaPDF02 = zeros(31,1);
gammaPDF03 = zeros(31,1);
gammaPDF04 = zeros(31,1);

gammaCDF01 = zeros(31,1);
gammaCDF02 = zeros(31,1);
gammaCDF03 = zeros(31,1);
gammaCDF04 = zeros(31,1);

mu01 = 0.5;
mu02 = 2;
mu03 = 2;
mu04 = 1;

r01 = 2;
r02 = 3;
r03 = 7;
r04 = 7;

for x = 0:30
  gammaPDF01(x+1) = mu01*(mu01*x)^(r01-1)*exp(-mu01*x)/factorial(r01);
  gammaPDF02(x+1) = mu02*(mu02*x)^(r02-1)*exp(-mu02*x)/factorial(r02);
  gammaPDF03(x+1) = mu03*(mu03*x)^(r03-1)*exp(-mu03*x)/factorial(r03);
  gammaPDF04(x+1) = mu04*(mu04*x)^(r04-1)*exp(-mu04*x)/factorial(r04);

  for n = 0:(r01-1)
    gammaCDF01(x+1) = gammaCDF01(x+1) + exp(-mu01 *x)*(mu01*x)^n;
  endfor

    for n = 0:(r02-1)
    gammaCDF02(x+1) = gammaCDF02(x+1) + exp(-mu02 *x)*(mu02*x)^n;
  endfor

    for n = 0:(r03-1)
```

```
      gammaCDF03(x+1) = gammaCDF03(x+1) + exp(-mu03 *x)*(mu03*x)^n;
    endfor
        for n = 0:(r04-1)
      gammaCDF04(x+1) = gammaCDF04(x+1) + exp(-mu04 *x)*(mu04*x)^n;
      endfor

    gammaCDF01(x+1) = 1 - gammaCDF01(x+1);
    gammaCDF02(x+1) = 1 - gammaCDF02(x+1);
    gammaCDF03(x+1) = 1 - gammaCDF03(x+1);
    gammaCDF04(x+1) = 1 - gammaCDF04(x+1);
    endfor

figure(1)



subplot(3,2,1)
plot(0:30, gammaPDF01, 'k', 0:30, gammaCDF01, '--k')
title('$\mu = 0.5, r = 2$','Interpreter','latex')
set(gcf,'Color',[1 1 1])

subplot(3,2,2)
plot(0:30, gammaPDF02, 'k', 0:30, gammaCDF02, '--k')
title('$\mu = 2, r = 3$','Interpreter','latex')

subplot(3,2,3)
plot(0:30, gammaPDF03, 'k', 0:30, gammaCDF03, '--k')
ylim([0, 1])
title('$\mu = 2, r = 7$','Interpreter','latex')

subplot(3,2,4)
plot(0:30, gammaPDF04, 'k', 0:30, gammaCDF04, '--k')
ylim([0, 1])
legend({"PDF","CDF"})
legend("boxoff")
legend("location", "eastoutside")
title('$\mu = 1, r = 7$','Interpreter','latex')

subplot(3,2,5)
```

```
plot(0:30, gammaPDF03, 'k', 0:30, gammaCDF03, '--k')
title('$\mu = 2, r = 7$','Interpreter','latex')

subplot(3,2,6)
plot(0:30, gammaPDF04, 'k', 0:30, gammaCDF04, '--k')
title('$\mu = 1, r = 7$','Interpreter','latex')

print('-dpdflatex', './img/hw05_gamma.tex', '-S300,300');


figure(2)

subplot(3,1,1)
plot(0:50, expPDF01, 'k', 0:50, expCDF01, '--k')
title('$\lambda = 0.5$','Interpreter','latex')
set(gcf,'Color',[1 1 1])

subplot(3,1,2)
plot(0:50, expPDF02, 'k', 0:50, expCDF02, '--k')
legend({"PDF","CDF"})
legend("boxoff")
legend("location", "eastoutside")
title('$\lambda = 1$','Interpreter','latex')

subplot(3,1,3)
plot(0:50, expPDF03, 'k', 0:50, expCDF03, '--k')
title('$\lambda = 2$','Interpreter','latex')

print('-dpdflatex', './img/hw05_exp.tex', '-S300,300');
```