Homework #13

Isaac Ayala Lozano

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Problem. Explain why a system can be completely characterized with its impulse response.

A LTI system is fully characterized by knowing its impulse response $h(t)$ because any input to the system can be represented as a sum of impulses. Likewise, the output of the system can also be represented as the sum of all impulse responses corresponding to the inputs.

$$x(t) = \sum_{i=0}^{M} \alpha_i \delta(t - k_i) \rightarrow y(t) = \sum_{i=0}^{M} \alpha_i h(t - k_i) \tag{1}$$

We begin by assuming that the system $\mathfrak{F}$ is a Linear Time Invariant (LTI) system. As such, the superposition principle applies. This implies that the system has the following properties:

$$\begin{aligned} \mathfrak{F}(x_1 + x_2) &= \mathfrak{F}(x_1) + \mathfrak{F}(x_2) \quad \text{Additivity} \\ \mathfrak{F}(\alpha\, x) &= \alpha \mathfrak{F}(x) \quad \text{Homogeneity} \end{aligned}$$

For a given system $\mathfrak{F}$, the output of the system is defined as $y = \mathfrak{F}\{x\}$. For an input x_n, there will be a corresponding output y_n.

Consider now the unit impulse function δ. The function is defined as

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

If the unit impulse function is given as an input ($x = \delta(t)$) to the system $\mathfrak{F}$, an output known as the unit impulse response $y = h(t)$ will be obtained. Due to the principle of superposition, we know that if the input is shifted in time or scaled by a constant then the output will change accordingly.

$$\delta(t-k) \rightarrow h(t-k)$$
$$\alpha\delta(t) \rightarrow \alpha h(t)$$

Likewise, due to additivity the following is also true

$$\alpha_1\delta(t-k_1) + \alpha_2\delta(t-k_2) \rightarrow y = \alpha_1 h(t-k_1) + \alpha_2 h(t+k_2)$$

An input $x(t)$ can be represented as a sum of impulse functions, each scaled accordingly for each instant of time of the function.

$$x(t) = \sum_{i=0}^{M} \alpha_i \delta(t-k_i)$$

Likewise, the output of the system to an input $x(t)$ can be represented as the sum of the outputs for each impulse function.

$$y(t) = \sum_{i=0}^{M} \alpha_i h(t-k_i)$$

As such, for a LTI system it is possible to fully characterize its behaviour by simply knowing the impulse response of the system.