Homework #04: Random Variable and Discrete Distribution Functions

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1 Problem

1. Plot the behaviour of the Binomial, Geometric and Poisson distributions for the following values
   - Binomial distribution: $n \in [10, 20, 30]$ and $p \in [0.1, 0.3, 0.5]$
   - Geometric distribution: $p \in [0.1, 0.3, 0.5]$
   - Poisson distribution: $\lambda \in [1, 2, 3]$
2. Show that the Poisson distribution is a good approximation to the Binomial distribution for large values of n and small values of p.

2 Solution

2.1 Plots

Figure 1 shows the plots for the Binomial distribution.

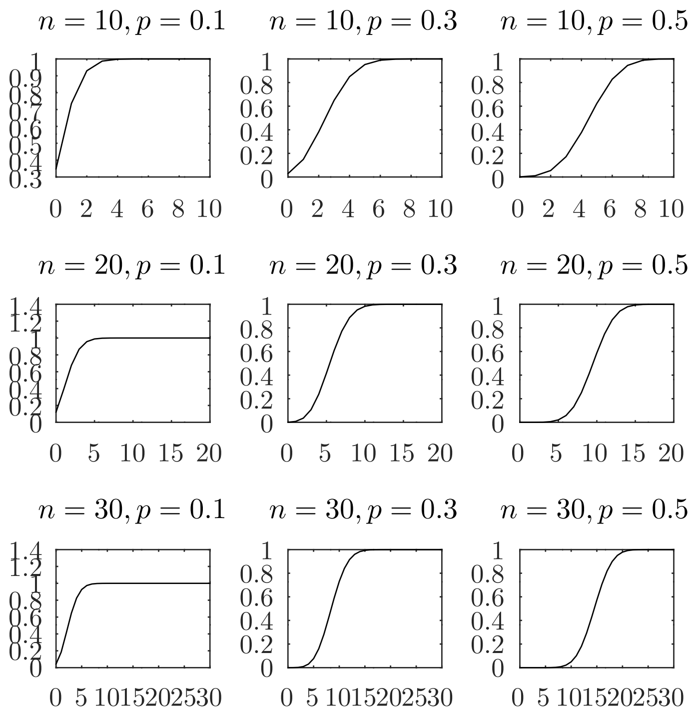

Figure 1: Binomial distribution.

Figure 2 shows the plots for the Geometric distribution.

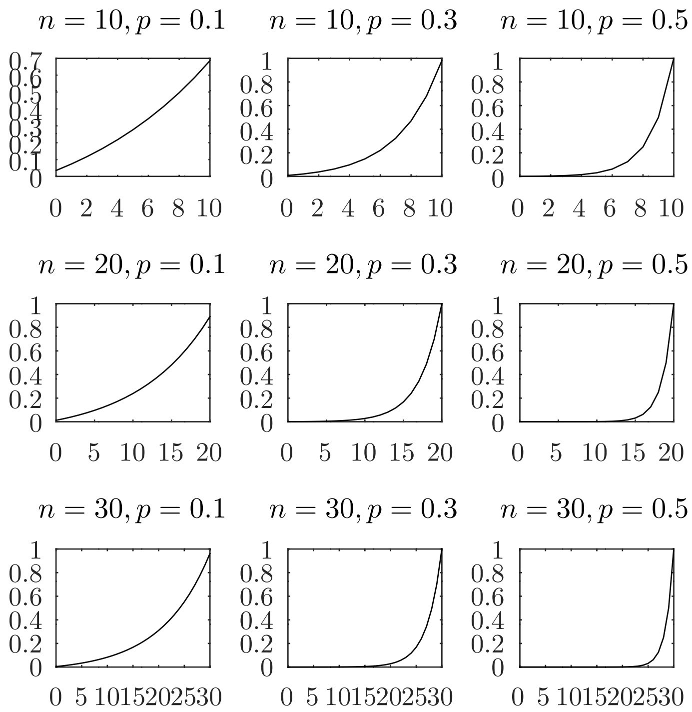

Figure 2: Geometric distribution

Figure 3 shows the plots for the Poisson distribution.

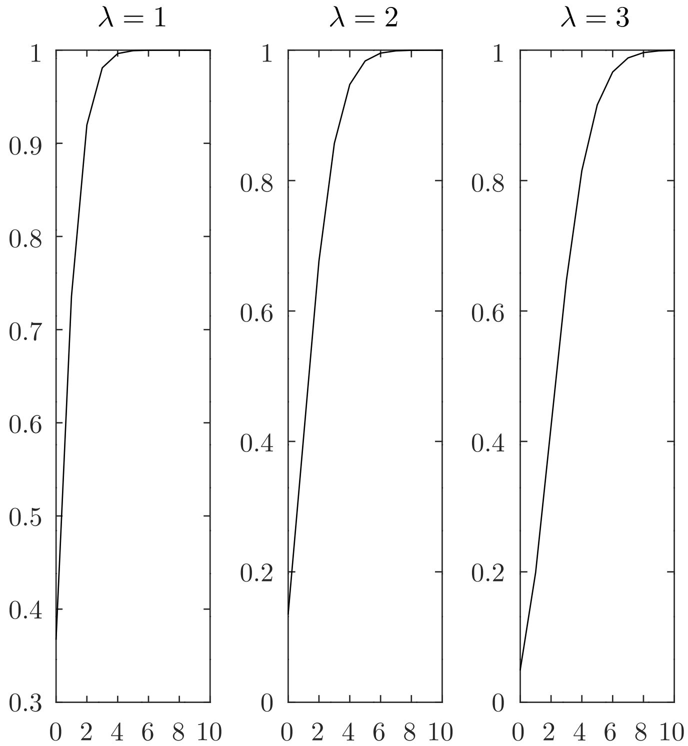

Figure 3: Poisson distribution

2.2 Proof

From [1] we obtain the equations for the Binomial and Poisson distributions, equations (1) and (2) respectively.

$$P(X) = \binom{n}{i} p^i (1-p)^{n-i} \tag{1}$$

$$P(X) = \exp(-\lambda)\frac{\lambda^i}{i!} \tag{2}$$

Let $\lambda = np$ such that (1) can written as follows

$$\binom{n}{i} p^i (1-p)^{n-i} = \frac{n!}{(n-i)!i!}\left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}$$

Rewrite the last term of the equation.

$$\frac{n!}{(n-i)!i!}\left(\frac{\lambda}{n}\right)^i \frac{(1-\lambda/n)^n}{(1-\lambda/n)^i} \tag{3}$$

Recall that for large values of n the term $(1-\lambda/n)^n$ is approximately equal to $exp(-\lambda)$. Thus the equation above can be expressed as

$$\frac{n!}{(n-i)!i!}\left(\frac{\lambda}{n}\right)^i \frac{exp(-\lambda)}{(1-\lambda/n)^i} \tag{4}$$

The term $\frac{n!}{(n-i)!i!}$ can be expanded as follows

$$\frac{n!}{(n-i)!i!} = \frac{n*(n-1)!}{(n-i)!i!} = \frac{n*(n-1)*(n-2)!}{(n-i)!i!}$$

This expansion permits us to rewrite the term as

$$\frac{n!}{(n-i)!i!} = \frac{n*(n-1)*(n-2)\ldots(n-i+1)*(n-i)!}{(n-i)!i!}$$

Notice how this new expression permits us to eliminate $(n-i)!$ from the equation.

$$\frac{n!}{(n-i)!i!} = \frac{n*(n-1)*(n-2)\ldots(n-i+1)}{i!}$$

Equation (4) can now be rewritten as

$$\frac{n(n-1)(n-2)\ldots(n-i+1)}{i!}\frac{\lambda^i}{n^i}\frac{exp(-\lambda)}{(1-\lambda/n)^i} \tag{5}$$

For large values of n, the term $(1-\lambda/n)^i$ is approximately equal to one. This simplifies the equation once again.

$$\frac{n(n-1)(n-2)\dots(n-i+1)}{i!}\frac{\lambda^i}{n^i}exp(-\lambda) \tag{6}$$

Exchanging the denominators in the fractions we obtain

$$\frac{n(n-1)(n-2)\dots(n-i+1)}{n^i}\frac{\lambda^i}{i!}exp(-\lambda) \tag{7}$$

Expanding the polynomial equation from the terms $n(n-1)\dots(n-i+1)$

$$\begin{cases} n(n-1) & = n^2 - n \\ n(n-1)(n-2) & = n^3 - 3n^2 + 2n \\ \vdots & \\ n(n-1)(n-2)\dots(n-i+1) & = n^i - in^{i-1} + (i-1)n^{i-2}\dots + n \end{cases}$$

Rewriting the first term of the equation and evaluating the limit as n goes to infinity we have

$$\lim_{n\to\infty}\frac{n^i - \beta_1 n^{i-1} + \beta_2 n^{i-2}\dots + \beta_n n}{n^i} = 1 \tag{8}$$

Equation (7) is now the equation for the Poisson distribution due to the result of the limit shown above. We conclude that for large values of n and small values of p, the Poisson distribution can approximate the result of the Binomial distribution.

$$P(X) = \binom{n}{i}(p)^i(1-p)^{n-i} \approx \frac{\lambda^i}{i!}exp(-\lambda) \tag{9}$$

References

[1] S.M. Ross. *Introduction to Probability Models*. Elsevier Science, 2006. ISBN: 9780123756879. URL: `https://books.google.com.mx/books?id=0yDAZf1TfJEC`.

A Código Octave

```
close all;
clear all;
clc;
clf;
fmt = {"horizontalalignment", "center", "verticalalignment", "middle"};

%% n in [10, 20, 30]
%% p in [0.1, 0.3, 0.5]
%% lambda in [1, 2, 3]

figure(1)
n01 = 10;
n02 = 20;
n03 = 30;

p01 = 0.1;
p02 = 0.3;
p03 = 0.5;

binomial11 = zeros(11,1);
binomial12 = zeros(11,1);
binomial13 = zeros(11,1);

binomial21 = zeros(21,1);
binomial22 = zeros(21,1);
binomial23 = zeros(21,1);

binomial31 = zeros(31,1);
binomial32 = zeros(31,1);
binomial33 = zeros(31,1);


for i = 0:30;
  if(i<11)
binomial11(i+1) = nchoosek(n01,i)*(p01)^i * (1-p01)^(n01-i);
binomial12(i+1) = nchoosek(n01,i)*(p02)^i * (1-p02)^(
```

```
n01-i);
binomial13(i+1) = nchoosek(n01,i)*(p03)^i * (1-p03)^(n01-i);
  endif

  if(i<21)
binomial21(i+1) = nchoosek(n02,i)*(p01)^i * (1-p01)^(n02-i);
binomial22(i+1) = nchoosek(n02,i)*(p02)^i * (1-p02)^(n02-i);
binomial23(i+1) = nchoosek(n02,i)*(p03)^i * (1-p03)^(n02-i);
endif

binomial31(i+1) = nchoosek(n03,i)*(p01)^i * (1-p01)^(n03-i);
binomial32(i+1) = nchoosek(n03,i)*(p02)^i * (1-p02)^(n03-i);
binomial33(i+1) = nchoosek(n03,i)*(p03)^i * (1-p03)^(n03-i);

endfor

cummulativeBinomial11 = cumsum(binomial11);
cummulativeBinomial12 = cumsum(binomial12);
cummulativeBinomial13 = cumsum(binomial13);

cummulativeBinomial21 = cumsum(binomial21);
cummulativeBinomial22 = cumsum(binomial22);
cummulativeBinomial23 = cumsum(binomial23);

cummulativeBinomial31 = cumsum(binomial31);
cummulativeBinomial32 = cumsum(binomial32);
cummulativeBinomial33 = cumsum(binomial33);

subplot(3,3,1)
plot(0:10, cummulativeBinomial11, 'k')
title('$n = 10, p = 0.1$','Interpreter','latex')
set(gcf,'Color',[1 1 1])

subplot(3,3,2)
```

```
plot(0:10, cummulativeBinomial12, 'k')
title('$n = 10, p = 0.3$','Interpreter','latex')

subplot(3,3,3)
plot(0:10, cummulativeBinomial13, 'k')
title('$n = 10, p = 0.5$','Interpreter','latex')

subplot(3,3,4)
plot(0:20, cummulativeBinomial21, 'k')
title('$n = 20, p = 0.1$','Interpreter','latex')

subplot(3,3,5)
plot(0:20, cummulativeBinomial22, 'k')
title('$n = 20, p = 0.3$','Interpreter','latex')

subplot(3,3,6)
plot(0:20, cummulativeBinomial23, 'k')
title('$n = 20, p = 0.5$','Interpreter','latex')

subplot(3,3,7)
plot(0:30, cummulativeBinomial31, 'k')
title('$n = 30, p = 0.1$','Interpreter','latex')

subplot(3,3,8)
plot(0:30, cummulativeBinomial32, 'k')
title('$n = 30, p = 0.3$','Interpreter','latex')

subplot(3,3,9)
plot(0:30, cummulativeBinomial33, 'k')
title('$n = 30, p = 0.5$','Interpreter','latex')

print('-dpdflatex', './img/hw04_binomial.tex', '-S300,300');

figure(2)

geometric11 = zeros(11,1);
geometric12 = zeros(11,1);
geometric13 = zeros(11,1);

geometric21 = zeros(21,1);
```

```
geometric22 = zeros(21,1);
geometric23 = zeros(21,1);

geometric31 = zeros(31,1);
geometric32 = zeros(31,1);
geometric33 = zeros(31,1);


for i = 0:30;
  if(i<11)
geometric11(i+1) = p01 * (1-p01)^(n01-i);
geometric12(i+1) = p02 * (1-p02)^(n01-i);
geometric13(i+1) = p03 * (1-p03)^(n01-i);
  endif

  if(i<21)
geometric21(i+1) = p01 * (1-p01)^(n02-i);
geometric22(i+1) = p02 * (1-p02)^(n02-i);
geometric23(i+1) = p03 * (1-p03)^(n02-i);
endif

geometric31(i+1) = p01 * (1-p01)^(n03-i);
geometric32(i+1) = p02 * (1-p02)^(n03-i);
geometric33(i+1) = p03 * (1-p03)^(n03-i);

endfor

cummulativeGeometric11 = cumsum(geometric11);
cummulativeGeometric12 = cumsum(geometric12);
cummulativeGeometric13 = cumsum(geometric13);

cummulativeGeometric21 = cumsum(geometric21);
cummulativeGeometric22 = cumsum(geometric22);
cummulativeGeometric23 = cumsum(geometric23);

cummulativeGeometric31 = cumsum(geometric31);
cummulativeGeometric32 = cumsum(geometric32);
cummulativeGeometric33 = cumsum(geometric33);

subplot(3,3,1)
plot(0:10, cummulativeGeometric11, 'k')
```

```
title('$n = 10, p = 0.1$','Interpreter','latex')
set(gcf,'Color',[1 1 1])

subplot(3,3,2)
plot(0:10, cummulativeGeometric12, 'k')
title('$n = 10, p = 0.3$','Interpreter','latex')

subplot(3,3,3)
plot(0:10, cummulativeGeometric13, 'k')
title('$n = 10, p = 0.5$','Interpreter','latex')

subplot(3,3,4)
plot(0:20, cummulativeGeometric21, 'k')
title('$n = 20, p = 0.1$','Interpreter','latex')

subplot(3,3,5)
plot(0:20, cummulativeGeometric22, 'k')
title('$n = 20, p = 0.3$','Interpreter','latex')

subplot(3,3,6)
plot(0:20, cummulativeGeometric23, 'k')
title('$n = 20, p = 0.5$','Interpreter','latex')

subplot(3,3,7)
plot(0:30, cummulativeGeometric31, 'k')
title('$n = 30, p = 0.1$','Interpreter','latex')

subplot(3,3,8)
plot(0:30, cummulativeGeometric32, 'k')
title('$n = 30, p = 0.3$','Interpreter','latex')

subplot(3,3,9)
plot(0:30, cummulativeGeometric33, 'k')
title('$n = 30, p = 0.5$','Interpreter','latex')

print('-dpdflatex', './img/hw04_geometric.tex', '-S300,300');

figure(3)
lambda01 = 1;
lambda02 = 2;
```

```
lambda03 = 3;
poisson01 = zeros(11,1);
poisson02 = zeros(11,1);
poisson03 = zeros(11,1);
for i = 0:10;
  poisson01(i+1)= exp(- lambda01) * (lambda01^i)/factorial(i);
  poisson02(i+1)= exp(- lambda02) * (lambda02^i)/factorial(i);
  poisson03(i+1) = exp(- lambda03) * (lambda03^i)/factorial(i);
endfor

cummulativePoisson01 = cumsum(poisson01);
cummulativePoisson02 = cumsum(poisson02);
cummulativePoisson03 = cumsum(poisson03);

subplot(1,3,1)
plot(0:10, cummulativePoisson01, 'k')
title('$\lambda = 1$','Interpreter','latex')
set(gcf,'Color',[1 1 1])

subplot(1,3,2)
plot(0:10, cummulativePoisson02, 'k')
title('$\lambda = 2$','Interpreter','latex')

subplot(1,3,3)
plot(0:10, cummulativePoisson03, 'k')
title('$\lambda = 3$','Interpreter','latex')

print('-dpdflatex', './img/hw04_poisson.tex', '-S300,300');
```