1 Lecture 2 - Exercise 8

Calculate the velocity, speed and acceleration for each position vector.

1.1 $\vec{r}(t) = (\cos(\omega t), \mathrm{e}^{\omega t})$

The first derivative of the vector gives us

$$\vec{v}(t) = (-\omega \sin(\omega t), \omega \mathrm{e}^{\omega t})$$

The norm of the velocity vector gives us the speed.

$$\|\vec{v}(t)\| = \sqrt{\omega^2 \sin^2(\omega t) + \omega^2 \mathrm{e}^{2\omega t}}$$

The second derivative results in

$$\vec{a}(t) = (-\omega^2 \cos(\omega t), \omega^2 \mathrm{e}^{\omega t})$$

Figure 1 shows the plots for all three vectors with conditions $\omega = 1$, and $t \in [0, 2\pi]$.

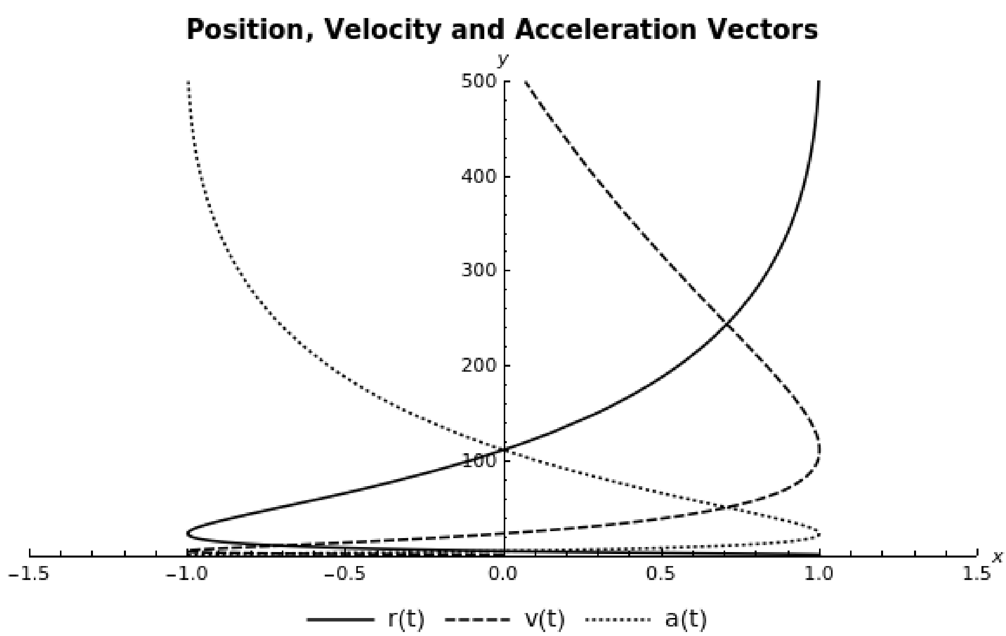

Figure 1: Position, Velocity and Acceleration plots for $\vec{r}(t) = (\cos(\omega t), \mathrm{e}^{\omega t})$

1.2 $\vec{r}(t) = (\cos(\omega t - \phi), \sin(\omega t - \phi))$

Using the following trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

the position vector can be rewritten as follows

$$\vec{r} = (\cos\omega t\cos\phi + \sin\omega t\sin\phi, \sin\omega t\cos\phi - \cos\omega t\sin\phi)$$

The velocity, speed and acceleration are as follows

$$\begin{aligned}\vec{v}(t) &= (-\omega\cos\phi\sin\omega t + \omega\sin\phi\cos\omega t, \omega\cos\phi\cos\omega t + \omega\sin\phi\sin\omega t)\\ &= (-\omega\sin(\omega t + \phi), \omega\cos(\omega t + \phi))\end{aligned}$$

$$\|\vec{v}(t)\| = \sqrt{\omega^2\sin^2(\omega t + \phi) + \omega^2\cos^2(\omega t + \phi)}$$

$$\begin{aligned}\vec{a}(t) &= (-\omega^2\cos\phi\cos\omega t - \omega^2\sin\phi\sin\omega t, -\omega^2\cos\phi\sin\omega t + \omega^2\sin\phi\cos\omega t)\\ &= (-\omega^2\cos(\omega t - \phi), -\omega^2\sin(\omega t - \phi))\end{aligned}$$

Figure 2 shows the plotted movement equations for the system. Notice that all three curves overlap when evaluated with $\omega = 1$, $t \in [0, 2\pi]$ and $\phi = \frac{\pi}{4}$.

1.3 $\vec{r}(t) = (c\cos^3(t), c\sin^3(t))$

Apply the chain rule to obtain the derivatives of $\vec{r}(t)$.

$$\vec{v}(t) = (-3c\cos^2(t)\sin(t), 3c\sin^2(t)\cos(t))$$

$$\|\vec{v}(t)\| = \sqrt{9c^2\cos^4(t)\sin^2(t) + 9c^2\sin^4(t)\cos^2(t)}$$

$$\vec{a}(t) = (-3c\cos^3(t) - 6c\sin^2(t)\cos(t), -3c\sin^3(t) + 6c\cos^2(t)\sin(t))$$

All three vectors are plotted in figure 3 with $t \in [0, 2\pi]$, and $c = 1$.

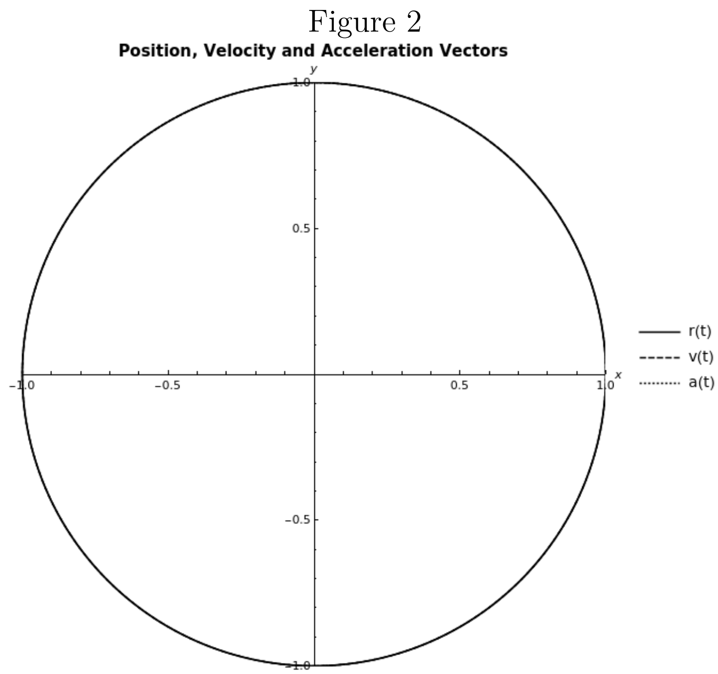

Figure 2: Position, Velocity and Acceleration plots for $\vec{r}(t) = (\cos(\omega t - \phi), \sin(\omega t - \phi))$

1.4 $\vec{r}(t) = (c(t - \sin t), c(1 - \cos t))$

$$\vec{v}(t) = (c(1 - \cos(t)), c\sin(t))$$

$$\|\vec{v}(t)\| = \sqrt{c^2 - c^2\cos^2(t) + c^2\sin^2(t)}$$

$$\vec{a}(t) = (c\sin(t), c\cos(t))$$

Figure 4 shows the plots for all three equations evaluated with $c = 1$ and $t \in [0, 2\pi]$.

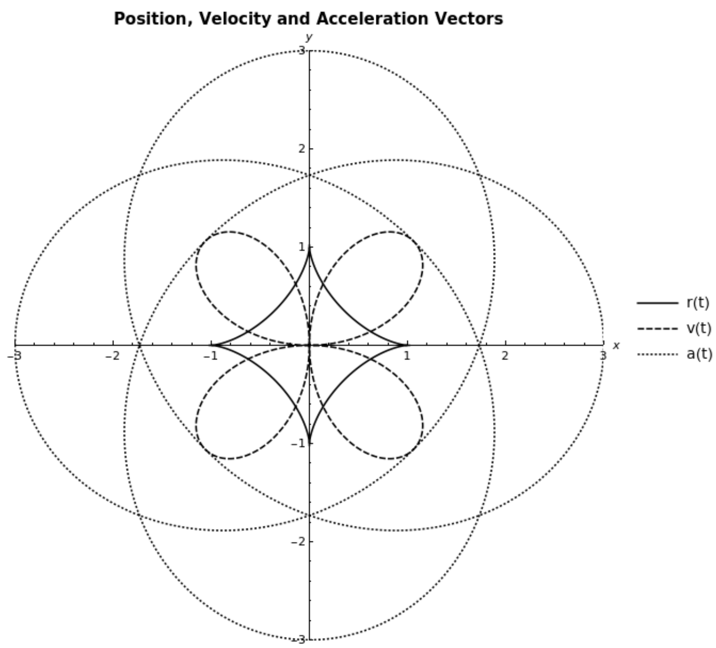

Figure 3: Position, Velocity and Acceleration plots for $\vec{r}(t) = (c\cos^3(t), andc\sin^3(t))$.

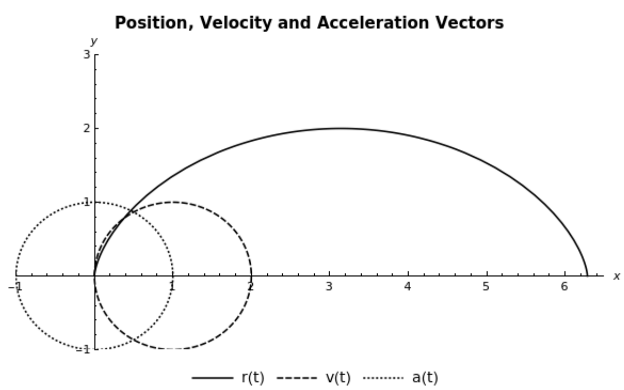

Figure 4: Position, Velocity and Acceleration plots for $\vec{r}(t) = (c(t - \sin t), c(1 - \cos t))$