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An introduction to functional and array programming

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Abstract

This book, which means to serve as an introduction to functional and array programming utilises KamilaLisp, a Lisp-inspired dynamically typed programming language that borrows many ideas from Haskell, Standard ML, APL, Scheme and others. It is a functional programming language with a strong emphasis on array programming, and it is designed to be used in a wide range of applications. Its design stems from previous iterations, which featured more orthodox Lisp syntax, lazy evaluation and symbolic computation with an emphasis on mathematical programming.

Foreword

KamilaLisp, the language described in the book, has its origins in its previous iterations, v0.1 and MalbolgeLISP. The first iteration of MalbolgeLISP (v1.0) was released in August 2020 and had very few of the features that distinguish KamilaLisp today. The second iteration of MalbolgeLISP (v1.1) was released in July 2021 and was the subject of much attention due to its unusual choice of implementation language. The final MalbolgeLISP (v1.2) was released in September 2021, sharing many ideas and features with KamilaLisp. Unfortunately, the effort required to implement it continued to grow. The codebase was becoming increasingly disorganised and difficult to maintain. In addition, the implementation of new complicated features was increasingly difficult due to the choice of implementation language. The first version of KamilaLisp, a MalbolgeLISP-inspired Lisp dialect, appeared in December 2021, featuring lazy evaluation, sophisticated numerical and symbolic operations, and many original design choices borrowed from MalbolgeLISP. Progress on the language has stalled due to significant performance problems with the Java Virtual Machine, the overhead of lazy evaluation, and some questionable decisions made at the core of the interpreter. After realising this rather early, an effort to create a C++ rewrite of the original code base began towards the end of 2021, but was quickly abandoned due to problems with garbage collection, efficient memory management, and C++ lacking certain qualities that would otherwise make it a good language in which to implement an interpreter for KamilaLisp. In August 2022 I started working on KamilaLisp v0.2, a Java rewrite of the original codebase that does not have some of the problematic features of the previous versions. This book, in the hope of being useful to the determined reader who wants to learn about KamilaLisp, describes my experiences in implementing a complex language runtime using Java, and also describes the language itself. It is assumed that the reader has an elementary knowledge of functional programming or of the APL programming language.

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Glossary

In this book, the following assumptions are made:

- *Folding* a list ω with a function $\alpha\alpha$ is equivalent to putting $\alpha\alpha$ between every element of ω , also given the identity element of $\alpha\alpha$. One could assume binding to the left $((\omega[1] \alpha\alpha \omega[2]) \alpha\alpha \omega[3])$, or to the right $(\omega[1] \alpha\alpha (\omega[2] \alpha\alpha \omega[3]))$, hence the names *fold-right* and *fold-left*. Unlike reduction, the result of *folding* an empty array is defined and equal to the identity element.
- *Scanning* a list ω with a function $\alpha\alpha$ is equivalent to mapping a fold over prefixes of a list. This definition of a scan has a $O(n^2)$ complexity regardless of the fold type. A more efficient, $O(n)$ definition of a *scan-left* exists (fold with intermediate steps).
- *Outer product* of function $\alpha\alpha$ ($\circ . \alpha\alpha$) and arrays α and ω is the application of $\alpha\alpha$ between every pair of elements from α and ω . The resulting list has a depth given by $\rho, \text{Ö}\rho$.
- *Inner product* of functions $\alpha\alpha$, $\omega\omega$ and lists α and ω ($\alpha\alpha . \omega\omega$) is equivalent to folding the list obtained by putting $\omega\omega$ between corresponding pairs of α and ω with $\alpha\alpha$.
- A *higher-order function* is a function that takes another function as its argument.
- A *lambda expression* (in the context of KamilaLisp) is an anonymous function with static (lexical) scoping, meaning that it can see all the variables bound by its lexical ancestor, unlike to dynamic scoping implemented by older LISPs where a function can see all the variables bound by its caller(s). In both cases, the first bound variable found is used, allowing to shadow variables.
- A *dyad* is a two argument function.
- A *monad* is generally a single argument function. The book also uses it in a context of an abstract data constructor implementing the *bind* and *unit* functions.
- *Replicating* a list ω according to list α is copying each element of ω a given number of times (specified by the corresponding element of α or assumed to be zero). In KamilaLisp, α can be a scalar, in which case the contents of ω are catenated to each other α times. If ω is also a scalar, it's repeated α times to form a list.
- *Filtering* a list ω with function $\alpha\alpha$ is equivalent to mapping $\alpha\alpha$ on every element of ω (assuming $\alpha\alpha$ returns a boolean value, i.e. 0 or 1), and replicating ω with the result of the mapping $(\omega / \text{¨}\alpha\alpha \text{¨}\omega)$.
- *Rotating* a list ω by α elements is equivalent to dropping α elements from ω , and joining them with ω (dyadic ϕ , or more illustratively, (\downarrow, \uparrow)).
- *Zippping* two arrays is the act of forming a list of pairs via the juxtaposition of corresponding elements from the given arrays.
- *Flattening* a list is the act of decreasing the list's depth by one level, enlisting all of the elements from the topmost sublists into a resultant list.
- *Mapping* a function $\alpha\alpha$ over a list ω is the act of processing each element of ω with the function $\alpha\alpha$ to produce a resultant list of equivalent length.

- *Partial application* is defined as fixing a number of arguments to a function, yielding an anonymous function of smaller arity.
- *Iteration* of a function over an argument is defined as evaluating the function over its result starting with the initial argument until a condition is satisfied. Sometimes the condition is numeric, or is defined as a dyadic function between the previous and current result (just `while`). A fixed point combinator could be implemented using partial application of deep equality (`*≡`). Iteration is faster and consumes less resources than recursion, assuming no tail call optimisation.
- *Function composition* (or `@/•`, as KamilaLisp calls it) is taking an arbitrary amount of functions where each function operates on the results of the function before, except the last function, which gets all the arguments passed to the anonymous function yielded by composition.
- *Selfie* is a higher order function that duplicates the argument to the function it takes (e.g. `*~`, computes the square of its argument).
- *Commute* swaps the order of two arguments to a function. $\alpha \text{ f } \omega \iff \omega \text{ f } \alpha$.
- A *variadic function* is a function that takes an arbitrary number of arguments.
- *Currying* is the technique of translating the evaluation of a function that takes multiple arguments into evaluating a sequence of functions, each with a single argument.

Chapter 1

Initial considerations

This chapter discusses the basics of KamilaLisp. Throughout this book, the author will use the KamilaLisp interpreter to check and execute the declarations of a program one by one. The emphasis will be put on list processing and mathematical functions to form elementary understanding of the language.

1.1. Programs and variables

A KamilaLisp program is a sequence of declarations, which are executed in the order they are written. The first program presented in this book is shown below:

```
--> def a (+ (* 2 3) 2)
8
--> def b (* 5 a)
40
```

It consists of two declarations. The first declaration binds the identifier `a` to the integer 8, and the second declaration binds the identifier `b` to the integer 40, which follows from the intuitive understanding of the arithmetic operations. To the reader not accustomed with Lisp-like syntax, every element of the syntax tree that would otherwise be implicitly grouped by a language with usual arithmetical precedence rules is explicitly grouped by parentheses to form a list. The resulting values of variables can be determined as follows:

```
--> ?a
8
--> ?b
40
```

The question mark is a sign for the KamilaLisp interpreter not to evaluate the entire input as an expression, but rather, to query the value of what it refers to.

Every list besides the empty list (usually written as `'()` or alternatively `nil`) has a *head* defined as the first element of it. When a Lisp program is evaluated, the *head* of the current list is assumed to be a callable value, while the rest of the list (also called the *tail*) is assumed to be a list of arguments.

Because the list of arguments to a function (the *tail*) is evaluated before its applied to the *head*, a perceptive reader could point out a potential issue - *How to introduce list literals in the code?* This question is indeed well-founded, since the list literal would be evaluated in order to pass it parameter to some callable object, hence the tail of the literal would be applied to its head, thus behaving undesirably and almost certainly raising an error. Every Lisp dialect addresses this issue in the same way using the quoting mechanism. Simply put, the quote prevents a list from being evaluated. To observe this behaviour, introduce two more functions called `car` and `cdr` to obtain respectively the *head* and *tail* of a list:

```
--> car '(1 2 3)
1
```

```
--> cdr '(1 2 3)
(2 3)
```

This may alert the observant reader once again - *What if I want to create a list out of a set of expressions?* This question is once again well founded - because the quote stops something from being evaluated, then surely the expressions inside of it will stay untouched too. The answer is simple - use the **tie** function as follows:

```
--> tie 1 2 3
(1 2 3)
--> tie (+ 2 2) (/ 6 3) (+ 2 3)
(4 2 5)
```

KamilaLisp follows scoping rules familiar from other programming languages, such as Scheme or C++ - *static scoping* (also called *lexical scoping*), where an attempt is initially made to resolve a variable in the current scope. If this approach fails, the variable is resolved in the scope of its lexical ancestors until either the interpreter finds an environment where the variable is bound, or raises an error regarding an unbound variable. Additionally, variables may be *shadowed*, as demonstrated below:

```
--> def my-list '(1 2 3)
(1 2 3)
--> car my-list
1
--> def my-list (cdr my-list)
(2 3)
--> car my-list
2
```

However, it is not possible to shadow pre-defined variables and functions in the global scope:

```
--> def car 5
RuntimeException thrown in thread 1dbd16a6:
    def can not shadow or redefine built-in bindings.
    at entity def 1:1
    at def primitive function
```

To fully exercise lexical scoping, the language needs to provide a way of binding names inside a specific block of code (unlike *def* which binds names in the global scope). This can be accomplished in a variety of ways, the most straight-forward one being the *let* construct. The *let* construct binds a list of name/value pairs, and evaluates the body of the construct in the context of the newly created environment. The syntax of the *let* construct is demonstrated by the following example:

```
--> def a 5
5
--> def b 6
6
--> + a b
11
--> let ((a 10) (b 15)) (+ a b)
25
```

1.2. Functions and lambda expressions

Functions are the core component of KamilaLisp. They are first-class objects, which means that they can be passed as arguments to other functions, returned from functions, and assigned to variables. The syntax of a function declaration is as follows:

```
--> defun square (x) (* x x)
      (λ x . (* x x))
```

The function that has just been declared is called **square**, and it takes one argument called **x**. The body of the function is the expression `(* x x)`. The function returns the value of their expression, which is simply the square of the argument. Notice that when defining a monadic function, a pair of parentheses can be omitted for brevity:

```
--> defun square x (* x x)
      (λ x . (* x x))
```

Since **square** is now bound in the global scope, it can be applied to an argument. The code below binds the result of application of the number 5 to the function **square** to the variable **a**:

```
--> def a (square 5)
      25
```

Functions do not need to be named. They can be introduced in the code *anonymously* using the *lambda* construct, which opens up many new possibilities. For example, the declaration of the function **square** can be rewritten as follows:

```
--> def square (lambda x (* x x))
      (λ x . (* x x))
```

Furthermore, multivariate lambda expressions could serve as a replacement for the *let* construct:

```
--> let ((a 10) (b 15)) (+ a b)
      25
--> (lambda (a b) (+ a b)) 10 15
      25
```

Since functions are first-class in KamilaLisp, it is now also possible to return them from functions and take them as arguments. The following example demonstrates these programming techniques using the *lambda* construct:

```
--> ; Returns a function that adds a given number to its argument.
--> ; The technique demonstrated is often called "currying".
--> defun make-adder (x) (lambda y (+ x y))
      (λ x . (λ y . (+ x y)))
--> def add-5 (make-adder 5)
      (λ y . (+ 5 y))
--> add-5 10
      15
--> ; Returns a string explaining the value of a function at point.
--> defun explain (f x) (str:format "The value of f(x) for x={?x} is {f x}")
      (λ f x . (str:format "The value of f(x) for x={?x} is {f x}"))
--> explain square 4.5
      The value of f(x) for x=4.5 is 20.25
```

When writing complex functions, it is of particular interest to keep the partial results obtained during the execution of the function. This can be achieved by cascading the *let* construct, or using the *let-seq* construct. The following examples implement a function that raises its first argument to the eighth power¹:

```
--> defun p8 x (let-seq (
...   (def y (* x x))
...   (def z (* y y))
...   (* z z)))
(λ x . (let-seq ((def y (* x x)) (def z (* y y)) (* z z))))
--> p8 4
65536
--> defun p8 x (let ((y (* x x))) (let ((z (* y y))) (* z z)))
(λ x . (let ((y (* x x)) (z (* y y))) (* z z)))
--> p8 4
65536
--> defun p8 x (let ((y (* x x)) (z (* y y))) (* z z))
(λ x . (let ((y (* x x)) (z (* y y))) (* z z)))
--> p8 4
65536
```

Notice that despite using **def**, **defun**, etc..., the *let-seq* construct does not create any bindings in the global scope - the bindings are local to the block.

Function calls in KamilaLisp are performed using a *call stack*. The call stack is a data structure inside of the interpreter that keeps track of the functions that are currently being executed. When a function is called, a new *stack frame* is created and pushed onto the stack. When the function returns, the stack frame is popped from the stack - this way, the interpreter knows where to return to after the function yields a value. When an exception happens, the interpreter will present the user with a *stack trace*, which is a list of functions that were executed before the exception was thrown, for example:

```
--> defun f x (/ 1 0) ; Oops! Division by zero!
(λ x . (/ 1 0))
--> defun g x (f x)
(λ x . (f x))
--> defun h x (g x)
(λ x . (g x))
--> h 10
ArithmeticException thrown in thread 5e82df6a:
  Division by zero
at entity h 1:1
at (λ x . (g x)) 1:9
at entity g 1:12
at (λ x . (f x)) 1:9
at entity f 1:12
at (λ x . (/ 1 ...)) 1:9
at entity / 1:12
at / primitive function
```

¹Many reimplementations of common built-in functions mentioned in the book that are present in KamilaLisp are rather suboptimal and demonstrated only for the sake of completeness. The programmer is urged to use the optimised predefined routines when possible instead.

1.3. Conditional expressions and comparisons

The comparison operators in KamilaLisp don't differ significantly from the ones present in other, perhaps more orthodox programming languages (such as C). It is worth noting that equality is checked using the `=` function, inequality is checked for using the `/=` function, while the `<=>` function is the so-called *three way comparison* operator, which returns -1, 0 or 1 respectively if the first argument is less than, equal to, or greater than the second argument.

KamilaLisp provides a number of conditional expressions, which are used to control the flow of execution. The most basic one is the *if* construct, which takes three arguments - a condition, an expression to be evaluated if the condition is true, and an expression to be evaluated if the condition is false. The syntax of the *if* construct is demonstrated by the following example:

```
--> defun my-abs x (if (< x 0) (- x) x)
(λ x . (if (< x 0) (- x) x))
--> my-abs -5
5
--> my-abs 5
5
--> my-abs 0
0
```

The *if* construct is a special case of the *cond* construct, which takes a list of pairs of conditions and expressions. The first condition that evaluates to true is used to evaluate the corresponding expression. The syntax of the *cond* construct is demonstrated by the following reimplementation of the three-way comparison operator:

```
--> defun compare (x y) (cond ((< x y) -1) ((> x y) 1) (0))
(λ x y . (cond ((< x y) -1) ((> x y) 1) (0)))
--> compare 5 10
-1
--> compare 10 5
1
--> compare 5 5
0
```

1.4. Recursive functions

Recursion is a powerful technique extensively used in functional programming. Its role in KamilaLisp is admittedly not as important as in other functional programming languages, since the language provides a number of other often more wieldy techniques for solving the same problems (e.g. using array programming). However, it is still worth mentioning that KamilaLisp provides a number of tools for writing recursive functions.

The following function implements the *factorial* function in a recursive manner:

```
--> defun factorial (n) (if (= n 0) 1 (* n (factorial (- n 1))))
(λ n . (if (= n 0) 1 (* n (factorial (- n 1)))))
--> factorial 5
120
```

Since KamilaLisp supports arbitrary precision integers, the factorial function can be applied to arbitrarily large numbers. However, at some point, this function will *overflow its call stack*, meaning that the number of recursive calls will exceed the maximum call stack size. This is a common problem in recursive functions, and it can be solved by using the *tail recursion* technique. Tail recursion is a special case of recursion, where the recursive call is the last expression in the function body. In this case, the stack frame of the current function can be reused for the recursive call, which means that the stack will not grow indefinitely. To make use of this technique, the factorial function needs to be altered so that the recursive call is the last expression in the function body:

```
--> defun factorial (n) (let-seq (
...   (defun f' (n acc) (if (= n 0) acc (f' (- n 1) (* n acc))))
...   (f' n 1)))
(λ n . (let-seq ((defun f' (n acc) (if (= n 0) acc (f' (- n 1) (* n acc)))) (f' n 1))))
--> factorial 5
120
```

The factorial function defines a helper function that has an accumulator argument, which is used to store the intermediate results of the computation. The helper function is called recursively, and the accumulator argument is updated with the result of the multiplication. The *let-seq* construct is used to define the helper function, so that it is not visible outside the factorial function. There is one more step to make this function tail-recursive: redefine the self-referential call to the helper function:

```
--> defun factorial (n) (let-seq (
...   (defun f' (n acc) (if (= n 0) acc (&0 (- n 1) (* n acc))))
...   (f' n 1)))
(λ n . (let-seq ((defun f' (n acc) (if (= n 0) acc (&0 (- n 1) (* n acc)))) (f' n 1))))
--> factorial 5
120
```

The self-referential call was replaced by `&0`, which in simple terms is a reference to the current function. This is a special case of the `&` operator, which is used to refer to functions by nesting level in the source code, akin to de Bruijn indices. The `&0` operator refers to the current function, `&1` refers to the function (anonymous or named) that is the lexical ancestor the current function, and so on. This way, the factorial function no longer errors when applied to large numbers, since the stack does not grow indefinitely:

```
--> factorial 1000
402387260077093773543702433923003985719374864210714
632543799910429938512398629020592044208486969404800
; [...]
```

Another, closely related function to the factorial function that can be implemented using tail recursion is the power function. The following function implements the power function in a simple recursive manner:

```
--> defun power (x n) (if (= n 0) 1 (* x (power x (- n 1)))))
(λ x n . (if (= n 0) 1 (* x (power x (- n 1)))))
--> power 2 10
1024
```

Once again, the problem is that the stack will overflow when the power function is applied to large numbers. To prevent this and speed up the computation, the power function needs to be first transformed so that the recursive call is the last expression in the function body. Consider the following *trace* of the power function applied to the arguments 2 and 4:

```
--> power 2 4
(* 2 (power 2 3))
(* 2 (* 2 (power 2 2)))
(* 2 (* 2 (* 2 (power 2 1))))
(* 2 (* 2 (* 2 (* 2 (power 2 0)))))
```

```

(* 2 (* 2 (* 2 2)))
(* 2 (* 2 4))
(* 2 8)
16

```

After transforming the function to use an accumulator:

```

--> defun power (x n) (let-seq (
...   (defun f' (x n acc) (if (= n 0) acc (f' x (- n 1) (* x acc))))
...   (f' x n 1)))

```

When this implementation of the power function is applied to the same arguments as before, the following trace is produced:

```

--> power 2 4
(f' 2 4 1)
(f' 2 3 2)
(f' 2 2 4)
(f' 2 1 8)
(f' 2 0 16)
16

```

Notice, that the size of the expression *does not grow*, hence the function could in theory be rewritten as a simple loop with bounded storage space requirements, a concept more familiar from imperative programming languages such as C or C++. The final improvement that needs to be applied is actually telling the interpreter to perform tail call optimisation in this case, using the self-referential call `&0`:

```

--> defun power (x n) ((
...   lambda (x n acc) (if (= n 0) acc (&0 x (- n 1) (* x acc)))) x n 1)
(λ x n . ((λ (x n acc) (if (= n 0) acc (&0/syn x (- n 1) (* x acc)))) x n 1))
--> power 2 4
16

```

1.5. Function composition

Function composition is a a common core concept in functional programming languages, an emphasis on which is placed in KamilaLisp. Using the `@` operator, it is possible to compose two or more functions into a single function. An example follows:

```

--> defun f x (* x 2)
(λ x . (* x 2))
--> defun g x (+ x 1)
(λ x . (+ x 1))
--> f 2
4
--> g 2
3
--> f@g 2
6

```

The usefulness of function composition might be hard to appreciate at first. In the end, $f@g\ x \Leftrightarrow f\ (g\ x)$,

however, the function returned by the @ operator does not have to be immediately applied - it can be bound to a name and used later. Without the composition operator, it would be necessary to introduce a *lambda* to achieve the same result as `f@g` - that is, `lambda x (f (g x))`. The difference between manually composing two functions to create a new function and using the @ operator is how they treat arguments. The @ operator variant of the same expression does not refer to the arguments of the composed functions (and thus does not manually relay an indeterminate amount of them to the innermost function), creating the basic building block for *point-free programming*.

Of course, it is possible to compose arbitrarily many functions by using the @ operator multiple times. The following example demonstrates the composition of three functions:

```
--> defun f x (* x 2)
(λ x . (* x 2))
--> defun g x (+ x 1)
(λ x . (+ x 1))
--> defun h x (- x 3)
(λ x . (- x 3))
--> f@g@h 2
0
```

Another form of function composition notoriously used in the APL family of programming languages is the so-called *fork* (the μ -recursive composition operator). Generally speaking, it is sometimes of special interest to preprocess the arguments using different functions (*reductees*), and then funnel the results into a single function (*reductor*). For instance, the arithmetic mean is defined as the quotient (*reductor*) of the sum and length of a list (*reductees*), while a palindrome is defined as a string for which the equality (*reductor*) between it and its reverse (*reductees*).

To implement a sum function using a fork, it is necessary to define a function that sums a list beforehand using the `tally` function which returns the length of a list. A non-zero value is considered truthy, hence the following definition of the *sum* function:

```
--> defun sum (l) (if (tally l) (+ (car l) (sum (cdr l))) 0)
(λ l . (if (tally l) (+ (car l) (sum (cdr l))) 0))
--> sum '(1 6 2 3)
12
```

There are many alternative ways to implement it. One problem the reader may stumble upon is `= 'nil 'nil` returning `'nil`. This behaviour is reasoned by the fact that equality *vectorises* over lists, meaning that equality is tested element-wise. To determine structural equality, it is advised to use the `same` function. The following example demonstrates this behaviour:

```
--> = '(1 2 3) '(1 2 4)
(1 1 0)
--> same '(1 2 3) '(1 2 4)
0
--> same '(1 2 3) '(1 2 3)
0
```

Hence arguing for a simpler implementation of the *sum* function:

```
--> defun sum (l) (if (same l '()) 0 (+ (car l) (sum (cdr l))))
(λ l . (if (same l '()) 0 (+ (car l) (sum (cdr l)))))
--> sum '(1 6 2 3)
12
```

The arithmetic mean function is thusly introduced as follows:

```
--> [/ sum tally] '(1 6 2 3 4)
3.2
```

Of course, it could also be bound to a name without applying the fork instantaneously:

```
--> def avg [/ sum tally]
[/ sum tally]
--> avg '(1 6 2 3 4)
3.2
```

The key merits of point-free programming are:

- The code is more concise.
- The code is often more readable.
- The code is faster, because there is no need to allocate a stack frame.
- The code does not need to bind any names.

The lack of argument naming gives the point-free paradigm a reputation of being obtuse, hence the humorously used epithet "pointless style"². However, the point-free paradigm is not necessarily obscure. In fact, it is often more readable than the equivalent imperative code and favoured by many languages, such as Haskell, APL, J, PostScript, Forth, Factor, jq and the Unix shell³.

1.6. Partial application and μ -recursive functions

Most functional programming languages, such as OCaml and Haskell automatically *curry* functions by default, that is, allow applying functions to fewer arguments than they are defined to take. This is a useful feature, because it allows for the creation of new functions by binding some of the arguments of a function to a value. This mechanism is called *partial application*. KamilaLisp supports partial application of functions but it does not happen by default, since unlike OCaml and Haskell, KamilaLisp supports variadic functions. Recall the following example given earlier in the book:

```
--> ; Returns a function that adds a given number to its argument.
--> ; The technique demonstrated is often called "currying".
--> defun make-adder (x) (lambda y (+ x y))
(λ x . (λ y . (+ x y)))
--> def add-5 (make-adder 5)
(λ y . (+ 5 y))
--> add-5 10
15
```

Using partial application, the example can be rewritten as follows:

```
--> defun add-x n $(+ n)
```

²<http://hdl.handle.net/1822/2869>

³The pipe operator is essentially point-free function composition. The Unix shell has more interesting properties from a category theoretic perspective, due to its extensive use of monads. The pipe operator is an implementation of `bind`, `cat` is in reality monadic `return`. Moreover, a pair of these operations, adding `<` and `>`, satisfy the monadic laws - `cat f | cmd` is the same as `cmd <f` (left-hand identity); `cmd | cat` is the same as `cmd` (right-hand identity); `c1 | (c2 | c3)` is the same as `(c1 | c2) | c3`.

```
(λ n . $(+ n))
--> def add-5 (add-x 5)
$(+ n)
--> add-5 6
11
```

Partial application is particularly useful in conjunction with various kinds of function composition, as it allows for the vast majority of functions that are defined in terms of other functions to be defined in a point-free manner. Another example would be a function that returns the remainder of a number when divided by two.

```
--> defun mod-2 (n) (mod n 2)
(λ n . (mod n 2))
--> mod-2 10
0
--> mod-2 11
1
```

It is not immediately clear how to define this function in a point-free manner, since partial application will apply the arguments in order as given, and the argument that needs to be partially applied here is actually the *second* argument to `mod`. This problem can be solved in a two ways - the first one is to use the commute operator `^` which reverses the argument order to a function:

```
--> def mod-2 $(^mod 2)
$['^mod, '2]
--> mod-2 5
1
--> mod-2 6
0
```

Another, perhaps more elegant solution is to use partial application placeholders:

```
--> def mod-2 $(mod _ 2)
$['mod, _, '2]
--> mod-2 5
1
--> mod-2 6
0
```

This way, the `_` placeholder indicates that the first argument given to the resulting function should be put in its place. The placeholder can be used multiple times and it is required that all placeholder slots are filled when applying the already partially applied function.

Variadic functions were briefly mentioned earlier in the book. Simply speaking, they are functions that take an arbitrary number of arguments. `+` is a good example of a variadic function, since when applied to more than two arguments, it just sums everything:

```
--> + 1 2 3 4 5
15
```

It is possible to define custom variadic functions in KamilaLisp using special argument syntax. To recall, `lambda (x y) (code)` takes two arguments - *x* and *y*. This function can be made variadic by prepending the *last argument* with an ellipsis - `lambda (x ...xs) (code)`. The last argument is bound to a list of all the remaining arguments. This has an interesting property: while the aforementioned function takes *at least* one

argument (because `...xs` can be empty), it can be modified to take any amount of arguments, even zero - `lambda ...xs (code)`. To demonstrate this behaviour, the following function will obtain the arithmetic average of all its arguments, but will refuse to be called with zero arguments:

```
--> defun avg (x ...xs) (/ (+ x (sum ...xs)) (+ 1 (tally ...xs)))
(lambda x ...xs . (/ (+ x (sum ...xs)) (+ 1 (tally ...xs))))
--> avg 1 6 2 3 4
3.2
--> avg 5
5
--> avg
TypeError thrown in thread 15eb5ee5:
    Expected at least 1 arguments to `(lambda x ...xs . (/ (+ x (sum ...xs)) ...))'.
    at entity avg 1:1
    at (lambda x ...xs . (/ (+ x (sum ...xs)) ...)) 1:11
```

Equipped with the power of variadic functions and recursion, the next natural step is to define μ -recursive functions. Consider the following *basic* μ -recursive functions:

- For all natural numbers i, k where $0 \leq i \leq k$, the projection function (sometimes also called the identity function when $i = 0$) is defined as $P_i^k(x_0, \dots, x_k) = x_i$. The projection function is a built-in operator in KamilaLisp, where `#a` is equivalent to P_a^k .
- For each natural n and k , the constant function C_n^k is defined as $C_n^k(x_0, \dots, x_k) = x_n$. This is easily implemented using the previously discussed projection function as `$(#0 n)`.
- For each natural n , the successor function S_n is defined as $S_n(x) = x + 1$. This is easily implemented using just partial application - `$(+ 1)`.

The μ -recursive composition operator (also called the substitution operator) defined for an m -ary function $h(x_0, \dots, x_m)$ and exactly m n -ary functions $g_0(x_0, \dots, x_n), \dots, g_m(x_0, \dots, x_n)$ as $h \circ (g_0, \dots, g_m) = f$ where $f(x_0, \dots, x_n) = h(g_0(x_0, \dots, x_n), \dots, g_m(x_0, \dots, x_n))$. This definition is essentially equivalent to KamilaLisp *forks*⁴ - `[h g_0 ... g_m] x_0 ... x_k`.

The μ -recursive primitive recursion operator $\rho(g, h) = f$ for k -ary function $g(x_0, \dots, x_k)$, $k + 2$ -ary function $h(y, z, x_0, \dots, x_k)$ and $k + 1$ -ary function f yields the following piecewise function:

$$f(a, x_0, \dots, x_k) = \begin{cases} g(x_0, \dots, x_k) & \text{if } x = 0 \\ h(a - 1, f(a, x_0, \dots, x_k), x_0, \dots, x_k) & \text{if } x \neq 0 \end{cases}$$

The KamilaLisp implementation of this concept is slightly more involved, requiring the `lift` function to apply a function on a existing variadic parameter pack:

```
--> defun mu-prim-rec (g h) (
...   lambda (a ...xs) (
...     if (= a 0)
...       (lift g ...xs)
...       (lift $(h (- a 1) (lift $(mu-prim-rec g h) ...xs)) ...xs)))
```

The μ -recursive *minimization* operator is less demanding to implement. Intuitively, minimisation seeks, beginning the search from 0 and proceeding upwards, the smallest argument that causes the function to return zero; if there is no such argument, or if one encounters an argument for which f is not defined, then the search never terminates, and is not defined for the argument.

⁴A general version of APL 3-trains; <https://aplwiki.com/wiki/Train#3-trains>

```
--> defun mu-min f (lambda ...xs (let-seq (
...   (defun mu-min-iter i (if (lift f ...xs) (&0 (S i)) i))
...   (mu-min-iter 0))))
```

1.7. Iteration

Given a function f , the n -fold application of f to x is usually denoted in mathematics as $f^n(x)$. For instance, $f^0(x) = x$, $f^1(x) = f(x)$, and $f^2(x) = f(f(x))$. More generally, a recursive relation can be defined as $f^k(x) = f(f^{k-1}(x))$ for $k \geq 1$. Given an *iteration* of a function $f^n(x)$, the function f is called the *step function*, n is called the order of iteration and x is called the *starting value*. In KamilaLisp, iteration is introduced using the **while** higher order function which, in the presently most useful form to the mathematical definition, takes three arguments - the starting value, the order of iteration and the step function.

Consider the successor function **defun s x (+ x 1)**. To provide a basic example, addition of two numbers **+ a b** can be implemented as an iteration of order b of the successor function **s** with the initial value a :

```
--> defun add (a b) (while a b $(+ 1))
(λ a b . (while a b $(+ 1)))
--> add 3 4
7
```

Recall the power function and its previously discussed, tail-recursive definition:

```
--> defun power (x n) ((
...   lambda (x n acc) (if (= n 0) acc (&0 x (- n 1) (* x acc)))) x n 1)
(λ x n . ((λ (x n acc) (if (= n 0) acc (&0/syn x (- n 1) (* x acc)))) x n 1))
```

Since tail recursion is essentially equivalent to iteration, the power function can be implemented as an iteration of order $n - 1$ of the multiplication function ***** with the initial value x :

```
--> defun power (x n) (while x (- n 1) $(* x))
(λ x n . (while x (- n 1) $(* x)))
--> power 2 4
16
```

Iteration using **while** and other higher order functions is described later on in the book. As a tool, iteration is more suitable to certain problems over recursive (functional) or array programming approaches⁵.

Of course, iteration with a fixed number of steps is the simplest form of iteration that covers many usages of the **for** loop from other more orthodox languages, such as C. However, iteration is often conditional and depends on some particular predicate. Consider the Collatz conjecture, one of the most famous unsolved problems in mathematics. The conjecture asks whether iterating this function as many times as needed will eventually reach the number 1:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ 3n + 1 & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

Providing a formal proof of the Collatz conjecture is beyond the scope of this book, but KamilaLisp makes it easy to test the conjecture for a finite set of numbers⁶. Start by implementing the function f in code and then use it in a recursive approach and a tail-recursive approach:

⁵For instance, computing the *Convex Hull* using the Graham scan algorithm, which has better asymptotic complexity than the more naive Jarvis scan as easily implemented in array fashion

⁶Even though the Collatz conjecture has been proven using computers for extremely large values of n , this is not enough evidence to claim that the Collatz function is likely true - consider the Polyá conjecture, for which the initial *disproof by counterexample* quoted a number estimated to be around $n = 1.845 \times 10^{361}$

```
--> defun collatz (n) (if (= (mod n 2) 0) (/ n 2) (+ (* 3 n) 1))
(λ n . (if (= (mod n 2) 0) (/ n 2) (+ (* 3 n) 1)))
--> defun collatz-rec (n) (if (= n 1) 1 (collatz-rec (collatz n)))
(λ n . (if (= n 1) 1 (collatz-iter (collatz n))))
--> defun collatz-tail (n) (if (= n 1) 1 (&0 (collatz n)))
(λ n . (if (= n 1) 1 (&0 (collatz n))))
```

After testing the functions for a few small numbers (say, $n < 1000$), it is clear that the conjecture holds for them. To rewrite this tail-recursive function using iteration, it is required to use the alternate definition of **while**. The second argument to **while** can be an integer as demonstrated previously, however, it can also be a function that returns a boolean value. The iteration will continue as long as the function returns true. The new function **collatz-whl** will iterate the **collatz** function until n reaches 1:

```
--> defun collatz-whl (n) (while n $(/= 1) collatz)
(λ n . (while n $(/= 1) collatz))
```

Another possible way to use **while** is to omit the predicate argument and return a two element list of whether to continue iteration (true/false) and the new value, however this approach is not demonstrated due to its lack of utility. It might be of particular interest to determine the number of iterations and the numbers that have been reached in the process. This can be done using the **partial-while** function which, as the name suggests, iterates a function and yields a list of partial results that have been obtained before the final result:

```
--> defun collatz-list (n) (partial-while n $(/= 1) collatz)
(λ n . (partial-while n $(/= 1) collatz))
--> collatz-list 15
(46 23 70 35 106 53 160 80 40 20 10 5 16 8 4 2 1)
```

Another valuable comparison to be made between recursion, tail recursion and iteration is the implementation of fibonacci numbers. Despite being implemented in KamilaLisp already as the function **fib**, consider the following reimplementations:

```
--> defun fibr (n) (if (< n 2) n (+ (fibr (- n 1)) (fibr (- n 2))))
(λ n . (if (< n 2) n (+ (fibr (- n 1)) (fibr (- n 2)))))
--> fibr 5
5
--> fibr 10
55
```

To derive a tail-recursive form of this function, it is necessary to use two accumulators:

```
--> defun fibt (n) ((
...    lambda (n a b) (if (= n 0) a (&0 (- n 1) b (+ a b)))) n 0 1)
(λ n . ((lambda (n a b) (if (= n 0) a (&0/syn (- n 1) b (+ a b)))) n 0 1))
--> fibt 10
55
```

The iterative version of this function trivially follows, however, let's **assume** that it is *not possible* to implement the iterative or tail-recursive version of this function - so the only implementation available a slow function that overflows the stack often. While in practice this is not the case, it is a good exercise to consider different ways in which the naive fibonacci function be improved.

Notice that the fibonacci function is considerably expensive to compute, yet it takes just a number and returns just a number, so it is a good candidate for memoization. The memoization function **memo** takes a function and

returns a memoized version of it. The memoized version of the function will store the results of previous calls and return them if the same arguments are passed to the function again. This is a very useful technique to improve the performance of functions that are expensive to compute and are called with the same arguments repeatedly via a technique akin to dynamic tabulation of a function. The memoized version of the fibonacci function is, to no surprise, created as follows:

```
--> def fibr (memo fibr)
memo$(λ n . (if (< n 2) n (+ (fibr (- n 1)) (fibr (- n 2)))))
--> fibr 10
55
--> fibr 20
6765
--> fibr 30
832040
--> fibr 50
12586269025
-->
```

The memoized version still has some problems (e.g. it will still overflow the stack if the cache is not big enough) but it is a good improvement over the original version. The built-in function `fib` in KamilaLisp uses the Binet formula to compute the fibonacci numbers, which is a much faster method than the naive recursive approach.

One nuisance that needs to be pointed out is that in functions and lambda expressions, names are updated ad-hoc and usually looked up when needed to allow the programmer to utilise some interesting concepts such as mutual recursion. This has a few consequences, for instance:

```
--> defun f (x) (+ x a)
(λ x . (+ x a))
--> def a 5
5
--> f 5
10
```

Another much more serious consequence is that the memoized version of the fibonacci function will not quite work as expected if it is defined under a different name (e.g. as `def fibm (memo fibr)`). The function `fibm` will quickly return the result of `fibm 10` if this value was explicitly asked for before, but not for `fibm 11` - because the underlying function, `fibr`, is still recursing and the memoized version of it is simply not being used.

1.8. Exceptions

Exceptions are the main error handling mechanism in KamilaLisp, unlike in Haskell/Rust (which usually make use of optional types) or OCaml (which supplies exceptions alongside optional types). Exceptions usually propagate the stack and exhibit worse performance characteristics in the exceptional case, but they are much easier to use and are more familiar to programmers coming from other languages. Additionally, they exhibit better performance in the non-exceptional case.

Using the `raise` function, the arithmetic average function could be rewritten to throw an exception if the list is empty:

```
--> defun avg (l) (if (same l 'nil) (raise "empty list") (/ (sum l) (tally l)))
(λ l . (if (same l 'nil) (raise "empty list") (/ (sum l) (tally l))))
--> avg '(1 2 3 4 5)
3
--> avg '()
RaiseError thrown in thread 6b927fb:
```

```

    empty list
at entity avg 1:1
at (λ l . (if (same l 'nil) ...)) 1:11
at entity if 1:16
at if primitive function
at entity raise 1:34
at raise primitive function

```

The exception thrown by `avg` and other kinds of exceptions (e.g. the one thrown by `/ 1 0`) can be caught using the `try-catch` function. For instance, it can be employed to return the length of the error message instead of the average in the exceptional case, since the `error` variable is bound in the catch block supplying the error handler with the exception's message:

```

--> try-catch (avg '()) (tally error)
299

```

Of course, there is an extension to this system that allows the catch handler to distinguish different kinds of failures, as well as an extension to `raise` that lets the programmer supply a custom exception type. An example of this follows:

```

--> defun frobnicate (x) (cond
...   ((< x 10) (raise 'domain_error "Input too large."))
...   ((> x 10) (raise 'domain_error "Input too small."))
...   ((= x 0) (raise 'arithmetic_error "Division by zero."))
...   ((= x 1) (raise 'logic_error "Unimplemented for x=1."))
...   (/ 1 x))

```

The invocation and subsequent handling of the exception is as follows:

```

--> defun stringify-type error-id (cond
...   ((same error-id 'domain_error) "Domain")
...   ((same error-id 'arithmetic_error) "Arithmetic")
...   ((same error-id 'logic_error) "Logic"))
--> try-catch (frobnicate 0) (str:format
...   "Caught a {stringify-type error-id} error of length {tally error}.")
Caught a Domain error of length 357.

```


Chapter 2

Data structures

This chapter will go through the most common data structures and how they are implemented in KamilaLisp. Many data structures in KamilaLisp are defined in terms of each other - for instance, all the functions that generally operate on sets can also be used on lists, since sets constitute a special case of lists. This chapter will also describe elementary array programming in KamilaLisp, which is usually the preferred by many programmers way to solve complex problems quickly.

2.1. Basic list operations

Lists are one of the most important data structures used in functional programming and every major programming language provides means of finite sequence storage. KamilaLisp is no different and a special emphasis is put on list and array processing, the basic building block of dataflow programming.

As mentioned before, every list besides the empty list contains a *head* (the first element, `car`) and a *tail* (the last element, `cdr`). The tail of a list is always another list, even if it is empty. The empty list literal is introduced in the code as `'nil` or `'()`.

Using `car` and `cdr` it is possible to define a basic, non-tail recursive function that yields the length of a list:

```
--> defun length l (if (same l 'nil) 0 (+ 1 (length (cdr l))))  
(λ l . (if (same l 'nil) 0 (+ 1 (length (cdr l)))))
```

A generalised version of this function that handles scalar values, variadic application and strings is available as `tally`¹:

```
--> tally '(1 2 3) '(4 5)  
(3 2)  
--> tally '(1 2 3 4 5)  
5  
--> tally "abcde"  
5  
--> tally 5  
1  
--> tally  
0
```

Individual elements may be prepended to a list using the `cons` function:

```
--> cons 6 'nil  
(6)  
--> cons 5 (cons 6 'nil)
```

¹*tally* - to count or calculate something

```
(5 6)
--> cons 1 '(2 3)
(1 2 3)
```

Hence, one could define a countdown function as follows:

```
--> defun countdown x (if x (cons x (countdown (- x 1))) '(0))
(λ x . (if x (cons x (countdown (- x 1))) '(0)))
--> countdown 5
(5 4 3 2 1 0)
```

Once again, a general result of this function is available in KamilaLisp as the **range** function:

```
--> range 5
(0 1 2 3 4)
--> range 5 10
(5 6 7 8 9)
--> range 10 5
(10 9 8 7 6)
--> range 5 -5
(5 4 3 2 1 0 -1 -2 -3 -4)
```

List concatenation in KamilaLisp is accomplished using the **append** function. The **append** function is of course variadic and accepts an empty parameter list:

```
--> append '(1 2 3) '(4 5)
(1 2 3 4 5)
--> append '(1 2 3) '(4 5) '(6 7)
(1 2 3 4 5 6 7)
--> append
'nil
--> append 'nil 'nil
'nil
--> append "Tomato" "sauce"
Tomatosauce
```

Prefixes and suffixes of lists may be extracted using the **take** and **drop** functions as follows:

```
--> take 3 '(1 2 3 4 5)
(1 2 3)
--> drop 3 '(1 2 3 4 5)
(4 5)
--> take 3 '(1 2 3)
(1 2 3)
--> drop 3 '(1 2 3)
--> take 3 'nil
[]
[]
[]
--> drop 3 'nil
--> take 5 '(1 2 3)
(1 2 3 0 0)
```

The **take** and **drop** functions also accept negative argument, which changes the direction of the operation:

```
--> take -3 "KamilaLisp is Fun"
Fun
```

More generally, *all* prefixes and suffixes of a list are extracted using the **prefixes** and **suffixes** functions:

```
--> prefixes '(1 2 3 4 5)
((1) (1 2) (1 2 3) (1 2 3 4) (1 2 3 4 5))
--> suffixes '(1 2 3 4 5)
((1 2 3 4 5) (2 3 4 5) (3 4 5) (4 5) (5))
--> suffixes "Lisp"
("Lisp" "isp" "sp" "p")
```

Going back to the **take** function, it is easy to notice that when the list is shorter than expected, the resultant list is simply padded with zeroes. This may not be the desired behaviour, thus a variant of **take** called **cycle** is provided. The **cycle** function takes a list and a number and returns a list of the same length as the number, where the elements are taken from the list in a cyclic manner:

```
--> cycle 5 '(1 2 3)
(1 2 3 1 2)
--> cycle 3 '(1 2 3)
(1 2 3)
--> cycle 2 '(1 2 3)
(1 2)
--> cycle 1 '(1 2 3)
(1)
--> cycle 0 '(1 2 3)
--> cycle -1 '(1 2 3)
RuntimeException thrown in thread 1dbd16a6:
      cycle: negative length
      at entity cycle 1:1
      at cycle primitive function
--> cycle 'nil 5
--> cycle "abc" 5
abcab
```

The **replicate** function ubiquitously used in APL and Haskell is also available in KamilaLisp, except its domain is extended to scalar values:

```
--> replicate 3 5
(5 5 5)
--> replicate 5 '(1 2 3)
(1 2 3 1 2 3 1 2 3 1 2 3 1 2 3)
--> replicate 5 "Kamila"
KamilaKamilaKamilaKamilaKamila
--> replicate 0 5
--> replicate 5 'nil
--> replicate '(1 2 3) '(4 5 6)
(4 5 5 6 6 6)
```

KamilaLisp also provides a few functions for altering the *order* of elements in a list. The **reverse** function reverses the order of elements in a list:

```
--> reverse '(1 2 3 4 5)
(5 4 3 2 1)
--> reverse "KamilaLisp"
psiLalimaK
```

The `rotate` function, as the name suggests, takes a list and a number and returns a list where the elements are rotated by the number:

```
--> rotate 2 '(1 2 3 4 5)
(3 4 5 1 2)
--> rotate -2 '(1 2 3 4 5)
(4 5 1 2 3)
--> rotate -4 "KamilaLisp"
LispKamila
--> rotate 1 'nil
--> rotate 0 'nil
```

Finally, the `list:shuffle` function will take a list and return a list with the same elements, but in a random order:

```
--> list:shuffle '(1 2 3 4 5)
(2 5 3 1 4)
--> list:shuffle "KamilaLisp"
LpLsikmaia
```

2.2. Sorting, searching and indexing

KamilaLisp uses a special syntax for indexing into lists. The syntax is as follows:

```
--> def x '(1 5 2 3 4)
(1 5 2 3 4)
--> ?x$[0]
1
```

This syntax is very confusing when demonstrated in isolation. First, indexing returns a value without a function call involved, so it is mandatory to tell the interpreter that the intent is to obtain the value of an object, hence the `?` prefix in the REPL. It is possible to index a list using a list, as follows:

```
--> ?x$[1 3 4]
(5 3 4)
```

The indexing function loops over the list it has received and returns respectively the first, third and fourth items of a list, all packed together into a single array. This is a very powerful feature, as it allows for a very concise syntax for extracting elements from a list, applying permutations and even sorting, as demonstrated later in the book. Of course, indexing can also be done using an expression:

```
--> ?x$[random 5]
3
```

Another closely related functionality related to indexing is searching. The `index-of` takes a value and a list and returns the index of the first occurrence of the value in the list:

```
--> index-of 5 '(9 8 6 5 4 7 2 3)
3
--> index-of 5 '(9 8 6 4 7 2 3)
-1
```

2.3. Rank

KamilaLisp lists can have *rank*, which is a measure of their nesting, usually interpreted in the context of how many dimensions they could have. For example, a doubly nested list can be interpreted as a matrix:

```
--> ?'((1 2) (3 4))
[[1 2]
 [3 4]]
```

Since a matrix usually has two axes, matrices (lists of lists of scalars) have rank 2. A vector is a list of scalar values, so it has rank 1. A scalar value has rank 0. The rank of an object can be computed using the `rank` function:

```
--> rank '((1 2) (3 4))
2
--> rank '(1 2 3 4)
1
--> rank 0
0
```

One interesting thing to consider are *ragged list*. A *ragged list* is defined as a list, whose elements have different ranks. For example, the following list is a ragged list:

```
--> ?'((1 2) (3 4) 5)
```

Since the first and second elements of it have ranks 1 (vectors; lists of scalars), the last element has rank 0 (a scalar). The rank of a ragged list is computed as if the maximum of ranks of a list was considered and the result is negated:

```
--> rank '((1 2) (3 4) 5)
-2
```

2.4. Elementary higher order functions

KamilaLisp provides a wide range of higher order functions for manipulating lists. Many them can be defined using recursion, however almost all of them are guaranteed to terminate, while recursion in general case does not. The use of list processing functions that constitute the core of array programming is highly encouraged over recursion, because they tend to be more concise, easier to understand and less error prone.

The most used function is a built-in function takes a function and a list and applies the function to each element of the list, returning a list of the results. Define a successor function and map it over a list by prepending a single colon before the function name:

```
--> def s $(+ 1)
$['+', '1]
--> :s '(1 2 3 4 5)
(2 3 4 5 6)
```

The `map` function (which is the more familiar name of this construct, predominantly called that in Haskell and OCaml) has a few nuisances that are worth mentioning. First, it is possible to apply it to a non-list argument and an empty list:

```
--> :s 5
(6)
--> :s 'nil
-->
```

The functor returned by `:` has the same arity as the function it is applied to, hence it is possible for it to act as a `zipWith` operation known from e.g. Haskell:

```
--> :+ '(1 2 3) '(4 5 6)
(5 7 9)
--> :+ '(1 2 3) '(4 5 6 7)
(5 7 9)
--> :+ '(1 2 3 4) '(4 5 6)
(5 7 9)
--> :+ '(1 1 1) '(2 2 2) '(3 3 3)
(6 6 6)
```

Of course, it is possible to specify *invariant arguments* to the function - the invariant arguments are constant arguments that are always supplied to the function being mapped, while other arguments over which `map` can iterate are changing:

```
--> :+ 5 '(5) '(1 2 3 4 5)
(11 12 13 14 15)
```

An important observation to be made is that it is possible to apply a function to a list of lists by stacking the comma operator multiple times also abusing the invariant arguments to write a function that forms tuples from the elements in a two-dimensional matrix:

```
--> def mat '((4 3) (3 4))
[[4 3]
 [3 4]]
--> ::cons 5 mat
(((5 4) (5 3)) ((5 3) (5 4)))
```

The colon operator may not be general enough to be suitable for all uses. For example, it may be desirable to create a *pervasive function* - a function which automatically applies itself to all the scalar values in a list. The built-in functions such as `+`, `-` or `ln` are pervasive by default, but for example the `reverse` function is not:

```
--> reverse '("hi" "hello") ("kamila" "lisp")
(("kamila" "lisp") ("hi" "hello"))
```

Since strings are generally considered scalar values by KamilaLisp (however, this is not the case in other array programming languages such as APL), applying it *on depth zero* yields the following results:

```
--> reverse%[0] '("KamilaLisp" "is") "fun!"
(("psiLalimaK" "si") "!nuf")
```

The function `reverse` was ran on every object of rank zero of the list. If it is desirable reverse vectors (lists of scalars), the `reverse` function should be applied *on depth one*:

```
--> reverse%[1] '((1 2) 3 4)
((2 1) 3 4)
```

To give another example, to reverse the rows of a list of matrices, the function should be applied *on depth two*:

```
--> reverse%[2] '(((1 2) (3 4)) ((5 6) (7 8)))
((2 1) (4 3))
```

Of course, since the depth operator is a *generalisation* of the colon operator (mapping), it is possible to use it to map a function over a list of lists. In this case, the depth specifier must be negative:

```
--> writeln%[-1] '("Hello" "world!")
```

The smaller negative number, the more times the map function is applied:

```
--> cons%[-2] mat 5
(((4 5) (3 5)) ((3 5) (4 5)))
--> ::cons mat 5
(((4 5) (3 5)) ((3 5) (4 5)))
```

An important thing to note is that the depth operator subtly differs from the colon operator in the variadic case. The colon operator will determine the shape of the result ad-hoc, regardless of argument order:

```
--> ::cons mat 5
(((4 5) (3 5)) ((3 5) (4 5)))
--> ::cons 5 mat
(((5 4) (5 3)) ((5 3) (5 4)))
```

The depth operator, however, will always infer the shape from its first argument, potentially leading to unexpected results:

```
--> cons%[-2] mat 5
(((4 5) (3 5)) ((3 5) (4 5)))
--> cons%[-2] 5 mat
[[5 4]]
```

This behaviour significantly differs from the behaviour of the depth operator in other languages, such as APL², where the depth operator is restricted to only two arguments, making it feasible to try dynamically determining the shape of the result. In KamilaLisp the shape is inferred from the first argument, since the operator's complexity would grow by a large magnitude as a result of it being a generalisation to an arbitrary amount of arguments. Additionally, the complex inferring rule in APL-like languages is not very useful in practice and leads to some design shortcomings.

To explore this topic further, it is necessary to demonstrate that the KamilaLisp depth operator accepts multiple depth values. For example, the following program will apply the function to objects of rank one extracted from the first array, and the objects obtained by descending once into the second array:

```
--> defun f (x y) (str:format "{?x}, {?y}")
(λ x y . (str:format "{?x}, {?y}"))
--> f%[1 -1] '((1 2) (3 4)) '(6 5 (4 3) 2 1)
("(1 2), 6" "(3 4), 5")
```

²[https://aplwiki.com/wiki/Depth_\(operator\)](https://aplwiki.com/wiki/Depth_(operator))

Notice that the depth operator makes an attempt to salvage the situation arising due to the fact that the lists are of different sizes by trimming the longer list to the size of the shorter one (which is not done by APL). To make a more fair comparison with APL, consider the following program instead:

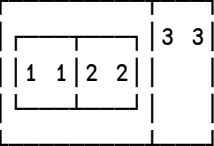
```
--> f%[-2 0] '((1 2) 3) '((1 2) 3)
(("1, 1" "2, 2") "3, 3")
```

Since the depth operator simply extracts the objects of the specified rank from the arguments, it does not pay attention to the shape of other arguments, so scalars for the second argument to f can be extracted also in this case:

```
--> f%[-2 0] '((1 2) 3) '(1 2 3)
(("1, 1" "2, 2") "3, 3")
```

Since APL determines the shape in a more "clever" (also way slower and more convoluted) way, this behaviour can not be achieved:

```
((1 2) 3) ({αω} ⍳ -2 0) (1 2) 3
```



```
((1 2) 3) ({αω} ⍳ -2 0) 1 2 3
LENGTH ERROR
((1 2)3)({α ω} ⍳ -2 0)1 2 3
      ^
```

Finally, the depths list can be defined as a result of an expression:

```
--> f%[[tie + -] 1] '((1 2) (3 4)) '(6 5 (4 3) 2 1)
(("(1 2), 6" "(3 4), 5")
```

2.5. Folding, scanning and filtering

2.6. Searching and partitioning

2.7. Sorting and permutations

2.8. Sets

2.9. Hashmaps

2.10. Using glyphs