Sensor Data Fusion

Exercise 2

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DATA
FUSION Lab

Today

- Lecture review
- Questions
- Homework - Distance measurements to walls
- Problem - Normal equation
- Example - Camera
- Homework - GPS

Questions

What are the assumptions for least squares?

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{e}$$

$$\mathbf{H} \in \mathbb{R}^{m \times n}, \quad m \geq n, \quad rank(\mathbf{H}) = n$$

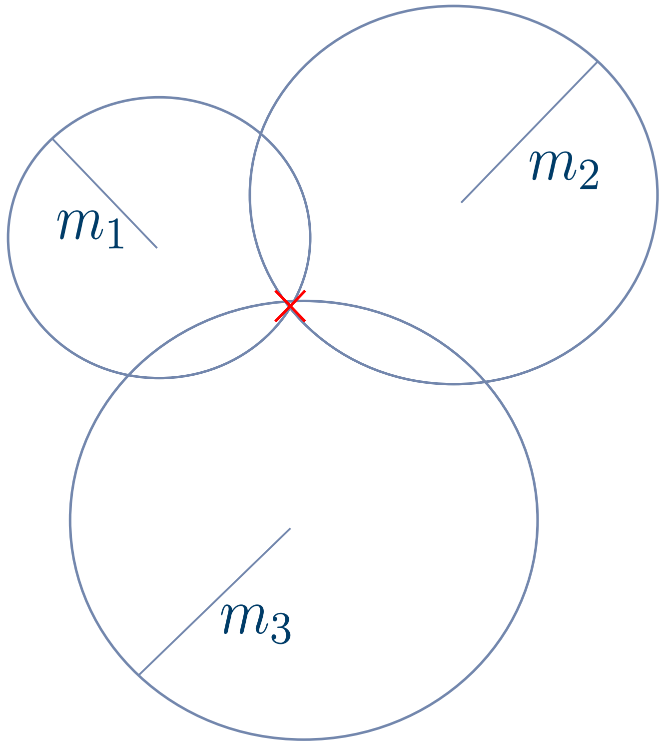
Questions

What is the objective/cost function of least squares?

$$
\begin{aligned}
\mathbf{y} &= \mathbf{H}\mathbf{x} + \mathbf{e} \\
\mathbf{x}^{LS} &= \underset{\mathbf{x}}{argmin}\, \underbrace{||\mathbf{y} - \mathbf{H}\mathbf{x}||^2_{\mathbf{W}}}_{G(\mathbf{x})} \\
G(\mathbf{x}) &= (\mathbf{y} - \mathbf{H}\mathbf{x})^{\mathrm{T}}\mathbf{W}(\mathbf{y} - \mathbf{H}\mathbf{x}) \\
G'(\mathbf{x}^{LS}) &\stackrel{!}{=} 0 \\
\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{y} &= \mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H}\mathbf{x}^{LS} \\
\mathbf{x}^{LS} &= (\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{y}
\end{aligned}
$$

Questions

How does trilateration work?



Distance measurements to arbitrary walls

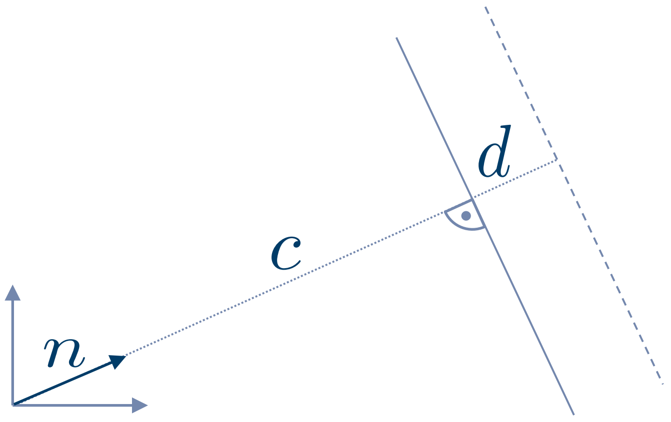
A wall in 2D space can be described using a normal vector $n = \begin{bmatrix} n_1 & n_2 \end{bmatrix}^{\mathrm{T}}$ (with $||n||_2 = 1$) and a scalar c

$$n_1 x_1 + n_2 x_2 = c$$

which holds true for all $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}}$ on the wall. Note that c in this case describes the shortest distance from the origin to the wall. So any shift d of c would describe all points within a distance d of the wall (see Figure below). So the formula

$$n_1 x_1 + n_2 x_2 = c + d$$

holds true for all x with distance d to the wall in the direction of n.



Distance measurements to arbitrary walls

If the given vector $\hat{n}$ is not a normal vector, so we have

$$\hat{n}_1 x_1 + \hat{n}_2 x_2 = \hat{c}$$

with $||\hat{n}||_2 \neq 1$, we need to normalize it first. This would result in the distance formula being

$$\frac{\hat{n}_1 x_1 + \hat{n}_2 x_2 - \hat{c}}{||\hat{n}||_2} = d$$

Homework 1

The objective is to estimate the two-dimensional object location $\boldsymbol{x} = \left[x_1, x_2\right]^T \in \mathbb{R}^2$ using (noise-corrupted) distance measurements $d^i \in \mathbb{R}$ to N walls. The location of the i-th wall is given in normal form

$$n_1^i \cdot x_1^w + n_2^i \cdot x_2^w = c^i \quad .$$

Assume n^i points to the half space where the object is located. Given are four walls with corresponding measurements:

i	n_1^i	n_2^i	c^i	distance d^i
1	-5	-1	-45	4.7
2	-1	-8	-70	5.2
3	-1	9	5	5.5
4	8	-1	7	4.5

a) Please write a function which visualizes walls and measurements using different colors.
b) Formulate a linear measurement equation $y^i = \mathbf{H}^i x + e^i$, which relates the measurement to the i-th wall with $\boldsymbol{x}$ and the error e^i. In the same manner, formulate a measurement equation that relates N walls, i.e., the 1-st to N-th walls, with $\boldsymbol{x}$ and $\boldsymbol{e}$. Note that the n^i vectors are not normalized.

Homework 1

c) Could you calculate the unique location for the first case in 1b), if not, please explain. If an unique location could be obtained, which requirements are needed?

d) Based on the measurement equation formulated in 1b), write a function which calculates the least squares solutions based on the measurements.
Using the function you implemented, calculate the least squares solutions $\hat{\boldsymbol{x}}_{12}$, $\hat{\boldsymbol{x}}_{34}$ and $\hat{\boldsymbol{x}}_{1234}$ based on the measurements(y_1, y_2), (y_3, y_4) as well as (y_1, y_2, y_3, y_4).

e) Given the true location x, implement a function which calculates the estimation error e using Euclidean norm, e.g.,
$e_{12} = ||\hat{\boldsymbol{x}}_{12} - \boldsymbol{x}||$, $e_{34} = ||\hat{\boldsymbol{x}}_{34} - \boldsymbol{x}||$, $e_{1234} = ||\hat{\boldsymbol{x}}_{1234} - \boldsymbol{x}||$
Assuming the true location $\boldsymbol{x} = [5,\ 5]^T$, calculate e_{12}, e_{34} and e_{1234}. What can you observe?

Homework 1: Solution

The objective is to estimate the two-dimensional object location $\boldsymbol{x} = \left[x_1, x_2\right]^T \in \mathbb{R}^2$ using (noise-corrupted) distance measurements $d^i \in \mathbb{R}$ to N walls. The location of the i-th wall is given in normal form

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$$d^i = \frac{n_1^i x_1 + n_2^i x_2 - c^i}{\sqrt{(n_1^i)^2 + (n_2^i)^2}} + e^i$$

The distance d^i to wall i

$$d^i\sqrt{(n_1^i)^2 + (n_2^i)^2} + c^i = n_1^i x_1 + n_2^i x_2 + e^i$$

$$\underbrace{d^i\sqrt{(n_1^i)^2 + (n_2^i)^2} + c^i}_{y^i} = \underbrace{\begin{bmatrix} n_1 & n_2 \end{bmatrix}}_{\mathbf{H}^i} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + e^i$$

Distances to N walls

$$\underbrace{\begin{bmatrix} d^1\sqrt{(n_1^1)^2 + (n_2^1)^2} + c^1 \\ \vdots \\ d^i\sqrt{(n_1^i)^2 + (n_2^i)^2} + c^i \\ \vdots \\ d^N\sqrt{(n_1^N)^2 + (n_2^N)^2} + c^N \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} n_1^1 & n_2^1 \\ \vdots & \vdots \\ n_1^i & n_2^i \\ \vdots & \vdots \\ n_1^N & n_2^N \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} e^1 \\ \vdots \\ e^i \\ \vdots \\ e^N \end{bmatrix}}_{e}$$

c) Could you calculate the unique location for the first case in 1b), if not, please explain. If an unique location could be obtained, which requirements are needed?

It is not possible to get an exact solution because we have 2 unknowns and only 1 formula. The exact solution can be calculated if $rank(\mathbf{H}) = 2$.

d) Based on the measurement equation formulated in 1b), write a function which calculates the least squares solutions based on the measurements. Using the function you implemented, calculate the least squares solutions $\hat{\boldsymbol{x}}_{12}$, $\hat{\boldsymbol{x}}_{34}$ and $\hat{\boldsymbol{x}}_{1234}$ based on the measurements(y_1, y_2), (y_3, y_4) as well as (y_1, y_2, y_3, y_4).

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$e_{12} = ||\hat{\boldsymbol{x}}_{12} - \boldsymbol{x}||$, $e_{34} = ||\hat{\boldsymbol{x}}_{34} - \boldsymbol{x}||$, $e_{1234} = ||\hat{\boldsymbol{x}}_{1234} - \boldsymbol{x}||$
Assuming the true location $\boldsymbol{x} = [5,\ 5]^T$, calculate e_{12}, e_{34} and e_{1234}. What can you observe?

Problem – Normal Equation

Consider a linear measurement equation

$$y = \mathbf{H}x + e$$

with measurement vector $y \in \mathcal{R}^3$, state vector $x \in \mathcal{R}^2$, error vector $e \in \mathcal{R}^3$ and measurement matrix $\mathbf{H} \in \mathcal{R}^{3\times 2}$, where $\mathbf{H} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $y = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

- Visualize the column space spanned by $\mathbf{H}$ using a three-dimensional plot.
- Illustrate the least squares estimate and the error vector in the three-dimensional plot. What is the meaning of the normal equation?
- Now assume $\mathbf{H} = \begin{pmatrix} 2 & 3 \\ 2 & 3 \\ 0 & 0 \end{pmatrix}$. Can you still calculate the LS estimate?

Solution – Normal Equation

$$\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$$

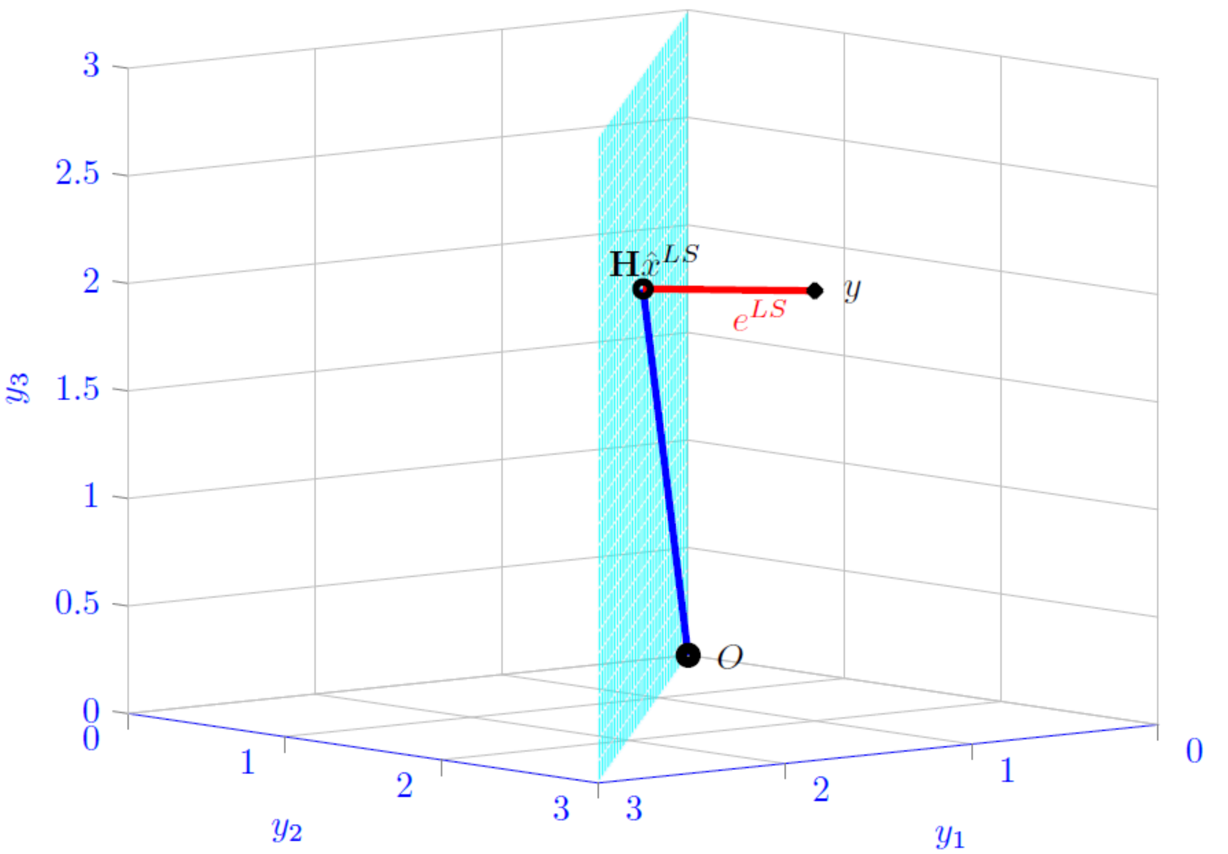
$$\begin{aligned} \hat{x}^{\text{LS}} &= (\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}} y \\ &= \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 0.75 \\ 1 \end{bmatrix} \end{aligned}$$

$$y^{\text{proj}} = \mathbf{H} x^{\text{LS}} = \begin{bmatrix} 1.5 \\ 1.5 \\ 2 \end{bmatrix}$$

$$e^{\text{LS}} = y - \mathbf{H} x^{\text{LS}} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix}$$

$$c_1^{\mathrm{T}} e^{\text{LS}} = 0$$

$$c_2^{\mathrm{T}} e^{\text{LS}} = 0$$



Solution – Normal Equation

$$y^{\text{proj}} = x_1 \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

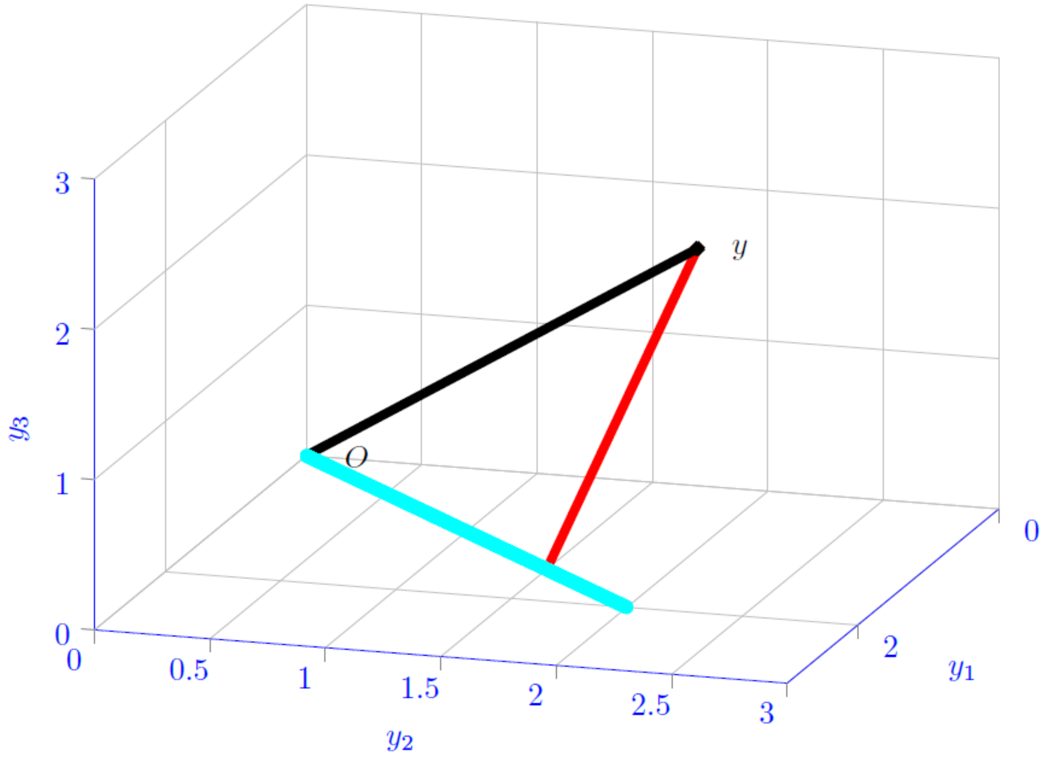
No unique solution, but set one element of x to 0 and calculate the other using only part of $\mathbf{H}$.

$$y = \hat{\mathbf{H}} x_1 + e_1$$

$$\hat{\mathbf{H}} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$x_1^{\text{LS}} = (\hat{\mathbf{H}}^{\mathrm{T}} \hat{\mathbf{H}})^{-1} \hat{\mathbf{H}}^{\mathrm{T}} y = 0.75$$

$$x_2 = 0$$



Example - Camera

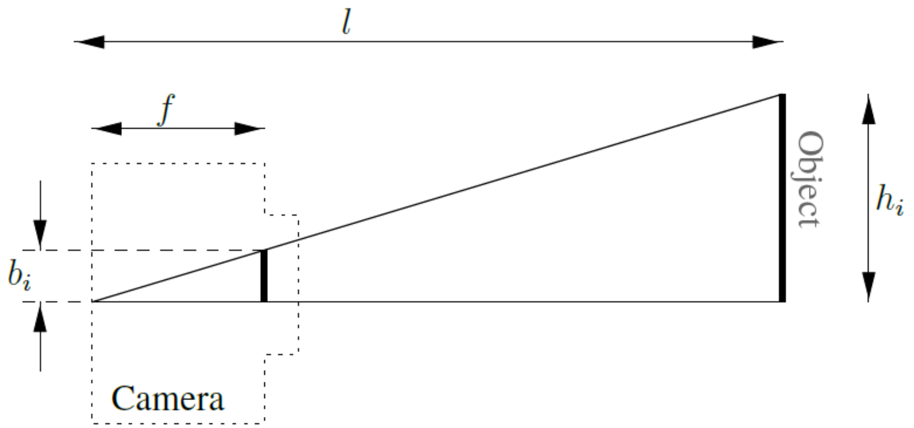
The objective is to determine the distance l from a camera to a planar object. For this purpose, features with height h_i are mapped according to

$$b_i = \frac{f}{l} h_i$$

to the image b_i, where $f = 4mm$ is the focal length. The pixel size is $2\mu m$. The following heights are measured

Feature height (cm)	1	5	15	25	50
Number of pixels	4	12	31	49	98

- Compute the distance l with an unweighted Least Squares (LS) estimator.



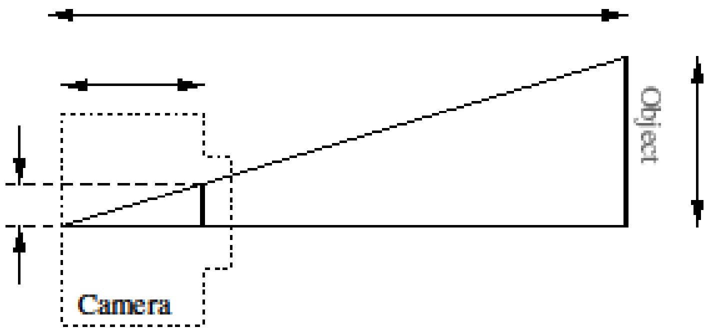
Example - Camera

Given:

$$h = \begin{bmatrix} 1 \\ 5 \\ 15 \\ 25 \\ 50 \end{bmatrix} \times 1 \times 10^{-2} m, \quad b = \begin{bmatrix} 4 \\ 12 \\ 31 \\ 49 \\ 98 \end{bmatrix} \times 2 \times 10^{-6} m, \text{ and } f = 4 \times 10^{-3} m$$

$$b_i = \frac{f}{l} h_i$$

Want: l



$$b_i = \frac{f}{l} h_i$$

$$h_i = \frac{b_i}{f} l$$

$$\underbrace{h}_{y} = \underbrace{\frac{1}{f} b}_{\mathbf{H}} \underbrace{l}_{x} + e$$

Example - Camera

$$x^{\mathsf{LS}} = (\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}y$$

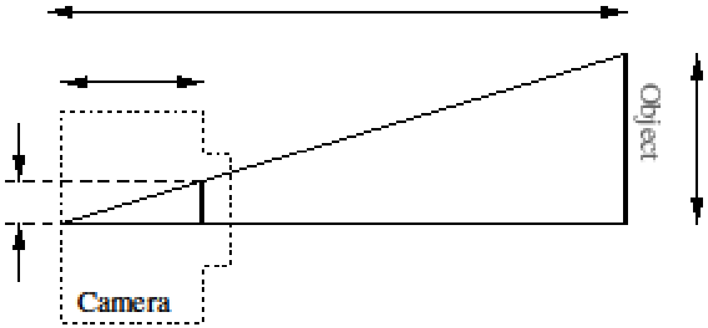
$$x^{\mathsf{LS}} = (\left(\frac{2\times 10^{-6}}{4\times 10^{-3}}\right)^2 \begin{bmatrix}4 & 12 & 31 & 49 & 98\end{bmatrix}\begin{bmatrix}4\\12\\31\\49\\98\end{bmatrix})^{-1}\frac{2\times 10^{-8}}{4\times 10^{-3}}\begin{bmatrix}4 & 12 & 31 & 49 & 98\end{bmatrix}\begin{bmatrix}1\\5\\15\\25\\50\end{bmatrix}$$

$$x^{\mathsf{LS}} = (\left(\frac{10^{-3}}{2}\right)^2(16+144+961+2401+9604))^{-1}\frac{10^{-5}}{2}(4+60+465+1225+4900)$$

$$x^{\mathsf{LS}} = \left(\frac{13126}{4000000}\right)^{-1}\frac{6654}{200000}$$

$$x^{\mathsf{LS}} = \frac{0.03327}{0.0032815}$$

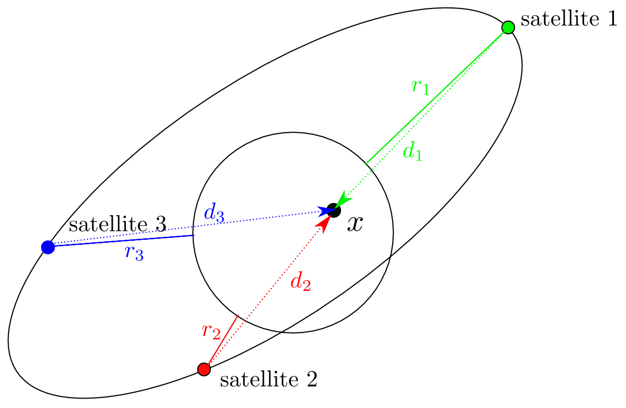
$$x^{\mathsf{LS}} \approx 10$$

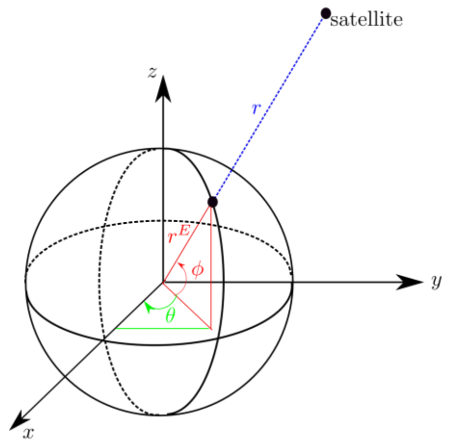


GPS consists of 24 satellites in orbit (20200km above mean sea level). Each satellite broadcasts its location (in spherical coordinates $[\theta, \phi, r]^T$) plus the emission time (see figures below).
A GPS device receives at time $t = 0$s the following four satellite signals:

$$p_1 = [0°, 40°, 20200\text{km}]^T, \qquad t_1 = -67.603\text{ms},$$
$$p_2 = [10°, 20°, 20200\text{km}]^\text{T}, \qquad t_2 = -70.102\text{ms} ,$$
$$p_3 = [10°, -10°, 20200\text{km}]^\text{T}, \qquad t_3 = -78.690\text{ms} ,$$
$$p_4 = [-10°, -20°, 20200\text{km}]^\text{T}, \qquad t_4 = -82.942\text{ms}.$$

Assume that the speed-of-light is $c = 3 \cdot 10^8 \text{m s}^{-1}$ and the Earth is an ideal sphere with radius $r^\text{E} = 6370$km.





Homework 2

a) Write a function which converts sphere coordinates (θ, ϕ, r) into Cartesian coordinates (x, y, z)
Hint:

The x coordinates could be obtained according to $x = (r^{\mathsf{E}} + r)\cos\phi\cos\theta$;

You could use `math.radians(...)` to convert degrees into radians.

Using the function you implemented, calculate the Cartesian coordinates of these four satellites.

b) Please calculate the distance between satellites and GPS device.
Using the distance measurements, form the measurement equation like we did in the lecture.

c) Reformulate the non-linear measurement equation in 2b) into a linear measurement equation and calculate the least squares estimate.

Homework 2

d) Write a function which converts the Cartesian coordinates into sphere coordinates. Use the function you implemented, calculate the longitude and latitude of the GPS device, and find out where it is using using *GPS Visualizer*:

http://www.gpsvisualizer.com/map?form=google

For this purpose, use the format

```
type,latitude,longitude
W,PHI,THETA
```

where `PHI` and `THETA` are the calculated coordinates in degrees.