Sensor Data Fusion

Exercise 1

Prof. Dr.-Ing. Marcus Baum
M.Sc. Kolja Thormann

www.fusion.informatik.uni-goettingen.de

GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

DATA
FUSION Lab

Today

- Lecture review
- Questions
- Math review
- Problem - Weight matrix visualization
- Homework

Questions

Why do we need multiple sensors? Name and explain two sensor fusion concepts.

Sensors have advantages and disadvantages. Using mutliple different sensors can balance these out.
Two concepts are competitive fusion (multiple sensors measure the same space to increase accuracy) and complementary fusion (mutliple sensors measure different spaces to increase completeness).

Questions

What is dead reckoning and what is its main problem?

Dead reckoning means determining your position based on previous knowledge and your motions. As this prediction can hold errors, these errors accumulate over time. Therefore, we need external sensors.

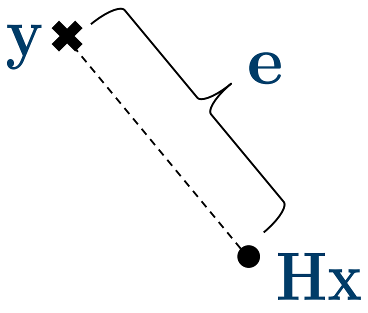
Questions

What is a measurement equation? Could you write down the linear measurement equation in a general form and explain the components in it?

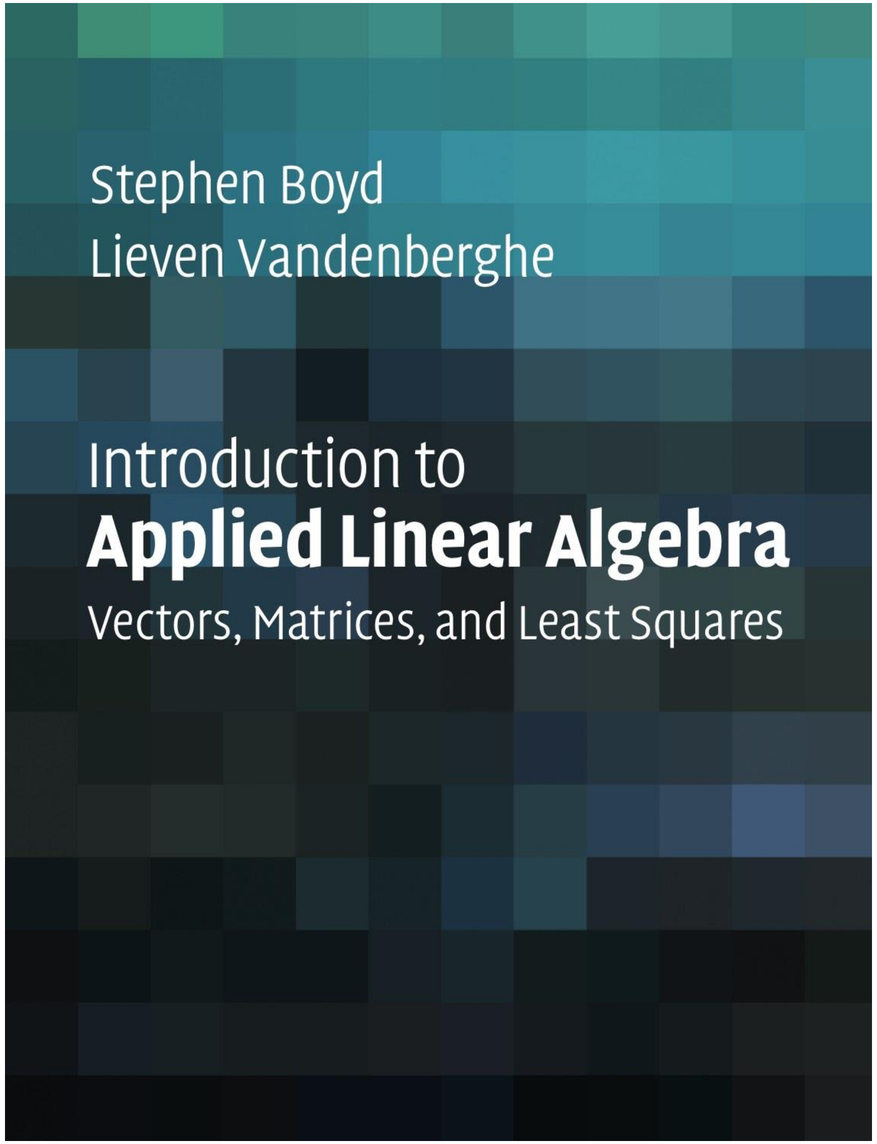
The measurement equation projects the state onto the measurement space and describes the measurement as the sum of the projected state and an error.

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{e}$$

with state $\mathbf{x} \in \mathcal{R}^n$, measurement $\mathbf{y} \in \mathcal{R}^m$, measurement matrix $\mathbf{H} \in \mathcal{R}^{m \times n}$, and error $\mathbf{e} \in \mathcal{R}^m$.



Additional Literature



Introduction to Applied Linear Algebra – Vectors, Matrices, and Least Squares

Stephen Boyd and Lieven Vandenberghe

Cambridge University Press

Online available:
https://web.stanford.edu/~boyd/vmls/

Additional Literature

Petersen, Kaare Brandt, and Michael Syskind Pedersen. "**The matrix cookbook**." *Technical University of Denmark* 7.15 (2008): 510.

Review: Positive Definite Matrices

- A symmetric matrix $\mathbf{W} \in \mathbb{R}^{n \times n}$ is positive definite if

$$z^{\mathrm{T}} \mathbf{W} z > 0$$

  for all non-zero $z \in \mathbb{R}^n$.

- Equivalent condition: All its eigenvalues are positive
- A positive semi-definite matrix $\mathbf{W}$ can be written as

$$\mathbf{W} = \mathbf{C}\mathbf{D}\mathbf{C}^{\mathrm{T}} \quad ,$$

  where $\mathbf{C} \in \mathbb{R}^{n \times n}$ is a rotation matrix (orthogonal matrix), i.e., $\mathbf{C}^{\mathrm{T}}\mathbf{C} = \mathbf{C}\mathbf{C}^{\mathrm{T}} = \mathbf{I}$ and $\mathbf{D}$ is a diagonal matrix with the eigenvalues

- Positive definite matrices are invertible

Review: Full Rank Matrices

- For a matrix $\mathbf{H} \in \mathbb{R}^{m\times n}$
  - rank$(\mathbf{H})$ is defined as the max. number of independent columns (or eq. rows)
  - rank$(\mathbf{H}) \leq \min\{m, n\}$
- $\mathbf{H}$ has full (column) rank if rank$(\mathbf{H}) = n$ and $n \leq m$
  - Kernel/Nullspace $\mathcal{N}(\mathbf{H}) = \{\mathbf{0}\}$, i.e., $\mathbf{H}x = \mathbf{0}$ iff $x = \mathbf{0}$
  - $\mathbf{H}^{\mathrm{T}}\mathbf{H} \in \mathbb{R}^{n\times n}$ is positive definite, Proof: $x^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{H}x = (\mathbf{H}x)^{\mathrm{T}}\mathbf{H}x = ||\mathbf{H}x||^2 > 0$
  - $\mathbf{H}^{\mathrm{T}}\mathbf{H}$ is invertible (follows from positive definite)

Review: Matrix Calculus – Denominator Layout

- The derivative of a scalar function w.r.t. a vector is a column vector
- The derivative of a vector function

$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$ w.r.t. a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is written as

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}.$$

(The transpose of the Jacobian)

Jacobian and Hessian

Assume a function $f(x) \in \mathbb{R}^m$ depending on a vector $x \in \mathbb{R}^n$ (f is differentiable at x). The Jacobian $\mathbf{J}$ is defined as the $m \times n$ matrix of all first-order partial derivatives of f by x:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix} = \left(\frac{\partial f}{\partial x}\right)^{\mathrm{T}}$$

Example: Given the function $f(\begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}}) = \begin{bmatrix} x_1 + 2x_2 & x_1^2 + x_2 \end{bmatrix}^{\mathrm{T}}$, the Jacobian would be

$$\mathbf{J} = \begin{bmatrix} 1 & 2 \\ 2x_1 & 1 \end{bmatrix}$$

Jacobian and Hessian

Assume a function $f : \mathbb{R}^n \to \mathbb{R}$. The Hessian $\mathbf{H}$ is defined as the $n \times n$ square matrix of all second-order partial derivatives of f by x:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_n} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \frac{\partial^2 f}{\partial x_n x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Example: Given a function $f(\begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}}) = 3x_1^2 + x_1 x_2 + x_2^3$, the Hessian would be

$$\mathbf{H} = \begin{bmatrix} 6 & 1 \\ 1 & 6x_2 \end{bmatrix}$$

Review: Matrix Calculus – Some Identities

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

For $c, x \in \mathbb{R}^n$ and symmetric $\mathbf{A} \in \mathbb{R}^{n\times n}$, we have

1. $\frac{\partial}{\partial x} c^{\mathrm{T}} x = \frac{\partial}{\partial x} x^{\mathrm{T}} c = c$
   (scalar-by-vector = column vector)
2. $\frac{\partial}{\partial x} x^{\mathrm{T}} \mathbf{A} x = 2\mathbf{A}x$
   (scalar-by-vector = column vector)
3. $\frac{\partial}{\partial x} \mathbf{A} x = \frac{\partial}{\partial x} x^{\mathrm{T}} \mathbf{A} = \mathbf{A}$
   (vector-by-vector = matrix)

Examples

1. $\frac{\partial}{\partial x} c^{\mathrm{T}} x = \frac{\partial}{\partial x} x^{\mathrm{T}} c = c$
   (scalar-by-vector = column vector)

Set $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}}$, $c = \begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}$, and $f(x) = x^{\mathrm{T}} c$. We have

$$f(x) = x_1 + 2x_2$$
$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}} = c$$

2. $\frac{\partial}{\partial x} x^{\mathrm{T}} \mathbf{A} x = 2\mathbf{A} x$
   (scalar-by-vector = column vector)

Set $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $g(x) = x^{\mathrm{T}} \mathbf{A} x$. We have

$$g(x) = x_1(x_1 + 2x_2) + x_2(2x_1 + 6x_2) = x_1^2 + 4x_1x_2 + 6x_2^2$$
$$\frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 2x_1 + 4x_2 & 4x_1 + 6x_2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix} x = 2\mathbf{A} x$$

Trace

- Given is a positive definite matrix $\mathbf{C} \in \mathbb{R}^{n\times n}$.
- The trace of $\mathbf{C}$ is the sum of its main diagonal, but also the sum of its eigenvalues.

$$tr\{C\} = tr\{RDR^{\mathrm{T}}\} = tr\{(RD)R^{\mathrm{T}}\} = tr\{R^{\mathrm{T}}RD\} = tr\{ID\} = tr\{D\}$$

- Per definition, all eigenvalues of $\mathbf{C}$ are non-negative. So we have $tr\{C\} > 0$.
- Given a full rank matrix $\mathbf{R} \in \mathbb{R}^{m\times n}$ with $rank\{\mathbf{R}\} = m$, $\mathbf{RCR}^{\mathrm{T}}$ is still positive definite. So we still have $tr\{\mathbf{RCR}^{\mathrm{T}}\} > 0$.

Assume non-zero $z \in \mathbb{R}^m$.

$$z^{\mathrm{T}}\mathbf{RCR}^{\mathrm{T}}z = (\underbrace{\mathbf{R}^{\mathrm{T}}z}_{x})^{\mathrm{T}}\mathbf{C}\,\underbrace{\mathbf{R}^{\mathrm{T}}z}_{x} > 0 \text{ for all non-zeros } x \in \mathbb{R}^n$$

Full column rank $\mathbf{R}^{\mathrm{T}}$ gives us $x = \mathbf{R}^{\mathrm{T}}z = 0 \iff z = 0$
As we only regard non-zero z, x is also non-zero.

Review: Weighted Norm

- Semi-positive definite weighting matrix $\mathbf{W}$
- Weighted Norm:

$$||e||^2_{\mathbf{W}} = e^{\mathrm{T}}\mathbf{W}e$$

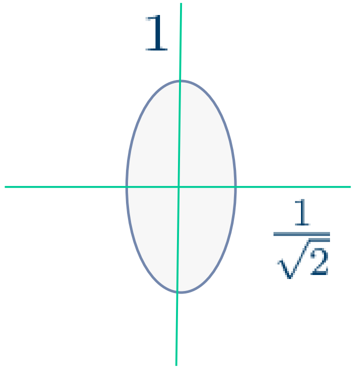
- Example: For $m = 2$,
  $\mathbf{W} = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}$ and $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$, yields

$$||e||^2_{\mathbf{W}} = w_1 e_1^2 + w_2 e_2^2$$

- Higher weight means more effect on estimate

Visualization:

$w_1 = 2,\ w_2 = 1$



$||e||^2_{\mathbf{W}} = 1$

Problem - Weight Matrix Visualization

Assume semi-positive definite weight matrix with eigenvalue decomposition $\mathbf{W} = \mathbf{R}\mathbf{D}\mathbf{R}^{\mathrm{T}}$. We get a rotation matrix with $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ and a diagonal matrix $\mathbf{D} = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}$.

- Now visualize the weight matrix as before using $w_1 = 4, w_2 = 0.25$ and $\mathbf{R} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$ with $\alpha = \frac{\pi}{4}$
- Next, take the weight matrix $\mathbf{W}_2 = \begin{bmatrix} 2.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix}$. Apply eigenvalue decomposition to obtain the elements of $\mathbf{D}$ and calculate $\mathbf{R}$ depending on α.
  Reminder:
  Eigenvalue λ with $\det(\mathbf{W} - \lambda\mathbf{I}_2) = 0$; $\sin(\alpha) = \cos(\alpha) \implies \alpha = \frac{\pi}{4}$
- Use $\mathbf{D}$ and $\mathbf{R}$ to visualize $\mathbf{W}_2$ like before.

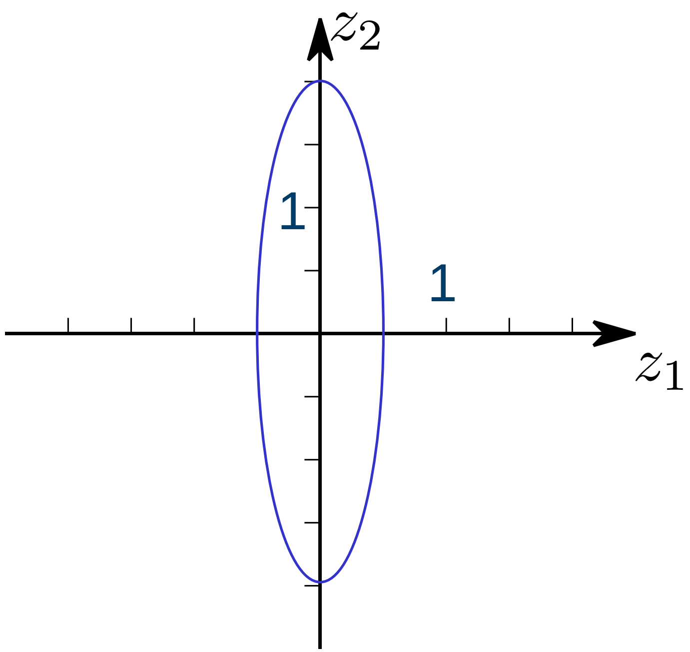
Solution 1: Weight Matrix Visualization

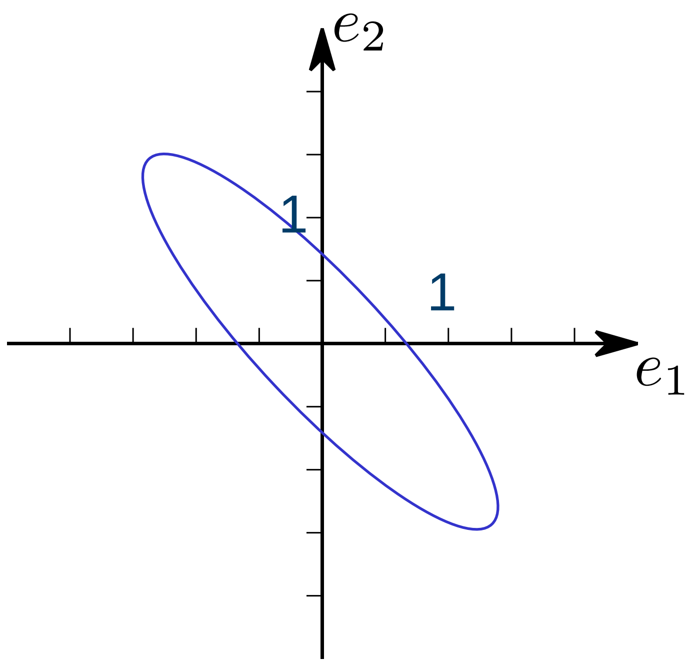
$w_1 = 4$, $w_2 = 0.25$ and $\mathbf{R} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$ with $\alpha = \frac{\pi}{4}$

$$||e||^2_{\mathbf{W}} = e^{\mathrm{T}}\mathbf{W}e = e^{\mathrm{T}}\mathbf{R}\mathbf{D}\mathbf{R}^{\mathrm{T}}e = (\mathbf{R}^{\mathrm{T}}e)^{\mathrm{T}}\mathbf{D}\mathbf{R}^{\mathrm{T}}e$$

$$z = \mathbf{R}^{\mathrm{T}}e$$

$$e = \mathbf{R}z$$





Solution 1: Weight Matrix Visualization

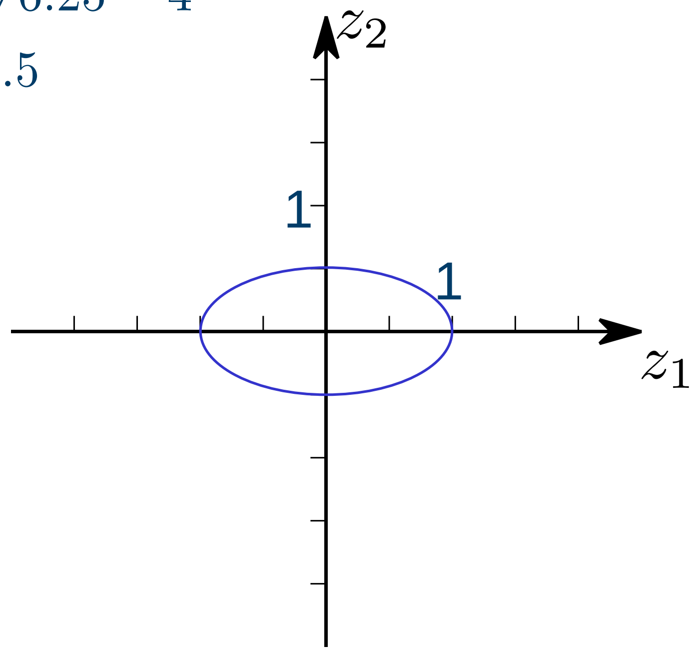
$$\mathbf{W}_2 = \begin{bmatrix} 2.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix}$$

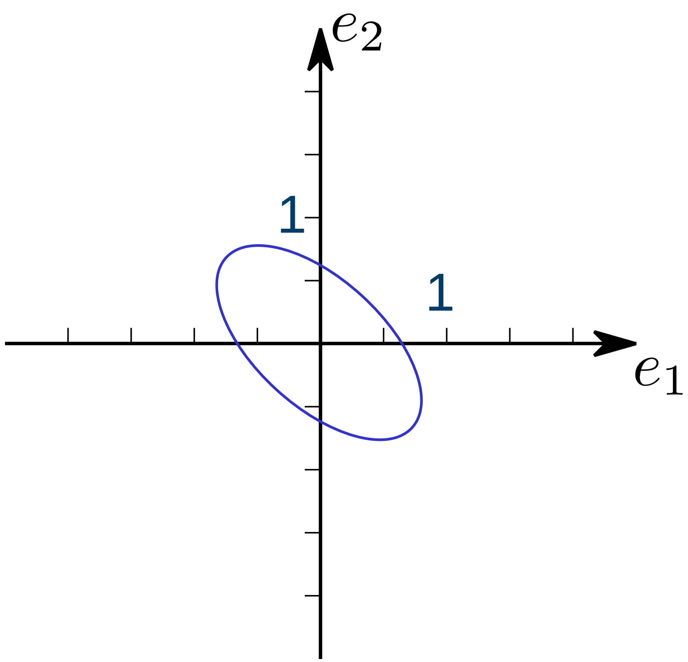
$$det(\mathbf{W}_2 - \lambda\mathbf{I}) = 0$$
$$(2.5 - \lambda)^2 - 1.5^2 = 0$$
$$\lambda^2 - 5\lambda + 6.25 - 2.25 = 0$$
$$\lambda^2 - 5\lambda + 4 = 0$$
$$\lambda_{1/2} = 2.5 \pm \sqrt{6.25 - 4}$$
$$\lambda_{1/2} = 2.5 \pm 1.5$$
$$\lambda_1 = 1$$
$$\lambda_2 = 4$$

$$\mathbf{R} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$\mathbf{W}_2 \begin{bmatrix} a & c \end{bmatrix}^{\mathrm{T}} = \lambda_1 \begin{bmatrix} a & c \end{bmatrix}^{\mathrm{T}}$$

$$2.5a + 1.5c = a$$
$$1.5a = -1.5c$$
$$a = -c$$
$$\cos(\alpha) = -\sin(\alpha)$$
$$\cos(-\alpha) = \sin(-\alpha)$$
$$\alpha = -\frac{\pi}{4}$$

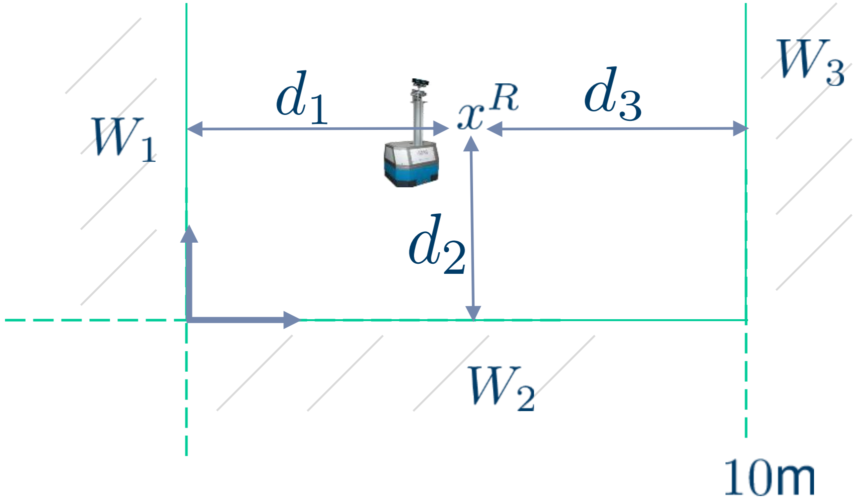




Example – Distance measurements to walls

We measure $d_1 = 4$m, $d_2 = 3$m, and $d_1 = 7$m. Calculate the Least Squares solution.

$$\underbrace{\begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}}_{=:y} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{=:\mathbf{H}} x^R + \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}}_{=:e}$$



$$\begin{aligned} x^{\text{LS}} &= (\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}} y \\ &= \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3.5 \\ 3 \end{bmatrix} \end{aligned}$$

Distance measurements to arbitrary walls

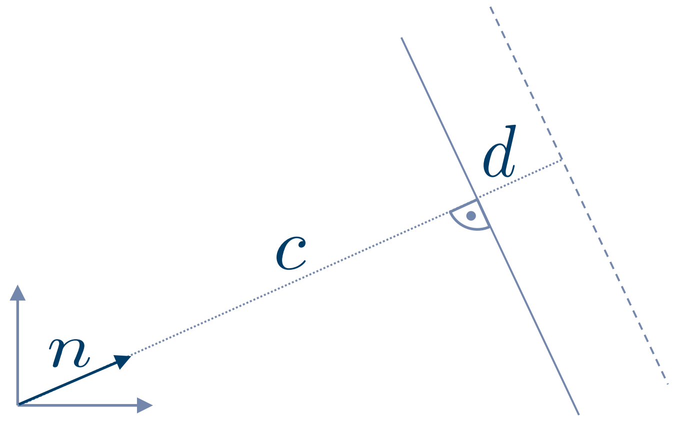
A wall in 2D space can be described using a normal vector $n = \begin{bmatrix} n_1 & n_2 \end{bmatrix}^\mathrm{T}$ (with $||n||_2 = 1$) and a scalar c

$$n_1 x_1 + n_2 x_2 = c$$

which holds true for all $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\mathrm{T}$ on the wall. Note that c in this case describes the shortest distance from the origin to the wall. So any shift d of c would describe all points within a distance d of the wall (see Figure below). So the formula

$$n_1 x_1 + n_2 x_2 = c + d$$

holds true for all x with distance d to the wall in the direction of n.



Distance measurements to arbitrary walls

If the given vector $\hat{n}$ is not a normal vector, so we have

$$\hat{n}_1 x_1 + \hat{n}_2 x_2 = \hat{c}$$

with $||\hat{n}||_2 \neq 1$, we need to normalize it first. This would result in the distance formula being

$$\frac{\hat{n}_1 x_1 + \hat{n}_2 x_2 - \hat{c}}{||\hat{n}||_2} = d$$

Homework 1

The objective is to estimate the two-dimensional object location $\boldsymbol{x} = \left[x_1, x_2\right]^T \in \mathbb{R}^2$ using (noise-corrupted) distance measurements $d^i \in \mathbb{R}$ to N walls. The location of the i-th wall is given in normal form

$$n_1^i \cdot x_1^w + n_2^i \cdot x_2^w = c^i \quad .$$

Assume n^i points to the half space where the object is located. Given are four walls with corresponding measurements:

i	n_1^i	n_2^i	c^i	distance d^i
1	-5	-1	-45	4.7
2	-1	-8	-70	5.2
3	-1	9	5	5.5
4	8	-1	7	4.5

a) Please write a function which visualizes walls and measurements using different colors.
b) Formulate a linear measurement equation $y^i = \mathbf{H}^i x + e^i$, which relates the measurement to the i-th wall with $\boldsymbol{x}$ and the error e^i. In the same manner, formulate a measurement equation that relates N walls, i.e., the 1-st to N-th walls, with $\boldsymbol{x}$ and $\boldsymbol{e}$. Note that the n^i vectors are not normalized.

Homework 1

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

c) Could you calculate the unique location for the first case in 1b), if not, please explain. If an unique location could be obtained, which requirements are needed?

d) Based on the measurement equation formulated in 1b), write a function which calculates the least squares solutions based on the measurements.
Using the function you implemented, calculate the least squares solutions $\hat{\boldsymbol{x}}_{12}$, $\hat{\boldsymbol{x}}_{34}$ and $\hat{\boldsymbol{x}}_{1234}$ based on the measurements(y_1, y_2), (y_3, y_4) as well as (y_1, y_2, y_3, y_4).

e) Given the true location $\boldsymbol{x}$, implement a function which calculates the estimation error e using Euclidean norm, e.g.,
$e_{12} = ||\hat{\boldsymbol{x}}_{12} - \boldsymbol{x}||$, $e_{34} = ||\hat{\boldsymbol{x}}_{34} - \boldsymbol{x}||$, $e_{1234} = ||\hat{\boldsymbol{x}}_{1234} - \boldsymbol{x}||$
Assuming the true location $\boldsymbol{x} = [5,\ 5]^T$, calculate e_{12}, e_{34} and e_{1234}. What can you observe?