Sensor Data Fusion

Exercise 4

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DATA
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Today

- Lecture review
- Questions
- Homework 3 solution
- Theory - TDOA closed-form
- Problem 4.1 - Silly estimator
- Problem 4.2 - Polynomial Regression
- Homework 4 presentation

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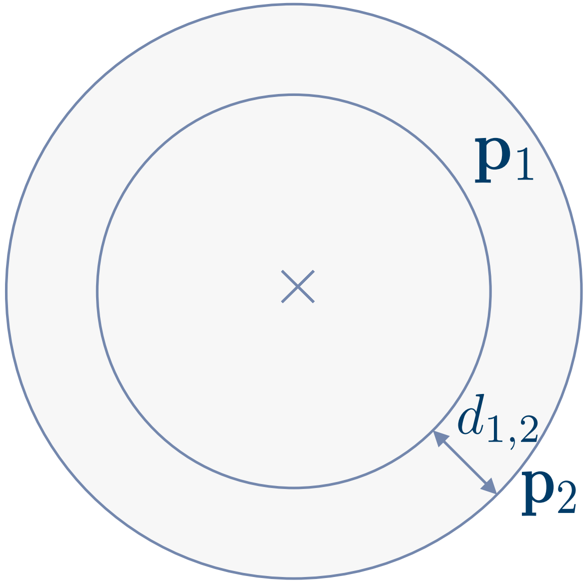
Can you explain how Time-difference-of-Arrival works?



$$d_{1,2} = ||\mathbf{x} - \mathbf{p}_2|| - ||\mathbf{x} - \mathbf{p}_1|| + e$$

What is an estimator?

An estimator is a function $\theta_y : \mathbb{R}^m \to \mathbb{R}^n$ providing an estimate for the desired variable $\mathbf{x}$ based on an observation $\mathbf{y}$. A linear estimator has the form

$$\theta_y = \mathbf{K}\mathbf{y} + \mathbf{b} \ .$$

The quality of an estimator can be measured using the mean squared error (MSE)

$$MSE(\theta_y) = \mathrm{E}[||\theta_y - \mathbf{x}||^2] \ .$$

Questions

How can the MSE be decomposed?

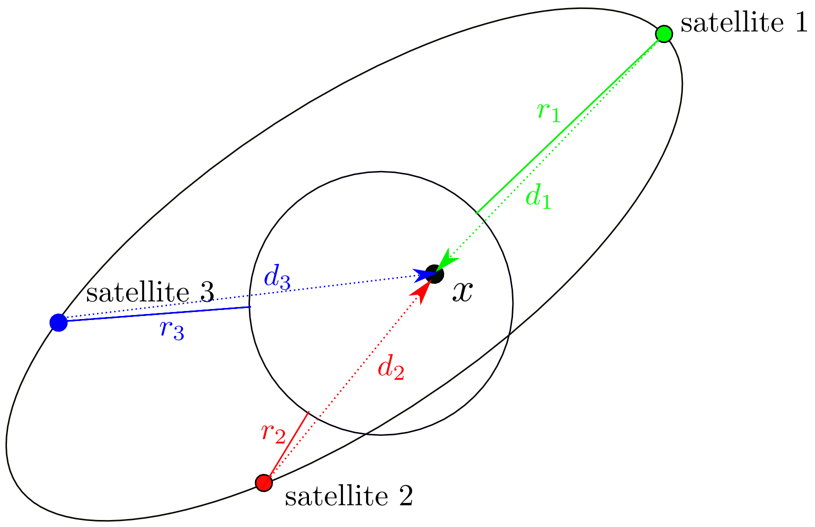
Note: Fisher approach, so no statistical information about $\mathbf{x}$.

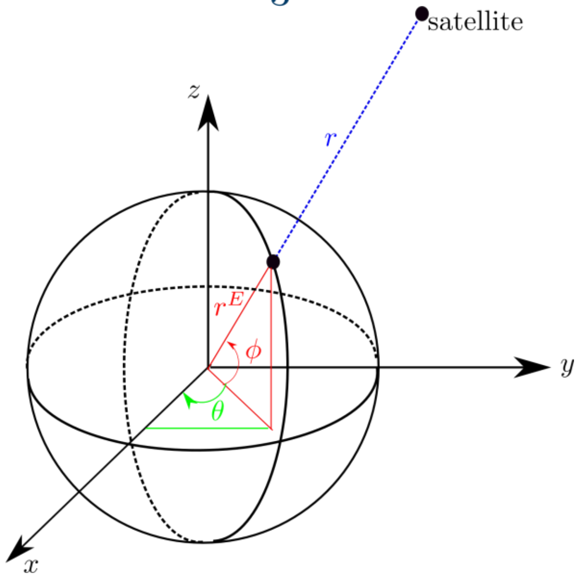
$$\begin{aligned}
\mathrm{E}[||\theta_y - \mathbf{x}||^2] &= \mathrm{E}[\mathrm{Tr}\,(\theta_y - \mathbf{x})(\theta_y - \mathbf{x})^{\mathrm{T}}] \\
&= \mathrm{Tr}\,\mathrm{E}[\theta_y^2 - 2\theta_y\mathbf{x} + \mathbf{x}^2] \\
&= \mathrm{Tr}\,\mathrm{E}[\theta_y^2] - 2\,\mathrm{E}[\theta_y]\mathbf{x} + \mathbf{x}^2 + \mathrm{E}[\theta_y]^2 - \mathrm{E}[\theta_y]^2 \\
&= \mathrm{Tr}\,\mathrm{E}[\theta_y^2] - \mathrm{E}[\theta_y]^2 + \mathrm{Tr}\,\mathrm{E}[\theta_y]^2 - 2\,\mathrm{E}[\theta_y]\mathbf{x} + \mathbf{x}^2 \\
&= \underbrace{\mathrm{Tr}\,\mathrm{Cov}[\theta_y]}_{\text{Covariance}} + \underbrace{||\,\mathrm{E}[\theta_y] - \mathbf{x}||^2}_{\text{Bias}}
\end{aligned}$$

Homework 3: Solution

Look again at the GPS setting of last homework (sattelites are 20200km above mean sea level, each satellite broadcasts its location in spherical coordinates $[\theta, \phi, r]^T$ plus the emission time (see figures below), assume that the speed-of-light is $c = 3 \cdot 10^8 \, \text{m s}^{-1}$ and the Earth is an ideal sphere with radius $r^\text{E} = 6370\text{km}$)
This time, the device only receives the first three sattelite signals at time $t = 0\text{s}$:

$$p_1 = [0°, 40°, 20200\text{km}]^T, \qquad t_1 = -67.603\text{ms},$$
$$p_2 = [10°, 20°, 20200\text{km}]^\text{T}, \qquad t_2 = -70.102\text{ms},$$
$$p_3 = [10°, -10°, 20200\text{km}]^\text{T}, \qquad t_3 = -78.690\text{ms}.$$





Homework 3: Solution

a) Would the reformulation from last time still work? If not, explain why.

b) Instead of reformulating the measurement equation into a linear equation, implement the Gauss-Newton method to calculate the least squares solution.

c) Alternatively, use the Bancroft solution to determine the position of the GPS device.

Trilateration: Bancroft Solution (1)

- Squared meas. equation:

$$\begin{aligned} d_i^2 &= ||x - p_i||^2 + e_i^* \\ &= (x_1 - p_{i,1})^2 + (x_2 - p_{i,2})^2 + e_i^* \\ &= -2x_1 p_{i,1} - 2x_2 p_{i,2} + ||p_i||^2 + R^2 + e_i^* \end{aligned}$$

with $R^2 := ||x||^2 = (x_1)^2 + (x_2)^2$

- Linear measurement equation for given R^2:

$$y = \mathbf{H}_1 x + \mathbf{H}_2 R^2 + e^*$$

with

$$y = \begin{bmatrix} d_1^2 - ||p_1||^2 \\ \vdots \\ d_m^2 - ||p_m||^2 \end{bmatrix} \quad , \quad \mathbf{H}_1 = \begin{bmatrix} -2p_{i,1} & -2p_{i,2} \\ \vdots & \vdots \\ -2p_{m,1} & -2p_{m,2} \end{bmatrix} \quad , \quad \mathbf{H}_2 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Trilateration: Bancroft Solution (2)

- Least squares solution for a fixed R^2:

$$\begin{aligned} x^{LS}(R^2) &= (\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1)^{-1}\mathbf{H}_1^{\mathrm{T}}(y - \mathbf{H}_2 R^2) \\ &= z_1 + R^2 z_2 \end{aligned}$$

with $z_1 := (\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1)^{-1}\mathbf{H}_1^{\mathrm{T}} y$ and $z_2 := -(\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1)^{-1}\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_2$

- What is R^2?

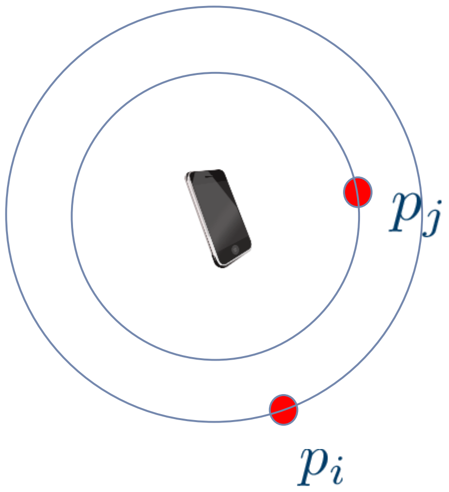
$$\begin{aligned} R^2 &= ||x^{LS}(R^2)||^2 \\ &= (z_1 + R^2 z_2)^{\mathrm{T}} \cdot (z_1 + R^2 z_2) \end{aligned}$$

- Solve the following quadratic equation for R^2:

$$0 = z_1^{\mathrm{T}} z_1 + z_1^{\mathrm{T}} z_2 R^2 + R^2 z_2^{\mathrm{T}} z_1 + (R^2)^2 z_2^{\mathrm{T}} z_2 - R^2$$

TDOA: Problem Formulation (2D)

Desired: Sender position $x = \left[x_1, x_2\right]^T \in \mathbb{R}^2$



Given: $\binom{m}{2}$ TDOAs to m landmarks:

- position $p_i = \left[p_{i,1}, p_{i,2}\right]^T \in \mathbb{R}^2$
- time difference $\Delta T_{i,j}$
- distance difference $d_{i,j} = c \cdot \Delta T_{i,j}$ with wave speed c

with $i, j \in \{1, \ldots, m\}$

Measurement equation:

$$d_{i,j} = \underbrace{||x - p_i||}_{D_i} - \underbrace{||x - p_j||}_{D_j} + e_i$$

TDOA: Closed-Form Solution (1)

- Define

$$(R^x)^2 := ||x||^2 = (x_1)^2 + (x_2)^2$$

and

$$(R_i^p)^2 := ||p_i||^2 = (p_{i,1})^2 + (p_{i,2})^2$$

- Assume $p_1 = [0, 0]^T$; hence, $D_1 = R^x$ and $R_1^p = 0$.
- From

$$\begin{aligned} (d_{i,1} + R^x)^2 &= ||x - p_i||^2 \\ &= (R^x)^2 + (R_i^p)^2 - 2p_i^T x \end{aligned}$$

we get

$$2(p_i)^T x + 2d_{i,1}R^x = (R_i^p)^2 - d_{i,1}^2$$

TDOA: Closed-Form Solution (2)

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- Stacked measurement equation

$$y = \mathbf{H}_1 x + \mathbf{H}_2 R^x + e^* \quad ,$$

where

$$y = \begin{bmatrix} (R_2^p)^2 - d_{2,1}^2 \\ \vdots \\ (R_m^p)^2 - d_{m,1}^2 \end{bmatrix}, \; \mathbf{H}_1 = \begin{bmatrix} 2p_2^T \\ \vdots \\ 2p_m^T \end{bmatrix}, \; \mathbf{H}_2 = \begin{bmatrix} 2d_{2,1} \\ \vdots \\ -2d_{m,1} \end{bmatrix}$$

- Linear least squares solution: $x^{LS}(R^x)$
- Use $(R^x)^2 = ||x^{LS}(R^x)||^2$ to get R^x

Problem 4.1 – Silly Estimator

Given is a random variable $y \sim \mathcal{N}(x, 1)$. The objective is to estimate the deterministic mean x based on a single observation y. We consider two estimators:

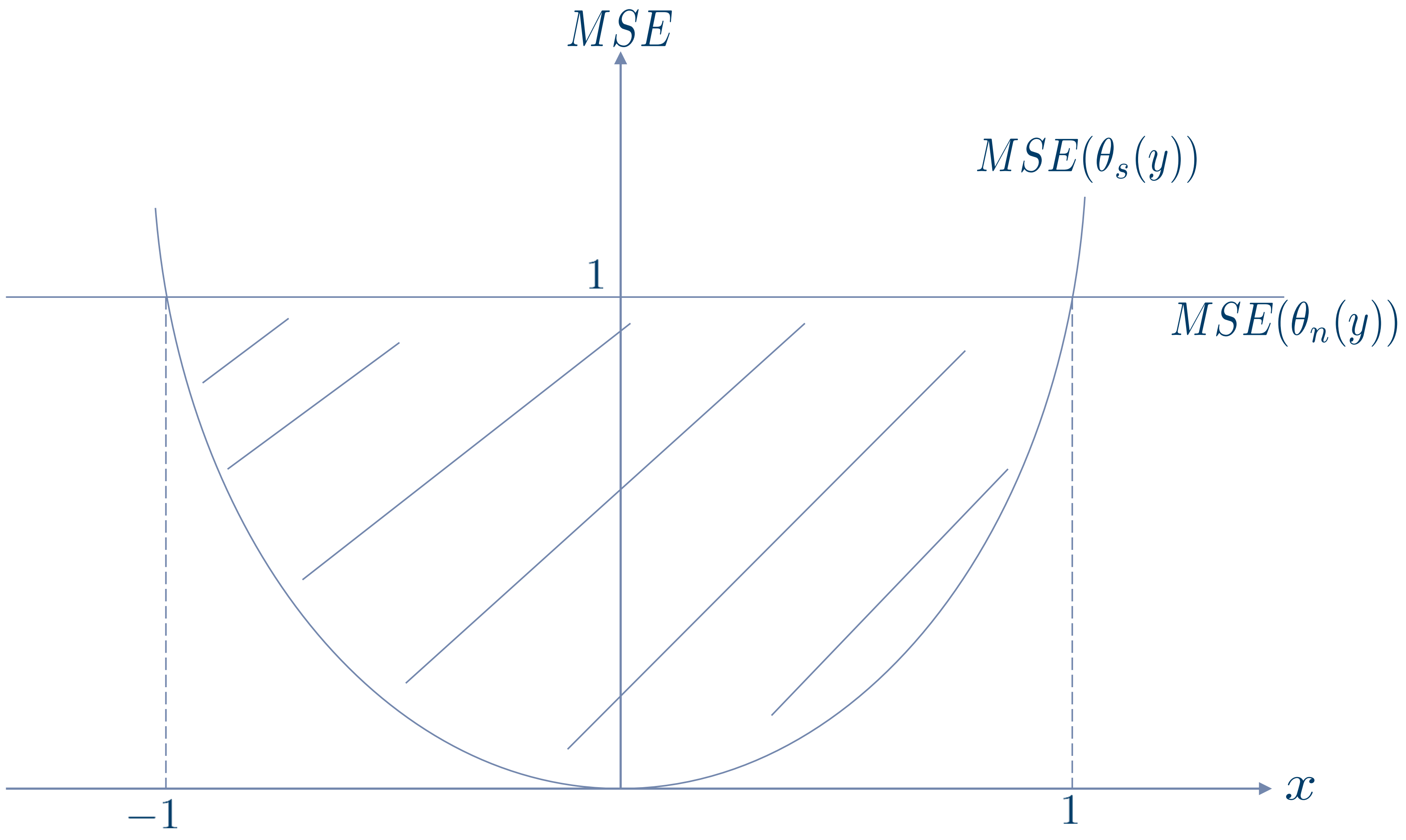
- Natural estimator: $\theta_n(y) = y$
- Silly estimator: $\theta_s(y) = 0$

a) Calculate the bias, variance, and MSE of both estimators.

b) In which cases outperforms the silly estimator the natural estimator?

$$
\begin{aligned}
\theta_n :& \beta = \mathrm{E}[\theta_n(y)] - x = x - x = 0 \\
& \mathrm{Var}[\theta_n(y)] = \mathrm{Var}[y] = 1 \\
& MSE(\theta_n(y)) = 1 + 0 = 1 \\
\theta_s :& \beta = \mathrm{E}[\theta_s(y)] - x = 0 - x = -x \\
& \mathrm{Var}[\theta_s(y)] = 0 \\
& MSE(\theta_s(y)) = 0 + ||-x||^2 = x^2
\end{aligned}
$$

Problem 4.1: Solution



Problem 4.2 - Polynomial Regression

Given are m data pairs $(y_i, x_i) \in \mathbb{R}^2$, $i = 1, \ldots, m$. The objective is to find a polynomial function

$$f_a(x) := a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

with $a = \left[a_0, \ldots, a_n\right]^T$ that fits the data as good as possible, i.e., $\sum_i ||y_i - f_a(x_i)||^2$ should be minimal.

- Formulate the fitting problem as a linear Least Squares (LS) problem, i.e., write down the linear measurement equation for this problem.
- Consider the values

i	1	2	3	4	5
x_i	1	2	5	7	8
y_i	4	1	2	3	4

  please perform a quadratic curve fit.
- Now we would like to perform a line fit. Assume the fitting passes through the origin, i.e., the equation of this line is $y = kx$. Given the two measured points $P_1 = \begin{bmatrix}1.7 & 0\end{bmatrix}^\mathrm{T}$, $P_2 = \begin{bmatrix}2.8 & 1\end{bmatrix}^\mathrm{T}$, please find the line using least squares. Visualize the fitted line and the column space spanned by the measurement matrix.

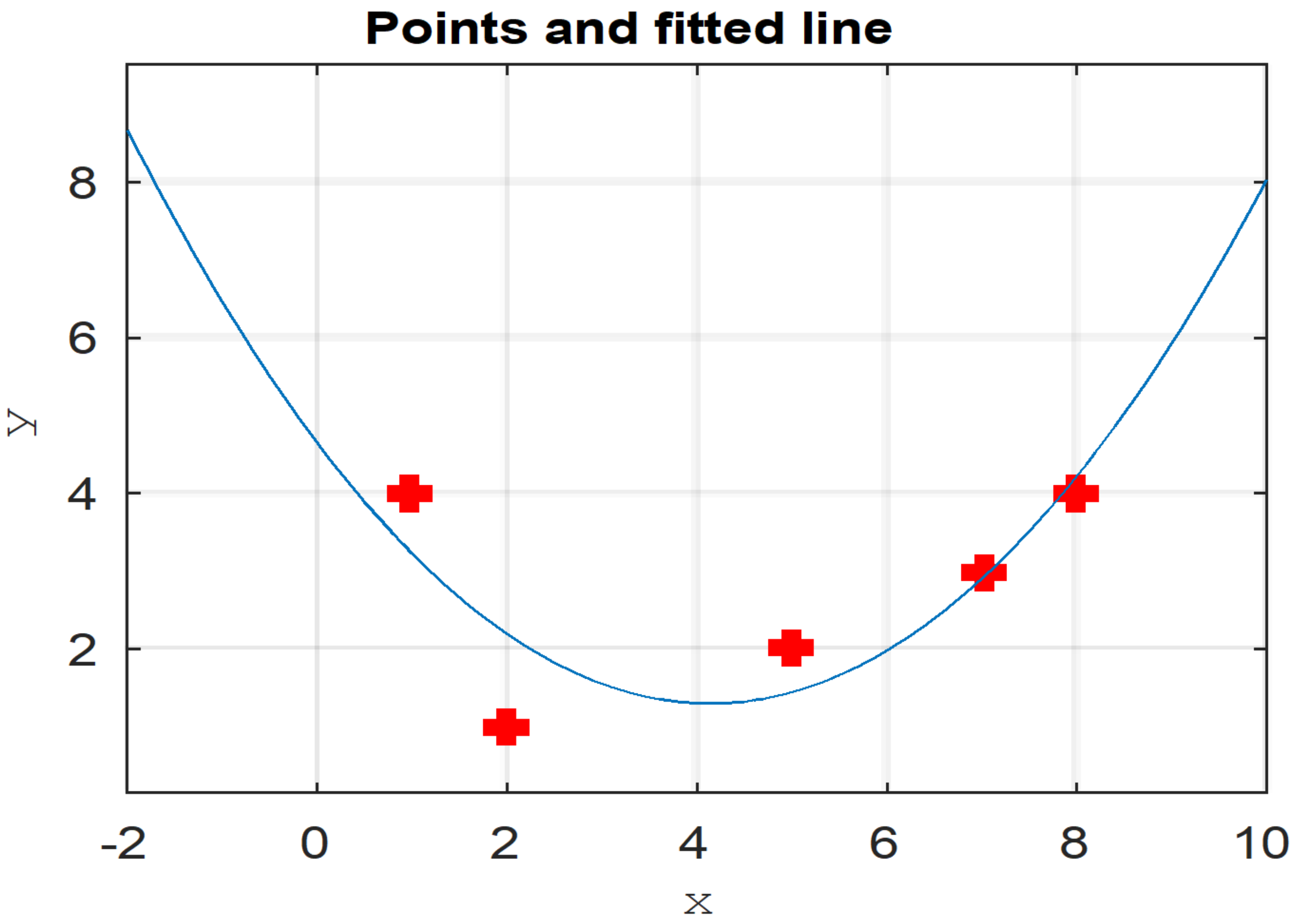
Problem 4.2: Solution

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1 & x_1 & \dots & x_1^n \\ 1 & x_2 & \dots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \dots & x_m^n \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 5 & 25 \\ 1 & 7 & 49 \\ 1 & 8 & 64 \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \qquad a^{\mathsf{LS}} = (\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}} y = \begin{bmatrix} 4.64740 \\ -1.61753 \\ 0.19557 \end{bmatrix}$$

$$f_a(x) = 4.64740 - 1.61753x + 0.19557x^2$$

Problem 4.2: Solution



Problem 4.2: Solution

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.7 \\ 2.8 \end{bmatrix} k + e$$

$$k^{\mathsf{LS}} = \left(\begin{bmatrix} 1.7 & 2.8 \end{bmatrix} \begin{bmatrix} 1.7 \\ 2.8 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1.7 & 2.8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$k^{\mathsf{LS}} = 0.26095$$

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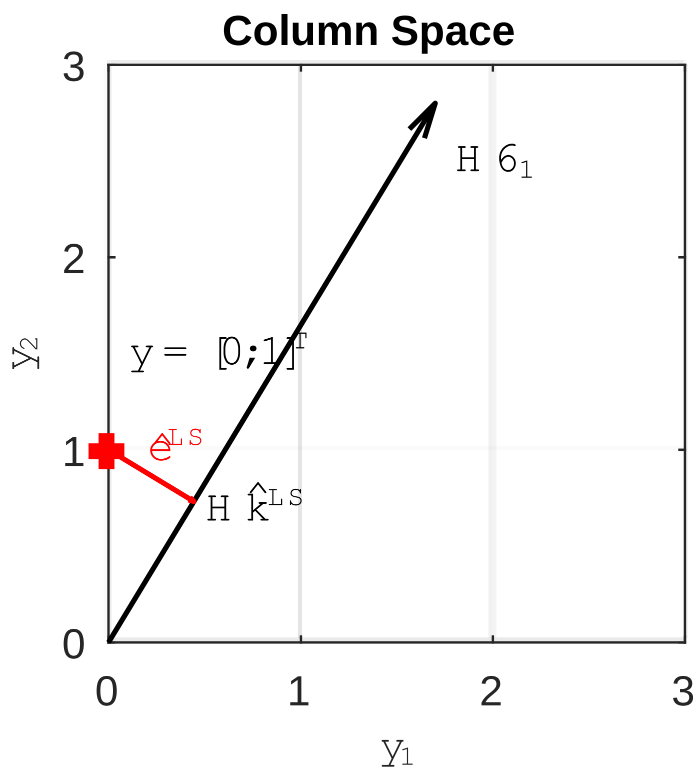
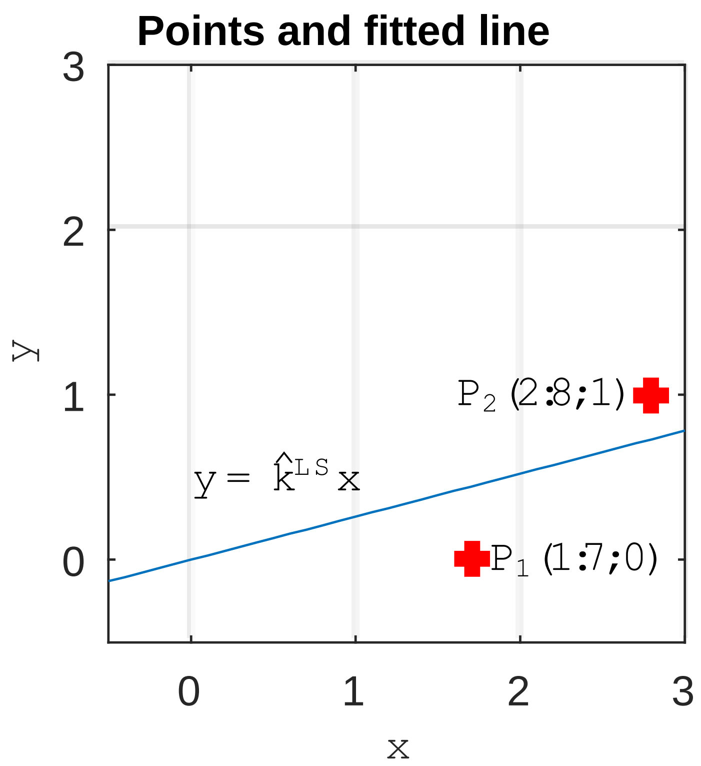
Problem 4.2: Solution





Homework 4

Given are two one-dimensional (uncorrelated) sensor readings $y_1 \in \mathbb{R}$ and $y_2 \in \mathbb{R}$ of a one-dimensional location $x \in \mathbb{R}$. The variances of the errors are $\sigma_1^2 = 2$ for the first and $\sigma_2^2 = 3$ for the second sensor.

a) Formulate the (joint) measurement equation for the fusion of the two measurements.

b) Write a function that simulates $m = 1000$ measurements using the measurement model in a) with true $x = 2$.

c) For each simulated measurement, calculate the weighted least squares (WLS) solution using $W^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & c \end{pmatrix}$ with $c = 1, ..., 10$. Which value for c provides the BLUE estimate?

d) Derive an analytic formula for the variance of the weighted least squares solution, using the W providing the BLUE estimate.

e) Check if the analytical solution coincides with the empirical variance of the WLS solution from c).