Sensor Data Fusion

Exercise 7

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Today

- Lecture review
- Questions
- Homework 6 solution
- Example - Optimal Bayesian estimator
- Problem 7 - MMSE Estimator
- Homework 7 presentation

Questions

Why is it difficult to obtain the optimal Bayesian estimator?

- No closed-form solution (in general)
- complete prior and likelihood information required
- Kalman filter is optimal Bayesian filter if
  - Linearity is given
  - All distributions are Gaussian

Questions

What happens in the Kalman filter update for **H=I** if the noise of the prior is much higher than the measurement noise and vice versa?

Regard the update formula:

$$x_{k+1|k+1} = x_{k+1|k} + \underbrace{\mathbf{C}_{k+1|k}\mathbf{H}^{\mathrm{T}}(\overbrace{\mathbf{H}\mathbf{C}_{k+1|k}\mathbf{H}^{\mathrm{T}} + \mathbf{R}}^{\mathbf{S}})^{-1}}_{\mathbf{K}}(y - \mathbf{H}x_{k+1|k})$$

With the prior noise $\mathbf{C}_{k+1|k}$ being much larger then the measurement noise $\mathbf{R}$, $\mathbf{KH}$ gets close to being the identity matrix, so that the state is subtracted and the update will only use the measurement to set the new state. Vice versa, $\mathbf{S}^{-1}$ would be close to zero, leaving only the prior state.

Questions

What can be said about the posterior distribution if all other distributions involved are Gaussian?

The posterior $p(x|y)$ is also Gaussian.

Homework 6: Solution

Consider again the setup of the Silly Estimator exercise with $x \sim \mathcal{N}(1, 1)$ and $y = x + e$ with $e \sim \mathcal{N}(0, 1)$.

a) Write a function which calculates a possible value for x based on its distribution and then takes a measurement from that value using the measurement noise.

b) Consider the natural estimator $\theta_n(y) = y$. Calculate the empirical mean square error of 1000 runs using this estimator and compare it with the analytic solution.

c) Repeat the process with the silly estimator $\theta_s(y) = 0$.

From exercise 6:

$$\text{BMSE}(\theta_n(y)) = \sigma_e^2 = 1$$

$$\text{BMSE}(\theta_s(y)) = \sigma_x^2 + \mu_x^2 = 2$$

Example: Optimal Bayesian Estimator

We want to find θ_y^{opt} minimizing

$$\text{BMSE}(\theta_y) = \int \underbrace{\int ||\theta_y - \mathbf{x}||^2 p(\mathbf{x}|\mathbf{y})\text{d}\mathbf{x}}_{\mathrm{E}[||\theta_y - \mathbf{x}||^2|\mathbf{y}]} p(\mathbf{y})\text{d}\mathbf{y}$$

As $p(\mathbf{y})$ is always positive, we only need to minimize the inner integral for all $\mathbf{y}$.

$$\begin{aligned}
&\mathrm{E}[(\theta_y^{opt} - \mathbf{x})^{\mathrm{T}}(\theta_y^{opt} - \mathbf{x})|\mathbf{y}] \\
&\quad = \text{E}[(\theta_y^{opt} - \mathbf{c} + \mathbf{c} - \mathbf{x})^{\mathrm{T}}(\theta_y^{opt} - \mathbf{c} + \mathbf{c} - \mathbf{x})|\mathbf{y}] \\
&\quad = \text{E}[(\theta_y^{opt} - \mathbf{c})^{\mathrm{T}}(\theta_y^{opt} - \mathbf{c}) + (\theta_y^{opt} - \mathbf{c})^{\mathrm{T}}(\mathbf{c} - \mathbf{x}) + (\mathbf{c} - \mathbf{x})^{\mathrm{T}}(\theta_y^{opt} - \mathbf{c}) \\
&\qquad\quad + (\mathbf{c} - \mathbf{x})^{\mathrm{T}}(\mathbf{c} - \mathbf{x})|\mathbf{y}] \quad |\mathbf{c} = \mathrm{E}[\mathbf{x}|\mathbf{y}] \\
&\quad = \mathrm{E}[(\theta_y^{opt} - \mathrm{E}[\mathbf{x}|\mathbf{y}])^{\mathrm{T}}(\theta_y^{opt} - \mathrm{E}[\mathbf{x}|\mathbf{y}])|\mathbf{y}] \\
&\qquad\quad + \mathrm{E}[(\mathrm{E}[\mathbf{x}|\mathbf{y}] - \mathbf{x})^{\mathrm{T}}(\mathrm{E}[\mathbf{x}|\mathbf{y}] - \mathbf{x})|\mathbf{y}] \quad |\theta_y^{opt} = \mathrm{E}[\mathbf{x}|\mathbf{y}] \\
&\quad = \mathrm{E}[(\mathrm{E}[\mathbf{x}|\mathbf{y}] - \mathbf{x})^{\mathrm{T}}(\mathrm{E}[\mathbf{x}|\mathbf{y}] - \mathbf{x})|\mathbf{y}]
\end{aligned}$$

Example: Optimal Bayesian Estimator

Unbiasedness:

$$\begin{aligned}
\mathrm{E}_{\mathbf{y}}[\mathrm{E}[\mathbf{x}|\mathbf{y}]] &= \int \mathrm{E}[\mathbf{x}|\mathbf{y}]p(\mathbf{y})\mathrm{d}\mathbf{y} \\
&= \int\int \mathbf{x}p(\mathbf{x}|\mathbf{y})\mathrm{d}\mathbf{x}p(\mathbf{y})\mathrm{d}\mathbf{y} \\
&= \int \mathbf{x}p(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= \mathrm{E}[\mathbf{x}] \\
\mathrm{E}[\theta_y^{opt} - \mathbf{x}] &= \underbrace{\mathrm{E}_{\mathbf{y}}[\mathrm{E}[\mathbf{x}|\mathbf{y}]]}_{\mathrm{E}[\mathbf{x}]} - \mathrm{E}[\mathbf{x}] \\
&= \mathbf{0}
\end{aligned}$$

Problem 7 – MMSE Estimation

Consider a two-dimensional joint density

$$p(x,y) = \begin{cases} 2 & \text{for } [x,y] \in S \\ 0 & \text{otherwise} \end{cases}$$

where the set S is a triangle specified by the points $[0,0]$, $[1,0]$, and $[1,1]$.

a) What is the conditional distribution $p(x|y)$?

b) What is the MMSE estimator of x given y?

c) What is the conditional MSE associated to the MMSE estimator?

d) What is the unconditional MSE associated to the MMSE estimator?

If $x \sim \mathcal{U}(a,b)$, we have

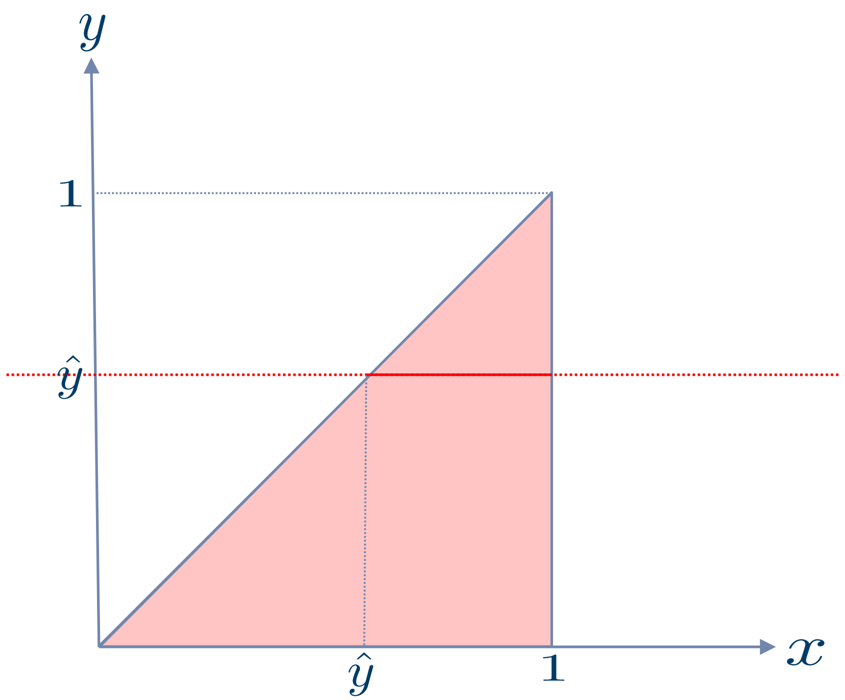
- $p(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$
- $\mathrm{E}[x] = \frac{1}{2}(a+b)$
- $\mathrm{Var}[x] = \frac{1}{12}(b-a)^2$

Problem 7: Solution

a) $p(x|y) = \begin{cases} \frac{1}{1-y} & \text{for } x \in [y, 1] \\ 0 & \text{otherwise} \end{cases}$

b) $\theta_y = \mathrm{E}[x|y] = \frac{1}{2}(y+1)$

c) $\mathrm{MSE}(\theta_y|y) = \mathrm{E}[(x - \mathrm{E}(x|y))^2|y] = \mathrm{Var}[x|y] = \frac{1}{12}(1-y)^2$



Problem 7: Solution

d)

$$
\begin{aligned}
\text{MSE}(\theta_y) &= \mathrm{E}[||x - \theta_y||^2] \\
&= \mathrm{E}_y[\mathrm{E}[||x - \mathrm{E}[x|y]||^2|y]] \\
&= \mathrm{E}_y[\text{MSE}(\theta_y|y)] \\
&= \mathrm{E}_y[\frac{1}{12}(1-y)^2] \\
&= \int \frac{1}{12}(1-y)^2 \boxed{p(y)} dy \\
&= \int_0^1 \frac{1}{12}(1-y)^2 2(1-y) dy \\
&= -\frac{1}{6}\frac{1}{4}(1-y)^4|_0^1 \\
&= \frac{1}{24}
\end{aligned}
$$

$$
p(x,y) = \begin{cases} 2 \ , & \text{for } y \in [0,1] \ , x \in [y,1] \\ 0 \ , & \text{otherwise} \end{cases}
$$

$$
\begin{aligned}
p(y) &= \int p(x,y)dx \\
&= \begin{cases} \int_y^1 2dx \ , & y \in [0,1] \\ 0 \ , & \text{otherwise} \end{cases} \\
&= \begin{cases} 2(1-y) \ , & y \in [0,1] \\ 0 \ , & \text{otherwise} \end{cases}
\end{aligned}
$$

Homework 7

Assume a robot in 2D-space. Its position is modeled as a Gaussian random variable. The prior has $\hat{\mathbf{x}}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$ and

$$\mathbf{C}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad .$$

a) A sensor measures the robot's true position. Formulate and implement a measurement equation assuming independent zero-mean Gaussian noise with

$$\mathbf{R} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \quad .$$

b) Implement a function which samples a true position of $\mathbf{x}$ from the prior and then generates a measurement from the true position.

c) Implement the Kalman update formula to calculate the posterior distribution.

Homework 7

d) Now, assume the sensor will provide 5 measurements in a row. Use the Kalman filter update formulas to update the robot's state recursively.

e) Visualize the robot's covariance matrix as an ellipse and observe how it changes with each update.