Sensor Data Fusion

Exercise 5

Prof. Dr.-Ing. Marcus Baum
M.Sc. Kolja Thormann

www.fusion.informatik.uni-goettingen.de

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DATA
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Today

- Lecture review
- Questions
- Homework 4 solution
- Example - LS as an estimator
- Problem 5 - non-Gaussian Noise
- Homework 5 presentation

Questions

What is BLUE?

- Quality of an estimator measured via MSE
- Goal: Minimum MSE Estimator
- Problem: unknown $\mathbf{x}$
- Solution: assume unbiasedness

$$MSE(\theta_y) = \operatorname{Tr}\operatorname{Cov}[\theta_y] \cancel{+ ||\operatorname{E}[\theta_y] - \mathbf{x}||^2}$$

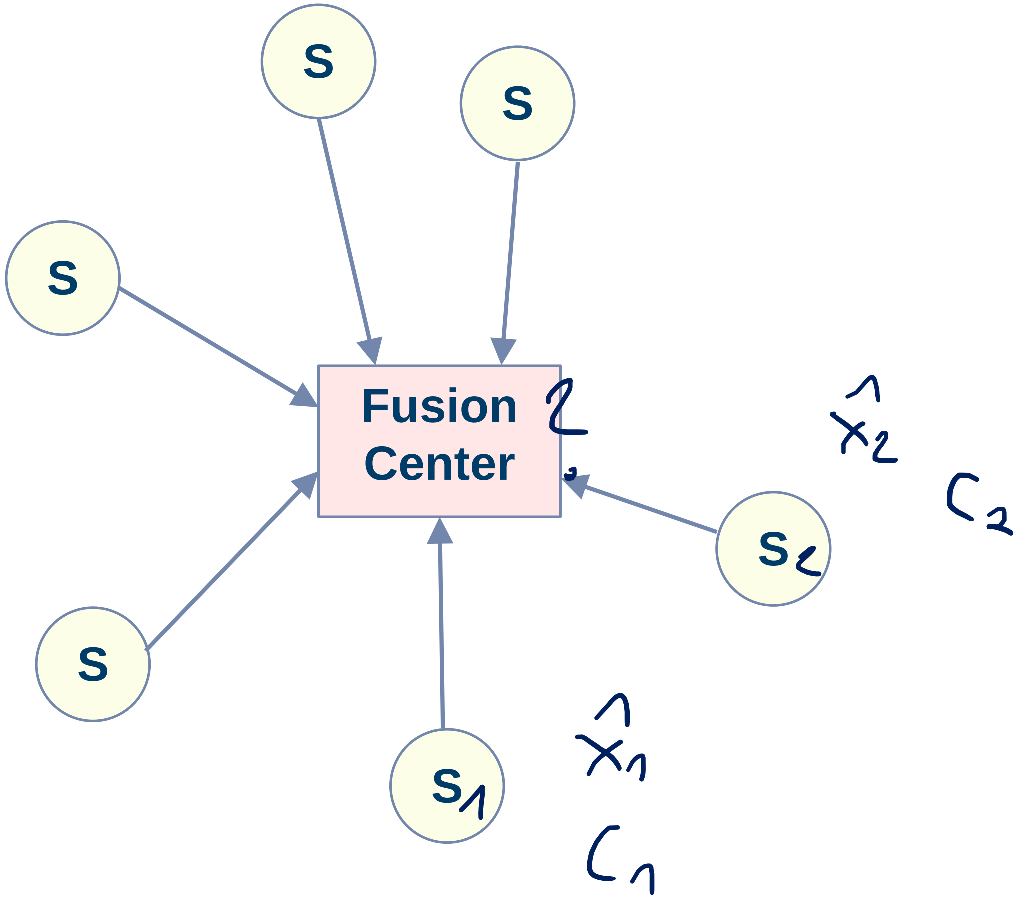
- New goal: Minimum Variance Unbiased (MVU) Estimator
- If MVU estimator is linear, we have the Best Linear Unbiased Estimator (BLUE)

Questions

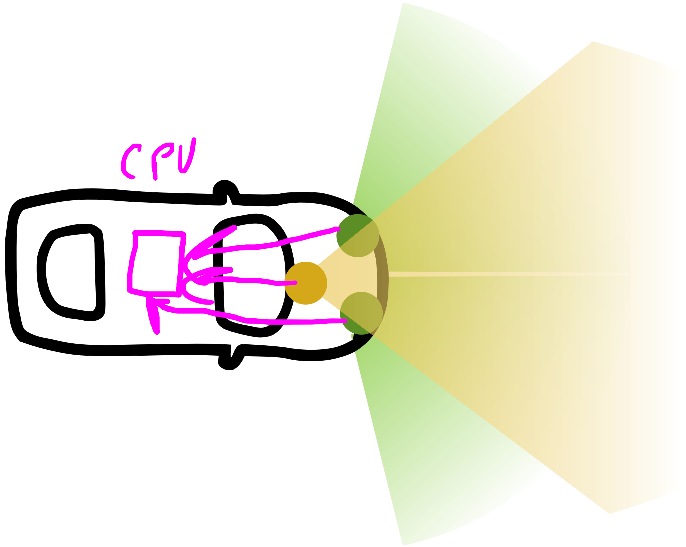
What are types of sensor network topologies?

Centralized Fusion Architecture

PREVIOUS LECTURE

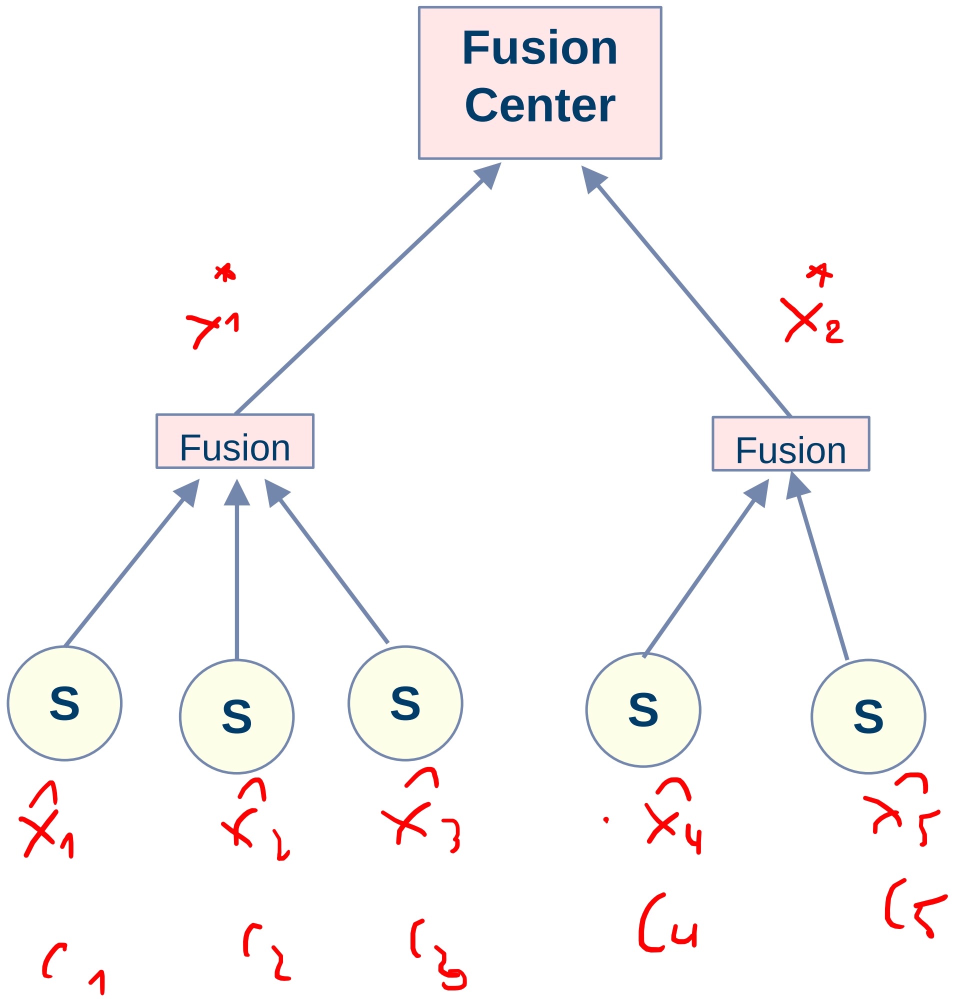


- Central fusion center
- Sensors communicate sensor da to fusion center



Distributed Fusion Architecture

PREVIOUS LECTURE

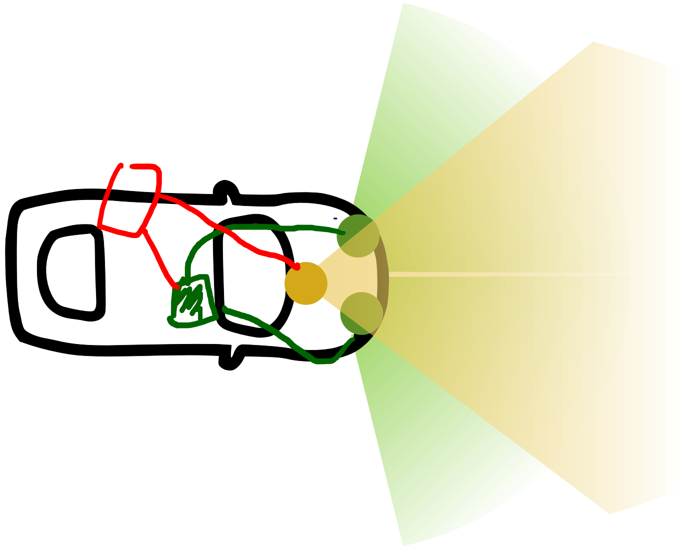


Properties:

1. Several computing nodes
2. Data is processed on nodes
3. Central node required for final result

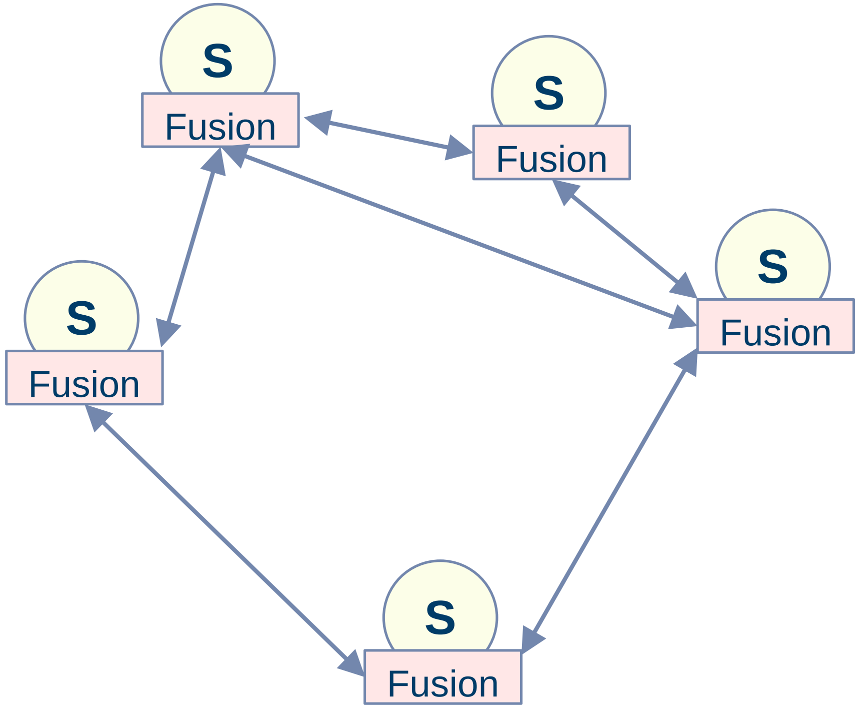
Advantages:

- Scalable



Decentralized Fusion

PREVIOUS LECTURE



Properties:
1. No central fusion center
2. Each node fuses data
2. Node-to-node communication
3. Network topology unknown to sensors

Advantages:
- Scalable
- Dynamic topology
- Robust against failures

Challenges:
- Rumor problem, i.e., double-counting of information

Questions

How can unknown correlation between estimates be handled?

Find uncorrelated ellipse encapsulating correlated one.

$$\tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{C}^{xx} & \mathbf{C}^{xy} \\ \mathbf{C}^{yx} & \mathbf{C}^{yy} \end{bmatrix}$$

$$\mathbf{C} > \tilde{\mathbf{C}}$$

$$\mathbf{C} = \begin{bmatrix} \frac{1}{0.5-\alpha}\mathbf{C}^{xx} & 0 \\ 0 & \frac{1}{0.5+\alpha}\mathbf{C}^{yy} \end{bmatrix}$$

Homework 4: Solution

Given are two one-dimensional (uncorrelated) sensor readings $y_1 \in \mathbb{R}$ and $y_2 \in \mathbb{R}$ of a one-dimensional location $x \in \mathbb{R}$. The variances of the errors are $\sigma_1^2 = 2$ for the first and $\sigma_2^2 = 3$ for the second sensor.

a) Formulate the (joint) measurement equation for the fusion of the two measurements.

b) Write a function that simulates $m = 1000$ measurements using the measurement model in a) with true $x = 2$.

c) For each simulated measurement, calculate the weighted least squares (WLS) solution using $W^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & c \end{pmatrix}$ with $c = 1, ..., 10$. Which value for c provides the BLUE estimate?

d) Derive an analytic formula for the variance of the weighted least squares solution, using the W providing the BLUE estimate.

e) Check if the analytical solution coincides with the empirical variance of the WLS solution from c).

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a) Formulate the (joint) measurement equation for the fusion of the two measurements.

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{H} x + \underbrace{\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}}_{e}$$

$$e \in \mathcal{N}(0, C)$$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

b) Write a function that simulates $m = 1000$ measurements using the measurement model in a) with true $x = 2$.

c) For each simulated measurement, calculate the weighted least squares (WLS) solution using $W^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & c \end{pmatrix}$ with $c = 1, ..., 10$. Which value for c provides the BLUE estimate?

$$x^{LS} = \underbrace{(H^\intercal W H)^{-1} H^\intercal W}_{K} y = (\frac{1}{2} + \frac{1}{c})^{-1} \cdot (\frac{y_1}{2} + \frac{y_2}{c})$$

Using $c = 3$, we get the BLUE estimate, because then we get $W^{-1} = C$.

d) Derive an analytic formula for the variance of the weighted least squares solution, using the W providing the BLUE estimate.

$$
\begin{aligned}
Cov\{x^{LS}\} &= KCK^\intercal \\
&= (H^\intercal W H)^{-1} H^\intercal W C W H (H^\intercal W H)^{-1} \\
&= (H^\intercal C^{-1} H)^{-1} H^\intercal C^{-1} C C^{-1} H (H^\intercal C^{-1} H)^{-1} \\
&= (H^\intercal C^{-1} H)^{-1} H^\intercal C^{-1} H (H^\intercal C^{-1} H)^{-1} \\
&= (H^\intercal C^{-1} H)^{-1}
\end{aligned}
$$

e) Check if the analytical solution coincides with the empirical variance of the WLS solution from c).

Example – LS as an estimator

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- General: $\theta_y = \mathbf{K}\mathbf{y} + \mathbf{b}$
- Unbiased: $\mathrm{E}[\mathbf{K}\mathbf{y} + \mathbf{b}] = \mathrm{E}[\mathbf{K}\mathbf{y}] + \mathbf{b} = \mathbf{K}\mathbf{H}\mathbf{x} + \mathbf{b} \overset{!}{=} \mathbf{x} \iff \mathbf{K}\mathbf{H} = \mathbf{I} \wedge \mathbf{b} = \mathbf{0}$
- Therefore, we have

$$\mathbf{x}^{LS} = \underbrace{(\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W}}_{\mathbf{K}}\,\mathbf{y}$$

$$\mathbf{K}\mathbf{H} = (\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H} = \mathbf{I}$$

Example – LS as an estimator

- Gauss-Markov: If the error is zero mean with covariance $\mathbf{C}$, then Least Squares is BLUE if $\mathbf{W} = \mathbf{C}^{-1}$
- Assume $\mathbf{K}^{LS}$ to be calculated using $\mathbf{W} = \mathbf{C}^{-1}$ and any different $\mathbf{K} = \mathbf{K}^{LS} + \mathbf{B}$
- The different $\mathbf{K}$ must still fullfill the unbiased condition

$$
\begin{aligned}
\mathbf{K}\mathbf{H} &= \mathbf{I} \\
\underbrace{\mathbf{K}^{LS}\mathbf{H}}_{\mathbf{I}} + \mathbf{B}\mathbf{H} &= \mathbf{I} \\
\mathbf{B}\mathbf{H} &= \mathbf{0}
\end{aligned}
$$

Example – LS as an estimator

- The MSE given the new $\mathbf{K}$ is then the trace of

$$\begin{aligned}
\mathrm{Cov}[\mathbf{K}\mathbf{y}] &= \mathbf{K}\mathbf{C}\mathbf{K}^{\mathrm{T}} \\
&= \underbrace{\mathbf{K}^{LS}\mathbf{C}(\mathbf{K}^{LS})^{\mathrm{T}}}_{\mathrm{Cov}[\mathbf{K}^{LS}\mathbf{y}]} + \cancel{\mathbf{K}^{LS}\mathbf{C}\mathbf{B}^{\mathrm{T}}} + \cancel{\mathbf{B}\mathbf{C}(\mathbf{K}^{LS})^{\mathrm{T}}} + \underbrace{\mathbf{B}\mathbf{C}\mathbf{B}^{\mathrm{T}}}_{\geq 0} \\
&\geq \mathrm{Cov}[\mathbf{K}^{LS}\mathbf{y}] \\
\mathbf{B}\mathbf{C}(\mathbf{K}^{LS})^{\mathrm{T}} &= \underbrace{\mathbf{B}\cancel{\mathbf{C}\mathbf{C}^{-1}}\mathbf{H}}_{\mathbf{0}}(\mathbf{H}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{H})^{-1} \\
&= \mathbf{0}
\end{aligned}$$

Example – LS as an estimator

Four independent sensors measure the temperature in a fridge. Their results are

Sensor	1	2	3	4
Measurement	6	3	1	2

Calculate the unweighted least squares solution.

$$\begin{bmatrix} 6 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} x + e$$

$$x^{LS} = \frac{12}{4} = 3$$

Example – LS as an estimator

To improve the estimate, the sensors errors are determined. They are zero-mean with variances

Sensor	1	2	3	4
Variance	2	1	1	0.66

Calculate the result of the BLUE.

$$\mathbf{W} = \mathbf{C}^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.66 \end{bmatrix}^{-1}$$

$$x_{\text{BLUE}}^{LS} = (\frac{1}{2} + 1 + 1 + \frac{3}{2})^{-1}(\frac{6}{2} + 3 + 1 + 1.5 \cdot 2) = \frac{10}{4} = 2.5$$

Problem 5 – Non-Gaussian Noise

Consider a one-dimensional state $x \in \mathbb{R}$ and measurements $y_k \in \mathbb{R}, k = 1, \ldots, m$ with linear measurement equation

$$y_k = x + e_k$$

where $e_k \sim \text{Exp}(\mu)$ is exponential distributed according to

$$p(e_k) = \begin{cases} \frac{1}{\mu} e^{-\frac{1}{\mu} e_k} & e_k \geq 0 \ , \\ 0 \quad e_k < 0 \ . \end{cases}$$

where $\mu > 0$ is the scale parameter and e is the base of the natural logs.

Hint: For exponential distribution e_k with scale parameter μ, we have $\text{E}[e_k] = \mu$ and $\text{Var}[e_k] = \mu^2$

a) What is the BLUE of x for $m = 1$?

b) What is the variance of the BLUE of x for $m = 1$?

c) What is the BLUE and its variance of x for $m > 1$?

Problem 5: Solution

a) Reformulate the measurement equation

$$\underbrace{y_k - \mu}_{y_k^*} = x + \underbrace{e_k - \mu}_{e_k^*} \quad ,$$

to get a zero-mean error. Then, apply least squares as the BLUE with $\mathbf{H} = 1$ to get

$$\hat{x}_1^{\text{BLUE}} = y_k - \mu$$

b) As $\hat{x}_1^{\text{BLUE}} = x + e_k^*$ and $\text{E}[\hat{x}_1^{\text{BLUE}}] = x$, its variance is $\text{Var}[\hat{x}^{\text{BLUE}}] = \text{Var}[e_k^*] = \mu^2$.

c) For the multidimensional measurement case, we again stack the measurement equations, resulting in

$$\hat{x}_m^{\text{BLUE}} = \frac{\sum_{k=1}^m y_k}{m} - \mu \quad ,$$

with variance $\text{Var}[\hat{x}_m^{\text{BLUE}}] = \frac{\mu^2}{m}$.

Homework 5 – Fusion under unknown correlation

Consider a 1D state x and estimates x_1 and x_2. The errors of the estimates are correlated with

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \sim \mathcal{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.8\sqrt{1 \cdot 4} \\ 0.8\sqrt{1 \cdot 4} & 4 \end{bmatrix})$$

a) Write a function which visualizes the joint covariance matrix. Use it to draw the matrix along with the column space in the measurement space. Then assume the correlation coefficient 0.8 from the description is unknown and define a covariance which ignores it. Draw that covariance and calculate and visualize the solution of the BLUE.

b) Now write a function which calculates a matrix which is bigger than the input matrix, determined by a parameter α as in the lecture. Calculate the BLUE estimate for different α values and determine which works best. Visualize the results.