Sensor Data Fusion

Exercise 3

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Today

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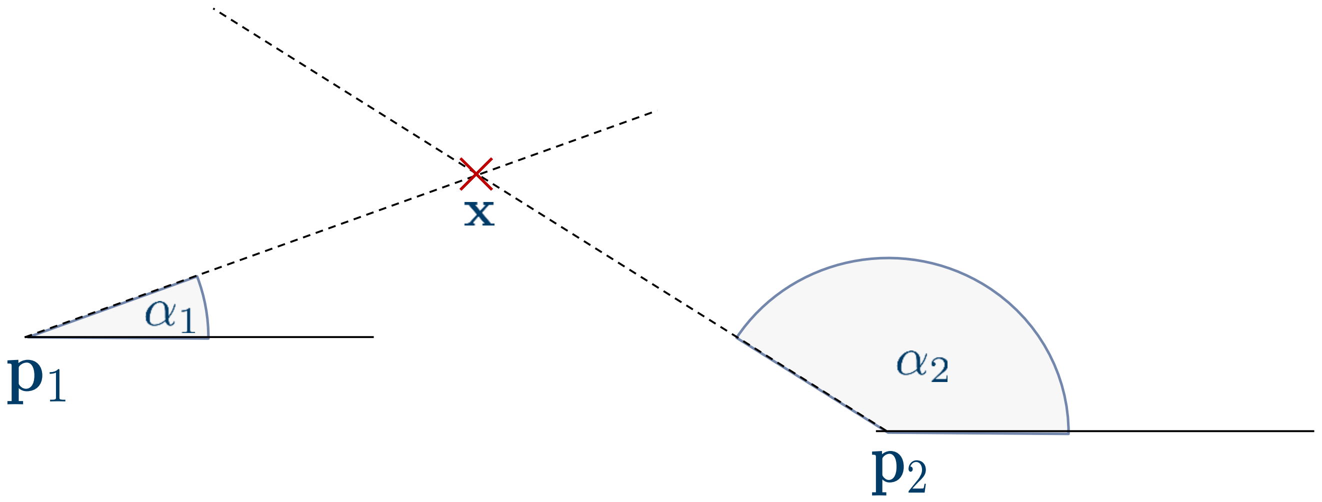
- Lecture review
- Questions
- Homework 2 solution
- Review statistics
- Theory - Bancroft solution and closed-form solution for triangulation
- Problem 3 - Triangulation
- Homework 3 presentation

Questions

Can you explain how Triangulation works? Why can't you use the normal equation to solve it?

$$\alpha_i = \text{atan2}(x_2 - p_2, x_1 - p_1) + e_i$$



As the measurement equation is non-linear, we are unable to apply least square formula. We can still solve the problem by applying a closed-form solution or using iterative optimization.

Questions

Can you explain the steps of the Gauss-Newton algorithm?

Gauss-Newton Method

Linear approximation of $r(x)$ around $\hat{x}^{(l)}$

- **Objective:**

$$x^{LS} = \arg\min_x ||r(x)||^2 \quad \approx \quad \arg\min_x ||r(\hat{x}^{(l)}) + \mathbf{J}_l(x - \hat{x}^{(l)})||^2$$

- **Algorithm:**
  - **Step 1:** Choose initial estimate $\hat{x}^{(1)}$, $l = 1$
  - **Step 2:** Use linear LS to get $\hat{x}^{(l+1)}$:

$$\begin{aligned}\hat{x}^{(l+1)} &= (\mathbf{J}_l^{\mathrm{T}}\mathbf{J}_l)^{-1}\mathbf{J}_l^{\mathrm{T}} \left(\mathbf{J}_l\hat{x}^{(l)} - r(\hat{x}^{(l)})\right) \\ &= \hat{x}^{(l)} - (\mathbf{J}_l^{\mathrm{T}}\mathbf{J}_l)^{-1}\mathbf{J}_l^{\mathrm{T}}\, r(\hat{x}^{(l)})\end{aligned}$$

  - **Step 3:** Goto **Step 2** until convergency is reached

Questions

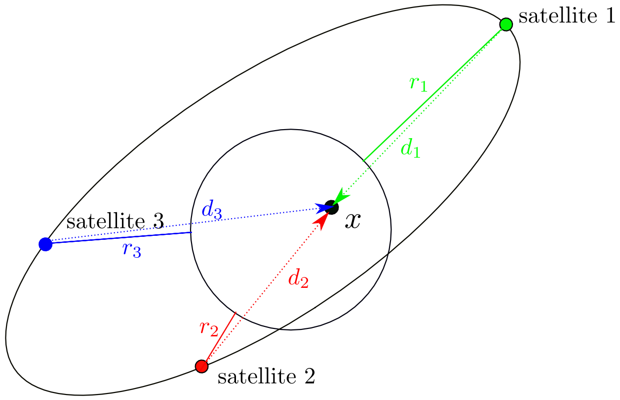
What is a typical use for closed-form approximations?

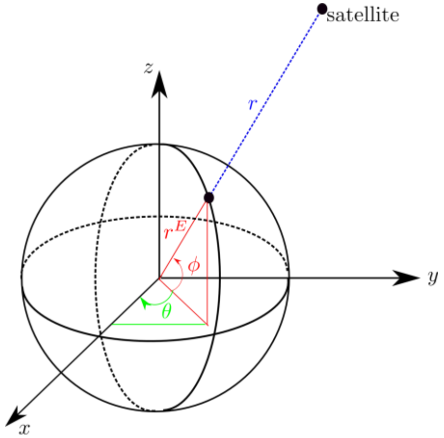
A closed-form approximation can be used as an initial guess for an iterative optimization method.

GPS consists of 24 satellites in orbit (20200km above mean sea level). Each satellite broadcasts its location (in spherical coordinates $[\theta, \phi, r]^T$) plus the emission time (see figures below).
A GPS device receives at time $t = 0$s the following four satellite signals:

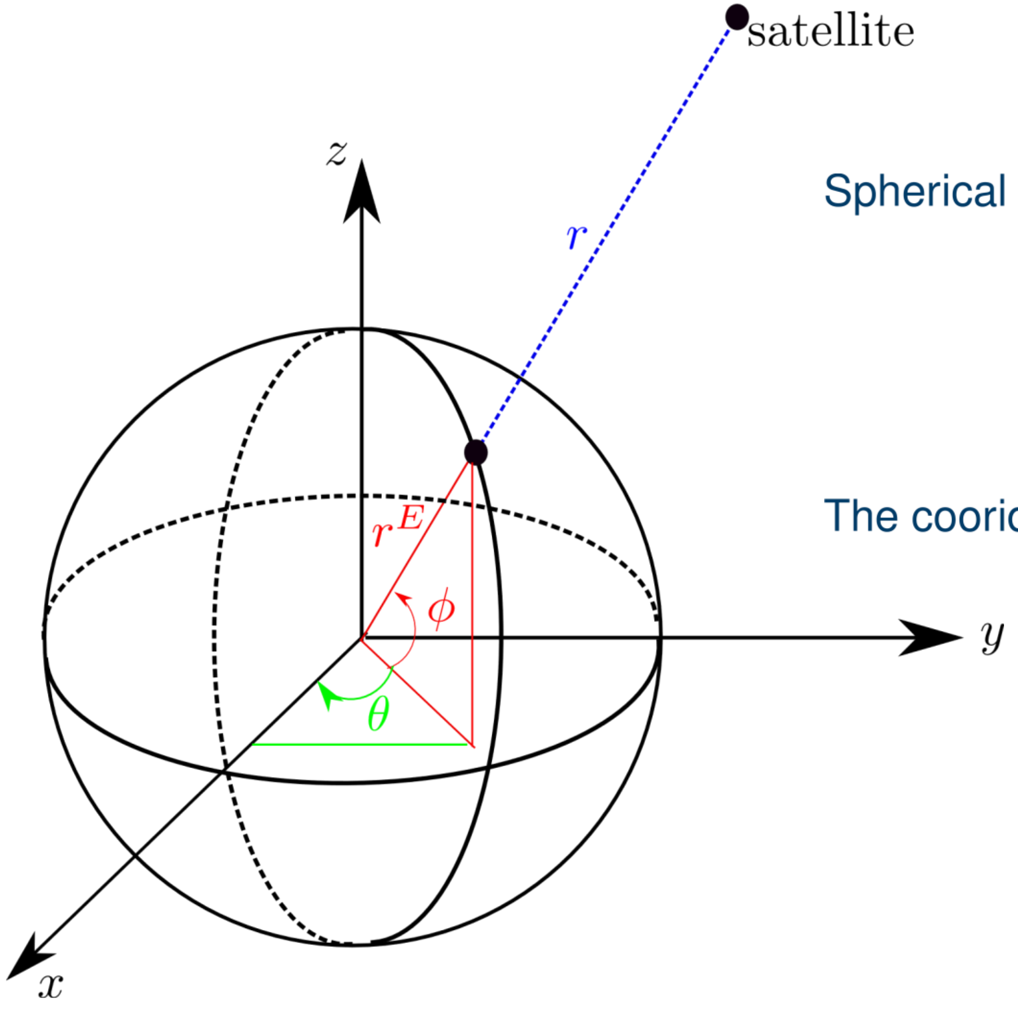
$$
\begin{aligned}
p_1 &= [0^\circ, 40^\circ, 20200\text{km}]^T, & t_1 &= -67.603\text{ms},\\
p_2 &= [10^\circ, 20^\circ, 20200\text{km}]^\mathrm{T}, & t_2 &= -70.102\text{ms}\ ,\\
p_3 &= [10^\circ, -10^\circ, 20200\text{km}]^\mathrm{T}, & t_3 &= -78.690\text{ms}\ ,\\
p_4 &= [-10^\circ, -20^\circ, 20200\text{km}]^\mathrm{T}, & t_4 &= -82.942\text{ms}.
\end{aligned}
$$

Assume that the speed-of-light is $c = 3 \cdot 10^8 \text{m s}^{-1}$ and the Earth is an ideal sphere with radius $r^\mathsf{E} = 6370$km.





a) Write a function which converts sphere coordinates (θ, ϕ, r) into Cartesian coordinates (x, y, z)



Spherical cooridnates could be converted into Cartesian coordinates following

$$
\begin{aligned}
x &= (r^{\mathsf{E}} + r) \cos\phi \cos\theta \\
y &= (r^{\mathsf{E}} + r) \cos\phi \sin\theta \\
z &= (r^{\mathsf{E}} + r) \sin\phi
\end{aligned}
$$

The cooridinates of satellites are

$$
\begin{aligned}
P_1 &= [20354, 0, 17079]^T \\
P_2 &= [24588, 4336, 9087]^T \\
P_3 &= [25769, 4544, -4614]^T \\
P_4 &= [24588, -4336, -9088]^T
\end{aligned}
$$

Homework 2: Solution

b) Please calculate the distance between satellites and GPS device.
Using the distance measurements, form the measurement equation like we did in the lecture.

Get the distances between satellites and GPS device,

$$\begin{aligned} d_1 &= c \cdot |t_1| &= 3 \times 10^8 \times 67.603 \times 10^{-3} &= 20281\text{km} \\ d_2 &= c \cdot |t_2| &= 3 \times 10^8 \times 70.102 \times 10^{-3} &= 21031\text{km} \\ d_3 &= c \cdot |t_3| &= 3 \times 10^8 \times 78.690 \times 10^{-3} &= 23607\text{km} \\ d_4 &= c \cdot |t_4| &= 3 \times 10^8 \times 83.942 \times 10^{-3} &= 24883\text{km} \end{aligned}$$

Measurement equation

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} ||\boldsymbol{x} - P_1|| \\ ||\boldsymbol{x} - P_2|| \\ ||\boldsymbol{x} - P_3|| \\ ||\boldsymbol{x} - P_4|| \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

is non-linear but it could not be sloved with linear least squares directly by reformulation.

c) Reformulate the non-linear measurement equation in 2b) into a linear measurement equation and calculate the least squares estimate.

Squared measurement equation:

$$\begin{aligned} d_i^2 &= ||\boldsymbol{x} - P_i||^2 + e_i^* \\ &= (\boldsymbol{x}_x - P_{i,x})^2 + (\boldsymbol{x}_y - P_{i,y})^2 + (\boldsymbol{x}_z - P_{i,z})^2 + e_i^* \\ &= \boldsymbol{x}_x^2 + \boldsymbol{x}_y^2 + \boldsymbol{x}_z^2 - 2\boldsymbol{x}_x P_{i,x} - 2\boldsymbol{x}_y P_{i,y} - 2\boldsymbol{x}_z P_{i,z} + P_{i,x}^2 + P_{i,y}^2 + P_{i,z}^2 + e_i^* \quad , \end{aligned}$$

where e_i^* is a new error term subsuming the transformed error e_i.
Substracting d_4^2 using d_1^2, we have,

$$\begin{aligned} d_1^2 - d_4^2 = & ||\boldsymbol{x}||^2 - ||\boldsymbol{x}||^2 - 2\begin{bmatrix} P_{1,x} - P_{4,x} & P_{1,y} - P_{4,y} & P_{1,z} - P_{4,z} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_x \\ \boldsymbol{x}_y \\ \boldsymbol{x}_z \end{bmatrix} \\ & + ||P_1||^2 - ||P_4||^2 + e_1^* - e_4^* \end{aligned}$$

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c) cont.

Moving $||P_1||^2 - ||P_4||^2$ to the left side of the equation, we have

$$\underbrace{d_1^2 - d_4^2 - ||P_1||^2 + ||P_4||^2}_{:=\boldsymbol{y}_{1,4}} = \underbrace{-2\begin{bmatrix} P_{1,x} - P_{4,x} & P_{1,y} - P_{4,y} & P_{1,z} - P_{4,z} \end{bmatrix}}_{:=\mathbf{H}_{1,4}} \underbrace{\begin{bmatrix} \boldsymbol{x}_x \\ \boldsymbol{x}_y \\ \boldsymbol{x}_z \end{bmatrix}}_{\boldsymbol{x}}$$

$$+ e_1^* - e_4^*$$

In the similar manner we could have

$$\begin{bmatrix} \boldsymbol{y}_{1,3} \\ \boldsymbol{y}_{2,4} \\ \boldsymbol{y}_{3,4} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{1,4} \\ \mathbf{H}_{2,4} \\ \mathbf{H}_{3,4} \end{bmatrix} \boldsymbol{x} + \boldsymbol{e}$$

e) Write a function which visualizes

- earth,
- distance measurement from each satellite as a sphere centred at each satellite.

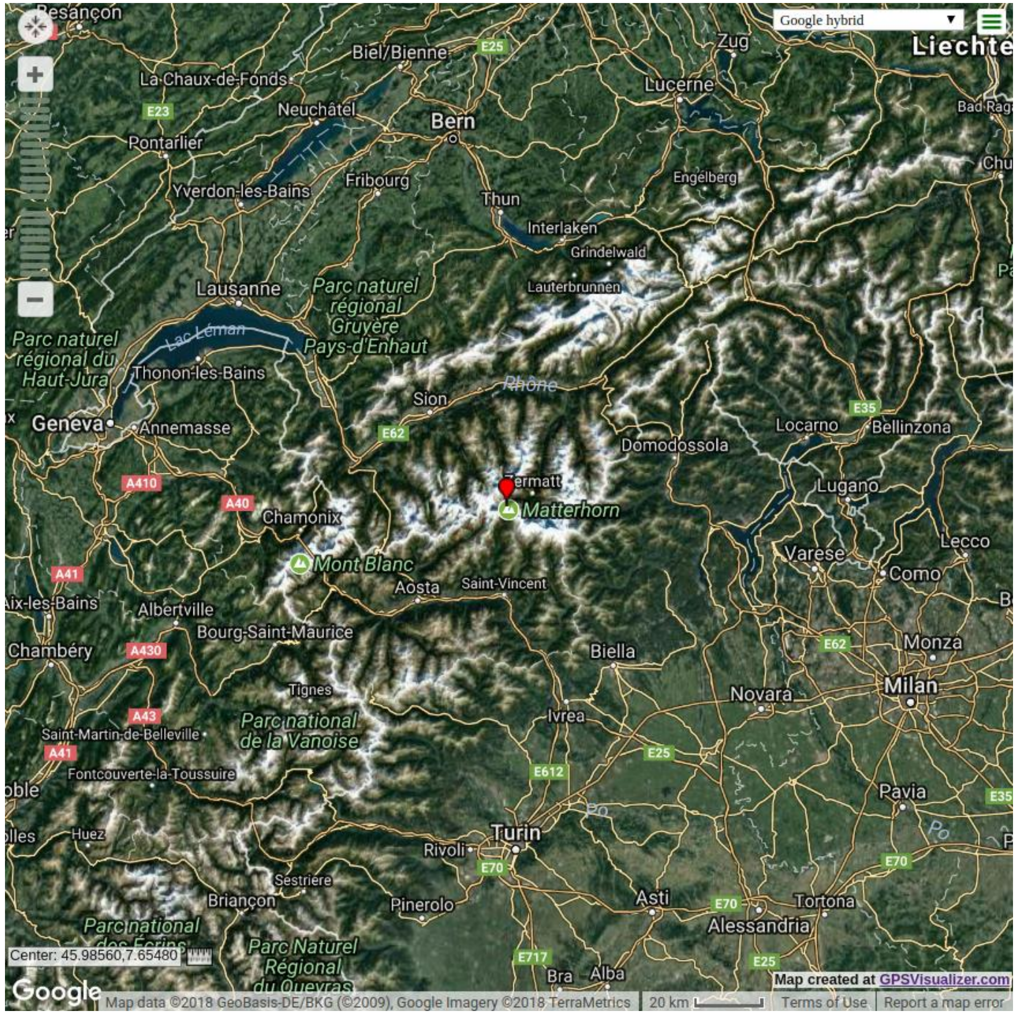
Rotate the figure you get, and find out the two intersection points of these distance measurements.

f) Write a function which converts the Cartesian coordinates into sphere coordinates. Use the function you implemented, calculate the longitude and latitude of the GPS device, and find out where it is using using *GPS Visualizer*:

http://www.gpsvisualizer.com/map?form=google

Homework 2: Solution

f) cont.



Real Random Variable

- Random variable $\boldsymbol{x} \in \mathbb{R}$:
  Random experiment whose outcome is a associated with a real number.

- Cumulative distribution function (CDF):

$$F_{\boldsymbol{x}}(x) = \text{Prob}(\boldsymbol{x} \leq x)$$

  where Prob is the probability measure

- $F_{\boldsymbol{x}}(x)$ monotone increasing and $\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to +\infty} F(x) = 1$

- Probability density function $p_{\boldsymbol{x}}(x)$:

$$F_{\boldsymbol{x}}(x) = \text{Prob}(\boldsymbol{x} \leq x) = \int_{-\infty}^{x} p_{\boldsymbol{x}}(u)\, du$$

- If $F_{\boldsymbol{x}}(x)$ differentiable:

$$p_{\boldsymbol{x}}(x) = \frac{d}{dx} F_{\boldsymbol{x}}(x)$$

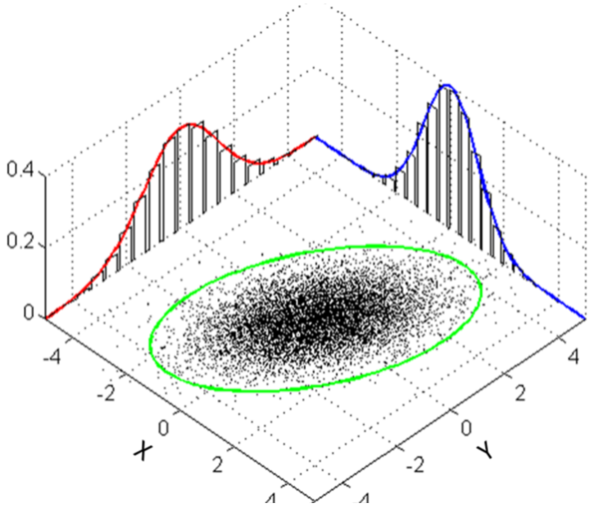
Real Random Vector (2D)

- Random vector $\boldsymbol{x} = \left[\boldsymbol{x}_1, \boldsymbol{x}_2\right]^T \in \mathbb{R}^2$: Column vector of scalar RVs
- Joint CDF function and joint PDF:

$$\text{Prob}(\boldsymbol{x}_1 < x_1, \boldsymbol{x}_2 < x_2) = F_{\boldsymbol{x}}(x) \quad = \quad \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} p_{\boldsymbol{x}}(u_1, u_2)\, du_1 du_2.$$

- If $F_{\boldsymbol{x}}(x)$ differentiable: $p(x) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x)$
- Marginal density:
  $p_{\boldsymbol{x}_1}(x_1) = \int p_{\boldsymbol{x}}(x_1, x_2)\, dx_2.$
- Independence: $p_{\boldsymbol{x}}(x_1, x_2) = p_{\boldsymbol{x}_1}(x_1) \cdot p_{\boldsymbol{x}_2}(x_2)$
- Conditioning on $\boldsymbol{x}_2$:

$$p_{\boldsymbol{x}_1}(x_1 | x_2) = \frac{p_{\boldsymbol{x}}(x_1, x_2)}{p_{\boldsymbol{x}_2}(x_2)}$$



Moments

- Expectation:

$$\mathrm{E}[\boldsymbol{x}] = \int x p(x) dx = \begin{bmatrix} \mathrm{E}[\boldsymbol{x}_1] \\ \mathrm{E}[\boldsymbol{x}_2] \end{bmatrix} \text{ with } \mathrm{E}[\boldsymbol{x}_i] = \int x_i p(x_i) dx_i$$

- Variance:
  $\mathrm{Var}[\boldsymbol{x}_i] = \mathrm{E}[(\boldsymbol{x}_i - \mathrm{E}[\boldsymbol{x}_i])^2]$

- Covariance:
  $\mathrm{Cov}[\boldsymbol{x}_1, \boldsymbol{x}_2] = \mathrm{E}[(\boldsymbol{x}_1 - \mathrm{E}[\boldsymbol{x}_1]) \cdot (\boldsymbol{x}_2 - \mathrm{E}[\boldsymbol{x}_2])]$

- Covariance matrix for n-dim. RV:

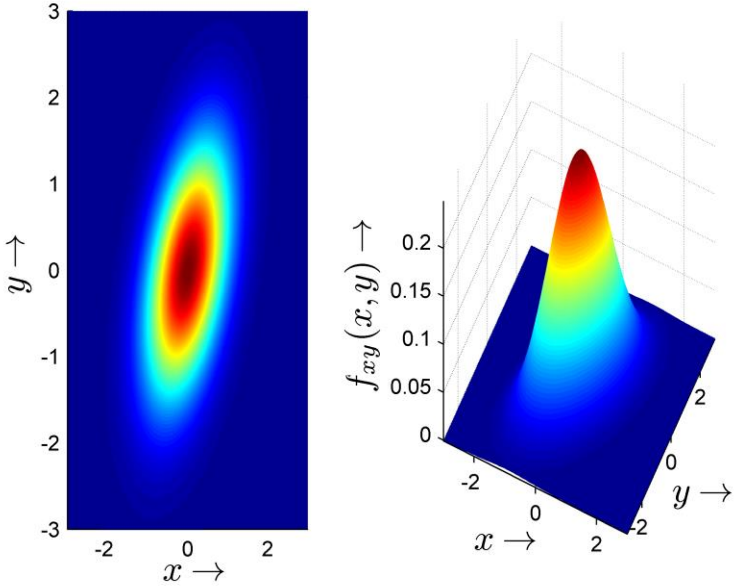
$$\begin{aligned} \mathrm{Cov}[\boldsymbol{x}] &= \mathrm{E}[(\boldsymbol{x} - \mathrm{E}[\boldsymbol{x}]) \cdot (\boldsymbol{x} - \mathrm{E}[\boldsymbol{x}])^T] \\ &= \begin{bmatrix} \mathrm{Var}[\boldsymbol{x}_1] & \mathrm{Cov}[\boldsymbol{x}_1, \boldsymbol{x}_2] & \dots & \mathrm{Cov}[\boldsymbol{x}_1, \boldsymbol{x}_n] \\ \mathrm{Cov}[\boldsymbol{x}_2, \boldsymbol{x}_1] & \mathrm{Var}[\boldsymbol{x}_2] & & \mathrm{Cov}[\boldsymbol{x}_2, \boldsymbol{x}_n] \\ \vdots & & \ddots & \vdots \\ \mathrm{Cov}[\boldsymbol{x}_n, \boldsymbol{x}_1] & \dots & & \mathrm{Var}[\boldsymbol{x}_n] \end{bmatrix} \end{aligned}$$

Multivariate Gaussian Distribution

n-dimensional Gaussian distribution, i.e., $\boldsymbol{x} \sim N(\mu, \mathbf{C})$

- Mean $\mu \in \mathbb{R}^n$
- Covariance matrix $\mathbf{C} \in \mathbb{R}^{n \times n}$

$$p(x) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(x-\mu)^T \mathbf{C}^{-1}(x-\mu)\right)$$



Linear Transformation of a RV (Expectation)

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- $\boldsymbol{x} \in \mathbb{R}^n$
- $\mathrm{E}[\boldsymbol{x}] = \hat{x}$
- $\mathrm{Cov}[\boldsymbol{x}] = \mathbf{C}$
- $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$

$$
\begin{aligned}
\mathrm{E}[\mathbf{A}\boldsymbol{x} + b] &= \int_{\mathbb{R}^n} (\mathbf{A}x + b)p(x)\,dx \\
&= \mathbf{A}\int_{\mathbb{R}^n} xp(x)\,dx + b\underbrace{\int_{\mathbb{R}^n} p(x)dx}_{=1} \\
&= \mathbf{A}\hat{x} + b \quad .
\end{aligned}
$$

Linear Transformation of a RV (Covariance)

$$
\begin{aligned}
\mathrm{Cov}[\mathbf{A}\boldsymbol{x} + b] &= \mathrm{E}\{(\mathbf{A}\boldsymbol{x} + b - (\mathbf{A}\hat{x} + b))\,(\mathbf{A}\boldsymbol{x} + b - (\mathbf{A}\hat{x} + b))^T\} \\
&= \mathrm{E}\{(\mathbf{A}\boldsymbol{x} - \mathbf{A}\hat{x})\,(\mathbf{A}\boldsymbol{x} - \mathbf{A}\hat{x})^T\} \\
&= \mathrm{E}\{\mathbf{A}\,(\boldsymbol{x} - \hat{x})\,(\boldsymbol{x} - \hat{x})^T\,\mathbf{A}^T\} \\
&= \mathbf{A}\,\mathrm{Cov}[\boldsymbol{x}]\mathbf{A}^T \\
&= \mathbf{A}\mathbf{C}\mathbf{A}^T
\end{aligned}
$$

Important: If x is Gaussian $\mathbf{A}x + b$ is Gaussian as well (no proof)

Interpretation: Covariance Matrix (2D)

- Eigenvalue decomposition of SPD matrix $\mathbf{C}$:

$$\mathbf{C} = \mathbf{R}\mathbf{D}\mathbf{R}^T$$

with diagonal matrix $\mathbf{D}$ and rotation matrix $\mathbf{R}$

$$\Rightarrow \quad p(x) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{R}^T(x-\mu))^T \mathbf{D}^{-1} (\mathbf{R}^T(x-\mu))\right)$$

- Interpretation of $p(x) = c$?
- With $\mathbf{D} = \mathrm{diag}(\sigma_1^2, \sigma_2^2)$ and $z := \mathbf{R}^T(x-\mu)$

$$z^T(\mathbf{D})^{-1}z = \tilde{c} \quad \Leftrightarrow \quad \frac{1}{\sigma_1^2}z_1^2 + \frac{1}{\sigma_2^2}z_2^2 = \tilde{c}$$

which is the equation of a scaled ellipse.

Nonlinear Least Squares

Nonlinear Measurement Equation with Additive Noise:

$$y = h(x) + e$$

Assumptions:

- Setting 1
- Overdetermined, $m > n$

Objective:

$$x^{LS} = \arg\min_x ||\underbrace{y - h(x)}_{e}||^2_{\mathbf{W}}$$

General Solution Approaches:

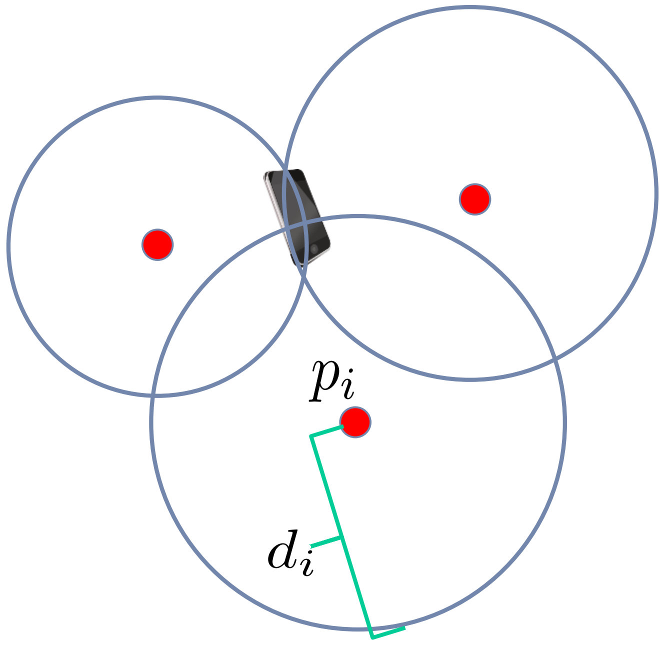
- Iterative optimization
- Closed-form approximation by reformulation and linear LQ

Trilateration: Problem Formulation (2D)

- **Desired:** Receiver location $x \in \mathbb{R}^2$
- **Given:** Distances to m landmarks:
  - Position $p_i = \left[p_{i,1}, p_{i,2}\right]^T \in \mathbb{R}^2$
  - Distance $d_i \in \mathbb{R}$ to landmark i
- **Indiv. measurement equation:**

  $d_i = ||x - p_i|| + e_i$

  with measurement error $e_i \in \mathbb{R}$.
- **Stacked measurement equation:**



$$\underbrace{\begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}}_{=:y} = \underbrace{\begin{bmatrix} ||x - p_1|| \\ \vdots \\ ||x - p_m|| \end{bmatrix}}_{=:h(x)} + \underbrace{\begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}}_{=:e}$$

Trilateration: Bancroft Solution (1)

- Squared meas. equation:

$$\begin{aligned} d_i^2 &= ||x - p_i||^2 + e_i^* \\ &= (x_1 - p_{i,1})^2 + (x_2 - p_{i,2})^2 + e_i^* \\ &= -2x_1 p_{i,1} - 2x_2 p_{i,2} + ||p_i||^2 + R^2 + e_i^* \end{aligned}$$

with $R^2 := ||x||^2 = (x_1)^2 + (x_2)^2$

- Linear measurement equation for given R^2:

$$y = \mathbf{H}_1 x + \mathbf{H}_2 R^2 + e^*$$

with

$$y = \begin{bmatrix} d_1^2 - ||p_1||^2 \\ \vdots \\ d_m^2 - ||p_m||^2 \end{bmatrix}, \quad \mathbf{H}_1 = \begin{bmatrix} -2p_{i,1} & -2p_{i,2} \\ \vdots & \vdots \\ -2p_{m,1} & -2p_{m,2} \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Trilateration: Bancroft Solution (2)

- Least squares solution for a fixed R^2:

$$\begin{aligned} x^{LS}(R^2) &= (\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1)^{-1}\mathbf{H}_1^{\mathrm{T}}(y - \mathbf{H}_2 R^2) \\ &= z_1 + R^2 z_2 \end{aligned}$$

  with $z_1 := (\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1)^{-1}\mathbf{H}_1^{\mathrm{T}} y$ and $z_2 := -(\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1)^{-1}\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_2$

- What is R^2?

$$\begin{aligned} R^2 &= ||x^{LS}(R^2)||^2 \\ &= (z_1 + R^2 z_2)^{\mathrm{T}} \cdot (z_1 + R^2 z_2) \end{aligned}$$

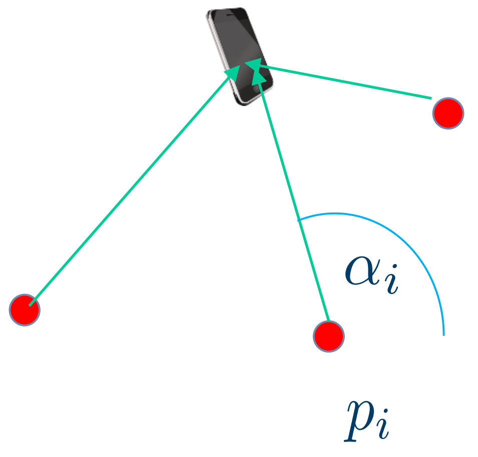
- Solve the following quadratic equation for R^2:

$$0 = z_1^{\mathrm{T}} z_1 + z_1^{\mathrm{T}} z_2 R^2 + R^2 z_2^{\mathrm{T}} z_1 + (R^2)^2 z_2^{\mathrm{T}} z_2 - R^2$$

Triangulation: Problem Formulation (2D)

- **Desired:** Cartesian object position $x = \left[x_1, x_2\right]^T \in \mathbb{R}^2$
- **Given:** m angular measurements to m landmarks:
  - Position $p_i = \left[p_{i,1}, p_{i,2}\right]^T \in \mathbb{R}^2$
  - Angle $\alpha_i \in [-\pi, \pi]$ to landmark i

  for $i = 1, \ldots, m$



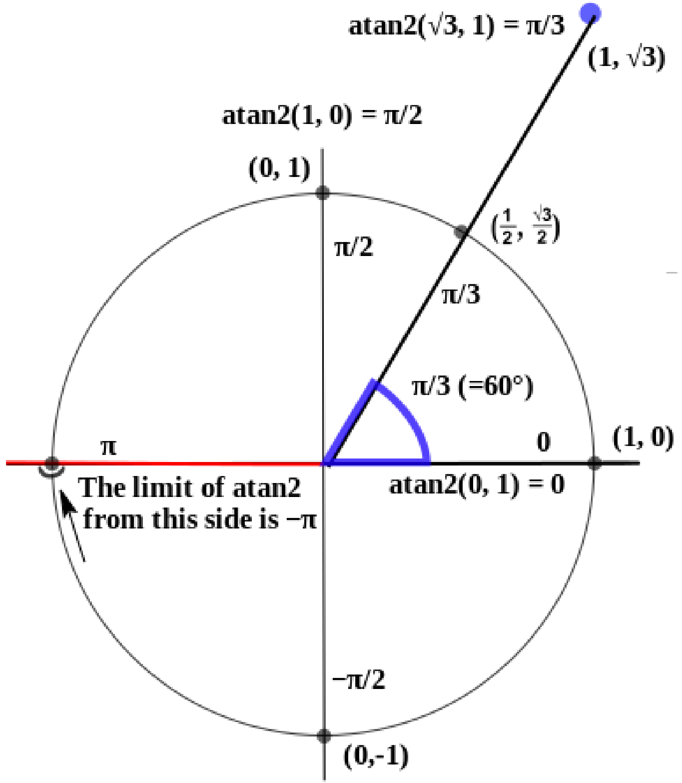
- **Stacked measurement equation:**

$$\underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{=:y} = \underbrace{\begin{bmatrix} \text{atan2}(x_2 - p_{1,2}, x_1 - p_{1,1}) \\ \vdots \\ \text{atan2}(x_2 - p_{m,2}, x_1 - p_{m,1}) \end{bmatrix}}_{=:h(x)} + \underbrace{\begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}}_{=:e}$$

Function atan2

- Four quadrant tangent inverse
- Returns values in $(-\pi, \pi]$
- Standard inverse of tangent returns only values in $(-\pi/2, \pi/2)$

$$\operatorname{atan2}(y, x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$



https://en.wikipedia.org/wiki/Atan2

Closed-Form Solution using Linear LQ

- **Reformulation:**

$$(x_1 - p_{i,1}) \cdot \mathrm{tan}(\alpha_i) = x_2 - p_{i,2}$$

- **Linear measurement equation:**

$$\underbrace{\begin{bmatrix} p_{1,1}\mathrm{tan}(\alpha_1) - p_{1,2} \\ \vdots \\ p_{m,1}\mathrm{tan}(\alpha_m) - p_{m,2} \end{bmatrix}}_{=:y} = \underbrace{\begin{bmatrix} \mathrm{tan}(\alpha_1) & -1 \\ \vdots & \vdots \\ \mathrm{tan}(\alpha_m) & -1 \end{bmatrix}}_{=:\mathbf{H}} x + \underbrace{\begin{bmatrix} e_1^* \\ \vdots \\ e_m^* \end{bmatrix}}_{=:e^*}$$

- **Problem:**
  Measurement part of the measurement matrix, which introduces additional errors.

Problem 3 - Triangulation

Assume you receive the following angular measurements in radian from two transmitters:

$$\alpha_1 = \frac{\pi}{4}, \mathbf{p}_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$$
$$\alpha_2 = \frac{3\pi}{4}, \mathbf{p}_2 = \begin{bmatrix} 4 & 0 \end{bmatrix}^{\mathrm{T}}$$

Calculate your position.
Hint: $\tan(0) = 0$, $\tan(\frac{\pi}{4}) = 1$, $\tan(\frac{3\pi}{4}) = -1$.

Visualize the measurements to confirm your results.

Next, assume a third measurement is received

$$\alpha_3 = 0, \mathbf{p}_3 = \begin{bmatrix} 0 & 2.5 \end{bmatrix}^{\mathrm{T}}$$

Update the estimate of your position.

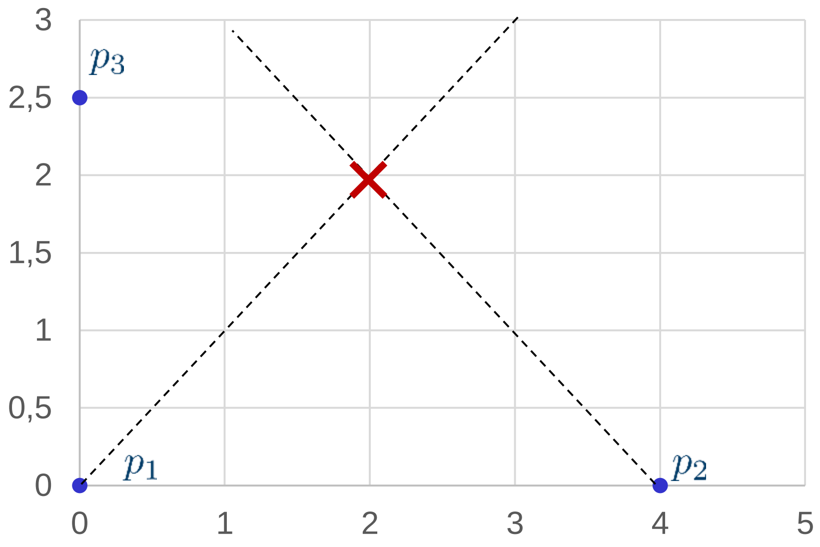
Problem 3: Solution

Reformulating, we get

$$\mathbf{y} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

Applying least squares:

$$\mathbf{x}^{LS} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$



Linear measurement equation:

$$\underbrace{\begin{bmatrix} p_{1,1}\tan(\alpha_1) - p_{1,2} \\ \vdots \\ p_{m,1}\tan(\alpha_m) - p_{m,2} \end{bmatrix}}_{=:y} = \underbrace{\begin{bmatrix} \tan(\alpha_1) & -1 \\ \vdots & \vdots \\ \tan(\alpha_m) & -1 \end{bmatrix}}_{=:\mathbf{H}} x + \underbrace{\begin{bmatrix} e_1^* \\ \vdots \\ e_m^* \end{bmatrix}}_{=:e^*}$$

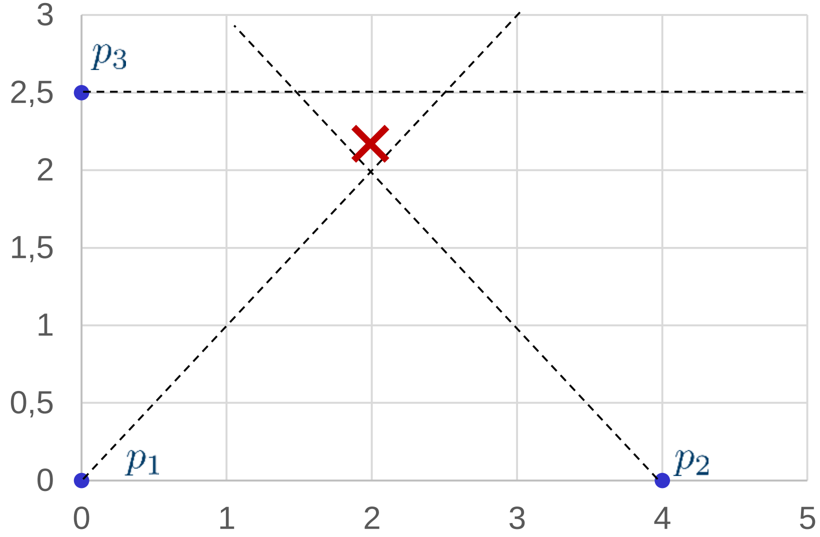
Problem 3: Solution

Adding the new measurement, we get

$$\mathbf{y} = \begin{bmatrix} 0 \\ -4 \\ -2.5 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix}$$

Applying least squares:

$$\mathbf{x}^{LS} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6.5 \end{bmatrix} \approx \begin{bmatrix} 2 \\ 2.17 \end{bmatrix}$$

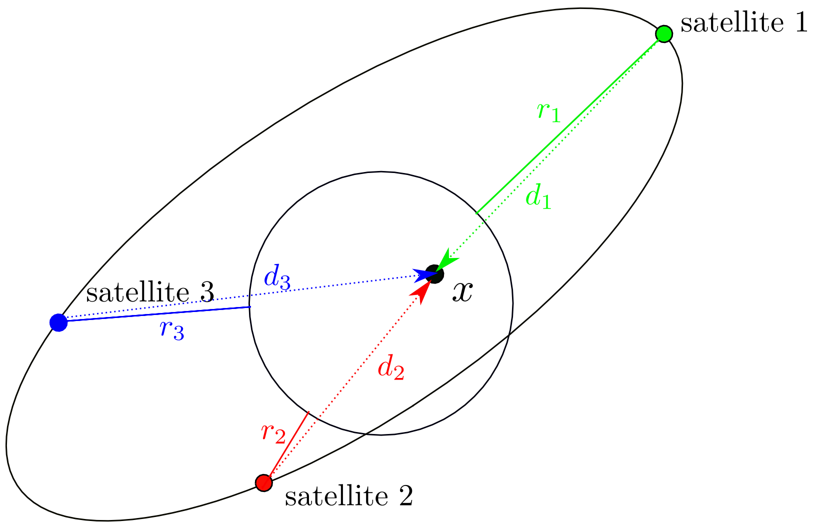


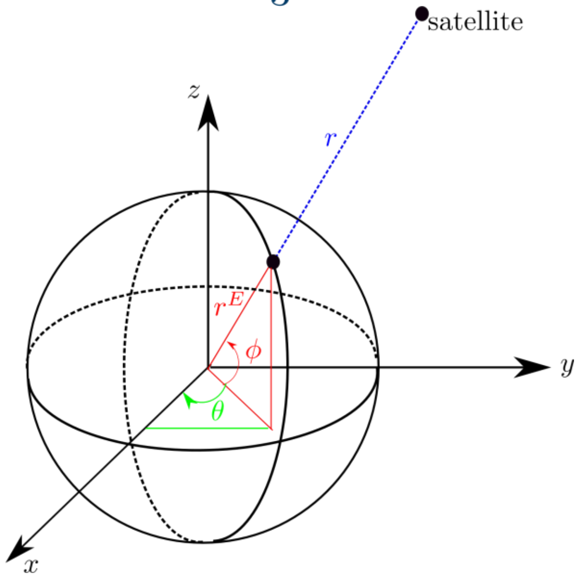
Homework 3

Look again at the GPS setting of last homework (sattelites are 20200km above mean sea level, each satellite broadcasts its location in spherical coordinates $[\theta, \phi, r]^T$ plus the emission time (see figures below), assume that the speed-of-light is $c = 3 \cdot 10^8 \mathrm{m\,s}^{-1}$ and the Earth is an ideal sphere with radius $r^{\mathrm{E}} = 6370\text{km}$)

This time, the device only receives the first three sattelite signals at time $t = 0\text{s}$:

$$p_1 = [0°, 40°, 20200\text{km}]^T, \qquad t_1 = -67.603\text{ms},$$
$$p_2 = [10°, 20°, 20200\text{km}]^{\mathrm{T}}, \qquad t_2 = -70.102\text{ms},$$
$$p_3 = [10°, -10°, 20200\text{km}]^{\mathrm{T}}, \qquad t_3 = -78.690\text{ms}.$$





Homework 3

a) Would the reformulation from last times still work? If not, explain why.

b) Instead of reformulating the measurement equation into a linear equation, implement the Gauss-Newton method to calculate the least squares solution.
Hint:
1. you will need to use symbolic python
2. you will need to use sym.Matrix instead of np.array for most variables: convert np arrays into sympy as necessary, the two packages don't mix well
3. sympy provides .jacobian(...) for calculating the Jacobian
4. sympy provides .subs(...) for symbolic substitution

c) Alternatively, use the Bancroft solution to determine the position of the GPS device.
Hint: numpy provides numpy.roots() to find the roots of a polynomial