Sensor Data Fusion

Exercise 13

Prof. Dr.-Ing. Marcus Baum
M.Sc. Kolja Thormann

GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

DATA
FUSION

Today

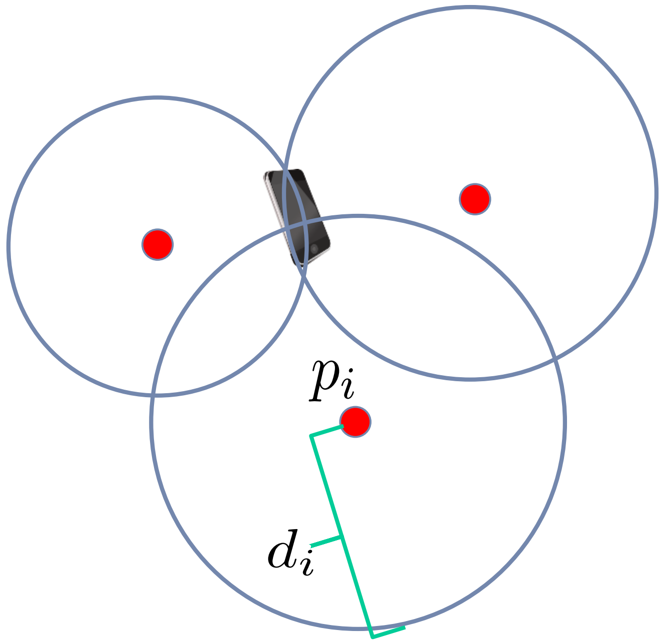
- Repitition: Bancroft
- Sample exam 2

Trilateration: Problem Formulation (2D)

- **Desired:** Receiver location $x \in \mathbb{R}^2$
- **Given:** Distances to m landmarks:
  - Position $p_i = \left[p_{i,1}, p_{i,2}\right]^T \in \mathbb{R}^2$
  - Distance $d_i \in \mathbb{R}$ to landmark i
- **Indiv. measurement equation:**

  $d_i = ||x - p_i|| + e_i$

  with measurement error $e_i \in \mathbb{R}$.
- **Stacked measurement equation:**



$$\underbrace{\begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}}_{=:y} = \underbrace{\begin{bmatrix} ||x - p_1|| \\ \vdots \\ ||x - p_m|| \end{bmatrix}}_{=:h(x)} + \underbrace{\begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}}_{=:e}$$

Trilateration: Bancroft Solution (1)

- Squared meas. equation:

$$\begin{aligned} d_i^2 &= ||x - p_i||^2 + e_i^* \\ &= (x_1 - p_{i,1})^2 + (x_2 - p_{i,2})^2 + e_i^* \\ &= -2x_1 p_{i,1} - 2x_2 p_{i,2} + ||p_i||^2 + R^2 + e_i^* \end{aligned}$$

  with $R^2 := ||x||^2 = (x_1)^2 + (x_2)^2$

- Linear measurement equation for given R^2:

$$y = \mathbf{H}_1 x + \mathbf{H}_2 R^2 + e^*$$

  with

$$y = \begin{bmatrix} d_1^2 - ||p_1||^2 \\ \vdots \\ d_m^2 - ||p_m||^2 \end{bmatrix} \; , \; \mathbf{H}_1 = \begin{bmatrix} -2p_{i,1} & -2p_{i,2} \\ \vdots & \vdots \\ -2p_{m,1} & -2p_{m,2} \end{bmatrix} \; , \; \mathbf{H}_2 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Trilateration: Bancroft Solution (2)

- Least squares solution for a fixed R^2:

$$\begin{aligned} x^{LS}(R^2) &= (\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1)^{-1}\mathbf{H}_1^{\mathrm{T}}(y - \mathbf{H}_2 R^2) \\ &= z_1 + R^2 z_2 \end{aligned}$$

with $z_1 := (\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1)^{-1}\mathbf{H}_1^{\mathrm{T}}y$ and $z_2 := -(\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1)^{-1}\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_2$

- What is R^2?

$$\begin{aligned} R^2 &= ||x^{LS}(R^2)||^2 \\ &= (z_1 + R^2 z_2)^{\mathrm{T}} \cdot (z_1 + R^2 z_2) \end{aligned}$$

- Solve the following quadratic equation for R^2:

$$0 = z_1^{\mathrm{T}} z_1 + z_1^{\mathrm{T}} z_2 R^2 + R^2 z_2^{\mathrm{T}} z_1 + (R^2)^2 z_2^{\mathrm{T}} z_2 - R^2$$

Sample Exam 2

Sample Exam 2

What are the assumptions for least squares?

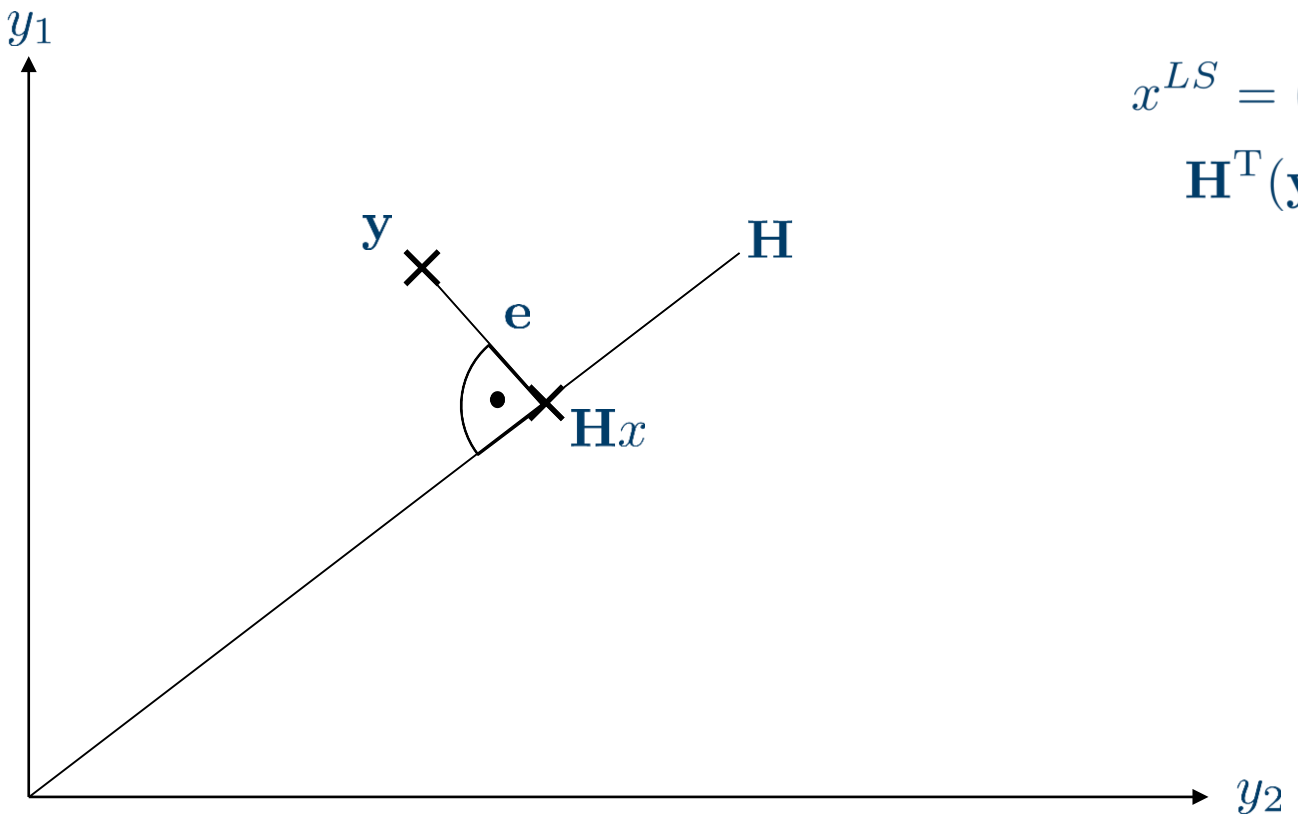
$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{e}$$

$$\mathbf{H} \in \mathbb{R}^{m \times n}, \quad m \geq n, \quad rank(\mathbf{H}) = n$$

Sample Exam 2

Can you explain the normal equation intuitively?

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{H}} x + \mathbf{e}$$



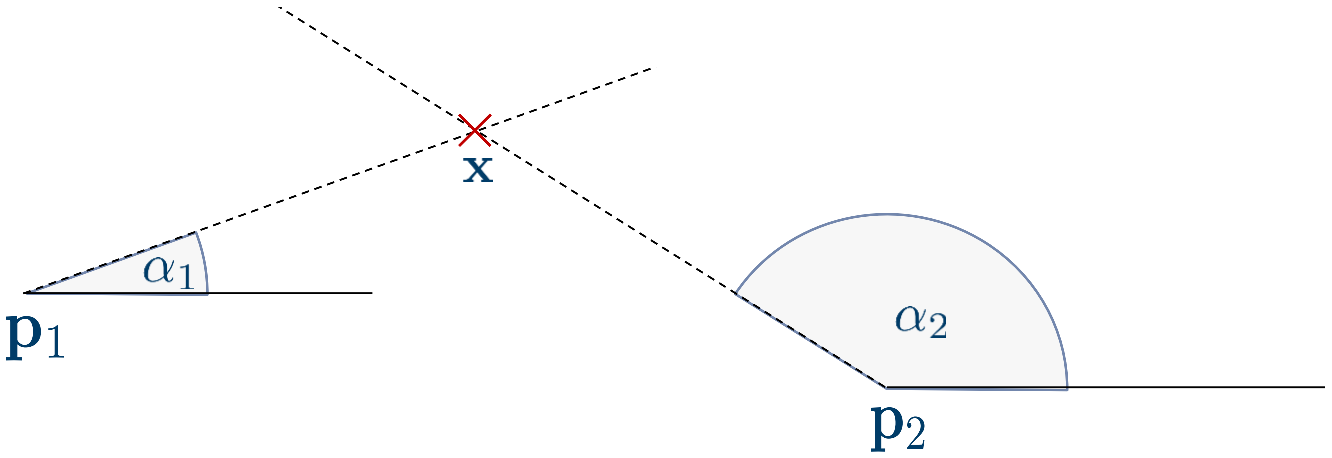
for $\mathbf{W} = \mathbf{I}$, we have

$$x^{LS} = (\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{y}$$

$$\mathbf{H}^{\mathrm{T}}(\mathbf{y} - \mathbf{H}x) = 0$$

Sample Exam 2

Can you explain how Triangulation works? Why can you not use the normal equation to solve it?



As the measurement equation is non-linear, we are unable to apply least square formula. We can still solve the problem by applying a closed-form solution or using iterative optimization.

Sample Exam 2

What options are there to solve non-linear least squares?

You can use iterative optimization, which are precise but suffer from local minima, or closed-form solutions, which are computationally efficient but can be bad if the noise is too high. A closed-form approximation can be used as an initial guess for an iterative optimization method.

Sample Exam 2

What is an estimator?

An estimator is a function $\theta_y : \mathbb{R}^m \to \mathbb{R}^n$ providing an estimate for the desired variable $\mathbf{x}$ based on an observation $\mathbf{y}$. A linear estimator has the form

$$\theta_y = \mathbf{K}\mathbf{y} + \mathbf{b} \ .$$

The quality of an estimator can be measured using the mean squared error (MSE)

$$MSE(\theta_y) = \mathrm{E}[||\theta_y - \mathbf{x}||^2] \ .$$

Sample Exam 2

How can the MSE be decomposed in the Fisher approach?

Note: Fisher approach, so no statistical information about $\mathbf{x}$.

$$
\begin{aligned}
\mathrm{E}[||\theta_y - \mathbf{x}||^2] &= \mathrm{E}[\mathrm{Tr}\left((\theta_y - \mathbf{x})(\theta_y - \mathbf{x})^{\mathrm{T}}\right)] \\
&= \mathrm{Tr}\left(\mathrm{E}[\theta_y^2 - 2\theta_y \mathbf{x} + \mathbf{x}^2)\right] \\
&= \mathrm{Tr}\left(\mathrm{E}[\theta_y^2] - 2\,\mathrm{E}[\theta_y]\mathbf{x} + \mathbf{x}^2 + \mathrm{E}[\theta_y]^2 - \mathrm{E}[\theta_y]^2\right) \\
&= \mathrm{Tr}\left(\mathrm{E}[\theta_y^2] - \mathrm{E}[\theta_y]^2\right) + \mathrm{Tr}\left(\mathrm{E}[\theta_y]^2 - 2\,\mathrm{E}[\theta_y]\mathbf{x} + \mathbf{x}^2\right) \\
&= \underbrace{\mathrm{Tr}\left(\mathrm{Cov}[\theta_y]\right)}_{\text{Covariance}} + \underbrace{||\,\mathrm{E}[\theta_y] - \mathbf{x}||^2}_{\text{Bias}}
\end{aligned}
$$

Sample Exam 2

What is the optimal Bayesian estimator and how can you compute it?

- $\theta_y^{opt} = \mathrm{E}[\mathbf{x}|\mathbf{y}]$
- Given $p(\mathbf{x})$, $p(\mathbf{y}|\mathbf{x})$
- $\mathrm{E}[\mathbf{x}|\mathbf{y}] = \int \mathbf{x}\underline{p(\mathbf{x}|\mathbf{y})}\mathrm{d}\mathbf{x}$

likelihood prior

- $p(\mathbf{x}|\mathbf{y}) = \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{y})}$

posterior

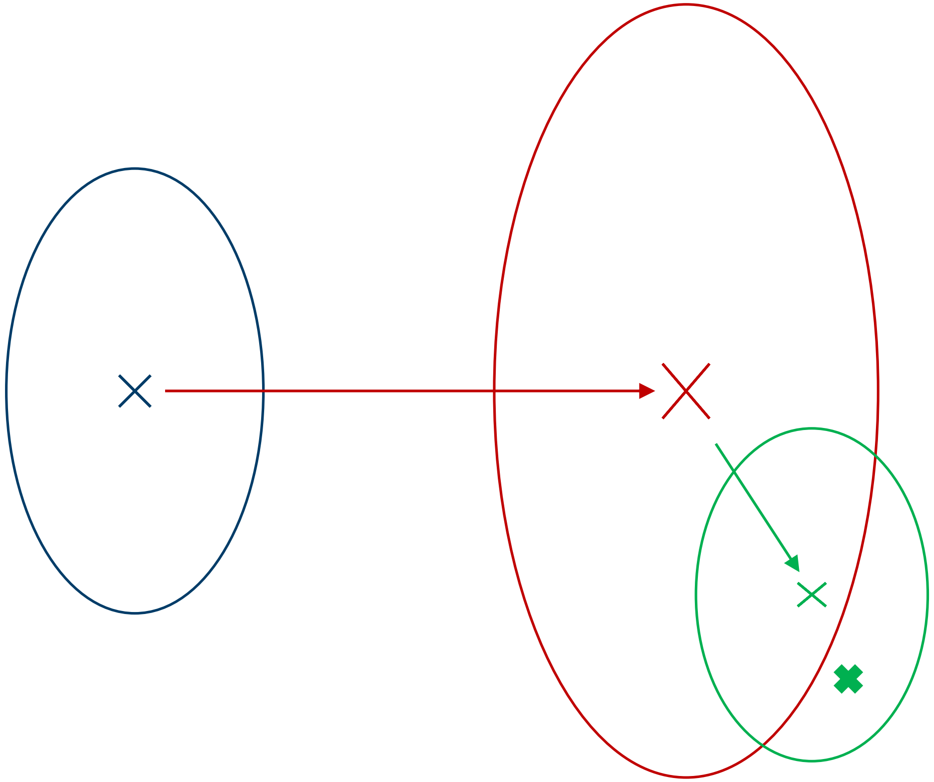
Sample Exam 2

What can be said about the posterior distribution if all other distributions involved are Gaussian?

The posterior $p(x|y)$ is also Gaussian.

Sample Exam 2

Can you intuitively explain the Kalman filter?



Prediction

Update

Sample Exam 2

What happens in the Kalman filter update for **H=I** if the noise of the prior is much higher than the measurement noise and vice versa?

- Special cases for $\mathbf{H} = \mathbf{I}$, i.e.,

$$\hat{x}_{k+1} = \hat{x}_k + \mathbf{C}_k(\mathbf{C}_k + \mathbf{C}_{ee})^{-1}(y_k - \hat{x}_k)$$
$$\mathbf{C}_{k+1} = \mathbf{C}_k - \mathbf{C}_k(\mathbf{C}_k + \mathbf{C}_{ee})^{-1}\mathbf{C}_k$$

$$\hat{x}_{k+1} = \hat{x}_k + \mathbf{C}_{k+1}\mathbf{C}_{ee}^{-1}(y_k - \hat{x}_k)$$
$$\mathbf{C}_{k+1} = (\mathbf{C}_k^{-1} + \mathbf{C}_{ee}^{-1})^{-1}$$

- $\mathrm{Tr}[\mathbf{C}_{ee}] \to \infty$

$$\hat{x}_{k+1} = \hat{x}_k$$
$$\mathbf{C}_{k+1} = \mathbf{C}_k$$

- $\mathrm{Tr}[\mathbf{C}_k] \to \infty$

$$\hat{x}_{k+1} = y_k$$
$$\mathbf{C}_{k+1} = \mathbf{C}_{ee}$$

Sample Exam 2

What is the idea of the EKF? Describe it using the time update formulas.

- Setup: nonlinear transition function $\mathbf{x}_{k+1} = a(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}$ with $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$
- Problem: we require a transition matrix $\mathbf{A}_k$ to transform the state covariance
- Solution: linearization around estimate $\hat{\mathbf{x}}_k$

$$a(\mathbf{x}_k, \mathbf{u}_k) \approx a(\hat{\mathbf{x}}_k, \mathbf{u}_k) + \mathbf{A}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k),\ \mathbf{A}_k = \frac{\partial a(\mathbf{x}_k, \mathbf{u}_k)}{\partial \mathbf{x}_k}$$

$$\hat{\mathbf{x}}_{k+1} = a(\hat{\mathbf{x}}_k, \mathbf{u}_k)$$
$$\mathbf{C}_{k+1} = \mathbf{A}_k \mathbf{C}_k \mathbf{A}_k^{\mathrm{T}} + \mathbf{Q}$$

Sample Exam 2

What are disadvantages of the EKF?

- Might diverge for severe nonlinearities
- Need to compute Jacobians

Sample Exam 2

How can you classify uncertain information?

- Imprecise information, e.g., more than ..., an interval, ...
- Affected by random effect, e.g., probability funtion
- Vague information, e.g., approximate values
- Erroneous information, e.g., systematic errors

Sample Exam 2

What is the difference between the random set based method to the classical approach?

The random set based methods replace the likelihood by a generalized likelihood, which describes the probability of the measurement belonging to a set. Each state x is then mapped to a set in the measurement space.