Sensor Fusion: Formelsammlung

Trilateration: closed form solution

$$d_1^2 = \|x - p_1\|^2 = \text{~~~~~~}$$ first square

$$d_m^2 = \|x - p_m\|^2 = \text{~~~~~~}$$

subtract to remove x^2

=> stacked measurement equation $y^* = H^* x + e^*$ $(e^* \neq e)$

$$y^* = \begin{bmatrix} d_1^2 - d_m^2 - \|p_1\|^2 + \|p_m\|^2 \\ \vdots \\ d_{m-1}^2 - d_m^2 \|p_{m-1}\|^2 + \|p_m\|^2 \end{bmatrix}$$

$$H^* = 2 \cdot \begin{bmatrix} p_{m,1} - p_{1,1} & p_{m,2} - p_{1,2} \\ \vdots & \vdots \\ p_{m,1} - p_{m-1,1} & p_{m,2} - p_{m-1,2} \end{bmatrix}$$

=> Linear LS solution possible

Multilateration

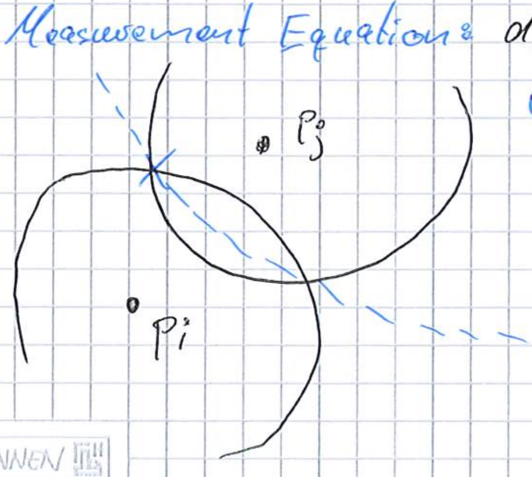
- Given landmark positions: p
- Time-Difference-of-Arrival (TDoA measurements)

$$\Delta T_{i,j} = t_i - t_j$$
$$= t_i + \cancel{b} - (t_j + \cancel{b})$$

unknown clock bias cancels out

- Measurement Equation: $d_{i,j} = \|x - p_i\| - \|x - p_j\| + e_j$

↳ results in Hyperbola



Sensor Fusion: Formelsammlung

(Weighted) Least squares:

- State: ~~Quantity~~ $x \in \mathbb{R}^n$ (unknown, desired)
- Measurement $y \in \mathbb{R}^m$ (given)
- Meas. Error $e \in \mathbb{R}^m$ (unknown)

- Linear Measurement equation: $y = Hx + e$
- Assumptions: $m \geq n$ and $\text{rank}(H) = n$
- Objective find an x^{LS} so that: $x^{LS} = \arg\min_x \underbrace{\|y - Hx\|_W^2}_{G(x)}$
- Cost function: $G(x) = (y - Hx)^T W (y - Hx)$
- $\underbrace{\text{Normal equation}}_{G'(x^{LS}) \overset{!}{=} 0}$: $H^T W y = H^T W H x$

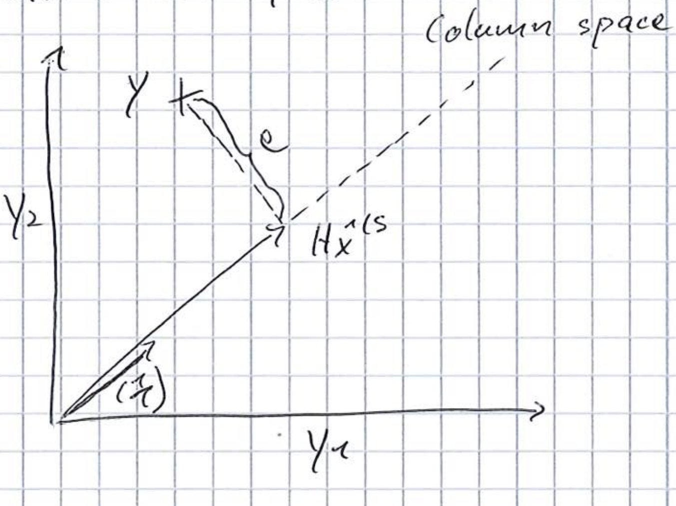
  $\hookrightarrow x^{LS} = (H^T W H)^{-1} H^T W y$ } as H is full rank

Normal Equation - Intuition: Example

- state $x \in \mathbb{R}$
- Meas. $y \in \mathbb{R}^2$

Meas. Equation:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$



Review - Positive Definite Matrices

- symmetric matrix: $A = A^T$
- positive matrix: all eigenvalues positive
- definite matrix: all eigenvalues greater 0

-> a positive definite matrix W can be written as $W = CDC^T$

where $C \in \mathbb{R}^{n \times n}$ is a rotation matrix ($C^T C = CC^T = I$)

D is a diagonal matrix of Eigenvalues

Extended Kalman Filter: Time Update

- Linearization at current estimate $\hat{x}_{k-1}$

$$a(x_{k-1}, u_k) \approx a(\hat{x}_{k-1}, u_k) + A_k (x_{k-1} - \hat{x}_{k-1})$$

with Jacobi Matrix: $A_k = \frac{\partial a}{\partial x_{k-1}} (\hat{x}_{k-1}, u_k)$

- Kalman Filter prediction:

$$\hat{x}_{k|k-1} = a(\hat{x}_{k-1}, u_k)$$

$$C^{xx}_{k|k-1} = A_k C^{xx}_{k-1} A^T_k + \underbrace{C^{ww}_k}_{\text{system noise}}$$

EKF: Measurement Update

- linearization at point $\hat{x}_{k|k-1}$

$$h_k(x_k) \approx h_k(\hat{x}_{k|k-1}) + H_k (x_k - \hat{x}_{k|k-1})$$

with Jacobi Matrix: $H_k = \frac{\partial h_k}{\partial x_k} (\hat{x}_{k|k-1})$

- Measurement Update

$$\hat{x}_k = \hat{x}_{k|k-1} + K_k (y_k - h(\hat{x}_{k|k-1}))$$

$$C^{xx}_k = C^{xx}_{k|k-1} - K_k H_k C^{xx}_{k|k-1}$$

$$K_k = C^{xx}_{k|k-1} H^T_k (H_k C^{xx}_{k|k-1} H^T_k + C^{vv}_k)^{-1}$$

Sensor Data Fusion:

Question: What happens in the Kalman filter update for $H=1$ if the noise of the prior is much higher than the measurement noise and vice versa?

→ Regard the update formula

$$x_{k+1|k+1} = x_{k+1|k} + \underbrace{C_{k+1|k} H^T \overbrace{(H C_{k+1|k} H^T + R)}^{S}{}^{-1}}_{K} (y - H x_{k+1|k})$$

With the ~~prior noise~~ prior noise $C_{k+1|k}$ being much larger than the measurement noise R, KH gets close to being the identity matrix, so that the state is subtracted & the update will only use ~~the~~ the measurement to set the new state. Vice versa, S^{-1} would be close to zero, leaving only the prior state.

Sensor Fusion

Bayesian Approach:

- prior: $p(x)$
- likelihood: $p(y|x)$
- joint probability density function: $p(x,y) = p(y|x) \cdot p(x)$

Bayesian Mean Square Error:

$$BMSE(\theta_y) = E_{x,y}\{\|\underbrace{\theta_y - x}_{\epsilon}\|^2\}$$

$$= \iint \|\theta_y - x\|^2 \, p(x,y) \, dx \, dy$$

$$= \iint \|\theta_y - x\|^2 \, p(y|x) \, dy \cdot p(x) \, dx$$

Bayesian Bias: $\beta = E_{x,y}\{\epsilon\} = E(x) - E(\theta_y)$

↳ unbiased: $E_{x,y}\{\epsilon\} = 0$

$$BMSE(\theta_y) = Tr(Cov(\epsilon)) + Tr\, \beta\beta^T$$

$\beta = E(\theta_y) - E(x)$

Optimal Bayesian Estimator:

- $\theta_y^{opt} = E(x|y)$
- given $p(x), p(y|x)$
- $E(x|y) = \int x \cdot p(x|y) \, dx$
- $p(x|y) = \dfrac{p(y|x) \cdot p(x)}{p(y)}$ ← prior

likelihood

evidence

↑ Different notations on the back of the page

Problem:

- no general closed form solution (Integral in higher Dim is difficult)
- Complete description of prior & likelihood required.

Sensor Fusion:

Uncertain (Imperfect) Information:

- Imprecise Information:
  - expressed through / as a set of possible values (crisp set)
  - Example:
    - more than 270 m / in the interval from 300-350 m
- aleatory uncertainty: Information affected by variability due to random effects
  - typically expressed by a probability distribution
  - Example: Probability Mass Function (PMF) whose support is {321m, 322m, 323m} with the probabilities 0.4, 0.5 & 0.1
- epistimic uncertainty: erroneous information
  - Model mismatch situation
  - Systemic errors
- vague information
  - typically expressed as a fuzzy set
  - Example (vague & imprecise): The height is at least approximately 325m

Fuzzy Sets

- Fuzzy sets are sets with elements that have a degree of membership, i.e. a membership function can take any value in (0,1)

Stochastic Least Squares: ①

- linear LS is a linear estimator: $\theta_Y = K_Y + h$

$$\theta_Y = \underbrace{(H^T W H)^{-1} H^T W}_{=K_{LS}} Y$$

- linear LS is unbiased $K_{LS} H = (H^T W H)^{-1} H^T W H = 1$

- Is linear LS also the best linear unbiased estimator?

↳ BLUE if $W = C^{-1}$ (siehe Gauss Markov Theorem)

Trilateration: Bancroft solution $d_i^2 = \|x - p_i\|^2 + e_i^*$

$$Y = H_1 x + H_2 R^2 + e^*$$

$$Y = \begin{pmatrix} d_1^2 - \|p_1\|^2 \\ \vdots \\ d_m^2 - \|p_m\|^2 \end{pmatrix}; \quad H_1 = \begin{pmatrix} -2p_{1,1} & -2p_{1,2} \\ \vdots & \vdots \\ -2p_{m,1} & -2p_{m,2} \end{pmatrix}; \quad H_2 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$x^{LS}(R^2) = (H_1^T H_1)^{-1} H_1^T (Y - H_2 R^2)$$

$$= z_1 + R^2 z_2$$

$$R^2 = \|x^{LS}(R^2)\|^2$$

$$= (z_1 + R^2 z_2)^T \cdot (z_1 + R^2 z_2)$$

$$\vdots$$

lösbar

Sensor Fusion: Formelsammlung

Linearization of multivariate Functions:

- $g: \mathbb{R}^n \to \mathbb{R}^n$ with component functions $g = [g_1 \dots g_m]^T$
- Linearization Point $\hat{x} \in \mathbb{R}^n$
- Linearization: $g(x) \approx g(\hat{x}) + G(x - \hat{x})$

with Jacobi Matrix:

$$G_{\hat{x}} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\hat{x}) & \dots & \frac{\partial g_1}{\partial x_n}(\hat{x}) \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1}(\hat{x}) & \dots & \frac{\partial g_m}{\partial x_n}(\hat{x}) \end{pmatrix}$$

Gauss-Newton Method

- Objective:

$$x^{LS} = \arg\min_x \|\underbrace{v(x)}_{y - h(x)}\|^2 \approx \arg\min_x \|v(\hat{x}^l) + J_l (x - \hat{x}^l)\|^2$$

- Algorithm:
  - Step 1: Choose initial estimate $\hat{x}^l$, $l = 1$
  - Step 2: Use linear LS to get $\hat{x}^{(l+1)}$:

$$\hat{x}^{(l+1)} = \hat{x}^{(l)} - (J_l^T J_l)^{-1} J_l^T v(\hat{x}^{(l)})$$

  - Step 3: Goto Step 2 until convergency is reached

(Herleitung) für x^{LS}:

$(J_l^T J_l)^{-1} J_l^T \cdot J_l = I$

$$\arg\min_x \|v(\hat{x}^l) + J_l (x - \hat{x}^l)\|^2 = \| J_l \cdot \hat{x}^l - v(x^l) - J_l \cdot x \|^2$$

$$\Rightarrow \underbrace{J_l \cdot \hat{x}^l - v(\hat{x}^l)}_{\hat{y}} = \underbrace{J_l}_{H} \cdot x$$

$$\hat{x}^{l+1} = (J_l^T J_l)^{-1} J_l^T (J_l \hat{x}^l - v(\hat{x}^l))$$

$$= \hat{x}^l - (J_l^T J_l)^{-1} J_l^T v(\hat{x}^l)$$