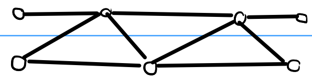
Graph of joints and rods



- Each rod wants to keep its endpoints at prescribed distance. The deformation energy is approximately:

$$\mathcal{E}(x) = \sum_{ij\in E} \left(|x_i - x_j| - \ell_{ij} \right)^2$$

- Assume we have a minimizer x, now expand $\mathcal{E}$ around that point:

$$\mathcal{E}(x+\Delta x) \approx \mathcal{E}(x) + \sum_i \Delta x_i \, \partial_i \mathcal{E}(x) + \frac{1}{2} \sum_{ij} \Delta x_i \Delta x_j \, \partial_i \partial_j \mathcal{E}(x)$$

We may ignore the constant contribution. At the minimizer have: $\partial_i \mathcal{E}(x) = 0$.
Compute now derivatives:

$d_{ij} := |x_i - x_j|$ (and: $|x_i - x_j| = \ell_{ij}$)

$$\partial_i \mathcal{E}(x) = \sum_{j:(i,j)\in E} 2 \left(d_{ij} - \ell_{ij} \right) \frac{x_i - x_j}{d_{ij}}, \qquad \frac{x_i - x_j}{d_{ij}} = \partial_i d_{ij}$$

$$\partial_i \partial_j \mathcal{E}(x) = 0 \quad \text{if } (i,j) \notin E,\ i \neq j$$

otherwise: $(i,j) \in E$:

$$= 2 \, \frac{x_j - x_i}{d_{ij}} \cdot \frac{x_i - x_j}{d_{ij}}$$

$$\partial_i^2 \mathcal{E}(x) = \sum_{j:(i,j)\in E} 2 \, \frac{(x_i - x_j)^2}{d_{ij}^2}$$

Careful: these things need to be properly interpreted in higher dimensions

redo computations in higher dimensions:

$$\partial_{ir} \mathcal{E}(x) = \sum_{j:(i,j)\in E} 2 \left(d_{ij} - \ell_{ij} \right) \frac{x_{ir} - x_{jr}}{d_{ij}}$$

$(i,j) \in E$:

$$\partial_{ir} \partial_{js} \mathcal{E}(x) = 2 \, \frac{x_{js} - x_{is}}{d_{ij}} \cdot \frac{x_{ir} - x_{jr}}{d_{ij}}$$

$$\partial_{ir} \partial_{is} \mathcal{E}(x) = \sum_{j:(i,j)\in E} 2 \, \frac{(x_{is} - x_{js})(x_{ir} - x_{jr})}{d_{ij}^2}$$

call this matrix $= -L$

Effective local energy

$$\mathcal{E}(\Delta x) = -\frac{1}{2} \Delta x^T L \Delta x$$

to simulate loads, apply linear external potential:

$$\mathcal{E}(\Delta x) = -\frac{1}{2} \Delta x^T L \Delta x + V^T \Delta x$$

optimality condition: $L \Delta x = V$
careful: L is blind to translations and rotations
so V must be mapped to right subspace.

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alternative: apply force to some vertices and find best solution
while deformation at some other vertices is fixed

$$\Delta x = \begin{array}{cl} \left(\begin{array}{c} \\ \end{array} \right) & \begin{array}{l} \text{fixed to } 0 \\ \Delta \hat{x} \end{array} \end{array} \qquad L = \begin{array}{cl} \begin{array}{cc} \text{fixed} & \text{free} \end{array} & \\ \left(\begin{array}{c|c} & B \\ \hline B^T & \hat{L} \end{array} \right) & \begin{array}{l} \text{fixed} \\ \text{free} \end{array} \end{array}$$

$$\mathcal{E}(\Delta \hat{x}) = -\frac{1}{2} \begin{pmatrix} 0 \\ \Delta \hat{x} \end{pmatrix}^T \begin{pmatrix} \ddots & B \\ B^T & \hat{L} \end{pmatrix} \begin{pmatrix} 0 \\ \Delta \hat{x} \end{pmatrix} + \hat{V}^T \Delta \hat{x}$$

$$= -\frac{1}{2} \Delta \hat{x}^T \hat{L} \Delta \hat{x} + \hat{V}^T \Delta \hat{x}$$

(information about fixated points is encoded in diagonal of $\hat{L}$)