1 Appendix A: Generalized Integration By Parts

$$\begin{aligned}
\frac{d}{dx}\prod_{i=1}^{n} f_i(x) &= f_1(x)\frac{d}{dx}\prod_{i=2}^{n} f_i(x) + \left(\prod_{i=2}^{n} f_i(x)\right)\frac{d}{dx}f_1(x) \\
&= \left(\prod_{i=2}^{n} f_i(x)\right)\frac{d}{dx}f_1(x) + f_1(x)\left[\left(\prod_{i=3}^{n} f_i(x)\right)\frac{d}{dx}f_2(x) + f_2(x)\frac{d}{dx}\prod_{i=3}^{n} f_i(x)\right] \\
&= \left(\prod_{i=2}^{n} f_i(x)\right)\frac{d}{dx}f_1(x) + f_1(x)\left(\prod_{i=3}^{n} f_i(x)\right)\frac{d}{dx}f_2(x) + f_1(x)\,f_2(x)\frac{d}{dx}\prod_{i=3}^{n} f_i(x)
\end{aligned}$$

From this, we can define the derivative of the generalized product to be a sum of terms

$$\frac{d}{dx}\prod_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} F_{i-1}(x)\left(\prod_{i'=i+1}^{n} f_{i'}(x)\right)\frac{d}{dx}f_i(x)$$

Where

$$\begin{aligned}
&F_n(x) \equiv f_n(x)\,f_{n-1}(x)\cdots f_1(x) \\
&F_0(x) \equiv 1
\end{aligned}$$

This can be simplified as follows

$$\frac{d}{dx}\prod_{i=1}^{n} f_i(x) = \sum_{i=1}^{n}\left(\prod_{i'\neq i}^{n} f_{i'}(x)\right)\frac{d}{dx}f_i(x)$$

Recall the derivation of integration by parts

$$\begin{aligned}
\frac{d}{dx}f(x)\,g(x) &= g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x) \\
\int\frac{d}{dx}f(x)\,g(x) &= \int g(x)\frac{d}{dx}f(x) + \int f(x)\frac{d}{dx}g(x) \\
f(x)\,g(x) &= \int g(x)\frac{d}{dx}f(x) + \int f(x)\frac{d}{dx}g(x) \\
\int g(x)\frac{d}{dx}f(x) &= f(x)\,g(x) - \int f(x)\frac{d}{dx}g(x)
\end{aligned}$$

Or, in the more common notation

$$\int gf' = fg - \int fg'$$

Next, using the same argument, we derive a generalization for any number of products in the intgrand.

$$\int \frac{d}{dx} \prod_{i=1}^{n} f_i(x) = \int \sum_{i=1}^{n} \left(\prod_{i' \neq i}^{n} f_{i'}(x) \right) \frac{d}{dx} f_i(x)$$
$$\prod_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} \int \left(\prod_{i' \neq i}^{n} f_{i'}(x) \right) \frac{d}{dx} f_i(x)$$

From the expression on the right of the equal sign, we can extract any chosen element of the sum and move it to the left hand side.

$$\int \left(\prod_{i' \neq \gamma}^{n} f_{i'}(x) \right) \frac{d}{dx} f_\gamma(x) = \prod_{i=1}^{n} f_i(x) - \sum_{i \neq \gamma}^{n} \int \left(\prod_{i' \neq i}^{n} f_{i'}(x) \right) \frac{d}{dx} f_i(x) \tag{1}$$

This provides us with a means of reasoning about the products in the integrand of the kinetic energy function.