Maxwell: Mathematical Documentation

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Contents

1 Overview

This document contains documentation and development of the mathematical model used in simulating the Maxwell device. This is the short code name for the linear accelerator gun that this project is meant to produce. I will refer to the unmoving device that contains the accelerator coils as the "accelerator" and the moving coil as the "projectile". The combined device is referred to as the Maxwell device. The focus of this mathematical model and its accompanying computational model, is to understand the relationship between the design variables of the device and its performance. Loosely speaking these design variables are

V_{AC}	The voltage of the supply capacitor array.	C_{AC}	The capacitance of the capacitor array.
N_{AC}	The number of accelerator coils.	R_{ACO}	The outer radius of the accelerator coils.
D_{AC}	The center to center spacing of accelerator coils.	R_{ACI}	The inner radius of the accelerator coils.
σ_{AC}	The conductivity of accelerator coil material.	L_{AC}	The length of the accelerator coils.
c_{AC}	The specific heat of accelerator coil material.	r_{AC}	The radius of accelerator coil wire.

for the accelerator and

V_{PC}	The voltage of the projectile capacitor array.	C_{PC}	The capacitance of the projectile capacitor.
σ_{PC}	The conductivity of projectile coil material.	R_{PCO}	The outer radius of the projectile coil.
c_{PC}	The specific heat of projectile coil material.	R_{PCI}	The inner radius of the projectile coil.
r_{PC}	The radius of projectile coil wire.	L_{PC}	The length of the projectile coil.
m_P	The mass of the projectile.		

for the projectile. Some of these variables act as constraints on eachother. For example, R_{ACI} must be greater than or equal to $R_{PCO} + \delta_p$ where δ_p is the thickness of the material surrounding the projectile coil. In addition to these variables, the coefficient of friction between the projectile and the interior of the accelerator must be considered. The effects of drag on the projectile during acceleration should also be taken into account. In accounting for drag, I will adopt a simple model that considers only linear and quadratic drag. The associated variables will then be

a	The linear coefficient of drag.	b	The quadratic coefficient of drag.
Γ	The drag force per unit contacting surface area per unit normal force per unit velocity.		

The most important dependent variables that will be extracted from the simulation are

v_P	The muzzle velocity of the projectile.	E_P	The muzzle energy of the projectile.
α	The efficiency of conversion from electric to kinetic energy.		

1.1 Motivation and Challenges

The goal of this model is to confidently determine the design variables necessary to produce a device that is both effective in use and cost effective to produce. A complicated simulation of the electrodynamics with some consideration of the thermodynamics and kinetics of the system is necessary to do this. As a result, a full treatment of classical electrodynamics will be used. Additionally, simple modeling of the heating of the coils and the frictional forces on the projectile will be used. The results of repeated simulations should allow for the construction of a list of necessary parts and determination of the time and money necessary to build a device.

1.2 Simulation Scope

This simulation will use a full classical treatment of Maxwell's equations to simulate the electrodynamics of the system. The geometry of the coils will be considered in an approximate, but fairly accurate manner. The projectile will have a single translational degree of freedom. Future improvements on the simulation may involve inducing a spin on the projectile and considering torque forces outside of the axis of motion (due to imperfect coil geometry). The temperature of the coils after sequential firings will be considered. The temperature dependence of the conductivity of the coil material may also be integrated into the final system of differential equations. The drag and friction forces that the projectile experiences during acceleration will be approximated with a simple model.

The ultimate goal of this project is to develop an accurate system of differential equations that describes the system outlined above. This will be converted into a numerical method for solving the equation. This method will be implemented, likely in Python or Julia, and used to simulate the Maxwell device.

1.3 Reference Format

In this document I will reference several books. In this section, I will define the shorthand for referencing and the book that it refers to.

Introduction to Electrodynamics, Fourth Edition, David J. Griffiths
I will reference this as (EM G, p. 100, eq. 4.5.6) where the page number is 100 and the equation number is 4.5.6.

Classical Mechanics, John R. Taylor
I will reference this as (CM T, p. 100, eq. 4.5.6) where the page number is 100 and the equation number is 4.5.6.

Thermal physics textbook not yet determined. Probably Reif for something like that.

2 Component Modeling

This section is concerned with the mathematical model used to define the shape and electrical characteristics of the primary components in the system. These are, in order of descending complexity, the coils, the power transistors and the capacitors. All other parts of the system can be treated in a more simple manner and are not involved in the simulation. These include but are not limited to the sensors in the barrel, the firing circuitry, the barrel material, the power supply and the projectile circuitry.

2.1 Coils

The most accurate way of modeling the coils would involve characterizing them as a corkscrew shaped continuum of material comprised of a deformed cylinder. This creates substantial complications in the modeling and is unnecessary in the limit where the radius of the wire is significantly smaller than the other dimensions of the coil. As such, coils will be treated as a series of concentric rings stacked vertically, with appropriate spacing between rings, reflecting the radius of the wire they are comprised of. In the simulation, they will be treated as line currents. The path of a single loop of wire can be described as

$$\begin{aligned} r\left(u\right) &= r_\ell & \quad r\left(u\right) &= \sqrt{r_\ell^2 + z_\ell^2} & \quad x\left(u\right) &= r_\ell \cos u \\ \varphi\left(u\right) &= u & \theta\left(u\right) &= \tan^{-1}\left(r_\ell / z_\ell\right) & y\left(u\right) &= r_\ell \sin u \\ z\left(u\right) &= z_\ell & \phi\left(u\right) &= u & z\left(u\right) &= z_\ell \\ u &\in [0,\, 2\pi] & u &\in [0,\, 2\pi] & u &\in [0,\, 2\pi] \end{aligned}$$

parametrically in cylindrical coordinates (left), spherical coordinates (middle) and cartesian coordinates (right). Here u is an arbitrary parameter. If one wishes to describe this loop implicitly, rather than parametrically, it can be written

$$\begin{aligned} x^2 + y^2 &= r_\ell^2 \\ z &= z_\ell \end{aligned}$$

in cartesian coordinates. Here r_ℓ is the radius of the loop and z_ℓ is the displacement of the loop above the z axis. The Cartesian coordinates used are right handed, with the cylindrical $\varphi = 0$ (spherical $\phi = 0$) corresponding to the x axis. The current density produced by a coil can be written,

$$\mathbf{J}_\ell\left(r,\, \varphi,\, z\right) = I\delta\left(r - r_\ell\right)\delta\left(z - z_\ell\right)\hat{\varphi} \tag{1}$$

$$\mathbf{J}_\ell\left(r,\, \theta,\, \phi\right) = I\delta\left(r - \sqrt{r_\ell^2 + z_\ell^2}\right)\delta\left(\theta - \tan^{-1}\left[r_\ell / z_\ell\right]\right)\hat{\phi} \tag{2}$$

$$\mathbf{J}_\ell\left(x,\, y,\, z\right) = I\delta\left(r - \sqrt{r_\ell^2 + z_\ell^2}\right)\delta\left(\theta - \tan^{-1}\left[r_\ell / z_\ell\right]\right)\left(\cos\phi\,\hat{y} - \sin\phi\,\hat{x}\right) \tag{3}$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} \qquad r = \sqrt{x^2 + y^2 + z^2} \qquad \phi = \tan^{-1}(y/x)$$

where δ is the Dirac delta function. These are necessary because this model characterizes wires as line currents. In an abstract sense, a loop will be denoted ℓ_r^z, which can be read as a loop of radius r at height z. A coil of wire with wire radius r_w, length l_c, inner radius r_i and outer radius r_o can be described as a superposition of these loops. If we treat the lower bound of the loop as being at $z = z_l$ and the loop axis (the normal vector of the loop plane) being aligned with the z axis, we can write a coil as

$$\mathcal{C} = \sum_{k=1}^{(r_o - r_i)/2r_w} \left\{ \sum_{j=1}^{l_c/2r_w} \ell_{r_i + r_w/2 + 2r_w(k-1)}^{2r_w(j-1)+z_l} \right\} \tag{4}$$

In general, this means that the behavior of a coil $\mathcal{C}$ as a whole can be treated as the superposition of the behaviors of a sum of loops ℓ_r^z. This sum is of course, nonsensical if l_c is not an integral multiple of r_w and $r_o - r_i$ is not an integral multiple of r_w. In order to construct a differential equation to model this system, we need several important functions. These are

1. The magnetic field produced by a loop ℓ_r^z, as a function of the current I running through it.
2. The total induced EMF (voltage) around a loop, as a function of the magnetic field surrounding it.
3. The force on a loop as a function of the magnetic field surrounding it.

Armed with these two functions, we can fully characterize the magnetic fields produced by a coil and the inductive kick-back it experiences due to a change in its own magnetic field and due to the magnetic field of the projectile coil.

2.1.1 Magnetic Field Produced by a Loop

The field produced by a line current is given generally by,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_\ell \frac{d\mathbf{l}' \times \hat{\boldsymbol{\delta}}}{\delta^2} \tag{5}$$
$$\boldsymbol{\delta} = \mathbf{r} - \mathbf{r}'$$

(EM G, p. 224, eq. 5.34)

where $\boldsymbol{\delta}$ is the displacement vector from the point where the current is measured to the point where the magnetic field is measured. $d\mathbf{l}'$ is an infinitesimal vector pointing in the direction of the current. From this general expression, we can generate a more specific expression for the integral in the case of the loop defined in this section (ℓ_r^z). First, I will recast eq. 3 as a current vector expression.

$$\mathbf{I}_\ell(x, y, z) = \begin{cases} I\cos\phi\,\hat{y} - I\sin\phi\,\hat{x} & z = z_\ell,\ r = r_\ell \\ 0 & \text{otherwise} \end{cases}$$
$$\sin\phi = \sin\tan^{-1}(y/x) = \frac{y}{x\sqrt{y^2/x^2 + 1}}$$
$$\cos\phi = \cos\tan^{-1}(y/x) = \frac{1}{\sqrt{y^2/x^2 + 1}}$$

With this, we can rewrite eq. 5 more explicitly

$$
\begin{aligned}
d\mathbf{l}' &= r_\ell d\varphi' \hat{\varphi} = -r_\ell \sin\varphi' d\varphi' \hat{x} + r_\ell \cos\varphi' d\varphi' \hat{y} \\
\hat{\boldsymbol{\delta}} &= \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \\
\mathbf{r}-\mathbf{r}' &= (x-x')\,\hat{x} + (y-y')\,\hat{y} + (z-z')\,\hat{z} \\
&= (x-x')\,\hat{x} + (y-y')\,\hat{y} + (z-z_\ell)\,\hat{z} \\
&= (x-r_\ell\cos\varphi')\,\hat{x} + (y-r_\ell\sin\varphi')\,\hat{y} + (z-z_\ell)\,\hat{z} \\
|\mathbf{r}-\mathbf{r}'| &= \sqrt{(x-x')^2 + (y-y')^2 + (z-z_\ell)^2} \\
&= \sqrt{(x-r_\ell\cos\varphi')^2 + (y-r_\ell\sin\varphi')^2 + (z-z_\ell)^2} \\
\therefore \hat{\boldsymbol{\delta}} &= \left[(x-r_\ell\cos\varphi')^2 + (y-r_\ell\sin\varphi')^2 + (z-z_\ell)^2\right]^{-1/2} \left[(x-r_\ell\cos\varphi')\,\hat{x} + (y-r_\ell\sin\varphi')\,\hat{y} + (z-z_\ell)\,\hat{z}\right] \\
d\mathbf{l}' \times \hat{\boldsymbol{\delta}} &= \delta^{-1}\left[-r_\ell\sin\varphi' d\varphi'\hat{x} + r_\ell\cos\varphi' d\varphi'\hat{y}\right] \times \left[(x-r_\ell\cos\varphi')\,\hat{x} + (y-r_\ell\sin\varphi')\,\hat{y} + (z-z_\ell)\,\hat{z}\right] \\
&= \delta^{-1}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -r_\ell\sin\varphi' d\varphi' & r_\ell\cos\varphi' d\varphi' & 0 \\ x-r_\ell\cos\varphi' & y-r_\ell\sin\varphi' & z-z_\ell \end{vmatrix} \\
&= \delta^{-1}\left(r_\ell\cos\varphi' d\varphi'\right)(z-z_\ell)\,\hat{x} \\
&+ \delta^{-1}(z-z_\ell)\left(r_\ell\sin\varphi' d\varphi'\right)\hat{y} \\
&+ \delta^{-1}\left[\left(-r_\ell\sin\varphi' d\varphi'\right)(y-r_\ell\sin\varphi') - \left(r_\ell\cos\varphi' d\varphi'\right)(x-r_\ell\cos\varphi')\right]\hat{z}
\end{aligned}
$$

Writing the full expression for the magnetic field in terms of its components,

$$
\begin{aligned}
B_x &= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{z r_\ell\cos\varphi' - z_\ell r_\ell\cos\varphi'}{\delta^3}\,d\varphi' \\
B_y &= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{z r_\ell\sin\varphi' - z_\ell r_\ell\sin\varphi'}{\delta^3}\,d\varphi' \\
B_z &= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{r_\ell^2\sin^2\varphi' + r_\ell^2\cos^2\varphi' - y r_\ell\sin\varphi' - x r_\ell\cos\varphi'}{\delta^3}\,d\varphi' \\
&= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{r_\ell^2 - y r_\ell\sin\varphi' - x r_\ell\cos\varphi'}{\delta^3}\,d\varphi' \\
\delta^3 &= \left[(x-r_\ell\cos\varphi')^2 + (y-r_\ell\sin\varphi')^2 + (z-z_\ell)^2\right]^{3/2} \\
&= \left[x^2 - 2x r_\ell\cos\varphi' + r_\ell^2\cos^2\varphi' + y^2 - 2y r_\ell\sin\varphi' + r_\ell^2\sin^2\varphi' + (z-z_\ell)^2\right]^{3/2} \\
&= \left[r_\ell^2 + x^2 + y^2 - 2r_\ell\left(x\cos\varphi' + y\sin\varphi'\right) + (z-z_\ell)^2\right]^{3/2}
\end{aligned}
$$

To summarize this concisely,

$$
B_x = (z-z_\ell)\,\frac{\mu_0 I r_\ell}{4\pi}\int_0^{2\pi}\frac{\cos\varphi'}{\delta^3}\,d\varphi' \tag{6}
$$

$$
B_y = (z-z_\ell)\,\frac{\mu_0 I r_\ell}{4\pi}\int_0^{2\pi}\frac{\sin\varphi'}{\delta^3}\,d\varphi' \tag{7}
$$

$$
B_z = \frac{\mu_0 I r_\ell}{4\pi}\left\{ r_\ell\int_0^{2\pi}\frac{1}{\delta^3}\,d\varphi' - y\int_0^{2\pi}\frac{\sin\varphi'}{\delta^3}\,d\varphi' - x\int_0^{2\pi}\frac{\cos\varphi'}{\delta^3}\,d\varphi' \right\} \tag{8}
$$

$$
\delta^3 = \left[r_\ell^2 + x^2 + y^2 - 2r_\ell\left(x\cos\varphi' + y\sin\varphi'\right) + (z-z_\ell)^2\right]^{3/2} \tag{9}
$$

These expressions are extremely easy to evaluate numerically. This provides us with a method of quickly determining the magnetic field vector due to a current carrying loop with current I radius r_ℓ and displacement along the z axis of z_ℓ. In order to determine the magnetic field produced by a coil, we simply evaluate these expressions for each case of r_ℓ and z_ℓ in eq. 4.

2.1.2 Induced EMF in a Loop

Starting with Faraday's law

$$
\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\int_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot d\mathbf{a} &= -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{a}
\end{aligned}
$$

Applying Stoke's theorem,

$$
\begin{aligned}
\int_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot d\mathbf{a} &= \int_\ell \mathbf{E} \cdot d\mathbf{l} \\
V_\ell &= \int_\ell \mathbf{E} \cdot d\mathbf{l} \\
\therefore \int_\ell \mathbf{E} \cdot d\mathbf{l} &= -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{a} \\
\therefore V_\ell &= -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{a}
\end{aligned}
$$

Where V_ℓ is the induced voltage around the entire loop and the current through the loop is given by $I = V_\ell / R$, where R is the resistance of the loop. Since Stoke's theorem applies for any arbitrary surface S, we can choose this to be the most simple surface that bounds the loop. That is, the surface formed by a plane, bounded by the loop (a circle). We can write this integral in cylindrical coordinates,

$$
\begin{aligned}
&-\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{a} \\
d\mathbf{a} &= \hat{\mathbf{n}} da = \hat{\mathbf{n}} r \, d\varphi \, dr = \hat{z} r \, d\varphi \, dr \\
\mathbf{B} \cdot d\mathbf{a} &= B_z r \, d\varphi \, dr \\
\int_S \mathbf{B} \cdot d\mathbf{a} &= \int_0^{r_\ell} \int_0^{2\pi} B_z \left(r \cos\varphi, \, r \sin\varphi, \, z_\ell\right) r \, d\varphi \, dr \\
\therefore V_\ell &= -\frac{\partial}{\partial t} \int_0^{r_\ell} \int_0^{2\pi} B_z \left(r \cos\varphi, \, r \sin\varphi, \, z_\ell\right) r \, d\varphi \, dr \\
B_z &= B_z \left(x, \, y, \, z\right)
\end{aligned}
$$

To summarize, the induced EMF around a loop is,

$$
V_\ell = -\frac{\partial}{\partial t} \int_0^{r_\ell} \int_0^{2\pi} B_z \left(r \cos\varphi, \, r \sin\varphi, \, z_\ell\right) r \, d\varphi \, dr \tag{10}
$$

$$
B_z = B_z \left(x, \, y, \, z\right) \tag{11}
$$

In order to convert this into the current through a coil, we need to sum the EMF around all of the loops and divide this my the sum of the resistance of all of the loops (because the loops are effectively in series).

$$
I_{\mathcal{C}} = \frac{\sum_{i=1}^{N_l} V_\ell^i}{\sum_{i=1}^{N_l} R_\ell^i} \tag{12}
$$

where N_l is the number of loops in a coil, V_ℓ^i denotes the EMF around the ith coil and R_ℓ^i denotes the resistance of the ith coil.

2.1.3 Force on A Loop Due to Magnetic Field

The force on a loop due to a magnetic field can be derived from the Lorentz force law. Since the loops are made of neutral wire, we can write

$$\mathbf{F} = q\left(\cancel{\mathbf{E}} + \mathbf{v}\times\mathbf{B}\right) = q\mathbf{v}\times\mathbf{B} \tag{13}$$

Next, we need to convert this to an expression in terms of the current through a wire and the average velocity of the electrons in the wire. Current is defined as an amount of charge passing through a point per unit time (coulombs per second in SI units).

$$I = \frac{dQ}{dt}$$

where dQ is effectively charge per unit length times the distance traveled

$$\begin{aligned} dQ &= \lambda\, dx \\ I &= \lambda \frac{dx}{dt} \\ I &= \lambda v \end{aligned}$$

where λ is the charge per unit length in the wire. From this, we can write,

$$\begin{aligned} d\mathbf{F} &= q\mathbf{v}\times\mathbf{B} \\ &= \lambda\mathbf{v}\, dl\times\mathbf{B} \\ &= I\, d\mathbf{l}\times\mathbf{B} \end{aligned}$$

Where $d\mathbf{l}$ is an infinitesimal vector pointing in the direction of the current. Next,

$$\begin{aligned} d\mathbf{F} &= I\, d\mathbf{l}\times\mathbf{B} \\ \mathbf{F} &= I\int_\ell d\mathbf{l}\times\mathbf{B} \end{aligned}$$

Where the integral is over the entire loop. Writing this explicitly, for the case of the loop defined in §2.1,

$$\begin{aligned} d\mathbf{l} &= r_\ell d\varphi\hat{\varphi} = -r_\ell \sin\varphi\, d\varphi\hat{x} + r_\ell\cos\varphi\, d\varphi\hat{y} \\ \mathbf{B} &= B_x\hat{x} + B_y\hat{y} + B_z\hat{z} \\ d\mathbf{l}\times\mathbf{B} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -r_\ell\sin\varphi d\varphi & r_\ell\cos\varphi\, d\varphi & 0 \\ B_x & B_y & B_z \end{vmatrix} \\ &= B_z r_\ell \cos\varphi\, d\varphi\hat{x} + B_z r_\ell \sin\varphi\, d\varphi\hat{y} - \left[B_y\sin\varphi + B_x\cos\varphi\right] r_\ell\, d\varphi\hat{z} \end{aligned}$$

$$F_x = I r_\ell \int_0^{2\pi} B_z\left(r_\ell\cos\varphi,\, r_\ell\sin\varphi,\, z_\ell\right)\cos\varphi\, d\varphi \tag{14}$$

$$F_y = I r_\ell \int_0^{2\pi} B_z\left(r_\ell\cos\varphi,\, r_\ell\sin\varphi,\, z_\ell\right)\sin\varphi\, d\varphi \tag{15}$$

$$F_z = -I r_\ell \left\{\int_0^{2\pi} B_y\left(r_\ell\cos\varphi,\, r_\ell\sin\varphi,\, z_\ell\right)\sin\varphi\, d\varphi + \int_0^{2\pi} B_x\left(r_\ell\cos\varphi,\, r_\ell\sin\varphi,\, z_\ell\right)\cos\varphi\, d\varphi\right\} \tag{16}$$

$$B_i = B_i\left(x,\, y,\, z\right) \quad \forall\, i\in\{x,\, y,\, z\}$$

2.1.4 Summary

We now have expressions for the magnetic field produced by a loop with a constant current, the voltage induced in a loop due to the magnetic field around it and the force on a current carrying loop due to the magnetic field around it. These equations can be inserted into an equation of the form of eq. 4 in order to calculate the field, induced voltage and force on a coil comprised of multiple loops. It should be noted that the fact that these fields propagate at the speed of light (rather than instantaneously) has been neglected. This is a good approximation because the device is small and the field pulses are very slow relative to the speed of light.