Perfect Sets

April 14, 2016 **Meesala Raviteja**

Perfect Set: A set P in which every point x $\in$ P is a limit point and every limit point 'y' of P $\in$ P. Prove that P is uncountable in $\mathbb{R}$.

Proof: Lets take apart a few things here that need to be proved.

1. How do we prove that a set is uncountable in $\mathbb{R}$?

   Consider any countable subset of the given set. If one could prove that any arbitrary countable subset K of the given set S is a proper subset of S, then we can state that S is uncountable. Hence we must look to do something similar here, i.e. given a countable subset we must be able to produce x | x $\in$ S and x $\notin$ K.

2. Are there any examples of perfect sets in $\mathbb{R}$?

   Consider any closed interval [a,b] $\in \mathbb{R}$. It is a perfect set.

3. What properties does a perfect set posses?

   Perfect sets are closed. Also, since every point is a limit point (and hence there can be no isolated points), we can think of seeking a dense subset of P. What if we are able to seek a countable dense subset of P? (we can do so for our example above) Can we claim it will be closed? If it is closed then the theorem is not true. (verify this!) Hence we can search for a cauchy in that countable subset E that may not converge in E.

We now start our proof by considering any arbitrary countable subset of P. Call this E. Since the set is countable we can represent the set as $\{e_n\}$ where $e_i \in$ E $\forall$ i $\in \mathbb{N}$. We are now looking for a cauchy, that doesn't converge in E. Consider e_1 , and label it as $\overline{e_1}$. We construct an open ball V_1 around e_1 , with $\delta_1 = 1$ Infinitely many points lie in this, but $\exists$ a minimum 'k' | $e_k \in V_1$. Label this as $\overline{e_2}$ and pick $\delta_2 <$ min { 1/2 , | $\overline{e_1} - \overline{e_2}$ min }. We get our V_2. Note that $\overline{e_1} \notin V_2$. We proceed iteratively now. We construct an open ball around $\overline{e_i}$ such that $\delta_i <$ min { $1/i$, | $\overline{e_i} - \overline{e_{i-1}}$ min } The construction ensures that $\overline{e_i}$ form a cauchy. Also we remark that $\overline{e_i} \notin V_{i+1}$. (Implies the sequence cannot converge to $\overline{e_i}$)

We can now claim that this cauchy wont converge in E. For if it converged to some e then e = e_k for some k $\in \mathbb{N}$. But we have already showed that $e_k \notin V_{k+1}$.