Finite Element Model for a Timoshenko Beam

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INTRODUCTION:

THE TIMOSHENKO BEAM THOERY

The Timoshenko beam theory was developed by Ukrainian-born scientist and engineer Stephen Timoshenko early in the 20th century. The model takes into account shear deformation and rotational inertia effects, making it suitable for describing the behavior of short beams or beams subject to high-frequency excitation when the wavelength approaches the thickness of the beam. The resulting equation is of 4th order, but unlike ordinary beam theory, that is the Euler–Bernoulli beam theory, there is also a second order spatial derivative present.

Physically, taking into account the added mechanisms of deformation effectively lowers the stiffness of the beam, while the result is a larger deflection under a static load and lower predicted Eigen frequencies for a given set of boundary conditions. The latter effect is more noticeable for higher frequencies as the wavelength becomes shorter, and thus the distance between opposing shear forces decreases.

ASSUMPTIONS INVOLVED:

In reality, the beam cross-section deforms somewhat like what is shown in *Figure1a*. This is particularly the case for deep beams, i.e., those with relatively high cross sections compared with the beam length, when they are subjected to significant shear forces. Usually the shear stresses are highest around the neutral axis, which is where; consequently, the largest shear deformation takes place. Hence, the actual cross section curves. Instead of modeling this curved shape of the cross-section, the Timoshenko beam theory retains the assumption that the cross section remains plane during bending.

- The material is linear elastic according to Hooke's law.
- Plane sections remain plane.

The consideration of the shear deformation of the cross section, as shown in the following figure relaxes the assumption that the plane section remains normal to the neutral axis.

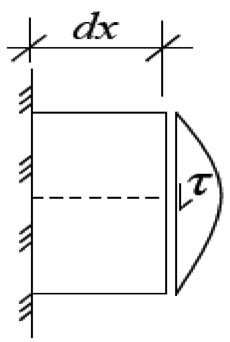

Figure 1a: Real Shear Distribution along cross section

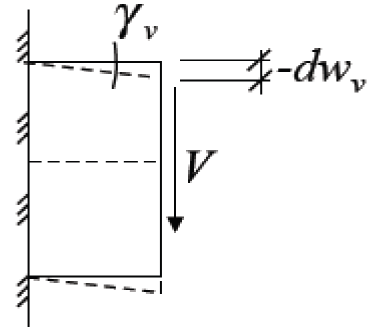

Figure 1b: Average Shear Strain considered (Assumption of timoshenko beam)

DIFFERENCE BETWEEN THE EULER BERNOULLI BEAM THEORY AND THE TIMOSHENKO BEAM THEORY:

The two key assumptions in the Euler Bernoulli beam theory are:

- The material is linear elastic according to Hooke's law.
- Plane sections remain plane and perpendicular to the neutral axis.

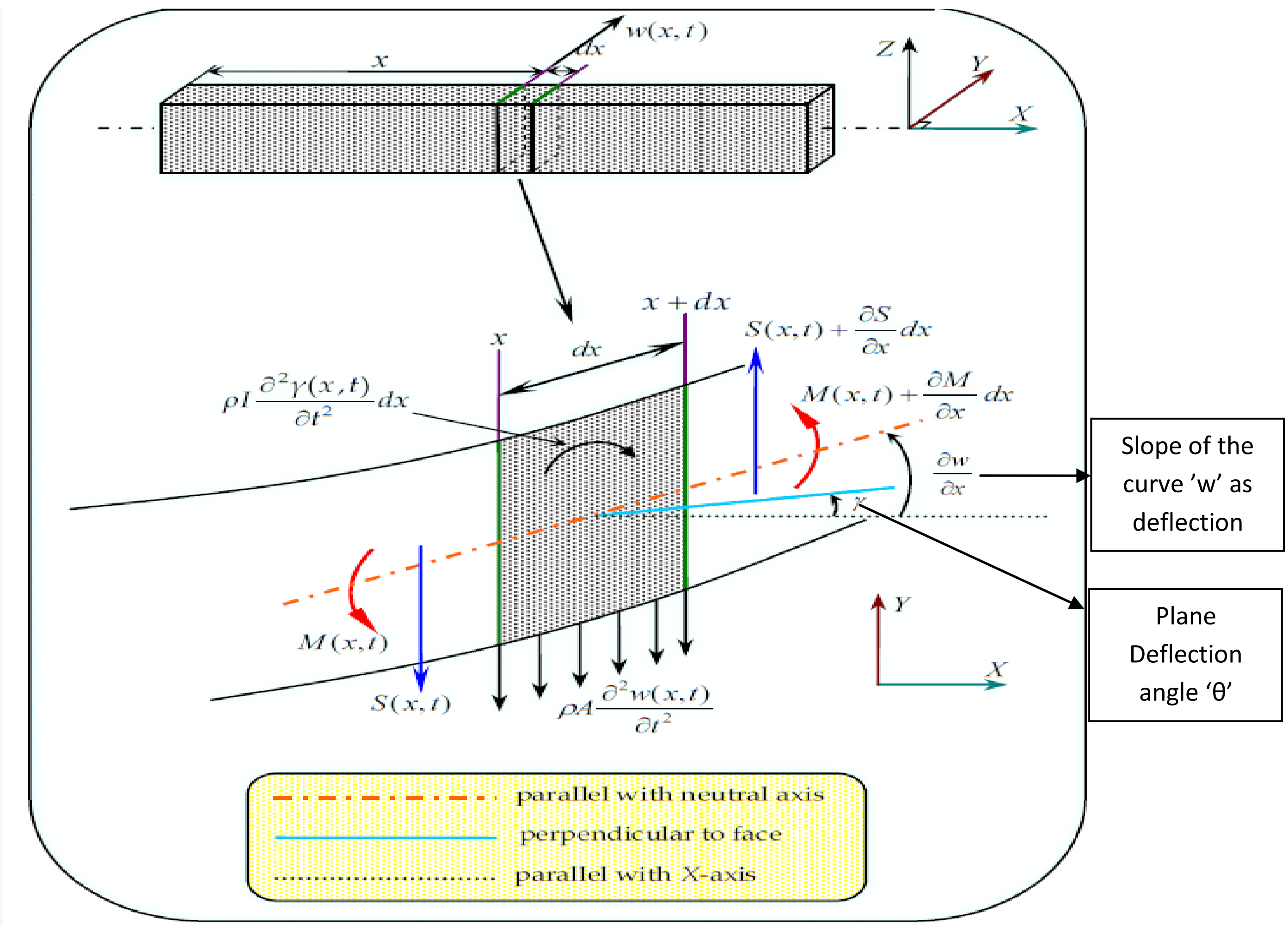

Figure 2a: A Timoshenko beam, with the shear deformation and slope indicated.

CONVERGENCE TO THE EULER BERNOULLI BEAM THEORY

If the shear modulus (G) of the beam material approaches infinity and thus the beam becomes rigid in shear and if rotational inertia effects are neglected, Timoshenko beam theory converges towards ordinary beam theory.

PROBLEM STATEMENT:

The main objective of the project was to study the dynamic characteristics of the Timoshenko beam, by finding the natural frequencies of a typical beam of a given cross sectional area and mechanical properties.

The natural frequencies are to be derived for various boundary conditions, and a parametric study of the same had to be done varying all possible mechanical properties.

This has been done using the finite element method, by deriving the necessary mass and stiffness matrix, for a beam with 2 degrees of freedom at every node, there by finding the Eigen values, which lead to the natural frequencies of a beam.

GOVERNING DIFFERENTIAL EQUATIONS:

Considering the free body diagram given below:

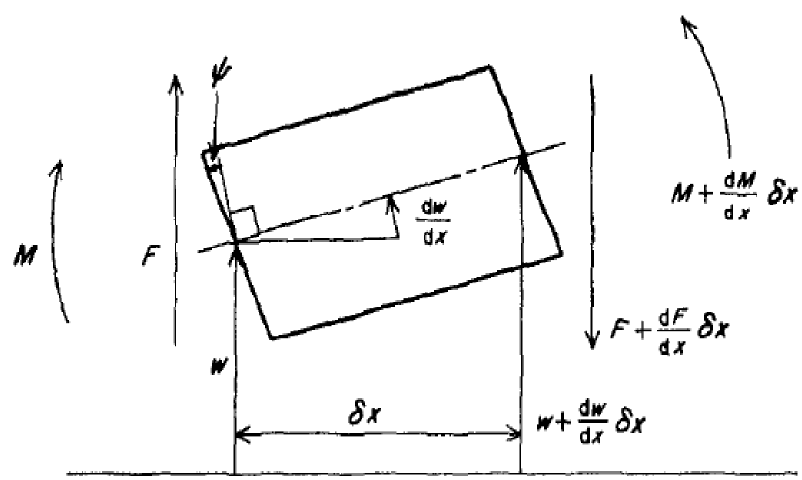

Figure 3.1: Forces and moments on an infinitesimal element of a beam.

The shear angle, ψ, is measured as positive in an anticlockwise direction from the normal to the mid surface of the outer face of the beam.

The static equilibrium equations are:

$\frac{\partial M}{\partial x} = F$ (Balancing moments about) [Eq: 3.1]

$\frac{\partial F}{\partial x} = 0$ (Balancing forces in Transverse direction) [Eq: 3.2]

Where, F is the Shear force experienced along the cross section, and M is the bending moment

Exact solution for the beam vibration problem:

The governing differential equation for the beam vibration problem:

$$\rho A\left(\frac{\partial^2 y}{\partial t^2}\right)+\frac{\partial F}{\partial x}=0$$

$$\rho I_z\left(\frac{\partial^2 \theta}{\partial t^2}\right)= EI_z\left(\frac{\partial^2 \theta}{\partial x^2}\right)+ GKA(\psi)$$

$$F = GKA(\psi)$$

$$\psi = \theta-\frac{\partial y}{\partial x}$$

Assuming: $y(x,t) = f_1(x)e^{iwt}$, $\theta(x,t) = f_2(x)e^{iwt}$

$$-\omega^2\rho A f_1(x)+ GAK(f_1''(x)-f_2'(x))=0$$

$$-\omega^2\rho I_z f_2(x)=EI\big(f_2''(x)\big)+ GAK(f_1'(x)-f_2(x))$$

Form of the solution: $f_1(x)=C_1\cosh(\beta x)+C_2\sinh(\beta x)+C_3\cos(\beta x)+C_4\sin(\beta x)$

$$f_2(x)=D_1\cosh(\beta x)+D_2\sinh(\beta x)+D_3\cos(\beta x)+D_4\sin(\beta x)$$

8 equations 10 unknowns;

Substituting the forced, natural boundary conditions: 2 or more equations

Condition for a non trivial solution (trivial solution is $C_i, D_i = 0$) to exist:

All 10 equations must be consistent: *we find different values of ω for which the above condition is satisfied.*

The above 10 equations are non-linear equations; hence solved by approximating the terms.

Euler Bernoulli beam and the Timoshenko beam are 2 approximated theories that provide solutions to this vibration problem.

However, the solution to the above equations gives the exact solutions for a vibrating beam.

THE PLAIN STRESS SOLUTION OF THE BEAM:

If the above beam problem; is considered as a plain stress problem ($\{\sigma_z, \sigma_{zx}, \sigma_{zy}\} = 0$):
The governing stress strain relations are:

$$\sigma_x = E(\frac{\partial u}{\partial x} + \vartheta \frac{\partial v}{\partial y}) \quad \text{[Eq: 10.1]}$$

$$\sigma_{xy} = G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right); [\gamma_{xy}] \quad \text{[Eq: 10.2]}$$

$$\sigma_y = E(\frac{\partial v}{\partial y} + \vartheta \frac{\partial u}{\partial x}) \quad \text{[Eq: 10.3]}$$

The equilibrium equations are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad \text{[Eq: 10.4]}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \quad \text{[Eq: 10.5]}$$

(Assuming v = 0)

Using moment curvature formula; $\sigma_x = c_1 y + c_2$; $(c_1, c_2\ functions\ of\ x)$

We get

$\sigma_{xy} = c_3 y^2 + c_4 y + c_5$; [Eq: 10.6.1]

$\gamma_{xy} = \sigma_{xy}\ (y)/G$; [Eq: 10.6.2]

Because the Timoshenko theory assumes that the plane sections remain plane even after application of load; the above shear strain is assumed to be a constant, proportional to the average shear strain of the cross section.

Shear force: $F = \sigma_{xy} * A = GK\psi$;

Where, K is the shear coefficient of a given cross section.

The value of K is chosen such that the solution obtained for the Timoshenko beam converges to the exact beam theory as closely as possible.

In solving for the natural frequencies, a beam of length L and a rectangular cross section with width 'h' and thickness 't' is assumed along with the following data:

$\vartheta = 0.3$ (Poisons ratio) h/L = 0.08 (Thickness length ratio) K = 0.85 (best for a rectangular cross section)

Instead of ω (natural frequency) a reduced frequency = $\omega\sqrt{(\rho A L^2/(EI)}$ is calculated reducing the parameters required effectively that affect the frequency, hence referred as reduced frequency

THE FEM MODEL OF THE BEAM:

FEM (Finite Element Method):

Finite element method is a numerical technique for finding approximate solutions to boundary value problems. It uses variational methods (the Calculus of variations) to minimize an error function and produce a stable solution.

A feature of FEM is that it is numerically stable, meaning that errors in the input and intermediate calculations do not accumulate and cause the resulting output to be meaningless.

The main applications of the FEM are:

- To find equilibrium (or) steady state equation.
- Solve Eigen value problem.
- Propagation or time dependent problem.

Step involved in a solution by a finite element method [for the above problem]:

1. Discretization of the structure into finite elements, based on the geometry
2. Selection of a proper interpolation model, coherent with the governing differential equations
3. Derivation of the mass and the stiffness matrix, using an appropriate variational principle
4. Assembling the element equations to obtain the equilibrium equation for the whole domain
5. Computing the solution for the unknown nodal displacements, forces

Step 1:

The Timoshenko beam considered, is a linear element. The beam is divided into n number of linear elements, and natural frequencies are calculated individually. Finally, a suitable number of elements are chosen based on the convergence of the natural frequencies obtained.

Step 2:

From the *figure 3.1* and *figure 2.1* the following relations can be established:

$$F/A = GK\psi \qquad \text{[Eq: 4.1]}$$

$$\theta = \frac{\partial y}{\partial x} + \psi \qquad \text{[Eq: 4.2]}$$

The stress-strain relationship in bending is (using Moment curvature formula):

$$\frac{M}{EI} = \frac{\partial^2 y}{\partial x^2} + \frac{\partial \psi}{\partial x} \qquad \text{[Eq: 5.1]}$$

Combining equations [Eq: 3.1] [Eq: 3.2] [Eq: 4.1] [Eq: 4.2] [Eq: 5.1]

Assume:

$y(x) = \frac{\alpha_1}{6}x^3 + \frac{\alpha_2}{2}x^2 + \alpha_3 x + \alpha_4$ *(The interpolation model assumed for the problem)* [Eq: 5.2]

We get:

$F(x) = EI * \alpha_1$ [Eq: 5.3]

$M(x) = \text{EI}^*(\alpha_1 x + \alpha_2)$ [Eq: 5.4]

Step 3:

The above equations can now be applied by substituting the end conditions as shown in the *figure 5.1*:

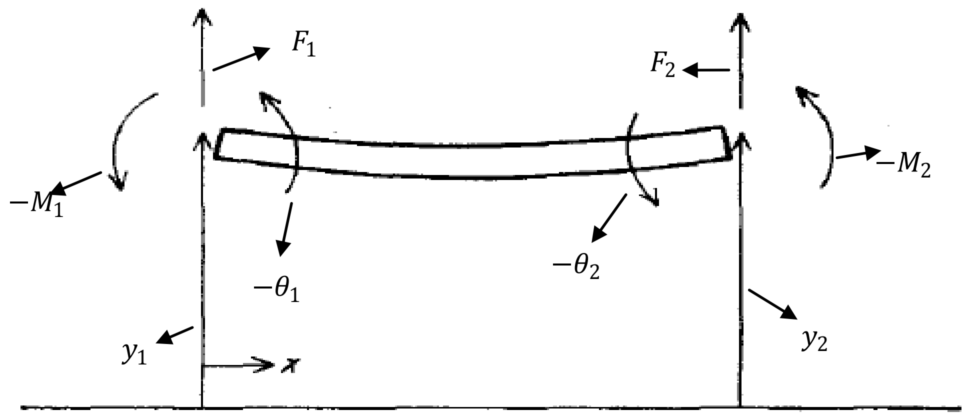

Figure 5.1 Generalized Forces and displacement on a finite element

$F_1 = EI * \alpha_1$ [Eq: 5.5.1]

$M_1 = EI * (\alpha_2)$ [Eq: 5.5.2]

$F_2 = -EI * \alpha_1$ [Eq: 5.5.3]

$M_2 = -EI * (\alpha_1 l + \alpha_2)$ [Eq: 5.5.4]

$\psi = EI * (\alpha_1)/(GKA)$ [Eq: 5.5.5]

$y_1 = \alpha_4$ [Eq: 6.1.1]

$y_2 = \frac{\alpha_1}{6} l^3 + \frac{\alpha_2}{2} l^2 + \alpha_3 l + \alpha_4$ [Eq: 6.1.2]

$\theta_1 = \alpha_3 + \psi$ [Eq: 6.1.3]

$\theta_2 = \frac{\alpha_1}{2} l^2 + \alpha_2 l + \alpha_3 + \psi$ [Eq: 6.1.4]

Derivation of the Mass and Stiffness matrix:

The Total energy of the system is given by the following equation:

$$\Pi = T + V \qquad \text{[Eq: 6.2]}$$

Where,

$$T = \frac{1}{2}\int_0^l (\rho A \left(\frac{\partial y}{\partial t}\right)^2 + \rho I_z (\frac{\partial \theta}{\partial t})^2) dx \qquad \text{[Eq: 6.3.1]}$$

$$V = \frac{1}{2}\int_0^l (F\psi + M(x)\left(\frac{\partial \theta}{\partial x}\right)^2) dx \qquad \text{[Eq: 6.3.2]}$$

Substituting the equations [Eq: 5.2], [Eq: 6.1.1-1.4]:

We get, $\Pi = f(y_1, y_2, \theta_1, \theta_2, \dot{y}_1, \dot{y}_2, \dot{\theta}_1, \dot{\theta}_2)$

Now using the Euler Lagrange equation (*The Variational Principle employed*) over the functional Π

$$\frac{\partial \Pi}{\partial w_j} - \frac{d(\frac{\partial \Pi}{\partial w_j})}{dt} = 0 \quad \text{where } w_j = \{y_1, y_2, \theta_1, \theta_2\} \qquad \text{[Eq: 6.4.1 – 6.4.4]}$$

The above set of equations represents the dynamic set of system equations.

From the above we get the equations in a matrix form, which can be represented as given below:

$$[m]\{\ddot{w}_j\} + [k]\{w_j\} = 0 \qquad \text{[Eq: 6.5]}$$

[m] represents the mass matrix; and [k] represents the stiffness matrix

The above matrix achieved is the local stiffness matrix. In order to solve for the complete beam one has to compute the global stiffness matrix.

**The matrix [m] and [k] are given separately at the end of this report.*

Step 4:

The local stiffness matrix and the mass matrix are added up together, using a computer simulation, with the help of *MATLAB R2007a*.

Once the global stiffness and the global mass matrix are calculated; the natural frequencies of the above beam element can be derived in the following way:

Assume: $w_j = w_j(x)e^{i\omega t}$

We get:

$$(-\omega^2[M_m] + [K_k])\{w_j\} = 0$$

This is an Eigen value problem in $'\omega^{2'}$, which can be obtained.

Step 5:

For a static problem, the kinetic energy of the system becomes zero, and the potential energy is equal to the work done by external loading.

Using the Castigliano's theorem (*special case of Euler Lagrangian*):

$$\frac{\partial V}{\partial x_i} = Q_i$$

The final equation for the static beam is:

$$[K_k]\{w_j\} = \{Q_j\}$$

where Q_j is the external load applied on the beam.

Expression for Kinetic Energy (in terms of matrices):

$$T = \frac{1}{2}\rho A\omega^2\{\alpha_i\}^T[X]^T[M][X]\{\alpha_i\}$$

Expression for potential energy

$$V = \frac{1}{2}EI\{\alpha_i\}^T[X]^T[K][X]\{\alpha_i\}$$

[X] Represents the transformation matrix from $\{\alpha_i\}$ to the local displacements of an element

RESULTS:

Initially a computer code was written to solve the natural frequencies of the beam with a given set of mechanical properties.

The computer code was written such that the natural frequencies could be found out for 'n' number of elements, where n is input given as a parameter.

The following graph (*figure 9.1*) was plotted, to study the convergence of the beam model towards the first natural frequency of the beam and to find the number of elements to be considered to do a parametric study on the natural frequency and solve it for any static/dynamic problem.

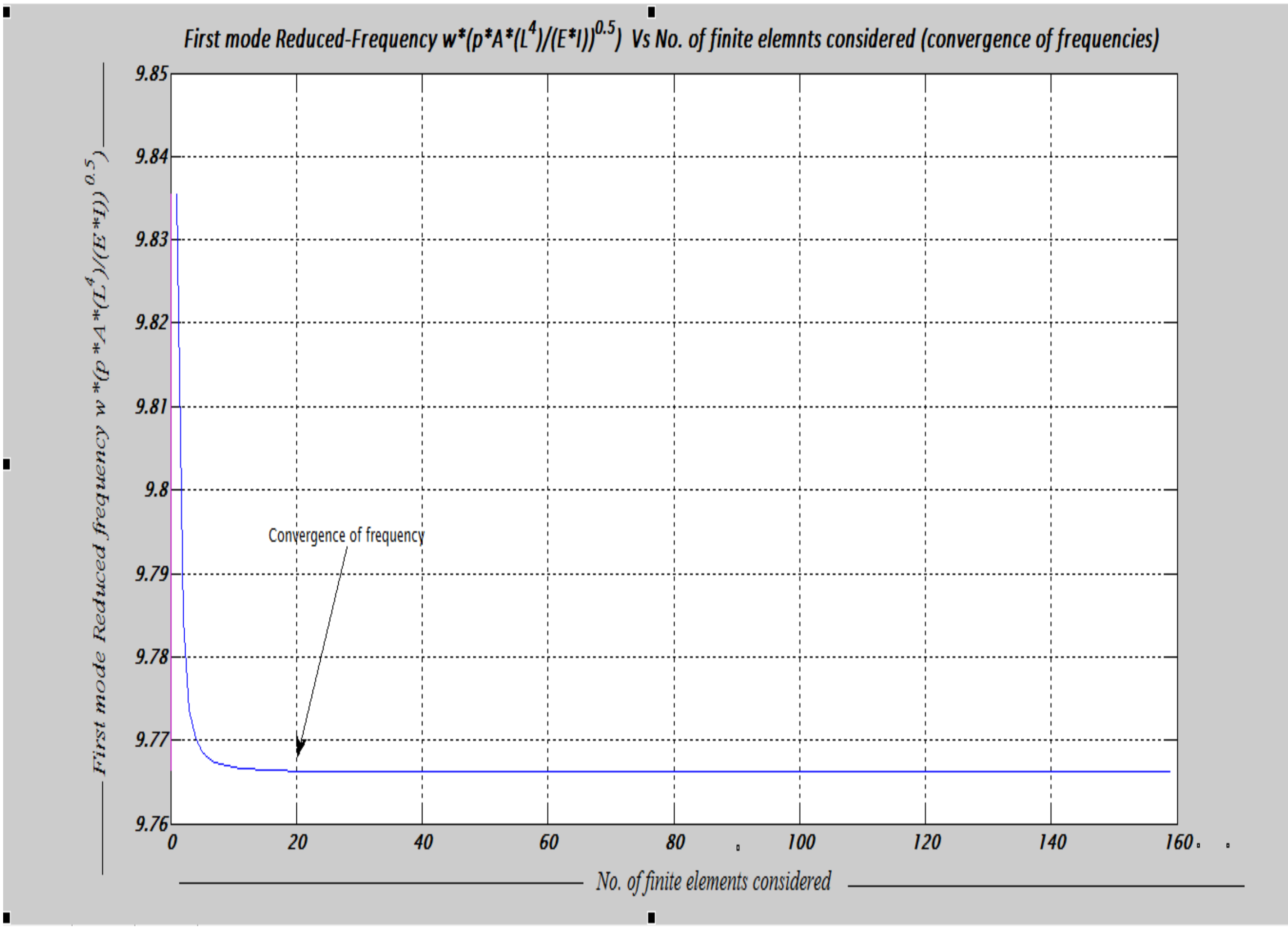

Figure 9.1: Graph indicating the convergence of natural frequency after 20 elements

Observation:

The above plot was generated for the case of a simply supported beam. It can be seen from the graph that when a 20 – element beam model is considered, the natural frequency is in admissible range. Henceforth all other calculations for the considered beam were studied considering a 20 element model.

Plots for the case of a simply supported beam; considering up to 20 elements of the beam, each of equal size, varying the value of K (Shear coefficient) alone for a given beam

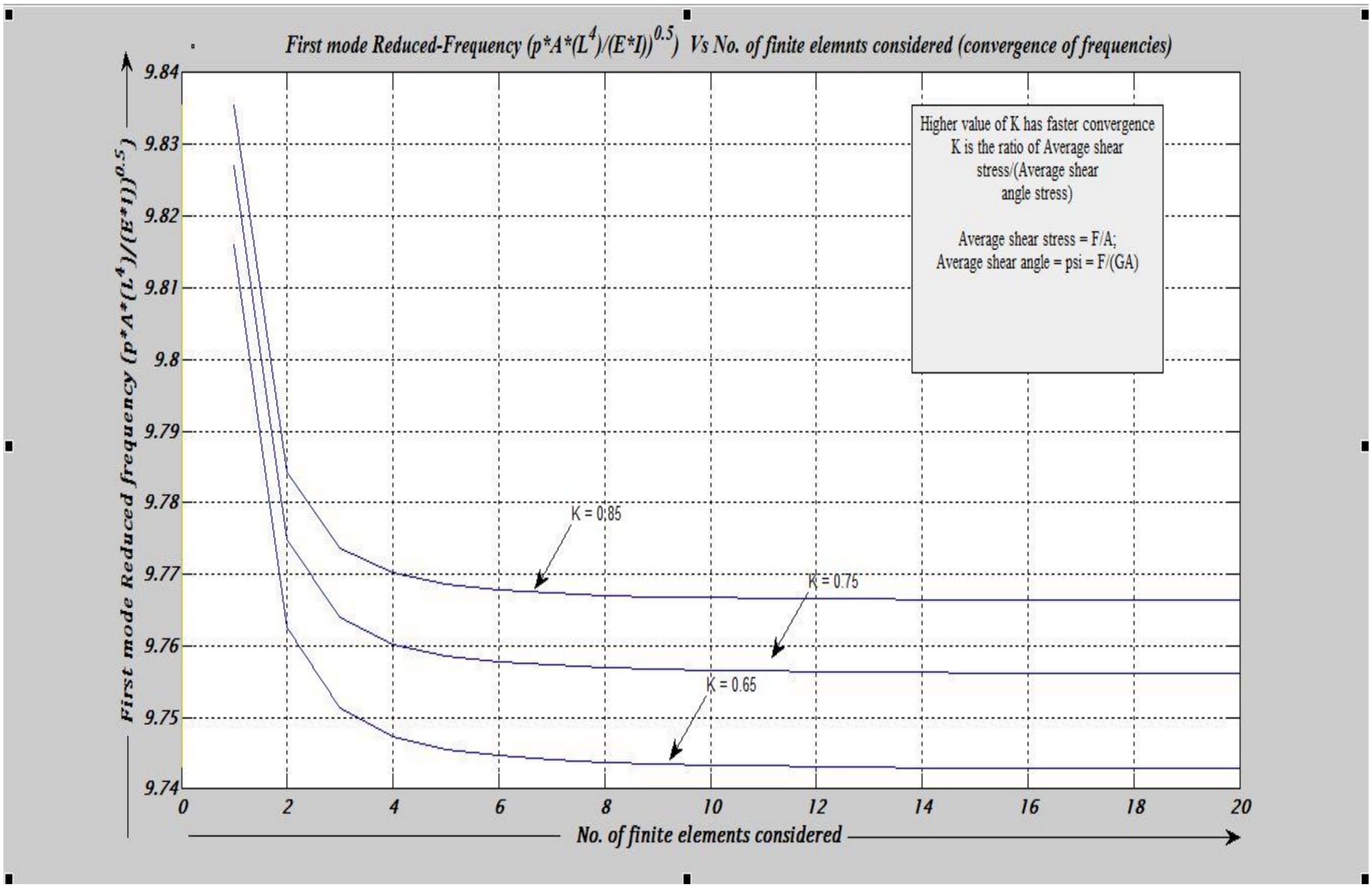

Figure 10.1: Graph indicating the convergence of the first mode natural (reduced) frequency; for various values of the shear coefficient. K = {0.65, 0.75, 0.85} for a simply supported beam

Observation:

It was verified from this graph that at K = 0.85; for the given mechanical properties, the first mode natural frequencies converge faster than when obtained for other values of K.

Plots for the case of a cantilever beam; considering up to 20 elements of the beam, each of equal size, varying the value of K (Shear coefficient) alone for a given beam

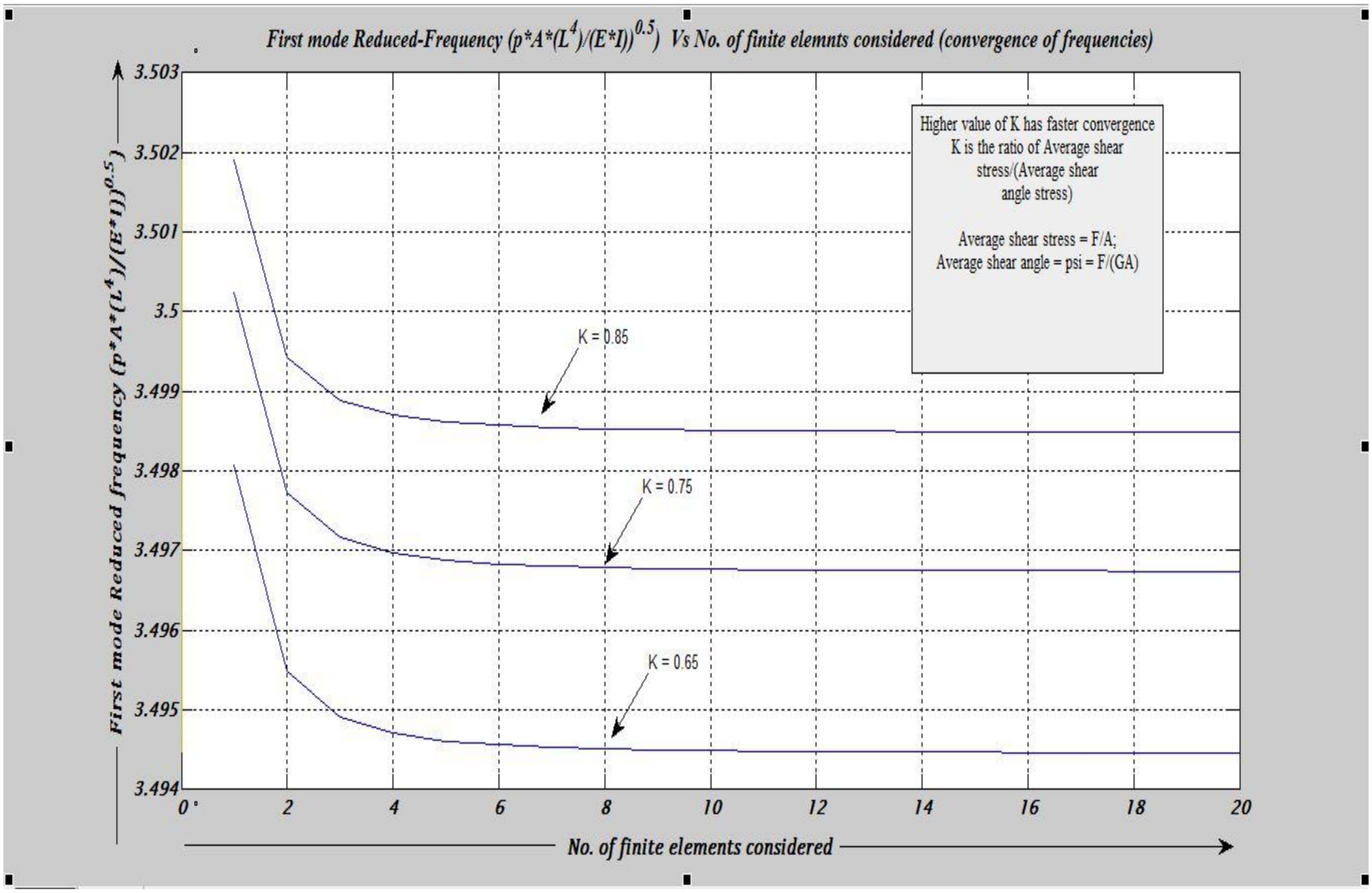

Figure 11.1: Graph indicating the convergence of the first mode natural (reduced) frequency; for various values of the shear coefficient. K = {0.65, 0.75, 0.85} for a cantilever beam

Observation:

It was verified from this graph that at with an increase in the value of K(shear coefficient); for the given mechanical properties, the first mode natural frequencies converge faster.

Plots for the variation of natural frequencies with respect to the h/L ratio; considering up to 20 elements of the beam model, each of equal size, for both the (1) simply supported beam and the (2) cantilever beam for a given mechanical beam properties*

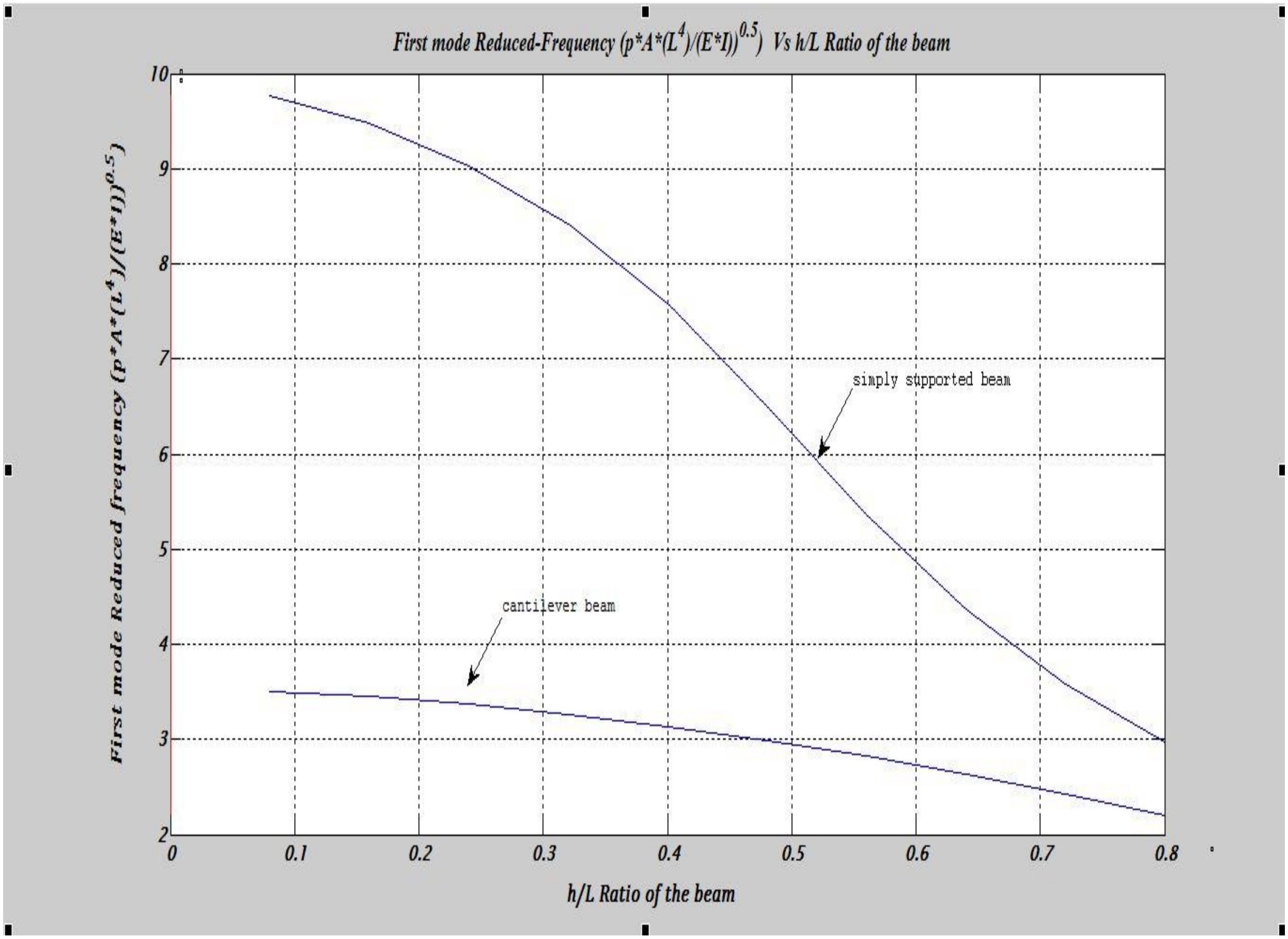

Figure 12.1: Graph indicating the variation of the first mode natural (reduced) frequency; for various values of the Thickness-Length ratio for both the cases (a) simply supported beam, (b) cantilever beam.

Observation:

It can be seen from the plot, the first mode natural (reduced) frequency decreases with increase in the thickness of the beam, with respect to the length of the beam.

The thickness/Length ratio directly also indicates the ratio between 'I' and 'A*L^2' for the considered beam, so the variation of reduced frequency with (A*L^2/I) would indicate the same conclusion as from the above graph.

Plot for a comparison between the Euler-Bernoulli beam theory with the Timoshenko beam theory, for the case of simply supported beam.

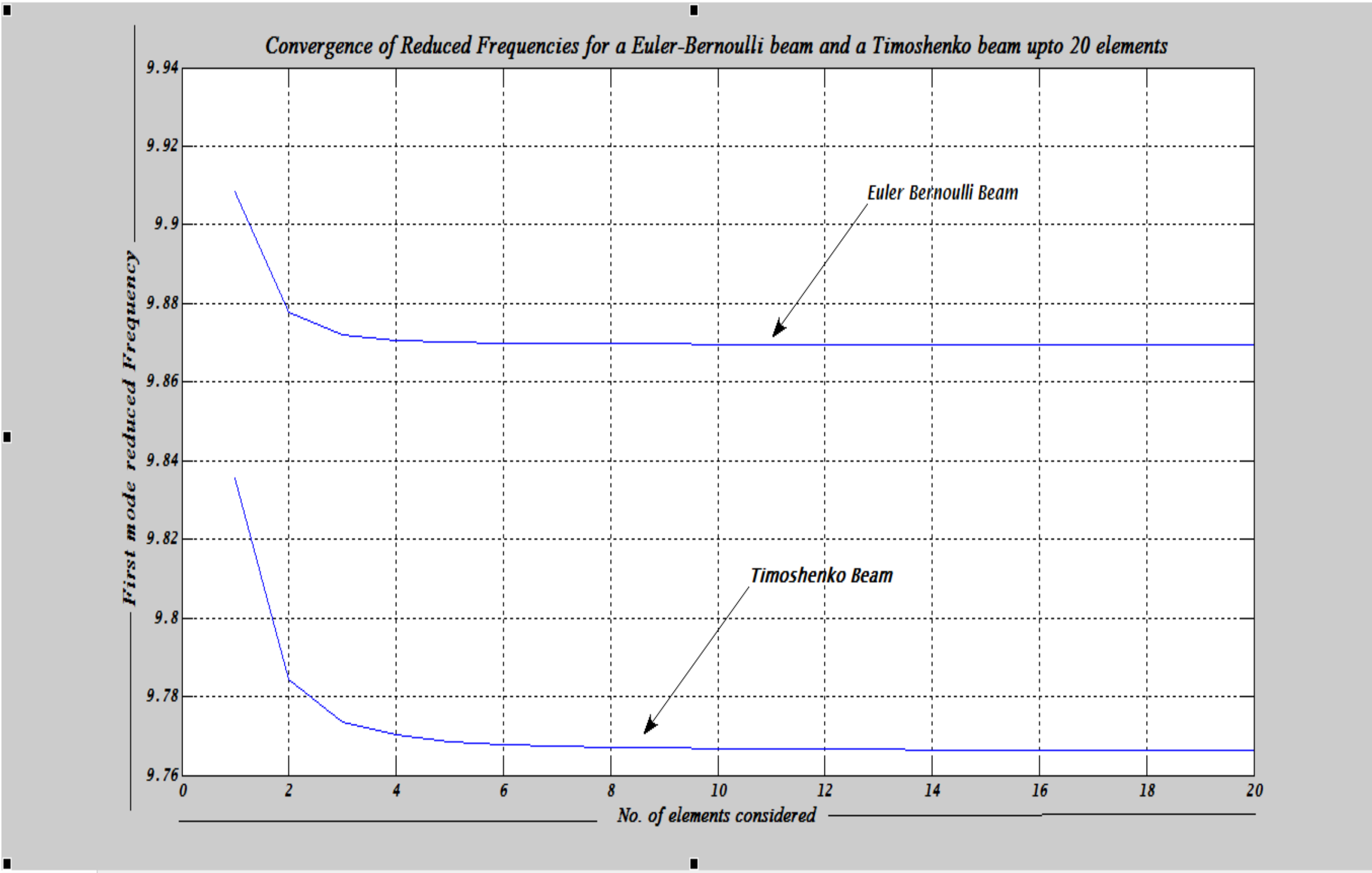

Figure 13.1: Graph indicating the variation of the first mode natural (reduced) frequency; for a Timoshenko beam and a Euler-Bernoulli beam for the case of a simply supported beam.

Observation:

As expected the frequencies for the Timoshenko Beam are lower when compared with the Euler-Bernoulli beam. This indicates that a Timoshenko beam exhibits more rigidity than the Euler-Bernoulli beam.

Plot of Variation of reduced natural frequency, with variation in the ratio of thickness/length of the beam for the case of a simply supported beam.

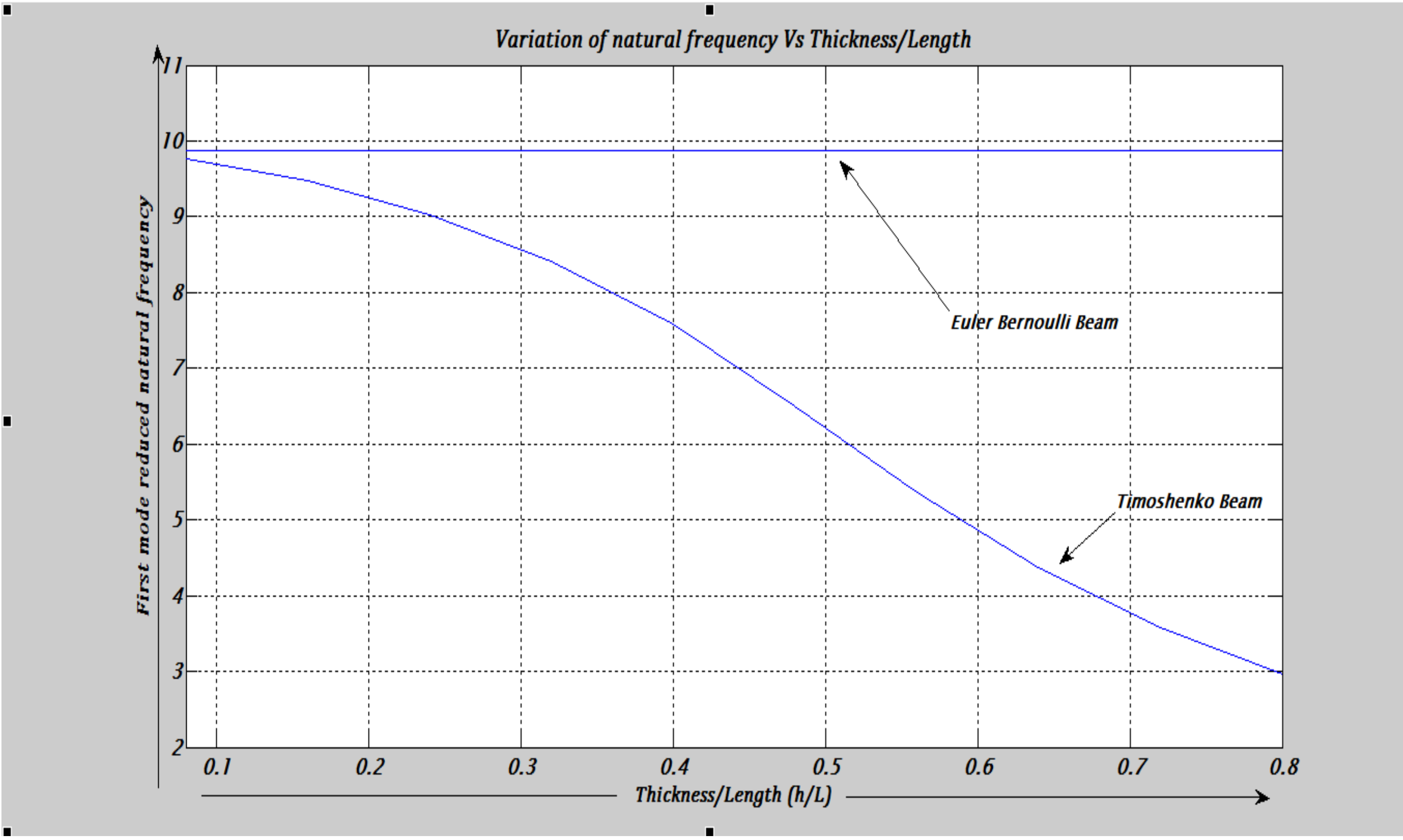

Figure 14.1: Graph indicating the variation of the first mode natural (reduced) frequency; for a Timoshenko beam and an Euler-Bernoulli beam for the case of a simply supported beam with the variation in the thickness/length ratio

Observation:

It is observed that for higher values of thickness/length ratio; the reduced frequency of the Timoshenko beam has decreased. The graph indicates that when the ratio of thickness/Length approaches a higher value, the beam behaves as a rigid structure.

Comparison between the natural frequencies (reduced), obtained for a Timoshenko beam and a Euler-Bernoulli beam

		Reduced Natural Frequencies for the two boundary conditions			
Mode number	No. of DOF	Timoshenko beam		Euler Bernoulli beam	
		Simply supported	Cantilever	Simply supported	Cantilever
1	6	9.7686	3.4986	9.8701	3.5160
2	6	38.0745	21.3346	39.5104	22.0399
3	6	83.3833	57.8693	89.1770	61.8101
4	6	146.9474	110.9557	159.7802	121.6810
1	10	9.7669	3.4985	9.8697	3.5160
2	10	37.9456	21.3090	39.4826	22.0352
3	10	81.7582	57.2709	88.8739	61.7129
4	10	137.5841	106.3908	158.1753	121.0171
1	16	9.7664	3.4985	9.8696	3.5160
2	16	37.9148	21.3028	39.4791	22.0346
3	16	81.3966	57.1376	88.8338	61.6997
4	16	135.5174	105.4171	157.9547	120.9202
1	20	9.7663	3.4985	9.8696	3.5160
2	20	37.9088	21.3016	39.4787	22.0345
3	20	81.2391	57.1129	88.8295	61.6982
4	20	135.1418	105.2425	157.9306	120.9095

Plot of displacement at the free end of a cantilever beam, when a sinosudially oscillating force of magnitude 500kN was applied, with a frequency w= 2.8 kHz (within range of first mode frequency)

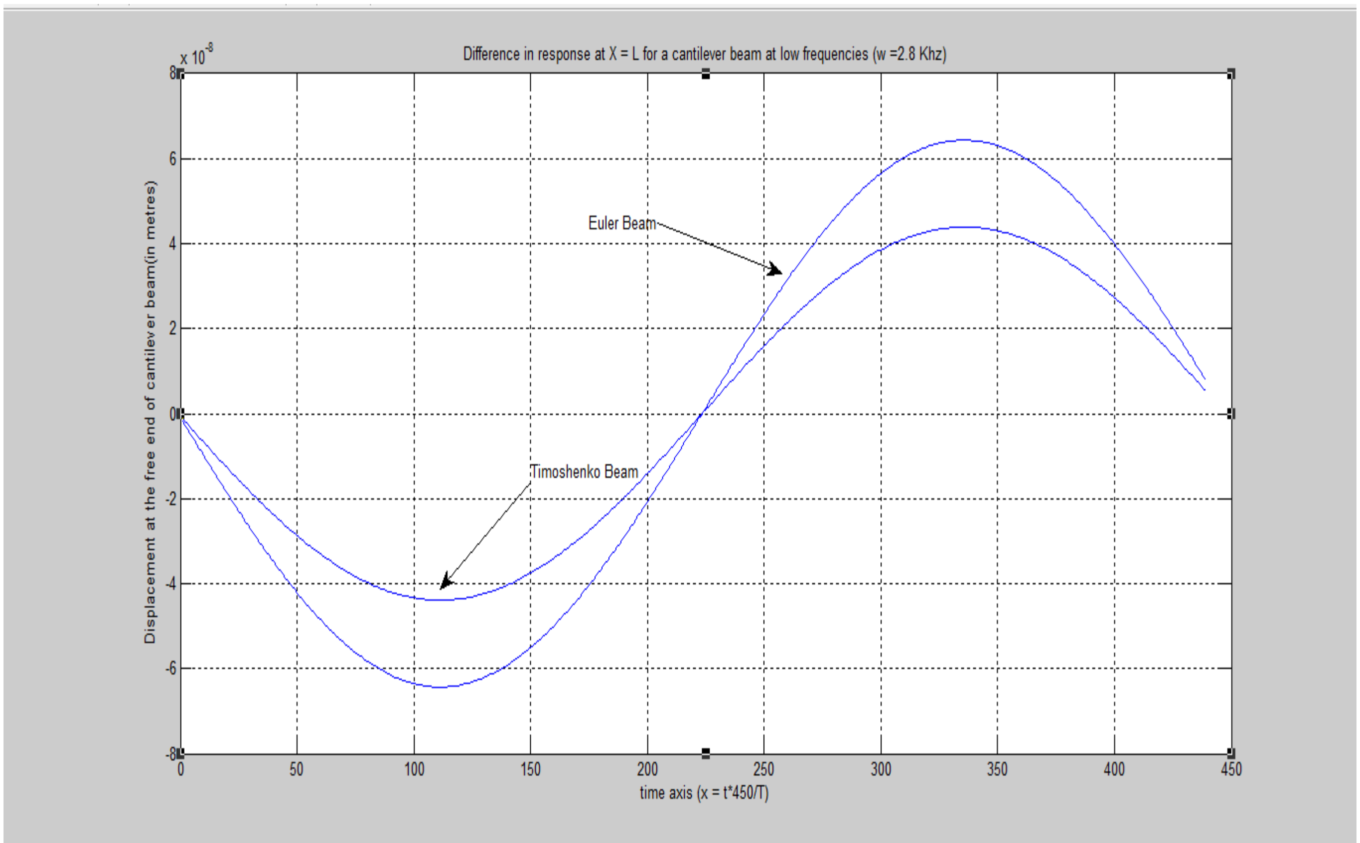

Figure 15.1: Graph indicating the displacement at the free end of a cantilever beam for one complete time period

At low frequencies (w is closer to first mode natural frequency; considering a steel beam of length 120m), both the Euler Bernoulli and the Timoshenko beam, respond similarly

Plot of displacement at the free end of a cantilever beam, when a sinosudially oscillating force of magnitude 500kN was applied, with a frequency w= 28 MHz (within range of first mode frequency)

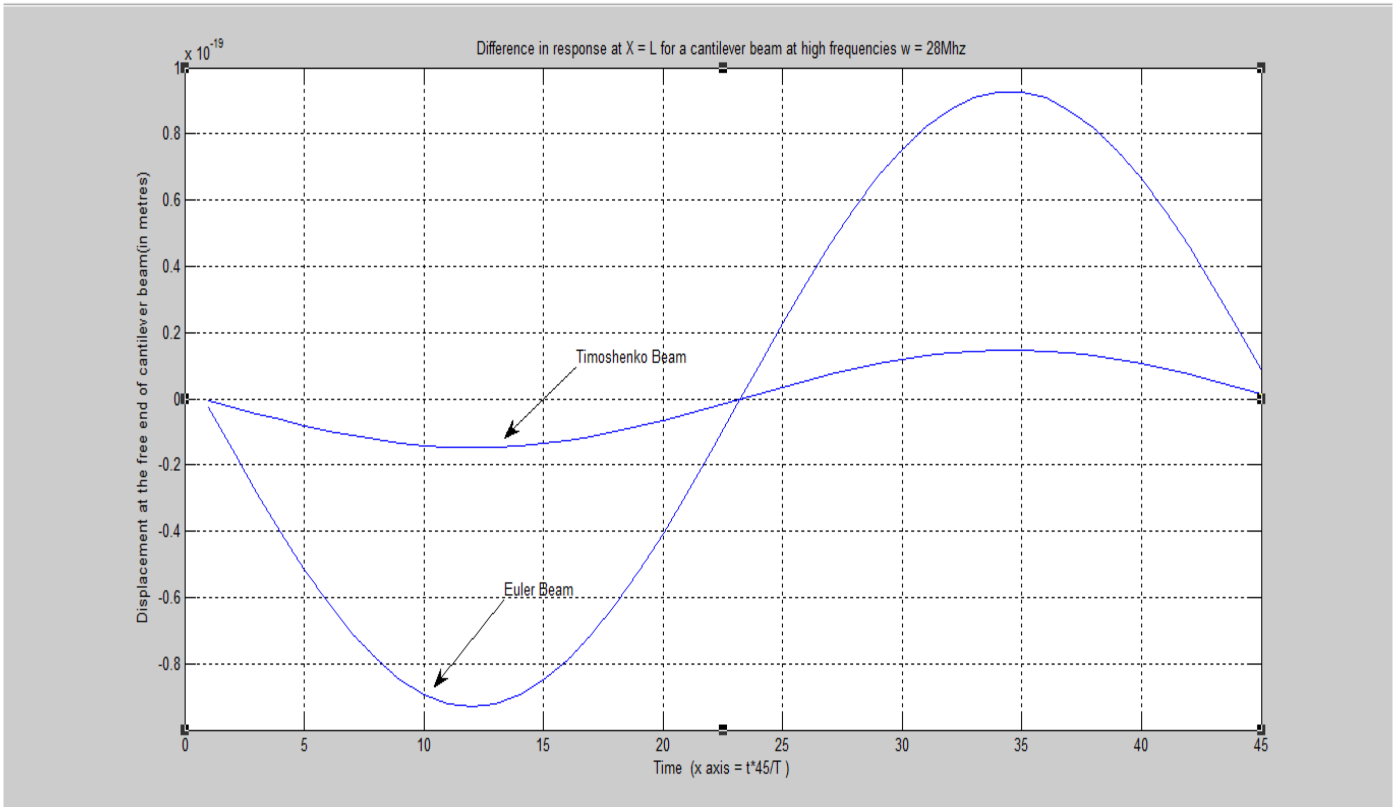

Figure 16.1: Graph indicating the displacement at the free end of a cantilever beam for one complete time period

At high frequencies (w = 28 MHz), the difference in the response by a Euler Bernoulli beam and the Timoshenko beam is significantly different, as some part of energy is consumed by shear deformation of cross section

Mode shapes (for 2nd and 3rd mode of simply supported beam)

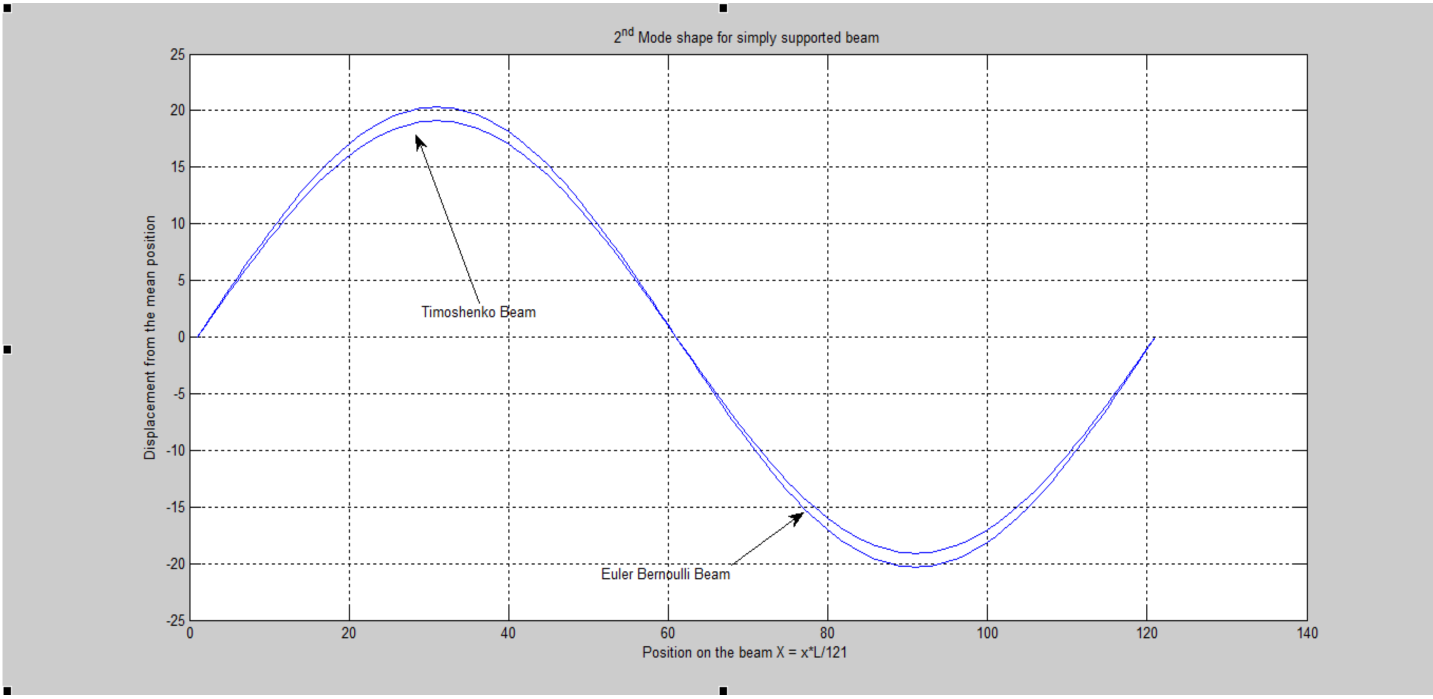

Figure 17.1: Graph indicating the 2nd mode shapes of the simply supported beam

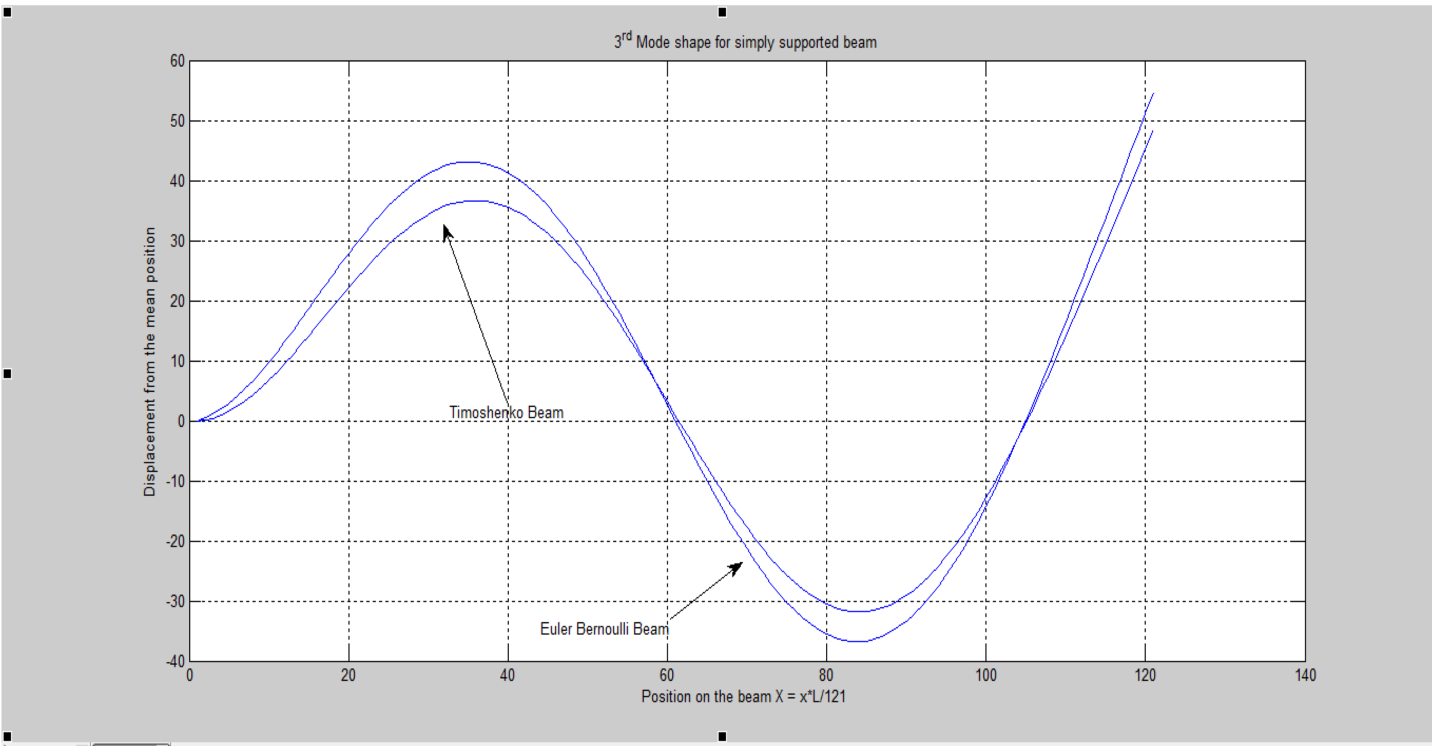

Figure 17.2: Graph indicating the 3rd mode shapes of the simply supported beam

Mode shapes (for 2nd and 3rd mode of Cantilevered beam)

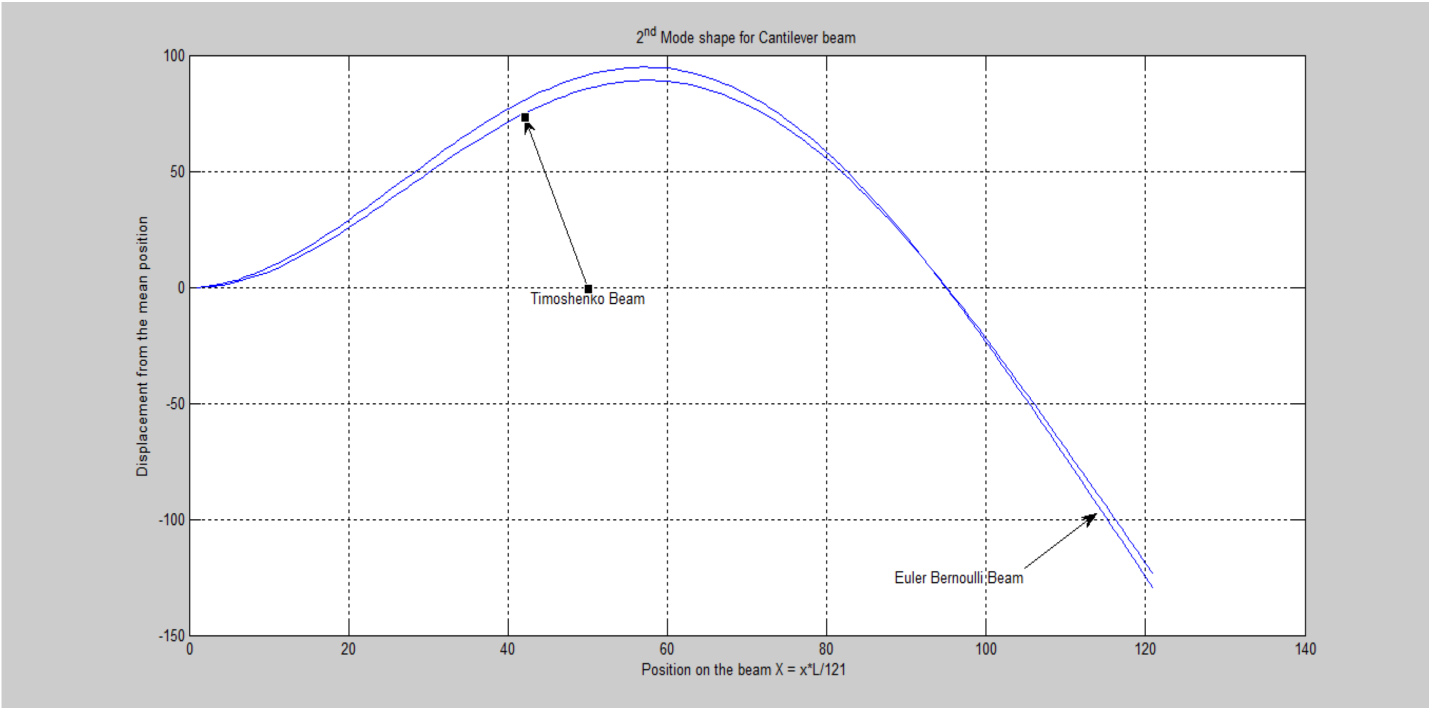

Figure 18.1: Graph indicating the 2^{nd} mode shapes of the cantilever beam

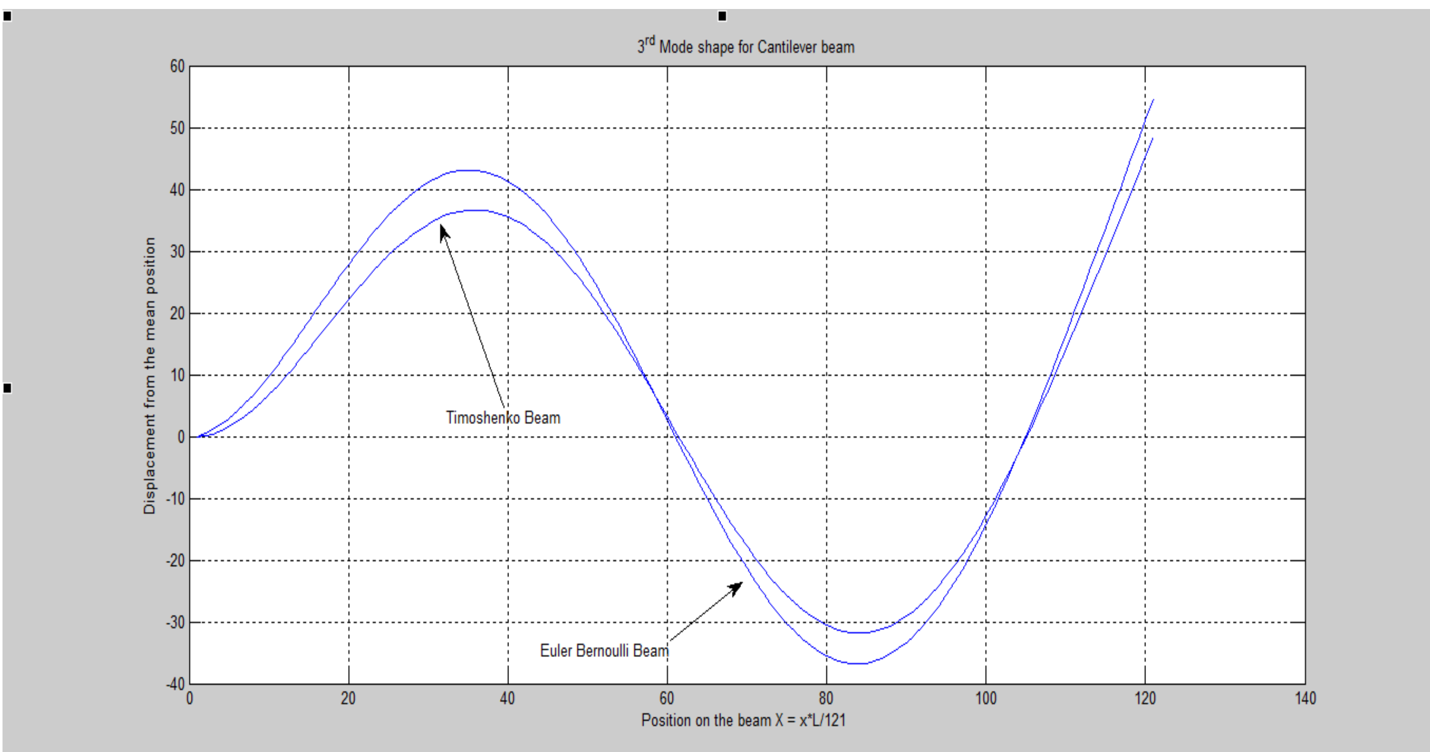

Figure 18.2: Graph indicating the 3^{rd} mode shapes of the cantilever beam

Conclusion:

A beam element which includes the effects of shear deformation and rotary inertia has been presented which converges when a 20 element model is considered.

It has also been shown that how the natural frequencies of the beam vary as per both the theories available. This has been established by varying the geometry of the beam

Further, with the finite element model developed, it has also been established that for forced vibrations of the beam at high frequencies, the response of the beam has been significantly different. However at very high frequencies both the theories would be unsatisfactory because, the above theories are only the approximated solutions at different loading conditions.

The finite element code developed consists of the mass matrix and the stiffness matrix, which can further be used to find displacements, reaction forces for any kind of loading problem for a uniform beam, by specifying the proper boundary conditions and the external forces.

Abstract

The Euler Bernoulli beam theory was one of the earliest theory, based on simplifying the theory of elasticity, which provides the means to calculate the relation between the load and the deflection on a beam. However, for thicker beams a more improved theory had to be considered, which accounts for the shear deformation experienced along the cross section of the beam. Thus the Timoshenko beam theory was proposed, considering the rotary effects of the beam and the shear deformations at the cross section, though retaining some parts of the assumptions made in the Euler-Bernoulli beam. In the present report, a finite element model has been developed for a Timoshenko beam. When an acceptable number of elements required for convergence was obtained, a comparison was made with the Euler Bernoulli beam with variation in the geometry, making the beam thicker and the general response of the beam at higher frequencies thereby verifying the advantage of the Timoshenko beam theory over the former theory, with the help of the finite element model developed.

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Appendix

$[K]$ = [$\frac{L^3EI}{6}+\frac{kL}{2}$, $\frac{L^2EI}{4}$,0 ,0 ;
$\frac{L^2EI}{4}$, $\frac{LEI}{2}$, 0, 0 ;
0 , 0, 0, 0;
0 , 0, 0, 0]

$[M]$ = [$\frac{L^7}{252}+\frac{h^2L^5}{240}+\frac{h^2L^3k}{36}+\frac{h^2k^2L}{12}$, ... ;

$\frac{L^6}{72}+\frac{h^2L^4}{48}+\frac{h^2L^2k}{24}$, $\frac{L^5}{20}+\frac{h^2L^3}{36}$,.......;

$\frac{L^5}{30}+\frac{h^2L^3}{72}+\frac{h^2Lk}{12}$, $\frac{L^4}{8}+\frac{h^2L^2}{24}$, $\frac{L^5}{3}+\frac{h^2L}{12}$, ... ;

$\frac{L^4}{24}$, $\frac{L^3}{6}$, $\frac{L^2}{2}$, L]

[M] is symmetric matrix

[X]*rt = [$-\frac{L^2}{2},-\frac{L^2}{2},-\frac{L^1}{1},\frac{L^1}{1}$;

$\frac{L^3}{3}+kL,\ \frac{L^3}{6}-kL,\frac{L^2}{2},-\frac{L^2}{2}$;

$-\frac{L^4}{12}+k\frac{L^2}{2},k\frac{L^2}{2},k\frac{L}{2},-k\frac{L}{1}$;

0, 0,$-\frac{L^4}{12}+k\frac{L^2}{2}$, 0]

rt = $-\frac{L^4}{12}+k\frac{L^2}{2}$

k = EI/GKA

A – Area of cross section
I – Moment of inertia = $bh^3/12$
E – Young's Modulus
υ – Poisons ratio
L – Length of the beam
h – Thickness of the beam
ρ – Density of the beam
ω – Natural frequency
y – Translational displacement
θ – Angular displacement