Tata Institute of Fundamental Research
Centre for Applicable Mathematics
Functional Analysis
Assignment 1
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1. If X is a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, then any two norms on X are equivalent. Also prove that any finite dimensional normed linear space is a Banach space.

2. Let $X = C^1[0,1]$, then X is vector space over $\mathbb{R}$ or $\mathbb{C}$ w.r.t. the usual addition and scalar multiplication. Check whether X is a normed linear space in the following cases. Also check whether they are Banach spaces.

$$(a)\quad ||f|| = \sup_{x\in[0,1]} |f'(x)|. \qquad (b)\quad ||f|| = \left(\int_0^1 |f(x)|^p dx\right)^{1/p}, \quad 0 < p < \infty.$$

$$(c)\quad ||f|| = \int_0^{1/2} |f(x)|dx + \sup_{x\in[0,1]} |f'(x)|. \quad (d)\quad ||f|| = \sup_{x\in[0,1]} \big(|f(x)| + |f'(x)|\big).$$

$$(e)\quad ||f|| = \Big(\int_0^1 (|f(x)|^p + |f'(x)|^p)dx\Big)^{1/p} \quad 1 \le p < \infty. \quad (f)\quad ||f|| = \max_{x\in[0,1]}\{|f(x)|, |f'(x)|\}.$$

3. Let $C_c(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$ with compact support and $C_0(\mathbb{R})$ is the space of all continuous functions on $\mathbb{R}$ such that for any $\epsilon > 0$ there exists a compact subset K of $\mathbb{R}$ such that

$$|f(x)| < \epsilon \quad \forall\ x \in \mathbb{R} \setminus K.$$

Now for $f \in C_c(\mathbb{R})$ or $C_0(\mathbb{R})$ define

$$||f||_\infty := \sup_{x\in\mathbb{R}} |f(x)|.$$

Then prove the following:

(a) $\big(C_c(\mathbb{R}), ||.||_\infty\big)$ *is a normed linear space.*
(b) $\big(C_0(\mathbb{R}), ||.||_\infty\big)$ *is a Banach space.*
(c) $\big(C_c(\mathbb{R}), ||.||_\infty\big)$ *is dense in* $\big(C_0(\mathbb{R}), ||.||_\infty\big)$.

4. Let $1 \le p \le \infty$ and l^p denotes

$$l^p := \Big\{(x_n)_{n=1}^\infty : \sum_{n=1}^\infty |x_n|^p < \infty, x_n \in \mathbb{C}\Big\}, \quad 1 \le p < \infty$$
$$l^\infty := \Big\{(x_n)_{n=1}^\infty : x_n \in \mathbb{C},\ \sup_{n\in\mathbb{N}} |x_n| < \infty\Big\}.$$

Then $\left(l^p, ||.||_p\right)$ is a Banach space over $\mathbb{C}$ w.r.t. usual addition and scalar multiplication, where $||.||_p$ is given by

$$||(x_n)||_p = \begin{cases} \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sup_{n\in\mathbb{N}}(|x_n|), & p = \infty. \end{cases}$$

Prove that for $1 \le p < \infty$, the space l^p is separable, but l^∞ is not separable.

5. Let X, Y be a topological vector spaces. Assume that X has a countable local base at 0. Prove that a mapping $f : X \to Y$ is continuous iff it is sequentially continuous. Sequentially continuous means: $x_n \to x$ in $X \Rightarrow f(x_n) \to f(x)$ in Y.

6. Suppose X is a vector space. Then prove the following statements:

   (a) A is convex iff $(s+t)A = sA + tA$, for all positive scalar s, t.

   (b) Every union (and intersection) of balanced sets is balanced.

   (c) Every intersection of convex sets is convex.

   (d) If A and B are convex (or balanced) then $A + B$ is convex (or balanced)

7. Suppose X is a topological vector space. Then prove the following statements:

   (a) Convex hull of every open set is open.

   (b) If A and B are bounded then $A + B$ is bounded.

   (c) If A and B are compact then $A + B$ is bounded.

   (d) If A and B are compact then $A + B$ is compact.

   (e) If A is compact and B is closed, then $A + B$ is closed.

   (f) Sum of two closed sets may fail to be closed.

   (g) A set B is bounded iff every countable subset of B is bounded.

   (h) $\overline{x + A} = x + \bar{A}$.

   (i) $A + B^\circ \subset (A + B)^\circ$.

   (j) Let A be a convex subset of X with a nonempty interior, then

      i. $\overline{A} = \overline{A^\circ}$.

      ii. $A^\circ = (\overline{A})^\circ$.

   (k) Every Cauchy sequence in X is bounded.

   (l) If X is locally bounded then X is metrizable.

8. Let X be a vector space over $I\!R$ or $\mathbb{C}$. Let A be a convex balanced absorbing set. Let P_A be the Minkowski function of A. Then the following are equivalent

   (a) If A contains any linear subspace Y, then $Y = \{0\}$.

   (b) P_A is norm on X.