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Functional Analysis Homework 1

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Question 1. If X is a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, then any two norms on X are equivalent. Also prove that any finite dimensional normed linear space is a Banach space.

Proof. Consider any 2 norms over X. Let them be n_1 and n_2 . Since X is a finite dimensional linear space, we can seek a finite basis $\{x_i\}_{i=1}^{n}$. Now, since n_1 and n_2 are norms, we have that $\{n_1(x_i)\}_{i=1}^{n}$ and $\{n_2(x_i)\}_{i=1}^{n}$ are finite and positive.
$\implies$ we can find $c_1 > 0$ and $c_2 > 0$ such that

$$Max(\{n_2(x_i)\}_{i=1}^{n}) < c_1 * n_1(x_i)$$

$$Max(\{n_1(x_i)\}_{i=1}^{n}) < c_2 * n_2(x_i)$$

Consider any x ın X; then $n_1(x) = n_1(\sum_{i=1}^{n} p_i * x_i)$ (by the definition of basis)
Using the definition of norm we then have that:

$$n_1(x) = \sum_{i=1}^{n} |p_i| * n_1(x_i) \leq n * |p| * Max(n_1(x_i))$$

$$\leq n * |p| * c_2 * \sum_{i=1}^{n} |p_i| * n_2(x_i) \text{ (using the above inequalities)}$$

$$\implies n_1(x) \leq n * |p| * c_2 * n_2(x) \leq M_1 * n_2(x)$$

Similarly we can prove that

$$\implies n_2(x) \leq n * |p| * c_1 * n_1(x) \leq M_2 * n_1(x)$$

Hence, we have established that the norms are equivalent and since they have been chosen arbitrarily we have that any two norms are equivalent.

X is a Banach space

Consider any Cauchy sequence $\{y_n\} \in$ X. Each y_m can be represented as:

$$y_m = (\sum_{i=1}^{n} p_i^m * x_i)$$

It follows that p_i^m is a cauchy(for a given i) if y_m is, for each i. And since p_i^m is in $\mathbb{R}$ or $\mathbb{C}$, we know that this sequence(cauchy) converges to some p_i in $\mathbb{R}$ or $\mathbb{C}$ accordingly.
It can then be verified by using the definition of norm, that y_m converges to $\sum_{i=1}^{n} p_i * x_i)$ $\square$

Question 2. Let X = $C^1[0,\ 1]$, then X is vector space over $\mathbb{R}$ or $\mathbb{C}$ w.r.t. the usual addition and scalar multiplication. Check whether X is a normed linear space in the following cases. Also check whether they are Banach spaces.

a. $\|\text{f}\| = \sup_{x\in[0,1]} |f'|$
Consider the constant function f = c for some c≠0 in [0,1], $\implies$ f'(x) = 0 in [0,1].
$\implies$ $\|\text{f}\| = 0$; but f $\neq 0$ in $C^1[0,1]$
Hence, this does not define a norm ($\|\text{f}\| = 0$ does not necessarily imply f = 0) $\square$

b. $\|\text{f}\| = (\int_0^1 |f(x)|^p)^{\frac{1}{p}}$

$\|f\|$ defines a norm for p $\geq$ 1, as a consequence of Minkowski's inequality. For p < 1, we have the counter example $f, g \in C^1[0,1]$ as follows:

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ (1/2 - x)^2 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$g(x) = \begin{cases} (1/2 - x)^2 & 0 \leq x \leq \frac{1}{2} \\ 0 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

We obtain that, since f, g are symmetric wrt. $x = 1/2$;

$$\|f + g\| = 2^{1/p}\|f\| = 2^{1/p}\|g\|$$

$$\implies \|f + g\| \geq \|f\| + \|g\| \text{ ; when p} < 1$$

Hence, the triangular inequality is violated whenever

To check if it's a banach space for p $\geq$ 1.
We will be using the result that

$$\left(\int_x^y |f(x)|^p\right)^{\frac{1}{p}} \leq \left(\int_0^1 |f(x)|^p\right)^{\frac{1}{p}}$$

for $0 \leq x \leq y \leq 1$ and

$$\left(\int_0^1 |f(x)|^p\right)^{\frac{1}{p}} \leq \left(\int_0^1 |g(x)|^p\right)^{\frac{1}{p}}$$

whenever $f \leq g \in$ [0,1] (These are trivially true when given f and g are continuous functions)
Note that $\|f\| \leq \sup_{x\in[0,1]} |f| \implies$ A cauchy sequence in the sup norm is always a cauchy sequence in the given norm. Since C[0,1] is complete in the sup norm, consider any cauchy sequence (for ex. polynomials) converging to a continuous function that is not differentiable everywhere. Define one such function, f as follows:

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 1-x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Clearly, f is continuous but not differentiable (at x = $\frac{1}{2}$). We can construct a sequence of polynomials p_n (the advantage of the choice being that they are continuously differentiable) uniformly converging to f in [0,1]. Note that these p_n will be cauchy in the l_p sense above as well. But f is not in $C^1[0,1]$. The proof is complete by realizing the below claim.
Claim: *p_n cannot converge to any continuous function, other than f in the l_p sense.*
Assume $p \neq f$ be the other desired continuous function. Then $p(x) \neq f(x)$ for at least one x $\in$ (0,1). Let $M = |f(x) - p(x)|$. Since f and p are continuous functions we have that $\exists$ a δ neighborhood around x such that $|f(y) - p(y)| > M/2$ whenever $|y - a| < \delta$. We then have the following:

$$\left(\int_{x-\delta}^{x+\delta} |f(x) - p(x)|^p\right)^{\frac{1}{p}} \leq \left(\int_0^1 |f(x) - p(x)|^p\right)^{\frac{1}{p}}$$

But,

$$\left(\int_{x-\delta}^{x+\delta} |f(x) - p(x)|^p\right)^{\frac{1}{p}} \geq M * \delta/2$$

Using,

$$\left(\int_{x-\delta}^{x+\delta} |f(x) - p(x)|^p\right)^{\frac{1}{p}} \leq \left(\int_{x-\delta}^{x+\delta} |f(x) - p_n(x)|^p\right)^{\frac{1}{p}} + \left(\int_{x-\delta}^{x+\delta} |p(x) - p_n(x)|^p\right)^{\frac{1}{p}}$$

and since p_n converges to f in l_p sense, we have,

$$\left(\int_{x-\delta}^{x+\delta} |p(x) - p_n(x)|^p\right)^{\frac{1}{p}} \geq \left(\int_{x-\delta}^{x+\delta} |f(x) - p(x)|^p\right)^{\frac{1}{p}} \geq M * \delta/2$$

$\therefore p_n$ does not converge to p in the l_p sense. Since f is not in $C^1[0,1]$, the cauchy sequence p_n in $C^1[0,1]$ does not converge with the l_p norm. Hence, it is not a Banach space. $\square$

c. $\|f\| = \int_0^{\frac{1}{2}} |f(x)| + \sup_{x\in[0,1]} |f'(x)|$.
$\|f\|$ defines a norm.

i) $\|c.f\| = |c|\|f\|$
We will use the following properties of $C^1[0,1]$

$$\Big(\int_a^b |cf(x)|\Big) = |c|\Big(\int_a^b |f(x)|\Big)$$

and

$$(cf(x))' = c(f'(x))$$

ii) $\|f+g\| \leq \|f\| + \|g\|$
Follows because,

$$\Big(\int_a^b |f(x)+g(x)|\Big) \leq \Big(\int_a^b |f(x)|\Big) + \Big(\int_a^b |g(x)|\Big)$$

and

$$|(f'(x)+g'(x))| \leq |f'(x)| + |g'(x)|$$

iii) $\|f\| = 0 \implies f = 0$
Let $f \neq 0$, then $\exists$ x such that $|f(x)|$ ¿ ϵ for some ϵ ¿ 0. Further, we can find a δ ¿ 0 neighborhood around x such that $|f(y)| > \epsilon/2$ whenever $|y-x| < \delta$
If $x < 1/2$, then we have that

$$\Big(\int_0^{1/2} |f(x)|\Big) \geq \epsilon * \delta/2 > 0$$

else, we have that $f(y) = 0$ for any $y \leq \frac{1}{2}$ and $f(x) > 0$. Then, by mean value theorem we have that $\exists$ c $\in$ [y,x] such that $|f'(c)| = |\frac{f(x)-f(y)}{x-y}| > 0$.
$\implies sup_{x\in[0,1]}|f'(x)| > 0$
$\therefore f \neq 0, \implies \|f\| \neq 0$

Note that, $\sup_{x\in[0,1]} |f'(x)| < \|f\|$. $\implies$ if f_n is cauchy in the given norm, then it is cauchy in the sup-norm.
Now, consider any cauchy sequence f_n w.r.t the given norm, then f_n' converges uniformly. And since the sup-norm is complete w.r.t continuous functions, we can find a g such that f_n' converges uniformly to g in the sup-norm.
Using the fundamental theorem of calculus, we know that f_n can be written as:

$$f_n(x) = \Big(\int_0^x f_n'(x)\Big) + c_n$$

$$|f_n(x) - f_m(x)| \leq \Big(\int_0^x |f_n'(x) - f_m'(x)|\Big) + |c_n - c_m|$$

It follows that f_n converges pointwise iff c_n is cauchy. Since c_n is cauchy in $\mathbb{R}$, it will converge to some c $\in \mathbb{R}$. Hence, defining

$$f(x) = \left(\int_0^x g^(x) \right) + c$$

it follows that f_n converges to f in the sup sense and $f_n^{'}$ converges to ($f^{'} = g$) in the sup sense. And that every cauchy sequence converges in $C^1[0,1]$ w.r.t the given norm. $\therefore$ the norm defines a banach space ☐

d. $\|f\| = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|$.

$\|f\|$ defines a norm.

i) $\|c.f\| = |c|\|f\|$
We will use the following properties of $C^1[0,1]$

$$(cf(x)) = c(f(x)), (cf(x))' = c(f'(x)),$$

ii) $\|f+g\| \leq \|f\| + \|g\|$
Follows because,

$$|(f(x)+g(x))| \leq |f(x)| + |g(x)|$$
$$|(f'(x)+g'(x))| \leq |f'(x)| + |g'(x)|$$

iii) $\|f\| = 0 \implies f = 0$
Let $f \neq 0$, then $\exists$ x such that $|f(x)|$ ¿ ϵ
$\implies sup_{x\in[0,1]}|f(x)| > 0$
$\therefore f \neq 0, \implies \|f\| \neq 0$

Note that, $\sup_{x\in[0,1]} |f'(x)| < \|f\|$. $\implies$ if f_n is cauchy in the given norm, then it is cauchy in the sup-norm.
Now, consider any cauchy sequence f_n w.r.t the given norm, then $f_n^{'}$ converges uniformly. And since the sup-norm is complete w.r.t continuous functions, we can find a g such that $f_n^{'}$ converges uniformly to g in the sup-norm.
Using the fundamental theorem of calculus, we know that f_n can be written as:

$$f_n(x) = \left(\int_0^x f_n^{'}(x) \right) + c_n$$

$$|f_n(x) - f_m(x)| \leq \left(\int_0^x |f_n^{'}(x) - f_m^{'}(x)| \right) + |c_n - c_m|$$

It follows that f_n converges pointwise iff c_n is cauchy. Since c_n is cauchy in $\mathbb{R}$, it will converge to some c $\in \mathbb{R}$. Hence, defining

$$f(x) = \left(\int_0^x g(x) \right) + c$$

it follows that f_n converges to f in the sup sense and $f_n^{'}$ converges to $(f^{'} = g)$ in the sup sense. $\therefore$ the norm defines a banach space $\square$

e. $\|f\|$ defines a norm (since p $\geq$ 1) as a consequence of Minkowski's inequality.

To check if its a banach space:

Consider the sequence

$$f_n = \frac{1}{1 + (2x)^n}$$

They are point-wise convergent to 1 for x $\in$ [0,1/2) and point-wise convergent to 0 for x $\in$ (1/2,1]. $\implies$ f_n are cauchy in the l_p sense. Now define:

$$g_n(x) = \left(\int_0^x f_n(x) \right)$$

Since f_n are cauchy in the l_p sense, and are continuous, it follows that g_n is continuous and $\{g_n\}$ is cauchy in the sup-norm. $\implies$ $\{g_n\}$ is cauchy in the l_p sense. Further $g_n' = f_n$ is also cauchy in the l_p sense. $\implies$ g_n is cauchy as per the given norm. But, $\exists$ no continuous function f such that f_n converges to f in the l_p sense. If f_n converges to f with respect to the given norm, then f_n converges to f in C[0,k] and C[1-k,1] $\forall$ 0 ¡ k ¡ 1/2. But as per the claim proved in part(b) we will have that f = 0 in [0,1/2) and f = 1 in (1/2,1]. $\implies$ f cannot be continuous at x = 1/2. Contradiction. $\implies$ The space is not Banach.

$\square$

f. $\|f\| = Max(\sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|)$.

The result that this is a norm and defines a banach space follows from the following. The triangular inequality of the norm, also using part(d) follows from

$$\{ \sup_{x\in[0,1]} |f(x)|, \sup_{x\in[0,1]} |f'(x)|\} \leq \|f\|$$

We get equivalence of the two norms as

$$\|f\| \leq \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)| \leq 2 * \|f\|$$

$\square$

Question 3.

a. $C_c(\mathbb{R})$ is a linear space.

If f,g $\in C_c(\mathbb{R})$ then so is f + g, as let K_f and K_g,their supports, are compact, so is their sum (from question 7). Since supp(f + g) $\subset$ $(K_f + K_g)$ and support is a closed set, supp(f+g) is compact.

If f has compact support, so does αf for any scalar α, as f(x) = 0 $\implies$ αf(x) = 0

$\|f\|_\infty$ **is a norm**

i. If f $\neq 0$ in $C_c(\mathbb{R})$, then $|f(x)| > 0$ for some x $\in \mathbb{R}$. $\implies \|f\|_\infty > 0$

ii. $|\alpha f(x)| = |\alpha||f(x)|$. Since f is compactly supported sup is well defined.

iii . Consider h = f + g; Then supp(h) $\subset$ supp(f) + supp (g). Since supp(f,g,h) lie on compact set (union of their supports) the triangular inequality follows. □

b. $C_0(\mathbb{R})$ is a linear space.
If f,g $\in C_c(\mathbb{R})$ then so is f + g, as let K_f and K_g,be compact sets outside which $|f|$ and $|g| < \epsilon/2$, . Since sum of compact sets is compact, say K, have that $|(f+g)| < \epsilon$ outside K.
Similarly αf $\in C_0\mathbb{R}$ whenever f is. Since $\exists$ compact set K_f such that $|f| < \epsilon/|alpha|$, we have that $|\alpha f| < \epsilon$ outside K_f.
Since $C_0(\mathbb{R})$ is a linear space and sup-norms are well defined, it follows that $\|f\|_\infty$ defines a norm.

To prove it is banach or otherwise.
Consider any cauchy sequence $\{f_n\}$. Note that, this sequence is cauchy over every compact set in the sup-norm. $\implies$ Over compact sets [-n,n] (any compact set can be covered by these compact sets), we can find a continuous function f^n to which these functions uniquely converge. $\implies$ the extensions of f^n to f^{n+1} ,such that the above sequences converge over their respective compact sets,are unique and so are the restrictions. Further we have that the convergence is also point-wise over these sets. $\implies$ if we define f as the point-wise limit(exists as they are Cauchy) of f_n,then we only need to prove that $f \in C_0(\mathbb{R})$.
Consider any ϵ ¿ 0 Since f_n is cauchy, choose N such that $|f_N - f_m| < \epsilon/2$ $\forall$ m > N. Now, $\exists$ a compact set K such that $|f_N| < \epsilon/2$ outside K. $\implies |f_m| < \epsilon$ $\forall$ m > N outside K. $\implies |f| < \epsilon$ (as its defined as a point-wise limit of f_n) outside K. $\implies f \in C_0(\mathbb{R})$. □

c. Consider f $\in C_0(\mathbb{R})$. Let K_1 be a compact set such that $|f| < \epsilon$ outside K_1. Choose n such that $K_1 \subset$ [-n,n] and be a compact set $|f| < \epsilon/2$ outside K_2. Choose m ¿ n such that $K_2 \subset$ [-m,m]. Lets now define a function g as follows:

$$g(x) = \begin{cases} 1 & -n \leq x \leq n \\ (m - |x|)/(m-n) & n \leq x \leq m \\ 0 & |x| \geq m \end{cases}$$

Note that $g \in C_c\mathbb{R}$. Hence, so is fg. Further, $|fg - f| = 0$ over [-n,n]. and $|fg - f| \leq |f|$ outside. $\implies |fg - f| \leq \epsilon$ outside. $\implies \|fg - f\|_\infty \leq \epsilon$
$\therefore C_c\mathbb{R}$ is dense in $C_0\mathbb{R}$

□

Question 4. Prove that for $1 \leq$ p $< \infty$, the space l_p is separable, but l_∞ is not separable.

Proof.

Lets consider the case for p = 1;
Consider the following set of sequences $S_k = \{(x_n) : x_i = 0$ if i > k ; x_i ın $\mathbb{Q}$ otherwise$\}$
It follows that S_k is countable for each k; hence their union taken over k $\in \mathbb{N}$ (represented as S henceforth) is also countable
Now consider any given $(x_n) \in l_1$;
$\implies$ we have that $\sum_{i=n}^{\infty} |x_i| < \epsilon$ for some n $\in \mathbb{N}$
For each x_i for $i < n$we can find q_i (that is rational) such that $|x_i - q_i| < \epsilon/n$ Note that the sequence defined by $\{x_m : x_i = q_i fori < nelsex_i = 0\ \} \in$ S.
Further, $\sum_{i=1}^{\infty} |x_i - q_i| < 2 * \epsilon$ Hence, we have that S(a countable set) is dense in l_1. The same arguments can be repeated for any $1 \leq p < \infty$, given that

$$\sum_{i=1}^{\infty} |x_i|^p < \infty \iff \{\sum_{i=1}^{\infty} |x_i|^p\}^{\frac{1}{p}} < \infty$$

For p = ∞;
Assume $\exists$ a countable dense subset A.
$\implies$ we can represent it as a sequence a_n; where each $a_i \in l_\infty$
Define a sequence q_i as follows:

$$q_i = 0 \text{ if } (|a_i^i| > 1);$$

$$q_i = 2 \text{ otherwise}$$

Note that $(q_i) \in l_\infty$. Further $\|a_j - q_i\| > 1$ $\forall$ j (as $|a_j^j - q_j| > 1$ by definition $\forall$ j) .
$\implies$ A(or any countable subset) is not dense in l_∞
$\implies$ l_∞ is not separable ☐

Question 5. Let X, Y be a topological vector spaces. Assume that X has a countable local base at 0. Prove that a mapping f : X $\to$ Y is continuous iff it is sequentially continuous. Sequentially continuous means: $x_n \to$ x in X $\implies$ f(x_n) $\to$ f(x) in Y

Proof. Assume f is continuous. Consider any neighborhood V of f(x) in Y. Let x_n be a sequence such that $x_n \to$ x in X. Then $f^{-1}(V)$ is a neighborhood of x $\in$ X (as f is continuous). $\implies$ we can find N such that $x_n \in f^{-1}(V)$ [by the definition of convergence] if n > N.
$\therefore f(x_n) \in$ V f $\forall n >$ N. The same arguments holds for any neighborhood V of f(x).
$\implies$ $f(x_n) \to$ f(x) in Y

Converse: Assume f is discontinuous. $\implies$ $\exists$ a open set in Y such that U = $f^{-1}(Y)$ is not open in X.
Let x $\in$ U be s.t. $\nexists$ a open set of x $\in$ U. (we can ask for at least one such x as U is not open)
Since X has a countable base, and is a topological vector space, so does x(by translation).
$\implies$ for each open set x + V_n $\exists$ a x_n such that $x_n \notin$ U.
Clearly, since V_n constitutes the base, we have that $x_n \to$ x $\in$ U.
But $f(x_n) \not\to f(x)$ as no $f(x_n) \in$ V (as $x_n \notin$ U)

Since V is an open neighborhood of $f(x)$ in Y, we have that f is discontinuous (not sequentially continuous at x)

$\square$

Question 6. X is a vector space.

a. A is convex iff (s + t)A = sA + tA, for all positive scalar s, t.
If A is convex, then we know that for positive s,t

$$\frac{s}{s+t}A + \frac{t}{s+t}A = A$$

$$\implies sA + tA = (s+t)A \text{ multiplication scalar on both sides}$$

The converse follows by dividing the above equality by (s+t) on both sides
we will have that :

$$\frac{s}{s+t}A + \frac{t}{s+t}A = A \forall \text{ positive s,t}$$

$\implies$ A is convex (by definition) $\square$

b. Consider an arbitrary union S of balanced sets $\{B_i\}$;

$$S = \cup B_i$$

Consider any x $\in$ S. Then x $\in$ B_i for some index i;
$\implies$ α x $\in$ B_i for | alpha | $\leq$ 1 as B_i is balanced.
$\implies$ α x $\in$ S which implies that S is balanced.
Now, for arbitrary intersection T of $\{B_i\}$

$$T = \cap B_i$$

Consider any x $\in$ T. Then x $\in$ B_i for all i;
$\implies$ $\alpha x \in B_i$ for $|alpha| \leq 1$ as B_i is balanced and this is true for each B_i.
$\implies$ α x $\in$ T which implies that T is balanced. $\square$

c. Consider an arbitrary intersection S of convex sets $\{B_i\}$;

$$S = \cup B_i$$

Consider any x,y $\in$ S. Then x,y $\in$ B_i for each i;
$\implies$ s*x + (1-s)*y $\in$ B_i for each B_i, whenever s $\in$ [0,1] as it is convex
$\implies$ s*x + (1-s)*y $\in$ S, whenever s $\in$ [0,1]. $\therefore$ S is convex. $\square$

d. Given A and B are convex.
Let S = A + B. Then consider any x,y $\in$ S.
Then x can be written as x = x_A (which is in A) + x_B (which is in B). Similarly y = y_A + y_B

Now consider: s*x + (1-s)*y (can be expanded using distributive property as) =
(s*(x_A) + (1-s)*y_A) + (s*(x_B) + (1-s)*y_B) ;
Note that (s*(x_A) + (1-s)*y_A) $\in$ A as A is convex (s $\in$ [0,1]) and
(s*(x_B) + (1-s)*y_B) $\in$ B as B is convex (s $\in$ [0,1]).
$\implies$ s*x + (1-s)*y $\in$ S whenever x,y $\in$ S and s $\in$ [0,1] $\therefore$ S is convex
Given A and B are bounded.
Let S = A + B. Then consider any x$\in$ S.
Then x can be written as x = x_A (which is in A) + x_B (which is in B).
Now consider: $\alpha x = \alpha(x_A) + \alpha(x_B)$; ($|\alpha| \leq 1$)
Note that $\alpha(x_A) \in$ A as A is bounded and similarly $\alpha(x_B) \in$ B
$\implies$ $\alpha(x_A) + \alpha(x_B) = \alpha x \in$ S $\therefore$ S is bounded $\square$

Question 7. X is a topological vector space.

b. Given A and B are bounded. Let V be any 0 neighborhood. Then we can find a balanced neighborhood W such that W $\subset$ V. Now, given A we can find $t_A > 0$ such that A $\subset$ t_AW , as A is bounded similarly we can find $t_B > 0$ such that B $\subset$ t_BW.
$\implies$ A + B $\subset$ t_AW + t_BW $\subset$ $(t_A + t_B)$W (as W is balanced). Since W $\subset$ V, we have $(t_A + t_B)$W $\subset$ $(t_A + t_B)$V
$\implies$ (A + B) $\subset$ $(t_A + t_B)$V $\therefore$ A + B is bounded $\square$

c. Given A and B are compact. We know that compact sets are bounded in X (i.e. a topological vector space)
From part (b) we have that some of 2 bounded sets is bounded.
$\implies$ A + B is bounded $\square$

d. Given A and B are compact. Since addition is a continuous operation, we have that A + B is compact if A and B are compact $\square$

e. Given A is closed and B is compact. We will try to construct an open set V around x $\notin$ A+B that does not not intersect A+B to prove the result. Consider any x $\notin$ A+B.
$\implies$ x - B $\notin$ A. x - B is compact ({x} and B are compact and using the result in (d)) and A is closed.
$\implies$ we can find an open neighborhood V s.t. (x-B+V) $\cap$ (A + V) = $\emptyset$
Consider the following open cover of B with open sets x_B + V; where x_B are elements of B. Since B is compact we have a finite sub-cover such that B $\subset \cup_{i=1}^{n} b_i + V$ for some $b_i \in$ B. Now, from above we have that (x-B+V) $\cap$ A = $\emptyset$
$\implies$ $(x - B + V + b_1) \cap (A + b_1 + V) = \emptyset$
$(x - B + V + b_1)$ is open(because V is) and contains x, denote this is V_1
Similarly we can construct till V_n, and let $V_x = \cap_{i=1}^{n} V_i$, V_X is open as its a finite intersection of open sets and contains x.
Further, we have that $V_x \cup (\cap_{i=1}^{n}(A + b_i + V)) = \emptyset$. Now since B $\subset \cup_{i=1}^{n} b_i + V$ we have that
(A + B) $\subset \cap_{i=1}^{n}(A + b_i + V)$
$\implies$ $V_x \cap$ (A+B) = $\emptyset$. Since V_x is open and contains x we are done. $\therefore$ A+B is closed if A is closed and B is compact. $\square$

f. Consider the following 2 closed sets $\in \mathbb{R}$.

$$A = \{n + \frac{1}{n}; \forall n > 0 \in \mathbb{N}\}$$

$$B = \{-n + \frac{1}{n}; \forall n > 0 \in \mathbb{N}\}$$

Clearly A and B are closed as they donot have any limit points. Consider a sequence $c_n = a_n + b_n =$ n + $\frac{1}{n} - n + \frac{1}{n} = \frac{2}{n}$; this converges to 0.
Let a = i + $\frac{1}{i} \in$ A and b = $-j + \frac{1}{j} \in$ B.
Then a + b = i - j + $(\frac{1}{i} + \frac{1}{j}) \neq 0$ for any i,j $\in \mathbb{N}$. $\implies$ $0 \notin$ A + B.
But 0 is a limit point of A+B as from above. Hence A + B is not closed. Since $\mathbb{R}$ is a topological vector space, we have a counterexample. $\square$

g. Given A is bounded it is trivial that all its subsets(countable or otherwise) are bounded. Let's assume A is not bounded. $\implies$ $\exists$ a neighborhood V such that for each n $\in \mathbb{N}$, we have a $x_n \in$ A s.t. $x_n \notin$ nV. (We can always choose distinct x_n for this given V) This countable subset(as x_n are distinct) S of A is not bounded, because $\exists$ no $t > 0$, such that S $\subset$ kV whenever $k > t$ as a direct consequence of the above construction of S.
$\therefore$ S is the required subset of A which is not bounded if A is not. $\square$

h. We will use the fact that the closure of a set is the "smallest" closed set containing the set itself. Consider any point y $\in$ x + $\bar{A}$. Let V be any neighborhood of y $\in$ X. y - x $\in \bar{A}$. $\implies$ V - x , a neighborhood of (y-x), intersects with A.
$\implies$ V intersects A + x. $\implies$ y $\in \overline{A+x}$. $\implies$ x + $\bar{A} \subset \overline{A+x}$.
Now since $\bar{A}$ is closed, so is x + $\bar{A}$. And x + A $\subset$ x + $\bar{A}$ (which is closed)
$\implies$ x + $\bar{A} = \overline{A+x}$ $\square$

i. We will use the fact the interior of a set is the "largest" open set contained in the given set. It is trivial that A + $B^\circ \subset$ A + B. Further, since B° is open, so is A + B°.
Hence, A + $B^\circ \subset (A+B)^\circ$. $\square$

j.

i. Since $\overline{A^\circ}$ is closed , it is sufficient to show that A $\subset \overline{A^\circ}$. As $\overline{A}$ is smallest closed set containing A, and since $A^\circ \subset$ A, it follows that $\overline{A^\circ} = \overline{A}$.
Since A° is non-empty, let $y \in A^\circ$. $\implies$ $\exists$ a 0 neighborhood V_y such that $y + V_y \subset A^\circ$.
Let x $\in$ A, we will try to prove that x $\in \overline{A^\circ}$.
Now, since $y + V_y$ is open and A is convex (so is A°), we have that $tx + (1-t)(y + (V_y))$ is open in A (for $0 \leq t < 1$), hence $tx + (1-t)(y + (V_y)) \in A^\circ$. In particular, $tx + (1-t)y \in A^\circ$ for $0 \leq t < 1$.
Now, consider any neighborhood of x: $V_x = V + x$, and we can find a bounded neighborhood W $\subset$ V.
$\implies$ we can find $s > 0$ such that t(y-x) $\in$ W $\forall$ $0 < t < s$. In other words, W + x intersects the set (1-t)x + t(y) for $0 < t < s$
$\implies$ V_x intersects the set $\{tx + (1-t)(y + (V_y))\}$ for some $0 < t < 1$

$\implies V_x$ intersects A°.
$\implies$ x $\in \overline{A^\circ}$.

ii. This follows from (i) by observing that $(\overline{A^\circ})^\circ = A^\circ(as A^\circ$, being open, is the largest open set contained in its closure). And since $\overline{A^\circ} = \overline{A}$, we have that $(\overline{A})^\circ = (\overline{A^\circ})^\circ = A^\circ$ ☐

k. Let $\{x_n\}$ be a cauchy sequence. Consider any 0 neighborhood V. Then we can find a balanced neighborhood W $\subset$ of V. Now since W is a neighborhood, by definition of Cauchy we can find an N such that $x_n - x_N \in$ W $\forall n > N$. Since W is balanced, we can find $s > 0$ such that $\{x_1, x_2 ..., x_N\} \in$ tW whenever $t > s$. $\implies x_n \in tW + W \forall n \in \mathbb{N}$ $\implies \{x_n\} \subset (t+1)W \subset (t+1)V \therefore \{x_n\}$ is bounded. ☐

Question 8. Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Let A be a convex balanced absorbing set. Let P_A be the Minkowski function of A.

Proof. Assume (a) is true, $\implies \exists$ no $x \neq 0 \in$ X such that $P_A(x) = 0$. Else, x $\in$ A (as A is balanced absorbing set). Further, we have that $\alpha x \in$ A $\forall$ α in $\mathbb{R}$ or $\mathbb{C}$ as $P_A(\alpha x) = |\alpha| P_A(x) = 0$. $\implies$ A contains the linear subspace $\{x \neq 0\}$. Contradiction.

Assume (a) is not true, Let Y be the linear subspace containing elements other than 0. Let x $\in$ Y $\subset$ A. $\implies$ $\alpha x \in$ A $\forall$ α in $\mathbb{R}$ or $\mathbb{C}$. $\implies$ By the definition of P_A, we have that $P_A(x) = 0$ ($P_A(x) < 1/n$ as $nx \in$ A), although x $\neq 0 \in$ X. $\implies$ P_A is not a norm. ☐