The Annals of Mathematical Statistics
1972, Vol. 43, No. 2, 659-662

NOTES ON THE COVARIANCE MATRIX OF A RANDOM, NESTED ANOVA MODEL

BY LYNN ROY LAMOTTE

University of Kentucky

A convenient representation of the covariance matrix for a general k-level random, nested AOV model is obtained, as well as an expression for its inverse and determinant.

1. Introduction. A convenient representation for the covariance matrix of a random, nested analysis of variance model is obtained in Section 2. In Section 3 a representation of V^{-1} is derived and in Section 4 an expression for $|V|$ is derived.

2. Representation of the model. Let J_{ij} be a column vector of 1's having n_{ij} rows. Let

$$K_i = \operatorname{diag}(J_{i1}, J_{i2}, \cdots, J_{in_i}) \tag{1}$$

be a matrix having n_i columns and $\sum_j n_{ij}$ rows, $i = 1, \cdots, k$, and $n_{i+1} = \sum_j n_{ij}$, $n_0 = 0$. A p-way random, nested analysis of variance model may be represented by Y_{p+1} where

$$Y_1 = Z_1 \tag{2}$$

and

$$Y_{i+1} = K_i Y_i + Z_{i+1}\,, \qquad i = 1, \cdots, p\,, \tag{3}$$

and

$$V(Z_i) = \sigma_i^2 I \tag{4}$$

and

$$\operatorname{Cov}(Z_i, Z_j) = 0\,, \qquad i \neq j\,. \tag{5}$$

For example, if $p = 1$, $n_1 = 2$, $n_{11} = 2$, $n_{12} = 3$, $n_2 = 5$,

$$\begin{aligned} Y_2 &= K_1 Y_1 + Z_2 \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix} + \begin{bmatrix} Z_{21} \\ Z_{22} \\ Z_{23} \\ Z_{24} \\ Z_{25} \end{bmatrix} = \begin{bmatrix} Z_{11} + Z_{21} \\ Z_{11} + Z_{22} \\ Z_{12} + Z_{23} \\ Z_{12} + Z_{24} \\ Z_{12} + Z_{25} \end{bmatrix}. \end{aligned} \tag{6}$$

which, with a slight change of notation, corresponds to the usual representation $y_{ij} = a_i + b_{ij}$.

Received February 19, 1971; revised September 1, 1971.

Let $V_i = \text{Cov}(Y_i)$ and $V = V_{p+1}$. Then

$$V_1 = \sigma_1^2 I \tag{7}$$

and

$$V_{i+1} = \sigma_{i+1}^2 I + K_i V_i K_i' , \qquad i = 1, \cdots, p . \tag{8}$$

The dimensions of V_i are $n_i \times n_i$, $i = 1, \cdots, p$, and V is $n \times n$ where $n = n_{p+1}$ is the number of components of Y.

3. The inverse of V. Let $\sigma^2 = \sigma_{p+1}^2$, $\gamma_i = \sigma_i^2/\sigma^2$, $B_i = V_i/\sigma^2$, $i = 1, \cdots, p+1$. Then

$$V = \sigma^2(I + K_p B_p K_p') \tag{9}$$

and

$$B_i = \gamma_i I + K_{i-1} B_{i-1} K_{i-1}' , \qquad i = 1, \cdots, p , \tag{10}$$

where $B_0 = 0$.

By defining

$$C_{p+1} = I , \tag{11}$$

$$D_{p+1} = I , \tag{12}$$

$$D_i^{-1} = (I + \gamma_i K_i' D_{i+1}^{-1} K_i)^{-1} K_i' D_{i+1}^{-1} K_i , \qquad i = 1, \cdots, p , \tag{13}$$

$$C_i = (K_i' D_{i+1}^{-1} K_i)^{-1} K_i' D_{i+1}^{-1} C_{i+1} , \qquad i = 1, \cdots, p , \tag{14}$$

$$C_0 = 0 , \tag{15}$$

and

$$A_i = C_i' D_i^{-1} C_i - C_{i-1}' K_{i-1}' D_i^{-1} K_{i-1} C_{i-1} , \quad i = 1, \cdots, p+1 , \tag{16}$$

it may be shown that

$$(\sigma^{-2} V)^{-1} = A_{p+1} + A_p + \cdots + A_1 \tag{17}$$

by defining

$$R_i = C_i'(I + K_i' D_{i+1}^{-1} K_i B_i)^{-1} K_i' D_{i+1}^{-1} K_i C_i , \quad i = 1, \cdots, p, R_0 = 0 , \tag{18}$$

noting that

$$(\sigma^{-2} V)^{-1} = A_{p+1} + R_p \tag{19}$$

and showing that

$$R_i = A_i + R_{i-1} , \qquad i = 1, \cdots, p . \tag{20}$$

Alternative expressions for A_i are

$$A_i = (C_i - K_{i-1} C_{i-1})' D_i^{-1} (C_i - K_{i-1} C_{i-1}) \tag{21a}$$

$$= C_i'(D_i^{-1} - D_i^{-1} K_{i-1}(K_{i-1}' D_i^{-1} K_{i-1})^{-1} K_{i-1}' D_i^{-1}) C_i . \tag{21b}$$

It follows from (21a) that each A_i is nonnegative definite and

$$\operatorname{rank}(A_i) = \operatorname{rank}(D_i) - \operatorname{rank}(D_{i-1}) . \tag{22}$$

It should be noted that D_i is diagonal, so that the only inverses involved in the above expressions are inverses of diagonal matrices. Generally, the inverses used above should be taken as generalized inverses. For definiteness, it will be assumed that if D is diagonal, D^{-1} will be taken as a diagonal array with $d^i = 1/d_i$ if $d_i \neq 0$ and $d^i = 0$ if $d_i = 0$.

Applying (17) results in a partition of $Y'V^{-1}Y$ into $p + 1$ n.n.d. quadratics analogous to the partition appearing in the usual ANOVA table. If $n_{ij} = m_i$, $j = 1, \cdots, n_i$, for some m_i, $i = 1, \cdots, p$ (that is, if the setup is balanced), then each A_i is the product of a scalar function of $\sigma^2, \sigma_p^2, \cdots, \sigma_i^2$, and a matrix which does not depend on $\sigma^2, \sigma_i^2, i = 1, \cdots, p$. Finally, (11)—(17) describe an efficient iterative computational procedure for inverting V for specified values of $\sigma^2, \sigma_i^2, i = 1, \cdots, p$.

4. The determinant of V. With notation as in Section 3,

$$\begin{aligned} |\sigma^{-2}V| &= |I + K_pB_pK_p'| \\ &= |I + K_p'K_pB_p| \\ &= |I + \gamma_pK_p'K_p + K_p'K_pK_{p-1}B_{p-1}K_{p-1}'| \\ &= |K_p'K_pD_p||I + D_p^{-1}K_{p-1}B_{p-1}K_{p-1}'| \\ &= |K_p'K_pD_p||I + K_{p-1}'D_p^{-1}K_{p-1}B_{p-1}| \\ &= |K_p'K_pD_p||K_{p-1}'D_p^{-1}K_{p-1}D_{p-1}||I + K_{p-2}'D_{p-1}^{-1}K_{p-2}B_{p-2}| \\ &\;\;\vdots \\ &= \textstyle\prod_{i=1}^p |K_i'D_{i+1}^{-1}K_iD_i| \\ &= \textstyle\prod_{i=1}^p |I + \gamma_iK_i'D_{i+1}^{-1}K_i| \end{aligned} \tag{23}$$

since

$$K_i'D_{i+1}^{-1}K_iD_i = I + \gamma_iK_i'D_{i+1}^{-1}K_i . \tag{24}$$

5. An example. The results of the previous sections will be applied to the covariance matrix of the balanced two-way nested layout. Let: $p = 2$; $m_0 = n_1$; $n_{1j} = m_1, j = 1, \cdots, m_0$; $n_{2j} = m_2, j = 1, \cdots, m_0m_1$. Denote by $A \times B$ the Kronecker product of two matrices A and B. The notation I_m will denote an $m \times m$ identity matrix and J_m an m-vector of 1's. Then

$$K_1 = I_{m_0} \times J_{m_1} , \tag{25}$$

$$K_2 = I_{m_0m_1} \times J_{m_2} , \tag{26}$$

$$D_2^{-1} = \frac{m_2}{m_2\gamma_2 + 1} I_{m_0m_1} , \tag{27}$$

(28) $$D_1^{-1} = \frac{m_1 m_2}{m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1} I_{m_0} ,$$

(29) $$C_2 = m_2^{-1} I_{m_0 m_1} \times J'_{m_2} ,$$

(30) $$C_1 = \frac{1}{m_1 m_2} I_{m_0} \times J'_{m_1 m_2} ,$$

(31) $$C_2 - K_1 C_1 = m_2^{-1} I_{m_0} \times (I_{m_1} - m_1^{-1} J_{m_1} J'_{m_1}) \times J'_{m_2} ,$$

(32) $$C_3 - K_2 C_2 = I_{m_0} \times I_{m_1} \times (I_{m_2} - m_2^{-1} J_{m_2} J'_{m_2}) ,$$

so that, using (21a),

(33) $$A_3 = I_{m_0} \times I_{m_1} \times (I_{m_2} - m_2^{-1} J_{m_2} J'_{m_2}) ,$$

(34) $$A_2 = \frac{1}{m_2(m_2\gamma_2 + 1)} I_{m_0} \times (I_{m1} - m_1^{-1} J_{m_1} J'_{m_1}) \times J_{m_2} J'_{m_2} ,$$

(35) $$A_1 = \frac{1}{m_1 m_2 (m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1)} I_{m_0} \times J_{m_1} J'_{m_1} \times J_{m_2} J'_{m_2} .$$

If the components of Y are denoted by y_{ijk}, $i = 1, \cdots, m_0$, $j = 1, \cdots, m_1$, $k = 1, \cdots, m_2$, then

(36) $$Y' A_3 Y = \sum_{i,j,k} (y_{ijk} - \bar{y}_{ij\cdot})^2 ,$$

(37) $$Y' A_2 Y = \frac{1}{m_2 \gamma_2 + 1} \sum_{i,j,k} (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot})^2 ,$$

(38) $$Y' A_1 Y = \frac{1}{m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1} \sum_{i,j,k} \bar{y}_{i\cdot\cdot}^2 .$$

If μ is a scalar then

(39) $$\begin{aligned} &\sigma^2 (Y - \mu J)' V^{-1} (Y - \mu J) \\ &\quad = \sum_{i,j,k} (y_{ijk} - \bar{y}_{ij\cdot})^2 + \frac{1}{m_2\gamma_2 + 1} \sum_{i,j,k} (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot})^2 \\ &\qquad + \frac{1}{m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1} [\textstyle\sum_{i,j,k} (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2 + (\bar{y}_{\cdot\cdot\cdot} - \mu)^2] . \end{aligned}$$

To find $|V|$, note that

(40) $$\begin{aligned} |I + \gamma_2 K_2' D_3^{-1} K_2| &= |(m_2\gamma_2 + 1) I_{m_0 m_1}| \\ &\doteq (m_2 \gamma_2 + 1)^{m_0 m_1} , \end{aligned}$$

(41) $$\begin{aligned} |I + \gamma_1 K_1' D_2^{-1} K_1| &= \left| \frac{m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1}{m_2 \gamma_2 + 1} I_{m_0} \right| \\ &= \left(\frac{m_1 m_2 \gamma_1 + m_2 \gamma_2 + 1}{m_2 \gamma_2 + 1} \right)^{m_0} , \end{aligned}$$

so that, using (23) and simplifying,

(42) $$|V| = (m_1 m_2 \sigma_1^2 + m_2 \sigma_2^2 + \sigma^2)^{m_0} (m_2 \sigma_2^2 + \sigma^2)^{m_0(m_1 - 1)} (\sigma^2)^{m_0 m_1 (m_2 - 1)} .$$