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SOLVING INTEGER MINIMUM COST FLOWS WITH SEPARABLE CONVEX COST OBJECTIVE POLYNOMIALLY

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A polynomial algorithm is described to solve minimum cost network flow problems with separable convex cost functions on the arcs and integrality restrictions on the flows. The proof generalizes the scaling approach used by Edmonds and Karp for proving polynomiality of the out-of-kilter method for ordinary (linear cost) network flows.

Key words: Nonlinear Network Flows, Quadratic Programming, Complexity of Algorithms, Combinatorial Optimization.

1. Introduction

Minimum cost flow problems with convex cost functions provide a natural extension of the classical network flow theory which appears useful in a number of important areas of applications.

Past work on these problems has been directed mainly towards solving them as continuous convex mathematical programs.

However the discrete version of the problem (i.e. when integrality restrictions are imposed on the flow variables) seems to have received but very little attention.

Here we show that the discrete convex separable minimum cost flow problem is indeed solvable polynomially in terms of the problem size (expressed as a function of N, number of nodes, M number of arcs and $\log_2 c_{\max}$, the maximum number of binary digits required to represent the capacities on the arcs). The approach suggested here generalizes the scaling technique used by Edmonds and Karp (1972) for proving polynomiality of the out-of-kilter method for ordinary (linear cost) network flows.

2. The polynomial algorithm

We denote by $G=[X, U]$ the given graph, where X is the set of nodes ($|X|=N$), and U the set of arcs ($|U|=M$). With each arc $u\in U$, we associate:

- two integer numbers b_u and c_u ($b_u \leq c_u$), respectively the lower and upper bound on the flow value ϕ_u on arc u;
- a real (convex continuously differentiable) function $\gamma_u(\phi_u)$ representing the cost of sending a flow ϕ_u on arc u.

The (continuous) convex cost flow problem on G can be written as

$$\text{Minimize} \quad \sum_{u \in U} \gamma_u(\phi_u)$$

(P) subject to

$$A \cdot \phi = 0, \tag{1}$$

$$b_u \leqslant \phi_u \leqslant c_u \quad (\forall u \in U) \tag{2}$$

where $A = (a_{iu})$, $i = 1, \ldots, N$, $u = 1, \ldots, M$, is the node-arc incidence matrix of the graph and where $\phi \in \mathbb{R}^M$ denotes the vector of components ϕ_u $(u \in U)$. We denote by (IP) the discrete version of the problem obtained by adding the restriction $\forall u \in U$, ϕ_u integer.

We introduce here the concept of pth order approximation of problem (P) (p integer$\geqslant 0$) as the piecewise linear cost network flow problem (PA[p]) obtained in the following way. For each arc $u \in U$ of G:

- the lower and upper capacity bounds $b_u^{(p)}$ and $c_u^{(p)}$ are given by

$$c_u^{(p)} = 2^p \left\lceil \frac{c_u}{2^p} \right\rceil, \qquad b_u^{(p)} = 2^p \left\lceil \frac{b_u}{2^p} \right\rceil$$

where $\lceil \;\; \rceil$ (resp: $\lfloor \;\; \rfloor$) denotes the least integer greater than or equal to (resp. the largest integer less than or equal to);

- the cost functions of problems (P) and (PA[p]) coincide on all abscissae $k \cdot 2^p$ for integer k, $\lfloor b_u/2^p \rfloor \leqslant k \leqslant \lceil c_u/2^p \rceil$.

Let $\bar{p} = \lceil \log_2 c_{\max} \rceil$, the least integer greater than or equal to $\log_2 c_{\max}$ where $c_{\max} = \max_{u \in U} \{\max\{|b_u|, |c_u|\}\}$.

Property 1. (*i*) *An optimal solution to* (PA[0]) *obtained by the out-of-kilter algorithm is an optimal integer solution to* (P).

(*ii*) *For each* $p < \bar{p}$ (*resp.* $p = \bar{p}$), *starting from an optimal solution to* (PA[p]) (*resp. from the zero flow and the zero dual variables*), *an optimal solution to* (PA[$p-1$]) *can be determined in time complexity* $0(MN^2)$ *by the out-of-kilter algorithm.*

Proof. (i) Since the kilter diagrams of (PA[0]) (see Lawler, 1976; Gondran and Minoux, 1979, Chapter 5) are staircase increasing functions with jumps at integer points, it can be shown that the out-of-kilter algorithm always provides an optimal solution to (PA[0]) which has integer components. Moreover, by construction, the cost functions of (PA[0]) and (P) take on the same values for each integral feasible flow, which proves (i). The proof of (ii) can be found in Minoux (1983). □

Property 1 suggests the following algorithm for solving (PA[0]), hence (IP).

Algorithm 1. (a) Solve (PA[$\bar{p}$]) by the out-of-kilter algorithm with the zero flow and the zero dual variables as starting solution.

(b) For $p = \bar{p} - 1$, $\bar{p} - 2, \ldots, 0$ successively, solve (PA[p]) by the out-of-kilter algorithm, with a pair of optimal primal and dual solutions to (PA[$p+1$]) as starting solution.

We are then in a position to state:

Theorem 1. *Algorithm 1 is polynomial and has time complexity* $O(\bar{p}MN^2)$ *where* $\bar{p} = \lceil \log_2 c_{\max} \rceil$ *is the maximum number of binary digits required to represent the capacities on the arcs.*

Proof. It follows immediately from Property 1. □

In a companion paper (see Minoux, 1983) the approach presented here is further extended to provide an exact polynomial algorithm for solving the continuous problem (P) in the case of *quadratic convex separable cost functions* on the arcs. To do this, it is shown that Algorithm 1 should not be stopped when $p = 0$ but has to be run for negative values of p until some required accuracy is reached (which usually results in a large number of extra iterations). Minimum quadratic cost flow problems thus provide a very uncommon example of mathematical programming problems which are usually easier to solve in integer than in continuous variables.

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