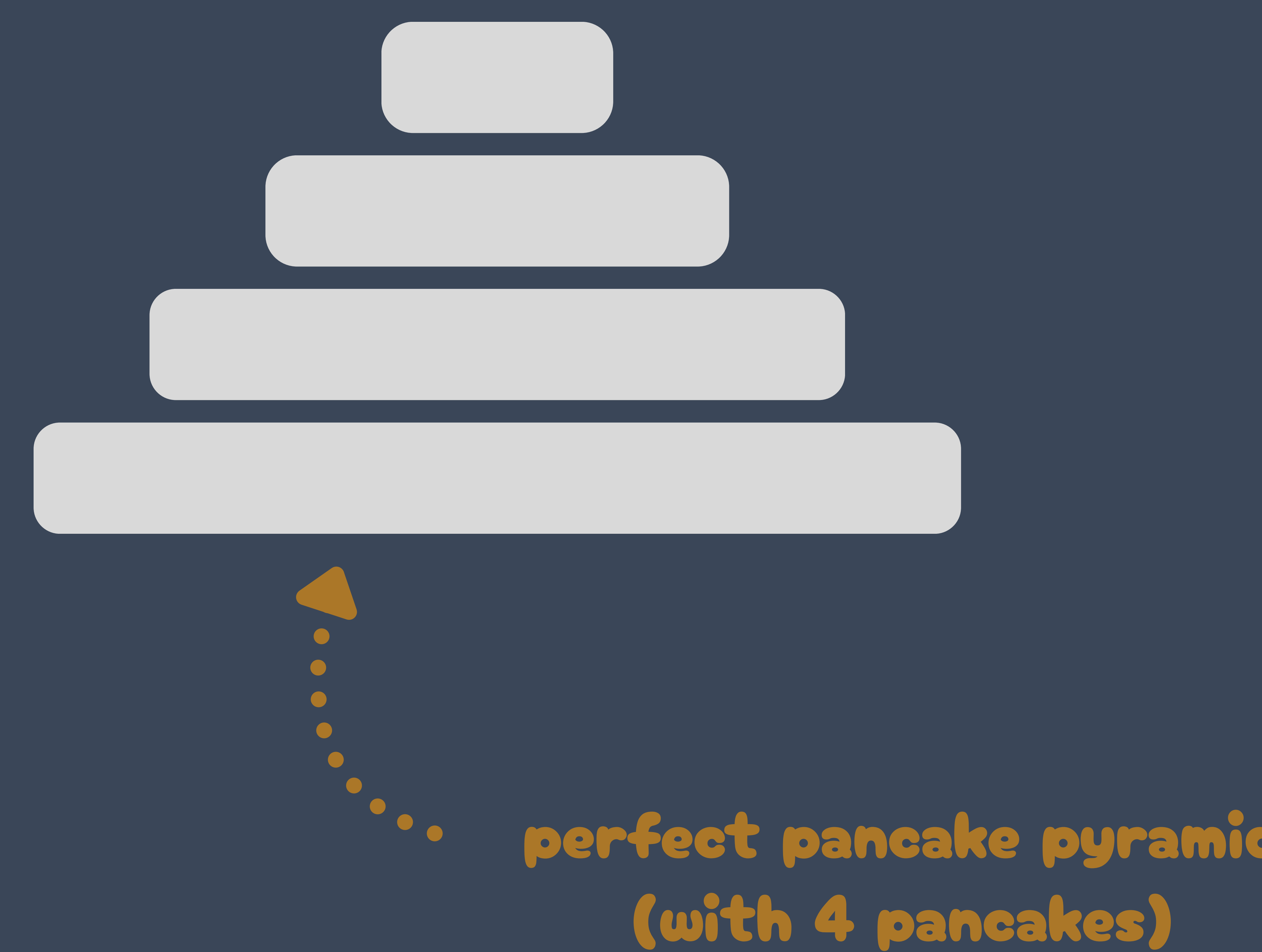
The Pancake Problem

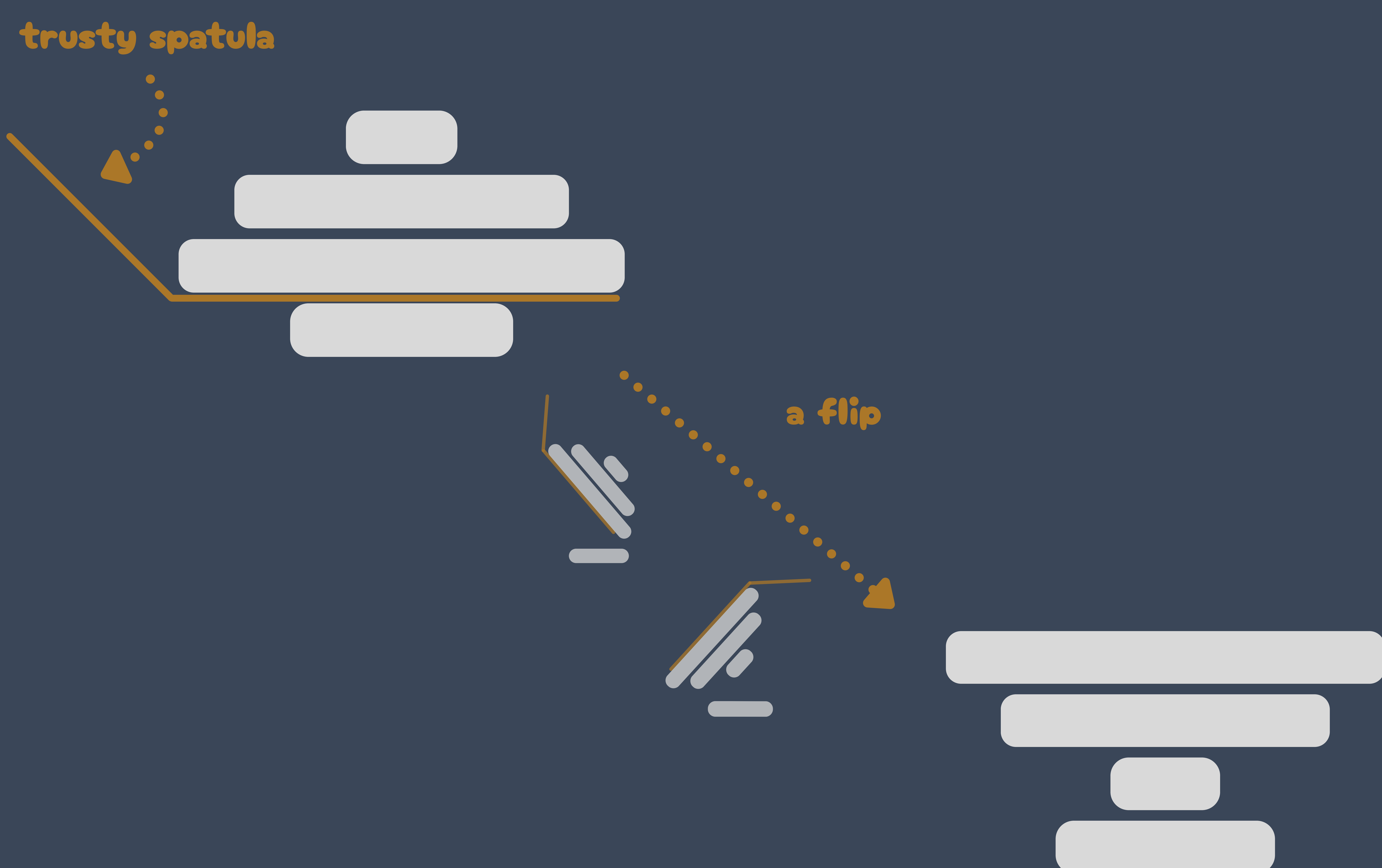
a guided exploration

Let's imagine you work at a breakfast restaurant with a raaaather picky chef.

The most popular dish at the restaurant is a stack of pancakes. But these aren't just any pancakes. These pancakes are served in the *perfect pancake pyramid*.

Each pancake is a different size, so the perfect pyramid has the largest pancake at the bottom, the second largest on top of that, and so on, with the smallest on top.





When the chef makes pancakes, they're always stacked out of order. How absurd!

Before the pancakes are served to customers, your job is to flip them back into the perfect pyramid.

But remember, the chef is picky. The *only* way you can reorder the pancake stack is by placing your trusty spatula under one of the pancakes and flipping over all pancakes above your spatula, together.

Each flip you perform reverses the order of all pancakes above the spatula, but leaves the pancakes below the spatula alone.

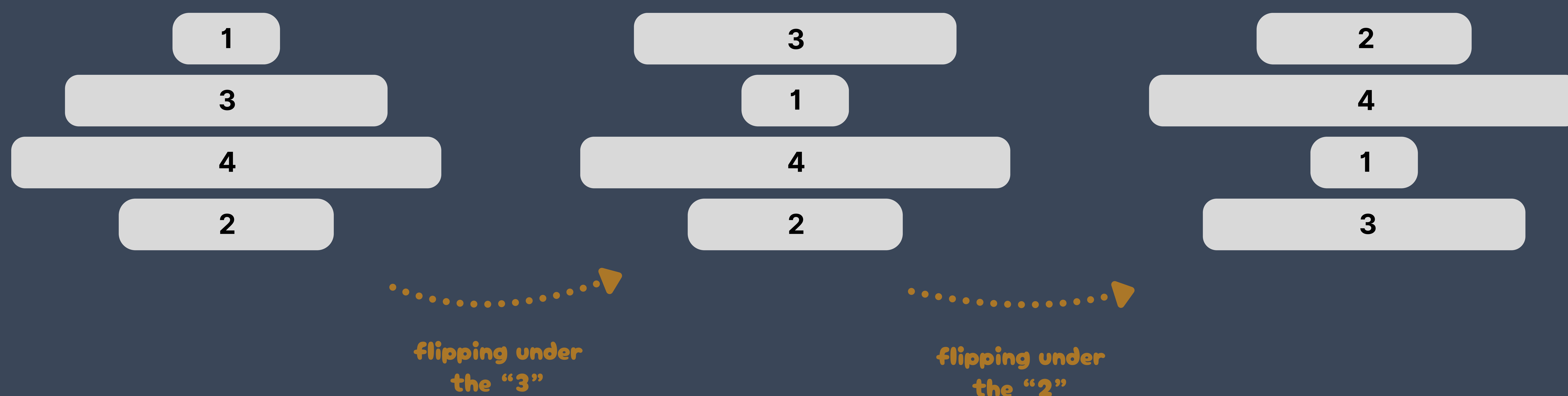
Let's practice!

Using your trusty spatula, can you turn the stack on the right in to the perfect pancake pyramid? Give it a try!

Keep track of your flips so you know how many it takes.

One way to start the progression of flips is like this.

Feel free to start your own way, too!



TIP

Don't want to draw the pancakes for every stack? Use Numbers!

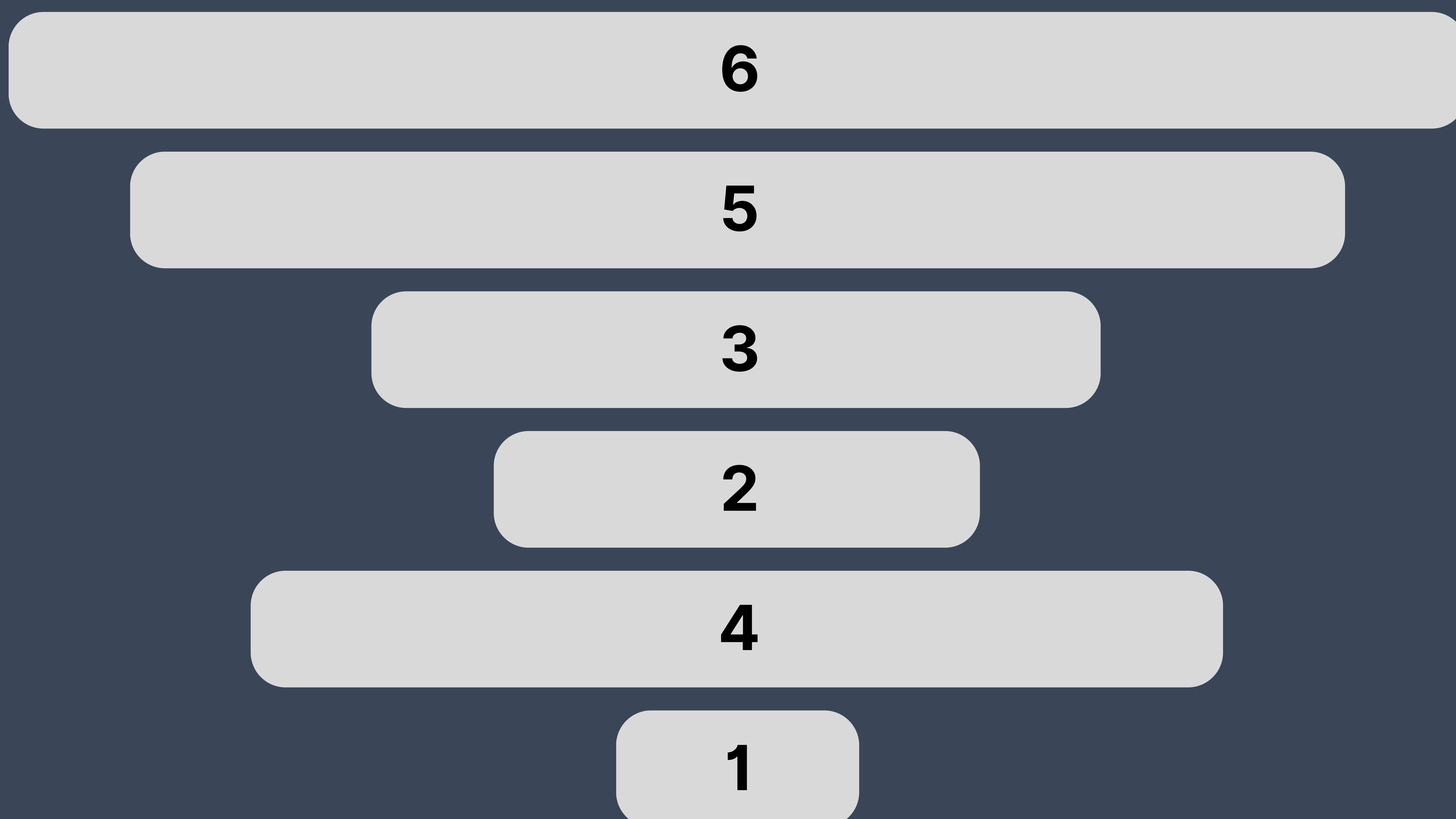
1	3	2
3	1	4
4	4	1
2	2	3

Great work!!

Let’s try to solve another one. The stack below has 6 pancakes. Can you turn it into the perfect pancake pyramid? Don’t forget to keep track of your flips!

You’re welcome to work on paper like we did for the “1342” stack a few minutes ago. If you’d prefer to work digitally, you can use flipjack, a browser-based sandbox for creating and solving stacks of pancakes.

In flipjack, click a pancake to flip the stack under that pancake. Your progression of flips appears in the history area on the right. Click a stack in your history to return to that stack. Use the controls on the left to change the display style of the pancakes, add or remove pancakes, reorder the stack, and more. Feel free to play around as much as you’d like!



Well done!! You’ve now solved at least two pancake stacks. The chef is very pleased and grateful for your efforts.

Being extremely picky, though, the chef has just one more request for you. It’s crucial to the chef that the flipping of the pancake stacks into perfect pyramids be done *as efficiently as possible*. So we need to use the fewest number of flips to make the pyramids. Thank goodness you kept track of your flips when solving the last two stacks!

How many flips did it take to solve the original 1342 stack? Do you think this is the fewest number of flips that could be used to solve that stack, or do you think there’s a more efficient solution?

In addition to being picky, the chef is also *skeptical*. This means we have to defend our answers and convince the chef that our solution method is actually the most efficient!

Let’s do this in two parts:

- First, determine whether our solution is the most efficient. You can do this in many different ways. Some people might choose to search for better solutions and (hopefully) come up empty-handed, while others might choose to detail every possible solution and (again, hopefully) see that ours is optimal. And others might use a completely different method. The choice is yours!

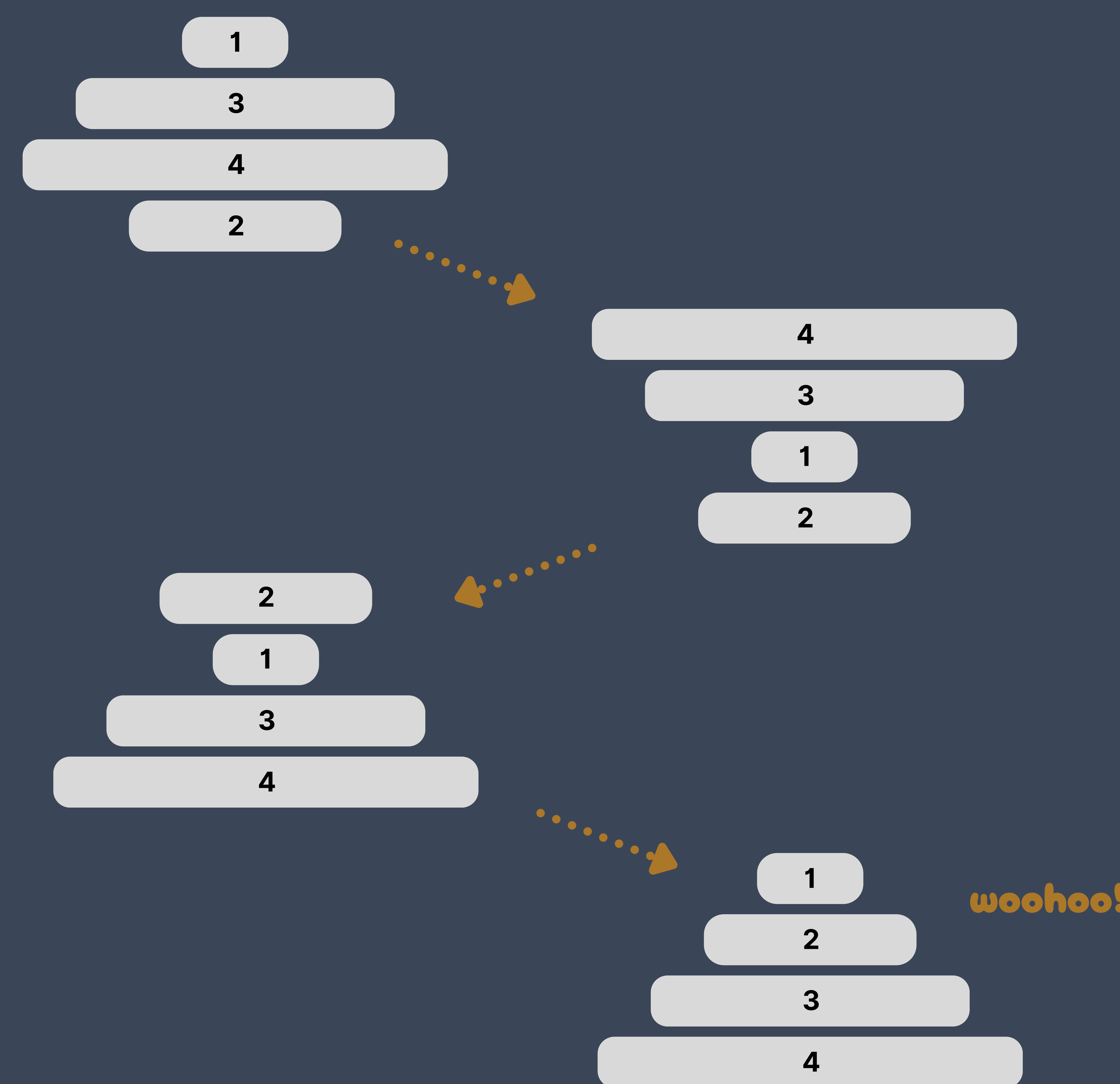
- Second, convince the skeptical chef that our solution *is* most efficient. Again, there are countless options. You could show that there are no available improvements to our solution. Or, if you detailed every possible solution, you could show that nothing else is quicker. And so on.

You’re welcome to use flipjack, if you find it helpful!

TIP

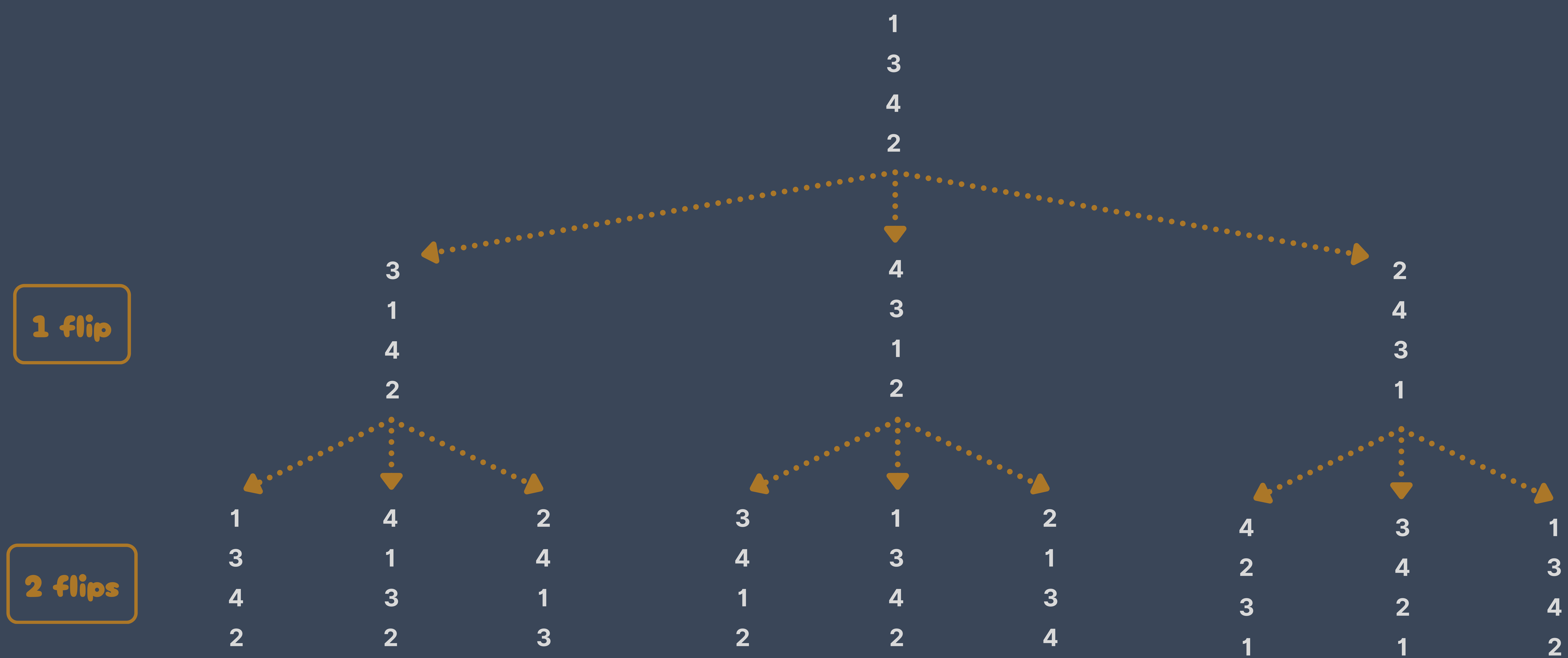
When finding efficient solutions, it can be helpful to visualize the entire solution progression, as shown below.

This is just one possible way of doing it. Please feel free to make up your own visualizations, too!!



It turns out that the minimum number of flips required to solve the 1342 stack is indeed 3 flips (the progression is shown on the previous page, if you're curious). If you managed to do this, great work!

One way of convincing any skeptic—including the ever-skeptical chef—is shown below. The logic proceeds like this: In order for a solution to have fewer flips than 3, it would need to have just 1 or 2 flips. The diagram below contains all possible stacks that can be obtained from 1342 in 1 and 2 flips. Since none of them is the perfect pancake pyramid, we can be absolutely certain that there is no solution in 1 or 2 flips, meaning that 3 is the minimum number of flips needed to solve the stack. That's an argument that will convince even the most reluctant of skeptics!



FOOD FOR THOUGHT:

We show every possible flip of each stack under the second, third, and fourth pancake. Why DON'T we include the flip under the first pancake?

With a completely solved 4-stack under our belt, let's repeat the process with a stack of 5 or even 6 pancakes. You could use the 6-stack you solved earlier, but you could also create your own 5-stack if jumping from 4 to 6 feels like too much. Whatever you choose, determine whether your original progression was minimal. Then construct an argument that convinces a skeptic of the efficiency of your solution (or a better one you find along the way).

A flipjack screenshot showing the optimal solution for 653241, which we solved earlier.

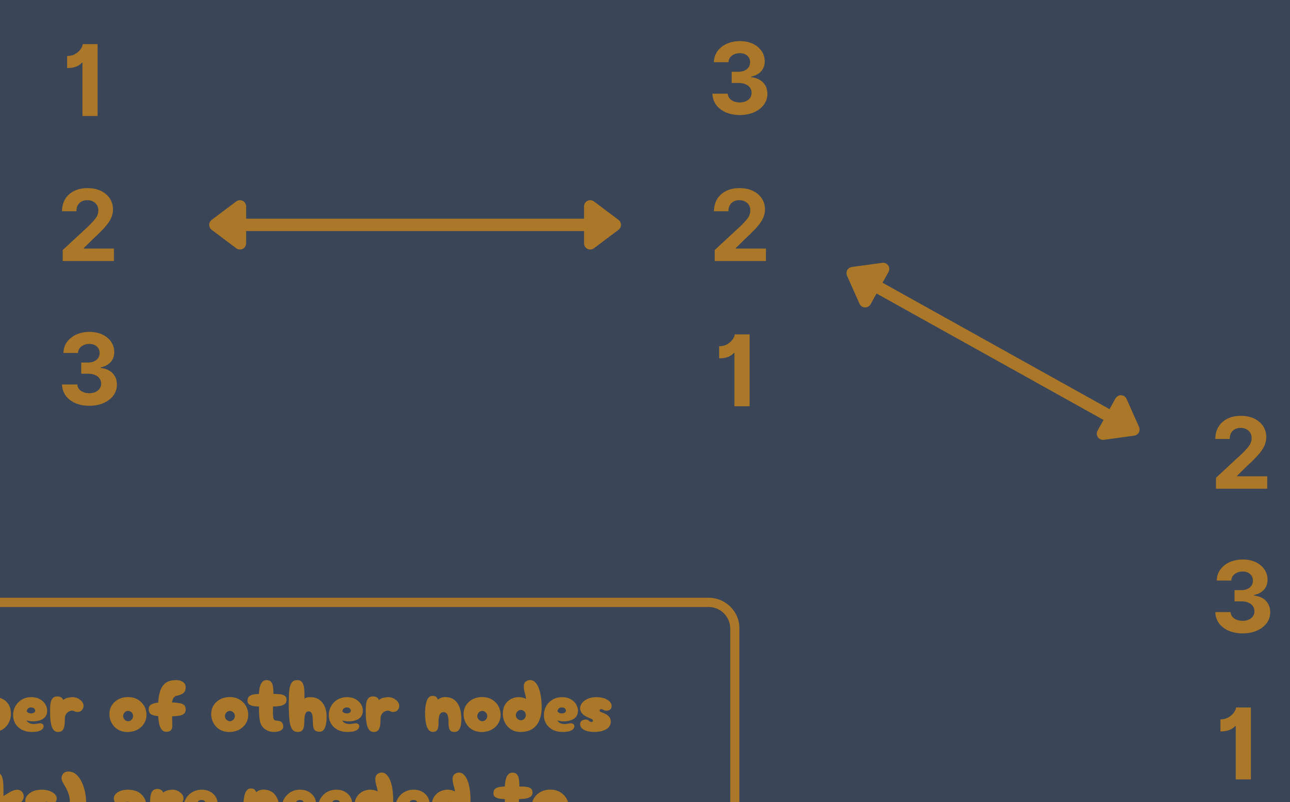
Whew! That's a lot of pancakes and flipping. Hopefully you've enjoyed playing in this mathematical sandbox!

Believe it or not, we've just barely scratched the surface of all the different kinds of math that we can learn from the pancake problem. There are dozens of directions we could go and so many questions we could explore. Some of these questions and avenues for exploration are included on the next page, labeled "Dig Deeper." Feel free to try any (or all) of them out! You're welcome to continue using flipjack in your explorations!

Dig Deeper

- How many unique stacks of 3 pancakes are there? What about 4 pancakes, then 5, 6, and so on? Can you find a formula for the number of unique stacks for *n* pancakes?

  - A pancake graph is a visual representation of related pancake stacks. Each pancake stack is a "node," and each flip is an "edge" connecting two stacks (nodes). Part of the pancake graph for 3 pancakes is shown at the right.
    - Can you finish the pancake graph for 3 pancakes?
    - Can you draw the whole pancake graph for 4?
    - What does a solution to a given stack look like on a pancake graph?
    - How could you use these pancake graphs to prove a solution optimal?



A number of other nodes (stacks) are needed to complete this pancake graph.

Can you find them?

- The official *Pancake Problem* in mathematics takes our initial work one step further and asks: "for any stack of *n* pancakes, what is the minimum number of flips required to solve the stack?" Mathematicians are particularly interested in the largest such minimum for a given size stack. So, for example, take all possible stacks of four pancakes and find the minimum needed to solve each one. Mathematicians are interested in the largest of those minimum values, both for 4 pancakes and all other positive integer numbers of pancakes. In fact, these largest minimums are actually called the *pancake numbers*. How cool!
  - See if you can find the pancake number for 3-stacks. Then try 4!
  - If you're feeling up to it, go for 5!

- Find an algorithm for solving *any* stack of pancakes. Your algorithm doesn't need to be the most efficient, it just needs to work for any possible stack.

Dig Deeper, cont'd

- A classic variation of the pancake problem is the *burnt* pancake problem. Your pancakes are now burnt on only one side (not both). The perfect burnt pancake pyramid has to have each pancake in order, as before, but now the burnt side of each pancake must be facing downwards. In the initial stack, the burnt side of each pancake may or may not be facing the correct way.
    - Make a few stacks of burnt pancakes and solve them. Start small to make it easier to identify patterns. You'll have to develop a way to identify the burnt side of the pancake in your visualizations.
    - How is solving the burnt version of the problem different than the original?
    - Can you find the burnt pancake numbers for stacks of 3, 4, or 5 pancakes? You might even want to start with 2 pancakes, because that's slightly less trivial than the original pancake number for a 2-stack.
    - How does the structure of burnt pancake graphs differ from original pancake graphs, particularly for the same number of pancakes?