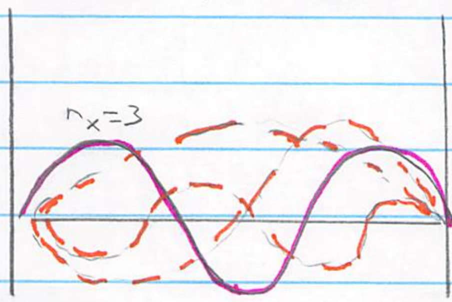
Quantum Gasses

- The formalism of the grand canonical ensemble is very useful for treating cases when the density of particles is high so that the interparticle spacing l_0 becomes comparable to λ_{th}

- In Quantum Mechanics we speak about modes or single particle orbitals / states. The momentum can be used to label these states



The momentum of the mode is

$$\vec{p} = \hbar \left(n_x \frac{\pi}{L}, n_y \frac{\pi}{L}, n_z \frac{\pi}{L} \right)$$

- The system has many modes. Each mode is an independent subsystem, sharing particles and energy from the other modes. Lets focus on one mode

- The single-particle energy of the mode is $\mathcal{E}(\vec{p})$. If there are n particles in the mode the energy of the subsystem is $n\,\mathcal{E}(p)$

this is not the density, nor is it n_x it is the number of particles in a mode.

- For bosonic particles n is arbitrary $n = 0, 1, 2, \ldots$
  For fermionic particles n is either $n = 0, 1$, i.e. occupied or unoccupied

Bosons:

$$\mathcal{Z}_p = \sum_{n=0}^{\infty} e^{-\beta(n\varepsilon - \mu n)} = \sum_{n=0}^{\infty} \left(e^{-\beta(\varepsilon-\mu)}\right)^n$$

$$\boxed{\mathcal{Z}_p = \frac{1}{1 - e^{-\beta(\varepsilon-\mu)}}}$$

This is the grand sum for one mode, in the boson case:

Then

$$\bar{n} = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln \mathcal{Z} = \frac{e^{-\beta(\varepsilon-\mu)}}{1 - e^{-\beta(\varepsilon-\mu)}}$$

$$\boxed{\bar{n}_{BE} = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1}}$$ (Bosons) ← Called

n_{BE} ← Bose-Einstein Statistics

The mean energy in a mode is

$$\bar{E}_p = \bar{n}\,\varepsilon = \frac{\varepsilon}{e^{\beta(\varepsilon-\mu)} - 1}$$

Fermions:

- There can only be either no or one particle in the mode

$$\boxed{\mathcal{Z}_p = 1 + e^{-\beta(\varepsilon - \mu)}}$$

This is for one mode

Then

$$\bar{n} = \text{Prob of One} = \frac{e^{-\beta(\varepsilon-\mu)}}{\mathcal{Z}} = \frac{e^{-\beta(\varepsilon-\mu)}}{1+e^{-\beta(\varepsilon-\mu)}}$$

$$\boxed{\bar{n}_{FD} = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}}$$

(Fermions)

Called n_{FD} Fermi-Dirac distribution

The mean energy is

$$\bar{\varepsilon} = \overline{n\varepsilon} = \bar{n}\varepsilon = \frac{\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

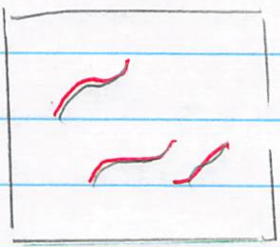
We will describe both of these distributions, n_{BE} and n_{FD} in greater detail.

Summary both in one; Top sign bosons, bottom fermions

$$\ln \mathcal{Z}_p = \mp \ln\left(1 \mp e^{-\beta(\varepsilon_p-\mu)}\right)$$

$$n = \frac{1}{e^{+\beta(\varepsilon_p-\mu)} \mp 1}$$

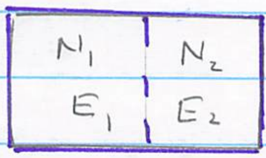
The Photon Gas



The oven glows red hot, lets calculate The number of photons in the oven.

- First note that photons can be created and destroyed, e.g. $e^+ + e^- \leftrightarrow \gamma + \gamma$ or $e \to e + \gamma$.

- Since photon number is not conserved, its chemical potential is zero. Proof:



$$dS = \frac{dU}{T} + \frac{\mu}{T} dN \quad \text{with } U_1 + U_2 = \text{const}$$

$$dS_{TOT} = dS_1 + dS_2$$

$$= \left(\frac{1}{T_1} - \frac{1}{T_2}\right) dU_1 + \frac{\mu_1}{T} dN_1 + \frac{\mu_2}{T} dN_2$$

In equilibrium $dS_{TOT} = 0$ so $T_1 = T_2$ and $\mu_1 = \mu_2 = 0$

Black Body Radiation and the Photon Gas:

Then the total number of photons in the box is

$$\boxed{N = \sum_{\substack{\text{all} \\ \text{modes}}} \bar{n}}$$

mean number of photons in mode

$$\bar{n} = \frac{1}{e^{\beta \varepsilon} - 1}$$

this depends on the energy of the mode.

- Each mode is Labelled by its momentum

$$\vec{p} = \hbar \vec{k} = \hbar \left(\frac{n_x \pi}{L}, \frac{n_y \pi}{L}, \frac{n_z \pi}{L} \right)$$

and there are two polarizations of the light for each momentum, $\boxed{\varepsilon = c|\vec{p}| = \hbar c k = \hbar \omega_p}$ is the energy.

$$\boxed{N = 2 \sum_{\vec{p}} \frac{1}{e^{\beta \varepsilon(p)} - 1}}$$

Spin from two polarizations

Now we need to covert the sum over modes to an integral. Since $L \to \infty$ the sum is

$$\sum_{n_x} \sum_{n_y} \sum_{n_z} \simeq \int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z = \frac{1}{8} \int_{-\infty}^{\infty} dn_x \int_{-\infty}^{\infty} dn_y \int_{-\infty}^{\infty} dn_z$$

Now $\quad dn_x = \left(\frac{L}{\pi \hbar}\right) dp_x \quad$ since $\quad p_x = \frac{\hbar n_x}{L}$

and similarly for x, y, z

So

$$\boxed{\sum_{n_x}\sum_{n_y}\sum_{n_z} \longrightarrow \int \frac{V\, d^3p}{(2\pi\hbar)^3} \quad \text{or} \quad \int \frac{V\, d^3p}{h^3}}$$

Thus we find

$$\boxed{N = 2V \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta\varepsilon(p)} - 1}} \qquad \text{where } \varepsilon(\vec{p}) = c|\vec{p}|$$

- Similarly the energy in the gas is

$$U = 2 \sum_{p} \frac{\varepsilon(p)}{e^{\beta\varepsilon(p)} - 1}$$

$$U = 2V \int \frac{d^3p}{(2\pi\hbar)^3} \frac{\varepsilon(p)}{e^{\beta\varepsilon(p)} - 1} \qquad \varepsilon(p) = c|\vec{p}|$$

- Now we need to do these integrals

$$\underbrace{\int d^3p}_{\text{spherical shell}} = \int p^2\, dp\, d\Omega_p = 4\pi p^2\, dp$$

- So

$$N = \frac{2V4\pi}{(2\pi\hbar)^3}\int_0^\infty p^2 dp \, \frac{1}{e^{cp/kT}-1}$$

characteristic momentum

define $u = p/p_0$ with $\boxed{p_0 = kT/c}$ and get

$$N = \frac{V p_0^3}{\pi^2} \underbrace{\int_0^\infty \frac{u^2 du}{e^u - 1}}$$

You can do this numerically

this a dimensionless integral and gives $2.404 = 2\zeta(3)$

Such integrals you dont know how to do, and they will be given.

$\zeta(x)$ is the "zeta" function

$$\boxed{\frac{N}{V} \simeq 0.244 \left(\frac{k_B T}{\hbar c}\right)^3}$$

And

$$U = \frac{2V\cdot 4\pi}{(2\pi\hbar)^3}\int_0^\infty p^2 dp \, \frac{cp}{(e^{cp/kT}-1)}$$

Again define $p_0 = kT/c$ and change variables

$$U = \frac{1}{\pi^2}\frac{V p_0^3}{\hbar^3} c p_0 \underbrace{\int_0^\infty \frac{u^3}{e^u - 1}}_{\pi^4/15}$$

dimensionless integral. Can be done analytically. But not too easily

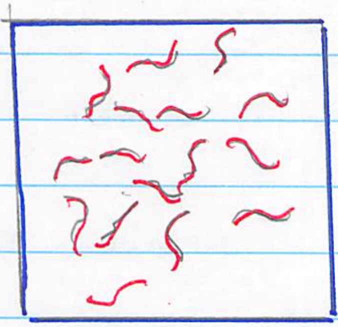
So the energy density is

$$u \equiv \frac{U}{V} = \left(\frac{kT}{\hbar c}\right)^3 kT \cdot \frac{\pi^2}{15}$$

$$\boxed{u = \left(\frac{kT}{\hbar c}\right)^3 kT \cdot 0.66}$$

$$u \propto T^4$$

Picture



- The energy of the typical photon is

$$E \sim kT$$

- The corresponding momentum is $p_0 \sim \frac{kT}{c}$ which has

wavelength $\lambda_0 \equiv \frac{\hbar}{p_0} = \frac{\hbar c}{kT}$. Thus the density of the photons is of order the

$$\boxed{\frac{N}{V} = \frac{0.244}{\lambda_0^3}}$$

The interpaticle spacing is of order the wavelength

$$\boxed{\left(\frac{N}{V}\right)^{1/3} = \frac{0.62}{\lambda_0}}$$