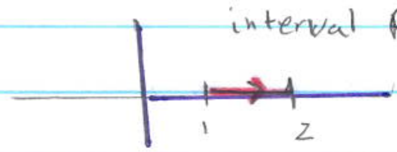
Change of Variables and Probability

- Consider the basic (oriented) integral

$$I = \int_1^2 x^2 \, dx$$

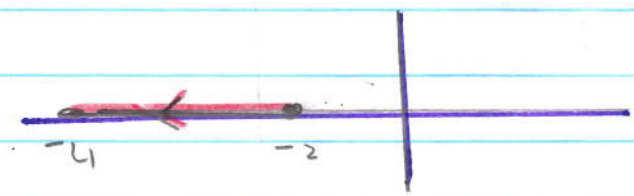


Picture

- Change vars $u = -2x$ or $x = -u/2$, $dx = -\frac{1}{2} du$

dx/du

$$I = \int_{-2}^{-4} \left(\frac{u}{2}\right)^2 \left(-\frac{1}{2}\right) du$$



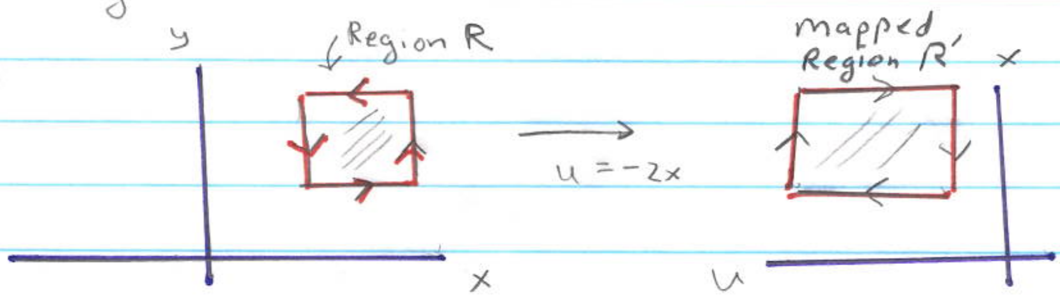
flip limits of integration

$$= \int_{-4}^{-2} \left(\frac{u}{2}\right)^2 \left(\frac{1}{2}\right) du$$

mapped oriented interval

$\left|\frac{dx}{du}\right|$

- The sign is a lot to bother with especially in higher dimension where it reflects the orientations



of the regions of integration

- Usually it is easier (and no worse) to use unoriented integrals, especially w probability

absolute value

$$I = \int_{[1,2]} dx \, x^2 = \int_{[-4,-2]} du \left|\frac{dx}{du}\right| \left(\frac{u}{2}\right)^2$$

So the change of variables formula can be written

$$\boxed{I = \int_R dx\, f(x) = \int_{R'} du \left|\frac{dx}{du}\right| f(x(u)) \quad \text{always works}}$$

region of integration

mapped region of integration

- Similarly if I have a probability $d\mathcal{P}$ to be between x and $x + dx$ and change of vars $u(x)$ or $x(u)$ then

$$\boxed{d\mathcal{P} = P_x(x)\, dx = P_x(x(u)) \left|\frac{dx}{du}\right| du}$$

- Thus, the probality density $P_u(u)$ is

$$\boxed{P_u(u) = P_x(x(u)) \left|\frac{dx}{du}\right|}$$

Ex:

normalizing const

$$P(x) = C x^2 \qquad x \in [1, 2]$$

Under change of vars $u = -2x$ we have:

$$P(u) = C\left(\frac{u}{2}\right)^2 \frac{1}{2} \qquad u \in [-4, -2]$$

3 Dimensions - Spherical Coordinates

- Suppose we have a probality distribution

$$d\mathcal{P} = P(x, y, z) \underbrace{dx\, dy\, dz}_{\equiv dV}$$

- for a partticle to have position in (x,y,z) to $(x+dx, y+dy,$ and $z+dz)$. Then what is the probility for it to have probability in $[r, r+dr]$, $[\theta, \theta+d\theta]$, $[\phi, \phi+d\phi]$
From the picture (next slide)

$$\partial V = dA\, dr = (r\, d\theta)(r\sin\theta\, d\phi)(dr)$$

$$= r^2 \sin\theta\, d\theta\, d\phi\, dr$$

- There is a mathematical way to do this. There is a map $(r, \theta, \phi) \to (x, y, z)$ $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

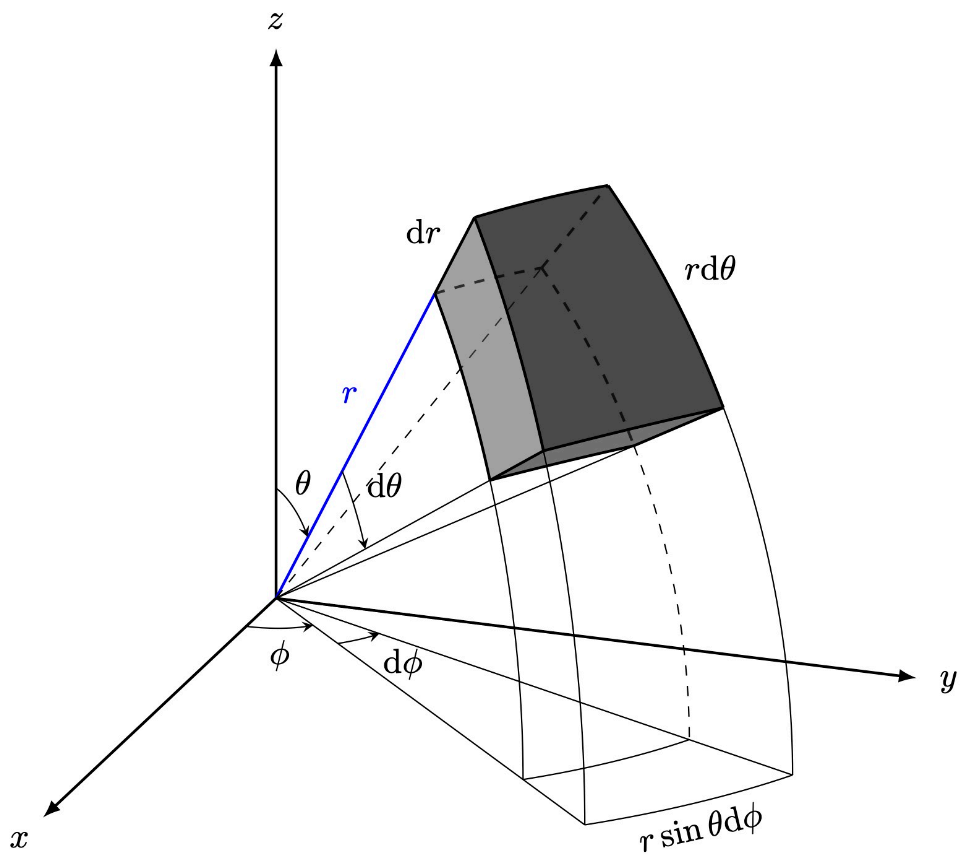
The jacobian of the map is

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

this is symbol means the matrix shown here

determinant of matrix

Spherical Coordinates



Volume and area elements

$$
\begin{aligned}
dV =& \mathrm{d}A\,\mathrm{d}r = (r\mathrm{d}\theta)\,(r\sin\theta\mathrm{d}\phi)\,(\mathrm{d}r) \\
=& r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}A =& (r\mathrm{d}\theta)(r\sin\theta\mathrm{d}\phi) \\
=& r^2\sin(\theta)\,\mathrm{d}\theta\mathrm{d}\phi
\end{aligned}
$$

Then

absolute value of jacobian determinant

$$dx\,dy\,dz = \left\| \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \right\| dr\,d\theta\,d\phi$$

$$= r^2 \sin\theta \; dr\,d\theta\,d\phi$$ (see slide for algebra)

So either way with $x(r,\theta,\phi)$

$$d\mathcal{P} = P(x,y,z) \cdot r^2 \sin\Theta \; dr\,d\theta\,d\phi$$

Example

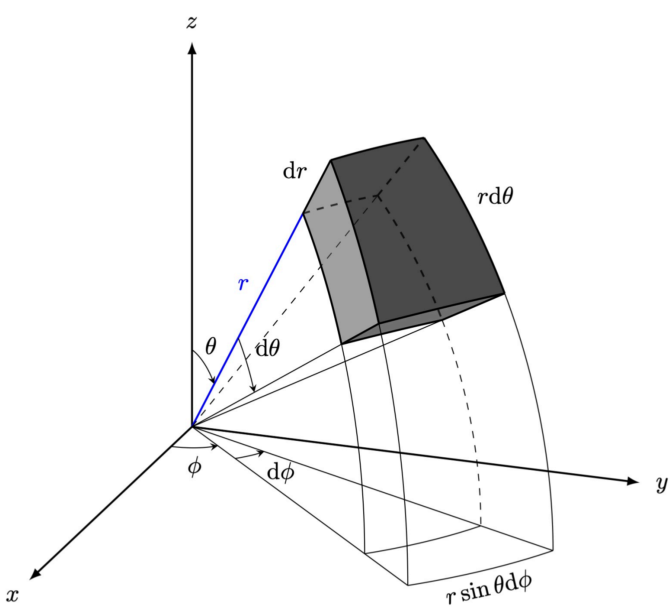
if $P(x,y,z) = Ce^{-(x^2+y^2+z^2)/2\sigma^2}$ then

$$d\mathcal{P} = Ce^{-(x^2+y^2+z^2)/2\sigma^2}\,dx\,dy\,dz$$

$$= Ce^{-r^2/2\sigma^2}\, r^2\, dr\, \sin\theta\, d\theta\, d\phi$$

So $P(r,\theta,\phi) = Ce^{-r^2/2\sigma^2}\, r^2 \sin\theta$ is the probility density for r,θ,ϕ.

Jacobian Determinant



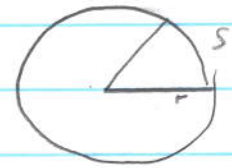
$$
\begin{aligned}
x =& r \sin\theta \cos\phi \\
y =& r \sin\theta \sin\phi \\
z =& r \cos\theta
\end{aligned}
$$

$$
\begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}
\end{vmatrix}
=
\begin{vmatrix}
\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\
\sin\theta\cos\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\
\cos\theta & -r\sin\theta & 0
\end{vmatrix}
= r^2 \sin\theta
$$

Solid Angle

- In 1 dimension specify an arc on a circle

$$\theta = \frac{s}{r}$$



- In 2-d we need to specify an area on sphere

$$\Omega = \frac{A}{r^2} \quad \text{so} \quad \Omega = 4\pi \text{ for a full sphere}$$

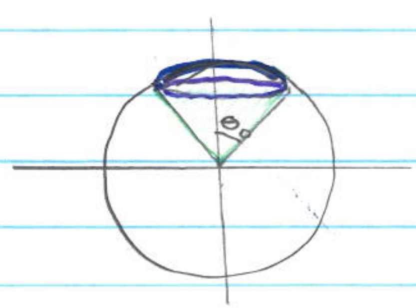
solid "angle" ← bad name we have to specify a spherical area, not really an "angle", but region on sphere. It is in range $[0, 4\pi]$

- From the picture disussed earlier

$$d\Omega \equiv \frac{dA}{r^2} = \frac{r^2 \sin\theta \, d\theta \, d\phi}{r^2} = \sin\theta \, d\theta \, d\phi$$

differential solid angle

- Excample find the solid angle of a cone:



$$\Omega = \int_{cone} \overbrace{\sin\theta \, d\theta \, d\phi}^{d\Omega}$$

$$= \int_0^{\theta_0} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$\Omega = 2\pi(1 - \cos\theta_0)$$

and for $\theta_0 = \pi$, $\cos\theta_0 = -1$ and $\Omega = 4\pi$

- Consider a particle which is confined to the surface of a sphere of radius R. Suppose that it is uniformly distributed over the area of the sphere. What is its probality distribution in θ, ϕ

$$d\mathcal{P} \propto dA \propto R^2 d\Omega \propto \sin\theta \, d\theta \, d\phi$$

The normalization constant is 4π

$$\boxed{d\mathcal{P} = \frac{d\Omega}{4\pi} = \frac{\sin\theta \, d\theta \, d\phi}{4\pi}}$$

probability density for a particle uniformly distributed over the sphere