Problem 1

a) • The kinetic energy is (5 points)

$$K = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2$$

There are two quadratic forms, so the number of dof = 2 and thus

$$\langle K \rangle = 2 \times \frac{1}{2}k_B T = \left\langle \frac{1}{2}mv^2 \right\rangle$$

So $\langle v^2 \rangle = \frac{2k_B T}{m}$ and $\sqrt{\langle v^2 \rangle} = \sqrt{\frac{2k_B T}{m}}$

b) The probability dirstribution is (10 points)

$$d\mathcal{P}_{x,y,v_x,v_y} = C\, dx\, dy\, dp_x\, dp_y\, e^{-mgy/kT} e^{-(p_x^2+p_y^2)/2mT}$$

This clearly factorizes into a probability distribution x, a dist for y, a dist for p_x, p_y. Thus

$$dP_y = C_y e^{-mgy/kT} dy$$

Normalizing:

$$\int_0^\infty dy\, C_y e^{-mgy/kT} = C_y \left[\int_0^\infty \frac{mg\,dy}{kT} e^{-mgy/kT} \right] \cdot \frac{kT}{mg} = 1$$

The integral is 1 leading to

$$C_y = \frac{mg}{kT}$$

and

$$d\mathcal{P}_y = \frac{mg}{kT} e^{-mgy/kT} dy$$

Then the average PE is

$$\langle mgy \rangle = \left[\int_0^\infty \frac{mg\,dy}{kT} e^{-mgy/kT} \frac{mgy}{kT} \right] kT$$

$$= kT \int_0^\infty du\, e^{-u}\, u = kT$$

So in total

$$\langle E \rangle = \langle KE \rangle + \langle PE \rangle$$

$$= kT + kT = 2kT$$

c) We have (10 points)

$$d\mathcal{P}_v = \frac{e^{-m(v_x^2+v_y^2)/2kT}}{2\pi\sigma_v^2} dv_x\, dv_y$$

$$\sigma_v^2 = \sqrt{\frac{kT}{m}} \sqrt{\frac{kT}{m}} = kT/m$$

Or

$$d\mathcal{P} = \frac{1}{\cancel{2\pi}\sigma_v^2} e^{-v^2/2\sigma_v^2} \cancel{2\pi} v\, dv$$

So if we want the prob, $v < v_{rms}$ or $v < \sqrt{kT/m}$

$$\mathcal{P}\left(v < \sqrt{\frac{2kT}{m}}\right) = \int_0^{v_{rms}} \frac{v\, dv}{\sigma^2} e^{-v^2/2\sigma_v^2}$$

Let $u = \frac{v^2}{2\sigma_v^2}$ $du = \frac{v\, dv}{\sigma_v^2}$ so $u(v_{rms}) = \frac{v_{rms}^2}{2\sigma_v^2} = 1$

$$\mathcal{P}\left(v < \sqrt{\frac{2kT}{m}}\right) = \int_0^1 du\, e^{-u} = -e^{-u}\Big|_0^1$$

$$= -e^{-1} + 1$$

Problem 2

a) $$Z = \sum_n e^{-E_n/kT}$$ (8 points)

$$Z = 1 + e^{-\Delta/kT} + e^{-5\Delta/kT}$$ 4+4 points

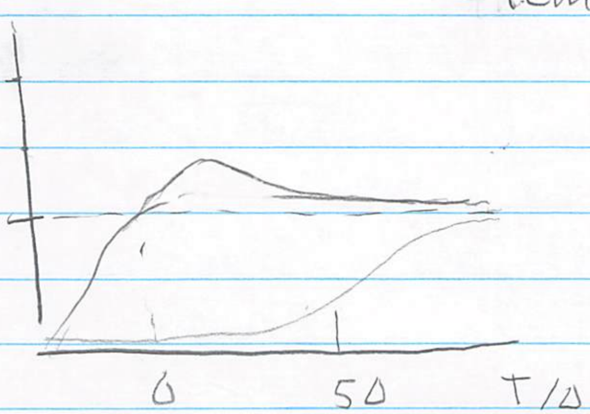
$$\langle E \rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{\Delta e^{-\Delta/kT} + 5\Delta e^{-5\Delta/kT}}{(1 + e^{-\Delta/kT} + e^{-5\Delta/kT})}$$

b) $$P_1 = \frac{e^{-\Delta/kT}}{(1 + e^{-\Delta/kT} + e^{-5\Delta/kT})}$$ (10 points)

4 points

$$P_2 = \frac{e^{-5\Delta/kT}}{(1 + e^{-\Delta/kT} + e^{-5\Delta/kT})}$$

i) So at low temperature P_1 + P_2 are zero, while at high temperature $P_0 = P_1 = P_2 = \frac{1}{3}$, all states are equally likely. The P_1 curve reaches its high temperature limit sooner



6 points

ii) Then taking P_2 (8 points)

$$P_2 = \frac{e^{-5\Delta\beta}}{(1 + e^{-\Delta\beta} + e^{-5\Delta\beta})} = \frac{N}{D}$$

Expanding the denominator D

$$D \simeq 3 - 6\Delta\beta = 3(1 - 2\Delta\beta)$$

Then

$$\frac{1}{D} \simeq \frac{1}{3}(1 + 2\Delta\beta)$$

While $N \simeq 1 - 5\Delta\beta$ so

$$P_2 \simeq \frac{N}{D} \simeq \frac{1}{3}(1 - 3\Delta\beta) \simeq \frac{1}{3} - \Delta\beta$$ 5 points

The correction will be small when $\Delta\beta \ll 1/3$, so
$T \gg 3\Delta/k_B$ or $T \gg 3(0.1\text{eV})/(0.025\text{eV}/300\text{K})$

or $T \gg 3600\,°\text{K}$ 3 points

Problem 3

a) We have (10 points)

$$P_a V_a^{\gamma} = P_b V_b^{\gamma} \quad \text{with} \quad \gamma = 5/3 \text{ for MAIG}$$

So

$$V_b = V_a \left(\frac{P_a}{P_b}\right)^{1/\gamma} = 1.0\,L \left(\frac{1.6b}{1.0b}\right)^{1/\gamma} = 27.6\,L$$

Then

$$P_b V_b = RT_b \Rightarrow T_b = \frac{P_b V_b}{R_b} = \frac{1.0b \cdot 27.6L}{8.32\ \frac{J}{K}}$$

Note $1\,bL = 100\,J$ so we find $T_b = 331\,°K$

b) Now (parts b and c 10 points)

① $W_{ba} \equiv W_{b \leftarrow a}$ $\qquad \Delta E_{ba} = \cancel{Q_{ba}} + W_{ba}$

$$\Delta E_{ba} = C_V (T_b - T_a) = W_{ba} = -855\,J$$

$$C_V = \frac{3}{2} R = 12.48\,J/°K$$

② $W_{cb} = W_{c \leftarrow b}$

$$\left.\begin{aligned} \Delta E_{cb} &= C_V (T_c - T_b) = -1016\,J \\ W_{cb} &= -p(V_c - V_b) = 678\,J \end{aligned}\right\} \quad \begin{aligned} Q_{cb} &= \Delta E_{cb} - W_{cb} \\ &= -1694\,J \end{aligned}$$

(3) $W_{ac} = W_{a \leftarrow c}$

$$\Delta E_{ac} = Q_{ac} + \cancel{W_{ac}}^{\,0}$$

$$C_V (T_a - T_c) = Q_{ac} = 1872 \text{ J}$$

So

$$\eta = \frac{|W_{net}|}{Q_{in}} = \frac{-(W_{ba} + W_{cb})}{Q_{ac}} =$$

$$= \frac{855\text{J} - 678\text{J}}{1872\text{J}} = 0.095$$

Note

$|W_{net}| = 177 \text{ J}$ $\qquad Q_{net} = Q_{ac} + Q_{cb} = 178 \text{ J}$

these should be same

c) $\Delta S_{ac} = \int_c^a \frac{dQ}{T} = \int_c^a C_V \frac{dT}{T} = C_V \ln \frac{T_a}{T_c} = 5.87 \frac{\text{J}}{°\text{K}}$

d) $\Delta S_{ba} = 0$ since $a \to b$ is adiabatic, $\Delta S_{TOT} = 0$ since it is a closed loop so ΔS_{cb} is $-\Delta S_{ac} = -5.87 \frac{\text{J}}{°\text{K}}$

parts c & d 10 points