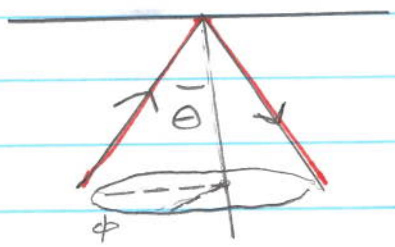
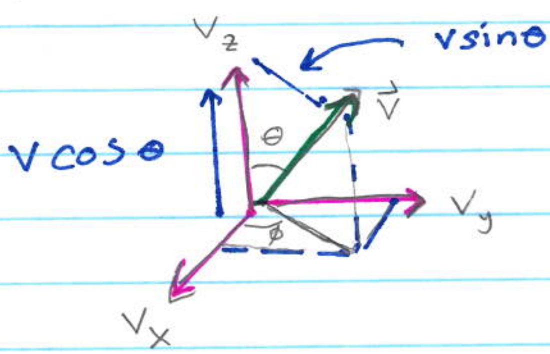
Pressure and Effusion



- Particles bounce of the walls with a variety of angles θ, ϕ a variety of speeds V each second. This gives a net force per area. We need the distribution of angles and speeds
- Then the velocity distribution is

$$d\mathcal{P} = \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{1}{2}mv^2/kT} \, dv_x \, dv_y \, dv_z$$

- So we can change Variables to speeds and angles $v = |\vec{v}|$, $v_z = v\cos\theta$ $v_x = v\sin\theta\cos\phi$ etc like before



$$d\mathcal{P} = \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{1}{2}mv^2/kT} \, v^2 \, dv \, \underbrace{d\Omega}_{\sin\theta \, d\theta \, d\phi}$$

- So we multiply and divide by 4π and write

$$d\mathcal{P} = P(v)\, dv\, \frac{d\Omega}{4\pi} \quad \text{where}$$

$$P(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, 4\pi v^2$$ is the distribution of speeds discussed before.

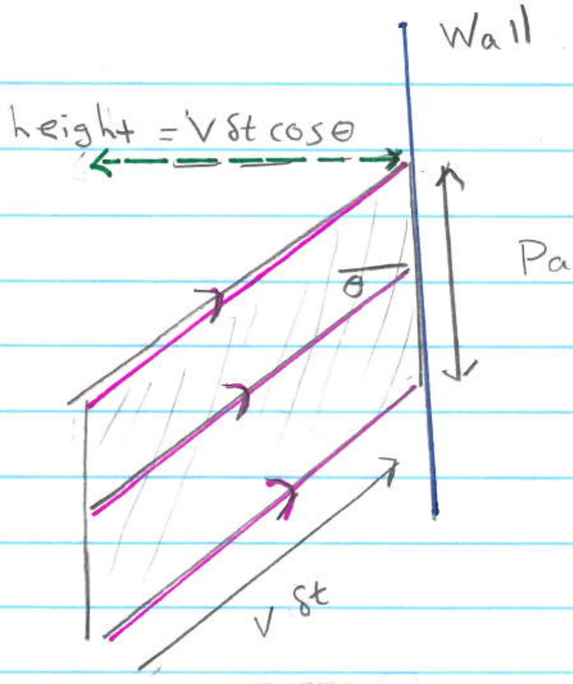
This gives the distribution of speeds and angles. The angular part is uniformly distributed over the sphere $d\Omega = \sin\theta\, d\theta\, d\phi / 4\pi$, as we could have hoped.

- To find the density of particles with speed $[v, dv]$ and angles $[\theta, \theta + d\theta]$, $[\phi, \phi + d\phi]$ we multiply by the total density of the system $n = \frac{N}{V}$

$$\boxed{dn = n\, d\mathcal{P} = n\, P(v)\, dv\, \frac{d\Omega}{4\pi}}$$

- Now consider a small patch of area A on the wall. We want to work out the flux Φ of particles striking the surface per dv, $d\theta$, $d\phi$

$$\boxed{\Phi \equiv \frac{\#\text{ of particles}}{\text{Area} \cdot \text{time}} \equiv \text{Flux}}$$



- The tube shows a swarm of particles in space flying with speed v, angles θ, ϕ. In a given δt all particles in the tube will strike the surface. The volume of the tube is

$$\text{Volume} = A \overbrace{v\,\delta t \cos\theta}^{\text{height}}$$

So the number per area per time in $dv, d\theta, d\phi$ is

$$\boxed{\frac{dN}{A\,\delta t} = \frac{\text{Volume} \cdot dn}{A\,\delta t} = n\,P(v)\,v\cos\theta\,\frac{dv\,d\Omega}{4\pi}}$$

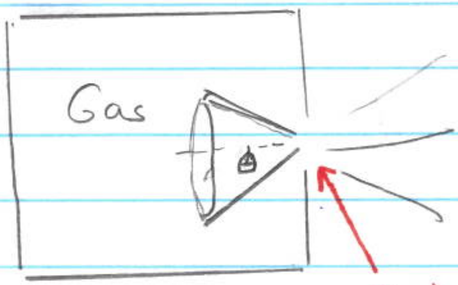
So

$d\Omega = \sin\theta\, d\theta\, d\phi$

$$d\Phi = n\,P(v)\,v\cos\theta\,\frac{dv\,d\Omega}{4\pi}$$

number of particles striking wall per area per time with $[v, v+dv]$ and angles $[\theta, \theta+d\theta]$ and $[\phi, \phi+d\phi]$

Effusion



- For example a slow bicycle leak. This is gas escaping through A

- Find the total number escaping per area per time

$$\Phi = \int d\Phi = \int_0^\infty dv \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \; n P(v) v \cos\theta \; \frac{\sin\theta}{4\pi}$$

from $\frac{d\Omega}{4\pi}$

- You will do the angular integrals in homework

$$\Phi = \int_0^\infty dv \; v P(v) \underbrace{\int_0^{\pi/2} \cos\theta \; \frac{1}{2} \sin\theta \, d\theta \int_0^{2\pi} \frac{d\phi}{2\pi}}_{1/4 \text{ from homework}}$$

$$\boxed{\Phi = \frac{1}{4} n \langle v \rangle \qquad \Phi = n \left(\frac{k_B T}{m}\right)^{1/2} \frac{1}{\sqrt{2\pi}}}$$

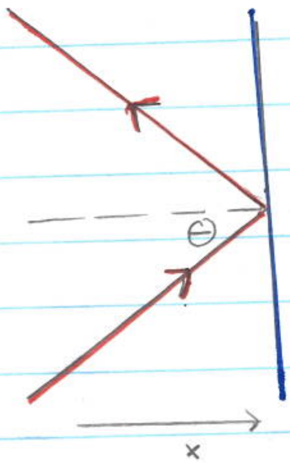
velocity

$$\langle v \rangle = \left(\frac{8 k_B T}{\pi m}\right)^{1/2}$$

- units: $\frac{\text{Number}}{m^3} \times \frac{m}{s} = \frac{\text{number}}{\text{Area time}}$ ✓

Pressure

We are now able to work out the pressure



- Each particle that strikes the walls experince a momentum transfer

$$\delta p_x = 2mv_x = 2mv\cos\theta$$

- The force, $\delta p_x/\delta t$, per area is

- $$\frac{dF}{A} = \delta p_x \frac{\delta N}{\delta t\, A} = 2mv\cos\theta \; d\Phi$$

So

$$p = \frac{\text{Force}}{\text{Area}} = n \int_{\text{particles hitting}} (2mv\cos\theta)\, \overbrace{n P(v)\, v\cos\theta\, dv \frac{\sin\theta\, d\theta\, d\phi}{4\pi}}^{= d\Phi}$$

$$= n \underbrace{\int_0^\infty 2mv^2 P(v)\, dv}_{2m\langle v^2\rangle} \underbrace{\int_0^{\pi/2} d\theta \cos^2\theta \frac{1\sin\theta}{2}}_{=1/6} \underbrace{\int_0^{2\pi} \frac{d\phi}{2\pi}}_{=1}$$

$$\boxed{p = \frac{1}{3} n m \langle v^2\rangle}$$

- So finally using $\langle v^2 \rangle = 3k_BT/m$ we have

$$\boxed{p = n k_B T}$$

In agreement with the ideal gas law $PV = Nk_BT$ or $p = nk_BT$

- We also note

$$\boxed{\begin{aligned} \Phi &= n\left(\frac{k_BT}{2\pi m}\right)^{1/2} \qquad \updownarrow \; p = nk_BT \\ \Phi &= \frac{p}{(2\pi m k_BT)^{1/2}} \end{aligned}}$$

We see that the flux $\Phi \propto \dfrac{1}{\sqrt{m}}$

- This was exploited in the manhattan project to separate $^{238}UF_6$ and $^{235}UF_6$ ← what we "want" for A-bomb.

$$\frac{\Phi_{235}}{\Phi_{238}} \simeq \left(\frac{m_{238}}{m_{235}}\right)^{1/2} \simeq \left(\frac{352}{348}\right)^{1/2} \simeq 1.006$$

so the lighter molecule effuses faster, it is moving faster and this causes the flux to be larger