1 Integrals you need to be able to use

First thing – get the units right! Integrals with wrong units loose plenty of points. Integrals with right units (but wrong numbers) loose almost nothing. Ask yourself what the units of the integral are before you start, and check at the end that you get the right units. You may want to switch to some dimensionless variables, which separates the untis part from doing the actual integral part.

Here are some integral we came across

- Integrals of the gaussian type

$$I_0 = \int_{-\infty}^{\infty} du e^{-u^2/2\sigma^2} = \sqrt{2\pi\sigma^2} \tag{1}$$

$$I_2 = \int_{-\infty}^{\infty} du e^{-u^2/2\sigma^2} u^2 = \sqrt{2\pi\sigma^2}\,\sigma^2 \tag{2}$$

$$I_4 = \int_{-\infty}^{\infty} du e^{-u^2/2\sigma^2} u^4 = \sqrt{2\pi\sigma^2}\,3\sigma^4 \tag{3}$$

  the odd moments vanish. The overall factor of σ is given by dimension. For example, I_4 has units of (meters)^5 (assuming u and σ have units of meters), so the integral is proportional to σ^5 which gives a quantity of units of meters^5.

- Integrals of the exponential type:

$$I_n = \int_0^{\infty} \mathrm{d}u\, e^{-x/\ell} u^n = (n!)\,\ell^{n+1} \tag{4}$$

  The factor of ℓ^{n+1} is the overall dimension of the integral (see above).

- A generalization of the exponential type integrals is those related to the Γ function.

$$I(z) = \int_0^{\infty} \frac{\mathrm{d}u}{u}\, u^z\, e^{-u/\ell} = \ell^z\, \Gamma(z) \tag{5}$$

  with z and arbitrary positive number. The overall factor of ℓ^z is by dimensions. I write $(du/u)u^z$ rather than $du\, u^{z-1}$ since du/u is dimensioneless, but this is idiosyncratic.

  This most often use case comes in the form

$$I = \int_0^{\infty} \mathrm{d}u\, u^{1/2} e^{-u/\ell} = \ell^{3/2}\,\Gamma(3/2) = \ell^{3/2}\,\frac{\sqrt{\pi}}{2} \tag{6}$$

  The overall factor of $\ell^{3/2}$ is by dimension. The last step uses the properties of the Γ function which you are expected to know (see next item). As an excercise you might evaluate $\int_0^{\infty} du u^{3/2} e^{-u/\ell}$.

- You need to know two properties of the Γ function:

  - Two values

$$\Gamma(1) = 1 \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{7}$$

– The factorial property (for any value of x).

$$\Gamma(x+1) = x\Gamma(x) \tag{8}$$

So that for positive integer $\Gamma(n) = (n-1)!$.

- If you need the area of a sphere embedded in d dimensions (probably not), I will give it

$$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1} \tag{9}$$

2 Taylor Series

You are expected to know the following Taylor series in addition to $\sin(x)$ and $\cos(x)$:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \mathcal{O}(x^3) \tag{10}$$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \mathcal{O}(x^4) \tag{11}$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \mathcal{O}(x^4) \tag{12}$$

$$\frac{1}{1+x} = 1 - x + x^2 + \mathcal{O}(x^3) \tag{13}$$

These get me through life. Here x is considered to be a small, dimensionless, number. The $\mathcal{O}(x^3)$ etc shows an estimate for the size of the terms that have been dropped. Taylor series can be combined, and integrated etc.