1 Integrals

Bose and Fermi:

$$\int_0^\infty \mathrm{d}x \, \frac{x}{e^x - 1} = \frac{\pi^2}{6} \tag{1}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^2}{e^x - 1} = 2\zeta(3) \simeq 2.404 \tag{2}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^3}{e^x - 1} = \frac{\pi^4}{15} \tag{3}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^4}{e^x - 1} = 24\,\zeta(5) \simeq 24.88 \tag{4}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^5}{e^x - 1} = \frac{8\pi^6}{63} \tag{5}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x}{e^x + 1} = \frac{\pi^2}{12} \tag{6}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^2}{e^x + 1} = \frac{3}{2}\,\zeta(3) \simeq 1.80309 \tag{7}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^3}{e^x + 1} = \frac{7\pi^4}{120} \tag{8}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^4}{e^x + 1} = \frac{45}{2}\,\zeta(5) \simeq 23.33 \tag{9}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^5}{e^x + 1} = \frac{31\pi^6}{252} \tag{10}$$

Gamma Function:

$$\Gamma(z) \equiv \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x \tag{11}$$

with specific results

$$\Gamma(z+1) = z\Gamma(z) \qquad \Gamma(n) = (n-1)! \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{12}$$

Gaussian Integrals:

$$I_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \mathrm{d}x \, e^{-x^2/2} x^n \tag{13}$$

with specific results

$$I_0 = 1 \quad I_2 = 0 \quad I_4 = 3 \quad I_6 = 15 \tag{14}$$

Problem 1. A nucleus as a fermi gas

Large nuclei can be treated as approximately "infinite" in size. This means that density of protons and neutrons within the nucleus approaches a constant, and in first approximation edge effects can be neglected. In the infinite volume limit the material is known as nuclear matter, and the density of the protons and neutrons is known as nuclear matter density.

Treat a nucleus as a ball of radius R with A nucleons[1]. The radius of a ball grows with $A^{1/3}$ as

$$R = (1.3 \times 10^{-15}\, m)\, A^{1/3} \tag{15}$$

Assume that the number of protons and the number of neutrons are equal.

(a) Compute the density of protons and the density neutrons.

(b) Show that the Fermi energy of protons is approximately 27 MeV.

Since we have assumed the number of protons and neutrons are equal, the Fermi energy of neutrons is also 27 MeV. In reality the number of neutrons is somewhat larger than the number of protons. Thus, the density of neutrons is higher, and the corresponding Fermi energy is somewhat higher.

(c) Show that energy per nucleon inside a nucleus is approximately 16 MeV. This is a reasonable estimate for the kinetic energy per volume in a nucleus.

[1] A nucleon is either a proton or neutron. Oxygen has eight protons and eight neutrons and has $A = 16$.

Problem 2. 2D Fermi gas

Consider a fermi gas of electrons in two dimensions.

(a) Show that the fermi momentum is

$$p_F = \hbar\sqrt{2\pi n} \tag{16}$$

(b) Show that the mean value of the debroglie wavelength divided by 2π, i.e. $\lambdabar \equiv \hbar/p$, is

$$\langle \lambdabar \rangle = \frac{2\hbar}{p_F} \tag{17}$$

Problem 3. Relativistic Degenerate Electron Gas

Consider an ultra-relativistic degenerate electron gas where $\epsilon \simeq cp$, and the electron mass can be neglected.

(a) Show that the Fermi Energy is related density by

$$\epsilon_F = \hbar c(3\pi^2 n)^{1/3} \tag{18}$$

where $n = N/V$.

(b) Compute the Fermi momentum p_F. Define the Fermi wavelength, $\lambdabar_F \equiv \hbar/p_F$. Explain qualitatively the dependence of $\lambdabar_F$ on the density $n = N/V$.

(c) Show that the total energy of the gas is

$$U = \frac{3}{4}N\epsilon_F \tag{19}$$

(d) Show that the pressure of the the gas is

$$\mathcal{P} = \frac{1}{3}\frac{U}{V} \tag{20}$$

and determine its dependence on density $n = N/V$. Compare your result to a classical ideal gas where $\mathcal{P} \propto n$ and a non-relativistic degenerate Fermi gas where $\mathcal{P} \propto n^{5/3}$