1 Integrals

Bose and Fermi:

$$\int_0^\infty \mathrm{d}x\,\frac{x}{e^x-1} = \frac{\pi^2}{6} \tag{1}$$
$$\int_0^\infty \mathrm{d}x\,\frac{x^2}{e^x-1} = 2\zeta(3) \simeq 2.404 \tag{2}$$
$$\int_0^\infty \mathrm{d}x\,\frac{x^3}{e^x-1} = \frac{\pi^4}{15} \tag{3}$$
$$\int_0^\infty \mathrm{d}x\,\frac{x^4}{e^x-1} = 24\,\zeta(5) \simeq 24.88 \tag{4}$$
$$\int_0^\infty \mathrm{d}x\,\frac{x^5}{e^x-1} = \frac{8\pi^6}{63} \tag{5}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x}{e^x+1} = \frac{\pi^2}{12} \tag{6}$$
$$\int_0^\infty \mathrm{d}x\,\frac{x^2}{e^x+1} = \frac{3}{2}\,\zeta(3) \simeq 1.80309 \tag{7}$$
$$\int_0^\infty \mathrm{d}x\,\frac{x^3}{e^x+1} = \frac{7\pi^4}{120} \tag{8}$$
$$\int_0^\infty \mathrm{d}x\,\frac{x^4}{e^x+1} = \frac{45}{2}\,\zeta(5) \simeq 23.33 \tag{9}$$
$$\int_0^\infty \mathrm{d}x\,\frac{x^5}{e^x+1} = \frac{31\pi^6}{252} \tag{10}$$

Gamma Function:

$$\Gamma(z) \equiv \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x \tag{11}$$

with specific results

$$\Gamma(z+1) = z\Gamma(z) \qquad \Gamma(n) = (n-1)! \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{12}$$

Gaussian Integrals:

$$I_n = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\,e^{-x^2/2}x^n \tag{13}$$

with specific results

$$I_0 = 1 \quad I_2 = 0 \quad I_4 = 3 \quad I_6 = 15 \tag{14}$$

Problem 1. Most energetic frequency band

(a) The energy density can be written

$$u = \int_0^\infty d\omega \, \frac{du}{d\omega} \tag{15}$$

where $du/d\omega$ is the energy per frequency interval $d\omega$. Using a graphical means show $du/d\omega$ is maximum for $\hbar\omega = 2.82kT$. What is the energy of a photon with this frequency for a black body of 5340 °K, which is approximately the surface temperature of the sun.

(b) The energy density can be written

$$u = \int_0^\infty d\lambda \, \frac{du}{d\lambda} \tag{16}$$

where $du/d\lambda$ is the energy per wavelength inteval $d\lambda$. Find $du/d\lambda$, and using a graphical method find the wavelength $\lambda_{\rm max}$ where $du/d\lambda$ is maximum. (You should find $hc/\lambda_{\rm max} \simeq 4.9\, k_B T$.) What is this wavelength in nm for a black body of 5340 °K, which is approximately the surface temperature of the sun. You should find that the wavelength corresponds to a yellow color.

Problem 2. Density of single particle states

In class we classified the modes (wave-functions) of a box by three quantum numers

$$\psi_{n_x n_y n_z}(x,y,z) \propto \sin(k_{n_x} x) \sin(k_{n_y} y) \sin(k_{n_z} z) \tag{17}$$

and we showed that if the box is large

$$\sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \sum_{n_z=1}^{\infty} \cdots \to \int \frac{V d^3 p}{(2\pi\hbar)^3} \cdots \tag{18}$$

(a) Show that the number of modes $g(k)dk$ with wavenumber k, between k and $k + dk$ is

$$g(k)dk = \frac{1}{2\pi^2} V k^2 dk \tag{19}$$

and determine the analogous formula in two dimensions. $g(k)$ is known as the density of (single particle) states. Assume that the particles are spinless, so that

$$\sum_{\rm modes} \cdots = \sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \sum_{n_z=1}^{\infty} \cdots \tag{20}$$

(b) The density of states is often expressed in terms of energy. For spinless non-relativistic particles (with $\epsilon(p) = p^2/2m$) show that the number of modes, $g(\epsilon)d\epsilon$, with energy between ϵ and $d\epsilon$. Show that the density of states in three dimensions is

$$g(\epsilon)d\epsilon = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\epsilon}\, d\epsilon \tag{21}$$

and find the analogous formula in two dimensions.

(c) In any number of dimensions, and for relativistic and non-relativistic particles, explain why the grand potential of a Bose and Fermi gas can be written

$$\Phi_G = \pm kT \int_0^\infty g(\epsilon) \ln\left(1 \mp e^{-\beta(\epsilon_p - \mu)}\right) \mathrm{d}\epsilon \tag{22}$$

By differentiation of Φ_G show that the number of particles is

$$N = \int_0^\infty g(\epsilon) \frac{1}{e^{\beta(\epsilon_p-\mu)} \mp 1} \mathrm{d}\epsilon \tag{23}$$

where the upper sign is for fermions and the lower sign is for bosons.

Determine $g(\epsilon)$ for a photon gas in three dimensions, and express the pressure of the photon gas as an integral. You will evaluate this integral in the next problem.

The photon has two polarization states. So there are two modes for every value of k

$$\sum_{\text{modes}} = 2 \sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \sum_{n_z=1}^{\infty} \ldots \tag{24}$$

Problem 3. Entropy per Photon

You should have found the pressure (or minus the grand potential per volume) of gas of photons. Ater recognizing that $\epsilon = \hbar\omega$, the result of problem 2 is

$$pV = \frac{V}{\pi^2 c^3} \int_0^\infty \omega^2 kT \ln\left(1 - e^{-\beta\hbar\omega}\right) \mathrm{d}\omega \tag{25}$$

(a) Integrate by parts to show that

$$p = \frac{\pi^2}{45} \left(\frac{kT}{\hbar c}\right)^3 kT \tag{26}$$

The necessary integrals are given below.

(b) Show that

$$d\Phi_G = -SdT - Nd\mu - pdV \tag{27}$$

and then by differentiating the pressure (or grand potential), that

$$S = 4\frac{pV}{T} \tag{28}$$

Using the result from class for the number of photons and show that the entropy per photon S/Nk_B is 3.6.

(c) Use the Gibbs-Duhem relation and the results for S and p in this problem to find the energy density of the system, $u = U/V$. Check your result by comparing with the method used in class.

Hint: What is the chemical potential of the photon gas?

Problem 4. Temperature of the Sun

The intensity (energy per area per time) of sunlight on earth is $I = 1\,\mathrm{kW}/m^2$. Show that the temperature of the sun is related to the intensity of the sunlight and the solid angle $\Omega_\odot$ that the suns takes up in our sky:

$$T_\odot = \left(\frac{I}{\sigma}\frac{\pi}{\Omega_\odot}\right)^{1/4} \tag{29}$$

Here σ is the Steffan Boltzmann constant and Ω is the solid angle subtended by the sun in our sky. Using

$$\Omega_\odot \simeq 6.8 \times 10^{-5} \tag{30}$$

evaluate $T_\odot$ numerically. I find $T_\odot \simeq 5340\,\mathrm{K}$.

Problem 5. Neutrino Gas

A neutrino is like a photon – it is neutral, (nearly) massless, and it has two spin states, But a neutrino is a fermion not a boson, and the modes can be in two possible states, either unoccupied or occupied by a particle, with corresponding energies 0 and ϵ. Consider a gas of neutrinos at temperature T and in a cubic box of volume $V = L^3$.

(a) Consider a single mode in the container with single particle energy ϵ. Determine the grand partition function for the mode and derive the Fermi-Dirac expression for the mean number of particles in the mode:

$$n_{FD}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \tag{31}$$

What are the probabilities of finding the mode unoccupied and occupied respectively?

(b) What is the energy and number of the neutrinos in the box? Explain each step carefully and assume $\mu = 0$.

(i) Compare the number of neutrinos in the box to the number of photons that would be in an equivalent box at the same temperature. Explain qualitatively why there are fewer neutrinos in the box.

You should find that the number of neutrinos is 3/4 the number of photons, but the energy is of the neutrinos is 7/8 of the photons.

(c) Determine the average de Broglie wavelength $\lambda \equiv h/p$ of the neutrino by integrating over the momenta. (Ans. $\lambda = 2.87\,\hbar c/kT$)

(d) What is the number of neutrinos striking the walls per unit time? Ans: $0.37\,L^2(kT/\hbar c)^3 c$.