Physics 306: Thermal Physics

Midterm Exam

Stony Brook University

Fall 2024

General Instructions:

You may use one page (front and back) of handwritten notes and a calculator. Graphing calculators are allowed. **No other materials may be used**.

1 Integrals

Gamma Function:

$$\Gamma(z) \equiv \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x \tag{1}$$

with specific results

$$\Gamma(z+1) = z\Gamma(z) \qquad \Gamma(n) = (n-1)! \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{2}$$

Gaussian Integrals:

$$I_n = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-x^2/2\sigma^2} x^n \tag{3}$$

with specific results

$$I_0 = 1 \quad I_2 = \sigma^2 \quad I_4 = 3\sigma^4 \quad I_6 = 15\sigma^6 \tag{4}$$

Other integrals:

$$\begin{gathered} \int e^u \, \mathrm{d}u = e^u + C \\ \int e^u \, u \, \mathrm{d}u = e^u (u-1) + C \\ \int e^u \, u^2 \mathrm{d}u = e^u \left(u^2 - 2u + 2\right) + C \end{gathered} \tag{5}$$

Here α is any real number (positive or negative).

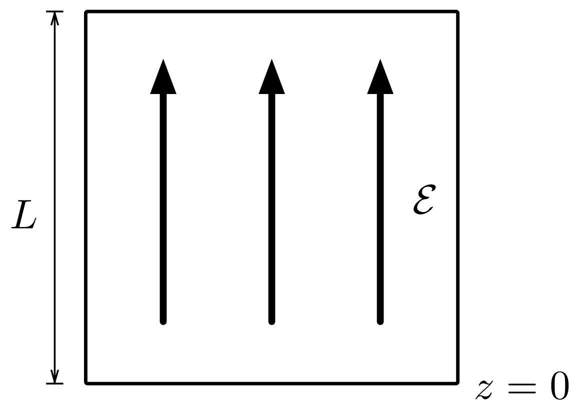
Problem 1. Plasma in an electric field

Consider a plasma made up of N electrons in a 3D box with volume $V = L^3$. Treat the plasma as an ideal gas maintained at temperature T, and ignore the Coulomb interaction between the electrons.

A uniform electric field of magnitude $\mathcal{E}$ is applied in the z-direction. Recall that the potential energy $V(z)$ of a negatively charged particle in a constant electric field is given by[1]

$$V(z) = q\mathcal{E}z \tag{6}$$

where q is the absolute value of the electron's charge.



Some potentially useful integrals are given on the first page.

(a) Determine the most probable speed for the electrons.

(b) Determine the charge density $\rho(z)$, i.e. the density of electrons times $-q$, as a function of z.
Hint: First determine the probability density of an electron having height z.

(c) What is the average energy of the plasma. Taylor expand your result in the limit that the electric field is weak to leading (non-zero) order in the field strength $\mathcal{E}$.

[1] The potential energy is the charge times the voltage. The voltage is $\Phi(z) = -\mathcal{E}z$ for a constant field.

Solution

(a) The speed distribution is

$$\frac{d\mathscr{P}}{dv} = P(v) = Ce^{-mv^2/2kT}v^2 \tag{7}$$

where C is an unimportant constant. Maximizing $P(v)$ by finding where the derivative is zero yield

$$P'(v) = Ce^{-mv^2/2kT}\left(\frac{mv^3}{kT} - 2v\right) = 0 \tag{8}$$

so

$$v_{\max} = \sqrt{2kT/m} \tag{9}$$

(b) The probability distribution in z is

$$\frac{d\mathscr{P}}{dz} \equiv P(z) = Ce^{-\beta V(z)} = Ce^{-\beta q\mathcal{E}z} = Ce^{-az} \tag{10}$$

where we defined, $a \equiv \beta q\mathcal{E}$. The normalization constant can be found by normalizing the probability

$$\int_0^L P(z)dz = \int_0^L Ce^{-az} = \frac{C}{a}\left(1 - e^{-aL}\right) = 1 \tag{11}$$

So

$$C = \frac{a}{1 - e^{-aL}} \tag{12}$$

Now we can find the probability per dz

$$\frac{d\mathscr{P}}{dz} \equiv P(z) = \frac{a\, e^{-az}}{1 - e^{-aL}}\,, \tag{13}$$

Mulitipying $d\mathscr{P}/dz$ by N would give the number of particles per dz. Dividing by the area L^2 and noting that $\mathrm{d}V = L^2\mathrm{d}z$ we find the electron density

$$\boxed{n(z) = \frac{dN}{dV} = \frac{1}{L^2}\left(\frac{dN}{dz}\right) = \frac{Na}{L^2}\,\frac{e^{-az}}{1 - e^{-aL}}\,.} \tag{14}$$

(c) The energy of an electron is

$$\bar{\epsilon} = \langle KE\rangle + \langle PE\rangle = \left\langle \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_y^2}{2m}\right\rangle + \langle PE\rangle = \frac{3}{2}kT + \langle PE\rangle \tag{15}$$

where we used the equipartition theorem to evaluate the kinetic energy. The average potential energy is

$$\langle PE\rangle = \int_0^L q\mathcal{E}z\, Ce^{-az}\mathrm{d}z = kTC\int_0^L az\, e^{-az}\,\mathrm{d}z \tag{16}$$

Performing the integral using the table, and putting in the value of C

$$\langle PE\rangle = kTC\left[-\frac{1}{a}e^{-az}(1+az)\right]_0^L \tag{17}$$

$$= kT\frac{C}{a}\left[1-e^{-aL}(1+aL)\right] \tag{18}$$

Putting in the value of C and multiplying by the total number of particles gives the total energy

$$\boxed{U = \frac{3}{2}NkT + NkT\left[\frac{1-e^{-aL}\,(1+aL)}{1-e^{-aL}}\right]} \tag{19}$$

The series expansion works as follows. Writing

$$e^{-aL} = 1-(aL)+\frac{1}{2}(aL)^2 \tag{20}$$

We find

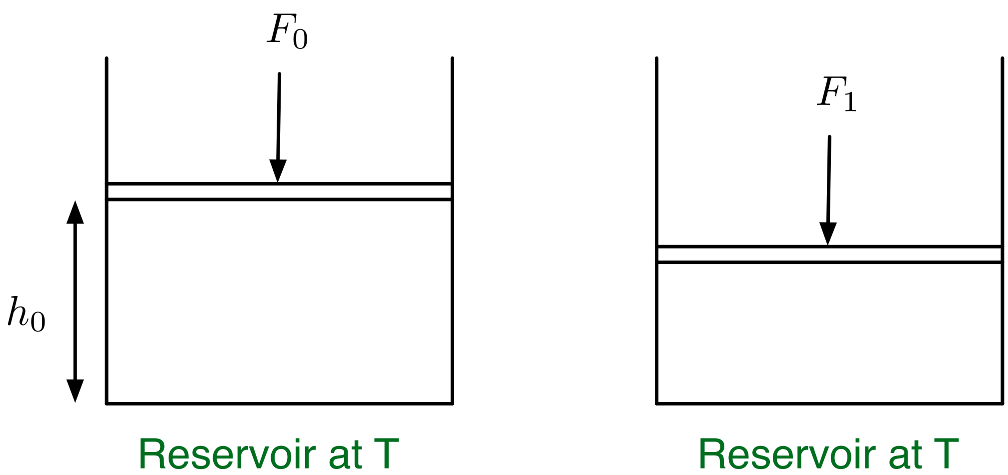
$$\langle PE\rangle = \frac{1}{\beta}\left[\frac{1-(1-(aL)+\frac{(aL)^2}{2})\,(1+(aL)}{aL}\right] \simeq kT\frac{aL}{2} \tag{21}$$

Then multiplying by N we find

$$U = \frac{3NkT}{2}+\frac{q\mathcal{E}L}{2} \tag{22}$$

Problem 2. A Sudden Force

A cylinder has cross-sectional area A and initial height h_0. The cylinder contains an ideal gas and is surrounded by a reservoir at temperature T. Initially, a piston applies a force F_0 on the gas in the cylinder. At time $t = 0$, the applied force is suddenly increased to F_1 and held constant. The cylinder then contracts until it reaches a new equilibrium volume. Do not assume that the gas is in equilibrium during the contraction.



(a) What is the final volume? What is the work by the external force done during the process? What is the heat that flows out of the cylinder during the process?

(b) Determine the change in entropy of the gas in the cylinder.

(c) What is the change in entropy of the universe ΔS_{univ}? Let $F_1 = F_0(1 + \delta)$. Make a series expansion of ΔS_{univ} for $\delta \ll 1$ to quadratic order in δ.

Solution

(a) Since $PV = NkT$ is a constant before and after, and since $pV = Fh$ we have

$$F_0 h_0 = F_1 h_1 = NkT \,, \tag{23}$$

or

$$\boxed{h_1 = h_0 \frac{F_0}{F_1} \,.} \tag{24}$$

The work done on the gas

$$\boxed{W_{in} = F_1 \Delta x = F_1 (h_0 - h_1) \,.} \tag{25}$$

Using the first law

$$\Delta U = Q_{\rm in} + W_{\rm in} = 0 \,, \tag{26}$$

Thus the heat that flows out is

$$\boxed{Q_{\rm in} = -W_{\rm in} \qquad Q_{\rm out} = -Q_{\rm in} = F_1 (h_0 - h_1) \,.} \tag{27}$$

(b) The entropy is

$$S = \text{const} + \frac{3}{2} Nk \ln U + Nk \ln V \,, \tag{28}$$

The temperature is fixed so $U_f = U_i$ and thus $\Delta S = S_f - S_i$ is

$$\Delta S = Nk \left[\ln(V_f) - \ln(V_i) \right] = Nk \ln(V_f/V_i) = Nk \ln(h_1/h_0) = Nk \ln(F_0/F_1) \,. \tag{29}$$

We note that $\Delta S < 0$ since $F_0/F_1 < 1$.

(c) The change in entropy of the universe is

$$\Delta S_{\rm univ} = \Delta S_{\rm sys} + \Delta S_R = \Delta S_{\rm sys} - \frac{Q_{\rm in}}{T} = Nk \ln(F_0/F_1) + \frac{F_1(h_0 - h_1)}{T} \,. \tag{30}$$

Now using the results from part (a) e.g. Eq. (24) and Eq. (23) we find

$$\boxed{\Delta S_{\rm univ} = Nk \left[\ln(F_0/F_1) + F_1/F_0 - 1 \right] \,.} \tag{31}$$

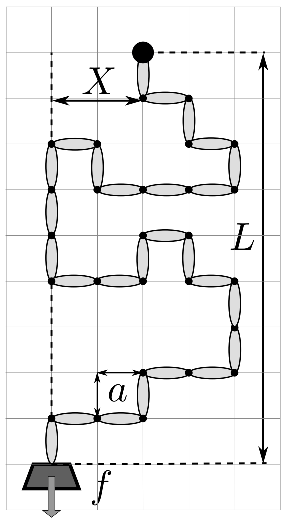
Writing $F_1 = F_0(1 + \delta)$ we expand

$$\Delta S_{\rm univ} = Nk \left[-\ln(1+\delta) + \delta \right] \simeq Nk \left[\left(-\delta + \frac{\delta^2}{2} \right) + \delta \right] \simeq Nk \, \frac{\delta^2}{2} \,. \tag{32}$$

We see that $\Delta S_{\rm univ}$ is positive as it must be.

Problem 3. A Model For an Elastic Band

In a simple model of an elastic polymer chain at temperature T, the polymer consists of N links, each of length a with $N \gg 1$. The first link is attached to a fixed point (the black dot), while the last link is pulled down by a force f in the z-direction.



Each link in the chain is independent and can point in one of four directions: Right (R), Left (L), Up (U), or Down (D). The links do not interact with each other and can cross without tangling. The displacement $(\Delta x, \Delta y)$ of the Right links is $(a, 0)$. The displacement of the Down link (D) is $(0, -a)$.

(a) What are the displacements $(\Delta x, \Delta y)$ of the Left and Up links[2]?

Because of the external force, the downward directed links have lower energy and are more probable. The energies of the four link configurations are tabulated below:

State	Energy
R	0
L	0
U	fa
D	$-fa$

(b) Determine the probability for a link be R, L, U, or D.

(c) In each step of the chain, the y position of the next link is displaced by Δy. Determine the mean Δy per link.

[2]Not a trick question. Just a question to make the setup clear.

(d) What is the mean length L in the y direction (see figure). What is the variance in L?

Hint: relate the variance in L to the variance of Δy.

(e) What approximately is the probability distribution for L?

Solution

(a) For left $(\Delta x, \Delta y) = (-a, 0)$. For up $(\Delta x, \Delta y) = (0, a)$.

(b) The partition function is

$$Z = \sum_{s \in RLUD} e^{-\beta\epsilon_s} = 1 + 1 + e^{-\beta f a} + e^{\beta f a} = 2 + 2\cosh(\beta f a)\,. \tag{33}$$

The probabilities are

$$P_R = P_L = \frac{1}{2 + 2\cosh(\beta f a)} \quad P_U = \frac{e^{-\beta f a}}{2 + 2\cosh(\beta f a))} \quad P_D = \frac{e^{\beta f a}}{2 + 2\cosh(\beta f a)}\,. \tag{34}$$

(c) The mean Δy is

$$\langle \Delta y \rangle = 0 \cdot P_R + 0 \cdot P_L + aP_U - aP_D = a(P_U - P_D) = -a\frac{\sinh(\beta f a)}{1 + \cosh(\beta f a)}\,. \tag{35}$$

(d) The mean length is

$$\overline{L} = N \langle \Delta y \rangle = -Na\frac{\sinh(\beta f a)}{1 + \cosh(\beta f a)}\,. \tag{36}$$

where the minus sign indicates a downward direction. The average of $(\Delta y)^2$ is

$$\left\langle (\Delta y)^2 \right\rangle = 0^2 P_R + 0^2 P_L + a^2 P_U + (-a)^2 P_D = \frac{a^2\cosh(\beta f a)}{1 + \cosh(\beta f a)}\,. \tag{37}$$

Using that

$$\delta y^2 = \left\langle (\Delta y)^2 \right\rangle - \langle \Delta y \rangle^2\,, \tag{38}$$

leads ultimately to the variance

$$\left\langle \delta y^2 \right\rangle = a^2 \left[\frac{\cosh(\beta f a)}{1 + \cosh(\beta f a)} - \frac{\sinh^2(\beta f a)}{(1 + \cosh(\beta f a))^2} \right] = \frac{a^2\cosh(\beta f a)}{(1 + \cosh(\beta f a))^2}\,, \tag{39}$$

where we used $\cosh^2(x) - \sinh^2(x) = 1$. The variance of the total length is just N times the variance for each link

$$\left\langle \delta L^2 \right\rangle = N \left\langle \delta y^2 \right\rangle\,. \tag{40}$$

We used that the variance of a sum is the sum of the variances for independently distribution quantities.

(e) The probability is Gaussian

$$d\mathscr{P} = \frac{1}{\sqrt{2\pi \left\langle \delta L^2 \right\rangle}} \exp(-(L - \bar{L})^2 / 2\left\langle \delta L^2 \right\rangle)\,. \tag{41}$$

This is the central limit theorem.