Physics 306: Thermal Physics

Final Exam

Stony Brook University

Spring 2024

General Instructions:

You may use one page (front and back) of handwritten notes and a calculator. Graphing calculators are allowed. **No other materials may be used**.

1 Integrals

Bose and Fermi:

$$\int_0^\infty \mathrm{d}x\,\frac{x}{e^x-1} = \frac{\pi^2}{6} \tag{1}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^2}{e^x-1} = 2\zeta(3) \simeq 2.404 \tag{2}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^3}{e^x-1} = \frac{\pi^4}{15} \tag{3}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^4}{e^x-1} = 24\,\zeta(5) \simeq 24.88 \tag{4}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^5}{e^x-1} = \frac{8\pi^6}{63} \tag{5}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x}{e^x+1} = \frac{\pi^2}{12} \tag{6}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^2}{e^x+1} = \frac{3}{2}\,\zeta(3) \simeq 1.80309 \tag{7}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^3}{e^x+1} = \frac{7\pi^4}{120} \tag{8}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^4}{e^x+1} = \frac{45}{2}\,\zeta(5) \simeq 23.33 \tag{9}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^5}{e^x+1} = \frac{31\pi^6}{252} \tag{10}$$

Gamma Function:

$$\Gamma(z) \equiv \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x \tag{11}$$

with specific results

$$\Gamma(z+1) = z\Gamma(z) \qquad \Gamma(n) = (n-1)! \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{12}$$

Gaussian Integrals:

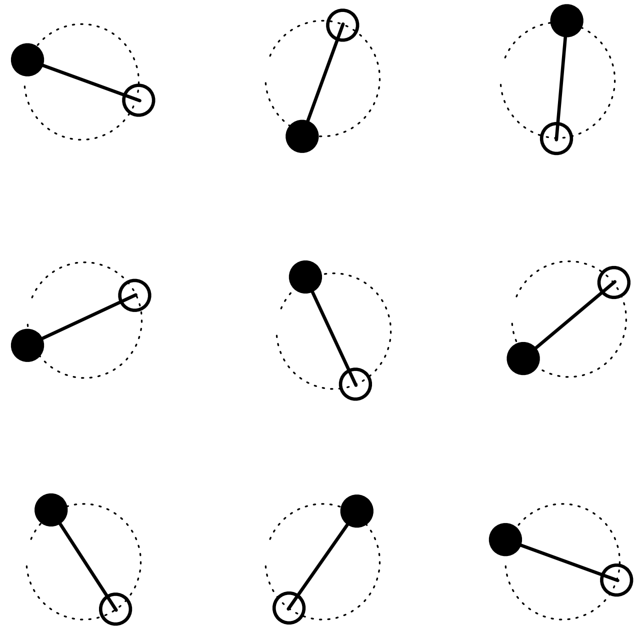
$$I_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \mathrm{d}x\, e^{-x^2/2\sigma^2} x^n \tag{13}$$

with specific results

$$I_0 = 1 \quad I_2 = 0 \quad I_4 = 3 \quad I_6 = 15 \tag{14}$$

Problem 1. 2D Rotors

A molecule in two spatial dimensions of consists of two non-identical atoms, each of mass M. The atoms lie in the xy plane and are separated from each other by a distance r_0. N such molecules are arranged on a lattice, as shown below. Each molecule is free to rotate in the xy plane around its lattice site, but is otherwise fixed to the site.



The classical energy of a molecule rotating in the xy plane is

$$\epsilon = \frac{1}{2} I \omega^2 = \frac{L_z^2}{2I} \tag{15}$$

where I is the moment of inertia of the molecule and $L_z = I\omega$ is the angular momentum around the z axis. Quantum mechanically, the angular momentum takes on discrete values in units of $\hbar$ labeled by an integer quantum number[1] m:

$$L_z = m\hbar \qquad m = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{16}$$

The positive and negative values of m describe counter-clockwise and clockwise rotations respectively, while $m = 0$ describes a molecule not rotating at all.

(a) Approximately evaluate the partition function of the molecule by including just the first two energy levels.

(i) Explain why this valid at low temperatures, and define what is meant by low in this context, i.e. low compared to what? Suppose that the atoms in the molecule each have a mass of a Hydrogen atom and assume the a bond length r_0 is a typical atomic distance, estimate the range temperature (in Kelvin where) the low temperature approximation is valid.

[1] Note the mass of the atoms M should not be confused with the integer m.

(b) Determine the mean energy and specific heat of the molecule using the low temperature approximation of (a)

(c) (i) Briefly explain why at high temperatures the sum over states m can be replaced with an integral. (ii) Evaluate the partition function with this approximation and the result result to find the mean energy the system. Explain the result for the energy using classical reasoning.

(d) Suppose that the molecules were free to move around in the xy plane as a an ideal gas. What is the specific heat of the gas at constant pressure in the low temperature approximation of part (b) in this case?

Problem 2. Electromagnetic pressure in a transmission line

The thermal fluctuations in a electromagnetic transmission line can be described as a gas of photons in a one dimensional box of size L at temperature T. In one dimension, the "volume" (V) of the box is simply the line length, L, and the "pressure" (P) is simply the force the photons exert on the ends of the transmission line, $\mathcal{F}$.

As photon number is not conserved, you may set the chemical potential to zero throughout this problem.

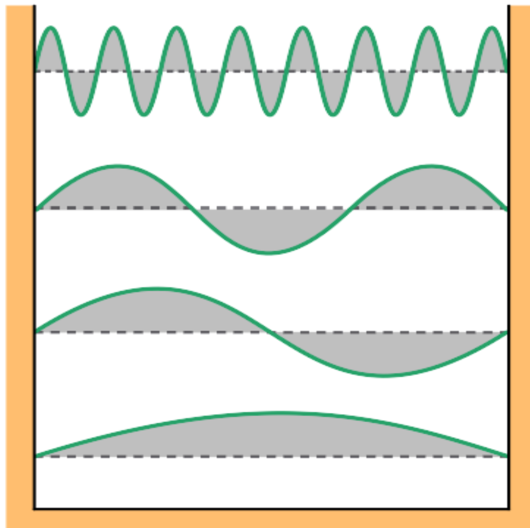
(a) (i) Starting from the first law of thermodynamics, show that force is related to the derivative of the grand potential[2], $\Phi_G \equiv U - TS$:

$$\left(\frac{\partial \Phi_G}{\partial L}\right)_T = -\mathcal{F}\,.$$

(ii) How could one express the force as a derivative of the energy U?

(b) Now consider just one Fourier mode in the wire labeled by ℓ, a positive integer. Some of the Fourier modes of the wire are shown below:

[2] Since the chemical potential is zero the grand potential and free energy are equal.



The energy of the mode $\varepsilon_\ell = cp_\ell$ where $p_\ell = \pi\hbar\ell/L$ is the magnitude of the momentum of the mode.

(c) (i) Derive grand potential for the ℓ-th mode and show that the force on the ends of the transmission from just this mode is

$$\mathcal{F}_\ell = \frac{1}{e^{\beta\varepsilon_\ell} - 1}\left(-\frac{d\varepsilon_\ell}{dL}\right) . \tag{17}$$

Give a qualitative interpretation of this formula.

(d) The total force on the ends of the wire is a sum of the forces from each mode

$$\mathcal{F} = \sum_\ell \mathcal{F}_\ell \tag{18}$$

By replacing the sum over with an appropriate integral determine the force exerted by the gas.

Problem 3. Absorbing walls

A canister of volume V contains a mono-atomic ideal of mass m. The canister is maintained temperature T and pressure P_0 through a small inlet valve. The walls of the container have N_0 absorbing sites. The energy of a site is 0 if no atoms are absorbed and $-\Delta$ if one atom is absorbed from the room.

(a) Derive the free energy the gas in the volume of the canister and use this result to derive the chemical potential of the gas as a function of the pressure P_0.

(b) (i) Determine the mean number of atoms absorbed by the traps. Express your result in terms of the ambient temperature T and pressure P_0. (ii) At fixed temperature, qualitatively sketch the mean number of atoms absorbed by the as a function of pressure at fixed temperature. At what pressure are half of the absorbing sites filled?

(c) If the pressure is increased from P_0 to P_1 how much heat flows from the surrounding gas into the traps. You may leave your result in terms of the chemical potentials μ_0 and μ_1 at the corresponding pressures, P_0 and P_1.

Hint: What is the relation between heat flow and entropy?

Problem 4. Relativistic corrections to a Fermi Gas

The energy of the electron is in general related to its momentum by the relativistic formula

$$\varepsilon(\mathbf{p}) = \sqrt{(mc^2)^2 + (cp)^2}\,. \tag{19}$$

If particles are non-relativistic we have

$$\frac{v}{c} \ll 1\,, \quad \text{or} \quad \frac{p}{mc} \ll 1\,. \tag{20}$$

(a) Show that for non-relativistic particles we have approximately

$$\varepsilon(\mathbf{p}) \simeq mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m(mc)^2} + \ldots \tag{21}$$

The leading term is the rest energy of the electron (a constant), the next term is the familiar $\frac{1}{2}mv^2$, and the next term is the first relativistic correction.

Now consider a gas of electrons at zero temperature in a three dimensional volume V and an electron density of n_e. The single particle energy of an electron is in general given by Eq. (19), and is approximated using part (a), Eq. (21).

(b) Determine the relation between the Fermi momentum and the density of electrons. Explain carefully your steps.

(c) Determine the energy U of the electron gas in the non-relativistic limit, including the first relativistic correction. Express your result in terms of the Fermi momentum.

(d) Determine the pressure of the gas including the first relativistic correction. At what density does the correction due to the relativistic term, change the leading result by 10%.