Manipulating Taylor Series

a) $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

This follows from the geometric series
$1/(1-u) = 1 + u + u^2 + \ldots$ with $u = -x$

Integrating

$$\int_0^x \frac{dx'}{(1+x')} = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$$

b)

c) For x large we have $e^{-x} \ll 1$

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}}$$

call $u = e^{-x} \ll 1$

$$\frac{1}{e^x - 1} = \frac{u}{(1-u)} = u\,(1 + u + u^2 + \ldots)$$

$$\boxed{\frac{1}{e^x - 1} = e^{-x}\left(1 + e^{-x} + e^{-2x} + O(e^{-3x})\right)}$$

d) $\dfrac{1}{e^x - 1}$

we expand $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$

$$\frac{1}{e^x-1} \simeq \frac{1}{x + x^2/2 + x^3/6} = \frac{1}{x}\,\frac{1}{(1 + x/2 + x^2/6 + O(x^3))}$$

calling, $u = x/2 + x^2/6$, we have

$$\frac{1}{e^x-1} \simeq \frac{1}{x}\left(\frac{1}{1+u} + O(x^3)\right)$$

$$\simeq \frac{1}{x}\left(1 - u + u^2 + O(x^3)\right)$$

$$\simeq \frac{1}{x}\left(1 - \left(\frac{x}{2} + \frac{x^2}{6}\right) + \left(\frac{x^2}{4}\right) + O(x^3)\right)$$

$$\boxed{\frac{1}{e^x-1} \simeq \frac{1}{x}\left(1 - \frac{x}{2} + \frac{x^2}{12}\right) + O(x^2)}$$

e) $\dfrac{1}{e^{-x}+1} \simeq 1 - e^{-x} + e^{-2x}$ set $u = e^{-x}$

f) $\log(1-e^{-x}) \simeq \log\left(1 - \left(1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6}\right)\right)$

$$\simeq \log x\left(1 - \frac{x}{2} + \frac{x^2}{6}\right) = \log x + \log\left(1 - \frac{x}{2} + \frac{x^2}{6}\right)$$

So

$$\log(1-e^{-x}) = \log(x) + \log(1 \underbrace{- \frac{x}{2} + \frac{x^2}{6}}_{\text{Call it } u}) + O(x^3)$$

Setting $u = -\frac{x}{2} + \frac{x^2}{6}$ we have

$$\log(1+u) = u - \frac{u^2}{2} + O(u^3)$$ with x of order u

So

$$\log(1-e^{-x}) \simeq \log(x) + \left(-\frac{x}{2} + \frac{x^2}{6}\right) - \frac{1}{2}\left(-\frac{x}{2}\right)^2 + O(x^3)$$

$$\boxed{\log(1-e^{-x}) \simeq \log x - \frac{x}{2} + \frac{x^2}{24} + O(x^3)}$$

Energy of SHO

a) We have

$$Z = \frac{1}{1 - e^{-\beta\hbar\omega_0}}$$

Then

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \log Z = +\frac{\partial}{\partial\beta} \log(1 - e^{-\beta\hbar\omega_0})$$

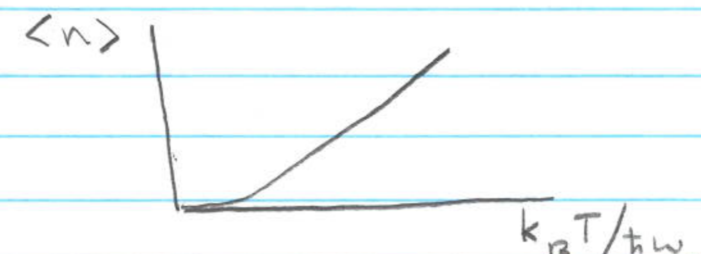
$$= \frac{1}{1 - e^{-\beta\hbar\omega_0}} e^{-\beta\hbar\omega_0} \hbar\omega_0$$

$$\boxed{\langle E \rangle = \frac{\hbar\omega_0}{e^{\beta\hbar\omega_0} - 1}}$$

b) Then

$$\frac{\langle E \rangle}{\hbar\omega_0} = \frac{1}{e^{\hbar\omega/kT} - 1} = \langle n \rangle$$

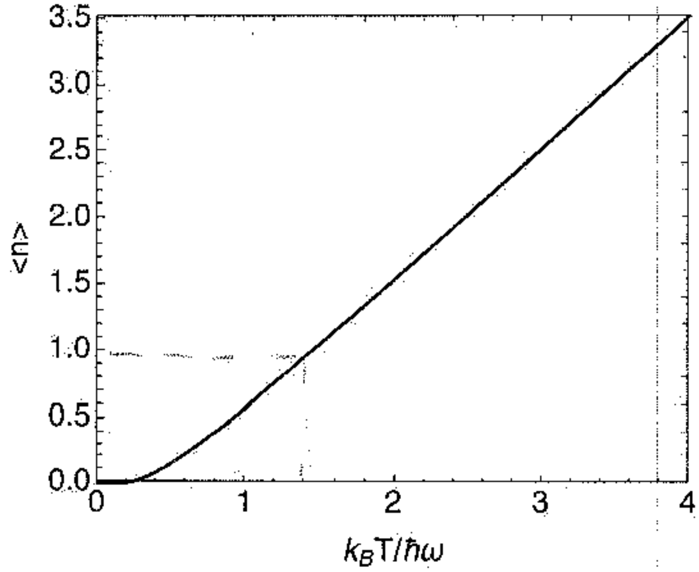
Then



Then from graph

$\langle n \rangle = 1$ when

$k_B T/\hbar\omega_0 = 1.45$



3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
<n>
0
1
2
3
4
k_B T/ħω

or

$$\boxed{k_B T = 1.45\,\hbar\omega_0}$$

c) Then a nice plot of $\langle E \rangle$ is given in the problem statement.

d) Using the series of problem 1

with $x \equiv \hbar\omega/kT$

at low temperature $kT \ll \hbar\omega_0$ then $x \gg 1$, and

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1-e^{-x}} \simeq e^{-x}(1 + e^{-x} + \dots)$$

And

$$\boxed{\langle E \rangle = \hbar\omega_0\, e^{-\beta\hbar\omega_0}(1 + e^{-\beta\hbar\omega_0} + \dots)}$$

At high temperature $x \ll 1$

$$\frac{1}{e^x - 1} \simeq \frac{1}{x} - \frac{1}{2}$$

$$\langle E \rangle = \hbar\omega_0\left(\frac{k_B T}{\hbar\omega_0} - \frac{1}{2}\right) \simeq k_B T\left(1 - \frac{\hbar\omega_0}{2kT}\right)$$

(e) At high temperature the number of quanta $\langle n \rangle$ is very large. In this regime $\langle n \rangle \gg 1$, quantum mechanics becomes continuous, $\frac{\Delta E}{E} \ll 1$, and it approaches classical mechanics

This is the Bohr correspondence principle

f) We have

i)
$$\boxed{U = N\left[\frac{5}{2}kT + \underbrace{\frac{\hbar\omega_0}{e^{\beta\hbar\omega_0}-1}}_{\text{this is } f_0(T)}\right]}$$

Then

ii)
$$C_V = \left(\frac{dU}{dT}\right)_V = N\left[\frac{5}{2}k + \frac{-\hbar\omega_0 e^{\beta\hbar\omega_0}}{(e^{\beta\hbar\omega_0}-1)^2}\hbar\omega_0 \frac{\partial}{\partial T}\frac{1}{kT}\right]$$

$$= N\left[\frac{5k}{2} + \frac{(\beta\hbar\omega_0)^2 e^{\beta\hbar\omega_0}}{(e^{\beta\hbar\omega_0}-1)}k\right]$$

$$C_V = Nk\left[\frac{5}{2} + \frac{(\beta\hbar\omega_0)^2 e^{\beta\hbar\omega_0}}{(e^{+\beta\hbar\omega_0}-1)^2}\right]$$

So

$$C_p = C_v + Nk_B$$

$$\boxed{C_p = Nk_B\left[\frac{7}{2} + \frac{(\beta\hbar\omega_0)^2}{(e^{\beta\hbar\omega_0} - 1)^2}\right]}$$

iiv) So we see that the model nicely captures the transition from $C_p = \frac{7}{2} = 3.5$ to $\frac{9}{2} = 4.5$

but misses the transition to $\frac{5}{2}$ at low temperatures

Problem 14.5

a) As described in class + 14.1

T_R = lake

T_S = ball

$$\Delta S = C \ln\left(\frac{T_R}{T_S}\right) + C\left(\frac{T_S}{T_R} - 1\right)$$

- So in the one plunge case $C = 1 \, k\frac{J}{°K}$ $T_S = 200K$

$T_R = 100K$

$$\Delta S = C \ln\left(\frac{1}{2}\right) + C(2-1)$$

$$\boxed{\Delta S = C\left[1 - \ln 2\right] = 0.3 \; kJ/°K}$$

b) In the two plunge case:

$$\underbrace{\Delta S_1}_{\text{plunge1}} = C \ln\left(\frac{T_R}{T_S}\right) + C\left(\frac{T_S}{T_R} - 1\right) \quad \text{with} \quad T_R = 150K, \; T_S = 200K$$

$$\boxed{\Delta S_1 = C\left[\ln \frac{3}{4} + \frac{1}{3}\right] = 0.046 \; kJ/°K}$$

And

$T_R' = 100K$

$T_S' = 150K$

$$\Delta S_2 = C \ln\left(\frac{T_R'}{T_S'}\right) + C\left(\frac{T_S'}{T_R'} - 1\right)$$

$$\boxed{\underbrace{\Delta S_2}_{\text{plunge 2}} = C \ln\left(\frac{2}{3}\right) + C\,\frac{1}{2} = 0.095 \; kJ/K}$$

$$\boxed{\Delta S_1 + \Delta S_2 = 0.14 \ kJ/^{o}K}$$

- For many steps

$$T_S = T_R + \Delta T \longleftarrow \text{step size in temperatur } \Delta T \ll T$$

And for one step:

$$\Delta S_{step} = C \log\left(\frac{T_R}{T_R + \Delta T}\right) + C\left(\frac{T_R + \Delta T}{T_R} - 1\right)$$

$$= C\left[\underbrace{-\log(1+x)}_{} + x\right] \quad \text{with } x = \frac{\Delta T}{T_R}$$

Taylor: $\log(1+x) \approx x - \frac{x^2}{2}$

$$\Delta S_{step} \simeq \frac{C x^2}{2}$$

So $$\boxed{\Delta S_{step} = \frac{C(\Delta T)^2}{2T_R^2} \longleftarrow \text{proportional to } \Delta T^2!}$$

So taking N steps:

$$\Delta S = \underbrace{\left(\sum_{steps} C \frac{\Delta T}{T_R^2}\right)}_{\substack{\text{this is} \\ \text{bounded, and approaches } \int \frac{C\, dT}{T^2}}} \Delta T \longrightarrow 0$$

this goes to zero

Phase Space Volume: Part II

a) $$V_{ps} = \int_{[E, E+\delta E]} \underline{d^3\vec{r}_1}\, d^3\vec{p}_1 \; \underline{d^3\vec{r}_2}\, d^3\vec{p}_2$$

$$= V^2 \; C_6 \; p^5 \; \delta p \qquad \text{(see lecture)}$$

We used that the area of the Sphere is $C_d \, p^{d-1}$ with $C_d = \dfrac{2\pi^{d/2}}{\Gamma(d/2)}$

From lecture:

$$\frac{\delta p}{p} = \frac{\delta E}{2E} \quad \text{with} \quad 2mE < p^2 < 2m(E+\delta E)$$

Yielding

$$V_{ps} = V^2 \, \frac{2\pi^3}{\Gamma(3)} \, p^6 \, \frac{\delta E}{2E}$$

$$\boxed{V_{ps} = V^2 (2mE)^3 \, \frac{\delta E}{2E}} \qquad p^2 = 2mE$$

c)

$$\Omega(E) = \frac{1}{N!} \int_{\text{access}} \frac{d^3r_1\, d^3p_1}{h^3} \cdots \frac{d^3r_N\, d^3p_N}{h^3}$$

over accessible States (configurations) with $E < \frac{p_1^2}{2m} + \cdots + \frac{p_N^2}{2m} < E + \delta E$

c) So performing the coordinate integrals first

$$\Omega(E) = \frac{V^N}{h^{3N}} \frac{1}{N!} \int_{\text{shell}} d^3p_1 \cdots d^3p_N$$

$$= \frac{V^N}{h^{3N}} \frac{1}{N!} C_{3N}\, p^{3N} \frac{\delta p}{p}$$

Now $\frac{\delta p}{p} = \frac{1}{2} \frac{\delta E}{E}$ and $p = (2mE)^{1/2}$ with

$C_{3N} = \frac{2\pi^{3N/2}}{\Gamma(3N/2)}$ we have

$$\Omega(E) = \frac{V^N}{h^{3N}} \frac{1}{N!} \frac{\cancel{2}\,\pi^{3N/2} (2mE)^{3N/2}}{\Gamma(3N/2)} \frac{\delta E}{\cancel{2}E}$$

$$\boxed{\Omega(E) = \frac{V^N}{N!} \left(\frac{2\pi m E}{h_0^2}\right)^{3N/2} \frac{1}{\Gamma(3N/2)} \frac{\delta E}{E}}$$

d) Using $N! = \left(\frac{N}{e}\right)^N$ and $\Gamma(3N/2) = \left(\frac{3N}{2e}\right)^{3N/2}$

we have

$$\Omega(E) = e^{N+3N/2} \left(\frac{V}{N}\right)^N \left(\frac{4\pi}{3} \frac{mE}{h_0^2 N}\right)^{3N/2} \frac{\delta E}{E}$$

Neglecting the $\delta E / E$ factor we find

$$\boxed{\Omega(E) = e^{5N/2} \left(\frac{V}{N}\right)^{N} \left(\frac{4\pi}{3} \frac{mE}{h_0^2 N}\right)^{3N/2}}$$

Thus

$$\boxed{S = k \ln \Omega = Nk \left[\ln \left(\frac{V}{N} \left(\frac{4\pi}{3} \frac{mE}{h_0^2 N} \right)^{3/2} \right) + \frac{5}{2} \right]}$$

Since

$$\lambda_{th} = \frac{h}{(2\pi m k T)^{1/2}} = \frac{h}{\left(\frac{4\pi}{3} m \frac{E}{N}\right)^{1/2}}$$ we have

$$S = Nk \left[\ln \left(\frac{V/N}{\lambda_{th}^3} \right) + \frac{5}{2} \right]$$