Last Time

- Discussed Magnetostatics

$$\nabla \times B = \vec{j}/c$$

$$\nabla \cdot B = 0$$

For steady current which is specified

$$\nabla \cdot \vec{j} = 0$$

After introducing the vector potential $\vec{B} = \nabla \times \vec{A}$
and the coulomb gauge $\nabla \cdot A = 0$

Find

$$\nabla \times (\nabla \times \vec{A}) = \vec{j}/c$$

$$-\nabla^2 \vec{A} + \nabla (\nabla \cdot \vec{A}) = \vec{j}/c$$
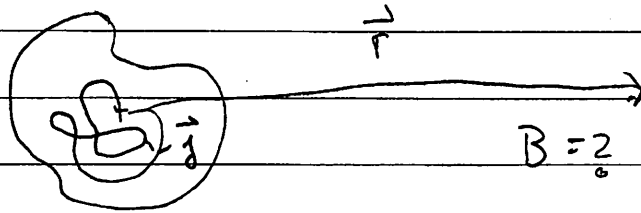
$$\boxed{-\nabla^2 \vec{A} = \vec{j}/c} \quad \checkmark$$

also useful
in a few cases

Then

$$\boxed{\vec{A}(\vec{r}) = \int d^3x \; \frac{\vec{j}(x)/c}{4\pi \; |\vec{r} - \vec{x}|}}$$

Can use this to solve problems

Expansion of the magnetic fields at great distances

$$\vec{r}$$

$$B = \frac{2}{c}$$

Start

$$A^i = \frac{1}{c}\int d^3x \; \frac{j^i(x)}{4\pi |\vec{r}-\vec{x}|^{1/2}}$$

$$\frac{1}{|\vec{r}-\vec{x}|} = \frac{1}{(r^2+x^2-2\vec{x}\cdot\vec{r})^{1/2}} \approx \frac{1}{r} + \frac{\vec{x}\cdot\vec{r}}{r^3} + \dots$$

Will find: no monopole

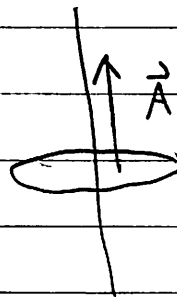$$A^i = 0 + \frac{(\vec{m}\times\hat{r})^i}{4\pi r^2}$$

where

$$\vec{m} = \frac{1}{2}\int \vec{x} \times \frac{\vec{j}(x)}{c} \, d^3x$$

is the magnetic moment

Expansion of magnetic fields pg. 2
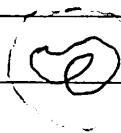
For a current loop in plane

$$d^3x \, \vec{J} = I \, d\vec{l}$$

and

$$\boxed{\vec{m} = \frac{(I/c)}{2} \int \vec{x} \times d\vec{l} = \frac{I}{c}\vec{A}}$$

← current makes closed loop

Prf

$$A^i = \frac{1}{r}\int \frac{d^3x}{4\pi} \frac{J^i_{(x)}}{c} + \left[\int d^3x \; J^i x^l\right] \frac{r_l}{4\pi r^3 c}$$

← from last page

$$\underbrace{\qquad\qquad}_{I^{il}}$$

Now Look at:

$$I^{il} = \frac{1}{c}\int d^3x \, J^i x^l$$

$$= \frac{1}{2c}\int d^3x \,(J^i x^l + J^l x^i) + \frac{1}{2c}\int (J^i x^l - J^l x^i)$$

$$\underbrace{\qquad\qquad}_{I_S^{il}} \qquad\qquad \underbrace{\qquad\qquad}_{I_A^{il}}$$

$$I_S^{il} = \frac{1}{2c}\int d^3x \; J^i x^l + J^l x^i$$

Since $\partial_m J^m = 0$ and $\partial_m x^i = \delta^i_m$

$$= \frac{1}{2c}\int d^3x \, \partial_m (J^m x^i x^l)$$

$$I_S^{il} = 0$$

Since J^m is bounded
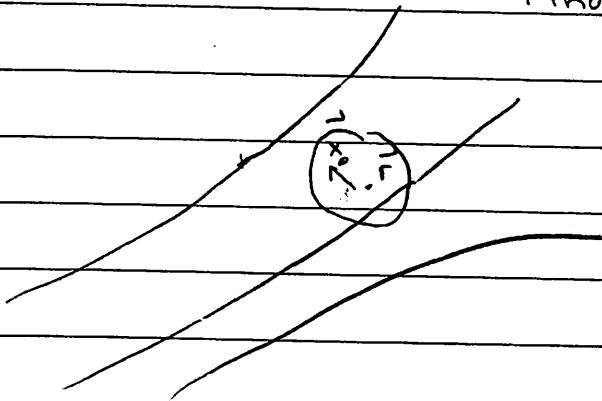
Forces on Current Distributions

- The

$$\vec{F} = \int \frac{\vec{J}(x)}{c} \times \vec{B}(\vec{x}) \; d^3x$$

Find:

$$\boxed{\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) = -\nabla U}$$

↑ magnetic moment

$$\boxed{U = -\vec{m}\cdot\vec{B}}$$

Prf

$$F_i = \int d^3x \; \varepsilon_{ijk} \; J^j_{(x)} B^k(x)$$

$$B^k(\vec{x}) = B^k(\vec{r}) + \delta x^\ell \partial_\ell B^k(\vec{r}) \qquad\qquad \delta x^\ell = x^\ell - r^\ell$$

Plugging in

$$F_i = \underbrace{\left[\int d^3x \; J^j \delta x^\ell\right]}_{\text{same as before } = -\varepsilon^{j\ell n} m_n} \varepsilon_{ijk} \; \partial_\ell B^k(\vec{r}) \qquad (\text{see two pages back})$$

$$F_i = -\varepsilon^{j\ell n} m_n \; \varepsilon_{ijk} \partial_\ell B^k(\vec{r})$$

Using

$$-\varepsilon^{jln}\varepsilon_{ijk} = \varepsilon^{jln}\varepsilon_{jik}$$

$$= \delta^{l}{}_{i}\delta^{n}{}_{k} - \delta^{l}{}_{k}\delta^{n}{}_{i}$$

We have

$$F_i = m_n \partial_\ell B^k [\delta^{l}{}_{i}\delta^{n}{}_{k} - \delta^{l}{}_{k}\delta^{n}{}_{i}]$$

$$= m_k \partial_i B^k - m_i \partial_k B^k$$

$$F_i = \partial_i (\vec{m}\cdot\vec{B}) \checkmark \qquad \overbrace{\nabla\cdot B = 0}$$

as claimed

From this we see that the potential energy of a magnetic dipole is

$$U = -\vec{m}\cdot\vec{B}$$

Finally we may take the derivative with respect to theta, (or go through another tortuous derivation), to show along the same lines that

$$\vec{\tau} = \vec{m} \times \vec{B}$$

i.e, that the magnetic moment tends to align itself @ the field