Problem 1. A cylinder with a split personality

5 points (a) Consider a charge distribution $\sigma(x, y)$ which is infintely extended in the z direction and has support in a finite region of the x, y plane. The potential satisfies the 2D Poisson equation

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \varphi(x, y) = \sigma(x, y) \tag{1}$$

Determine the potential at distances far from the charge density, *i.e.* determine the 2D analog of the Cartesian multipole expansion. Develop the expansion to quadrupole order, and define the appropriate 2D Cartesian monopole, dipole, and quadrupole moments.

6 points (b) Now consider an infinitely long cylinder of radius a. The top half of the cylinder is held at potential V_0 while the bottom half of the cylinder is held at $-V_0$. (The two halves are separated by a infintessimal gap). Determine a series expansion for the potential $\varphi(\rho, \phi)$ where $\rho = \sqrt{x^2 + y^2}$ and $\phi = \text{atan}(y/x)$ is the azimuthal angle.

4 points (c) Using the Cartesian definitions developed in part (b), determine the dipole moment (both magnitude and direction) from the long distance behavior of the potential determined in (c).

Problem 2. Angular momentum from an electromagnetic sphere

Consider an insulating sphere of radius a with charge Q distributed uniformly over its surface. The sphere has a uniform magnetization $\boldsymbol{M}_0 = M_0 \hat{\boldsymbol{z}}$ throughout its volume.

Recall that the magnetic field from magnetized sphere is constant on the interior with $\boldsymbol{B} = 2\boldsymbol{M}_0/3$, and takes a pointlike dipole form on the exterior with $\boldsymbol{m} = 4\pi a^3 \boldsymbol{M}_0/3$.

2 points (a) Write down the Coulomb gauge vector potential for this setup and verify its continuity across the surface of the sphere.

8 points (b) Calculate the angular momentum in the electromagnetic field.

8 points
broken down 2/3/3 (c) Assume that the magnetization is reduced to zero over a time τ:

$$M(t) = \begin{cases} M_0 & t < 0 \\ M_0\left(1 - \frac{t}{\tau}\right) & 0 < t < \tau \\ 0 & t > \tau \end{cases}$$

Determine the induced fields everywhere in space and evaluate their contribution to the torque on the charged sphere.

2 points (d) Discuss the relation between parts (b) and (c).

Problem 3. A current sheet in an ohmic medium

Consider an infinite sheet lying in the zy plane carrying surface current, $\boldsymbol{K} = K_0\, e^{-i\omega t}\, \hat{\boldsymbol{z}}$.

6 points broken down 1/3/2

(a) First consider carrying sheet in vacuum. Determine the magnetic and electric fields to lowest (non-trivial) order in the frequency.

   (i) Graph the amplitude of the electric field as a function of x including both positive and negative values of x.

   (ii) Do your results for the electromagnetic fields hold everywhere in space? Explain.

10 points broken down 3/2/2/2

(b) Now place the same current sheet in an ohmic medium of conductivity σ with $\sigma \gg \omega$. Determine the (real) electric and magnetic fields everywhere in space. Graph the amplitude of the electric field as a function of x including both positive and negative values of x.

Derive every step directly from the Maxwell equations and its boundary conditions. Take $\epsilon = \mu = 1$.

4 points

(c) Determine the energy dissipated per area per time for the setup described in part (b).