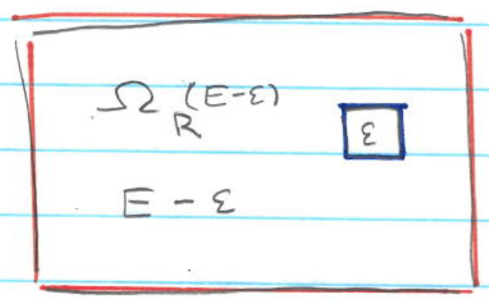
The Boltzmann Factor

- We can use notions of entropy to derive the Boltzmann Distribution

$$P_{\text{micro state } s} \propto e^{-\mathcal{E}_s / k_B T} \qquad \mathcal{E}_s \equiv \text{energy of microstate}$$

- Take a subsystem which is small compared to the total, interacting with a reservoir (the rest of the system):



$$\varepsilon \ll E$$

- The total system has energy E. Let us require that the subsystem be in <u>one</u> microstate with energy ε. The remaining system has energy $E-\varepsilon$. The probability of this configuation is

$$P(E-\varepsilon;\varepsilon) = \Omega_R(E-\varepsilon)\cdot 1 / \Omega(E) \leftarrow \text{constant}$$

(Before we had $\Omega_1(E_1)\,\Omega_2(E_2)$, now system 2 is in exactly one state). $\Omega_R(E-\varepsilon) = \Omega_1$ is the number of microstates associated with the reservoir, and $\Omega_2 = 1$. $\Omega(E)$ is constant since the energy is constant

- Take the log

$$\log P(\varepsilon) = \text{const} + \log \Omega_R(E-\varepsilon) + \cancel{\log 1}^{\,0}$$

- Now ε is small compared to the total system

$$\log \Omega_R(E-\varepsilon) = \log \Omega_R(E) - \left(\frac{\partial \log \Omega_R(E)}{\partial E}\right)\varepsilon$$

Or since

$$\frac{1}{k_B T} = \frac{\partial \log \Omega_R(E)}{\partial E}$$

- We have

$$\log P(\varepsilon) = \underbrace{\text{const} + \log \Omega_R(E)}_{\substack{\text{all const} \\ \text{indep of } \varepsilon}} - \frac{\varepsilon}{k_B T}$$

- So exponentiating

$$\boxed{P(\varepsilon) = (\text{Const})\, e^{-\varepsilon/k_B T}}$$

The constant can be found from normalization:

$$\sum_s P(\varepsilon_s) = 1 \quad \text{or} \quad \sum_s C e^{-\varepsilon_s/kT} = 1$$

$$C = \frac{1}{Z} \quad \text{with} \quad Z = \sum_s e^{-\varepsilon_s/kT}$$