1 Integrals

Bose and Fermi:

$$\int_0^\infty \mathrm{d}x \, \frac{x}{e^x - 1} = \frac{\pi^2}{6} \tag{1}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^2}{e^x - 1} = 2\zeta(3) \simeq 2.404 \tag{2}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^3}{e^x - 1} = \frac{\pi^4}{15} \tag{3}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^4}{e^x - 1} = 24\,\zeta(5) \simeq 24.88 \tag{4}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^5}{e^x - 1} = \frac{8\pi^6}{63} \tag{5}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x}{e^x + 1} = \frac{\pi^2}{12} \tag{6}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^2}{e^x + 1} = \frac{3}{2}\,\zeta(3) \simeq 1.80309 \tag{7}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^3}{e^x + 1} = \frac{7\pi^4}{120} \tag{8}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^4}{e^x + 1} = \frac{45}{2}\,\zeta(5) \simeq 23.33 \tag{9}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^5}{e^x + 1} = \frac{31\pi^6}{252} \tag{10}$$

Gamma Function:

$$\Gamma(z) \equiv \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x \tag{11}$$

with specific results

$$\Gamma(z+1) = z\Gamma(z) \qquad \Gamma(n) = (n-1)! \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{12}$$

Gaussian Integrals:

$$I_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \mathrm{d}x \, e^{-x^2/2} x^n \tag{13}$$

with specific results

$$I_0 = 1 \quad I_2 = 0 \quad I_4 = 3 \quad I_6 = 15 \tag{14}$$

Problem 1. A nucleus as a fermi gas

Large nuclei can be treated as approximately "infinite" in size. This means that density of protons and neutrons within the nucleus approaches a constant, and in first approximation edge effects can be neglected. In the infinite volume limit the material is known as nuclear matter, and the density of the protons and neutrons is known as nuclear matter density.

Treat a nucleus as a ball of radius R with A nucleons[1]. The radius of a ball grows with $A^{1/3}$ as

$$R = (1.3 \times 10^{-15}\, m)\, A^{1/3} \tag{15}$$

Assume that the number of protons and the number of neutrons are equal.

(a) Compute the density of protons and the density neutrons.

(b) Show that the Fermi energy of protons is approximately 27 MeV.

Since we have assumed the number of protons and neutrons are equal, the Fermi energy of neutrons is also 27 MeV. In reality the number of neutrons is somewhat larger than the number of protons. Thus, the density of neutrons is higher, and the corresponding Fermi energy is somewhat higher.

(c) Show that energy per nucleon inside a nucleus is approximately 16 MeV. This is a reasonable estimate for the kinetic energy per volume in a nucleus.

[1] A nucleon is either a proton or neutron. Oxygen has eight protons and eight neutrons and has $A = 16$.

Solution

(a) We note that $1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}$ is a femptometer, which is a common distance in nuclear physics. The density of protons and neutron is

$$n_p = n_n = \frac{A/2}{\frac{4}{3}\pi R^3} = \frac{3}{8\pi}\frac{1}{1.3^3}\frac{1}{(\mathrm{fm})^3} = 0.054\,\mathrm{fm}^{-3} \tag{16}$$

(b) The fermi energy is related to the density. For a system of non-relativistic fermions with $g = 2$ (spin up and spin down) we have

$$\epsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3} \tag{17}$$

Lets evaluate a typical energy scale for a distance of $r_0 = 1\,\mathrm{fm}$.

$$\frac{\hbar^2}{2mr_0^2} = \frac{(\hbar c)^2}{2(m_p c^2) r_0} = 20.7\,\mathrm{MeV} \tag{18}$$

We used $\hbar c = 197\,\mathrm{MeV\,fm}$ and note that $m_p c^2 = 938\,\mathrm{MeV}$. This $\hbar^2/(2mr_0^2)$ sets an energy scale for nuclear physics. Then

$$\epsilon_F = 20.7\,\mathrm{MeV}\left(3\pi^2 r_0^3/V\right)^{2/3} = 20.7\,\mathrm{MeV} \times 1.3724 \simeq 28\,\mathrm{MeV} \tag{19}$$

This is a little larger than the quoted result, but good enough to 4% accuracty.

(c) Then from class we have

$$\frac{U}{N} = \frac{3}{5}\epsilon_F \simeq 16.8\,\mathrm{MeV} \tag{20}$$

Problem 2. 2D Fermi gas

Consider a fermi gas of electrons in two dimensions.

(a) Show that the fermi momentum is

$$p_F = \hbar\sqrt{2\pi n} \tag{21}$$

(b) Show that the mean value of the debroglie wavelength divided by 2π, i.e. $\lambdabar \equiv \hbar/p$, is

$$\langle \lambdabar \rangle = \frac{2\hbar}{p_F} \tag{22}$$

Solution

(a) In two dimensions we have

$$N = \sum_{\text{modes}} n_{FD}(\epsilon) = 2A \int_0^{p_F=\sqrt{2m\mu_0}} \frac{d^2p}{(2\pi\hbar)^2} \times 1 \tag{23}$$

We used that that the Fermi-Dirac distribution $n_{FD}(\epsilon)$ is unity up to a maximum energy ϵ_F determined by the chemical potential. Above this energy distribution vanishes. We have

$$\epsilon_F = \frac{p_F^2}{2m} = \mu_0 \tag{24}$$

The integral is just that of a disk of radius p_F

$$N = \frac{A}{\pi^2}\pi\frac{p_F^2}{\hbar^2} \tag{25}$$

So solving for p_F in terms of the density we have

$$p_F = \hbar\sqrt{\pi(N/A)} \tag{26}$$

We also note that the chemical potential is related to the density via:

$$\mu_0 = \frac{p_F^2}{2m} = \frac{\hbar^2}{2m}\pi\,(N/A) \tag{27}$$

as claimed

(b) The average debroglie wavelength is

$$\langle \lambdabar \rangle = \frac{\sum_{\text{modes}} n_{FD}(\epsilon)\frac{\hbar}{p}}{\sum_{\text{modes}} n_{FD}(\epsilon)} \tag{28}$$

Converting the sum to an integral, and noting that all constants cancel beteween the numerator and denominator, we find:

$$\langle \lambdabar \rangle = \frac{\int_0^{p_F} pdp\,\left(\frac{\hbar}{p}\right)}{\int_0^{p_F} pdp} = \frac{\hbar p_F}{\frac{1}{2}p_F^2} = \frac{2\hbar}{p_F} = \left(\frac{2}{\pi n}\right)^{1/2} \tag{29}$$

Thus $\lambdabar$ is the same order of magnitude as the interparticle spacing $\ell_0 = (A/N)^{1/2}$.

$$\langle \lambdabar \rangle \simeq 0.8\ell_0 \tag{30}$$

Problem 3. Relativistic Degenerate Electron Gas

Consider an ultra-relativistic degenerate electron gas where $\epsilon \simeq cp$, and the electron mass can be neglected.

(a) Show that the Fermi Energy is related density by

$$\epsilon_F = \hbar c(3\pi^2 n)^{1/3} \tag{31}$$

where $n = N/V$.

(b) Compute the Fermi momentum p_F. Define the Fermi wavelength, $\lambdabar_F \equiv \hbar/p_F$. Explain qualitatively the dependence of $\lambdabar_F$ on the density $n = N/V$.

(c) Show that the total energy of the gas is

$$U = \frac{3}{4}N\epsilon_F \tag{32}$$

(d) Show that the pressure of the the gas is

$$\mathcal{P} = \frac{1}{3}\frac{U}{V} \tag{33}$$

and determine its dependence on density $n = N/V$. Compare your result to a classical ideal gas where $\mathcal{P} \propto n$ and a non-relativistic degenerate Fermi gas where $\mathcal{P} \propto n^{5/3}$

Solution

(a) We have to integrate

$$N = \sum_{\text{modes}} n_{FD}(\epsilon) \tag{34}$$

The fermi dirac distribution is unity until $\epsilon = \mu$ or when $cp = \mu$. It is zero beyond this point

$$N = 2V \int_0^{p_F} \frac{4\pi p^2 dp}{(2\pi\hbar)^3} = \frac{V}{3\pi^2}\left(\frac{p_F}{\hbar}\right)^3 \tag{35}$$

where $p_F = \mu/c = \epsilon_F/c$. So the density is related to fermi-momenum

$$n = \frac{1}{3\pi^2}\left(\frac{p_F}{\hbar}\right)^3 \tag{36}$$

Solving for ϵ_F we find

$$\epsilon_F = \hbar c \left(3\pi^2 n\right)^{1/3} \tag{37}$$

(b) We find

$$p_F = \frac{\epsilon_F}{c} \tag{38}$$

and

$$\lambda\!\!\!^{-}_F = \frac{\hbar}{p_F} = \left(3\pi^2 \frac{N}{V}\right)^{1/3} = 0.32\, \ell_0 \tag{39}$$

where we have defined the spacing between particles $\ell_0 = (V/N)^{1/3}$. So we see that in the degenerate regime the de Broglie wavelength is the same size as the interparticle spacing.

(c) We find the the energy

$$U = 2V \int_0^{p_F} \frac{d^3p}{(2\pi\hbar)^3} cp \tag{40}$$

$$= \frac{1}{\pi^2} c \int_0^{p_F} p^3 dp \tag{41}$$

$$= \frac{V}{4\pi^2} c p_F^4 \tag{42}$$

Since $\epsilon_F = cp_F$ and since the number is given by Eq. (35), we find

$$U = \frac{3}{4} N \epsilon_F \tag{43}$$

(d) We should differentiate the energy since

$$dU = TdS - \mathcal{P}dV + \mu dN \tag{44}$$

and thus

$$\mathcal{P} = -\left(\frac{\partial U}{\partial V}\right)_S \tag{45}$$

The fixed S part can be dropped since we are at zero temperature. The dependence on volume is hidden in the definition of the $\epsilon_F \propto (N/V)^{1/3}$. Unravelling the definitions we have

$$U = \kappa N^{4/3} V^{-1/3} \tag{46}$$

where κ is a constant. Differentiating with respect to volume give

$$\mathcal{P} = \frac{1}{3}\kappa N^{4/3} V^{-4/3} = \frac{1}{3}\frac{\kappa N^{4/3} V^{-1/3}}{V} = \frac{1}{3}\frac{U}{V} \tag{47}$$

We note that

$$\mathcal{P} = \frac{1}{3}\kappa \left(N/V\right)^{4/3} \tag{48}$$

So we see that $p \propto n^{4/3}$. As we will discuss in class a consequence of this dependence is that white dwarf stars become unstable when the reach the ultrarelativistic regime.