

Problem 1. Two State System

Consider an atom with only two states: a ground state with energy 0, and an excited state with energy Δ . Determine the mean energy $\langle \epsilon \rangle$ and variance in energy $\langle \delta \epsilon^2 \rangle$. Sketch the mean energy versus $\Delta/k_B T$.

Problem 2. Working with the speed distribution

Consider the Maxwell speed distribution

- (a) Evaluate the most probable speed v_* , i.e the speed where $P(v)$ is maximized. You should find $v_* = (2k_B T/m)^{1/2}$.
- (b) Determine the probability to have speed in a specific range, $v_* < v < 2v_*$. Follow the following steps:
 - (i) Write down the appropriate integral.
 - (ii) Change variables to a dimensionless speed $u = v/\sqrt{k_B T/m}$, i.e. u is the speed in units of $\sqrt{k_B T/m}$, and express the probability as an integral over u .
 - (iii) Write a short program (in any language) to evaluate the dimensionless integral, by (for example) dividing up the interval into 200 bins, and evaluate the integral with Riemann sums. You should find

$$\mathcal{P} \simeq 0.53 \tag{1}$$

Problem 3. Distribution of energies

The speed distribution is

$$d\mathcal{P} = P(v) dv \tag{2}$$

where $P(v) = (m/2\pi k_B T)^{3/2} e^{-mv^2/2k_B T} 4\pi v^2$.

- (a) Show that the probability distribution of energies $\epsilon = \frac{1}{2}mv^2$ is

$$d\mathcal{P} = P(\epsilon) d\epsilon \tag{3}$$

where

$$P(\epsilon) = \frac{2}{\sqrt{\pi}} \beta^{3/2} e^{-\beta\epsilon} \epsilon^{1/2} \tag{4}$$

Note: that the distribution of energies is independent of the mass, and recall $\beta = 1/k_B T$.

- (b) Compute the variance in energy using $P(\epsilon)$. Express all integrals in terms $\Gamma(x)$ (as given in the previous homework) – it is helpful to change to a dimensionless energy $u = \beta\epsilon$. You should find (after evaluating these Γ functions as in the previous homework) that

$$\langle (\delta \epsilon)^2 \rangle = \frac{3}{2} (k_B T)^2 \tag{5}$$

Problem 4. (Optional: Don't turn in) Change of variables

These take 1 sec each. Do them for yourself though. If you feel this is trivial – you're right.

- (a) Show that the distribution of momenta is

$$d\mathcal{P} = \left(\frac{1}{2\pi m k_B T} \right)^{3/2} e^{-p^2/2mk_B T} dp_x dp_y dp_z \quad (6)$$

where $p^2 = p_x^2 + p_y^2 + p_z^2$.

- (b) Show that

$$\int_{-\infty}^{\infty} dx f(x) = \int_{-\infty}^{\infty} du f(-u) \quad (7)$$

with $u = -x$.

- (c) Consider the probability distribution

$$d\mathcal{P} = f(\theta) \sin \theta d\theta \quad (8)$$

with $\theta \in [0, \pi]$. For the change of variables $u(\theta) = \cos \theta$ show that

$$d\mathcal{P} = f(\theta(u)) du \quad (9)$$

with $u \in [-1, 1]$, and $\theta(u) = \arccos(u)$.

Problem 5. Distributions on the sphere

- (a) Consider a particle with angular coordinates (θ, ϕ) which are uniformly distributed over a sphere of radius R , i.e. the probability per area is constant. Briefly explain (using the definition of solid angle) why $d\mathcal{P} = C d\Omega$ with C a normalizing constant. What is C and show that the probability distribution per $d\theta$ is $d\mathcal{P} = \frac{1}{2} \sin \theta d\theta$. Show that $\langle \cos^2 \theta \rangle_{\text{sphere}} = \frac{1}{3}$.
- (b) Consider a particle which is distributed (not uniformly) over the sphere

$$d\mathcal{P} = C(1 + \cos^2 \theta) \frac{d\Omega}{4\pi} \quad (10)$$

where C is a normalizing constant. This probability distribution is visualized in Fig. 1. Show that $C = 3/4$ and that the probability distribution for $u = \cos(\theta)$ is

$$d\mathcal{P} = \frac{3}{8}(1 + u^2) du \quad (11)$$

for $u \in [-1, 1]$.

- (c) Show for the probability distribution of (b) that $\langle \cos^2 \theta \rangle_{\text{sphere}} = \frac{2}{5}$ by averaging $\langle u^2 \rangle_u$.

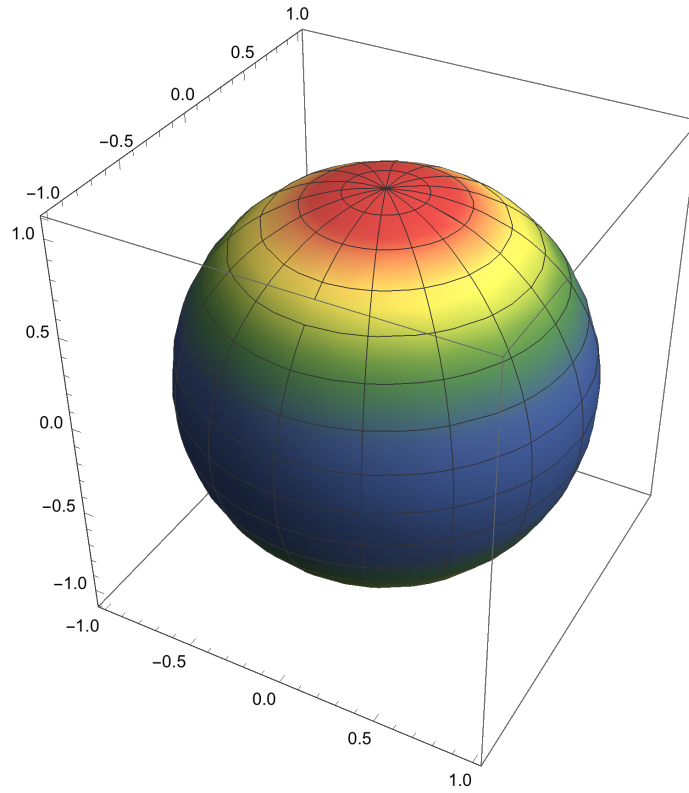


Figure 1: A visualization of a probability distribution on a sphere $\frac{d\mathcal{P}}{d\Omega} \propto (1 + \cos^2 \theta)$ given in problem 5. The colors are proportional to the probability per area or per solid angle (since $dA \propto d\Omega$ these are the same).

Problem 6. (Optional: Dont turn in) 2D World

Consider a mono-atomic ideal gas in a two dimensional world, so the velocities are labelled by $\mathbf{v} = (v_x, v_y)$.

- (a) Use Jacobians to show

$$dv_x dv_y = \left\| \frac{\partial(v_x, v_y)}{\partial(v, \theta)} \right\| dv d\theta = v dv d\theta \quad (12)$$

where $v_x = v \cos \theta$ and $v_y = v \sin \theta$. It is understood that these expressions are meant to be integrated over. The double bars mean determinant and then absolute value of the Jacobian matrix¹

$$\frac{\partial(v_x, v_y)}{\partial(v, \theta)} \equiv \begin{pmatrix} \frac{\partial v_x}{\partial v} & \frac{\partial v_x}{\partial \theta} \\ \frac{\partial v_y}{\partial v} & \frac{\partial v_y}{\partial \theta} \end{pmatrix} \quad (13)$$

- (b) Write down the (normalized) Maxwell velocity distribution, $d\mathcal{P}(v_x, v_y) = P(v_x, v_y) dv_x dv_y$, and, using the Jacobian of part (a) and an integral over θ , determine the (normalized) speed distribution

$$d\mathcal{P}(v) = P(v) dv \quad (14)$$

Describe the whole “Jacobian + integral” business with a picture.

You should find that all factors of π have canceled in your final expression for $P(v)$, and can optionally check that the distribution is correctly normalized.

- (c) Compute $\langle \frac{1}{2}mv^2 \rangle$ using the speed distribution. Is your result consistent with the equipartition theorem in two dimensions? Explain.
- (d) (Optional: Don't turn in) Consider the change of coordinates from $x = r \cos \theta$ and $y = r \sin \theta$. Construct the Jacobian matrix

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \quad (15)$$

The columns of the Jacobian form vectors

$$\mathbf{e}_r \equiv \frac{\partial x}{\partial r} \hat{\mathbf{i}} + \frac{\partial y}{\partial r} \hat{\mathbf{j}} \quad (16)$$

$$\mathbf{e}_\theta \equiv \frac{\partial x}{\partial \theta} \hat{\mathbf{i}} + \frac{\partial y}{\partial \theta} \hat{\mathbf{j}}. \quad (17)$$

Give a geometric interpretation of the vectors $\mathbf{e}_r dr$ and $\mathbf{e}_\theta d\theta$ and their respective lengths, dr , and $r d\theta$, by drawing the vectors with the at the position $(x, y) = (r \cos \theta, r \sin \theta)$.

¹Sometimes people use $\partial(v_x, v_y)/\partial(v, \theta)$ to mean the determinant of the Jacobian matrix. Our book uses this notation, as is described in appendix C.