SHO

a)

$$d\mathcal{P} = C\, e^{-H/k_B T}\, dx\, dp$$

$$= C\, e^{-p^2/2mkT}\, e^{-\frac{1}{2}m\omega_0^2 x^2/kT}\, dx\, dp$$

Let $\boxed{\sigma_x^2 = \frac{kT}{m\omega_0^2}}$ and $\boxed{\sigma_p^2 = mk_BT}$

So

this is a product: fcn(x) · fcn(p)

$$d\mathcal{P} = C\, e^{-x^2/2\sigma_x^2}\, e^{-p^2/2\sigma_p^2}\, dx\, dp$$

We know the normalization of a gaussian:

$$\int du\, e^{-u^2/2\sigma^2} = (2\pi\sigma^2)^{1/2}$$

Leading to $\int dx\, dp\, \text{fcn}(x)\, \text{fcn}(p) = \int dx\, \text{fcn}(x) \cdot \int dp\, \text{fcn}(p)$, or

$$\boxed{d\mathcal{P} = \frac{e^{-x^2/2\sigma_x^2}}{(2\pi\sigma_x^2)^{1/2}}\, \frac{e^{-p^2/2\sigma_p^2}}{(2\pi\sigma_p^2)^{1/2}}\, dx\, dp}$$ (see below)

$$d\mathcal{P} = P(x)\, dx\ \ P(p)\, dp$$

Integrating over $p$, determines $P(x) = \frac{e^{-x^2/2\sigma_x^2}}{(2\pi\sigma_x^2)^{1/2}}$

• In greater detail:

$$\int d\mathcal{P} = C \int e^{-x^2/2\sigma_x^2}\, dx \int e^{-p^2/2\sigma_p^2} = 1$$

$$= C\,(2\pi\sigma_x^2)^{1/2}\,(2\pi\sigma_p^2)^{1/2} = 1$$

$$C = \frac{1}{(2\pi\sigma_x^2)^{1/2}}\,\frac{1}{(2\pi\sigma_p^2)^{1/2}}$$

• Since the probability density factorizes, it is helpful to note that the probability to find $x$, without regard to $p$, is found by summing all possible values of $p$ (leaving $x$-fixed)

$$d\mathcal{P}_x = \int_p d\mathcal{P}_{x,p} = \int_p \frac{e^{-x^2/2\sigma_x^2}}{(2\pi\sigma_x^2)^{1/2}}\,\frac{e^{-p^2/2\sigma_p^2}}{(2\pi\sigma_p^2)^{1/2}}\, dx\, dp$$

$$= \frac{e^{-x^2/2\sigma_x^2}}{(2\pi\sigma_x^2)^{1/2}}\, dx\, \underbrace{\int_{-\infty}^{\infty} dp\, \frac{e^{-p^2/2\sigma_p^2}}{(2\pi\sigma_p^2)^{1/2}}}_{=1}$$

$$d\mathcal{P}_x = \frac{e^{-x^2/2\sigma_x^2}}{(2\pi\sigma_x^2)^{1/2}}\, dx \equiv P(x)\, dx$$

• Similarly a small calculation shows that:

$$d\mathcal{P}_p = \frac{e^{-p^2/2\sigma_p^2}}{(2\pi\sigma_p^2)^{1/2}}\, dp = P(p)\, dp$$

So

**b)** $$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx \; x^2 \, P(x)$$

Here $$d\mathcal{P}_x = P(x)\,dx = \frac{e^{-x^2/2\sigma_x^2}}{(2\pi\sigma_x^2)}\,dx$$

So

$$\boxed{\langle x^2 \rangle} = \int_{-\infty}^{\infty} \underline{dx} \; x^2 \, \frac{e^{-x^2/2\sigma_x^2}}{(2\pi\sigma_x^2)^{1/2}}$$

$$= \sigma_x^2 \int_{-\infty}^{\infty} du \; \frac{u^2 \, e^{-u^2/2}}{\sqrt{2\pi}} = \boxed{\sigma_x^2 = \frac{k_B T}{m\omega_0^2}}$$

Similarly

$$\boxed{\langle p^2 \rangle} = \int_{-\infty}^{\infty} dp \; p^2 \, \frac{e^{-p^2/2\sigma_p^2}}{(2\pi\sigma_p^2)^{1/2}} = \sigma_p^2 = \boxed{m k_B T}$$

the integral is the same as the x-case

**c)** Note

$$\langle K \rangle = \left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2m}\langle p^2 \rangle = \frac{1}{2m} mkT = \frac{1}{2}kT \checkmark$$

Similarly

$$\langle U \rangle = \frac{1}{2} m \omega_0^2 \langle x^2 \rangle = \frac{1}{2} m\omega_0^2 \left( \frac{kT}{m\omega_0^2} \right) = \frac{1}{2} kT$$

Then

$$\langle E \rangle = \langle U \rangle + \langle K \rangle = kT$$

Thus each oscillator consists of two dof.

$$2 \times \frac{1}{2} kT = kT$$

d) The energy is

$$U = N \left( \underbrace{3 \times \frac{1}{2} k_B T}_{\text{translations}} + \underbrace{2 \frac{1}{2} k_B T}_{\text{rotations}} + \underbrace{2 \times \frac{1}{2} k_B T}_{\text{vibrations}} \right)$$

$$= N \frac{7}{2} k_B T$$

$$C_V = \left( \frac{\partial U}{\partial T} \right)_V = \frac{7}{2} N k_B \qquad C_p = C_V + N k_B = \frac{9}{2} N k_B$$

So for one mole

$$C_p^{1 m^{l}} = N_A \frac{9}{2} k_B = \frac{9}{2} R$$

So at the highest temperature it agrees.

# Two State

- Then

$$Z = \sum_n e^{-\beta E_n}$$

$$-\frac{\partial Z}{\partial \beta} = \sum_n E_n e^{-\beta E_n}$$

$$-\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{\sum_n E_n e^{-\beta E_n}}{\sum_n e^{-\beta E_n}} = \langle E_n \rangle$$

- $Z = 1 + e^{-\beta \Delta}$

$$-\frac{\partial Z}{\partial \beta} = e^{-\beta \Delta} \Delta \quad \text{and} \quad \langle E \rangle = \frac{\Delta e^{-\beta \Delta}}{(1 + e^{-\beta \Delta})}$$

Problem 5

(b) $V = \frac{NkT}{p}$

$$\beta_p = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \frac{Nk}{pV} = \frac{1}{T}$$

(c) $$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = +\frac{1}{V}\left(\frac{NkT}{p^2}\right) = \frac{1}{p}$$

(d) $đQ = 0 \qquad dU = -p\,dV$

Then $C_V\,dT = -p\,dV$

So

$$dV = V\,\beta_p\,dT - V\,K_T\,dp$$

$$= -V\,\beta_p\,\frac{p}{C_V}\,dV - V\,K_T\,dp$$

$$\left(1 + \frac{\cancel{V}}{\cancel{T}}\frac{Nk\cancel{T}}{\cancel{V}C_V}\right)dV = -V\,K_T\,dp$$

$$\frac{C_p}{C_V}\,dV = -V\,K_T\,dp$$

So

$$\boxed{K_s = -\frac{1}{V}\left(\frac{dV}{dp}\right)_{adiab} = \frac{K_T}{\gamma}}$$

e) So

$$C_s = \left(\frac{\gamma}{\rho K_T}\right)^{1/2} = \left(\frac{\gamma n k T}{\rho}\right)^{1/2} = \left(\frac{\gamma k T}{m}\right)^{1/2}$$

$$K_T = \frac{1}{p} = \frac{1}{nkT}$$

So inserting avagadros number

$$\boxed{C_s = \left(\gamma \frac{N_A k T}{N_A m}\right)^{1/2}} = \left(\gamma \frac{RT}{m_{ml}}\right)^{1/2} = \left(\frac{1.4 \times 8.325 \times 300}{28g}\right)^{1/2}$$

molas mass of $N_2$ 28g

$$\boxed{= 353\,m/s}$$

$\gamma = \frac{7}{5}$ for diatomic molecules

f) The frequency of 440 Hz corresponds to a period of ~2 ms this is way to short for heat conduction in the gas.

# Otto Cycle

- Between $1 \to 2$ we have

$$\Delta U = \cancel{Q} + W_{12}$$

So

see next item

$$W_{12} = C_V \Delta T_{12} = C_V T_2 \left(1 - \frac{1}{r^{\gamma-1}}\right)$$

- Now $1 \to 2$ is adiabatic

$$T_2 = T_1 \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

So $\Delta T_{12} = T_1 (r^{\gamma-1} - 1)$

$\left(\frac{T_2}{T_1}\right) = r^{\gamma-1}$ so

$$\Delta T_{12} = T_2 \left(1 - \frac{1}{r^{\gamma-1}}\right)$$

- Then for $2 \to 3$

$$C_V \Delta T_{23} = Q + \cancel{W}^{0}$$

$$\Delta T_{23} = \frac{Q_{in}}{C_V}$$

- Then similarly for $3 \to 4$

$$\Delta U = W$$

$$T_4 = T_3 \left(\frac{V_3}{V_4}\right)^{\gamma-1} = T_3 \frac{1}{r^{\gamma-1}}$$

$$\Delta U = C_V (T_4 - T_3)$$

$$\Delta U = -C_V T_3 \left(1 - \frac{1}{r^{\gamma-1}}\right) = W_{34}$$

So

$$W_{net} = W_{34} + W_{12} = -\underbrace{C_V (T_3 - T_2)}_{= Q_{in}} \left(1 - \frac{1}{r^{\gamma-1}}\right)$$

Finally

$$\eta = \frac{-W_{net}}{Q_{in}} = \frac{+Q_{in}}{Q_{in}} \left(1 - \frac{1}{r^{\gamma-1}}\right)$$

$$\boxed{\eta = 1 - \frac{1}{r^{\gamma-1}}}$$

b) For 1mol

$$pV = Nk_B T$$

$$V = \frac{N_A k_B T}{p} = \frac{RT}{p} = \frac{8.32 \frac{J}{°K}\ 300°K}{10^5\ N/m^2}$$

$$= 25.L$$

Now the volume of the cylinder is 2.5L
So

$$\frac{2.5L}{25L} = \frac{n_{ml}}{1\ mol} \Rightarrow \boxed{n_{ml} = 0.1}$$

c) We use

$\boxed{1 \to 2}$ $T_f = T_i \left(\frac{V_i}{V_f}\right)^{\gamma-1}$ $p_f = p_i \left(\frac{V_i}{V_f}\right)^{\gamma}$

$T_2 = T_1 \, r^{\gamma-1}$ $p_2 = p_1 \, r^{\gamma}$ $V_2 = V_1/r$

$\underline{T_2 = 689 \, °K}$ $\underline{p_2 = 18.4 \, b}$ $\underline{V_2 = 0.3L}$

$\Delta U_{12} = W_{1\to 2} = C_v (T_f - T_i) = C_v T_i \, (r^{\gamma-1} - 1)$

Now $C_v T_i = U_i$, for 1 ml we have

$C_v^{1ml} = \frac{5}{2} R$ $C_v^{0.1ml} = 0.1 \times \frac{5}{2} R = 0.832 \frac{J}{°K} \times \frac{5}{2} = 2.08 \frac{J}{°K}$

$\underline{\Delta U_{12} = \Delta W_{12} = 810 J, \quad Q = 0}$

$\boxed{2 \to 3}$ Now

$\underline{\Delta U_{23} = Q_{in} + \cancel{W}^{\,0} = 2,200 J \qquad W = 0}$

$C_v (T_3 - T_2) = Q_{in}$ $Q_{in} = 0.1 \text{ mol} \times \frac{22000 J}{\text{mol}} = 2,200 J$

$\underline{T_3} = T_2 + \frac{Q_{in}}{C_v} = \underline{1747 \, °K}$

$$P_3 = \frac{N k_B T}{V}$$

$$\frac{P_3}{P_2} = \frac{T_3}{T_2}$$

$$P_3 = P_2 \frac{T_3}{T_2} = 46.5\ b$$

$$V_3 = 0.3L$$

$\boxed{3 \to 4}$

$$\Delta U = \cancel{Q}^{\,0} + W$$

$$\Delta U = C_V (T_f - T_i)$$

$$= - C_V T_i \left(1 - \frac{1}{r^{\gamma-1}}\right)$$

$$W_{34} = \Delta U = -2052 J$$

Adiabatic

$$T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1}$$

$$T_4 = T_3 \frac{1}{r^{\gamma-1}}$$

$$T_4 = 760\ °K$$

$\boxed{4 \to 1}$

$$\Delta U = Q + \cancel{W}^{\,0}$$

$$C_V (T_1 - T_4) = Q$$

$$\Delta U = Q = -958 J \qquad W = 0$$

$$\frac{P_4}{P_1} = \frac{T_4}{T_1} = 6.33\ b$$

$$V_4 = 2.5L$$

As a quick check we note

$$W_{net} = W_{12} + W_{34} = -1242\,J$$

$$Q_{in} = 2200\,J$$

So

$$\eta = \frac{|W_{net}|}{Q} = 0.56$$ which should be compared with $1 - \frac{1}{r^{\gamma-1}} = 1 - \frac{1}{8^{\gamma-1}} = 0.56$