1 Integrals

Bose and Fermi:

$$\int_0^\infty \mathrm{d}x \, \frac{x}{e^x - 1} = \frac{\pi^2}{6} \tag{1}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^2}{e^x - 1} = 2\zeta(3) \simeq 2.404 \tag{2}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^3}{e^x - 1} = \frac{\pi^4}{15} \tag{3}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^4}{e^x - 1} = 24\,\zeta(5) \simeq 24.88 \tag{4}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^5}{e^x - 1} = \frac{8\pi^6}{63} \tag{5}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x}{e^x + 1} = \frac{\pi^2}{12} \tag{6}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^2}{e^x + 1} = \frac{3}{2}\,\zeta(3) \simeq 1.80309 \tag{7}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^3}{e^x + 1} = \frac{7\pi^4}{120} \tag{8}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^4}{e^x + 1} = \frac{45}{2}\,\zeta(5) \simeq 23.33 \tag{9}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^5}{e^x + 1} = \frac{31\pi^6}{252} \tag{10}$$

Gamma Function:

$$\Gamma(z) \equiv \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x \tag{11}$$

with specific results

$$\Gamma(z+1) = z\Gamma(z) \qquad \Gamma(n) = (n-1)! \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{12}$$

Gaussian Integrals:

$$I_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \mathrm{d}x \, e^{-x^2/2} x^n \tag{13}$$

with specific results

$$I_0 = 1 \quad I_2 = 0 \quad I_4 = 3 \quad I_6 = 15 \tag{14}$$

Problem 1. Neutrino Gas

A neutrino is like a photon – it is neutral, (nearly) massless, and it has two spin states, But a neutrino is a fermion not a boson, and the modes can be in two possible states, either unoccupied or occupied by a particle, with corresponding energies 0 and ϵ. Consider a gas of neutrinos at temperature T and in a cubic box of volume $V = L^3$.

(a) Consider a single mode in the container with single particle energy ϵ. Determine the grand partition function for the mode and derive the Fermi-Dirac expression for the mean number of particles in the mode:

$$n_{FD}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \tag{15}$$

What are the probabilities of finding the mode unoccupied and occupied respectively?

(b) What is the energy and number of the neutrinos in the box? Explain each step carefully and assume $\mu = 0$.

(i) Compare the number of neutrinos in the box to the number of photons that would be in an equivalent box at the same temperature. Explain qualitatively why there are fewer neutrinos in the box.

You should find that the number of neutrinos is 3/4 the number of photons, but the energy is of the neutrinos is 7/8 of the photons.

(c) Determine the average de Broglie wavelength $\lambda \equiv h/p$ of the neutrino by integrating over the momenta. (Ans. $\lambda = 2.87\,\hbar c/kT$)

(d) (Not this year) What is the number of neutrinos striking the walls per unit time? Ans: $0.27\,L^2(kT/\hbar c)^3 c$.

Solution

(a) We have

$$\mathcal{Z}_G = \sum_{n=0,1} e^{-\beta(n\epsilon - \mu n)} = 1 + e^{-\beta(\epsilon-\mu)} \tag{16}$$

$$\tag{17}$$

Then the probability of being occupied and unoccupied are

$$P_0 = \frac{1}{1+e^{-\beta(\epsilon-\mu)}} \qquad P_1 = \frac{e^{-\beta(\epsilon-\mu)}}{1+e^{-\beta(\epsilon-\mu)}} \tag{18}$$

So the mean number is

$$n_{FD}(\epsilon) = P_0 \cdot 0 + P_1 \cdot 1 = \frac{e^{-\beta(\epsilon-\mu)}}{1+e^{-\beta(\epsilon-\mu)}} = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \tag{19}$$

(b) Detailed explanations are given in lecture notes. The number of particles and total energy are

$$N = \sum_{\text{modes}} n_{FD}(\epsilon) \tag{20}$$

$$U = \sum_{\text{modes}} n_{FD}(\epsilon)\epsilon \tag{21}$$

$$\tag{22}$$

We then have

$$\sum_{\text{modes}} = 2\int \frac{V d^3p}{(2\pi\hbar)^3} \tag{23}$$

where the leading 2 accounts for the spin of the neutrino[1] And, the energy is related to the momentum $\epsilon = cp$. So after integrating over angles $d^3p \Rightarrow 4\pi p^2 dp$

$$N = \frac{V}{\pi^2\hbar^3}\int_0^\infty p^2 dp \frac{1}{e^{\beta cp}+1} \tag{24}$$

$$U = \frac{V}{\pi^2\hbar^3}\int_0^\infty p^2 dp \frac{cp}{e^{\beta cp}+1} \tag{25}$$

Changing variables to a dimensionless $x = \beta cp$ we have and

$$N = \frac{V}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \int_0^\infty x^2 dx \frac{1}{e^x+1} \tag{26}$$

$$U = \frac{V}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 kT \int_0^\infty x^2 dx \frac{x}{e^x+1} \tag{27}$$

[1] In fact neutrinos have a left handed polarization. The neutrino with right-handed polarization is in fact an anti-neutrino. It could also be said that the 2 accounts for neutrinos and anti-neutrinos.

The remaining integrals can be done using the tables given in class.

$$\int_0^\infty \mathrm{d}x\, \frac{x^2}{e^x+1} = \frac{3}{4}\, 2\zeta(3) \simeq 1.80309 \tag{28}$$

$$\int_0^\infty \mathrm{d}x\, \frac{x^3}{e^x+1} = \frac{7}{8}\, \frac{\pi^4}{15} \tag{29}$$

So we find finally

$$N = V \left(\frac{kT}{\hbar c}\right)^3 \frac{3}{4} \times 0.243 \tag{30}$$

$$U = \frac{V}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3 \times \frac{7}{8}\frac{\pi^2}{15} \tag{31}$$

As claimed we find that the number of neutrinos is 3/4 the number of photons, while the energy of the neutrinos is 7/8 the energy of photons.

Neutrinos are fermions and their modes can either be unoccupied or occupied. Photons are bosons, and their modes can have $0, 1, 2, 3, 4, \ldots$ photons. Since the distribution of modes is the same, and the number of particles per mode is always greater for the bosonic case, i.e.

$$\frac{1}{e^x-1} > \frac{1}{e^x+1} \tag{32}$$

for all x, we must have fewer fermions in total.

(c) The debroglie wavelength of a particle with momentum p is

$$\lambda = \frac{h}{p} = 2\pi \frac{\hbar}{p} \tag{33}$$

Then

$$\langle \lambda \rangle = \frac{\sum_{\text{modes}} n_{FD}(\epsilon) \frac{2\pi\hbar}{p}}{\sum_{\text{modes}} n_{FD}(\epsilon)} \tag{34}$$

Replacing the sum with an integral, and noting that many factors cancel when taking the ratio we have

$$\langle \lambda \rangle = \frac{\int_0^\infty p^2 dp \frac{1}{e^{\beta c p}+1} \frac{2\pi\hbar}{p}}{\int_0^\infty p^2 dp \frac{1}{e^{\beta c p}+1}} \tag{35}$$

$$= 2\pi \left(\frac{\hbar c}{kT}\right) \times \frac{\int_0^\infty x^2 dx \frac{1}{e^x+1} \frac{1}{x}}{\int_0^\infty x^2 dx \frac{1}{e^x+1}} \tag{36}$$

$$= 2\pi \left(\frac{\hbar c}{kT}\right) \times \frac{\pi^2/12}{(3/2)\zeta(3)} \tag{37}$$

$$= 2.87 \left(\frac{\hbar c}{kT}\right) \tag{38}$$

where in passing to the second line we have defined $x = \beta c p$. Then we used the integrals in the front to evaluate the numerical factors arriving at the required result.

(d) We can use the results derived in class. You should go over this derivation. The number of neutrinos striking the wall per area per time is

$$\Phi_N == \frac{1}{4} n_\nu c \tag{39}$$

We note that in a different number of dimensions, you would need to rederive the result, and the factor 1/4 would need to be modified. So multiplying by the total surface area ($6L^2$) we find

$$\frac{dN_\nu}{dt} = \frac{6}{4} L^2 \left(\frac{kT}{\hbar c}\right)^3 c \times \frac{3}{4} \cdot 0.243 \tag{40}$$

which gives the required result

$$\frac{dN_\nu}{dt} = 0.27\, L^2 \left(\frac{kT}{\hbar c}\right)^3 c \tag{41}$$

Problem 2. Most energetic frequency band

(a) The energy density can be written

$$u = \int_0^\infty d\omega \, \frac{du}{d\omega} \tag{42}$$

where $du/d\omega$ is the energy per frequency interval $d\omega$. Using a graphical means show $du/d\omega$ is maximum for $\hbar\omega = 2.82kT$. What is the energy of a photon with this frequency for a black body of 5340 °K, which is approximately the surface temperature of the sun.

(b) The energy density can be written

$$u = \int_0^\infty d\lambda \, \frac{du}{d\lambda} \tag{43}$$

where $du/d\lambda$ is the energy per wavelength inteval $d\lambda$. Find $du/d\lambda$, and using a graphical method find the wavelength λ_{max} where $du/d\lambda$ is maximum. (You should find $hc/\lambda_{\mathrm{max}} \simeq 4.9\, k_B T$.) What is this wavelength in nm for a black body of 5340 °K, which is approximately the surface temperature of the sun. You should find that the wavelength corresponds to a yellow color.

Most energetic

a) $$U = 2\int \frac{V\,d^3p}{(2\pi)^3}\,\frac{\varepsilon}{e^{\beta\varepsilon}-1}$$

So writing $\varepsilon = \hbar\omega = cp$

$$d^3p = 4\pi p^2\,dp = \frac{\hbar^3}{c^3}\,\omega^2\,d\omega \cdot 4\pi$$

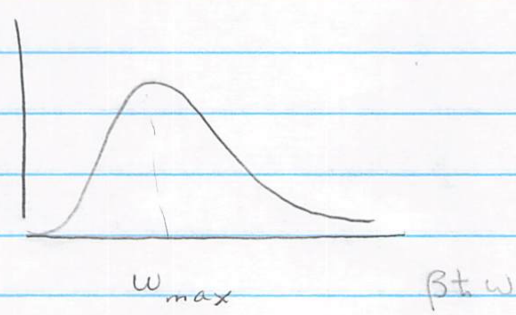
We find after algebra

$$\text{☆} \quad u = \frac{\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3\,d\omega}{e^{\beta\hbar\omega}-1}$$

So

$$\boxed{\frac{du}{d\omega} \propto \frac{\omega^3}{e^{\beta\hbar\omega}-1}}$$

Plotting this, we find a maximum (see next page) at:

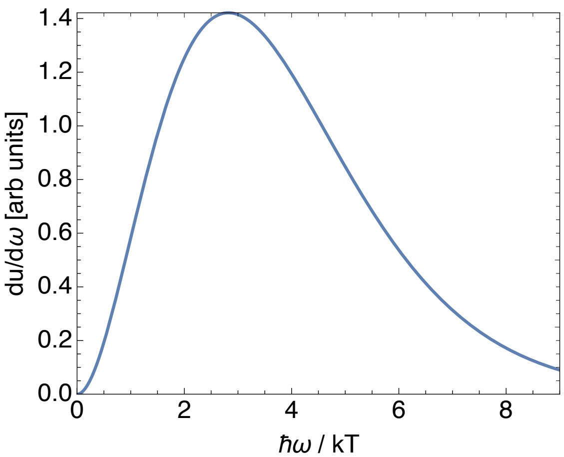


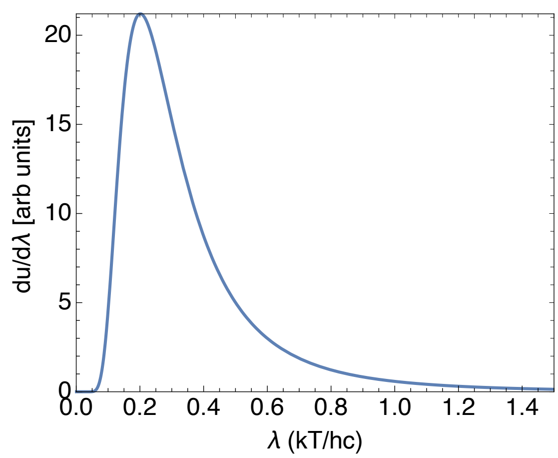
$\beta\hbar\omega \simeq 2.8$

So

$$\hbar\omega = 2.8\,k_B T = 2.8 \times \frac{0.025\,eV}{300\,°K}\,6000\,°K = 1.4\,eV$$

Spectral Density of Energy





From the plot we see that the spectral density $du/d\lambda$ is max when

$$\lambda \simeq 0.2\frac{hc}{kT}$$

Putting $k_B = 0.025\,\mathrm{eV}/300\,\mathrm{K}$ and $hc = 1240\,\mathrm{eVnm}$ I find for $T = 5340K$

$$\lambda \simeq 560\,\mathrm{nm} \qquad \text{Yellowish}$$

Problem 3. Density of single particle states

In class we classified the single-particle modes (or wave-functions) of a box by three quantum numbers:

$$\psi_{\ell_x \ell_y \ell_z}(x, y, z) \propto \sin(k_x x)\sin(k_y y)\sin(k_z z) \tag{44}$$

where

$$\vec{k} = (k_x, k_y, k_z) = \frac{\pi}{L}(\ell_x, \ell_y, \ell_z) \qquad \text{and} \qquad \vec{p} = \hbar\vec{k} \tag{45}$$

We showed that if the box is large

$$\sum_{\ell_x=1}^{\infty}\sum_{\ell_y=1}^{\infty}\sum_{\ell_z=1}^{\infty}\ldots \to \int \frac{V d^3p}{(2\pi\hbar)^3}\ldots \tag{46}$$

which you may wish to review.

(a) Show that the number of modes $g(k)dk$ with wavenumber magnitude k, between k and $k + dk$ is

$$g(k)dk = \frac{1}{2\pi^2}Vk^2dk \tag{47}$$

and determine the analogous formula in two dimensions [2]. $g(k)$ is known as the density of single particle states (or modes) . Assume that the particles are spinless, so that

$$\sum_{\text{modes}}\ldots = \sum_{\ell_x=1}^{\infty}\sum_{\ell_y=1}^{\infty}\sum_{\ell_z=1}^{\infty}\ldots \tag{48}$$

(b) The density of states is often expressed in terms of energy. For spinless non-relativistic particles (with $\epsilon(p) = p^2/2m$) show that the number of modes, $g(\epsilon)d\epsilon$, with energy between ϵ and $d\epsilon$. Show that the density of states in three dimensions is

$$g(\epsilon)d\epsilon = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\epsilon}\,d\epsilon \tag{49}$$

and find the analogous formula in two dimensions for non-relativistic particles[3]

(c) In any number of dimensions, and for relativistic and non-relativistic particles, explain why the grand potential of a Bose and Fermi gas can be written

$$\Phi_G = \pm kT\int_0^{\infty} g(\epsilon)\ln\left(1 \mp e^{-\beta(\epsilon_p-\mu)}\right)\mathrm{d}\epsilon \tag{50}$$

By differentiation of Φ_G show that the number of particles is

$$N = \int_0^{\infty} g(\epsilon)\frac{1}{e^{\beta(\epsilon_p-\mu)} \mp 1}\mathrm{d}\epsilon \tag{51}$$

[2] Answer: $g(k)dk = Akdk/2\pi$.
[3] Ans: $g(\epsilon)d\epsilon = Am\,d\epsilon/2\pi\hbar^2$.

where the upper sign is for fermions and the lower sign is for bosons.

Determine $g(\epsilon)$ for a photon gas in three dimensions, and express the pressure of the photon gas as an integral. You will evaluate this integral in the next problem.

The photon has two polarization states. So there are two modes for every value of k

$$\sum_{\text{modes}} = 2 \sum_{\ell_x=1}^{\infty} \sum_{\ell_y=1}^{\infty} \sum_{\ell_z=1}^{\infty} \cdots \tag{52}$$

Density of States

a) So

$$d\mathcal{N} = V\frac{d^3p}{(2\pi\hbar)^3}$$ = number of modes with momentum $\vec{p} = (p_x, p_y, p_z)$ in range

$[p_x; dp_x], [p_y; dp_y], [p_z; dp_z]$

This means $p_x < p_x' < p_x + dp_x$:

$$d^3p = 4\pi p^2 dp$$

And $p = \hbar k$, so

$$d\mathcal{N} = V\frac{4\pi}{(2\pi)^3} k^2 dk = \boxed{\frac{V}{2\pi^2} k^2 dk} \rightarrow \text{or } g(k) = \frac{Vk^2}{2\pi^2}$$

= number of modes with k in range $k < k' < k + dk$

b) In two dimensions

$$d\mathcal{N} = \frac{A\,d^2p}{(2\pi\hbar)^2} = A\frac{2\pi p\,dp}{(2\pi\hbar)^2} \quad \Big) \; p = \hbar k$$

$$\boxed{d\mathcal{N} = \frac{1}{2\pi} A\,k\,dk}$$

or $\boxed{g(k) = Ak/2\pi}$

So since

$$\mathcal{E}(p) = \frac{p^2}{2m} \qquad d\mathcal{E} = \frac{p}{m}\,dp$$

Then in 3D

$$d\mathcal{N} = \frac{V}{2\pi^2}\frac{p^2\,dp}{\hbar^3} = \frac{1}{2\pi^2}\frac{m\,p}{\hbar^3}\,d\mathcal{E} = \frac{1}{4\pi^2}\left(\frac{2m\,p}{\hbar^3}\right)d\mathcal{E}$$

$$= \boxed{\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\mathcal{E}}\,d\mathcal{E} = g(\mathcal{E})\,d\mathcal{E}}$$

grouping it like this is motivated by units:

In 2D:

$$d\mathcal{N}_{\vec{p}} = \frac{A\,d^2p}{(2\pi\hbar)^2} \qquad \frac{1}{\lambda_{typ}} \sim \frac{p_{typ}}{\hbar} \sim \left(\frac{2m}{\hbar^2}\right)^{1/2}\mathcal{E}^{1/2}$$

Integrating over the angles of p we have

$$d\mathcal{N}_p = \frac{A\,p\,dp}{2\pi\hbar^2}$$ now with

$$\mathcal{E} = \frac{p^2}{2m} \qquad d\mathcal{E} = \frac{p\,dp}{m}$$

$$d\mathcal{N}_{\mathcal{E}} = \frac{A\,m\,d\mathcal{E}}{2\pi\hbar^2}$$

$$\boxed{d\mathcal{N}_{\mathcal{E}} = \frac{A}{4\pi}\left(\frac{2m}{\hbar^2}\right)d\mathcal{E}}$$

c) Then the free energy of one mode is

$$\Phi_G^{\varepsilon} = - k_B T \ln \frac{1}{1 - e^{-\beta(\varepsilon - \mu)}}$$

where $\mathcal{Z}_p = \frac{1}{1 - e^{-\beta(\varepsilon-\mu)}}$ is the grand partition function of one mode for a boson

So

$$\Phi_G^{tot} = \sum_{modes} \Phi_G^{\varepsilon}$$

By definition the sum over modes becomes an integral over the mode density, $g(\varepsilon)\, d\varepsilon$

$$\sum_{modes} \ldots = \int g(\varepsilon)\, d\varepsilon \ldots$$

So

$$\boxed{\Phi_G = \int_0^{\infty} g(\varepsilon)\; k_B T \ln\left(1 - e^{-\beta(\varepsilon - \mu)}\right)}$$

For a fermion

$$\mathcal{Z}_p = 1 + e^{-\beta(\varepsilon - \mu)}$$

and

$$\Phi_G = \int_0^\infty g(\varepsilon)\, d\varepsilon \; -k_B T \ln(1 + e^{-\beta(\varepsilon-\mu)})$$

☆ Then for photon — spin of photons (or polarizations) = 2

$$d\mathcal{N}_{modes} = \int_{angles} \frac{2 V d^3 p}{(2\pi\hbar)^3} = \frac{1}{\pi^2 \hbar^3} V p^2 dp$$

Now $\varepsilon = cp$ so

$$d\mathcal{N} = \frac{1}{\pi^2} \frac{V}{(\hbar c)^3} \varepsilon^2 d\varepsilon = g(\varepsilon)\, d\varepsilon$$

And

$$\boxed{\Phi_G = \frac{V}{\pi^2 (\hbar c)^3} \int_0^\infty \varepsilon^2 \; k_B T \ln(1 - e^{-\beta(\varepsilon-\mu)})}$$

So

$\Phi_G = -pV$ So

$$pV = -\frac{1}{\pi^2 (\hbar c)^3} \int_0^\infty \varepsilon^2 k_B T \ln(1 - e^{-\beta(\varepsilon-\mu)})\, d\varepsilon$$

Problem 4. Entropy per Photon

You should have found the pressure (or minus the grand potential per volume) of gas of photons. Ater recognizing that $\epsilon = \hbar\omega$, the result of problem 2 is

$$pV = \frac{V}{\pi^2 c^3} \int_0^\infty \omega^2 kT \ln\left(1 - e^{-\beta\hbar\omega}\right) \mathrm{d}\omega \tag{53}$$

(a) Integrate by parts to show that

$$p = \frac{\pi^2}{45} \left(\frac{kT}{\hbar c}\right)^3 kT \tag{54}$$

The necessary integrals are given below.

(b) Show that

$$d\Phi_G = -SdT - Nd\mu - pdV \tag{55}$$

and then by differentiating the pressure (or grand potential), that

$$S = 4\frac{pV}{T} \tag{56}$$

Using the result from class for the number of photons and show that the entropy per photon S/Nk_B is 3.6.

(c) Use the Gibbs-Duhem relation and the results for S and p in this problem to find the energy density of the system, $u = U/V$. Check your result by comparing with the method used in class.

Hint: What is the chemical potential of the photon gas?

Entropy / Photon

- So setting $\varepsilon = \hbar\omega$ the results of the previous problem can be written

$$pV = -\frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\hbar^3 \int_0^\infty \underbrace{\omega^2 d\omega}_{dv} \; \underbrace{k_B T \ln(1-e^{-\beta\hbar\omega})}_{u}$$

- Now integrate by parts

$$dv = \omega^2 d\omega \qquad u = k_B T \ln(1-e^{-\beta\hbar\omega})$$

$$v = \frac{1}{3}\omega^3 \qquad du = k_B T \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} \beta\hbar\, d\omega$$

$$= \frac{\hbar\, d\omega}{e^{\beta\hbar\omega}-1}$$

So

$$\int_0^\infty u\, dv = uv\Big|_0^\infty - \int_0^\infty v\, du$$

this is zero since $u\big|_0 = 0$ and $v\big|_\infty = 0$

- And so

$$pV = \frac{V}{\pi^2 c^3}\hbar \int_0^\infty \frac{1}{3}\omega^3 d\omega \frac{\hbar}{e^{\beta\hbar\omega}-1}$$

So

$$\frac{\partial p}{\partial T}V = S$$

So since $pV = CT^4$

$$\boxed{S = \frac{\partial p}{\partial T}V = 4CT^3 = 4\frac{CT^4}{T} = 4\frac{pV}{T}}$$

$$\frac{S}{Nk_B} = \frac{4\pi^2/15}{0.245} = 3.6$$

So

$$U - TS + pV = \mu N \xrightarrow{} 0$$

Then

$$U = -pV + TS$$

$$U = -pV + 4pV = 3pV$$

So

$$\boxed{\frac{U}{V} = 3p \quad \text{as before}}$$

Problem 5. (Optional) Temperature of the Sun

See lecture: The intensity (energy per area per time) of sunlight on earth is $I = 1\,\mathrm{kW}/m^2$. Show that the temperature of the sun is related to the intensity of the sunlight and the solid angle $\Omega_\odot$ that the suns takes up in our sky:

$$T_\odot = \left(\frac{I}{\sigma}\frac{\pi}{\Omega_\odot}\right)^{1/4} \tag{57}$$

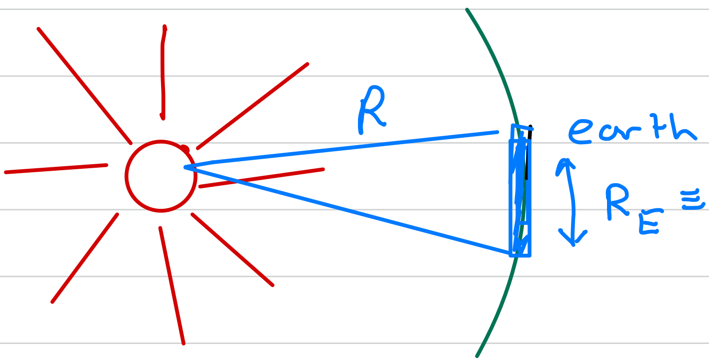
Here σ is the Steffan Boltzmann constant and Ω is the solid angle subtended by the sun in our sky. Using

$$\Omega_\odot \simeq 6.8 \times 10^{-5} \tag{58}$$

evaluate $T_\odot$ numerically. I find $T_\odot \simeq 5340\,\mathrm{K}$.

Example

The energy per area per second absorbed by the earth from the sun is $1 kW/m^2$. Estimate the temperature of the sun.



The sun emits a total energy per second, dU/dt, of

$$\frac{dU}{dt} = \sigma T^4 \, 4\pi R_{\odot}^2$$

This gets spread out over a sphere of radius R. So the energy per area on that sphere (which includes the earth) is

$$I = \frac{1}{A}\frac{dU}{dt} = \sigma T^4 \frac{4\pi R_{\odot}^2}{4\pi R^2} = \frac{1\,kW}{m^2}$$

So solving for T

$$T = \left(\frac{I}{\sigma}\frac{R^2}{R_{\odot}^2}\right)^{1/4} = \left(\frac{I}{\sigma}\pi\frac{R^2}{\pi R_{\odot}^2}\right)^{1/4} \equiv \left(\frac{I\pi}{\sigma\Omega}\right)^{1/4}$$

Here we have defined the "solid-angle" of the sun

$$\Omega \equiv \frac{A}{R^2} \equiv \frac{\text{Area of patch on sphere}}{(\text{radius})^2}$$

This can be measured with a protactor: $\Omega/4\pi$ is the fraction of the sky covered by the sun. $\Omega = 6.8 \times 10^{-5}$. Substituting Ω, and σ, we find

$$\boxed{T = 5340^\circ K}$$

pretty close!