Problem 1. The most energetic frequency interval and wavelength interval

(a) The energy density can be written

$$u = \int_0^\infty d\omega \, \frac{du}{d\omega} \tag{1}$$

where $du/d\omega$ is the energy per frequency interval $d\omega$. Using a graphical means show $du/d\omega$ is maximum for $\hbar\omega = 2.82 k_B T$. What is the energy of a photon with this frequency for a black body of 6000K, which is approximately the surface temperature of the sun.

(b) The energy density can be written

$$u = \int_0^\infty d\lambda \, \frac{du}{d\lambda} \tag{2}$$

where $du/d\lambda$ is the energy per wavelength inteval $d\lambda$. Find $du/d\lambda$ and where the wavelength where it is maximum. What is this wavelength in nm for a black body of 6000K, which is approximately the surface temperature of the sun. You should find $\lambda \simeq 4.95 hc/k_B T$.

Problem 2. Density of states $g(\epsilon)$

In class we classified the modes (wave-functions) of a box by three quantum numers

$$\psi_{n_x n_y n_z}(x, y, z) \propto \sin(k_{n_x} x) \sin(k_{n_y} y) \sin(k_{n_z} z) \tag{3}$$

and we showed that if the box is large

$$\sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \sum_{n_z=1}^{\infty} \ldots \to \int \frac{V d^3p}{(2\pi\hbar)^3} \ldots \tag{4}$$

(a) Show that the number of modes $g(k)dk$ with wavenumber k, between k and $k + dk$ is

$$g(k)dk = \frac{1}{2\pi^2} V k^2 dk \tag{5}$$

and determine the analogous formula in two dimensions. $g(k)$ is known as the density of (single particle) states. Assume that the particles are spinless, so that

$$\sum_{\text{modes}} \ldots = \sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \sum_{n_z=1}^{\infty} \ldots \tag{6}$$

(b) The density of states is often expressed in terms of energy. For spinless non-relativistic particles (with $\epsilon(p) = p^2/2m$) show that the number of modes, $g(\epsilon)d\epsilon$, with energy between ϵ and $d\epsilon$. Show that the density of states in three dimensions is

$$g(\epsilon)d\epsilon = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\epsilon}\, d\epsilon \tag{7}$$

and find the analogous formula in two dimensions.

(c) In any dimension explain why grand partition function of a Bose and Fermi gas can be written

$$\Phi_G = \pm k_B T \int_0^\infty g(\epsilon) d\epsilon \ln\left(1 \mp e^{-\beta(\epsilon_p - \mu)}\right) \tag{8}$$

where the upper sign is for fermions and the lower sign is for bosons. You may simply quote the grand partition function from the class notes.

Determine $g(\epsilon)$ for a photon gas in three dimensions, and express the pressure of the photon gas as an integral. You will evaluate this integral in the next problem.

The photon has two polarization states. So there are two modes for every value of k

$$\sum_{\text{modes}} = 2 \sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \sum_{n_z=1}^{\infty} \ldots \tag{9}$$

Hint: Recall the Gibbs-Duhem relation and the definition of $\Phi_G = U - TS - \mu N$.

Problem 3. Entropy per Photon

You should have found the pressure (or minus the grand potential per volume) of gas of photons. Ater recognizing that $\epsilon = \hbar\omega$, the result of problem 2 is

$$pV = \frac{V}{\pi^2 c} \int_0^\infty \omega^2 k_B T \ln\left(1 - e^{-\beta\hbar\omega}\right) \tag{10}$$

(a) Integrate by parts to show that

$$p = \frac{\pi^2}{45} \left(\frac{k_B T}{\hbar c}\right)^3 k_B T \tag{11}$$

The necessary integrals are given below.

(b) Show that

$$d\Phi_G = -SdT - Nd\mu - pdV \tag{12}$$

and then by differentiating the pressure (or grand potential), that

$$S = 4\frac{pV}{T} \tag{13}$$

Using the result from class for the number of photons and show that the entropy per photon S/N is 3.6.

(c) Use the Gibbs-Duhem relation and the previous result to find the energy density of the system, $u = U/V$. Check your result by comparing with the method used in class. You should find

$$u = 3pV \tag{14}$$

$$
\begin{aligned}
\int_0^\infty \mathrm{d}x\,\frac{x}{e^x-1} &= \frac{\pi^2}{6}\\
\int_0^\infty \mathrm{d}x\,\frac{x^2}{e^x-1} &= 2\zeta(3) \simeq 2.404\\
\int_0^\infty \mathrm{d}x\,\frac{x^3}{e^x-1} &= \frac{\pi^4}{15}\\
\int_0^\infty \mathrm{d}x\,\frac{x^4}{e^x-1} &= 24\,\zeta(5) \simeq 24.88\\
\int_0^\infty \mathrm{d}x\,\frac{x^5}{e^x-1} &= \frac{8\pi^6}{63}
\end{aligned}
$$

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$$
\begin{aligned}
\int_0^\infty \mathrm{d}x\,\frac{x}{e^x+1} &= \frac{\pi^2}{12}\\
\int_0^\infty \mathrm{d}x\,\frac{x^2}{e^x+1} &= \frac{3}{2}\,\zeta(3) \simeq 1.80309\\
\int_0^\infty \mathrm{d}x\,\frac{x^3}{e^x+1} &= \frac{7\pi^4}{120}\\
\int_0^\infty \mathrm{d}x\,\frac{x^4}{e^x+1} &= \frac{45}{2}\,\zeta(5) \simeq 23.33\\
\int_0^\infty \mathrm{d}x\,\frac{x^5}{e^x+1} &= \frac{31\pi^6}{252}
\end{aligned}
$$

Figure 1: A compendium of useful integrals of Bose and Fermi distributions

Problem 4. Density fluctuations of a classical gas

Consider a classical gas consisting a volume V. This volume V is a small part of the full container of volume V_0 which containes N_0 particles, i.e. it is the air just in front of your nose rather than the full room. Of course the number of particles in this subvolume is fluctuating all the time as particles move in and out of the subvolume

(a) Consider the grand canonical ensemble where the probability of a subsystem having energy ϵ_s and number of particles N_s is

$$P = \frac{e^{\beta(\epsilon_s - \mu N_s)}}{\mathcal{Q}}$$

. Show that

$$\langle N \rangle = \frac{1}{\mathcal{Q}} \left(\frac{1}{\beta} \frac{\partial}{\partial \mu} \right)^2 \mathcal{Q} \tag{15}$$

$$\langle N^2 \rangle = \frac{1}{\mathcal{Q}} \left(\frac{1}{\beta} \frac{\partial}{\partial \mu} \right)^2 \mathcal{Q} \tag{16}$$

Show also the variance in the number of particles

$$\sigma_N^2 \equiv \langle N^2 \rangle - \langle N \rangle^2 \tag{17}$$

is given by

$$\sigma_N^2 = \frac{1}{\mathcal{Q}} \frac{1}{\beta^2} \frac{\partial^2 \mathcal{Q}}{\partial \mu^2} - \left(\frac{1}{\mathcal{Q}} \frac{1}{\beta} \frac{\partial \mathcal{Q}}{\partial \mu} \right)^2 = \left(\frac{1}{\beta} \frac{\partial}{\partial \mu} \right)^2 \ln \mathcal{Q} \tag{18}$$

Show that the variance can be written in a variety of other ways

$$\sigma_N^2 = k_B T \left(\frac{\partial \langle N \rangle}{\partial \mu} \right\rangle = kTV \frac{\partial^2 p}{\partial \mu^2} \tag{19}$$

We will now use this to compute the fluctuations in the density of a classical ideal gas. Recall that the Bose-Einstein distribution is

$$n_{BE} = \frac{1}{e^{\beta(\epsilon(p) - \mu)} - 1} \tag{20}$$

and that the grand potential for one mode is

$$\Phi_G = k_B T \ln(1 - e^{-\beta(\epsilon(p) - \mu)}) \tag{21}$$

(b) Go through the derivativion of the Bose-Einstein distribution and the grand potential given in class. Recall that in a classical limit it is very unlikely that there will be more than one particle in a quantum state. Then, explain why the classical limit means that

$$e^{-\beta(\epsilon(p) - \mu)} \ll 1\,. \tag{22}$$

Show that in the classical limit we have

$$
\begin{aligned}
n_{BE} =& e^{-\beta(\epsilon-\mu)} \left(1 + e^{-\beta(\epsilon-\mu)} + \ldots\right) \,, & (23)\\
\Phi_G =& -k_B T e^{-\beta(\epsilon-\mu)} \left(1 + \tfrac{1}{2} e^{-\beta(\epsilon-\mu)} + \ldots\right) \,, & (24)
\end{aligned}
$$

where the second term is the first quantum correction arising from having two particles in one quantum state. We will ultimately neglect these quantum contributions, and just keep the leading term. You may wish to keep the leading term first and then work out the corrections.

(c) Use part (b) to show that the density and pressure in a classical gas are

$$p \simeq k_B T\, e^{\beta\mu} n_Q \left[1 + \frac{e^{\beta\mu}}{4\sqrt{2}}\right] \tag{25}$$

where $n_Q = 1/\lambda_{\rm th}^3$. Drop the subleading term in what follows.

(d) From your expression for pressure or grand potential ($pV = -\Phi_G$) determine the mean number of particles and the variance in the number of particles. You should find

$$\frac{\langle N \rangle}{V} = e^{\beta\mu} n_Q \tag{26}$$

Does this result agree with the canonical ensemble (HW12 problem 2)? Show also that

$$\sigma_N^2 = \langle N \rangle \tag{27}$$

The result is the expected one for a classical gas, where one expects that the uncertainty in N decreases with the number of particles.

$$\frac{\sigma_N}{\langle N \rangle} = \frac{1}{\sqrt{\langle N \rangle}} \tag{28}$$

Problem 5. Inversion of taylor series

This is needed for problem 6, but it is math that all physicists should know. Given a Taylor series of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \tag{29}$$

we want to solve for x as a function of y.

(a) Show that

$$x \approx \frac{y - a_0}{a_1} \tag{30}$$

(b) Show more generally that

$$x \simeq u - \left(\frac{a_2}{a_1}\right) u^2 \tag{31}$$

where $u \equiv (y - a_0)/a_1$. You will need this below.

Hint: Assume a taylor series

$$x = u + C_2 u^2 + \dots \tag{32}$$

and substitute into Eq. (29).

Remark: The first term is very easy to obtain, and is the most iportant in practice. Without approximation we have

$$u = x + \frac{a_2}{a_1} x^2 + \dots \tag{33}$$

Since x is almost u in a first approximation we can replace the x^2 with u^2 up to higher corrections

$$u \simeq x + \frac{a_2}{a_1} u^2 \tag{34}$$

(c) Use the methodology outlined above, especially the remark, to show

$$\sin^{-1}(y) = y + \frac{y^3}{3!} + \dots \tag{35}$$

(d) (Optional) Show more generally that

$$x = u - \left(\frac{a_2}{a_1}\right) u^2 + \left(2\left(\frac{a_2}{a_1}\right)^3 - \left(\frac{a_3}{a_1}\right)\right) u^3 + \dots \tag{36}$$

Problem 6. Almost Classical Gas

Recall that the Bose-Einstein distribution is

$$n_{BE} = \frac{1}{e^{\beta(\epsilon(p)-\mu)} - 1} \tag{37}$$

and that the grand potential for one mode is

$$\Phi_G = k_B T \ln(1 - e^{-\beta(\epsilon(p)-\mu)}) \tag{38}$$

(a) Go through the derivativion of the Bose-Einstein distribution and the grand potential given in class. Recall that in a classical limit it is very unlikely that there will be more than one particle in a quantum state. Then, explain why the classical limit means that

$$e^{-\beta(\epsilon(p)-\mu)} \ll 1\,. \tag{39}$$

Show that in the classical limit we have

$$n_{BE} = e^{-\beta(\epsilon-\mu)} \left(1 + e^{-\beta(\epsilon-\mu)} + \dots\right)\,, \tag{40}$$

$$\Phi_G = -\, k_B T e^{-\beta(\epsilon-\mu} \left(1 + \tfrac{1}{2} e^{-\beta(\epsilon-\mu)} + \dots\right)\,, \tag{41}$$

where the second term is the first quantum correction arising from having two particles in one quantum state.

(b) For a gas of non-relativistic particles $\epsilon(p) = p^2/2m$, start from the Bose-Einstein distribution and show that the density $n = N/V$ of these particles in the classical limit is approximately

$$n = e^{\beta\mu} n_Q \left(1 + \frac{e^{\beta\mu}}{2\sqrt{2}}\right) \tag{42}$$

where $n_Q = 1/\lambda_{\rm th}^3 = (2\pi m k_B T)^{3/2}/h^3$.

(c) Show that the chemical potential is approximately determined by the density via the relation

$$e^{\beta\mu} \simeq \frac{n}{n_Q}\left(1 - \frac{1}{2\sqrt{2}}\frac{n}{n_Q}\right) \tag{43}$$

Compare with problem the results of Problem 2 of Homeowrk 12. What is different and what is the same.

(d) Write down for the pressure of non-relativistic bose particles particles, $\epsilon(p) = p^2/2m$, discussed in class. Show that the pressure is

$$pV \simeq k_B T\, e^{\beta\mu} n_Q \left[1 + \frac{e^{\beta\mu}}{4\sqrt{2}}\right] \tag{44}$$

(e) Show that

$$pV \simeq N k_B T \left[1 - \frac{1}{4\sqrt{2}}\frac{n}{n_Q}\right] \tag{45}$$

(f) Repeat this problem for a fermion gas close to the classical limit.