Problem 1. Two State System

Consider an atom with only two states: a ground state with energy 0, and an excited state with energy Δ. Determine the mean energy $\langle\epsilon\rangle$ and variance in energy $\langle\delta\epsilon^2\rangle$. Sketch the mean energy versus Δ/k_BT.

Problem 2. Working with the speed distribution

Consider the Maxwell speed distribution

(a) Evaluate the most probable speed v_*, i.e the speed where $P(v)$ is maximized. You should find $v_* = (2k_BT/m)^{1/2}$.

(b) Determine the probability to have speed in a specific range, $v_* < v < 2v_*$. Follow the following steps:

  (i) Write down the appropriate integral.

  (ii) Change variables to a dimensionless speed $u = v/\sqrt{k_BT/m}$, i.e. u is the speed in units of $\sqrt{k_BT/m}$, and express the probability as an integral over u.

  (iii) Write a short program (in any language) to evaluate the dimensionless integral, by (for example) dividing up the interval into 200 bins, and evaluate the integral with Riemann sums. You should find

$$\mathscr{P} \simeq 0.53 \tag{1}$$

Problem 3. Distribution of energies

The speed distribution is

$$\mathrm{d}\mathscr{P} = P(v)\, dv \tag{2}$$

where $P(v) = (m/2\pi k_BT)^{3/2}\, e^{-mv^2/2k_BT} 4\pi v^2$.

(a) Show that the probability distribution of energies $\epsilon = \frac{1}{2}mv^2$ is

$$\mathrm{d}\mathscr{P} = P(\epsilon)\, d\epsilon \tag{3}$$

where

$$P(\epsilon) = \frac{2}{\sqrt{\pi}}\beta^{3/2}e^{-\beta\epsilon}\,\epsilon^{1/2} \tag{4}$$

Note: that the distribution of energies is independent of the mass, and recall $\beta = 1/k_BT$.

(b) Compute the variance in energy using $P(\epsilon)$. Express all integrals in terms $\Gamma(x)$ (as given in the previous homework) – it is helpful to change to a dimensionless energy $u = \beta\epsilon$. You should find (after evaluating these Γ functions as in the previous homework) that

$$\left\langle(\delta\epsilon)^2\right\rangle = \frac{3}{2}(k_BT)^2 \tag{5}$$

Problem 4. (Optional: Don't turn in) Change of variables

These take 1 sec each. Do them for yourself though. If you feel this is trivial – you're right.

(a) Show that the distribution of momenta is

$$\mathrm{d}\mathscr{P} = \left(\frac{1}{2\pi m k_B T}\right)^{3/2} e^{-p^2/2mk_BT} \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z \tag{6}$$

where $p^2 = p_x^2 + p_y^2 + p_z^2$.

(b) Show that

$$\int_{-\infty}^{\infty} dx\, f(x) = \int_{-\infty}^{\infty} du\, f(-u) \tag{7}$$

with $u = -x$.

(c) Consider the probability distribution

$$\mathrm{d}\mathscr{P} = f(\theta)\sin\theta \mathrm{d}\theta \tag{8}$$

with $\theta \in [0, \pi]$. For the change of variables $u(\theta) = \cos\theta$ show that

$$\mathrm{d}\mathscr{P} = f(\theta(u))\,\mathrm{d}u \tag{9}$$

with $u \in [-1, 1]$, and $\theta(u) = \arccos(u)$.

Problem 5. Distributions on the sphere

(a) Consider a particle with angular coordinates (θ, ϕ) which are uniformly distributed over a sphere of radius R, i.e. the probability per area is constant. Briefly explain (using the definition of solid angle) why $\mathrm{d}\mathscr{P} = C d\Omega$ with C a normalizing constant. Determine C and show that the probability distribution per $\mathrm{d}\theta$ is $\mathrm{d}\mathscr{P} = \frac{1}{2}\sin\theta d\theta$. Show that $\langle\cos^2\theta\rangle_{\text{sphere}} = \frac{1}{3}$.

(b) Consider a particle which is distributed (not uniformly) over the sphere

$$d\mathscr{P} = C(1 + \cos^2\theta)\frac{\mathrm{d}\Omega}{4\pi} \tag{10}$$

where C is a normalizing constant. This probability distribution is visualized in Fig. 1. Show that $C = 3/4$. Then show that the probability distribution in Eq. (10), implies that the probability distribution for $u = \cos(\theta)$ is

$$d\mathscr{P} = \frac{3}{8}(1 + u^2)\,\mathrm{d}u \tag{11}$$

for $u \in [-1, 1]$.

(c) Show for the probability distribution of (b) that $\langle\cos^2\theta\rangle_{\text{sphere}} = \frac{2}{5}$ by averaging $\langle u^2\rangle_u$.

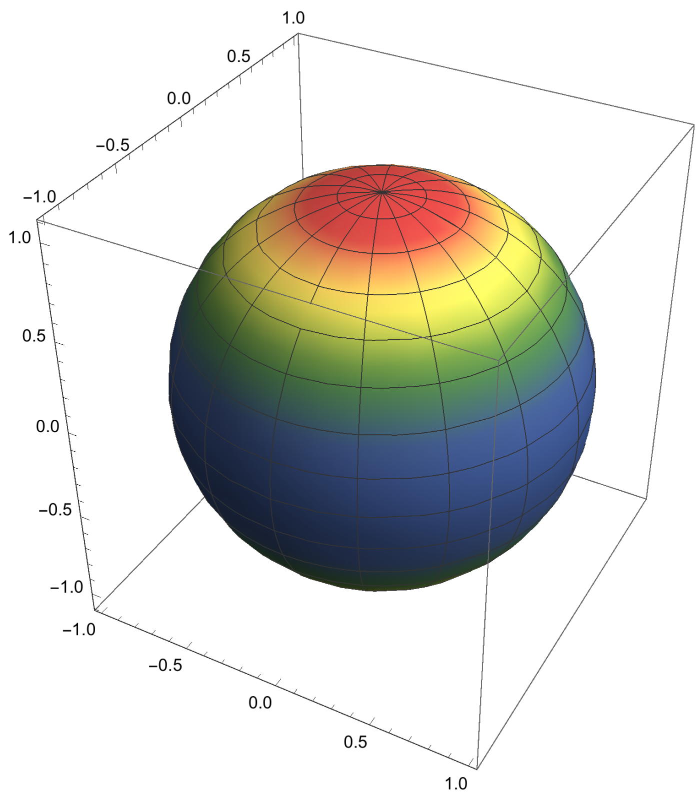

Figure 1: A visualization of a probability distribution on a sphere $\frac{d\mathscr{P}}{d\Omega} \propto (1+\cos^2\theta)$ given in problem 5. The colors are proportional to the probability per area or per solid angle (since $dA \propto d\Omega$ these are the same).

Two takeaway morals:

- Moral 1: $4\pi = 2 \cdot 2\pi$, the spherical measure factorizes,

$$\frac{\mathrm{d}\Omega}{4\pi} = P(\theta)\mathrm{d}\theta\, P(\phi)\mathrm{d}\phi = \left(\frac{1}{2}\sin\theta \mathrm{d}\theta\right)\left(\frac{1}{2\pi}\,\mathrm{d}\phi\right) \tag{12}$$

  So if you don't care abount the about the azimuthal direction and only want to know about the probability of θ, one has the measure $d\mathscr{P} = \frac{1}{2}\sin\theta \mathrm{d}\theta$.

- Moral 2: When integrating over the sphere it is almost always better to integrate over $u = \cos\theta$, so that $\mathrm{d}\Omega = \mathrm{d}u\,\mathrm{d}\phi$.

Problem 6. (Optional: Dont turn in) 2D World

Consider a mono-atomic ideal gas in a two dimensional world, so the velocities are labelled by $\boldsymbol{v} = (v_x, v_y)$.

(a) Use Jacobians to show

$$\mathrm{d}v_x \mathrm{d}v_y = \left\|\frac{\partial(v_x, v_y)}{\partial(v, \theta)}\right\| dv d\theta = \; v\,\mathrm{d}v\,\mathrm{d}\theta \tag{13}$$

where $v_x = v\cos\theta$ and $v_y = v\sin\theta$. It is understood that these expressions are meant to be integrated over. The double bars mean determinant and then absolute value of the Jacobian matrix[1]

$$\frac{\partial(v_x, v_y)}{\partial(v, \theta)} \equiv \begin{pmatrix} \frac{\partial v_x}{\partial v} & \frac{\partial v}{\partial \theta} \\ \frac{\partial v_y}{\partial v} & \frac{\partial y}{\partial \theta} \end{pmatrix} \tag{14}$$

(b) Write down the (normalized) Maxwell velocity distribution, $\mathrm{d}\mathscr{P}(v_x, v_y) = P(v_x, v_y)\mathrm{d}v_x \mathrm{d}v_y$, and, using the Jacobian of part (a) and an integral over θ, determine the (normalized) speed distribution

$$\mathrm{d}\mathscr{P}(v) = P(v)\mathrm{d}v \tag{15}$$

Describe the whole "Jacobian + integral" business with a picture.

You should find that all factors of π have canceled in your final expression for $P(v)$, and can optionally check that the distribution is correctly normalized.

(c) Compute $\left\langle \frac{1}{2}mv^2 \right\rangle$ using the speed distribution. Is your result consistent with the equipartition theorem in two dimensions? Explain.

(d) (Optional: Don't turn in) Consider the change of cooridnates from $x = r\cos\theta$ and $y = r\sin\theta$. Construct the Jacobian matrix

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \tag{16}$$

[1] Sometimes people use $\partial(v_x, v_y)/\partial(v, \theta)$ to mean the determinant of the Jacobian matrix. Our book uses this notation, as is described in appendix C.

The columns of the Jacobian form vectors

$$\boldsymbol{e}_r \equiv \frac{\partial x}{\partial r}\,\hat{\boldsymbol{\imath}} + \frac{\partial y}{\partial r}\,\hat{\boldsymbol{\jmath}} \tag{17}$$

$$\boldsymbol{e}_\theta \equiv \frac{\partial x}{\partial \theta}\,\hat{\boldsymbol{\imath}} + \frac{\partial y}{\partial \theta}\,\hat{\boldsymbol{\jmath}}. \tag{18}$$

Give a geometric interpretation of the vectors $\boldsymbol{e}_r dr$ and $\boldsymbol{e}_\theta d\theta$ and their respective lengths, $\mathrm{d}r$, and $r\mathrm{d}\theta$, by drawing the vectors with the at the position $(x, y) = (r\cos\theta, r\sin\theta)$.