Problem 1. Gaussian Integrals and moment generating functions

Consider a harmonic oscillator with potential energy $U(x) = \frac{1}{2}kx^2$. If the harmonic oscillator is subjected to an additional constant force f in the x direction its potential energy is $U(x, f) = \frac{1}{2}kx^2 - fx$. As we will see shortly, the probability to find the harmonic oscillator coordinate between x and $x + dx$ is

$$P(x)\mathrm{d}x = Ce^{-U(x,f)/k_BT}\mathrm{d}x\,. \tag{1}$$

This motivated people to study integrals of the form

$$I(f) \equiv C\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{1}{2}x^2+fx} \tag{2}$$

where f is a real number and C is a normalizing constant.

Consider integrals of the following form

$$I_n = \langle x^n \rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{1}{2}x^2} x^n \tag{3}$$

which come up a lot in this cours. There is a neat trick to evaluating evaluating the integrals I_n known as the moment generating function. Instead of considering I_n, consider

$$I(f) \equiv \left\langle e^{fx} \right\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{fx} \tag{4}$$

with f a fixed real number. Why would one ever want to do this? Well, if you expand the exponent

$$e^{fx} = 1 + fx + \frac{1}{2!}f^2x^2 \ldots \tag{5}$$

we can see that the Taylor series of $I(f)$ takes the form

$$I(f) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{1}{2}x^2}\left(1 + fx + \frac{1}{2!}f^2x^2 + \ldots\right) \tag{6}$$

$$= I_0 + I_1 f + I_2\frac{f^2}{2!} + I_3\frac{f^3}{3!} + \ldots \tag{7}$$

Thus knowing $I(f)$ amounts to knowing *all* I_n. Once simply needs to Taylor expand $I(f)$ in f and read off the coefficients in frount of f^n – that coefficient is $I_n/n!$. $\left\langle e^{fx} \right\rangle$ is known as the moment generating function since it "generates" integrals the moments $\langle x^n \rangle$. Now we only need to find $I(f)$

(a) (Optional) Show that

$$\boxed{I_0 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{1}{2}x^2} = 1} \tag{8}$$

Read the appendix in the book if you dont know how to do it.

(b) Show that

$$I(f) = e^{\frac{1}{2}f^2} \tag{9}$$

Hint: Complete the square

$$-\frac{1}{2}x^2 + fx = -\frac{1}{2}(x-f)^2 + \frac{1}{2}f^2 \tag{10}$$

and then do the integral by a change of variables.

(c) Show that

$$\langle x^2 \rangle = 1 \qquad \langle x^4 \rangle = 3 \qquad \langle x^6 \rangle = 15 \tag{11}$$

(d) For a distribution of the form

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \tag{12}$$

Determine $\langle x^2 \rangle$ and $\langle x^4 \rangle$. Do your results have the right dimensions?