2D-Debroglie Waves

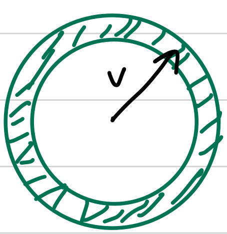
a) The probabity of velocity $\vec{v} = (v_x, v_y)$ is

$$d\mathcal{P}_{\vec{v}} \propto e^{-\frac{1}{2}mv^2/kT}\, dv_x\, dv_y$$

The normalization coefficient is found by requiring $\int d\mathcal{P}_{\vec{v}} = 1$

$$d\mathcal{P}_{\vec{v}} = \frac{1}{(2\pi\, kT/m)^{2/2}}\, e^{-\frac{1}{2}mv^2/kT}\, dv_x\, dv_y$$

For the probability of speed v, we dont care about the direction. We should sum over all configurations with speed (velocity magnitude) between v and $v+dv$



$$\boxed{d\mathcal{P}_v = \int_{shell} d\mathcal{P}_{\vec{v}} = \frac{m}{2\pi\, kT}\, e^{-\frac{1}{2}mv^2/kT}\, 2\pi v\, dv}$$

b) By the equipartion theorem

$$\left\langle \frac{1}{2} m v_x^2 \right\rangle = \left\langle \frac{1}{2} m v_y^2 \right\rangle = \frac{1}{2} kT$$

So

$$\frac{1}{2} m \langle v_x^2 + v_y^2 \rangle = \frac{1}{2} m \langle v^2 \rangle = kT$$

and thus

$$\boxed{v_{rms} = \sqrt{\langle v^2 \rangle} = \left(\frac{2kT}{m}\right)^{1/2}}$$

Numerically we insert Avagadro's # and use $N_A k = R$ so

$$V_{rms} = \left(\frac{2RT}{N_A m}\right)^{1/2} \sim \left(\frac{2.\ 8.32\ J/^\circ k\ \ 295^\circ K}{32\ g}\right)^{1/2}$$

$$\boxed{\simeq 391 \quad m/s}$$

We have taken room temperature $295^\circ K$, the mass of oxygen for 1 mole $\simeq 32g$. But anything reasonable is fine.

c) Maximizing $P(v) = C e^{-\frac{1}{2}mv^2/kT}\, v$ we have

$$P'(v) = C\ e^{-\frac{1}{2}mv^2/kT} - Ce^{-\frac{1}{2}mv^2/kT}\,\frac{mv}{kT}\cdot v = 0$$

$$= Ce^{-\frac{1}{2}mv^2/kT}\,(1 - mv^2/kT) = 0$$

So

$$\boxed{V_* = \sqrt{\frac{kT}{m}}}$$

And

$$\lambda_* = \frac{h}{mV_*} = \frac{h}{(m\,kT)^{1/2}}$$

Evaluating λ_* numerically we have

$$\boxed{\begin{aligned}\lambda_* &= \frac{hc}{(mc^2\,kT)^{1/2}} = \frac{1240\ eV\cdot nm}{(32\cdot 1GeV\ \ 0.025eV)^{1/2}} \\ &\simeq 0.04\ nm \simeq 0.4\ \text{\AA}\end{aligned}}$$

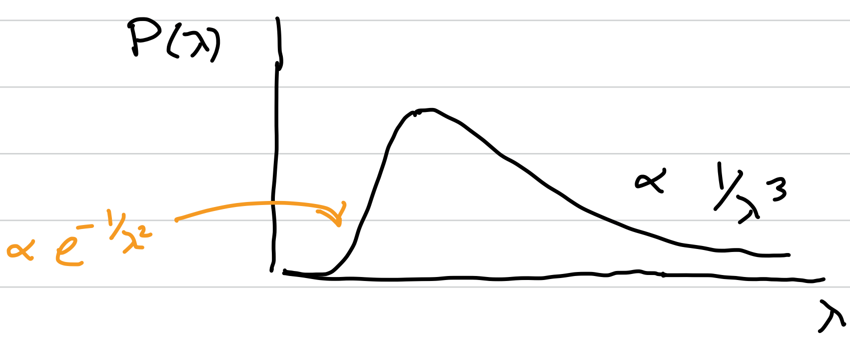
d) We make a change of variables $\lambda = \frac{h}{mv}$ $v = \frac{h}{m\lambda}$

Then $dv = \left|\frac{h}{m\lambda^2}\right| d\lambda$ when integrating

So $P(\lambda)\, d\lambda = P(v(\lambda)) \left|\frac{dv}{d\lambda}\right| d\lambda$

$$= \frac{m}{kT} e^{-\frac{1}{2} m (h/m\lambda)^2 / kT} \frac{h}{m\lambda} \frac{h}{m\lambda^2} d\lambda$$

$$= \left(\frac{\lambda^*}{\lambda}\right)^2 e^{-\frac{1}{2}(\lambda^*/\lambda)^2} \frac{d\lambda}{\lambda} \propto \frac{1}{u^3} e^{-\frac{1}{2} 1/u^2} du$$



Three State Paramagnet

a) $Z = e^{\beta\mu B} + 1 + e^{-\beta\mu B} = 1 + 2\cosh(\mu B\beta)$

$$\boxed{U_1 = -\frac{\partial \ln Z}{\partial \beta} = -\frac{2\sinh(\beta\mu B)}{(1 + 2\cosh(\beta\mu B))} \cdot \mu B}$$

☆ Note this is the mean energy per site $U_1 = U/N_A$

b) So for small x, $\cosh x = 1$ $\sinh x \simeq x$

$$\boxed{U_1 = \frac{-2\mu B}{1+2}\mu B = -\frac{2}{3}\frac{(\mu B)^2}{kT}}$$

So the specific heat per site is

$$C_{v1} = \frac{\partial U_1}{\partial T} = \frac{2}{3}\frac{(\mu B)^2}{kT^2} = k\left(\frac{\mu B}{kT}\right)^2 \cdot \frac{2}{3}$$

Multiplying by N_A to find the total specific heat we find

$$\boxed{C_V = \frac{2}{3}R\left(\frac{\mu B}{kT}\right)^2}$$

c) The ratio of Probabilities is

$$\frac{P_\downarrow}{P_\uparrow} = \frac{e^{-\varepsilon_\downarrow/kT}}{e^{-\varepsilon_\uparrow/kT}} = e^{-2\mu B/kT}$$

Requiring that this be 1/4 we find

$$e^{-2\mu B/kT} = \frac{1}{4} \implies \frac{2\mu B}{kT} = \ln 4$$

or

$$\boxed{kT = \frac{2\mu B}{\ln 4}}$$

So we have

$$e^{\mu B/kT} = \frac{1}{2}$$

So

$$\frac{P_0}{P_\downarrow} = \frac{1}{2} \quad \text{and} \quad \frac{P_\uparrow}{P_\downarrow} = \frac{1}{4} \quad \text{and} \quad P_\downarrow + P_\uparrow + P_0 = 1$$

So

$$\frac{1}{P_\downarrow} = 1 + \frac{1}{4} + \frac{1}{2} = \frac{7}{4} \implies \boxed{P_\downarrow = \frac{4}{7} \quad P_\uparrow = \frac{1}{7} \quad P_0 = \frac{2}{7}}$$

d) $$\frac{S}{k} = -\sum_i N_i \ln \frac{N_i}{N} = -N \sum_i \frac{N_i}{N} \ln\left(\frac{N_i}{N}\right)$$

$$= -N\left(\frac{4}{7}\ln\left(\frac{4}{7}\right) + \frac{2}{7}\ln\left(\frac{2}{7}\right) + \frac{1}{7}\ln\left(\frac{1}{7}\right)\right)$$

$$\boxed{\frac{S}{k} = N \cdot 0.9557}$$

e) $\langle u \rangle = P_{\uparrow} \varepsilon_{\uparrow} + P_{o} \varepsilon_{o} + P_{\downarrow} \varepsilon_{\downarrow}$

$$\left(\frac{\partial \langle u \rangle}{\partial B}\right)_{\text{fixed Prob}} = P_{\uparrow}\left(\frac{\partial \varepsilon_{\uparrow}}{\partial B}\right) + P_{o}\left(\frac{\partial \varepsilon_{o}}{\partial B}\right) + P_{\downarrow}\left(\frac{\partial \varepsilon_{\downarrow}}{\partial B}\right)$$

$$= \frac{4}{7} \cdot (-\mu) + 0 + \frac{1}{7}(\mu)$$

$$\boxed{\left(\frac{\partial \langle u \rangle}{\partial B}\right)_{\text{fixed}} = -\frac{3}{7}\mu}$$

Entropy Changes

a) $pV_L = N_L kT \implies \boxed{N_{He} = \frac{p\alpha V}{kT}}$

$pV_R = N_R kT \implies \boxed{N_{Ar} = \frac{p(1-\alpha)V}{kT}}$

b) $S = \text{const} + Nk\ln V + \frac{3}{2}Nk\ln E$

Now

$$\Delta S_{He} = N_{He}k\ln\frac{V_f}{V_i} = \frac{pV}{kT}\left[\alpha\ln\frac{1}{\alpha}\right]$$

$$\Delta S_{ar} = N_{Ar}k\ln\frac{V_f}{V_i} = \frac{pV}{kT}\left[(1-\alpha)\ln\frac{1}{1-\alpha}\right]$$

So

$$\boxed{\Delta S = \Delta S_{He} + \Delta S_{Ar} = \frac{pV}{kT}\left[-\alpha\ln\alpha - (1-\alpha)\ln(1-\alpha)\right]}$$

c) The number of Argon atoms would be twice larger as would ΔS_{Ar} leading to

$$\boxed{\Delta S = \frac{pV}{kT}\left[-\alpha\ln\alpha - 2(1-\alpha)\ln(1-\alpha)\right]}$$