Density @ Gravity

a) We have

$$d\mathcal{P} = C e^{-mgz/kT} \, dz \, dx \, dy \; e^{-p^2/2mT} \, dp_x \, dp_y \, dp_z$$

- We note that the probability factorizes

$$1 = \int d\mathcal{P} = C \int_0^\infty dz \, e^{-mgz/kT} \int_0^L dx\,dy \int e^{-p^2/2mT} \, d^3p$$

- Recognizing from the old result (in class)

$$\int d^3p \, e^{-p^2/2mT} \left(\frac{1}{2\pi m k_B T} \right)^{3/2} = 1$$

- We insert the factors

$$1 = C \, L^2 (2\pi m k T)^{3/2} \underbrace{\int_0^\infty dz \, e^{-mgz/kT}}_{= kT/mg} \underbrace{\int_0^L \frac{dx\,dy}{L^2}}_{=1} \underbrace{\int d^3p \, \frac{e^{-p^2/2mkT}}{(2\pi m k_B T)^{3/2}}}_{=1}$$

$$1 = C \, (2\pi m k T)^{3/2} \left(\frac{kT}{mg} \right)$$

So

$$\boxed{C = \frac{1}{(2\pi m k T)^{3/2}} \, \frac{1}{\ell} \qquad \text{with } \ell \equiv \frac{kT}{mg}}$$

• Then

$$d\mathcal{P} = \frac{e^{-z/\ell}}{\ell} dz \quad \frac{dx\,dy}{L^2} \quad \frac{e^{-p^2/2mkT}}{(2\pi mkT)^{3/2}} d^3p$$

The result factorizes

$$d\mathcal{P} = P(z)\,dz \quad P(x,y)\,dx\,dy \quad P(\vec{p})\,d^3p$$

• To find $P(z)$ we only need to sum over x, y, and $p_x, p_y\ p_z$ wich we don't want to know

$$d\mathcal{P}_z = \frac{e^{-z/\ell}}{\ell} dz \underbrace{\int \frac{dx\,dy}{L^2}}_{=1} \underbrace{\int d^3p \frac{e^{-p^2/2mkT}}{(2\pi mkT)^{3/2}}}_{=1}$$

So

$$\boxed{d\mathcal{P}_z = \overbrace{\frac{e^{-z/\ell}}{\ell}}^{P(z)} dz \qquad \ell \equiv \frac{kT}{mg}}$$

b) So

• $$\langle z \rangle = \int_0^\infty dz\, z\, P(z) = \int_0^\infty dz\, z \frac{e^{-z/\ell}}{\ell}$$

$$= \ell \int_0^\infty du\, u e^{-u} = \ell \cdot 1! = \ell$$

Esimating

$$\boxed{l = \frac{k_B T}{mg} = \frac{N_A k_B T}{N_A m g} = \frac{8.32\,J \times 300°K}{(28g)(9.8\,m/s^2)} = 9\,km}$$

The molar mass of air is 28 g.

c) • The density is proportional to $P(z)$

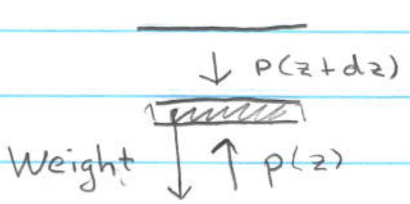
$$n(z) \propto P(z) \propto e^{-z/l}$$

So, $n(z) = C\, e^{-z/l}$, at $z = 0$ $n(0) = n_0$

So

$$\boxed{n(z) = n_0\, e^{-z/l}}$$

d) • Then



• the pressure difference needs to support the air's weight

$$\frac{dp}{dz} = -mg\, n(z)$$

So if $p = n(z)\, k_B T$

$$\frac{dp}{dz} = \frac{dn}{dz} k_B T$$

i.e.

$$\boxed{\frac{dn}{dz} = -\frac{mg}{kT}\, n}$$

For $\underline{n(z) = n_0 e^{-mgz/kT}}$ this is satisfied.

Phase Space Density

Then

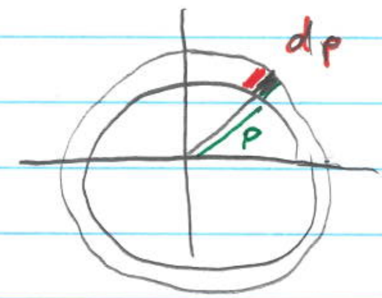
$$V_{ps} = \int_{[E,\, E+\delta E]} d^3q \, d^3p = L^3 V_p$$

momentum space volume

- The momenta are confined to a spherical shell

$$2mE < p^2 < 2m(E+\delta E)$$

- The volume of this shell is



$$V_p = 4\pi p^2 \, dp$$

Since $p^2 = 2mE$

$$2p\,dp = 2m\,dE$$

$$dp = \frac{m \cdot dE}{(2mE)^{1/2}} = (2mE)^{1/2} \frac{dE}{2E}$$

So

$$V_p = 4\pi (2mE)^{3/2} \frac{dE}{2E} \quad \text{and}$$

$$\boxed{V_{ps} = L^3 (2mE)^{3/2} \, 4\pi \, \frac{dE}{2E}}$$

Oscillator

- Then

a) $$\sum_n P_n = \sum_n C e^{-E_n/kT} = C \sum_n e^{-E_n/kT} = 1$$

$C = \frac{1}{Z}$

- So

$$Z = \sum_n e^{-E_n \beta}$$

$$Z = \sum_n e^{-\beta \hbar \omega_0 n}$$

we use $\frac{1}{1-u} = 1 + u + u^2 + \ldots\ldots$

b) $$Z = \sum_{n=0}^{\infty} u^n = \frac{1}{1-u} = \frac{1}{1 - e^{-\beta \hbar \omega_0}}$$

Then

$$\boxed{P_n = \frac{1}{Z} e^{-n\hbar\omega_0/kT} = e^{-n\hbar\omega_0\beta}\left(1 - e^{-\hbar\omega_0\beta}\right)}$$

- The plot is telling me that for low temperatures the oscillator is most likely in the ground state, while for high temperature is in its vibrational states

c) Then we see that when

$$\frac{\hbar\omega_0}{k_B T} \sim 1$$

for instance when $\hbar\omega_0 = kT$ 60% of atoms are in ground state and 40% are in vibrational states

- The atoms will begin to vibrate.

$$\hbar\omega_0 = \hbar c\, k_0 = 197\, eV\, nm\; 4400\, \frac{1}{cm}$$

$$h\omega_0 = 0.0866$$

- At room temperature $k_B T$ is $0.025\,eV$

So

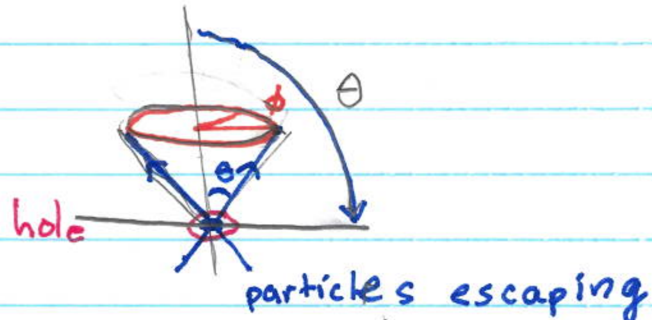
i.e. $\frac{1}{k_B} = \frac{300^\circ K}{0.025\,eV}$

$$T = \frac{0.086\,eV}{k_B} = \frac{0.086\,eV}{0.025\,eV} \cdot 300^\circ K$$

$$\boxed{T \simeq 1000^\circ K}$$

Escaping

a)



$$d\Phi = n\, P(v)\, dv \quad v\cos\theta \quad dv\, \frac{d\Omega}{4\pi}$$

see picture

- Integrating from $\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$, and v:

$$\Phi = \int_0^{\infty} n\, P(v)\, dv \cdot v \int_0^{\pi/2} \cos\theta \, \sin\theta \, d\theta \int_0^{2\pi} \frac{d\phi}{4\pi}$$

$$= n \langle v \rangle \left[-\frac{1}{2}\cos^2\theta \right]_0^{\pi/2} \cdot \frac{1}{2}$$

$$\Phi = n \langle v \rangle \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\boxed{\Phi = n \langle v \rangle \frac{1}{4}}.$$

b) Then we want

$\langle \cos\theta \rangle$

The, $d\Phi \propto \cos\theta \sin\theta \, d\theta$, and this serves as the probability distribution for escaping particles.

$$\boxed{d\mathcal{P} = C \cos\theta \sin\theta \, d\theta}$$

So

$$\boxed{\langle \cos\theta \rangle} = \frac{\int_0^{\pi/2} \cos\theta \times \cos\theta \sin\theta \, d\theta}{\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta} \qquad C = \frac{1}{\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta}$$

$$= \frac{-\frac{1}{3}\cos^3\theta \Big|_0^{\pi/2}}{-\frac{1}{2}\cos^2\theta \Big|_0^{\pi/2}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \boxed{2/3}$$

c) Then

$$\frac{dN}{dt} = -\underbrace{A}_{\text{area of hole}} \; \underbrace{\Phi}_{\text{number escaping per area per time}}$$

$$= -A \, \frac{1}{4} \underbrace{\frac{N}{V}}_{n} \underbrace{\left(\frac{8kT}{\pi m}\right)^{1/2}}_{\langle v \rangle} = -\frac{N}{\tau}$$

where $\tau = \frac{V}{A}\left(\frac{2\pi m}{kT}\right)^{1/2}$

So

$$\boxed{A = \frac{V}{\tau}\left(\frac{2\pi m}{kT}\right)^{1/2}}$$

- So using $N_A m_{air} = 28g$ $N_A k_b = 8.32\,J$

$V = 1L$ we have

$$A = \frac{(1L)}{3600s}\left(\frac{2\pi\ 28g}{8.32\frac{J}{°K} \times 300°K}\right)^{1/2}$$

$$= 2.33 \times 10^{-9}\, m^2$$

microns

- The size is $A = L^2$, $L = 48\,\mu m$

2D World

a) $\left|\dfrac{\partial(V_x, V_y)}{\partial(V,\theta)}\right| = \begin{pmatrix} \dfrac{\partial V_x}{\partial V} & \dfrac{\partial V_x}{\partial \theta} \\ \dfrac{\partial V_y}{\partial V} & \dfrac{\partial V_y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -V\sin\theta \\ \sin\theta & V\cos\theta \end{pmatrix}$

- The determinant is

$$V\cos^2\theta + V\sin^2\theta = V$$

i.e.

$$\left|\frac{\partial(v_x, v_y)}{\partial(v,\theta)}\right| dv\, d\theta = v\, dv\, d\theta$$

b) $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \cos\theta & -r\cos\theta \\ \sin\theta & r\sin\theta \end{pmatrix}$

- $\vec{e}_r\, dr = dr\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$, $|\vec{e}_r\, dr| = dr$
- $\vec{e}_\theta\, d\theta = d\theta\,(-r\cos\theta\,\hat{i} + r\sin\,\hat{j})$, $|\vec{e}_\theta\, d\theta| = r\, d\theta$

$\vec{e}_r \cdot \vec{e}_\theta = 0$

- $\vec{e}_r \cdot dr$ – is the displacement $d\vec{r}$ caused by increasing r to $r + dr$ (see θ example)

- $\vec{e}_\theta \, d\theta$ is the displacement caused by increasing θ to $\theta + d\theta$, for example

$$d\vec{r} = \vec{e}_\theta \, d\theta = \frac{\partial \vec{r}}{\partial \theta} \, d\theta$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

d) • We have

$$d\mathcal{P} = C \, e^{-mv_x^2/2kT} \, dv_x \, e^{-mv_y^2/2kT} \, dv_y$$

- The constant comes from

$$\int d\mathcal{P} = 1, \quad \text{we know} \quad \int dx \, \frac{e^{-x^2/2\sigma^2}}{(2\pi\sigma^2)^{1/2}} = \underline{1}$$

- Look at $e^{-V_x^2/2\sigma^2}$ with $\sigma^2 \equiv \frac{kT}{m}$, So

$$\boxed{d\mathcal{P}_{\vec{v}} = \frac{e^{-V_x^2/2\sigma^2}}{(2\pi\sigma^2)^{1/2}} \, dV_x \, \frac{e^{-v_y^2/2\sigma^2}}{(2\pi\sigma^2)^{1/2}} \, dV_y}$$

So since $dv_x \, dv_y = v \, dv \, d\theta$ and

$$\boxed{d\mathcal{P}_{v,\theta} = \frac{e^{-v^2/2\sigma^2}}{2\pi\sigma^2} \, v \, dv \, d\theta}$$

distribution of speed and angle

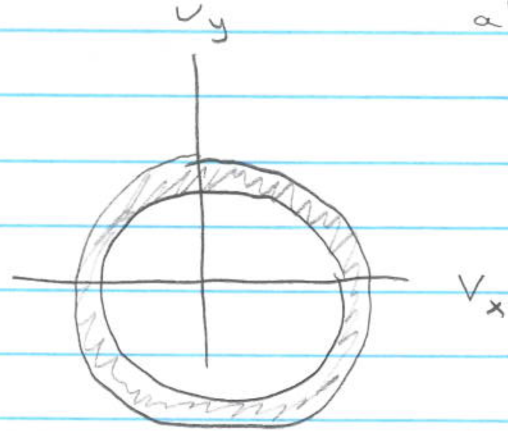
Then integrating over $\theta \in [0, 2\pi]$:

$$d\mathcal{P}_v = \int_{\theta \in [0,2\pi]} d\mathcal{P}_{v,\theta} = \frac{e^{-v^2/2\sigma^2}}{2\pi\sigma^2} \, v \, dv \int_0^{2\pi} d\theta$$

$$\boxed{d\mathcal{P}_v = \frac{e^{-v^2/2\sigma^2}}{\sigma^2} \, v \, dv}$$

distribution of speed v

We are integrating over a shell of configurations, all of which have speed between $v + dv$



• (e) So

$$\left\langle \frac{1}{2} m v^2 \right\rangle = \frac{1}{2} m \int_0^\infty \frac{dv}{\sigma^2} e^{-v^2/2\sigma^2} \, v \cdot v^2$$

• Changing vars, writing $u = v/\sigma$ we have

$$\left\langle \frac{1}{2} m v^2 \right\rangle = \frac{1}{2} m \sigma^2 \int_0^\infty \frac{dv}{\sigma} e^{-v^2/2\sigma^2} \frac{v}{\sigma} \frac{v^3}{\sigma^2}$$

$$= \frac{1}{2} m \sigma^2 \int_0^\infty du \, e^{-u^2/2} u^3$$

• This is very nearly the Γ fcn, let $x = \frac{u^2}{2}$
the $dx = u \, du$

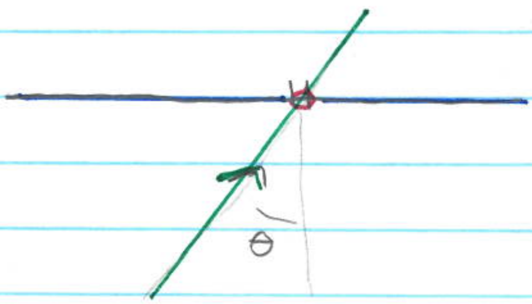
$$\left\langle \frac{1}{2} m v^2 \right\rangle = \frac{1}{2} m \sigma^2 \underbrace{\int_0^\infty dx \, e^{-x} \cdot 2x}_{2 \cdot 1!}$$

$$= m \sigma^2$$

$$= \cancel{m} \frac{k_B T}{\cancel{m}} = k_B T = 2 \times \frac{1}{2} k_B T$$

• This is the the equipartition theorem in 2D
The molecules have 2 dof, motion in x and y.

f)



We have simply (see lecture for more discussion)

$$\boxed{d\Phi = n P(v) \; v\cos\theta \; \frac{v\,dv\,d\theta}{2\pi}}$$

- Compare 2D and 3D

$$dv_x\, dv_y = v\, dv\, d\theta \qquad (2d)$$

$$dv_x\, dv_y\, dv_z = v^2 dv \; \sin\theta\, d\theta\, d\phi \qquad (3D)$$

- For particles uniformly distributed ion the circle, we have

$$d\mathcal{P} = \frac{d\theta}{2\pi} \qquad (2D)$$

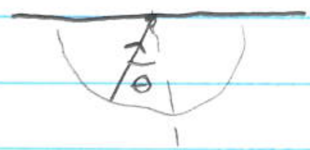
While for the sphere, we have

$$d\mathcal{P} = \frac{d\Omega}{4\pi} \qquad (3D)$$

- S_0 in 2D

$$d\Phi = n\,P(v)\,v\cos\theta\,\frac{dv\,d\theta}{2\pi}$$

Then total flux is



$$\left[\ \Phi = \int_v \int_{\theta\in[-\frac{\pi}{2},\pi/2]} n\,P(v)\,v\cos\theta\,dv\,\frac{d\theta}{2\pi}\right.$$

$$= n\langle v\rangle \times \int_{-\pi/2}^{\pi/2} \cos\theta\,\frac{d\theta}{2\pi} = \boxed{\frac{1}{\pi}\,n\langle v\rangle}$$

- Similarly see lecture!

$$\left[\ p = \int_v \int_{\theta \text{ in } [-\frac{\pi}{2},\frac{\pi}{2}]} \overbrace{(2mv\cos\theta)}^{\Delta p_x = \text{impulse}}\ \overbrace{n\,P(v)\,v\cos\theta\,\frac{dv\,d\theta}{2\pi}}^{d\Phi\ \leftarrow \text{flux}}\right.$$

$$= 2mn\int_0^\infty P(v)\,v^2\,dv \overbrace{\int_{-\pi/2}^{\pi/2} \cos^2\theta\,\frac{d\theta}{2\pi}}$$

$$= 2mn\langle v^2\rangle \cdot \frac{1}{4}$$

$$\boxed{= \frac{1}{2}\,mn\langle v^2\rangle}$$