Problem 1.   Central Limit Theorem and Random Walk

In a random walk, a collegiate drunkard starts at the origin and takes a step of size a, to the right with probability p and to the left with probability $1 - p$.

(a) Take $p = 1/2$, i.e. equal probability of right and left steps. Determine the probability of the drunkard having position X, i.e. $P(X)$, after three steps. Plot $P(X)$ where X can be one of $X = 0, \pm 1, \pm 2, \pm 3$. Note how your graph begins to approach a Gaussian after just three steps.

(b) Now keep p general. What is the mean and variance variance in the drunkard's position X after one step, and after two steps?

(c) After n steps (with $n \gg 1$) find his mean position $\langle X \rangle$, and the std. deviation in his position $\sigma_X = \sqrt{\langle \delta X^2 \rangle}$. Check your result by comparing with the figure below
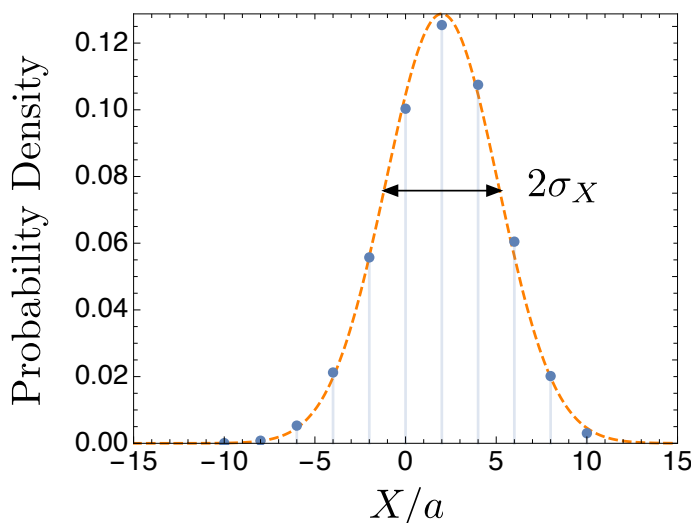

Figure 1:   Probability of our drunkard having position X after $n = 10$ steps (the blue points). Of course after 10 steps the drunkard will be between $-10 \dots 10$, and it is easy to show that he will be only at the even sites, i.e. $-10, -8, -6, \dots 10$. For $p = 0.6$, I find $\langle X \rangle = 2.0$. Twice the std deviation, $2\sigma_X$, is shown in the figure and is about six in this case. The orange curve is a gaussian (a.k.a the "bell-shaped" curve) approximation discussed in class and approximately agrees with the points – this is the central limit theorem. Recall that the central limit theorm says that if the number of steps n is large, the probability of X (a sum of n independent events) is approximately $P(x)\,dX \propto \exp(-(X - \langle X \rangle)^2/2\sigma_X^2)$. Evidently the gaussian approximation works well already for $n = 10$.

   Hint: X is a sum N independent events x_i where $x_i = \pm a$. Use results from class on the probability distribution of a *sum* of independent events.

(d) (Optional. Don't turn in) If p is very nearly $\frac{1}{2}$, say $p = 0.5001$, determine how many steps it will take before the mean value $\langle X \rangle$ is definitely different from zero. By

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"definitely" I mean that $\langle X \rangle$ is "more than two sigma" away from zero, $\langle X \rangle > 2\sigma_X$. If $p = \frac{1}{2} + \epsilon$ (with ϵ tiny), you should find (approximately) that

$$N_{\text{steps}} \simeq \frac{1}{\epsilon^2} \tag{1}$$

up to corrections of order ϵ. Here $p = \frac{1}{2} + \epsilon$ with $\epsilon = 0.0001$, how does the result scale with ϵ, e.g. if I where two half ϵ how would the number of required steps change?

Problem 2.   Combinatorics and The Stirling Approximation

(a) Consider a chain of 6×10^{23} atoms, laid out in a row. The atoms can be in two states, a ground state, and an excited state. $1/3$ of them are in the excited states. Using the Stirling approximation, show that the number of configurations with this number of excited states is approximately

$$\Omega = 10^{1.67 \times 10^{23}} \tag{2}$$

For instance, if the number of atoms is five, and the number of excited atoms (shown by the black circles) is 2, then two possible configurations are shown below.
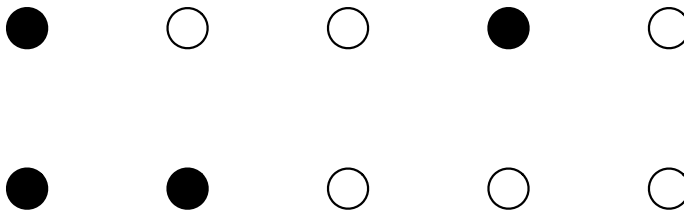
(b) Now work analytically. Assume there are N atoms laid out in a row. Assume that N_1 of them are in the ground state, and N_2 are in the excited state, with $N_1 + N_2 = N$. Show that the log of the number of configurations

$$\ln \Omega = -\sum_{i=1,2} N_i \ln(N_i/N) \tag{3}$$

$$= N \sum_{i=1,2} -P_i \ln P_i \tag{4}$$

In the last step we have recognized that the $N10/N$ is the probability that an atom will be in the ground state, and N_2/N is the probability that an atom will be in the excited state. If you get stuck look in the course notes.

Discussion: The log of the number of configurations $\ln \Omega$ is known as the entropy of the chain. Then entropy per site, i.e. $\ln \Omega/N$, is given by

$$\frac{\ln \Omega}{N} = \sum_i -P_i \ln P_i \tag{5}$$

which is known as the Shannon formula for the entropy of a probability distribution. The importance of these things will become clearer as the course progresses.

Problem 3. Parametrizing the EOS

The pressure as a function of temperature an volume, $p(T, V)$, or equivalently the volume as a function of temperature and pressure $V(T, p)$, is an important physical observable. Recall that its changes are parameterized by the measurables β_p and κ_T. Consider an ideal gas at temperature T with N particles

(a) Explain the physical meaning of the thermal expansion coefficient β_p and isothermal compressibility κ_T, and compute them for an ideal gas.

The first items only involved the EOS, $p(T, V)$. The next items also involves the energetics, so the specific heat and adiabatic index play a role. Assume that $U = c_0 T$ with c_0 a constant

(b) What is c_0 for a mono-atomic ideal gas? What about a diatomic ideal gas?

(c) Write down the specific heats C_p and C_v for a mono-atomic and diatomic gas.

(d) In class we said that for a general substance (and not necessarily an ideal gas) the specific heats C_p and C_V are are related by a formula which we will prove in full generality only later:

$$C_p = C_V + \frac{VT\beta_p^2}{\kappa_T} \, . \tag{6}$$

For an ideal gas (and one mole of substance) we have the following special case of this formula:

$$C_p = C_V + Nk_B \, . \tag{7}$$

Or

$$C_p^{1\text{ml}} = C_V^{1\text{ml}} + R \tag{8}$$

Show that Eq. (8) follows from Eq. (6) together with the results from parts (a).

(e) The *adiabatic* compressibility κ_S is the defined by[1]

$$\kappa_S \equiv \frac{-1}{V}\left(\frac{\partial V}{\partial p}\right)_{adiab} \tag{10}$$

This "*adiab*" means that as we change the pressure, the volume and temperature change, so that no heat flows, $đQ = 0$. Show for an ideal gas that

$$\kappa_S = \frac{\kappa_T}{\gamma} \tag{11}$$

We will show later that this result is not limited to an ideal gas.

Hint: You will need to recall that in an adiabatic change of pressure and volume, we have $pV^\gamma = $ const for an ideal gas.

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[1]The suffix S means adiabatic, $đQ = 0$. We will see that $đQ$ is related to the change in entropy S, $dS = đQ/T$. So S suffix means at fixed entropy.

$$\kappa_S \equiv \frac{-1}{V}\left(\frac{\partial V}{\partial p}\right)_S \equiv \frac{-1}{V}\left(\frac{\partial V}{\partial p}\right)_{adiab} \tag{9}$$

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(f) As discussed in class, the speed of sound is related to the compressibility[2]

$$c_s = \sqrt{\frac{B_S}{\rho}} \tag{12}$$

where the bulk modulus

$$B_S \equiv -V \left(\frac{\partial p}{\partial V} \right)_{adiab} \equiv \frac{1}{\kappa_S} \tag{13}$$

serves as a kind of spring constant for the material, and ρ is the mass per volume. Air is made of diatomic molecules, primarily (78%) diatomic nitrogen N_2. Determine the speed of sound of N_2 gas at $20^o C$ treating using only the ideal gas constant R and the fact that a nitrogen atom consists of 7 protons and 7 neutrons. Compare with the nominal value for the speed of sound in air. You should find favorable agreement.

(g) The frequency of the tuning note ($A440$) in the orchestra is 440 Hz. Explain qualitatively why it is the adiabatic compressibility κ_S, and not the isothermal one κ_T which is relevant for the speed of sound, by comparing the time scales of oscillation with a typical time scale for heat conduction.

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[2]I will not derive this. A good derivation at your level is given here. Unfortunately, this derivation uses the symbol κ for B_S, which for us (and indeed almost everyone) is $1/\kappa_S$!
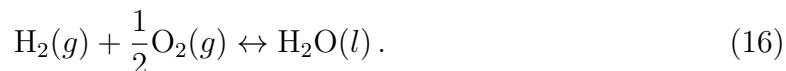
Problem 4.   Energy released during an combustion

(a) (Optional, but do it for yourself) Repeat the argument presented in class for the equation

$$dH = \dbar Q_{\text{in}} + V \, dp \qquad (14)$$

where $H = U + pV$ represents the enthalpy. Enthalpy is particularly useful when the pressure is constant then, leading to

$$dH = \dbar Q_{\text{in}} \qquad (15)$$

(b) Consider the combustion of Hydrogen gas:

$$H_2(g) + \frac{1}{2}O_2(g) \leftrightarrow H_2O(l). \qquad (16)$$

resulting in the formation of liquid water vapor. Tables of enthalpies for reactions are available in many books.

  (i) Look up in the accompanying data table the enthalpy of the products and reactants at 298 °K and standard pressure[3]. Determine the change in enthalpy, $\Delta H^{\ominus}$, for each mole of H_2O produced.

  (ii) Consider the reactants as ideal gasses, and treat the liquid product H_2O as having negligible volume compared to the gasses. Calculate the heat released during the combustion and the change in energy, $\Delta U^{\ominus} = U_{\text{final}} - U_{\text{initial}}$, per mole.

(c) Consider the reaction at

$$H(g) + H(g) \rightarrow H_2(g) \qquad (17)$$
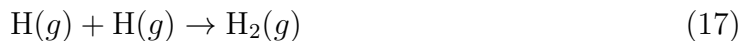
at NTP, which is accompanied by a large release of heat. Determine the energy of a bond between the two atoms in H_2 in eV.

Hint: First use the enthalpy data to determine the enthalpy change and heat out flow during the reaction. Use this to find ΔU for the reaction, treating all components as ideal gasses. The energy of a single H_2 molecule is its kinetic energy (translational and rotational) and its potential energies:

$$E_{H_2} = \text{KineticE} + \text{PotentialE} = \text{KineticE} - \Delta \qquad (18)$$

Here PE $= -\Delta$ is the binding energy (i.e. the bond energy) of the two atoms. (The negative sign indicates that the energy is lower when the two atoms are bound compared to when they are unbound. Δ is a positive value and is what we are trying to find.) The total energy U is the sum of kinetic and potential energies of the atoms. Use what we know about the kinetic energy of ideal gasses (both the mono-atomic and diatomic cases) to relate ΔU for one mol of H_2 produced to Δ.

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[3]This temperature and pressure is so-called Normal Temperature and Pressure (NTP) and denoted with a circle, i.e. $T^{\ominus}$, $p^{\ominus}$ and $H^{\ominus}$ denote the temperature, pressure, and enthalpy at NTP.