Physics 306: Thermal Physics

Final Exam

Stony Brook University

Spring 2022

General Instructions:

You may use one page (front and back) of handwritten notes and a calculator. Graphing calculators are allowed. **No other materials may be used**.

Problem 3.   An engine cycle

One mole of an ideal mono-atomic gas operates in an engine cycle shown below. Here the givens are the temperatures, $T_a = 400\,K$ and $T_c = 250\,K$, and the pressure, $P_c = 1\,\text{bar}$. These values are recorded in the table below.
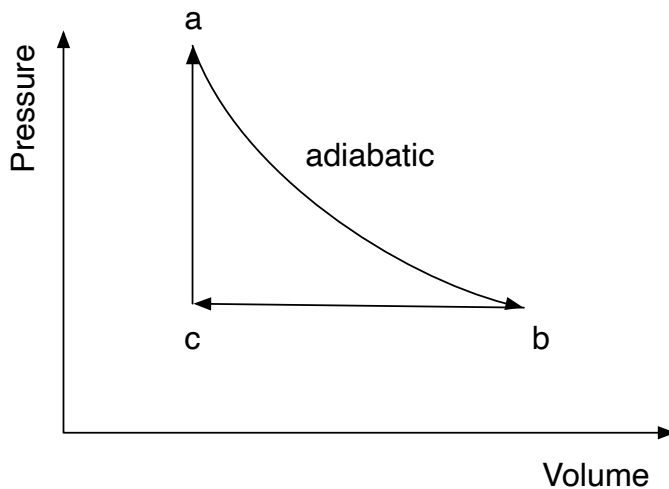
Using the ideal gas law $PV = nRT$, it is straightforward to show that the volumes at a and c are $V_a = V_c = 20.8\,\text{L}$, and that the pressure at a is $P_a = 1.6\,\text{bar}$, as recorded in the table below.

(a) Find the volume and temperature at b, completing the table shown below.

state	pressure	volume	temperature
a	1.6 bar	20.8 L	400 K
b	1.0 bar	?	?
c	1.0 bar	20.8 L	250 K

(b) Find the work done by the gas per cycle.

(c) Find the efficiency of the engine.

(d) Find the change in entropy of the gas from c to a.

(e) Using part (d), find the change in entropy from a to b and from b to c

Problem 3

a) We have   (10 points)

$$P_a V_a^\gamma = P_b V_b^\gamma \quad \text{with} \quad \gamma = 5/3 \text{ for MAIG}$$

So

$$V_b = V_a \left(\frac{P_a}{P_b}\right)^{1/\gamma} = 1.0L \left(\frac{1.6b}{1.0b}\right)^{1/\gamma} = 27.6L$$

Then

$$P_b V_b = RT_b \implies T_b = \frac{P_b V_b}{R_b} = \frac{1.0b \cdot 27.6L}{8.32 \frac{J}{K}}$$

Note $1 bL = 100 J$ so   we find $T_b = 331 °K$

b) Now   (parts b and c   10 points)

① $W_{ba} \equiv W_{b\leftarrow a}$ $\Delta E_{ba} = \cancel{Q_{ba}} + W_{ba}$

$$\Delta E_{ba} = C_V (T_b - T_a) = W_{ba} = -855 \text{ J}$$

② $W_{cb} = W_{c\leftarrow b}$ $C_V = \frac{3}{2} R = 12.48 \text{ J/}°K$

$$\Delta E_{cb} = C_V (T_c - T_b) = -1016 \text{ J} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \quad Q_{cb} = \Delta E_{cb} - W_{cb}$$

$$W_{cb} = -p(V_c - V_b) = 678 \text{ J} \qquad = -1694J$$

③ $W_{ac} = W_{a \leftarrow c}$

$$\Delta E_{ac} = Q_{ac} + \cancel{W_{ac}}^{0}$$

$$C_v (T_a - T_c) = Q_{ac} = 1872 \text{ J}$$

So

$$\eta = \frac{|W_{net}|}{Q_{in}} = \frac{-(W_{ba} + W_{cb})}{Q_{ac}} =$$

$$= \frac{855 \text{ J} - 678 \text{ J}}{1872 \text{ J}} = 0.095$$

Note

$$|W_{net}| = 177 \text{ J} \qquad Q_{net} = Q_{ac} + Q_{cb} = 178 \text{ J}$$

these should be same

c) $$\Delta S_{ac} = \int \frac{dQ}{T} = \int_c^a \frac{C_v \, dT}{T} = C_v \ln \frac{T_a}{T_c} = 5.87 \frac{\text{J}}{^\circ K}$$

d) $\Delta S_{ba} = 0$ since $a \rightarrow b$ is adiabatic. $\Delta S_{TOT} = 0$ since it is a closed loop so ΔS_{cb} is $-\Delta S_{ac} = -5.87 \frac{\text{J}}{^\circ K}$

→ parts c & d   10 points

③ $\quad W_{ac} = W_{a \leftarrow c}$

$$\Delta E_{ac} = Q_{ac} + W\!\!\!\!/_{ac}^{\,\,0}$$

$$C_v (T_a - T_c) = Q_{ac} = 1872 \text{ J}$$

So

$$\eta = \frac{|W_{net}|}{Q_{in}} = \frac{-(W_{ba} + W_{cb})}{Q_{ac}} =$$

$$= \frac{855\text{ J} - 678\text{ J}}{1872\text{ J}} = 0.095$$

Note

$$|W_{net}| = 177 \text{ J} \qquad\qquad Q_{net} = Q_{ac} + Q_{cb} = 178 \text{ J}$$

these should be same

c) $\quad \Delta S_{ac} = \int_c^a \frac{dQ}{T} = \int_c^a C_v \frac{dT}{T} = C_v \ln \frac{T_a}{T_c} = 5.87 \frac{\text{J}}{{}^\circ \text{K}}$

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$\rightsquigarrow$ parts c & d   10 points

Physics 306: Thermal Physics
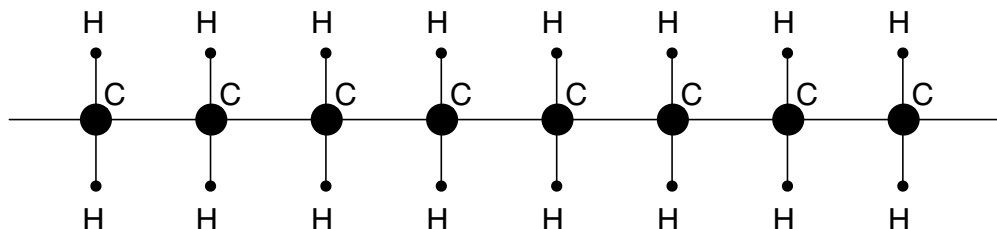
Second Midterm

Stony Brook University

Spring 2023

General Instructions:

You may use one page (front and back) of handwritten notes and a calculator. Graphing calculators are allowed. **No other materials may be used**.

Problem 1.   A Chain of Hydrogen Atoms

Consider a long chain of hydrogen atoms of mass m connected by a chain of N Carbon atoms. The Carbon atoms can be considered to be fixed in space[1]. Each hydrogen atom is harmonically bound to the Carbon with a spring constant k_0, and they are independent of each other[2]

6 pts (a) Derive the partition function of single hydrogen atom. Be explicit and explain your steps.

5 pts (b) Determine the mean vibrational energy and the variance in the vibrational energy of the chain of hydrogen atoms. Be explicit about your algebraic steps.

5 pts (c) Determine a Taylor series for the energy of part (b) in the high temperature limit, including both the leading term and the first correction. Use your series to answer the following: in the high temperature limit, if hydrogen is replaced by deuterium (which consists of a proton, a neutron, and an electron) what is the approximate difference in vibrational energies of the two system, $\Delta E = E_D - E_H$, at high temperatures.

Problem 2.   Entropy change in the mixing of hot and cold gasses

Consider two mono-atomic ideal gasses, Helium and Argon, separated by a divider which partitions a container of volume V into two equal parts. There are N_1 Helium atoms on the left of the divider, and N_2 Argon atoms on the right of the divider. The Helium atoms are initially at a temperature of T_1, while the Argon atoms are initially at a temperature of T_2. After the dividing wall is removed, the two gasses mix and ultimately equilibrate.

4 pts (a) Determine the final temperature of the system.

4 pts (b) Determine the change in entropy of the system resulting from the mixing process.

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[1]Carbon is twelve times heavier than hydrogen justifying this approximation.
[2]Recall that the resonant frequency is $\sqrt{k_0/m}$.

A chain of H-atoms

a)
$$Z = \sum_{states} e^{-\beta \varepsilon_s} = \sum_{n=0}^{\infty} e^{-\beta \hbar \omega_0 n} = \sum_{n=0}^{\infty} (e^{-\beta \hbar \omega})^n$$

$\varepsilon_n = n \hbar \omega_0$ for a harmonic oscillator

$$Z = \frac{1}{1 - e^{-\hbar \omega_0 \beta}}$$

we used the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

b) So $\ln Z = -\ln(1 - e^{-\beta \hbar \omega})$ and

$$\langle \varepsilon \rangle = -\frac{\partial}{\partial \beta} \ln Z = +\frac{\partial}{\partial \beta} \ln(1 - e^{-\beta \hbar \omega})$$

$$= \frac{1}{1 - e^{-\beta \hbar \omega}} (-e^{-\beta \hbar \omega})(-\hbar \omega)$$

$$\boxed{\langle \varepsilon \rangle = \frac{\hbar \omega \, e^{-\beta \hbar \omega}}{1 - e^{-\beta \hbar \omega}} = \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1}}$$

Then

$$\langle (\delta \varepsilon)^2 \rangle = -\frac{\partial \langle \varepsilon \rangle}{\partial \beta} = +\frac{\partial^2 \ln Z}{\partial \beta^2}$$

$$\boxed{\langle \delta \varepsilon^2 \rangle = \frac{\hbar \omega \, e^{\beta \hbar \omega}}{(e^{\beta \hbar \omega} - 1)^2}}$$

We have derived here the partition fcn and energy for a single atom. For N atoms

$$Z_N = Z_1^N \qquad \longleftarrow \quad \text{the atoms are distinguishable}$$

Each atom is at one site

$$\ln Z_N = N \ln Z_1, \qquad \text{on the chain.}$$

So we have that

$$\langle E \rangle = - \frac{\partial \ln Z_N}{\partial \beta} = N \langle \varepsilon \rangle$$

$$\langle \delta E^2 \rangle = + \frac{\partial^2 \ln Z_N}{\partial \beta^2} = N \langle \delta \varepsilon^2 \rangle$$

c) At high temperature $\beta \hbar \omega \ll 1$

$$\langle \varepsilon \rangle \simeq \frac{\hbar \omega}{\hbar \omega \beta \left(1 + \frac{\hbar \omega \beta}{2}\right)} \qquad e^{\beta \hbar \omega} = 1 + \hbar \omega \beta + \frac{(\hbar \omega \beta)^2}{2} + \; \cdots$$

$$\langle \varepsilon \rangle \simeq \frac{1}{\beta}\left(1 - \frac{\hbar \omega \beta}{2}\right)$$

$$\langle \varepsilon \rangle \simeq kT - \frac{\hbar \omega}{2}$$

So if we replace Hydrogen with deuterium

$$\Delta E = N\left(kT - \frac{\hbar \omega_D}{2}\right) - N\left(kT - \frac{\hbar \omega_H}{2}\right)$$

$$\Delta E = \frac{N\hbar}{2}(\omega_H - \omega_D)$$

where

$$\omega_D = \sqrt{\frac{k_o}{m_D}}$$

$$\omega_H = \sqrt{\frac{k_o}{m_H}}$$

Entropy change in the mixing of Hot & Cold Gasses

(a) Energy is conserved and the gasses ultimately have the same temperature.

$$E_1 + E_2 = E$$

$$\frac{3}{2} N_1 k T_1 + \frac{3}{2} N_2 k T_2 = \frac{3}{2}(N_1 + N_2) k T$$

$$N_1 T_1 + N_2 T_2 = T$$
$$\overline{(N_1 + N_2)}$$

b) The entropy of Each Gas is

$$S = Nk \ln V + \frac{3}{2} Nk \ln E + cast$$

$$\left.\begin{array}{c}\\ \end{array}\right\} E \propto T$$

$$S = Nk \ln V + \frac{3}{2} Nk \ln T + const$$

↖ a different constant

So

$$\Delta S = Nk \ln \frac{V_f}{V_i} + \frac{3}{2} Nk \ln T_f / T_i$$

Here we have two gasses:

$$\frac{V_f}{V_i} = \frac{V}{V/2} = 2 \qquad \frac{T_f}{T_i} = \frac{T}{T_1} \quad \text{for gas \#1}$$

$$\frac{T_f}{T_i} = \frac{T}{T_2} \quad \text{for gas \#2}$$

So

$$\Delta S = \Delta S_1 + \Delta S_2$$

$$\Delta S = (N_1 + N_2) k \ln 2 \quad + \quad \frac{3}{2} N_1 k \ln \frac{T}{T_1} + \frac{3}{2} N_2 k \ln \frac{T}{T_2}$$

See also homework for alternate solution
which applies to diatomic and non-ideal gasses

Physics 306: Thermal Physics

First Midterm

Stony Brook University

Spring 2023

General Instructions:

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Problem 1. Atoms 123

Twelve moles of atoms are placed in row. Each atom can be in one of three states: state 1, state 2, or state 3. Consider selecting two moles of atoms to be in state 1, four moles of atoms to be in 2, and, six moles of atoms to be in state 3.

(4 points)   (a) Explain why there are

$$W = \frac{N!}{N_1! N_2! N_3!}, \tag{1}$$

ways we can make this selection. Here $N = N_1 + N_2 + N_3$ is the total number atoms, and N_1, N_2 and N_3 are the number of atoms in each corresponding state, e.g. $N_1 = 2\,\mathrm{moles}$.

(4 points)   (b) Evaluate W numerically.

Problem 2. Energy distribution of a 2d world

Consider a mono-atomic ideal gas of mass m in two spatial dimensions at temperature T. The number of atoms in the box is N and the box has area L^2.

(4 points)   (a) What is average energy and the rms velocity of the atoms in the gas?

(b) Find the normalized probability distribution for speeds between v and $v + \mathrm{d}v$

(4 points)
$$\mathrm{d}\mathscr{P} = P(v)\mathrm{d}v, \tag{2}$$

and determine the corresponding energy distribution

(4 points)
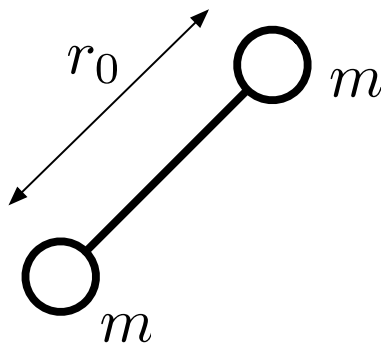$$\mathrm{d}\mathscr{P} = P(\epsilon)\mathrm{d}\epsilon. \tag{3}$$

(4 points)   (c) What is the variance in the energy $\langle \delta\epsilon^2 \rangle$?

Problem 3. Angular velocity of oxygen in air

(a) Consider an ideal diatomic molecular gas at temperature T. The atoms have mass m and are separated by bond length of r_0. Determine the root-mean-square angular velocity, $\omega_{\mathrm{rms}} = \sqrt{\langle \omega^2 \rangle}$.

(5 points)

(5 points)

(b) Estimate ω_{rms} numerically in Hz for diatomic Oxygen O_2 in this room. An Oxygen atom consists of eight protons and eight neutrons.

Problem 1

$$1\ 1\ 1\ 1\ 1\ 1 \quad , \quad 2\ 2\ 2 \quad , \quad 3\ 3\ 3\ 3$$

$$\underbrace{}_{N_1} \qquad \underbrace{}_{N_2} \quad \underbrace{}_{N_3}$$

a)

There are $N!$ rearrangements of the objects. $N_1!$ of these just rearranges the "ones" amongst themselves; $N_2!$ of these just rearrange the "twos" amongst themselves; and ditto for $N_3!$

So

$$W = \frac{N!}{N_1!\ N_2!\ N_3!}$$

b)

$$\ln W = \ln N! - \ln N_1! - \ln N_2! - \ln N_3!$$

$$= N \ln N - N - N_1 \ln N_1 + N_1 - N_2 \ln N_2 + N_2$$

$$- N_3 \ln N_3 + N_3$$

Now note $N = N_1 + N_2 + N_3$

and write $N_1 = p_1 N$, $N_2 = p_2 N$, and $N_3 = p_3 N$

with

$$p_1 = \frac{2}{12} \quad , \quad p_2 = \frac{4}{12} \quad , \quad p_3 = \frac{6}{12}$$

So

$$\ln W = N \ln N - p_1 N \ln (p_1 N) - p_2 N \ln (p_2 N) - p_3 N \ln (p_3 N)$$

So

$$\ln W = N \left(-p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 \right)$$

We will see much later the significance of this result.

So

$$\ln W = 12 \times 6 \times 10^{23} \left(-\frac{1}{6} \ln \frac{1}{6} - \frac{1}{3} \ln \frac{1}{3} - \frac{1}{2} \ln \frac{1}{2} \right)$$

$$\simeq 72 \times 10^{23} \, (1.01)$$

So

$$W \simeq e^{72 \times 10^{23}} \simeq 10^{31 \times 10^{23}}$$

Energy Distribution

a) By equipartion theorem

$\longleftarrow$ Two quadratic forms

$$\mathcal{E} = \frac{1}{2}m v_x^2 + \frac{1}{2}m v_y^2$$

$$\overline{\mathcal{E}} = 2 \times \frac{1}{2}k_B T = k_B T$$

So

$$\overline{\frac{1}{2}m \vec{v}^2} = k_B T \implies \overline{v^2} = \frac{2kT}{m}$$

and

$$\boxed{V_{rms} = \sqrt{\frac{2kT}{m}}}$$

b) We have

$$d\mathcal{P} = C e^{-\frac{1}{2}m v^2 / kT} \, 2\pi v \, dV$$

Since

$$\int d\mathcal{P} = 1 \implies C \int_0^\infty e^{-\frac{1}{2}m v^2 / kT} \, 2\pi v \, dV = 1$$

Let

$$C \, 2\pi \, \frac{kT}{m} \int_0^\infty e^{-m(\frac{1}{2}v^2)/kT} \underbrace{\frac{m}{kT} d\left(\frac{1}{2}v^2\right)}_{} = 1$$

$$= \int_0^\infty e^{-u} \, du = 1$$

So

$$C = \frac{m}{2\pi kT} \quad \text{and} \quad d\wp = \frac{m}{2\pi kT} e^{-\frac{1}{2}mv^2/kT} 2\pi v \, dv$$

Then we have

$$\varepsilon = \frac{1}{2}mv^2 \qquad d\varepsilon = mv \, dv$$

So

$$d\wp = \frac{1}{kT} e^{-\varepsilon/kT} d\varepsilon$$

So

c) $$\langle \varepsilon^2 \rangle = \int_0^\infty \frac{1}{kT} e^{-\varepsilon/kT} \varepsilon^2 \, d\varepsilon$$

$$= (kT)^2 \underbrace{\int_0^\infty e^{-\varepsilon/kT} \left(\frac{\varepsilon}{kT}\right)^2 d\left(\frac{\varepsilon}{kT}\right)}_{2! \; = \; \Gamma(3)}$$

So

$$\langle \varepsilon^2 \rangle = 2(kT)^2$$

$$\boxed{\langle \delta \varepsilon^2 \rangle} = \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2 = 2(kT)^2 - (kT)^2 = \boxed{(kT)^2}$$

Angular Velocity

a) $\langle K_{rot} \rangle = \langle \frac{1}{2} I (\omega_x^2 + \omega_y^2) \rangle = 2 \times \frac{1}{2} kT = kT$

So

$\omega^2 = \omega_x^2 + \omega_y^2$

Thus, since $I = m \left(\frac{r_0}{2}\right)^2 + m \left(\frac{r_0}{2}\right)^2 = \frac{1}{2} m r_0^2$

we have

$\frac{1}{4} m r_0^2 \langle \omega^2 \rangle = kT$

$$\boxed{\omega_{rms} = \sqrt{\langle \omega^2 \rangle} = \sqrt{\frac{4kT}{m r_0^2}}}$$

b) $r_0 \simeq 1 \overset{\circ}{A}$

mass of $O = 16 m_p$

So

$\omega_{rms} = \sqrt{\frac{4kT}{16 m_p r_0^2}} = \frac{1}{2 r_0} \sqrt{\frac{RT}{1g}} = \frac{1}{2 \overset{\circ}{A}} \sqrt{\frac{8.32 \, J/k \, (293 k)}{0.001 kg}}$

$N_A k = R$

$N_A m_p = 1g$ $1 \overset{\circ}{A} = 10^{-10} \, m$

$$\boxed{\omega_{rms} = .780 \times 10^{12} \, \frac{1}{s} \sim 1 \, THz}$$