1 Integrals

Bose and Fermi:

$$\int_0^\infty \mathrm{d}x \, \frac{x}{e^x - 1} = \frac{\pi^2}{6} \tag{1}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^2}{e^x - 1} = 2\zeta(3) \simeq 2.404 \tag{2}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^3}{e^x - 1} = \frac{\pi^4}{15} \tag{3}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^4}{e^x - 1} = 24\,\zeta(5) \simeq 24.88 \tag{4}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^5}{e^x - 1} = \frac{8\pi^6}{63} \tag{5}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x}{e^x + 1} = \frac{\pi^2}{12} \tag{6}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^2}{e^x + 1} = \frac{3}{2}\,\zeta(3) \simeq 1.80309 \tag{7}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^3}{e^x + 1} = \frac{7\pi^4}{120} \tag{8}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^4}{e^x + 1} = \frac{45}{2}\,\zeta(5) \simeq 23.33 \tag{9}$$

$$\int_0^\infty \mathrm{d}x \, \frac{x^5}{e^x + 1} = \frac{31\pi^6}{252} \tag{10}$$

Gamma Function:

$$\Gamma(z) \equiv \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x \tag{11}$$

with specific results

$$\Gamma(z+1) = z\Gamma(z) \qquad \Gamma(n) = (n-1)! \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{12}$$

Gaussian Integrals:

$$I_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-x^2/2} x^n \tag{13}$$

with specific results

$$I_0 = 1 \quad I_2 = 0 \quad I_4 = 3 \quad I_6 = 15 \tag{14}$$

Problem 1. A nucleus as a fermi gas

Large nuclei can be treated as approximately "infinite" in size. This means that density of protons and neutrons within the nucleus approaches a constant, and in first approximation edge effects can be neglected. In the infinite volume limit the material is known as nuclear matter, and the density of the protons and neutrons is known as nuclear matter density.

Treat a nucleus as a ball of radius R with A nucleons[1]. The radius of a ball grows with $A^{1/3}$ as

$$R = (1.3 \times 10^{-15}\, m)\, A^{1/3} \tag{15}$$

Assume that the number of protons and the number of neutrons are equal.

(a) Compute the density of protons and the density neutrons.

(b) Show that the Fermi energy of protons is approximately 27 MeV.

Since we have assumed the number of protons and neutrons are equal, the Fermi energy of neutrons is also 27 MeV. In reality the number of neutrons is somewhat larger than the number of protons. Thus, the density of neutrons is higher, and the corresponding Fermi energy is somewhat higher.

(c) Show that energy per nucleon inside a nucleus is approximately 16 MeV. This is a reasonable estimate for the kinetic energy per volume in a nucleus.

[1] A nucleon is either a proton or neutron. Oxygen has eight protons and eight neutrons and has $A = 16$.

Problem 2. (Optional) 2D Fermi gas

Consider a fermi gas of electrons in two dimensions.

(a) Show that the fermi momentum is

$$p_F = \hbar\sqrt{2\pi n} \tag{16}$$

(b) Show that the mean value of the debroglie wavelength divided by 2π, i.e. $\lambdabar \equiv \hbar/p$, is

$$\langle \lambdabar \rangle = \frac{2}{p_F} \tag{17}$$

Problem 3. Relativistic Degenerate Electron Gas

Consider an ultra-relativistic degenerate electron gas where $\epsilon \simeq cp$, and the electron mass can be neglected.

(a) Show that the Fermi Energy is related density by

$$\epsilon_F = \hbar\pi c(3n/\pi)^{1/3} \tag{18}$$

where $n = N/V$.

(b) Compute the Fermi momentum p_F. Define the Fermi wavelength, $\lambdabar_F \equiv \hbar/p_F$. Explain qualitatively the dependence of $\lambdabar_F$ on the density $n = N/V$.

(c) Show that the total energy of the gas is

$$U = \frac{3}{4}N\epsilon_F \tag{19}$$

(d) Show that the pressure of the the gas is

$$\mathcal{P} = \frac{1}{3}\frac{U}{V} \tag{20}$$

and determine its dependence on density $n = N/V$. Compare your result to a classical ideal gas where $\mathcal{P} \propto n$ and a non-relativistic degenerate Fermi gas where $\mathcal{P} \propto n^{5/3}$

Problem 4. Inversion of Taylor series

Given a Taylor series of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \tag{21}$$

we want to solve for x as a function of y.

(a) Show that

$$x \approx \frac{y - a_0}{a_1} \tag{22}$$

And that

$$u = x + \frac{a_2}{a_1} x^2 + \frac{a_3}{a_1} x^3 + \dots \tag{23}$$

where here and below

$$u \equiv \frac{y - a_0}{a_1} \tag{24}$$

Note: if x is close to zero, then y is close to a_0, and u is close to zero. x and u are the same order magnitude[2].

(b) Show more generally that

$$x \simeq u - \left(\frac{a_2}{a_1}\right) u^2 \tag{25}$$

Hint: Assume a Taylor series

$$x = u + C_2 u^2 + \mathcal{O}(u^3) \tag{26}$$

and substitute into Eq. (23) consistently keeping terms of order u^2 and discarding terms of order u^3 and higher. Then solve for C_2 by demanding that the coefficient of u^2 on the RHS of Eq. (23) vanishes as is required by the LHS of this equation.

Remark: The first term is very easy to obtain and is the most important in practice. Without approximation we have

$$u = x + \frac{a_2}{a_1} x^2 + \dots \tag{27}$$

Since x is almost u in a first approximation we can replace the x^2 with u^2 up to higher corrections

$$u \simeq x + \frac{a_2}{a_1} u^2 \tag{28}$$

(c) Use the methodology outlined above, especially the remark, to find an approximate expression for x as a function of y.

(i) Suppose that

$$y = \tan(x) \simeq x + \frac{x^3}{3} + \frac{2}{15} x^5 \tag{29}$$

Find x vs. y to order y^3. Assume that x is close to zero and include the first term beyond the leading term.

[2] We say that they are of the same order.

(ii) Consider

$$y = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \tag{30}$$

Find x as a function of y.

Hint: If x is small then $y \simeq 1$. So, define a variable $u \equiv (1 - y)$, which is of order x^2 at leading order. Start by finding x^2 vs. u. Then by taking a square root show that

$$x \simeq \sqrt{2u}\left(1 + \frac{u}{12} + O(u^2)\right) \tag{31}$$

(d) Show more generally that if y is given by the series expansion in Eq. (23) that

$$x = u - \left(\frac{a_2}{a_1}\right)u^2 + \left(2\left(\frac{a_2}{a_1}\right)^3 - \left(\frac{a_3}{a_1}\right)\right)u^3 + \ldots \tag{32}$$

where $u \equiv (y - a_0)/a_1$.

(e) The following comes up when analyzing a fermi gas at small temperatures (see Eq. 30.39 and Eq. 30.40 of Blundell). At *large* x we have a series expansion expressing y as a function of x

$$y = x^{3/2}\left[1 + C_1\frac{1}{x} + \ldots\right] \tag{33}$$

Show that x as a function of y reads

$$x = y^{2/3}\left[1 - \frac{2}{3}C_1\frac{1}{y^{2/3}} + \ldots\right] \tag{34}$$

In this case x is of order $y^{2/3}$, or equivalently, y is of order $x^{3/2}$.

Hint: Define a variable $u \equiv y^{2/3}$ as motivated by the leading result, and then solve for x vs u. Note u and x are both large and the same order of magnitude.

Problem 5. Almost Classical Gas

Recall that the Fermi-Dirac distribution is

$$n_{FD} = \frac{1}{e^{\beta(\epsilon(p)-\mu)}+1} \tag{35}$$

and that the grand potential for one mode is

$$\Phi_G = -k_B T \ln(1 + e^{-\beta(\epsilon(p)-\mu)}) \tag{36}$$

As we discussed in class no two fermions can be in the same quantum state. This leads to a replusion between fermions, increasing the pressure relative to the classical result. Conversely, two bosons can be in the same quantum state, and this reduces the pressure relative to the classical case.

We will compute this effect quantitatively when the gas is nearly classical. The pressure relative to the ideal gas pressure is shown in Fig. 1

(a) Recall that in a classical limit it is very unlikely that there will be more one particle in a quantum state and that most quantum states are empty. Qualitatively explain why the classical limit means that

$$e^{-\beta(\epsilon(p)-\mu)} \ll 1\,. \tag{37}$$

and show that in the classical limit we have

$$\begin{aligned} n_{FD} =& e^{-\beta(\epsilon-\mu)}\left(1 - e^{-\beta(\epsilon-\mu)} + \ldots\right)\,, \qquad (38)\\ \Phi_G =& -kTe^{-\beta(\epsilon-\mu)}\left(1 - \tfrac{1}{2}e^{-\beta(\epsilon-\mu)} + \ldots\right)\,, \qquad (39)\end{aligned}$$

where the second term in each case is the first quantum correction arising from forbidding two particles to be one quantum state.

(b) For a gas of non-relativistic particles $\epsilon(p) = p^2/2m$, start from the Fermi-Dirac distribution and show that the density $n = N/V$ of these particles in the classical limit is approximately

$$n = e^{\beta\mu} n_Q \left(1 - \frac{e^{\beta\mu}}{2\sqrt{2}}\right) \tag{40}$$

where[3] $n_Q = g/\lambda_{\rm th}^3 = g(2\pi mkT)^{3/2}/h^3$.

(c) Show that the chemical potential is approximately determined by the density via the relation

$$e^{\beta\mu} \simeq \frac{n}{n_Q}\left(1 + \frac{1}{2\sqrt{2}}\frac{n}{n_Q}\right) \tag{41}$$

(d) Write down for the pressure of non-relativistic fermionic particles particles, $\epsilon(p) = p^2/2m$, discussed in class. Show that the pressure is

$$pV \simeq k_B T\, e^{\beta\mu} n_Q \left[1 - \frac{e^{\beta\mu}}{4\sqrt{2}}\right] \tag{42}$$

[3]The factor of $g = 2$ in this definition of n_Q accounts for the two spin states.

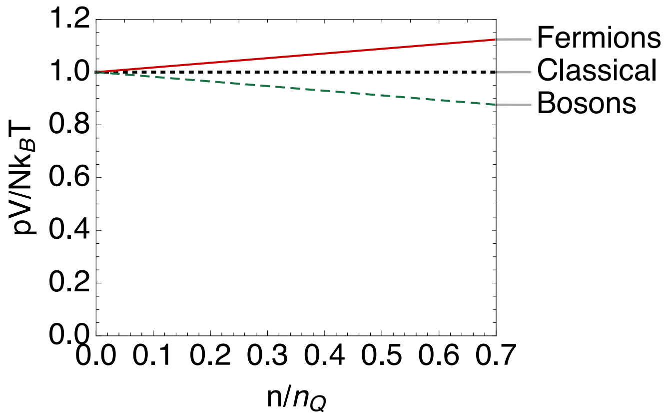

Figure 1: The pressure of a nearly classical gas as a function of density for fermions and bosons.

(e) Show that

$$pV \simeq Nk_BT \left[1 + \frac{1}{4\sqrt{2}} \frac{n}{n_Q}\right] \tag{43}$$

Thus we have determined the first virial coefficient for an almost classical gas. Qualitatively explain why the first quantum correction increases the classical pressure.

(f) (Optional) Repeat this problem for a gas of Bosons close to the classical limit.