Manipulating Taylos Series

a) $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

This follows from the geometric series $1/(1-u) = 1 + u + u^2 + \ldots$ with $u = -x$

Integrating

$$\int_0^x \frac{dx'}{(1+x')} = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$$

b)

$\boxed{c}$ For x large we have $e^{-x} \ll 1$

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}}$$

call $\underline{u = e^{-x}} \ll 1$

$$\frac{1}{e^x - 1} = \frac{u}{(1-u)} = u\,(1 + u + u^2 + \ldots)$$

$$\boxed{\frac{1}{e^x - 1} = e^{-x}\left(1 + e^{-x} + e^{-2x} + O(e^{-3x})\right)}$$

d) $\dfrac{1}{e^x - 1}$

we expand $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$

$$\frac{1}{e^x-1} \simeq \frac{1}{x + x^2/2 + x^3/6} = \frac{1}{x} \frac{1}{(1 + x/2 + x^2/6 + O(x^3))}$$

calling, $u = x/2 + x^2/6$, we have

$$\frac{1}{e^x-1} \simeq \frac{1}{x}\left(\frac{1}{1+u} + O(x^3)\right)$$

$$\simeq \frac{1}{x}\left(1 - u + u^2 + O(x^3)\right)$$

$$\simeq \frac{1}{x}\left(1 - \left(\frac{x}{2} + \frac{x^2}{6}\right) + \left(\frac{x^2}{4}\right) + O(x^3)\right)$$

$$\boxed{\frac{1}{e^x-1} \simeq \frac{1}{x}\left(1 - \frac{x}{2} + \frac{x^2}{12}\right) + O(x^2)}$$

e) $\dfrac{1}{e^{-x}+1} \simeq 1 - e^{-x} + e^{-2x}$ set $u = e^{-x}$

f) $\log(1 - e^{-x}) \simeq \log\left(1 - \left(1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6}\right)\right)$

$$\simeq \log x \left(1 - \frac{x}{2} + \frac{x^2}{6}\right) = \log x + \log\left(1 - \frac{x}{2} + \frac{x^2}{6}\right)$$

So

$$\log(1-e^{-x}) = \log(x) + \log(1\underbrace{- \frac{x}{2} + \frac{x^2}{6}}_{\text{Call it } u}) + O(x^3)$$

Setting $u = -\frac{x}{2} + \frac{x^2}{6}$ we have

$$\log(1+u) = u - \frac{u^2}{2} + O(u^3)$$ with x of order u

So

$$\log(1-e^{-x}) \simeq \log(x) + \left(-\frac{x}{2} + \frac{x^2}{6}\right) - \frac{1}{2}\left(-\frac{x}{2}\right)^2 + O(x^3)$$

$$\boxed{\log(1-e^{-x}) \simeq \log x - \frac{x}{2} + \frac{x^2}{24} + O(x^3)}$$

Energy of SHO

a) We have

$$Z = \frac{1}{1 - e^{-\beta\hbar\omega_0}}$$

Then

$$\langle E \rangle = -\frac{\partial}{\partial\beta}\log Z = +\frac{\partial}{\partial\beta}\log\left(1 - e^{-\beta\hbar\omega_0}\right)$$

$$= \frac{1}{1 - e^{-\beta\hbar\omega_0}}\, e^{-\beta\hbar\omega_0}\,\hbar\omega_0$$

$$\boxed{\langle E \rangle = \frac{\hbar\omega_0}{e^{\beta\hbar\omega_0} - 1}}$$

b) Then

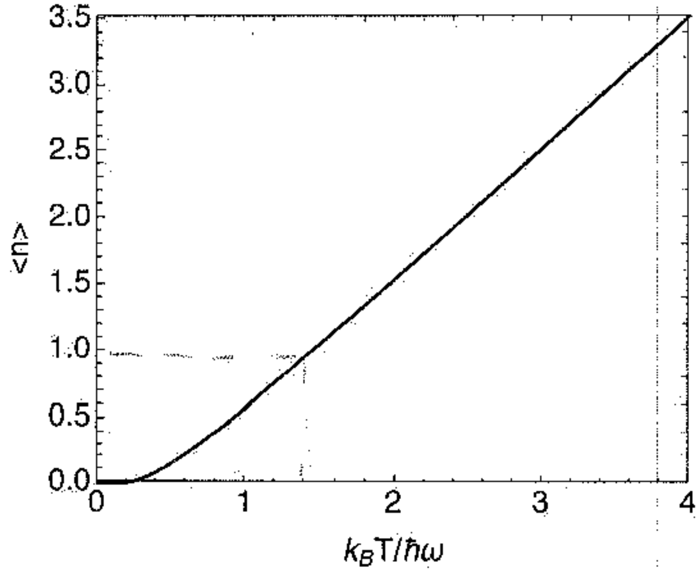
$$\frac{\langle E \rangle}{\hbar\omega_0} = \frac{1}{e^{\hbar\omega/kT} - 1} = \langle n \rangle$$

Then

Then from graph

$\langle n \rangle = 1$ when

$k_B T/\hbar\omega_0 = 1.45$



3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
<n>
0
1
2
3
4
k_B T/ħω

or

$$\boxed{k_B T = 1.45\,\hbar\omega_0}$$

c) Then a nice plot of $\langle E \rangle$ is given in the problem statement.

d) Using the series of problem 1

with $x \equiv \hbar\omega/kT$

at low temperature $kT \ll \hbar\omega_0$ then $x \gg 1$, and

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}} \simeq e^{-x}(1 + e^{-x} + \ldots)$$

And

$$\boxed{\langle E \rangle = \hbar\omega_0\, e^{-\beta\hbar\omega_0}(1 + e^{-\beta\hbar\omega_0} + \ldots)}$$

At high temperature $x \ll 1$

$$\frac{1}{e^x - 1} \simeq \frac{1}{x} - \frac{1}{2}$$

$$\langle E \rangle = \hbar\omega_0\left(\frac{k_B T}{\hbar\omega_0} - \frac{1}{2}\right) \simeq k_B T\left(1 - \frac{\hbar\omega_0}{2kT}\right)$$

(e) At high temperature the number of quanta $\langle n \rangle$ is very large. In this regime $\langle n \rangle \gg 1$, quantum mechanics becomes continuous, $\frac{\Delta E}{E} \ll 1$, and it appraoaches classical mechanics

This is the Bohr correspondence principle

f) We have

i) $$\boxed{U = N\underbrace{\left[\frac{5}{2}kT + \frac{\hbar\omega_0}{e^{\beta\hbar\omega_0}-1}\right]}_{\text{this is } f_0(T)}}$$

Then

ii) $$C_V = \left(\frac{dU}{dT}\right)_V = N\left[\frac{5}{2}k + \frac{-\hbar\omega_0 e^{\beta\hbar\omega_0}}{(e^{\beta\hbar\omega_0}-1)^2}\,\hbar\omega_0\,\frac{\partial}{\partial T}\frac{1}{kT}\right]$$

$$= N\left[\frac{5k}{2} + \frac{(\beta\hbar\omega_0)^2 e^{\beta\hbar\omega_0}}{(e^{\beta\hbar\omega_0}-1)}k\right]$$

$$C_V = Nk\left[\frac{5}{2} + \frac{(\beta\hbar\omega_0)^2 e^{\beta\hbar\omega_0}}{(e^{+\beta\hbar\omega_0}-1)^2}\right]$$

So

$$C_p = C_v + Nk_B$$

$$\boxed{C_p = Nk_B \left[\frac{7}{2} + \frac{(\beta \hbar \omega_0)^2}{(e^{\beta \hbar \omega_0} - 1)^2} \right]}$$

iiv) So we see that the model nicely captures the transition from $C_p = \frac{7}{2} = 3.5$ to $\frac{9}{2} = 4.5$

but misses the transition to $\frac{5}{2}$ at low temperatures