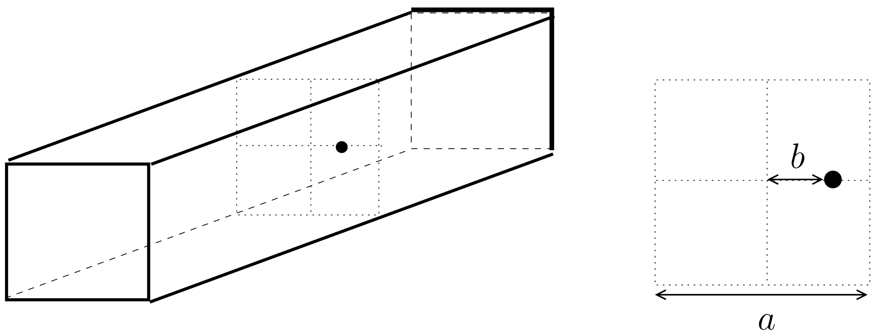
Do not hand in optional parts!

Problem 1. A point charge in a rectangular tube

Consider a point charge placed in an infinitely long grounded rectangular tube as shown below. The sides of the square cross sectional area of the tube have length a.



(a) (Optional) Show that the solutions to the *homogeneous* Laplace equation (i.e. without the extra point charge) are linear combinations of functions of the form

$$\Phi(k_x x)\,\Phi(k_y y)\,e^{\pm\kappa_z z} \qquad \text{where} \qquad \Phi(u) = \Big\{\cos(u) \text{ or } \sin(u) \tag{1}$$

for specific values of k_x, k_y and κ_z. Determine the allowed the values of k_x, k_y and κ_z and their associated functions.

(b) Now consider a point charge displaced from the center of the tube by a distance b in the x direction, i.e. the coordinatess of the charge are $\boldsymbol{r}_o = (x, y, z) = (b, 0, 0)$. Use the method of images to determine the potential. You will need an infinite number of image charges of both sign.

(c) As an alternative to the method of images, use a series expansion in terms of the homogeneous solutions of part (a) to determine the potential from the point charge described in part (b). You should find the form

$$\phi(\boldsymbol{r}; \boldsymbol{r}_o) = \frac{4q}{a^2} \sum_n \sum_m X_n(x) X_n(b)\, Y_m(y) Y_m(0) \frac{e^{-\kappa_{n,m}|z|}}{2\kappa_{n,m}} \tag{2}$$

where $X_n(x) = \Phi_n(k_n x)$ and $Y_m(y) = \Phi_m(k_m y)$ and $\kappa_{n,m} = \sqrt{k_n^2 + k_m^2}$.

(d) Determine the asymptotic form of the surface charge density, and the force per area on the walls of the rectangular tube far from the point charge, i.e. $z \gg a$. You should find that the force per area on the bottom plate (far from the charge) is

$$\frac{F^y}{A} \simeq \frac{q^2}{a^4} \cos^2(\pi x/a) \cos^2(\pi b/a)\, e^{-2\sqrt{2}\pi|z|/a}\,. \tag{3}$$