Problem 1

$$\underbrace{1\,1\,1\,1\,1}_{N_1},\ \underbrace{2\,2\,2}_{N_2},\ \underbrace{3\,3\,3\,3}_{N_3}$$

$\boxed{a)}$

There are $N!$ rearrangements of the objects. $N_1!$ of these just rearranges the "ones" amongst themselves; $N_2!$ of these just rearrange the "twos" amongst themselves; and ditto for $N_3!$

So
$$W = \frac{N!}{N_1!\,N_2!\,N_3!}$$

$\boxed{b)}$
$$\ln W = \ln N! - \ln N_1! - \ln N_2! - \ln N_3!$$
$$= N\ln N - N - N_1 \ln N_1 + N_1 - N_2 \ln N_2 + N_2$$
$$- N_3 \ln N_3 + N_3$$

Now note $N = N_1 + N_2 + N_3$

and write $N_1 = p_1 N$, $N_2 = p_2 N$, and $N_3 = p_3 N$
with
$$p_1 = \frac{2}{12}, \quad p_2 = \frac{4}{12}, \quad p_3 = \frac{6}{12}$$

So
$$\ln W = N\ln N - p_1 N \ln(p_1 N) - p_2 N \ln(p_2 N) - p_3 N \ln(p_3 N)$$

We will see much later the significance of this result,

So

$$\boxed{\ln W = N\left(-p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3\right)}$$

So

$$\ln W = 12 \times 6 \times 10^{23}\left(-\frac{1}{6}\ln\frac{1}{6} - \frac{1}{3}\ln\frac{1}{3} - \frac{1}{2}\ln\frac{1}{2}\right)$$

$$\simeq 72 \times 10^{23}\ (1.01)$$

So

$$\boxed{W \simeq e^{72\times 10^{23}} \simeq 10^{31\times 10^{23}}}$$

Energy Distribution

a) By equipartion theorem

$$\mathcal{E} = \frac{1}{2} m v_x^2 + \frac{1}{2} m v_y^2 \quad \leftarrow \text{Two quadratic forms}$$

$$\overline{\mathcal{E}} = 2 \times \frac{1}{2} k_B T = k_B T$$

So

$$\overline{\frac{1}{2} m \vec{v}^2} = k_B T \implies \overline{v^2} = \frac{2kT}{m}$$

and

$$\boxed{v_{rms} = \sqrt{\frac{2kT}{m}}}$$

b) We have

$$d\mathcal{P} = C e^{-\frac{1}{2} m v^2 / kT} \, 2\pi v \, dv$$

Since

$$\int d\mathcal{P} = 1 \implies C \int_0^\infty e^{-\frac{1}{2} m v^2 / kT} \, 2\pi v \, dv = 1$$

Let

$$C \, 2\pi \, \frac{kT}{m} \underbrace{\int_0^\infty e^{-m(\frac{1}{2} v^2)/kT} \frac{m}{kT} d\left(\tfrac{1}{2} v^2\right)}_{= \int_0^\infty e^{-u} du = 1} = 1$$

So

$$C = \frac{m}{2\pi kT} \quad \text{and} \quad \boxed{d\mathcal{P} = \frac{m}{2\pi kT} e^{-\frac{1}{2}mv^2/kT} 2\pi v\, dv}$$

Then we have

$$\varepsilon = \frac{1}{2} m v^2 \qquad d\varepsilon = m v\, dv$$

So

$$\boxed{d\mathcal{P} = \frac{1}{kT} e^{-\varepsilon/kT} d\varepsilon}$$

So

c) $$\langle \varepsilon^2 \rangle = \int_0^\infty \frac{1}{kT} e^{-\varepsilon/kT} \varepsilon^2 d\varepsilon$$

$$= (kT)^2 \underbrace{\int_0^\infty e^{-\varepsilon/kT} \left(\frac{\varepsilon}{kT}\right)^2 d\left(\frac{\varepsilon}{kT}\right)}_{2! = \Gamma(3)}$$

So

$$\langle \varepsilon^2 \rangle = 2(kT)^2$$

$$\boxed{\langle \delta\varepsilon^2 \rangle} = \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2 = 2(kT)^2 - (kT)^2 = \boxed{(kT)^2}$$

Angular Velocity

a) $\langle K_{rot} \rangle = \langle \frac{1}{2} I (\omega_x^2 + \omega_y^2) \rangle = 2 \times \frac{1}{2} kT = kT$

so

$$\omega^2 = \omega_x^2 + \omega_y^2$$

Thus, since $I = m\left(\frac{r_0}{2}\right)^2 + m\left(\frac{r_0}{2}\right)^2 = \frac{1}{2} m r_0^2$
we have

$$\frac{1}{4} m r_0^2 \langle \omega^2 \rangle = kT$$

$$\boxed{\omega_{rms} = \sqrt{\langle \omega^2 \rangle} = \sqrt{\frac{4kT}{m r_0^2}}}$$

b) $r_0 \simeq 1\,\mathring{A}$

mass of O $= 16\, m_p$

So

$$\omega_{rms} = \sqrt{\frac{4kT}{16 m_p r_0^2}} = \frac{1}{2r_0} \sqrt{\frac{RT}{1g}} = \frac{1}{2\mathring{A}} \sqrt{\frac{8.32\ J/k\ (293k)}{0.001\,kg}}$$

$$N_A k = R$$

$$N_A m_p = 1g \qquad 1\,\mathring{A} = 10^{-10}\,m$$

$$\boxed{\omega_{rms} = .780 \times 10^{12}\ \frac{1}{s} \sim 1\ THz}$$

High and Low

a) $Z = 2 + 3\, e^{-\beta\Delta}$

$$\boxed{\langle \varepsilon \rangle = \frac{1}{Z}\,\frac{-\partial Z}{\partial \beta} = \frac{3\Delta\, e^{-\beta\Delta}}{2 + 3\, e^{-\beta\Delta}}}$$

b) The probability to be in the first two states is

$$P_0 = \frac{2}{2+3e^{-\beta\Delta}} \quad \text{and} \quad \boxed{N_0 = N_A\left(\frac{2}{2+3e^{-\beta\Delta}}\right)}$$

while the probability to be in the higher state is

$$P_\Delta = \frac{3e^{-\beta\Delta}}{2+3e^{-\beta\Delta}} \quad \text{and} \quad \boxed{N_\Delta = N_A\left(\frac{3e^{-\beta\Delta}}{2+3e^{-\beta\Delta}}\right)}$$

So

$$\boxed{\delta N = N_0 - N_\Delta = N_A\left(\frac{2 - 3e^{-\beta\Delta}}{2 + 3e^{-\beta\Delta}}\right)}$$

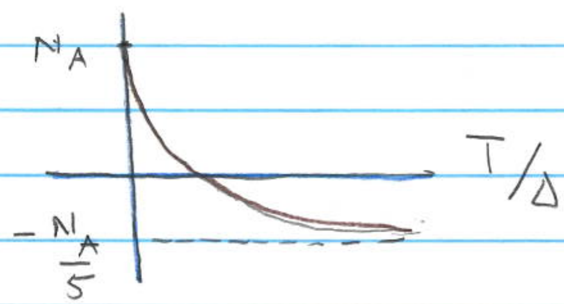
For $\beta\Delta \ll 1$, high temperature, all states are equally likely

$$\delta N = N_A\left(\frac{2-3}{2+3}\right) = -\frac{N_A}{5}$$

For low temperatures all states are in the ground state

$$\delta N = N_A\left(\frac{2-0}{2+0}\right) = N_A$$

A graph is shown below:



c) In the last part we must take the total # of atoms in the ground state and put them in the upper state

$$\boxed{E = N_0 \Delta = \frac{2 N_A \Delta}{2 + 3e^{-\beta\Delta}}}$$