Physics 306: Thermal Physics

Final Exam

Stony Brook University

Fall 2023

General Instructions:

You may use one page (front and back) of handwritten notes and a calculator. Graphing calculators are allowed. **No other materials may be used**.

1 Integrals

Bose and Fermi:

$$\int_0^\infty \mathrm{d}x\,\frac{x}{e^x-1} = \frac{\pi^2}{6} \tag{1}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^2}{e^x-1} = 2\zeta(3) \simeq 2.404 \tag{2}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^3}{e^x-1} = \frac{\pi^4}{15} \tag{3}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^4}{e^x-1} = 24\,\zeta(5) \simeq 24.88 \tag{4}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^5}{e^x-1} = \frac{8\pi^6}{63} \tag{5}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x}{e^x+1} = \frac{\pi^2}{12} \tag{6}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^2}{e^x+1} = \frac{3}{2}\,\zeta(3) \simeq 1.80309 \tag{7}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^3}{e^x+1} = \frac{7\pi^4}{120} \tag{8}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^4}{e^x+1} = \frac{45}{2}\,\zeta(5) \simeq 23.33 \tag{9}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^5}{e^x+1} = \frac{31\pi^6}{252} \tag{10}$$

Gamma Function:

$$\Gamma(z) \equiv \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x \tag{11}$$

with specific results

$$\Gamma(z+1) = z\Gamma(z) \qquad \Gamma(n) = (n-1)! \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{12}$$

Gaussian Integrals:

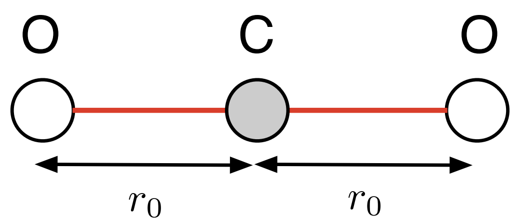
$$I_n = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{d}x\,e^{-x^2/2}x^n \tag{13}$$

with specific results

$$I_0 = 1 \quad I_2 = 0 \quad I_4 = 3 \quad I_6 = 15 \tag{14}$$

Problem 1. Carbon Dioxide

Carbon dioxide CO_2 consists of two oxygen atoms ${}^{16}_{8}O$ (8 protons and 8 neutrons) and a carbon atom ${}^{12}_{6}C$ as shown below. The spacing between the oxygen and carbon atoms is r_0 as shown below.



Consider CO_2 at room temperature T_0 and atmospheric pressure p_0. Assume that the molecule does not vibrate but only rotates as a whole around its center of mass. The rotations can be treated essentially classically.

(a) The rotational constant of CO_2 is[1], $\Delta \equiv \hbar^2/2I \simeq 0.048\,\text{meV}$.

  (i) Estimate r_0 and use your estimate to make an order of magnitude estimate for Δ. Is your estimate consistent with the experimental value: $\Delta \simeq 0.048\,\text{meV}$? Explain.

  (ii) Briefly describe the physical significance of Δ.

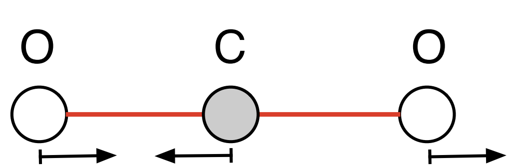
(b) Determine the energy per atom of the gas.

(c) Determine the entropy of the gas.

  (i) Evaluate the result for the S/Nk_B numerically.

  (ii) How would your result change for S/Nk_B if all the mass of CO_2 was concentrated at a single point?

The lowest vibrational mode of CO_2 involves the displacement of the central carbon atom which is opposite to the displacements of the two oxygen atoms – see below. The quantized oscillations of this mode are excited by a photon of energy $E = \hbar\omega = 83\,\text{meV}$.



(d) Determine the change in entropy per atom S/Nk_B of part (c) when the first vibrational mode is included. Evaluate this change numerically.

[1] Note that the units here are milli electron volts

Problem 2. Hot Gluon Plasma

A gluon is a lot like a photon – it is a massless relativistic boson and has two spin states. Like the photon, the gluon number is not conserved and thus its chemical potential is zero. Unlike the photon, the gluon has eight "colors" and thus there are a total of $16 = 2 \times 8$ internal states for every momentum, i.e. the degeneracy is sixteen as opposed to two. Consider a gas of gluons at temperature T and volume V

(a) Take a single mode with energy ε_1 and momentum p_1. Derive the grand partition partition function and show that the mean number of gluons per mode

$$n(\varepsilon_1) = \frac{1}{e^{\beta(\varepsilon_1 - \mu)} - 1} \tag{15}$$

You should keep $\mu \neq 0$ for this part and then set it zero in what follows.

(b) Derive the total number of gluons per volume n_g in the container carefully explaining each step.

(c) Determine the average wavelength of the gluons in the gas.

(d) Determine the distribution of wavelengths $\mathrm{d}n_g/\mathrm{d}\lambda$ of the gluons within gas.

Problem 3. N electrons and N sites

A system contains N independent sites and N electrons. At a given site there is one accessible orbital, but that orbital can be empty, occupied by one electron of either spin, or occupied by two electrons of opposite spin. The electrons can hop from site to site.

The site energy is zero if the site is either empty or singly occupied, and Δ if it is doubly occupied.

(a) (4 points) List all states and determine the grand partition function of the system.

(b) (8 points) Determine the chemical potential as a function of temperature. Check your work.

(c) (8 points) Calculate the mean energy and heat capacity of the sites.

Problem 4. Graphene

Graphene is a two-dimensional monolayer of carbon atoms which exhibits remarkable properties. The "carrier" electrons in graphene behave like relativistic particles with an effective speed of light, c_0. Indeed, the energy of a carrier electron with momentum p is $\varepsilon(p) = c_0 p$, where c_0 is approximately three hundred times less than the actual speed of light, $c_0 = c/300$.

The density of the carrier electrons is $n_e = 5 \times 10^{16}\,\mathrm{m}^{-2}$. Treat the carrier electrons as an electron gas in two spatial dimensions at zero temperature

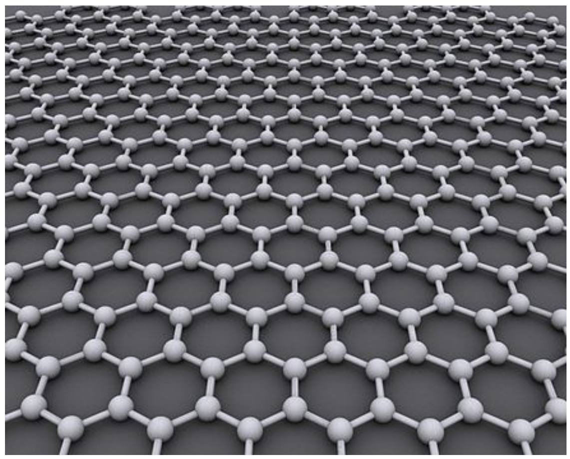


Figure 1: A two dimensional sheet of graphene. The carrier electrons move on this sheet forming a two-dimensional effectively relativistic gas, with effective "speed of light" c_0.

(a) Determine the relation between the Fermi-energy and the carrier density n_e.

  (i) Evaluate the Fermi-energy in electron volts. Is the zero temperature approximation justified? Explain.

(b) Determine average de Broglie wavelength of the electrons. Express your result in terms of the carrier density n_e.

(c) Compute the total energy per area of the electron gas.

(d) Determine the pressure due to the carrier electrons.