Problem 1. The most energetic frequency interval and wavelength interval

(a) The energy density can be written

$$u = \int_0^\infty d\omega \, \frac{du}{d\omega} \tag{1}$$

where $du/d\omega$ is the energy per frequency interval $d\omega$. Using a graphical means show $du/d\omega$ is maximum for $\hbar\omega = 2.82kT$. What is the energy of a photon with this frequency for a black body of 6000K, which is approximately the surface temperature of the sun.

(b) The energy density can be written

$$u = \int_0^\infty d\lambda \, \frac{du}{d\lambda} \tag{2}$$

where $du/d\lambda$ is the energy per wavelength inteval $d\lambda$. Find $du/d\lambda$, and using a graphical method find the wavelength where $du/d\lambda$ is maximum. (You should find $\lambda \simeq 4.9hc/kT$.) What is this wavelength in nm for a black body of 6000K, which is approximately the surface temperature of the sun.

Problem 2. 2D World

Consider a box of area L^2 in two dimensions at temperature T. Determine the number of photons and the energy in the box. The integrals on the last page should be helpful. A neutrino is like a photon, it has two spin states and its number is not conserved (so its chemical potential is zero). But the neutrino is a fermion not a boson. How would the answers for the number and energy in the box change in this neutrino case?

Problem 3. Temperature of the Sun

The intensity (energy per area per time) of sunlight on earth is $I = 1\,\mathrm{kW}/m^2$. Show that the temperature of the sun is related to the intensity of the sunlight and the solid angle $\Omega_\odot$ that the suns takes up in our sky:

$$T_\odot = \left(\frac{I}{\sigma} \frac{\pi}{\Omega_\odot} \right)^{1/4} \tag{3}$$

Here σ is the Steffan Boltzmann constant and Ω is the solid angle subtended by the sun in our sky i.e.

$$\Omega_\odot \simeq \frac{A_\odot}{D^2} \simeq \frac{\pi R_\odot^2}{D^2} \simeq 6.8 \times 10^{-5} \tag{4}$$

Here $R_\odot$ is the radius of the sun, and D is the distance between the earth and the sun. Evaluate $T_\odot$ numerically.

Problem 4. Density of states $g(\epsilon)$

In class we classified the modes (wave-functions) of a box by three quantum numers

$$\psi_{n_x n_y n_z}(x, y, z) \propto \sin(k_{n_x} x) \sin(k_{n_y} y) \sin(k_{n_z} z) \tag{5}$$

and we showed that if the box is large

$$\sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \sum_{n_z=1}^{\infty} \cdots \to \int \frac{V d^3 p}{(2\pi\hbar)^3} \cdots \tag{6}$$

(a) Show that the number of modes $g(k)dk$ with wavenumber k, between k and $k + dk$ is

$$g(k)dk = \frac{1}{2\pi^2} V k^2 dk \tag{7}$$

and determine the analogous formula in two dimensions. $g(k)$ is known as the density of (single particle) states. Assume that the particles are spinless, so that

$$\sum_{\text{modes}} \cdots = \sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \sum_{n_z=1}^{\infty} \cdots \tag{8}$$

(b) The density of states is often expressed in terms of energy. For spinless non-relativistic particles (with $\epsilon(p) = p^2/2m$) show that the number of modes, $g(\epsilon)d\epsilon$, with energy between ϵ and $d\epsilon$. Show that the density of states in three dimensions is

$$g(\epsilon)d\epsilon = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\epsilon}\, d\epsilon \tag{9}$$

and find the analogous formula in two dimensions.

(c) In any dimension explain why grand potential of a Bose and Fermi gas can be written

$$\Phi_G = \pm kT \int_0^{\infty} g(\epsilon) d\epsilon \ln\left(1 \mp e^{-\beta(\epsilon_p - \mu)}\right) \tag{10}$$

and the number of particles is

$$N = \int_0^{\infty} g(\epsilon) \frac{1}{e^{\beta(\epsilon_p - \mu)} \mp 1} \tag{11}$$

where the upper sign is for fermions and the lower sign is for bosons. You may simply quote the grand partition function from the class notes.

Determine $g(\epsilon)$ for a photon gas in three dimensions, and express the pressure of the photon gas as an integral. You will evaluate this integral in the next problem.

The photon has two polarization states. So there are two modes for every value of k

$$\sum_{\text{modes}} = 2 \sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \sum_{n_z=1}^{\infty} \cdots \tag{12}$$

Problem 5. Entropy per Photon

You should have found the pressure (or minus the grand potential per volume) of gas of photons. Ater recognizing that $\epsilon = \hbar\omega$, the result of problem 2 is

$$pV = \frac{V}{\pi^2 c^3} \int_0^\infty \omega^2 kT \ln\left(1 - e^{-\beta\hbar\omega}\right) \tag{13}$$

(a) Integrate by parts to show that

$$p = \frac{\pi^2}{45} \left(\frac{kT}{\hbar c}\right)^3 kT \tag{14}$$

The necessary integrals are given below.

(b) Show that

$$d\Phi_G = -SdT - Nd\mu - pdV \tag{15}$$

and then by differentiating the pressure (or grand potential), that

$$S = 4\frac{pV}{T} \tag{16}$$

Using the result from class for the number of photons and show that the entropy per photon S/Nk_B is 3.6.

(c) Use the Gibbs-Duhem relation and the previous result to find the energy density of the system, $u = U/V$. Check your result by comparing with the method used in class. You should find

$$u = 3pV \tag{17}$$

$$
\begin{aligned}
\int_0^\infty \mathrm{d}x\, \frac{x}{e^x-1} =& \frac{\pi^2}{6} \\
\int_0^\infty \mathrm{d}x\, \frac{x^2}{e^x-1} =& 2\zeta(3) \simeq 2.404 \\
\int_0^\infty \mathrm{d}x\, \frac{x^3}{e^x-1} =& \frac{\pi^4}{15} \\
\int_0^\infty \mathrm{d}x\, \frac{x^4}{e^x-1} =& 24\,\zeta(5) \simeq 24.88 \\
\int_0^\infty \mathrm{d}x\, \frac{x^5}{e^x-1} =& \frac{8\pi^6}{63}
\end{aligned}
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\begin{aligned}
\int_0^\infty \mathrm{d}x\, \frac{x}{e^x+1} =& \frac{\pi^2}{12} \\
\int_0^\infty \mathrm{d}x\, \frac{x^2}{e^x+1} =& \frac{3}{2}\,\zeta(3) \simeq 1.80309 \\
\int_0^\infty \mathrm{d}x\, \frac{x^3}{e^x+1} =& \frac{7\pi^4}{120} \\
\int_0^\infty \mathrm{d}x\, \frac{x^4}{e^x+1} =& \frac{45}{2}\,\zeta(5) \simeq 23.33 \\
\int_0^\infty \mathrm{d}x\, \frac{x^5}{e^x+1} =& \frac{31\pi^6}{252}
\end{aligned}
$$

Figure 1: A compendium of useful integrals of Bose and Fermi distributions