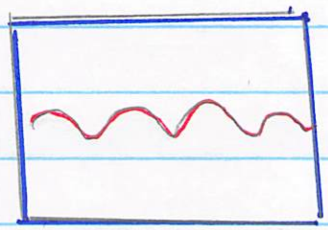
Last Time



- consider quantum particles in a box
- The waves are broken up into modes

$$\vec{p} = \hbar \vec{k} = \hbar \left(\frac{\pi n_x}{L}, \frac{\pi n_y}{L}, \frac{\pi n_z}{L} \right)$$

- Each mode is an independent subsystem sharing the available energy and particles. Temperature is a parameter describing how the energy is shared. μ is a parameter describing how the particles are shared

- The mean number of particles in a mode of single-particle energy $\varepsilon(p)$ is

$$\bar{n}_{BE}^{(p)} = \frac{1}{e^{\beta(\varepsilon(p) - \mu)} - 1} \longleftarrow \text{if particles are bosons}$$

$$\bar{n}_{FD}^{(p)} = \frac{1}{e^{\beta(\varepsilon(p) - \mu)} + 1} \longleftarrow \text{if particles are fermions}$$

- The energy in a mode is $E_p = n\,\varepsilon(p)$ and the mean energy of a mode is

$$\bar{E}_p = \bar{n}\,\varepsilon(p)$$

Black Body Radiation and the Photon Gas:

$$\mathcal{E}(p) = \frac{p^2}{2m} \qquad \text{non-relativist particles}$$

$$\mathcal{E}(p) = cp \qquad \text{light}$$

Then the total number of quantum particles in the box is

$$N = \sum_{\text{modes}} \frac{1}{e^{\beta(\mathcal{E}(p)-\mu)} \mp 1}$$

Photons

- Photons are bosons.
- $\mathcal{E}(p) = cp$, since they are relativistic & massless.
- $\mu = 0$ since photons are easily created and destroyed
- The sum over modes / states was:

$$\sum_{n_x} \sum_{n_y} \sum_{n_z} \ldots \longrightarrow \int \frac{V\, d^3p}{(2\pi\hbar)^3} \ldots = \int \frac{d^3r\, d^3p}{h^3} \ldots$$

- The two polarizations of photons gives "extra" an overall factor of 2:

$$\sum_{\text{modes}} \ldots \longrightarrow 2 \int \frac{V\, d^3p}{(2\pi\hbar)^3} \ldots$$

- Putting Together The ingrediants we found:

$$N = 2V \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{cp/kT} - 1}$$

- Looking at this integral, there is a characteristic energy scale for the photons

$$\boxed{E_0 \sim kT}$$

Then, $p_0 = \frac{E_0}{c} \sim \frac{kT}{c}$, and the typical vavelength is:

$$\boxed{\lambda\!\!\!^-_0 \equiv \frac{\hbar}{p_0} \equiv \frac{\hbar c}{kT}} \quad \text{with} \quad \lambda\!\!\!^- \equiv \frac{\lambda}{2\pi}$$

- Then doing integral we found:

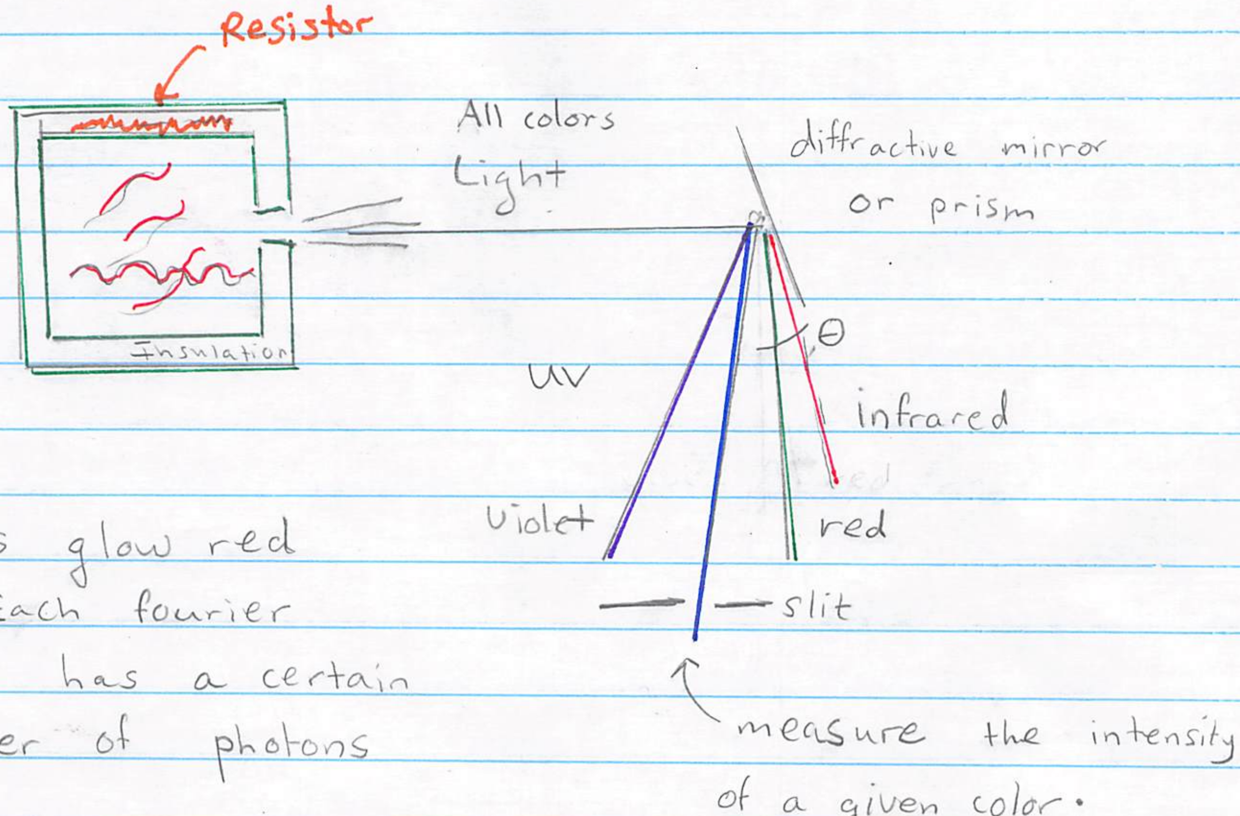
numerical integration

$$N = \underbrace{V\left(\frac{p_0}{\hbar}\right)^3}_{\text{units}} \# = V\left(\frac{kT}{\hbar c}\right)^3 0.244 \propto T^3$$

So the density of photons is $n_\gamma \equiv N/V$

$$n_\gamma \equiv \left(\frac{N}{V}\right) = 0.244 \left(\frac{kT}{\hbar c}\right)^3 = \frac{0.244}{\lambda\!\!\!^-_0{}^3}$$

Experiment



- Objects glow red hot. Each fourier mode has a certain number of photons
- Watch video of oven.
- We just showed that the photon density in the oven is

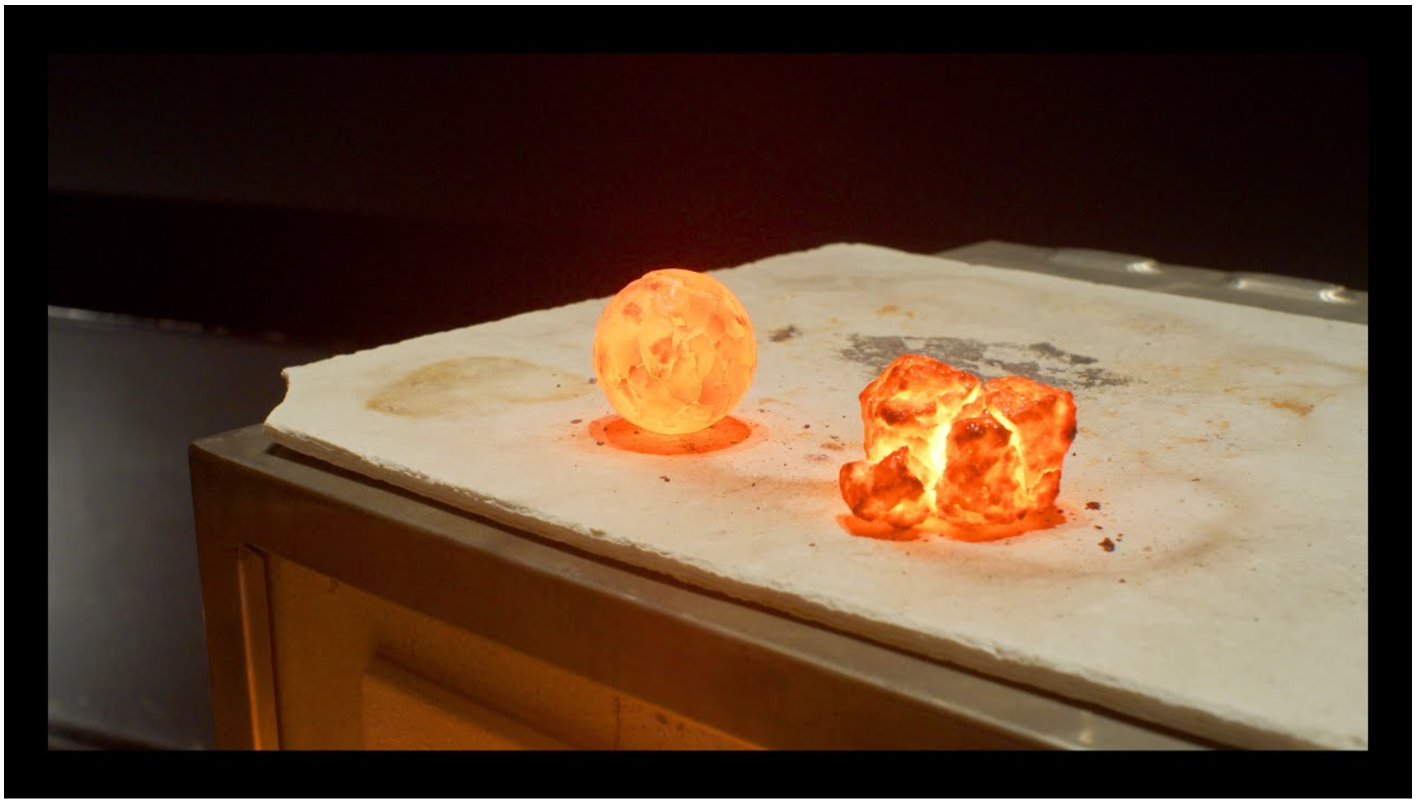
$$n_\gamma = 0.244 \left(\frac{kT}{\hbar c}\right)^3$$

Yields

- We can actually do much more: What is the number of photons per volume with frequency between ω and $\omega + d\omega$, i.e.

$$\frac{dn_\gamma}{d\omega} \qquad \text{note: } \omega = 2\pi f$$

Video: https://www.youtube.com/watch?v=Psvo_XEc784&t=5s



$$N = 2V \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{cp/kT} - 1}$$

$$d^3p = 4\pi p^2 dp$$

$$N = 2V \int_0^\infty \frac{4\pi p^2 dp}{(2\pi\hbar)^3} \frac{1}{e^{cp/kT} - 1}$$

$$n_\gamma = \frac{1}{\pi^2} \frac{1}{\hbar^3} \int_0^\infty \frac{p^2 dp}{e^{cp/kT} - 1}$$

So

$$dn_\gamma = \frac{1}{\pi^2} \frac{1}{\hbar^3} \frac{p^2 dp}{e^{cp/kT} - 1}$$

Now lets convert to frequency

$$\mathcal{E} = cp = \hbar\omega \quad , \quad p = \frac{\hbar\omega}{c} \quad , \quad dp = \frac{\hbar}{c} d\omega$$

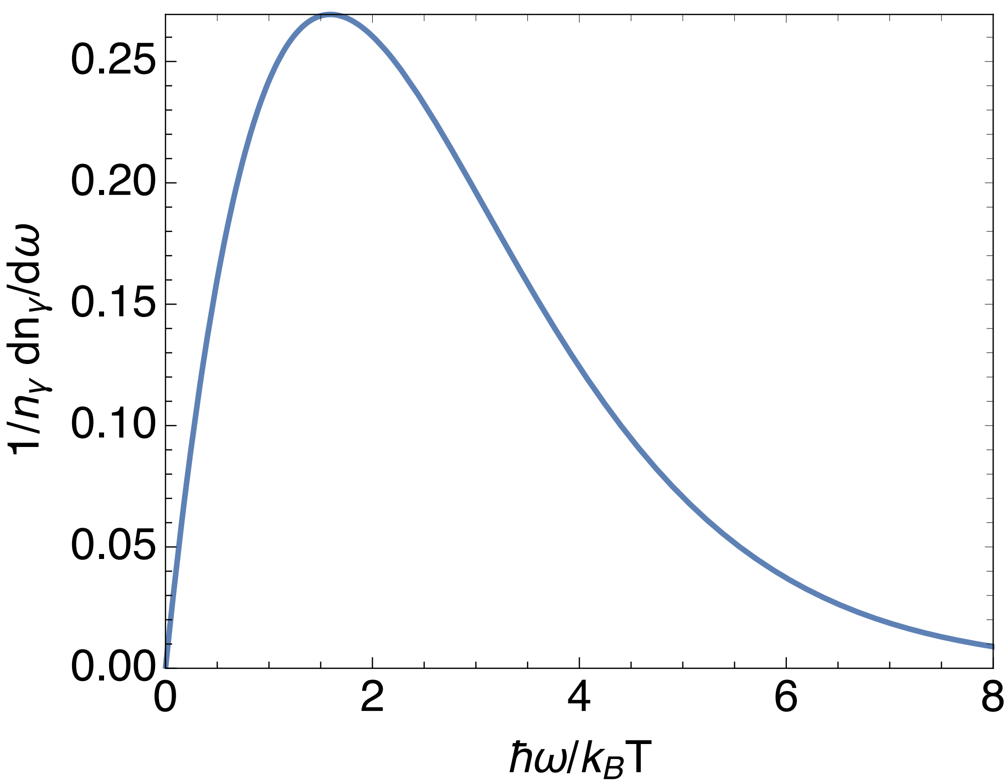
So with this change of variables:

$$\boxed{dn_\gamma = \frac{1}{\pi^2 c^3} \frac{\omega^2}{e^{\hbar\omega/kT} - 1} d\omega}$$

Now

$\frac{1}{n_\gamma}\left(\frac{dn_\gamma}{d\omega}\right)$ is shown on the slide

Photon Spectrum



$$\frac{dn_\gamma}{d\omega} \propto \frac{\omega^2}{e^{\hbar\omega/k_BT} - 1}$$

- The curve is maximized when

$$\frac{\hbar\omega}{k_B T} \simeq 1.6 \quad \text{so the}$$

So the most probable photon energy is $\hbar\omega \simeq 1.6\, k_B T$

Energy Of The Photon Gas

- The mean energy of the photon gas can be determined from the energies of each mode

$$E_p = \bar{n}_{BE}\, \mathcal{E}(p)$$

Then

$$E = \sum_{\text{moder}} n_{BE}\, \mathcal{E}(p)$$

$$= 2 \int \frac{V d^3p}{(2\pi\hbar)^3} \frac{\mathcal{E}(p)}{e^{\beta \mathcal{E}(p)} - 1} \qquad \text{use } d^3p = 4\pi p^2\, dp$$

and $\mathcal{E}(p) = cp$

$$= \frac{V}{\pi^2 \hbar^3} \int_0^\infty \frac{p^2\, dp\, (cp)}{e^{cp/kT} - 1}$$

- Again, measure p in units of $p_0 = kT/c$ & $cp_0 = kT$

$$E = V \left(\frac{p_0}{\hbar}\right)^3 c p_0 \left[\frac{1}{\pi^2} \int_0^\infty \frac{u^3\, du}{e^u - 1} \right]$$

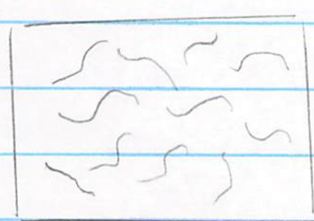
The integral can be done analytically

$$\frac{E}{V} = \left(\frac{p_0}{\hbar}\right)^3 c p_0 \left(\frac{\pi^2}{15}\right) \qquad \text{now} \quad c p_0 = k_B T$$

$$= \left(\frac{k_B T}{\hbar c}\right)^3 k_B T \left(\frac{\pi^2}{15}\right) \qquad \text{note} \quad \frac{\pi^2}{15} \simeq 0.66$$

- So the energy density is :

$$u = 0.66 \frac{k_B T}{\lambda_0^3} \propto T^4 \qquad \text{note} \quad u \sim n_\gamma k_B T$$



- Each photon volume of order λ_0^3 and energy of order $k_B T$

Energy Per Frequency

The energy per volume is

$$u = \frac{c}{\pi^2 \hbar^3} \int_0^\infty \frac{p^3 \, dp}{e^{cp/kT} - 1}$$

We want $\frac{du}{d\omega}$, the energy density per frequency

So

$$du = \frac{c}{\pi^2 \hbar^3} \frac{p^3}{e^{cp/kT} - 1} dp$$

Writing $p = \frac{\hbar \omega}{c}$ $dp = \frac{\hbar}{c} d\omega$ we find

$$du = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1} d\omega$$

So

$$\boxed{\frac{du}{d\omega} = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1}}$$

energy density per frequency

The Cosmic Microwave Background

- Many years ago, around 370,000 years after the big bang the electrons and protons recombined to make neutral hydrogen. The temperature was around 3000°K. Over the intervening 15 billion years the universe expanded, and the photons from that epoch got red shifted, effectively cooling off; every wavelength gets elongated by the same factor.

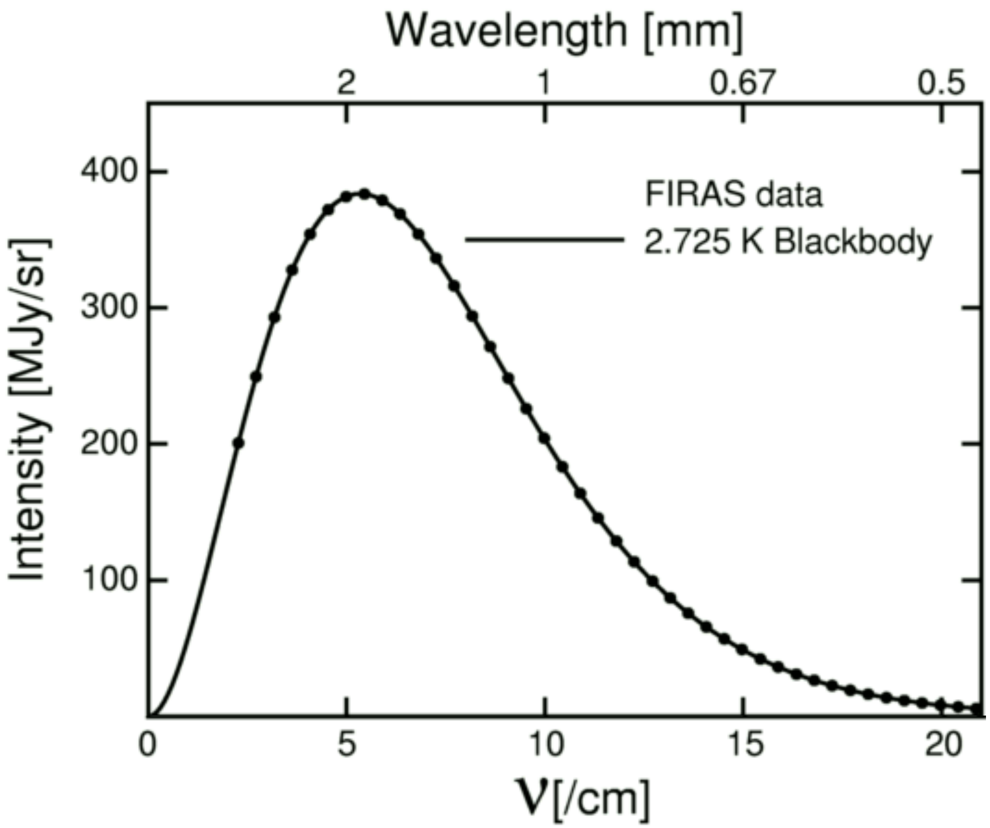
- What is observed is a background spectrum of microwave photons at a temperature of 2.725 °K

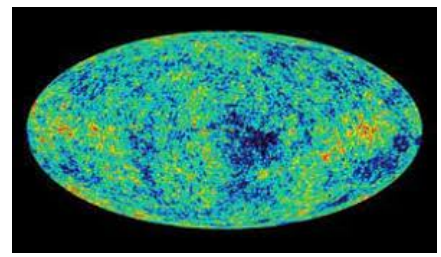
- What is observed in all directions of the sky is the best black body spectrum ever seen.

See Slide:

By fitting the blackbody curve we find $T = 2.725\ K$

The cosmic microwave background





The intensity proportional to :

$$I \propto \frac{\nu^3}{e^{h\nu/k_B T} - 1}$$

The frequency is ν and $h\nu = \hbar\omega$

The Pressure and Free energy

- The grand sum for one mode is

$$\mathcal{Z}_p = \frac{1}{1 - e^{-\beta(\varepsilon - \mu)}}$$

From here we derived $\bar{n} \propto \frac{\partial}{\partial \mu} \ln \mathcal{Z}$

- But we have

$$\Phi_G = -k_B T \ln \mathcal{Z}_p \quad \leftarrow \text{one mode}$$

- And so the total grand potential is

$$\Phi_G^{tot} = \sum_{modes} -k_B T \ln \mathcal{Z}_p$$

This is general

- $$\Phi_G^{TOT} = \sum_{modes} k_B T \ln\left(1 - e^{-\beta(\varepsilon(p) - \mu)}\right)$$

This is for the photon gas with $\mu = 0$ and $\varepsilon(p) = cp$

$$= 2V \int \frac{d^3p}{(2\pi\hbar)^3} k_B T \ln\left(1 - e^{-\beta c p}\right)$$

- Once we know Φ_G all else follows

$$d\Phi_G = -S\,dT - P\,dV - N\,d\mu$$

pressure!

Differentiating (or using that $\Phi_G = -pV$ for simple fluids)

$$\boxed{P = -\frac{\partial \Phi_G}{\partial V} = -2\int \frac{d^3p}{(2\pi\hbar)^3}\, k_B T \ln\left(1 - e^{-\beta c p}\right)}$$

So with $d^3p = 4\pi p^2\, dp$

$$P = \frac{4\pi}{(2\pi\hbar)^3} - \int_0^\infty p^2\, dp\, k_B T \ln\left(1 - e^{-\beta c p}\right)$$

- Now integrate by parts once

$$dv = p^2\, dp \qquad u = k_B T \ln\left(1 - e^{-\beta c p}\right)$$

$$v = \frac{1}{3}p^3 \qquad du = \frac{k_B T\, e^{-\beta c p}}{1 - e^{-\beta c p}}\,\beta c\, dp,$$

$$du = \frac{c\, dp}{e^{\beta c p} - 1}$$

So

$$P = \frac{4\pi}{(2\pi\hbar)^3}\int_0^\infty \frac{1}{3}p^3\, dp \times \frac{c}{e^{\beta c p} - 1}$$

- So comparison with the energy per volume gives

$$\boxed{P = \frac{1}{3}u}$$

- In words; The radiation pressure is $1/3$ of the energy density

$$p = \frac{1}{3}\left(\frac{k_B T}{\hbar c}\right)^3 k_B T \cdot \frac{\pi^2}{15}$$