Exact and Inexact Differentials

Take $f(x,y) = x\,y$

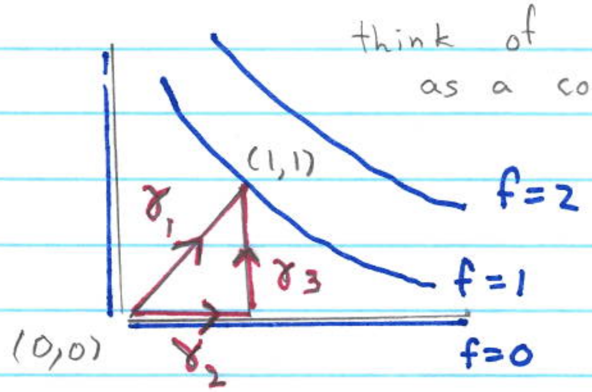
Then $df = y\,dx + x\,dy$ ← this exact representing a small change in f

- Then the integral:

$$\Delta f = \int_{\gamma_1} df = \int_{\gamma_2+\gamma_3} df = xy\Big|_{(0,0)}^{(1,1)}$$

$$= 1\cdot 1 - 0\cdot 0 = 1$$

think of f as a contour map



Is independent of path. It is the change in the contour level

- Note by $\int_{\gamma_1} df$ we mean a line integral. $\gamma_1(t)$ is a path, which is a map from $t \in [0,1] \longrightarrow (x(t), y(t))$ forming a path.

$$\int_{\gamma_1} df = \int_0^1 dt \; y(t)\frac{dx}{dt} + x(t)\frac{dy}{dt}$$

$\gamma_1 : t \longrightarrow (t, t)$

↑ this the path $\gamma_1(t)$ shown above.

$$= \int dt \; t\cdot 1 + t\cdot 1 = 1$$

A path $\gamma_1(t)$ is a one parameter map from $t \in [0,1]$ to the x,y plane: $t \mapsto (x(t), y(t))$. The path $\gamma_1(t)$ is shown above, and is $t \mapsto (t,t)$ mathematically.

- Now consider $dg = y\,dx$

Then for γ_1

$$g_{(1)} = \int_{\gamma_1} đg = \int_0^1 \underbrace{t}_{y(t)} \cdot \underbrace{1}_{\frac{dx}{dt}} \, dt = \frac{1}{2}$$

But for

$$g_{(2)+(3)} = \underbrace{\int_{\gamma_2} y\,dx}_{\substack{y=0 \\ \text{on } \gamma_2}} + \underbrace{\int_{\gamma_3} y\,dx}_{\substack{dx=0 \\ \text{on } \gamma_3}} = 0$$

- So the integral $\int đg$ depends on path and $đg$ is not exact. It is <u>not</u> the change in a function g. It is a small amount of something

- Suppose we have a differential

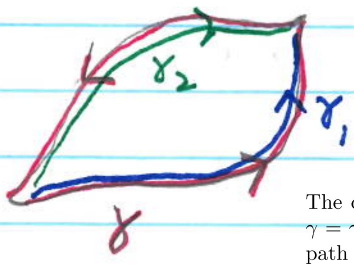
$$đh = F_x(x,y)\,dx + F_y(x,y)\,dy$$

How do we know if it is exact. It will be exact if it is curl free

$$\boxed{\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0}$$ exact differential

- Then the integral over any closed loop vanishes, implying path independence.

closed loop



The closed loop γ is a sum of two paths $\gamma = \gamma_1 \oplus (-\gamma_2)$, with $(-\gamma_2)$ meaning the path γ_2 with *reversed* direction.

$$\oint_\gamma dh = 0$$

or

$$\int_{\gamma_1} dh - \int_{\gamma_2} dh = 0$$

This is by the curl theorem:

$$\oint_\gamma dh = \int_{\text{area}} dx\, dx \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \quad \text{(always true)}$$

$$= 0 \quad \text{if} \quad \vec{F} \doteq (F_x, F_y) \text{ is curl free}$$

- If a differential dh is exact, it is the gradient of some function h

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$$

i.e. $$\boxed{\vec{F} = (F_x, F_y) = \vec{\nabla} h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right)}$$