Most energetic

a) $$U = 2\int \frac{V\,d^3p}{(2\pi)^3} \frac{\varepsilon}{e^{\beta\varepsilon}-1}$$

So writing $\varepsilon = \hbar\omega = cp$

$$d^3p = 4\pi p^2\,dp = \frac{\hbar^3}{c^3}\omega^2\,d\omega \cdot 4\pi$$

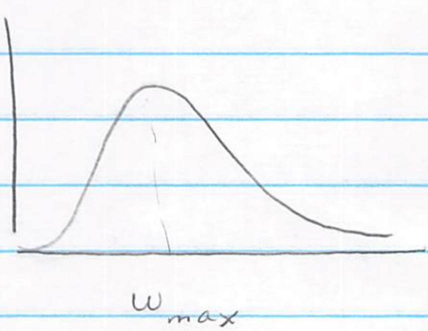
We find after algebra

$$\star \quad u = \frac{\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3\,d\omega}{e^{\beta\hbar\omega}-1}$$

So

$$\boxed{\frac{du}{d\omega} \propto \frac{\omega^3}{e^{\beta\hbar\omega}-1}}$$

Plotting this, we find a maximum (see next page) at:



$\beta\hbar\omega \simeq 2.8$

so

$$\hbar\omega = 2.8\,k_B T = 2.8 \times \frac{0.025\,eV}{300\,°K}\,6000\,°K = 1.4\,eV$$

b) Then

$$\omega = 2\pi f = \frac{2\pi c}{\lambda}$$

$$d\omega = d\lambda \left|\frac{d\omega}{d\lambda}\right| = d\lambda \left|\frac{-2\pi c}{\lambda^2}\right|$$

note absolute value for unoriented integrals as discussed previously

So substituting into eq ☆ above

$$u = \frac{\hbar}{\pi^2 c^3} (2\pi c)^4 \int_0^\infty \frac{d\lambda}{\lambda^5} \frac{1}{e^{\beta\hbar(2\pi)/\lambda} - 1}$$

Note $\beta \hbar \, 2\pi/\lambda = \beta \, hc/\lambda$ so

$$\boxed{\frac{du}{d\lambda} \propto \frac{1}{\lambda^5} \frac{1}{e^{\beta hc/\lambda} - 1}}$$

Plotting this gives (see next page)

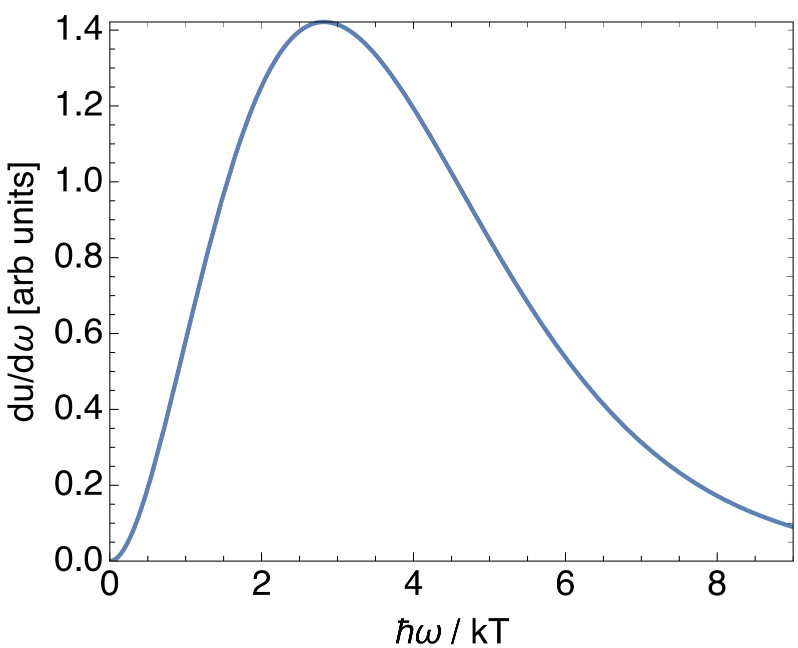
See next page for corrected solution

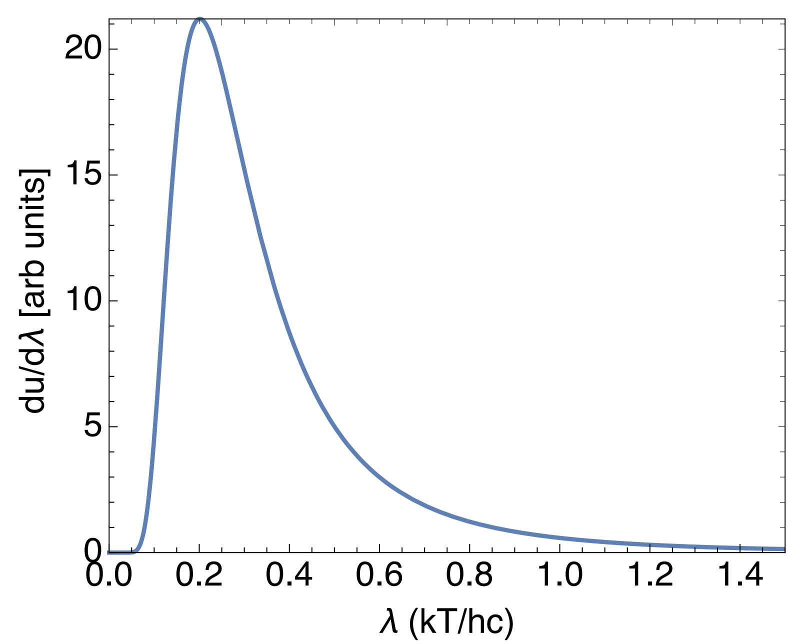
$\beta \frac{hc}{\lambda} = 0.20$ or $\lambda \simeq 5.0 \frac{hc}{k_B T}$

Infrared

So $\lambda = 5.0 \; \frac{1240\,\text{eV nm}}{.025\,\text{eV}} \; \frac{300°\text{K}}{6000°\text{K}} = 12{,}400\,\text{nm}$

Spectral Density of Energy





From the plot we see that the spectral density $du/d\lambda$ is max when

$$\lambda \simeq 0.2\frac{hc}{kT}$$

Putting $k_B = 0.025\,\mathrm{eV}/300\,\mathrm{K}$ and $hc = 1240\,\mathrm{eVnm}$ I find for $T = 5340K$

$$\lambda \simeq 560\,\mathrm{nm} \qquad \text{Yellowish}$$

2D World

The number of bosons

$$N = \sum_{\text{modes}} \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$$

mean number of bosons in a mode

- For photons, $\mu = 0$ and $\varepsilon(p) = cp$

$$\sum_{\text{modes}} \longrightarrow \underset{\uparrow}{2} \int \frac{L\,dp_x}{h} \frac{L\,dp_y}{h} = 2A \int \frac{d^2p}{(2\pi\hbar)^2}$$

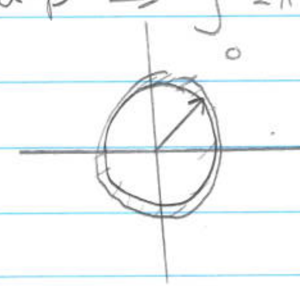
spin degeneracy

- Compare to 3D

$$\sum_{\text{modes}} \longrightarrow 2 \int \frac{V\,d^3p}{h^3}$$

- So

$$\int d^2p \rightarrow \int_0^\infty 2\pi p\,dp$$



$$N = 2A \int \frac{d^2p}{(2\pi\hbar)^2} \frac{1}{e^{\beta cp} - 1}$$

$$= 2A \int_0^\infty \frac{2\pi\,p\,dp}{(2\pi\hbar)^2} \frac{1}{e^{\beta cp} - 1}$$

$$N = \frac{A}{\pi} \frac{1}{\hbar^2} \int_0^\infty \frac{p\,dp}{e^{\beta cp} - 1}$$

The same tricks gives

$$U = \frac{A}{\pi}\left(\frac{kT}{\hbar c}\right)^2 kT \underbrace{\int_0^\infty \frac{x^2\,dx}{e^x - 1}}_{2.404}$$

So the energy density is

$$\boxed{\frac{U}{A} = \left(\frac{kT}{\hbar c}\right)^2 kT \cdot (0.765)}$$

- For the neutrino case the only thing that changes is the mean number of particles per mode

$$\boxed{\begin{aligned} \bar{n}_{boson} &= \frac{1}{e^{\beta\varepsilon} - 1} \\ \bar{n}_{fermion} &= \frac{1}{e^{\beta\varepsilon} + 1} \end{aligned}}$$

This is for $\mu = 0$

So

$$\frac{N}{A}_{neutrino} = \frac{1}{\pi}\left(\frac{kT}{\hbar c}\right)^2 \underbrace{\int_0^\infty \frac{x\,dx}{e^x + 1}}_{= \pi^2/12}$$

$$\boxed{\frac{N}{A} = A\left(\frac{kT}{\hbar c}\right)^2 \frac{\pi}{12}}$$

So we see that this is half the boson case.

So switch to dimensionless momentum

$$x \equiv \beta c p \quad \rightarrow \quad p\,dp = \frac{x\,dx}{(\beta c)^2}$$

And

$$N = \frac{A}{\pi \hbar^2} \frac{1}{(\beta c)^2} \underbrace{\int_0^\infty \frac{x\,dx}{e^x - 1}}_{\pi^2/6}$$

So the density of photons is

$$\boxed{\frac{N}{A} = \left(\frac{kT}{\hbar c}\right)^2 \frac{\pi}{6}}$$

$$\lambda_{th} \equiv \left(\frac{\hbar c}{kT}\right)$$ typical thermal wavelength

$$\frac{N}{A} = \frac{\pi}{6} \frac{1}{\lambda^2}$$

i.e. the area per photon A/N is of order the typical wavelength, $(kT/\hbar c)$, squared

- Similarly the energy is

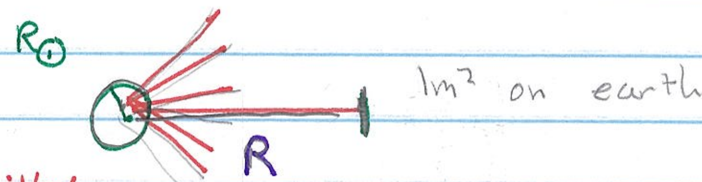
$$U = \sum_{\text{modes}} \frac{\varepsilon(p)}{e^{\beta\varepsilon(p)} - 1}$$ energy of mode × mean number per mode

$$U = 2A \int \frac{d^2p}{(2\pi\hbar)^2} \frac{cp}{e^{\beta c p} - 1} \sim \int \frac{2\pi p\,dp \cdot p}{e^{\beta c p} - 1}$$

- This is expected since there is only at most one fermion per mode while there can be many boson's permode

Ex,

- What is the Temperature of the Sun, given that $I = 1 kW/m^2$ of energy per area per second comes from sun?



total energy emitted

$$\frac{dU}{dt} = \sigma T^4 \; 4\pi R_\odot^2$$

R ≡ distance from sun to earth

$R_\odot$ ≡ Radius of sun

$$\left.\frac{1}{A}\frac{dU}{dt}\right|_{earth} = \sigma T^4 \frac{4\pi R_\odot^2}{4\pi R^2} = \sigma T^4 \frac{R_\odot^2}{R^2}$$

$\Omega_{sun} = 6.8 \times 10^{-5}$

the energy per area absorbed on earth.

$$I = \frac{\sigma T^4}{\pi}\left(\frac{\pi R_\odot^2}{R^2}\right)$$

this is the Solid angle of sun as seen on earth

$$T = \left(\frac{I\pi}{\sigma \Omega_{sun}}\right)^{1/4}$$

$\Omega_{sun} = \frac{A}{r^2}$

The dU/dt is spread out over area $4\pi R^2$

$$\boxed{T = 5342\,°K}$$

pretty close!

you measure this with a protractor

Density of States

a) So

$$d\mathcal{N} = V\frac{d^3p}{(2\pi\hbar)^3} = \text{number of modes with momentum } \vec{p} = (p_x, p_y, p_z) \text{ in range}$$

$[p_x; dp_x]$, $[p_y; dp_y]$, $[p_z; dp_z]$

This means $p_x < p_x' < p_x + dp_x$:

$$d^3p = 4\pi p^2\, dp$$

And $p = \hbar k$, so

$$d\mathcal{N} = V\frac{4\pi}{(2\pi)^3}k^2 dk = \boxed{\frac{V}{2\pi^2}k^2 dk} \rightarrow \text{or } g(k) = \frac{Vk^2}{2\pi^2}$$

= number of modes with k in range $k < k' < k + dk$

b) In two dimensions

$$d\mathcal{N} = \frac{A\,d^2p}{(2\pi\hbar)^2} = A\frac{2\pi p\,dp}{(2\pi\hbar)^2} \quad \Big) \; p = \hbar k$$

$$\boxed{d\mathcal{N} = \frac{1}{2\pi}A\,k\,dk}$$

or $\boxed{g(k) = Ak/2\pi}$

c) Then the free energy of one mode is

$$\Phi_G^{\varepsilon} = - k_B T \ln \frac{1}{1 - e^{-\beta(\varepsilon - \mu)}}$$

where $\mathcal{Z}_p = \frac{1}{1 - e^{-\beta(\varepsilon-\mu)}}$ is the grand partition function of one mode for a boson

So

$$\Phi_G^{tot} = \sum_{modes} \Phi_G^{\varepsilon}$$

By definition the sum over modes becomes an integral over the mode density, $g(\varepsilon)\,d\varepsilon$

$$\sum_{modes} \ldots = \int g(\varepsilon)\, d\varepsilon \ldots$$

So

$$\boxed{\Phi_G = \int_0^{\infty} g(\varepsilon)\; k_B T \ln\left(1 - e^{-\beta(\varepsilon - \mu)}\right)}$$

For a fermion

$$\mathcal{Z}_p = 1 + e^{-\beta(\varepsilon - \mu)}$$

So since

$$\mathcal{E}(p) = \frac{p^2}{2m} \qquad d\mathcal{E} = \frac{p}{m}\,dp$$

<u>Then in 3D</u>

$$d\mathcal{N} = \frac{V}{2\pi^2}\frac{p^2\,dp}{\hbar^3} = \frac{1}{2\pi^2}\frac{m\,p}{\hbar^3}\,d\mathcal{E} = \frac{1}{4\pi^2}\left(\frac{2m\,p}{\hbar^3}\right)d\mathcal{E}$$

$$= \boxed{\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\mathcal{E}}\,d\mathcal{E} = g(\mathcal{E})\,d\mathcal{E}}$$

grouping it like this is motivated by units:

<u>In 2D:</u>

$$d\mathcal{N}_{\vec{p}} = \frac{A\,d^2p}{(2\pi\hbar)^2} \qquad \frac{1}{\lambda_{typ}} \sim \frac{p_{typ}}{\hbar} \sim \left(\frac{2m}{\hbar^2}\right)^{1/2}\mathcal{E}^{1/2}$$

Integrating over the angles of p we have

$$d\mathcal{N}_p = \frac{A\,p\,dp}{2\pi\hbar^2}$$

now with

$$\mathcal{E} = \frac{p^2}{2m} \qquad d\mathcal{E} = \frac{p\,dp}{m}$$

$$d\mathcal{N}_{\mathcal{E}} = \frac{A\,m\,d\mathcal{E}}{2\pi\hbar^2}$$

$$\boxed{d\mathcal{N}_{\mathcal{E}} = \frac{A}{4\pi}\left(\frac{2m}{\hbar^2}\right)d\mathcal{E}}$$

and

$$\Phi_G = \int_0^\infty g(\varepsilon)\, d\varepsilon \quad -k_B T \ln\left(1 + e^{-\beta(\varepsilon - \mu)}\right)$$

☆ Then for photon $\quad$ spin of photons (or polarizations) $= 2$

$$d\mathcal{N}_{modes} = \int_{angles} 2 \, \frac{V\, d^3 p}{(2\pi\hbar)^3} = \frac{1}{\pi^2 \hbar^3} V p^2 dp$$

Now $\quad \varepsilon = cp \quad$ so

$$d\mathcal{N} = \frac{1}{\pi^2} \frac{V}{(\hbar c)^3} \varepsilon^2 d\varepsilon = g(\varepsilon)\, d\varepsilon$$

And

$$\boxed{\Phi_G = \frac{V}{\pi^2 (\hbar c)^3} \int_0^\infty \varepsilon^2 \, k_B T \ln\left(1 - e^{-\beta(\varepsilon - \mu)}\right)}$$

So

$$\Phi_G = -pV \quad \text{So}$$

$$pV = -\frac{1}{\pi^2 (\hbar c)^3} \int_0^\infty \varepsilon^2 k_B T \ln\left(1 - e^{-\beta(\varepsilon - \mu)}\right) d\varepsilon$$

Entropy / Photon

- So setting $\varepsilon = \hbar\omega$ the results of the previous problem can be written

$$pV = -\frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\hbar^3 \int_0^\infty \underbrace{\omega^2 d\omega}_{dv} \, \underbrace{k_BT \ln(1-e^{-\beta\hbar\omega})}_{u}$$

- Now integrate by parts

$$dv = \omega^2 d\omega \qquad u = k_BT \ln(1-e^{-\beta\hbar\omega})$$

$$v = \frac{1}{3}\omega^3 \qquad du = k_BT \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} \beta\hbar \, d\omega$$

$$= \frac{\hbar \, d\omega}{e^{\beta\hbar\omega}-1}$$

So

$$\int_0^\infty u \, dv = uv \Big|_0^\infty - \int_0^\infty v \, du$$

this is zero since $u\big|_0 = 0$ and $v\big|_\infty = 0$

- And so

$$pV = \frac{V}{\pi^2 c^3}\hbar \int_0^\infty \frac{1}{3}\omega^3 d\omega \frac{\hbar}{e^{\beta\hbar\omega}-1}$$

First we make the integral dimensionless

$$u = \beta \hbar \omega \qquad du = \beta \hbar \, d\omega$$

Then we have

$$pV = \frac{V}{\pi^2 c^3} \hbar \frac{1}{3} \frac{1}{(\beta \hbar)^4} \int_0^\infty \frac{u^4}{e^u - 1}$$

$$= V \left(\frac{k_B T}{\hbar c} \right)^3 k_B T \left[\frac{1}{3\pi^2} \int_0^\infty \frac{u^4 \, du}{e^u - 1} \right]$$

The integral is

$$\frac{\pi^4}{15} \qquad \text{(See attached Table)}$$

And so we find

$$\boxed{pV = V \left(\frac{k_B T}{\hbar c} \right)^3 k_B T \frac{\pi^2}{45}}$$

Note

$$d\Phi_G = -S\,dT - N\,d\mu - p\,dV$$

With $\Phi_G = -p(T,\mu) V$

So

$$\frac{\partial p}{\partial T} V = S$$

So since $pV = CT^4$

$$\boxed{S = \frac{\partial p}{\partial T} V = 4CT^3 = 4\frac{CT^4}{T} = 4\frac{pV}{T}}$$

So

$$U - TS + pV = \cancelto{0}{\mu N}$$

$$\frac{S}{Nk_B} = \frac{4\pi^2/15}{0.245}$$

$$= 3.6$$

Then

$$U = -pV + TS$$

$$U = -pV + 4pV = 3pV$$

So

$$\boxed{\frac{U}{V} = 3p \qquad \text{as before}}$$