Physics 306: Thermal Physics

Final Exam

Stony Brook University

Spring 2022

General Instructions:

You may use one page (front and back) of handwritten notes and a calculator. Graphing calculators are allowed. **No other materials may be used**.

Problem 1. Ideal gas in 2D

First consider a classical gas of N atoms of mass M at temperature T in two spatial dimensions. Each particle is free to move in the x, y directions, but is confined to an area A.

(a) Determine the partition function and free energy of the system.

(b) Starting from the first law of thermodynamics $dU = dQ + dW$, derive an expression for dF where F is the free energy.

(c) Use your result for the free energy and dF to derive an expression for the entropy of the gas.

Now consider the same 2D gas, consisting of N molecules. Each molecule is of mass M (as before), but now each molecule has internal energy states ϵ_s, so that the total energy of one molecule is

$$E_1 = \frac{p^2}{2M} + \epsilon_s \,. \tag{1}$$

The internal energy levels can take on two possible values: the ground state has energy $\epsilon_0 = 0$ and is not degenerate, while the excited energy level has energy $\epsilon_1 = \Delta$ and degeneracy g.

(d) Determine the entropy of the gas of molecules.

(e) Determine the entropy of the gas in the limits where $k_B T$ is low compared to Δ, and high compared to Δ. How do your limiting expressions compare to part (b)? In both limits, explain the similarities or differences with part (b) physically.

Problem 2. Classical partition function

A classical particle of mass m at temperature T moves in one dimension, in a potential well

$$V(x) = \alpha |x| \,, \tag{2}$$

where α is a constant with units of energy per length. The total energy is $p^2/2m + V(x)$.

(a) Determine the classical partition function by integrating over the coordinates and momenta.

(b) Determine the energy and the specific heat of the particle.

(c) Derive an expression for the entropy of the particle in the well. Explain physically the dependence of the entropy on the constant α.

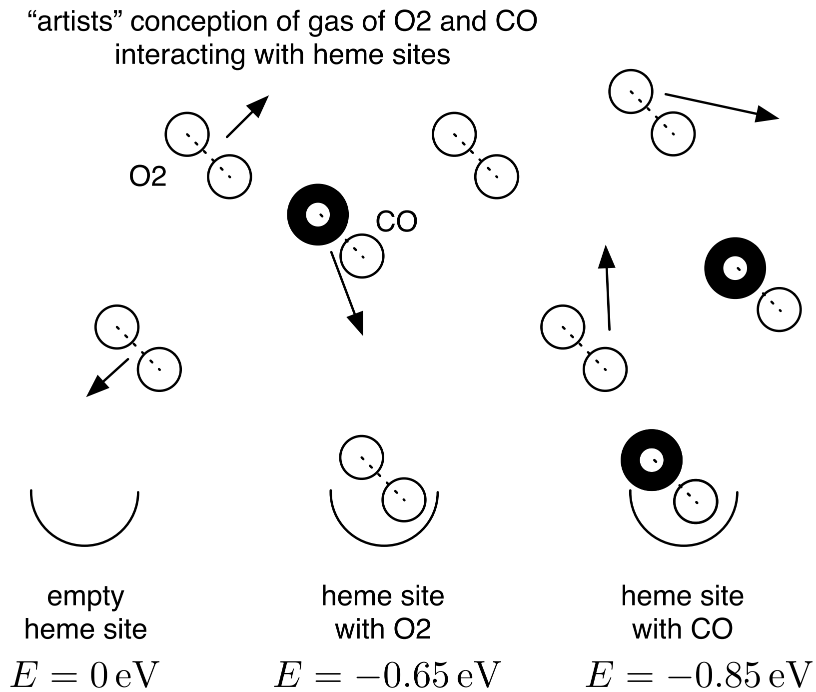

Figure 1: Artists conception of problem 3

Problem 3. Carbon Monoxide Poisoning

A model for carbon monoxide poisoning is the following. Consider a gas which is mixture of diatomic oxygen O_2 and carbon monoxide CO. The sites on the hemoglobin molecule can either be unoccupied, with energy $E = 0$, occupied by an O_2 molecule with energy $E = -0.65\,\mathrm{eV}$, or occupied by a carbon monoxide molecule with energy $E = -0.85\,\mathrm{eV}$, see figure. In this problem you will calculate the probability that the hemoglobin site will be occupied by an O_2 (what we want!). This depends on the concentration of O_2 and sensitively on the concentration of carbon monoxide.

The questions below refer to the surrounding O_2 gas at a temperature of $295\,\mathrm{K}$ and a pressure of $0.2\,\mathrm{bar}$. From the temperature and pressure of O_2, the corresponding concentration $n = N/V$ of the gas can be found, as can its quantum concentration[1], $n_Q \equiv \lambda_{\mathrm{th}}^{-3}$. The quantum concentration of CO can be found similarly. These values and the atomic numbers of the two atoms are given in the table below.

[1] λ_{th} is the thermal de Broglie wavelength.

quantity	value
T	295 K
p	0.2 bar
n	$0.005\,\mathrm{nm}^{-3}$
$(n_Q)_{O_2}$	$1.68 \times 10^5\,\mathrm{nm}^{-3}$
$(n_Q)_{CO}$	$1.37 \times 10^5\,\mathrm{nm}^{-3}$
atomic number O	16
atomic number C	12

(a) Explain the ratio of quantum concentrations for the two gasses, O_2 and CO.

(b) The CO and O_2 molecules in the surrounding gas rotate with moment of inertia I. Their rotational constants, i.e. $\Delta \equiv \hbar^2/2I$, are $\Delta_{CO} = 0.00024\,\mathrm{eV}$ and $\Delta_{O_2} = 0.00018\,\mathrm{eV}$ respectively. Show that the rotational constant of O_2 is roughly consistent with an order of magnitude estimate for Δ.

(c) Recall that the rotational energy levels are

$$\epsilon_{\mathrm{rot}} = \ell(\ell+1)\Delta \qquad \text{with} \qquad \ell = 0, 1, 2, \ldots \infty \tag{3}$$

and that the rotational partition function (i.e. an appropriate sum over these levels) is $Z_{\mathrm{rot}} \simeq k_B T/\Delta$ in a classical approximation. Estimate the typical value of ℓ for the CO gas. Based on this estimate how accurate is the classical approximation?

(d) Recall that the partition function of the classical diatomic gas is

$$Z_{\mathrm{tot}} = \frac{1}{N!}(Z_{\mathrm{trans}} Z_{\mathrm{rot}})^N \tag{4}$$

where $Z_{\mathrm{rot}} \equiv k_B T/\Delta$ with $\Delta = \hbar^2/2I$, and Z_{trans} describes the translational motion.

(i) Determine the chemical potential of the classical diatomic gas as a function of the concentration n and the rotational constant Δ.

(ii) Numerically evaluate the chemical potential μ_{O_2} of the O_2 gas.

(iii) Numerically evaluate the chemical potential μ_{CO} of the surrounding CO gas, assuming that the concentration of CO is a thousand times smaller than O_2.

(e) Now return to the hemoglobin sites. By considering the grand partition function of the site, determine the probability that the site is occupied by O_2. Evaluate this probability numerically, using the numerical results of previous parts.

(f) Determine how the probability of (e) would change if the concentration of CO was negligibly small.

$$
\begin{aligned}
\int_0^\infty \mathrm{d}x \, \frac{x}{e^x - 1} &= \frac{\pi^2}{6} \\
\int_0^\infty \mathrm{d}x \, \frac{x^2}{e^x - 1} &= 2\zeta(3) \simeq 2.404 \\
\int_0^\infty \mathrm{d}x \, \frac{x^3}{e^x - 1} &= \frac{\pi^4}{15} \\
\int_0^\infty \mathrm{d}x \, \frac{x^4}{e^x - 1} &= 24\,\zeta(5) \simeq 24.88 \\
\int_0^\infty \mathrm{d}x \, \frac{x^5}{e^x - 1} &= \frac{8\pi^6}{63}
\end{aligned}
$$

Figure 2: A compendium of useful integrals over Bose distributions

Problem 4. Photon gas in three dimensions

Consider a gas of photons in three dimensions at temperature T and volume V.

(a) Starting from the general expression for the grand partition function of a single mode, derive the Bose-Einstein expression for the mean number of particles in a mode with single particle energy ϵ

$$n_{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} - 1}$$

Assume that $\mu = 0$ in what follows.

(b) Derive an expression for the total number of photons per volume in the gas, explaining carefully each step. Some integrals are below.

(c) The density of the sun is $1.40\,\mathrm{g/cm}^3$ which is predominantly hydrogen. The sun may be treated as $5000^o\,\mathrm{K}$ black body:

- Estimate the number of photons per proton in the sun.
- Give a rough estimate of the typical photon wavelength.

(d) Determine the number of photons per volume with frequency less than ω_0, which we will call $n_<(\omega_0)$. Also determine the number of photons per volume per frequency interval, $dn/d\omega$.

(e) Find a series expansion for $dn/d\omega$ at high temperature. Determine both the leading term and the first subleading term in this expansion. Find an approximate expression for the number $n_<(\omega_0)$ from part (d) from this series.