Otto Cycle

This is the last part (which was the first part in prior years)

- Between $1 \to 2$ we have

$$\Delta U = \cancel{Q} + W_{12}$$

So

$$W_{12} = C_V \, \Delta T_{12} = C_V T_2 \left(1 - \frac{1}{r^{\gamma-1}}\right)$$

see next item

- Now $1 \to 2$ is adiabatic

$$T_2 = T_1 \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

So $\Delta T_{12} = T_1 (r^{\gamma-1} - 1)$

$\left(\frac{T_2}{T_1}\right) = r^{\gamma-1}$ so

$$\Delta T_{12} = T_2 \left(1 - \frac{1}{r^{\gamma-1}}\right)$$

- Then for $2 \to 3$

$$C_V \, \Delta T_{23} = Q + \cancel{W}^{0}$$

$$\Delta T_{23} = \frac{Q_{in}}{C_V}$$

- Then similarly for $3 \to 4$

$$\Delta U = W$$

$$T_4 = T_3 \left(\frac{V_3}{V_4}\right)^{\gamma-1} = T_3 \frac{1}{r^{\gamma-1}}$$

$$\Delta U = C_V (T_4 - T_3)$$

$$\Delta U = - C_V T_3 \left(1 - \frac{1}{r^{\gamma-1}}\right) = W_{34}$$

So

$$W_{net} = W_{34} + W_{12} = -\overbrace{C_V (T_3 - T_2)}^{= Q_{in}} \left(1 - \frac{1}{r^{\gamma-1}}\right)$$

Finally

$$\eta = \frac{-W_{net}}{Q_{in}} = \frac{+Q_{in}}{Q_{in}} \left(1 - \frac{1}{r^{\gamma-1}}\right)$$

$$\boxed{\eta = 1 - \frac{1}{r^{\gamma-1}}}$$

b) For 1mol

$$pV = Nk_B T$$

$$V = \frac{N_A k_B T}{p} = \frac{RT}{p} = \frac{8.32 \frac{J}{°K} \; 300°K}{10^5 \, N/m^2}$$

$$= 25.L$$

Now the volume of the cylinder is 2.5L

So

$$\frac{2.5L}{25L} = \frac{n_{ml}}{1 \text{ mol}} \Rightarrow \boxed{n_{me} = 0.1}$$

c) We use

$\boxed{1 \to 2}$ $$T_f = T_i \left(\frac{V_i}{V_f}\right)^{\gamma-1} \qquad p_f = p_i \left(\frac{V_i}{V_f}\right)^{\gamma}$$

$$T_2 = T_1\, r^{\gamma-1} \qquad p_2 = p_1\, r^{\gamma} \qquad V_2 = V_1/r$$

$$\underline{T_2 = 689\ ^\circ K} \qquad \underline{p_2 = 18.4\, b} \qquad \underline{V_2 = 0.3L}$$

$$\Delta U_{12} = W_{1\to 2} = C_v(T_f - T_i) = C_v T_i \,(r^{\gamma-1} - 1)$$

Now $C_v T_i = U_i$, for 1 ml we have

$$C_v^{1ml} = \frac{5}{2}R \qquad C_v^{0.1ml} = 0.1 \times \frac{5}{2}R = 0.832 \frac{J}{^\circ K} \times \frac{5}{2} = 2.08 \frac{J}{^\circ K}$$

$$\underline{\Delta U_{12} = \Delta W_{12} = 810 J, \qquad Q = 0}$$

$\boxed{2 \to 3}$ Now

$$\underline{\Delta U_{23} = Q_{in} + \cancel{W}^{0} = 2{,}200J \qquad W = 0}$$

$$C_v\,(T_3 - T_2) = Q_{in} \qquad Q_{in} = 0.1\ mol \times 22000\frac{J}{mol} = 2{,}200J$$

$$\underline{T_3} = T_2 + \frac{Q_{in}}{C_v} = \underline{1747\ ^\circ K}$$

$$P_3 = \frac{N k_B T}{V}$$

$$\frac{P_3}{P_2} = \frac{T_3}{T_2}$$

$$P_3 = P_2 \frac{T_3}{T_2} = 46.5\ b$$

$$V_3 = 0.3 L$$

$\boxed{3 \to 4}$

$$\Delta U = \cancel{Q}^{\,0} + W$$

$$\Delta U = C_V (T_f - T_i)$$

$$= - C_V T_i \left(1 - \frac{1}{r^{\gamma - 1}}\right)$$

$$W_{34} = \Delta U = -2052 J$$

Adiabatic

$$T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1}$$

$$T_4 = T_3 \frac{1}{r^{\gamma-1}}$$

$$T_4 = 760\ °K$$

$\boxed{4 \to 1}$

$$\Delta U = Q + \cancel{W}^{\,0}$$

$$C_V (T_1 - T_4) = Q$$

$$\Delta U = Q = -958 J \qquad W = 0$$

$$\frac{P_4}{P_1} = \frac{T_4}{T_1} = 6.33\ b$$

$$V_4 = 2.5 L$$

As a quick check we note

$$W_{net} = W_{12} + W_{34} = -1242\ J$$

$$Q_{in} = 2200\ J$$

So

$$\eta = \frac{|W_{net}|}{Q} = 0.56$$ which should be compared with $1 - \frac{1}{r^{\gamma-1}} = 1 - \frac{1}{8^{\gamma-1}} = 0.56$

Manipulating Taylor Series

a) $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

This follows from the geometric series
$1/(1-u) = 1 + u + u^2 + \ldots$ with $u = -x$

Integrating

$$\int_0^x \frac{dx'}{(1+x')} = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$$

b)

c) For x large we have $e^{-x} \ll 1$

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}}$$

call $u = e^{-x} \ll 1$

$$\frac{1}{e^x - 1} = \frac{u}{(1-u)} = u\,(1 + u + u^2 + \ldots)$$

$$\boxed{\frac{1}{e^x - 1} = e^{-x}\,(1 + e^{-x} + e^{-2x} + O(e^{-3x}))}$$

d) $\dfrac{1}{e^x - 1}$

we expand $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$

$$\frac{1}{e^x - 1} \simeq \frac{1}{x + x^2/2 + x^3/6} = \frac{1}{x} \frac{1}{(1 + x/2 + x^2/6 + O(x^3))}$$

calling, $u = x/2 + x^2/6$, we have

$$\frac{1}{e^x - 1} \simeq \frac{1}{x}\left(\frac{1}{1+u} + O(x^3) \right)$$

$$\simeq \frac{1}{x}(1 - u + u^2 + O(x^3))$$

$$\simeq \frac{1}{x}\left(1 - \left(\frac{x}{2} + \frac{x^2}{6}\right) + \left(\frac{x^2}{4}\right) + O(x^3)\right)$$

$$\boxed{\frac{1}{e^x - 1} \simeq \frac{1}{x}\left(1 - \frac{x}{2} + \frac{x^2}{12}\right) + O(x^2)}$$

e) $\dfrac{1}{e^{-x} + 1} \simeq 1 - e^{-x} + e^{-2x}$ set $u = e^{-x}$

f) $\log(1 - e^{-x}) \simeq \log\left(1 - \left(1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6}\right)\right)$

$$\simeq \log x\left(1 - \frac{x}{2} + \frac{x^2}{6}\right) = \log x + \log\left(1 - \frac{x}{2} + \frac{x^2}{6}\right)$$

So

$$\log(1-e^{-x}) = \log(x) + \log(1 \underbrace{- \frac{x}{2} + \frac{x^2}{6}}^{\text{Call it } u}) + O(x^3)$$

Setting $u = -\frac{x}{2} + \frac{x^2}{6}$ we have

$$\log(1+u) = u - \frac{u^2}{2} + O(u^3)$$ with x of order u

So

$$\log(1-e^{-x}) \simeq \log(x) + \left(-\frac{x}{2} + \frac{x^2}{6}\right) - \frac{1}{2}\left(-\frac{x}{2}\right)^2 + O(x^3)$$

$$\boxed{\log(1-e^{-x}) \simeq \log x - \frac{x}{2} + \frac{x^2}{24} + O(x^3)}$$

Energy of SHO

a) We have

$$Z = \frac{1}{1 - e^{-\beta \hbar \omega_0}}$$

Then

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z = +\frac{\partial}{\partial \beta} \log\left(1 - e^{-\beta \hbar \omega_0}\right)$$

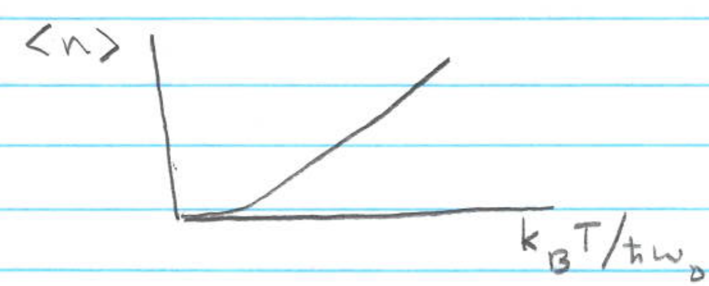
$$= \frac{1}{1 - e^{-\beta \hbar \omega_0}} e^{-\beta \hbar \omega_0} \hbar \omega_0$$

$$\boxed{\langle E \rangle = \frac{\hbar \omega_0}{e^{\beta \hbar \omega_0} - 1}}$$

b) Then

$$\frac{\langle E \rangle}{\hbar \omega_0} = \frac{1}{e^{\hbar \omega / kT} - 1} = \langle n \rangle$$

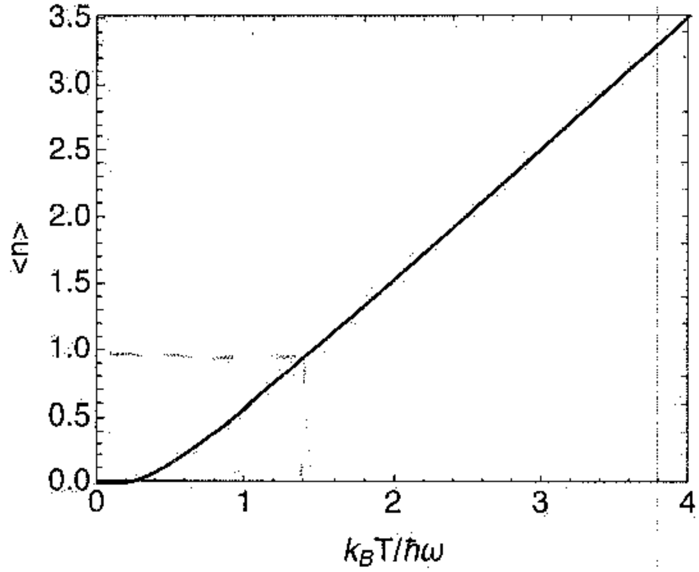
Then



Then from graph

$\langle n \rangle = 1$ when

$k_B T / \hbar \omega_0 = 1.45$



3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
<n>
0
1
2
3
4
$k_B T/\hbar\omega$

or

$$\boxed{k_B T = 1.45\, \hbar\omega_0}$$

c) Then a nice plot of $\langle E \rangle$ is given in the problem statement.

d) Using the series of problem 1

with $x \equiv \hbar\omega/kT$

at low temperature $kT \ll \hbar\omega_0$ then $x \gg 1$, and

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}} \simeq e^{-x}(1 + e^{-x} + \dots)$$

And

$$\boxed{\langle E \rangle = \hbar\omega_0\, e^{-\beta\hbar\omega_0}(1 + e^{-\beta\hbar\omega_0} + \dots)}$$

At high temperature $x \ll 1$

$$\frac{1}{e^x - 1} \simeq \frac{1}{x} - \frac{1}{2}$$

$$\langle E \rangle = \hbar\omega_0\left(\frac{k_B T}{\hbar\omega_0} - \frac{1}{2}\right) \simeq k_B T\left(1 - \frac{\hbar\omega_0}{2kT}\right)$$

e) At high temperature the number of quanta $\langle n \rangle$ is very large. In this regime $\langle n \rangle \gg 1$, quantum mechanics becomes continuous, $\frac{\Delta E}{E} \ll 1$, and it appraaches classical mechanics

This is the Bohr correspondence principle

f) We have

i) $$\boxed{U = N\Big[\underbrace{\frac{5}{2}kT + \frac{\hbar\omega_0}{e^{\beta\hbar\omega_0}-1}}_{\text{this is } f_0(T)}\Big]}$$

Then

ii) $$C_V = \left(\frac{dU}{dT}\right)_V = N\left[\frac{5}{2}k + \frac{-\hbar\omega_0 e^{\beta\hbar\omega_0}}{(e^{\beta\hbar\omega_0}-1)^2}\,\hbar\omega_0 \frac{\partial}{\partial T}\frac{1}{kT}\right]$$

$$= N\left[\frac{5k}{2} + \frac{(\beta\hbar\omega_0)^2 e^{\beta\hbar\omega_0}}{(e^{\beta\hbar\omega_0}-1)}k\right]$$

$$C_V = Nk\left[\frac{5}{2} + \frac{(\beta\hbar\omega_0)^2 e^{\beta\hbar\omega_0}}{(e^{+\beta\hbar\omega_0}-1)^2}\right]$$

So

$$C_p = C_v + Nk_B$$

$$\boxed{C_p = Nk_B \left[\frac{7}{2} + \frac{(\beta\hbar\omega_0)^2}{(e^{\beta\hbar\omega_0} - 1)^2} \right]}$$

iiv) So we see that the model nicely captures the transition from $C_p = \frac{7}{2} = 3.5$ to $\frac{9}{2} = 4.5$

but misses the transition to $\frac{5}{2}$ at low temperatures