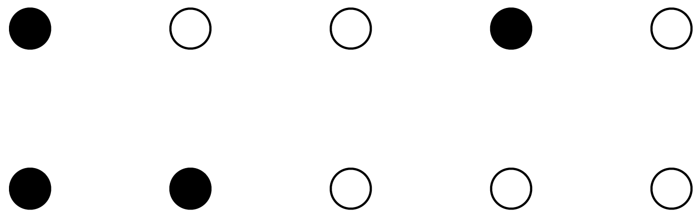
Problem 1. Entropy Revisited

Consider a collection of N atoms arranged on a line. The atoms can be in the ground state with probability P_0, and the excited state with probability P_1.

(a) Without approximation what is the total number configurations $\Omega(N_0, N_1)$ having N_0 atoms in the ground state and N_1 in the excited state. For instance, if the number of atoms is five, and the number of excited atoms (shown by the black circles) is 2, then two possible configurations are shown below



(b) Show that for N large:

$$\ln \Omega = N \left(-P_0 \ln P_0 - P_1 \ln P_1\right) \tag{1}$$

(c) How would you generalize this problem to an atom that can be in three states, A, B, C?

Problem 2. Entropy of Mixing form Gibbs Formula

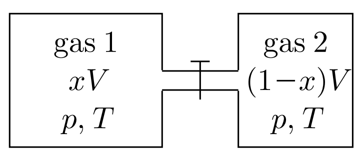


Fig. 14.6 Gas 1 is confined in a vessel of volume xV, while gas 2 is confined in a vessel of volume $(1-x)V$. Both gases are at pressure p and temperature T. Mixing occurs once the tap on the pipe connecting the two vessels is opened.

(a) Consider the entropy of mixing described in class Two gasses initially separated into two containers which have equal temperature and pressure. When the valve is openened the two gasses mix with each other and fill the entire container, as shown in the figure above. Follow the thermodynamic argument given in the text (Eq. 14.39) to show that the entropy produced during the process is

$$\Delta S = -\,N k_B \left[x \log x + (1-x)\log(1-x)\right] \tag{2}$$

(b) Use the Gibbs formula for the entropy

$$S = -k_B \sum_m P_m \log P_m \tag{3}$$

to rederive the result for ΔS.

Hint: For each molecule of the gasses define two states, i.e. in left container or in right container, and work out the probabilities both before and after the valve is openened, neglecting the momenum space.

Problem 3. Entropy of Paramagnets

Consider a paramagnet interacting with a heat bath at temperature T consisting of N independent spins with N large. Let $N_\uparrow$ denote the number of up spins , and $N_\downarrow$ the number of down spins. A magnetic field B points in the z direction, and the spins want to align with the magnetic field. The energy levels of each spin are $\mp\mu_B B$, where the spin up states have lower energy $-\mu_B B$, and the spin down states have an energy of $+\mu_B B$. So, the spin down states have a higher energy than the spin up states by an amount $\Delta = 2\mu_B B$. Since it is energetically favorable for the spin to be in the direction of B, it is the down spins that are excited. The mean number of excited atoms (spin down) is $\bar{n} = \bar{N}_\downarrow/N$. The total mean energy of the atoms is $\bar{E} = -\mu_B B(\bar{N}_\uparrow - \bar{N}_\downarrow)$.

(a) Work problem 20.5 from Blundell.

(b) Show that the mean number of down spins $n = N_\downarrow/N$ is related to the temperature

$$\frac{\Delta}{kT} = \ln\left(\frac{1-\bar{n}}{\bar{n}}\right) \tag{4}$$

(c) Determine the magnetization of the system M and sketch the magenetization versus B.

(d) Determine isothermal magnetic susceptibility $\chi_T(T, B)$. Show that for small fields $\chi_T \propto 1/T$.

(e) Determine the variance in the magnetization, $\langle (N_\uparrow - N_\downarrow)^2 \rangle$.

Problem 4. Paramagnets from the Microcanonial Ensemble

Now we will work through the paramagnet problem in the micro-canonical ensemble. Recall that in the microcanonical ensemble we are supposed to directly count the number of configurations (states) with a given total fixed energy E. The system is closed, meaning it does not exchange heat with the environment. This counting procedure determines the entropy, and from there all else can be determined.

(a) Describe the state of lowest possible energy (the ground state), and show that the energy of this state is $-\mu_B BN$. Let's define the *excitation* energy $\mathcal{E} = E - (-N\mu_B B)$, i.e. the energy *above* the ground state energy. Show that

$$\frac{\mathcal{E}}{N} = n\Delta \tag{5}$$

(b) Show that the total number of configurations with excitation energy $\mathcal{E}$ is

$$\Omega(\mathcal{E}) = \frac{N!}{N_\uparrow! N_\downarrow!} \tag{6}$$

where $N_\uparrow = N - N_\downarrow$ and $\mathcal{E} = Nn\Delta$

(c) Show that the entropy as a function of energy is

$$S(\mathcal{E}) = Nk_B \left[(1-n)\log(1-n) - n\log n\right] \tag{7}$$

(d) Using Eq. (7) show that the temperature of the system with a given $\mathcal{E}$ is related to the mean number of down arrows

$$\frac{\Delta}{kT} = \ln\left(\frac{1-n}{n}\right) \tag{8}$$

as deduced by the canonical approach.