Problem 1. Exponential distribution

A particle is created at time $t = 0$ and flies a distance x (greater than zero) before being destroyed. The probability of surviving up to a given distance between x and $x + \mathrm{d}x$ is

$$P(x)\mathrm{d}x = Ae^{-x/\ell}\mathrm{d}x \tag{1}$$

with $x > 0$. For parts (a), (b), (c) you should do the integrals yourself (showing your work) and dont use Mathematica. I suggest switching to some dimensionless variables i.e. x/ℓ before trying to do the integrals. In part (d) you will prove the boxed integral.

(a) Find the value of A that makes $P(x)$ a well defined normalized probability distribution with $\int_0^\infty \mathrm{d}x\, P(x) = 1$. What are the units of A?

(b) Show that the mean survival length is ℓ, i.e. show that $\langle x\rangle = \int_0^\infty \mathrm{d}x\, xP(x) = \ell$.

(c) Show that variance and std. deviation of the survival length are ℓ^2 and ℓ respectively.

(d) For simplicity set $\ell = 1$ in what follows. This is the equivalent to saying we will measure x in units of ℓ. Use the generating function method of a previous problem, and calculate $\left\langle e^{fx}\right\rangle$. Use the result to prove that

$$\langle x^n\rangle = n! \tag{2}$$

You will need to use the Taylor series

$$\frac{1}{1-u} = 1 + u + u^2 + \ldots \tag{3}$$

This problem establishes that

$$\boxed{n! = \int_0^\infty e^{-x}x^n\,\mathrm{d}x} \tag{4}$$

Problem 2. The Γ function

The $\Gamma(x)$ function can be defined as[1]

$$\boxed{\Gamma(x) = \int_0^\infty \mathrm{d}u e^{-u}u^{x-1}} \tag{5}$$

A plot of $\Gamma(x)$ is shown below. $\Gamma(n)$ provides a generalization of $(n-1)!$ when when n is not an integer, and even negative. It will come up a number of times in this course and is good to know.

(a) Explain briefly why $\Gamma(n) = (n-1)!$ for n integer.

[1] I like to write $\Gamma(x) = \int_0^\infty \frac{\mathrm{d}u}{u}\, e^{-u}u^x$, which makes the x is more explicit. Also the measure du/u is invariant under a homogeneous rescaling, e.g. under change of variables $u \to u' = \lambda u$ we have $du'/u' = du/u$.

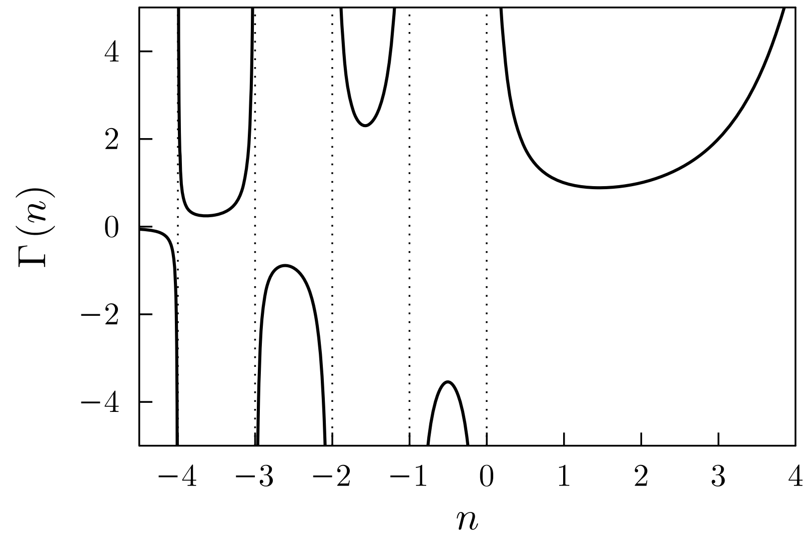


Fig. C.1 The gamma function $\Gamma(n)$ showing the singularities for integer values of $n \leq 0$. For positive, integer n, $\Gamma(n) = (n-1)!$.

Figure 1: Appendix C.2 of our book

(b) Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$? *Hint:* try a substitution $y = \sqrt{u}$.

The following identity is needed below.

$$\boxed{\Gamma(x+1) = x\Gamma(x)\,,} \tag{6}$$

or

$$x! = x \cdot (x-1)!\,, \tag{7}$$

but now x is a real number, and $x!$ is defined by $\Gamma(x+1)$.

(c) (Optional. Dont turn in) Use integration by parts to prove the identity.

(d) Use the results of this problem to show that $\Gamma(\frac{7}{2}) = 15\sqrt{\pi}/8$. What is the result numerically? 7/2 is between two integers. Show that $\Gamma(7/2)$ is between the appropriate factorials related to those two integers?

(e) The "area" (i.e. circumference) of a "sphere" in two dimensions (i.e. the circle) is $2\pi r$. The area of a sphere in three dimensions is $4\pi r^2$. A general formula for the area of the sphere in d dimensions is derived in the book is (the proof is simple, using what we know)

$$A_d(r) = \frac{2\pi^{d/2}}{(\frac{d}{2}-1)!} r^{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1} \tag{8}$$

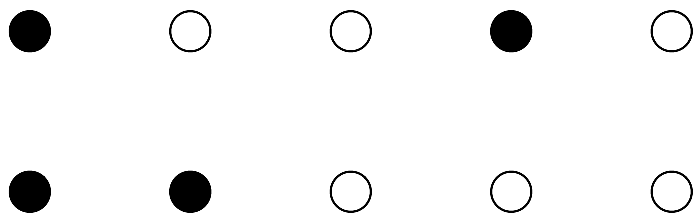
Show that this formula gives the familiar result for $d = 2$ and $d = 3$.

Problem 3. Combinatorics and The Stirling Approximation

Consider a chain of 6×10^{23} atoms, laid out in a row. The atoms can be in two states, a ground state, and an excited state. 1/3 of them are in the excited states. Show that the number of configurations with this number of excited states is approximately

$$10^{1.67\times 10^{23}} \tag{9}$$

For instance, if the number of atoms is five, and the number of excited atoms (shown by the black circles) is 2, then two possible configurations are shown below.



Problem 4. Central Limit Theorem and Random Walk

In a random walk, a collegiate drunkard starts at the origin and takes a step of size a, to the right[2] with probability p or to the left with probability $1-p$.

(a) What is the mean and variance variance in his position X after one step.

(b) After n steps find his mean position $\langle X \rangle$, and the std. deviation in his position $\sigma_X = \sqrt{\langle \delta X^2 \rangle}$. You can check your result by comparing with the figure below

Hint: X is a sum N independent events x_i where $x_i = \pm a$. Use results from class on the probability distribution of a *sum* of independent events.

(c) If p is very nearly $\frac{1}{2}$, say $p = 0.5001$, determine how many steps it will take before the mean value $\langle X \rangle$ is definitely different from zero. By "definitely" I mean that $\langle X \rangle$ is "more than two sigma" away from zero, $\langle X \rangle > 2\sigma_X$. If $p = \frac{1}{2} + \epsilon$ (with ϵ tiny), you should find (approximately) that

$$N_{\text{steps}} \simeq \frac{1}{\epsilon^2} \tag{10}$$

up to corrections of order ϵ.

Problem 5. Speeds of nitrogen gas

Two moles of nitrogen (N_2) are in a 6-L container at a pressure of 5 bar. Find the average kinetic energy of one molecule of the gas in electron volts and the root-mean-square velocity in m/s. I find that the energy and rms velocity are, 0.04 eV and 400 m/s. Is the kinetic energy $\frac{1}{2}mv^2$?

[2] The problem was originally worded as "to the left with probability p". If you already did it with left, that is fine. Both ways will be accepted. It makes only a minor difference.

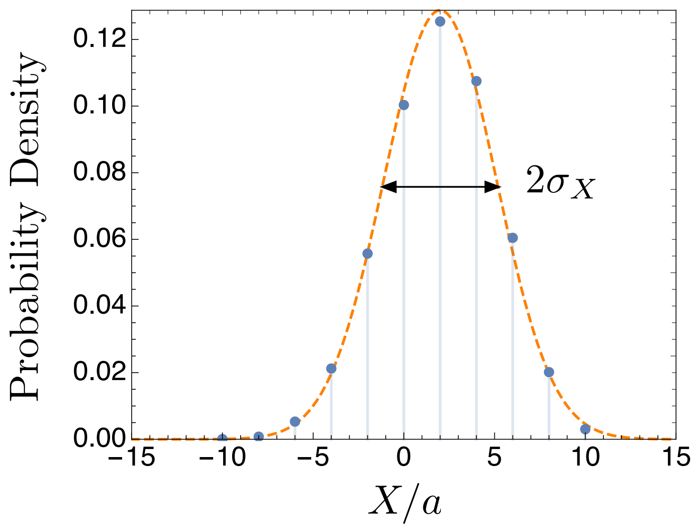


Figure 2: Probability of our drunkard having position X after $n = 10$ steps (the blue points). Of course after 10 steps the drunkard will be between $-10 \ldots 10$, and it is easy to show that he will be only at the even sites, i.e. $-10, -8, -6, \ldots 10$. For $p = 0.6$, I find $\langle X \rangle = 2.0$. Twice the std deviation, $2\sigma_X$, is shown in the figure and is about six in this case. The orange curve is a gaussian/bell shaped approximation discussed in class and approximately agrees with the points – this is the central limit theorem. Recall that the central limit theorm says that if the number of steps n is large, the probability of X (a sum of n independent events) is approximately $P(x)\,\mathrm{d}X \propto \exp(-(X - \langle X \rangle)^2/2\sigma_X^2)$. Evidently the gaussian approximation works well already for $n = 10$.