Most energetic

a) $$U = 2\int \frac{V\,d^3p}{(2\pi)^3}\,\frac{\varepsilon}{e^{\beta\varepsilon}-1}$$

So writing $\varepsilon = \hbar\omega = cp$

$$d^3p = 4\pi p^2\,dp = \frac{\hbar^3}{c^3}\,\omega^2\,d\omega \cdot 4\pi$$

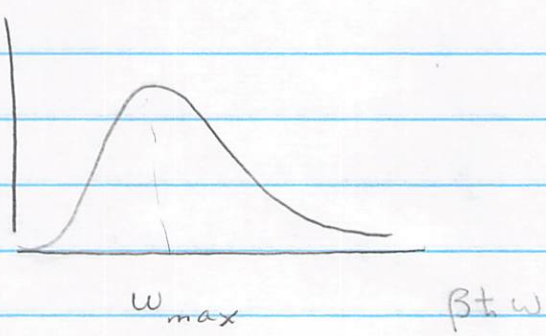
We find after algebra

☆ $$u = \frac{\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3\,d\omega}{e^{\beta\hbar\omega}-1}$$

So

$$\boxed{\frac{du}{d\omega} \propto \frac{\omega^3}{e^{\beta\hbar\omega}-1}}$$

Plotting this, we find a maximum (see next page)
at:



$\beta\hbar\omega \simeq 2{,}8$

so

$$\hbar\omega = 2.8\,k_B T = 2{,}8 \times \frac{0.025\,eV}{300\,°K}\,6000\,°K = 1.4\,eV$$

b) Then

$$\omega = 2\pi f = \frac{2\pi c}{\lambda}$$

$$d\omega = d\lambda \left|\frac{d\omega}{d\lambda}\right| = d\lambda \left|\frac{-2\pi c}{\lambda^2}\right|$$

note absolute value for unoriented integrals as discussed previously

So substituting into eq ☆ above

$$u = \frac{\hbar}{\pi^2 c^3} (2\pi c)^4 \int_0^\infty \frac{d\lambda}{\lambda^5} \frac{1}{e^{\beta\hbar(2\pi)/\lambda} - 1}$$

Note $\beta\hbar\, 2\pi/\lambda = \beta\, hc/\lambda$ so

$$\boxed{\frac{du}{d\lambda} \propto \frac{1}{\lambda^5} \frac{1}{e^{\beta hc/\lambda} - 1}}$$

Plotting this gives (see next page)

$$\beta \frac{hc}{\lambda} = 0.20 \quad \text{or} \quad \lambda \simeq 5.0 \frac{hc}{k_B T}$$

infrared

So $\lambda = 5.0 \; 1240\,\text{eV nm} \Big/ \frac{.025\,\text{eV} \; 6000°\text{K}}{300°\text{K}} = 12{,}400\,\text{nm}$

Density of States

a) So

$$d\mathcal{N} = V\frac{d^3p}{(2\pi\hbar)^3} = \text{number of modes with momentum } \vec{p} = (p_x, p_y, p_z) \text{ in range}$$

$[p_x; dp_x], [p_y; dp_y], [p_z; dp_z]$

This means $p_x < p_x' < p_x + dp_x$:

$$d^3p = 4\pi p^2\,dp$$

And $p = \hbar k$, so

$$d\mathcal{N} = V\frac{4\pi}{(2\pi)^3}k^2\,dk = \boxed{\frac{V}{2\pi^2}k^2\,dk} \rightarrow \text{or } g(k) = \frac{Vk^2}{2\pi^2}$$

= number of modes with k in range $k < k' < k + dk$

b) In two dimensions

$$d\mathcal{N} = \frac{A\,d^2p}{(2\pi\hbar)^2} = A\frac{2\pi p\,dp}{(2\pi\hbar)^2} \quad \Big) \; p = \hbar k$$

$$\boxed{d\mathcal{N} = \frac{1}{2\pi}A\,k\,dk}$$

or $\boxed{g(k) = Ak/2\pi}$

So since

$$\mathcal{E}(p) = \frac{p^2}{2m} \qquad d\mathcal{E} = \frac{p}{m}\,dp$$

<u>Then in 3D</u>

$$d\mathcal{N} = \frac{V}{2\pi^2}\frac{p^2\,dp}{\hbar^3} = \frac{1}{2\pi^2}\frac{m\,p}{\hbar^3}\,d\mathcal{E} = \frac{1}{4\pi^2}\left(\frac{2mp}{\hbar^3}\right)d\mathcal{E}$$

$$= \boxed{\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\mathcal{E}}\,d\mathcal{E} = g(\mathcal{E})\,d\mathcal{E}}$$

grouping it like this is motivated by units:

$$\frac{1}{\lambda_{typ}} \sim \frac{p_{typ}}{\hbar} \sim \left(\frac{2m}{\hbar^2}\right)^{1/2}\mathcal{E}^{1/2}$$

<u>In 2D:</u>

$$d\mathcal{N}_{\vec{p}} = \frac{A\,d^2p}{(2\pi\hbar)^2}$$

Integrating over the angles of p we have

$$d\mathcal{N}_p = \frac{A\,p\,dp}{2\pi\hbar^2}$$

now with

$$\mathcal{E} = \frac{p^2}{2m} \qquad d\mathcal{E} = \frac{p\,dp}{m}$$

$$d\mathcal{N}_{\mathcal{E}} = \frac{A\,m\,d\mathcal{E}}{2\pi\hbar^2}$$

$$\boxed{d\mathcal{N}_{\mathcal{E}} = \frac{A}{4\pi}\left(\frac{2m}{\hbar^2}\right)d\mathcal{E}}$$

and

$$\Phi_G = \int_0^\infty g(\varepsilon)\, d\varepsilon \quad -k_B T \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right)$$

★Then for photon

spin of photons (or polarizations) = 2

$$d\mathcal{N}_{modes} = \int_{angles} 2 \frac{V\, d^3p}{(2\pi\hbar)^3} = \frac{1}{\pi^2 \hbar^3} V p^2\, dp$$

Now $\varepsilon = cp$ so

$$d\mathcal{N} = \frac{1}{\pi^2} \frac{V}{(\hbar c)^3} \varepsilon^2\, d\varepsilon = g(\varepsilon)\, d\varepsilon$$

And

$$\boxed{\Phi_G = \frac{V}{\pi^2 (\hbar c)^3} \int_0^\infty \varepsilon^2 \quad k_B T \ln\left(1 - e^{-\beta(\varepsilon-\mu)}\right)}$$

So

$\Phi_G = -pV$ So

$$pV = -\frac{1}{\pi^2 (\hbar c)^3} \int_0^\infty \varepsilon^2 k_B T \ln\left(1 - e^{-\beta(\varepsilon-\mu)}\right) d\varepsilon$$

c) Then the free energy of one mode is

$$\Phi_G^{\varepsilon} = -k_B T \ln \frac{1}{1 - e^{-\beta(\varepsilon - \mu)}}$$

where $\mathcal{Z}_p = \frac{1}{1 - e^{-\beta(\varepsilon-\mu)}}$ is the grand partition function of one mode for a boson

So

$$\Phi_G^{tot} = \sum_{modes} \Phi_G^{\varepsilon}$$

By definition the sum over modes becomes an integral over the mode density, $g(\varepsilon)\, d\varepsilon$

$$\sum_{modes} \ldots = \int g(\varepsilon)\, d\varepsilon \ldots$$

So

$$\boxed{\Phi_G = \int_0^{\infty} g(\varepsilon)\; k_B T \ln\left(1 - e^{-\beta(\varepsilon - \mu)}\right)}$$

For a fermion

$$\mathcal{Z}_p = 1 + e^{-\beta(\varepsilon - \mu)}$$

Entropy / Photon

- So setting $\varepsilon = \hbar\omega$ the results of the previous problem can be written

$$pV = -\frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\hbar^3 \int_0^\infty \underbrace{\omega^2 d\omega}_{dv} \underbrace{k_B T \ln(1-e^{-\beta\hbar\omega})}_{u}$$

- Now integrate by parts

$$dv = \omega^2 d\omega \qquad u = k_B T \ln(1-e^{-\beta\hbar\omega})$$

$$v = \frac{1}{3}\omega^3 \qquad du = k_B T \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} \beta\hbar \, d\omega$$

$$= \frac{\hbar \, d\omega}{e^{\beta\hbar\omega}-1}$$

So

$$\int_0^\infty u \, dv = uv\Big|_0^\infty - \int_0^\infty v \, du$$

this is zero since $u\big|_0 = 0$ and $v\big|_\infty = 0$

- And so

$$pV = \frac{V}{\pi^2 c^3}\hbar \int_0^\infty \frac{1}{3}\omega^3 d\omega \frac{\hbar}{e^{\beta\hbar\omega}-1}$$

First we make the integral dimensionless

$$u = \beta \hbar \omega \qquad du = \beta \hbar d\omega$$

Then we have

$$pV = \frac{V}{\pi^2 c^3} \hbar \frac{1}{3} \frac{1}{(\beta\hbar)^4} \int_0^\infty \frac{u^4}{e^u - 1}$$

$$= V \left(\frac{k_B T}{\hbar c}\right)^3 k_B T \left[\frac{1}{3\pi^2} \int_0^\infty \frac{u^4 \, du}{e^u - 1}\right]$$

The integral is

$$\frac{\pi^4}{15} \qquad \text{(See attatched Table)}$$

And so we find

$$\boxed{pV = V \left(\frac{k_B T}{\hbar c}\right)^3 k_B T \frac{\pi^2}{45}}$$

Note

$$d\Phi_G = -S\,dT - N\,d\mu - p\,dV$$

With $\Phi_G = -p(T,\mu) V$

So

$$\frac{\partial p}{\partial T} = S$$

So since $\quad pV = CT^4$

$$\boxed{S = \frac{\partial p}{\partial T} = 4CT^3 = 4\frac{CT^4}{T} = 4\frac{pV}{T}}$$

So

$$U - TS + pV = \cancelto{0}{\mu N}$$

$$\frac{S}{Nk_B} = \frac{4\pi^2/15}{0.245}$$

$$= 3.6$$

Then

$$U = -pV + TS$$

$$U = -pV + 4pV = 3pV$$

So

$$\boxed{\frac{U}{V} = 3p \qquad \text{as before}}$$

Fluctuations

$$\mathcal{Z} = \sum_s e^{-\beta(\varepsilon_s - \mu N_s)} = \sum_s e^{-\beta \varepsilon_s} e^{\beta \mu N_s}$$

Then

$$\langle N \rangle = \frac{1}{\mathcal{Z}} \frac{1}{\beta} \frac{\partial}{\partial \mu} \mathcal{Z} = \frac{\sum_s e^{-\beta \varepsilon_s} e^{\beta \mu N_s} N_s}{\mathcal{Z}}$$

$$\langle N \rangle = \sum_s P_s N_s$$

Here we have recognized that the probability to be in a state with energy ε_s and number N_s is

$$P_s = \frac{e^{-\beta(\varepsilon_s - \mu N_s)}}{\mathcal{Z}}$$

Similarly

$$\langle N^2 \rangle = \sum_s P_s N_s^2 = \frac{1}{\mathcal{Z}} \sum_s e^{-\beta \varepsilon_s} e^{\beta \mu N_s} N_s^2$$

No

$$\left(\frac{1}{\beta} \frac{\partial}{\partial \mu}\right) \left(\frac{1}{\beta} \frac{\partial}{\partial \mu}\right) e^{\beta \mu N_s} = e^{\beta \mu N_s} N_s^2$$

So

$$\sigma_N^2 = \frac{1}{\beta}\frac{\partial}{\partial \mu}\left(\frac{-1}{\beta}\frac{\partial \Phi_G}{\partial \mu}\right)$$

$$\boxed{\sigma_N^2 = \frac{1}{\beta}\frac{\partial \bar{N}}{\partial \mu}}$$

This formula should be compared to the variance in the energy

$$\sigma_E^2 = \frac{\partial^2}{\partial \beta^2}(\ln Z)$$

$$\sigma_E^2 = -\frac{\partial \bar{E}}{\partial \beta}$$ note $$\bar{\sigma}_E = -\frac{\partial \ln Z}{\partial \beta}$$

b) So

$$\mathcal{Z}_{BE} = \sum_{n=0}^{\infty} e^{-\beta(n\varepsilon - \mu n)} = \frac{1}{1 - e^{-\beta(\varepsilon-\mu)}}$$

The terms where

$n \neq 0, 1$ are small only if $e^{-\beta(\varepsilon-\mu)} \ll 1$

For instance Look at the contribution from two particles in a mode to $\mathcal{Z}$:

$$e^{-\beta(2\varepsilon - \mu 2)} = e^{-2\beta(\varepsilon-\mu)} = \left(e^{-\beta(\varepsilon-\mu)}\right)^2$$

So

$$\langle N^2 \rangle = \frac{1}{\mathcal{Z}} \left(\frac{1}{\beta} \frac{\partial}{\partial \mu} \right)^2 \mathcal{Z}$$

So

$$\sigma_N^2 = \frac{1}{\mathcal{Z}} \left(\frac{1}{\beta} \frac{\partial}{\partial \mu} \right)^2 \mathcal{Z} - \left(\frac{1}{\mathcal{Z}} \frac{1}{\beta} \frac{\partial \mathcal{Z}}{\partial \mu} \right)^2$$

$$\sigma_N^2 = \left(\frac{1}{\beta} \frac{\partial}{\partial \mu} \right)^2 \ln \mathcal{Z}$$

Now $\ln \mathcal{Z} = -\frac{\Phi_G}{k_B T} = \frac{pV}{k_B T}$

$$\sigma_N^2 = k_B T \, V \frac{\partial^2 p}{\partial \mu^2}$$

Note also

$$d\Phi_G = -S\,dT - N\,d\mu - p\,dV$$

$$\frac{\partial \Phi}{\partial \mu} = N$$

The classical approximation assumes this is small compared to having one particle per mode:

$$\left(e^{-\beta(\varepsilon-\mu)}\right)^2 \ll e^{-\beta(\varepsilon-\mu)}$$

So we have

(1) $$n_{BE} = \frac{1}{e^{\beta(\varepsilon-\mu)}-1} = \frac{e^{-\beta(\varepsilon-\mu)}}{1-e^{-\beta(\varepsilon-\mu)}}$$

$$\simeq e^{-\beta(\varepsilon-\mu)}\left(1+e^{-\beta(\varepsilon-\mu)}+\dots\right)$$

Similarly

use $\ln(1+x) \simeq x - \frac{x^2}{2}$

(2) $$\Phi_G = k_B T \ln\left(1-e^{-\beta(\varepsilon-\mu)}\right)$$

$$\simeq -k_B T e^{-\beta(\varepsilon-\mu)} - \frac{1}{2}k_B T\left(e^{-\beta(\varepsilon-\mu)}\right)^2$$

c) Then

$$pV = \sum_{\text{modes}} -\Phi_G^{1\,\text{mode}}$$

use Eq. (2)

$$= \int \frac{V\,d^3p}{(2\pi\hbar)^3}\, k_B T e^{-\beta(\varepsilon-\mu)}\left(1+\frac{1}{2}e^{-\beta(\varepsilon-\mu)}\right)$$

Performing the integral

$$pV = V \int \frac{d^3p}{(2\pi\hbar)^3}\, k_B T\, e^{-p^2/2mkT}\, e^{\beta\mu}$$

$$\left[1 + \frac{1}{2} e^{-p^2/2mkT} e^{\beta\mu} \right]$$

$$= V \int \frac{d^3p}{(2\pi\hbar)^3}\, k_B T\, e^{-p^2/2mkT}\, e^{\beta\mu}$$

$$+ \frac{V}{2} \int \frac{d^3p}{(2\pi\hbar)^3}\, k_B T\, e^{-2p^2/2mkT}\, e^{2\beta\mu}$$

The Integrals are

$$I_1 = \int \frac{d^3p}{(2\pi\hbar)^3}\, e^{-p^2/2mkT} = n_Q \equiv \frac{(2\pi m k_B T)^{3/2}}{h^3}$$

$$I_1 = \int \frac{d^3p}{(2\pi\hbar)^3}\, e^{-2p^2/2mkT} = \frac{n_Q}{2\sqrt{2}} = \frac{(2\pi (m/2) k_B T)^{3/2}}{h^3}$$

So

$$pV = V k_B T\, e^{\beta\mu}\, n_Q \left[1 + \frac{e^{\beta\mu}}{4\sqrt{2}} \right]$$

Thus

$p = k_B T\, e^{\beta\mu}\, n_Q$ at leading order

So

$$\bar{N} = -\left(\frac{\partial \Phi_G}{\partial \mu}\right)_T = \frac{\partial (pV)}{\partial (\mu)}$$

$$\bar{N} = V k_B T \, e^{\beta\mu} \beta \, n_Q$$

$$\boxed{\left(\frac{\bar{N}}{V}\right) = e^{\beta\mu} n_Q}$$

this agrees with HW 12 prob 2.

d)

$$\sigma_N^2 = \frac{1}{\beta} \frac{\partial \bar{N}}{\partial \mu} = V \frac{1}{\beta} \frac{\partial (e^{\beta\mu})}{\partial \mu} n_Q$$

$$\boxed{\sigma_N^2 = V e^{\beta\mu} n_Q = N}$$

So

$$\frac{\sigma_N}{\langle N \rangle} = \frac{1}{\sqrt{\langle N \rangle}}$$