Physics 306: Thermal Physics

Final Exam

Stony Brook University

Spring 2023

General Instructions:

You may use one page (front and back) of handwritten notes and a calculator. Graphing calculators are allowed. **No other materials may be used**.

1 Integrals

Bose and Fermi:

$$\int_0^\infty \mathrm{d}x\,\frac{x}{e^x-1} = \frac{\pi^2}{6} \tag{1}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^2}{e^x-1} = 2\zeta(3) \simeq 2.404 \tag{2}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^3}{e^x-1} = \frac{\pi^4}{15} \tag{3}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^4}{e^x-1} = 24\,\zeta(5) \simeq 24.88 \tag{4}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^5}{e^x-1} = \frac{8\pi^6}{63} \tag{5}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x}{e^x+1} = \frac{\pi^2}{12} \tag{6}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^2}{e^x+1} = \frac{3}{2}\,\zeta(3) \simeq 1.80309 \tag{7}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^3}{e^x+1} = \frac{7\pi^4}{120} \tag{8}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^4}{e^x+1} = \frac{45}{2}\,\zeta(5) \simeq 23.33 \tag{9}$$

$$\int_0^\infty \mathrm{d}x\,\frac{x^5}{e^x+1} = \frac{31\pi^6}{252} \tag{10}$$

Gamma Function:

$$\Gamma(z) \equiv \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x \tag{11}$$

with specific results

$$\Gamma(z+1) = z\Gamma(z) \qquad \Gamma(n) = (n-1)! \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{12}$$

Gaussian Integrals:

$$I_n = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{d}x\, e^{-x^2/2}x^n \tag{13}$$

with specific results

$$I_0 = 1 \quad I_2 = 0 \quad I_4 = 3 \quad I_6 = 15 \tag{14}$$

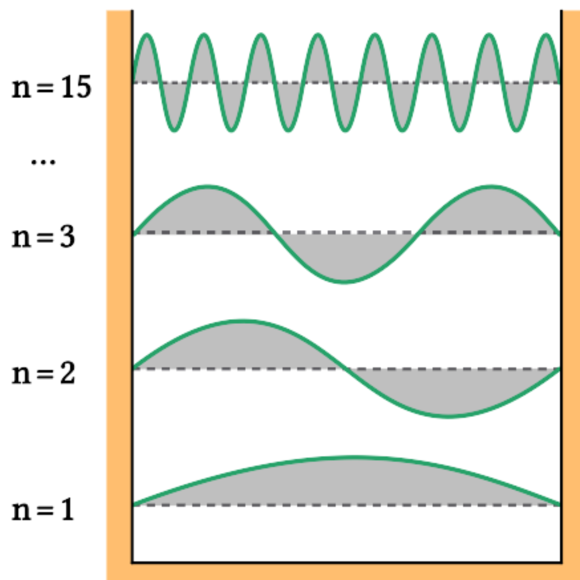
Problem 1. A thermal particle in a box

Recall that a quantum mechanical particle in a one dimensional box of size L has energy levels

$$E_n = n^2 \epsilon_0 \,, \qquad \epsilon_0 = \frac{\hbar^2 \pi^2}{2mL^2} \,, \tag{15}$$

with quantum wave functions labelled by n that are shown below

$$n = 1, 2, 3, \ldots \,. \tag{16}$$



(a) Approximately evaluate the partition function by including just the first two states $n = 1$ and $n = 2$.

  (i) Explain why this approximation is valid at low temperatures, and define what is meant by *low* in this context (i.e. low compared to what?). Given an electron at room temperature, what is the range of the parameters for this problem where the temperature can be considered low?

(b) Determine the mean energy with the same approximations as in (a).

(c) In order for the approximation of part (a) to be valid, the probability to be in the $n = 3$ state should be small. Determine this probability.

  Hint: You can use the partition function from (a) when evaluating this probability.

(d) In the high temperature limit the sum over states n can be replaced with an integral. Evaluate the partition function in this limit and explain the result using classical reasoning.

Problem 2. A classical relativistic gas

In class we studied a non-relativistic classical mono-atomic gas of N particles in a volume V at temperature T using partition functions and the canonical ensemble. The energy was $\epsilon = \frac{p^2}{2m}$. The same methods can be used for a classical ultra-relativistic gas. The only difference is the energy is now, $\epsilon(p) = cp$. Consider a three dimensional classical relativistic gas consisting of N particles in a volume V at temperature T.

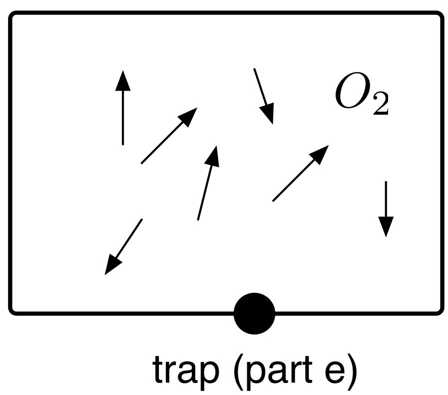
(a) Determine the partition function and the free energy.

(b) Determine the pressure. What are the similarities and differences to the non-relativistic case?

(c) Determine the entropy of the gas? What are the similarities and differences to the non-relativistic case?

Problem 3. O_2 in equilibrium

Consider a classical ideal gas in volume V at temperature T consisting of N molecules of diatomic oxygen. The moment of inertia of the O_2 molecule is I.



(a) Use the equipartition theorem to determine the energy of the gas and the specific heat at constant pressure. You may treat the rotational motion classically and neglect vibrations.

(b) Determine the free energy of the gas and show that the pressure satisfies the ideal gas law, $PV = NkT$.

(c) Determine the chemical potential, μ, and the "fugacity", $z_g \equiv e^{\beta\mu}$, as a function of temperature and pressure of the gas.

(d) Estimate z_g at room temperature and standard pressure. Assume $\hbar^2/2I = 0.00018\,\text{eV}$. Note that for O_2 the "quantum pressure" is[1]

$$p_Q \equiv k_B T n_Q \equiv \frac{k_B T}{\lambda_{\text{th}}^3} \equiv T\left(\frac{2\pi M kT}{h^2}\right)^{3/2} = 6.8 \times 10^6\,\text{bar}\,. \tag{17}$$

[1] λ_{th} is the thermal de Broglie wavelength discussed in class.

A trap on the wall of the container can absorb up to two O_2 molecules from the room. The energy of the trap is lowered from its unoccupied state by $-\epsilon_0$ if one O_2 is absorbed, and $-2\epsilon_0$ if two O_2 are absorbed.

(e) What is the mean number of O_2 molecules that are absorbed by the trap? Express your result as a function of the $u \equiv z_g e^{\beta\epsilon_0}$.

(f) For the conditions of part (d), estimate the binding energy ϵ_0 where the mean number of O_2 molecules in the trap is of order unity.

Problem 4. Neutrino Gas

A neutrino is like a photon – it is neutral, massless, and it has two spin states, But a neutrino is a fermion not a boson, and the modes can be in two possible states, either unoccupied or occupied by a particle, with corresponding energies 0 and ϵ. Consider a gas of neutrinos at temperature T and in a cubic box of volume $V = L^3$.

(a) Consider a single mode in the container with single particle energy ϵ. Determine the grand partition function for the mode and derive the Fermi-Dirac expression for the mean number of particles in the mode:

$$n_{FD}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \tag{18}$$

What are the probabilities of finding the mode unoccupied and occupied respectively?

(b) What is the energy and number of the neutrinos in the box? Explain each step carefully and assume $\mu = 0$.

(c) Determine the average de Broglie wavelength $\lambda \equiv h/p$ of the neutrino by integrating over the momenta.

(d) What is the number of neutrinos striking the walls per unit time?