Department of Atmospheric Sciences
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Chaos and Predictability/ HW3

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Question 1

For a dynamical system with third-order derivative,

$$x^{'''} = P(t, x, x^{'}, x^{''}) \tag{1}$$

where $x^{'''} = \frac{dx^{''}}{dt}$, we can find a Jacobian matrix J, which maps the initial states $x_{n_0} = x(t = 0, a, b, c)$ to the final states $x_{n_t} = x(t = n_t, a, b, c)$. The corresponding determinant can then be written as

$$\det(J) = \det \begin{pmatrix} \frac{dx}{da} & \frac{dx}{db} & \frac{dx}{dc} \\ \frac{dx^{'}}{da} & \frac{dx^{'}}{db} & \frac{dx^{'}}{dc} \\ \frac{dx^{''}}{da} & \frac{dx^{''}}{db} & \frac{dx^{''}}{dc} \end{pmatrix} \tag{2}$$

show that $\frac{d\det(J)}{dt} = \det(J)\frac{\partial P}{\partial x^{''}}$. (Hint: using the chain rule $\frac{dx^{'''}}{da} = \frac{\partial x^{'''}}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial x^{'''}}{\partial x^{'}}\frac{\partial x^{'}}{\partial a} + \frac{\partial x^{'''}}{\partial x^{''}}\frac{\partial x^{''}}{\partial a}$, also see Fig. 2 in here). The LE suggests that the time evolution of $\det(J)$ only depends on the highest-order differentiation.

Question 2

The **Liouville equation** describes how a probability density function (PDF) evolves over time for a system following a deterministic flow. In this case, the system's state X evolves according to:

$$dX = \Phi(X, t)\, dt \qquad \text{, equivalent to } \dot{X} = \Phi(X, t)$$

where $\Phi(X, t)$ is a deterministic drift term representing the system's flow. The Liouville equation for the PDF $p(X, t)$ is given by:

$$\frac{\partial p}{\partial t} = -\nabla \cdot (p\, \Phi) \tag{LE}$$

This equation indicates that the probability density is preserved as it follows the system's deterministic flow, and it's predictable if we have full knowledge of Φ.

However, if we introduce **additive noise**, the evolution of X is governed by a **stochastic differential equation** (SDE):

$$dX = \Phi(X, t)\, dt + \sigma\, dB_t$$

where σ represents the noise intensity, and B_t is standard Brownian motion (or Wiener process). In this case, the system has both deterministic and random components, and the

PDF $p(X, t)$ evolves according to the **Fokker-Planck equation** (or forward Kolmogorov equation):

$$\frac{\partial p}{\partial t} = -\nabla \cdot (p\,\Phi) + \frac{\sigma^2}{2}\,\Delta p$$

Here, the term $\frac{\sigma^2}{2}\,\Delta p$ represents the diffusion due to the noise, which accounts for the spreading of the probability density because of the random fluctuations.

Let us consider a simple example where the drift term is constant, specifically $\Phi(X, t) = 1$, and the system is in a one-dimensional phase space. Given these conditions,

(a) derive the corresponding Fokker-Planck equation for the probability density function $p(X, t)$,

assuming an additive noise with variance σ^2, obtain the following:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial X}p + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial X^2}$$

(b) Choose ONE of the following approaches to analyze the behavior of the solution as σ^2 varies:

You can either

1. Find an analytical solution (noting that this equation resembles the linear heat equation with an added drift term).
2. Conduct a numerical experiment with an appropriate numerical method. You may choose any reasonable initial data $p_0 = p(X, t = 0)$ as you want.

Also note that when $\sigma = 0$, the noise term is absent, making the system deterministic. In this case, the Fokker-Planck equation reduces to the Liouville equation.

Question 3

The Lorenz-84 system is a 3-dimensional dynamical system given as follows :

$$\begin{cases} \dot{x} &= -\,y^2 - z^2 - a\,x + aF \\ \dot{y} &= xy - bxz - y + G \\ \dot{z} &= bxy + xz - z \end{cases} \tag{L84}$$

where the parameters (a,b,F,G) = $(0.25, 4.0, 8.0, 1.0)$.

1. Please give the formula of Liouville equation in this case:

$$\rho(\mathbf{X}, \tau) = \rho(\Xi, 0)\exp\left(-\int_{t'=0}^{t=\tau} \boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\left(\mathbf{X}, t'\right)\Big|_{\mathbf{X}(\Xi,t')} dt'\right) \tag{3}$$

2. Implement this model, and give the initial condition under the following :

$$\begin{cases} \Xi[x, y, z] = [1, 0, 0], \quad \Sigma_0 = \mathrm{diag}[1, 1, 1] \\ \text{Initial condition: Sampling from } \mathcal{N}(\Xi, \Sigma_0) \\ \text{Initial PDF : } \mathcal{N}(\Xi, \Sigma_0) \end{cases} \tag{4}$$

   where $\mathcal{N}$ is the Gaussian distribution.

   Assign the initial distribution to those points, and update it using (3) when you simulating (L84), discuss what you see.

Question 4

Non-Modal Growth and Transient Dynamics of the Subtropical High

Non-modal growth theory characterizes the most transiently amplified structures in a dynamical system that has been linearized around a given mean state. This framework captures *locally unstable perturbations* that can experience finite-time growth even in an overall stable system. Following Tseng and Ho (2024, see NTU cool), the transient amplification of the Western North Pacific Subtropical High (WNPSH) can be effectively described using non-modal dynamics. In this exercise, students will reproduce key results from that study by following the steps below.

Data: Use the files provided in the shared Google Drive folder, which contain all datasets required for this analysis. (see NTU cool for the hands-on)

1. **Load Data** Read the pre-computed time series of the subtropical high index (`WNPSH index total`) from the `.npz` file, and the daily 500-hPa geopotential height (`z500`) from the provided NetCDF (`.nc`) file.

2. **Compute EOFs** Extract the June—July—August (JJA) subset of the `z500` data, and perform an Empirical Orthogonal Function (EOF) analysis. (A smaller regional domain may be selected to better capture subtropical high variability.)

3. **Construct the Linear Dynamical Operator** Let the state vector $\mathbf{x}(t)$ represent the time series of leading EOF principal components (PCs). Solve for the linear operator $\mathbf{G}$ using the normal equation such that

$$\mathbf{x}(t+1) = \mathbf{G}\,\mathbf{x}(t),$$

   where the columns of $\mathbf{x}$ contain the PC time series:

$$\mathbf{x} = \begin{bmatrix} PC_{1,t=1979/01/01} & PC_{1,t=1979/01/02} & \cdots \\ PC_{2,t=1979/01/01} & PC_{2,t=1979/01/02} & \cdots \\ \vdots & \vdots & \ddots \\ PC_{m,t=1979/01/01} & PC_{m,t=1979/01/02} & \cdots \end{bmatrix}.$$

4. **Project onto the Subtropical High Index** Regress the WNPSH index onto the EOF PCs:

$$\text{index} = \sum_i a_i\, PC_i.$$

   Construct a diagonal projection matrix:

$$\boldsymbol{\rho} = \begin{bmatrix} a_1 & 0 & \cdots \\ 0 & a_2 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_m \end{bmatrix},$$

   which maps the forecasted PC state back to the subtropical high index.

5. **Analyze Transient Growth** Compute the matrix

$$\mathbf{M} = \boldsymbol{\rho}\,\mathbf{G}\,\mathbf{G}^{\mathrm{T}}\,\boldsymbol{\rho}^{\mathrm{T}},$$

and solve for its eigenvalues and eigenvectors. The leading eigenvector represents the *optimal initial pattern* that maximizes variance growth over one time step.

6. **Examine Lag Dependence** Repeat the above eigenanalysis using powers of the operator,

$$\mathbf{G}_\tau = \mathbf{G}^\tau,$$

for different lags τ. Discuss how the leading pattern and corresponding growth rate evolve with increasing lag, and interpret their physical meaning for transient development of the WNPSH.

Remark

Maybe you already noticed that the formulation of (LE) is different from the equation (64) in the course material, states below :

$$\rho(\mathbf{X}, \tau) = \rho(\ , 0) \exp \left(- \int_{t'=0}^{t=\tau} \boldsymbol{\nabla} \cdot \boldsymbol{\Phi} \left(\mathbf{X}, t' \right) \Big|_{\mathbf{X}(\Xi, t')} dt' \right) \tag{64}$$

In fact, this two equation is the same equation but with different view point of observation, where (LE) is in Euler form, and (64) is in Lagrange form.

We notice that $\boldsymbol{\nabla} \cdot (p\Phi) = p\boldsymbol{\nabla} \cdot \Phi + \Phi \boldsymbol{\nabla} \cdot p$, Then (LE) becomes

$$\frac{\partial p}{\partial t} + \phi \boldsymbol{\nabla} \cdot p = p \boldsymbol{\nabla} \cdot \Phi \tag{5}$$

Now, consider $p = p(X, t) = p(X(t), t)$, where X is a function of t , means that X is changing with respect to time, Using chain rule, with summation over repeated indices, we have

$$\begin{aligned} \frac{\partial}{\partial t}_{\text{Lag.}} &= \frac{\partial}{\partial t} + \frac{\partial X_i}{\partial t} \frac{\partial}{\partial X_i} \\ &= \frac{\partial}{\partial t} + \Phi \cdot \frac{\partial}{\partial X_i} \\ &= \frac{\partial}{\partial t} + \Phi \boldsymbol{\nabla} \cdot \end{aligned}$$

which is exactly the operator at left hand side of (5). We now have

$$\frac{\partial p}{\partial t_{\text{Lag.}}} = -p \boldsymbol{\nabla} \cdot \Phi$$

$$\frac{\partial \log p}{\partial t} = -\boldsymbol{\nabla} \cdot \Phi$$

Solve this equation symbolically, and let $\Xi = X(t = 0)$, we have

$$\frac{p(X, \tau)}{p(\Xi, 0)} = \int_{t'=0}^{t'=\tau} -\boldsymbol{\nabla} \cdot \Phi \ \mathrm{d}t'$$

which is exactly the equation (64).