Department of Atmospheric Sciences [ATS]
National Taiwan University

Chaos and Predictability / HW1

Kai-Chih Tseng

2025 Fall Version

In this homework, you will learn the concept of "phase dependent" error growth. i.e., given the same initial error, the spread of error grows differently at different physical spaces. You will also learn that the error growth is not only a function of physical spaces but also a function of dynamical system itself.

Question 1

We know the Lorenz 63 model is written as

$$
\begin{aligned}
\frac{dx}{dt} &= \sigma(y - x) \\
\frac{dy}{dt} &= x(\rho - z) - y \\
\frac{dz}{dt} &= xy - \beta z
\end{aligned}
\tag{1}
$$

(a) Linearizing the system about the equilibrium solution x=0,y=0,z=0, analyze the stability for the time continuous case. Assume that σ=10, $\beta = \frac{8}{3}$, and ρ=24.74.

(b) Repeat the same stability analysis at different equilibrium point, what do you see?

(c) Following (a) and dropping the z term. Using a first-order forward scheme to discretize equation (1) and assume $x^{n+1} = \lambda x^n$, $y^{n+1} = \lambda y^n$. $\|\lambda\|$ is so-called amplification factor. Compare the results between the discrete version and the continuous version in (a). Suggest a good choice for the value of the time step. (i.e., when do both cases give you similar results?)

Question 2

Program the model twice, using 1st-order forward time-differencing in one case and the fourth- order Runge-Kutta method in the other. As in part (a), use the parameter settings σ=10, $\beta = \frac{8}{3}$ and ρ=24.74. Start the model from the initial condition (x,y,z)=(0.1,0,0), which is close to the equilibrium mentioned above. Experiment to determine the longest permissible time step with each scheme. Run the model long enough to see the butterfly. Plot the results for both schemes, and compare them.

Question 3

Characteristic time is defined as a time scale relevant to the processes being investigated. There's two method for defining characteristic time:

- For decaying system, we define it to be the efolding time.
- For oscillating problem, we define it to be the inverse of angular frequency.

1. Please find the characteristic time of this ODE:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x \tag{2}$$

2. Please find the charanteristic time of this ODE:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x + 9x = 0 \tag{3}$$

3. Finally, we consider the forcing that has different characteristic time:

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}t^2}x + 9x = k\sin\left(\frac{1}{k}t\right), \quad k \in \mathbb{R}^+ \\ x(0) = 0, x'(0) = 1 \end{cases} \tag{4}$$

Please solve this equation (either directly or numerically, for direct solution, Laplace transform night helps) and discuss the change of characteristic time as k varies. (Hint: Find the dominant term.)

Question 4

Start from QG-PV equation

$$\frac{\partial q}{\partial t} + J(\psi, q) = D \tag{5}$$

$$q = \beta y + \zeta + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}b\right) \tag{6}$$

After some calculation and linearization, we have the following equations :

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{q'^2} = -\overline{v'q'}\frac{\partial \overline{q}}{\partial y} + \overline{D'q'}. \tag{7}$$

$$\frac{\partial \overline{q}}{\partial t} = -\frac{\partial}{\partial y}\overline{v'q'} + \overline{D}. \tag{8}$$

Eq.(11) is the progonostic equation of variance $\overline{q'^2}$ (which we've derived in the course), and eq.(12) is the progonostic equation of mean state $\overline{q}$.

1. Suppose the following:
   - D = 0 everywhere
   - $$\begin{cases} \overline{v'q'} &= -K_q \partial_y \bar{q} \\ K_q &= K_0[\overline{q'^2}]^{1/2} \end{cases} \tag{9}$$

   The physical meaning of second line indicates the increasing PV induces stronger mixing.

   Please give the progonostic equation of mean and variance under this assumption.

2. Please simulate this equation numerically, under these assumption:

$$\begin{cases} L(\text{ domain length}) & = 2\pi \\ K_0 & = 0.1 \\ \bar{q}(0,y) & = \exp\Big(-(y-\pi)^2\Big) \\ \overline{q'^2}(0,y) & = 0.1\exp\Big(-(y-\pi)^2\Big) \\ \text{Boundary condition} & : \text{periodic} \end{cases} \tag{10}$$

(Hint: You might need to use high order of finite difference and rk4 to prevent numerical blowup. I used 6th order of finite difference on y and rk4 on t. Also, you may want to compute $\partial_y \bar{q}$ first, then compute $\partial_y K_q \partial_y \bar{q}$.)

3. Please follow the assumption in 2. , consider a time-dependent forcing :

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{q'^2} + \overline{v'q'}\frac{\partial \overline{q}}{\partial y} = -\exp\Big(-(y-\pi)^2/2\Big)\sin(0.3\pi t) \tag{11}$$

$$\frac{\partial \overline{q}}{\partial t} + \frac{\partial}{\partial y}\overline{v'q'} = -3\exp\Big(-(y-\pi)^2/2\Big)\sin(0.3\pi t) \tag{12}$$

Please simulate this equation, and discuss the behavior under this forcing. (Hint : from the perdpective of characteristic time.)