Department of Atmospheric Sciences
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**Chaos and Predictability/ HW2**

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### Question 1

We know that the Rayleigh-Benard convection (instability) arises from unstable background temperature profile which generates free convection to conquer the friction. Rayleigh number, defined as the ratio of buoyancy force to friction, is center to the measurement of instability.

(a) Following the link to install Dedalus on workstation (might need to install anaconda or miniconda first). Using the example to simulate Rayleigh-Benard convection. Testing low Rayleigh number ($\sim 10^1$) and high Rayleigh number case ($\sim 10^8$). Run the model long enough to have a dynamical equilibrium state. Plot the results and discuss the difference.

(b) Following question (a) and equation (1), plot the X(t) against Y(t). Discuss how the choice of Rayleigh number influences the periodicity of X(t) and Y(t). An exmaple of Fourier Transform can be found here DFT

$$\begin{cases} \zeta &= X(t)\sin(\frac{\pi a x}{H})\sin(\frac{\pi z}{H}) \\ T &= Y(t)\cos(\frac{\pi a x}{H})\sin(\frac{\pi z}{H}) - Z(t)\sin(2\frac{\pi z}{H}) \end{cases} \tag{1}$$

(a is defined as $\frac{H}{L}$, where L is the domain size along x-axis and H is the domain size along z-axis)

(c) Revisit the Lorenz 63 equations from Homework 1 together with Equation 1. Derive the tangent linear model (TLM) of the Lorenz—63 system around a chosen reference point. Compute the eigenvalues and eigenvectors of the TLM, and identify at least one eigenvalue that is positive (if not, choose another reference point). Using the corresponding eigenvector(s), reconstruct the spatial structure of the most unstable mode. Finally, explain why this mode is unstable based on its dynamical characteristics.

## Question 2

A generalized Lorenz-96 system is given as follows :

$$\frac{d}{dt}X_k = -X_{k-1}\left(X_{k-2} - X_{k+1}\right) - X_k + F - \left(\frac{hc}{b}\right)\sum_{j=0}^{J-1} Y_{j,k} \tag{2}$$

$$\frac{d}{dt}Y_{j,k} = -cbY_{j+1,k}\left(Y_{j+2,k} - Y_{j-1,k}\right) - cY_{j,k} + \frac{hc}{b}X_k - \left(\frac{\hat{h}\hat{c}}{\hat{b}}\right)\sum_{\ell=0}^{L-1} Z_{\ell,j,k} \tag{3}$$

$$\frac{d}{dt}Z_{\ell,j,k} = -\hat{c}\hat{b}\cdot cbZ_{\ell+1,j,k}\left(Z_{\ell+2,j,k} - Z_{\ell-1,j,k}\right) - \zeta Z_{\ell,j,k} + \frac{\hat{h}\hat{c}}{\hat{b}}Y_{j,k} \tag{4}$$

This system has 3 different time time scale, and $\hat{h}, \hat{b}, \hat{c}$ has the similar meaning as we see in the course here.

(a) Please prove that the energy of this system is decaying without forcing, where the energy is defined as follows :

$$E = \frac{1}{2}\left(\sum_{k}^{K} X_k^2 + \sum_{j,k}^{J,K} Y_{j,k}^2 + \sum_{\ell,j,k}^{L,J,K} Z_{\ell,j,k}^2\right) \tag{5}$$

and discuss the relations between decaying rate and parameters.

(Hint : for each term, $\frac{\mathrm{d}X^2}{\mathrm{d}t} = X\frac{\mathrm{d}X}{\mathrm{d}t}$. Calculate it, then take summation.)

(b) Please implement this model under the following setting, under initial conditions that gives non-trivial result:

$$\begin{cases} K = 40, J = 10, L = 10; \\ F = 10, c = 10, b = 10; \\ \hat{c} = 5, \hat{b} = 1, \zeta = 0.1 \end{cases} \tag{6}$$

and discuss your result.

(Hint: The settings above works for me, but you can change the setting of parameters if you wish to. Also, you may want to present your result like the link mentioned.)

(c) Please change the coupling coefficients $h, \hat{h}$, and discuss the predictability under different strength of coupling among different scales.