In [1]:
```python
import matplotlib.pyplot as plt
import pandas as pd
import numpy as np
import sympy as sp

from numpy import transpose as trans
from numpy.linalg import inv

np.set_printoptions(suppress=True)
```

Question 1

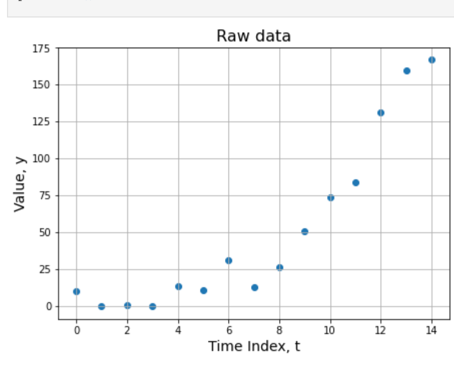
Importing the raw data into a dataframe so that it can be visualized

In [2]:
```python
data = pd.read_csv('dataHw1.csv', header=None).to_numpy()
time = data[:,[0]]
vals = data[:,[1]]
```

In [3]:
```python
time -= 1  # Assume the the variable t starts from 0
```

Plotting the data to get an idea of what it looks like

In [4]:
```python
plt.figure(figsize=(7, 5))
plt.scatter(time, vals)
plt.title('Raw data', fontsize=16)
plt.xlabel('Time Index, t', fontsize=14)
plt.ylabel('Value, y', fontsize=14)
plt.grid()
plt.savefig('pictures/q1/raw_data.jpg', bbox_inches='tight')
plt.show()
```



Q1.1

This fucntion takes in the matrix Φ corresponding to the number of parameters that the model has and returns a vector of estimated parameters, $\hat{\theta}$

In [5]:
```python
def theta_hat(Phi, targets=vals):
    """Takes in Phi and y(t), a vector of current and previous regressors and the real outputs, and returns the
    current value of the parameters
    """
#     The following block of code was implemented as a check to make sure the answers were consistent.
#     This implements equation 2.9 from the text

#     P = inv(trans(Phi) @ Phi)
#     row, col = Phi.shape
#     temp = np.zeros((col, 1))
#     for phi, y in zip(Phi, targets):
#         temp += np.array([trans(phi).reshape(-1,1) @ y]).reshape(-1, 1)
#     check = P @ temp

    return (inv(trans(Phi) @ Phi)) @ (trans(Phi) @ targets)  # Implementation of equation 2.6 from the text
```

The following block of code creates a matrix assuming 5 parameters. For each model, this matrix is sliced and only the columns that correspond to the number of parameters in that model are used

In [6]:
```python
Phi = np.zeros((1,5))
Phi[0][0] = 1
for t in np.arange(1, 15):
    Phi = np.concatenate((Phi, np.array([t**0, t**1, t**2, t**3, t**4]).reshape(1,5)), axis=0)
```

Model assuming that $\varphi^T = (t^0)$ and $\hat{\theta} = b_0$. Note that the function that is called by model p (theta_hat) takes in $\Phi(k)$ with p columns, for $p \,\epsilon \{1, 2, 3, 4, 5\}$ and returns the result of equation 2.6 in the textbook.

$$\hat{\theta}(k) = (\Phi(k)^T \Phi(k))^{-1} \Phi(k)^T Y(k)$$

Model 1

In [7]:
```python
Phi1 = Phi[:, 0].reshape(-1, 1)  # Slicing only the first column of Phi matrix
theta1 = theta_hat(Phi1)
```

Model 2

Model assuming that $\varphi^T = (t^0\; t^1)$ and $\hat{\theta} = (b_0\; b_1)^T$.

In [8]:
```python
Phi2 = Phi[:, 0:2]  # Slicing only the first two column of Phi matrix
theta2 = theta_hat(Phi2)
```

Model 3

Model assuming that $\varphi^T = (t^0\; t^1\; t^2)$ and $\hat{\theta} = (b_0\; b_1\; b_2)^T$.

In [9]:
```python
Phi3 = Phi[:, 0:3]  # Slicing first three column of Phi matrix
theta3 = theta_hat(Phi3)
```

Model 4

Model assuming that $\varphi^T = (t^0\; t^1\; t^2\; t^3)$ and $\hat{\theta} = (b_0\; b_1\; b_2\; b_3)^T$.

In [10]:
```python
Phi4 = Phi[:, 0:4]  # Slicing first four column of Phi matrix
theta4 = theta_hat(Phi4)
```

Model 5

Model assuming that $\varphi^T = (t^0\; t^1\; t^2\; t^3\; t^4)$ and $\hat{\theta} = (b_0\; b_1\; b_2\; b_3\; b_4)^T$.

In [11]:
```python
Phi5 = Phi[:, 0:5]  # Slicing first five column of Phi matrix
theta5 = theta_hat(Phi5)
```

This block below is dedicated strictly to organizing and printing the data.

In [12]:
```python
row1 = np.pad(theta1.reshape(-1,), (0, 4), mode='constant')
row2 = np.pad(theta2.reshape(-1,), (0, 3), mode='constant')
row3 = np.pad(theta3.reshape(-1,), (0, 2), mode='constant')
row4 = np.pad(theta4.reshape(-1,), (0, 1), mode='constant')
row5 = theta5.reshape(-1,)
parameters = np.array([row1,
                       row2,
                       row3,
                       row4,
                       row5])

data_1 = pd.DataFrame(data=parameters,
                      index=np.arange(1, 6),
                     columns=['b0', 'b1', 'b2', 'b3', 'b4'])
data_1.index.name = 'Model'
data_1
```

Out[12]:

Model	b0	b1	b2	b3	b4
1	51.435013	0.000000	0.000000	0.000000	0.000000
2	-31.106229	11.791606	0.000000	0.000000	0.000000
3	11.150564	-7.711529	1.393081	0.000000	0.000000
4	8.137128	-4.576673	0.813574	0.027596	0.000000
5	4.234310	3.497340	-1.992023	0.346007	-0.011372

Q1.2

The "estimates" function uses equation 2.1 to estimate the output of the system. That is,

$$\hat{y}(k) = \varphi^T(k)\theta(k)$$

The "square_error" function returns the square of the residual.

$$\big(y(k) - \hat{y}(k)\big)^2$$

In [13]:
```python
def estimates(Phi_func, theta_func):
    """Takes in the current value of the vector of regressors and the current value of the parameters and retur
    the estimated output"""
    # This function takes in the
    return (Phi_func.reshape(1, -1) @ theta_func).reshape(1,)

def square_error(y_func, epsilon):
    """Takes in the current value of the output and the systems estimate of the output and returns the square
    of the residual"""
    return np.array([np.square(y_func - epsilon)]).reshape(1,)
```

The for loop in the block below loops through all values of k and calculates all the estimates, $\hat{y}(k)$, and squares of the residuals, $\big(y(k) - \hat{y}(k)\big)^2$.

Following the for loop, the loss function is found using equation 2.2 from the textbook,

$$V(\hat{\theta}, k) = \frac{1}{2}\sum_{i=1}^{t}\big(y(k) - \hat{y}(k)\big)^2$$

In [14]:
```python
y1_lse, y2_lse, y3_lse, y4_lse, y5_lse = (0, 0, 0, 0, 0)
for t, y in zip(time, vals):
    if t == 0:
        y1 = estimates(Phi[int(t), 0:1], theta1)
        y2 = estimates(Phi[int(t), 0:2], theta2)
        y3 = estimates(Phi[int(t), 0:3], theta3)
        y4 = estimates(Phi[int(t), 0:4], theta4)
        y5 = estimates(Phi[int(t), 0:5], theta5)

    else:
        y1 = np.concatenate((y1, estimates(Phi[int(t), 0:1], theta1)), axis=0)
        y2 = np.concatenate((y2, estimates(Phi[int(t), 0:2], theta2)), axis=0)
        y3 = np.concatenate((y3, estimates(Phi[int(t), 0:3], theta3)), axis=0)
        y4 = np.concatenate((y4, estimates(Phi[int(t), 0:4], theta4)), axis=0)
        y5 = np.concatenate((y5, estimates(Phi[int(t), 0:5], theta5)), axis=0)

    y1_lse += square_error(y, estimates(Phi[int(t), 0:1], theta1))
    y2_lse += square_error(y, estimates(Phi[int(t), 0:2], theta2))
    y3_lse += square_error(y, estimates(Phi[int(t), 0:3], theta3))
    y4_lse += square_error(y, estimates(Phi[int(t), 0:4], theta4))
    y5_lse += square_error(y, estimates(Phi[int(t), 0:5], theta5))

loss = (1/2*np.array([y1_lse, y2_lse, y3_lse, y4_lse, y5_lse])).reshape(-1,)
```

In [15]:
```python
data_1['V'] = pd.Series(data=loss, index=data_1.index)
data_1
```

Out[15]:

Model	b0	b1	b2	b3	b4	V
1	51.435013	0.000000	0.000000	0.000000	0.000000	24103.092840
2	-31.106229	11.791606	0.000000	0.000000	0.000000	4637.216740
3	11.150564	-7.711529	1.393081	0.000000	0.000000	634.251379
4	8.137128	-4.576673	0.813574	0.027596	0.000000	612.440363
5	4.234310	3.497340	-1.992023	0.346007	-0.011372	563.290183

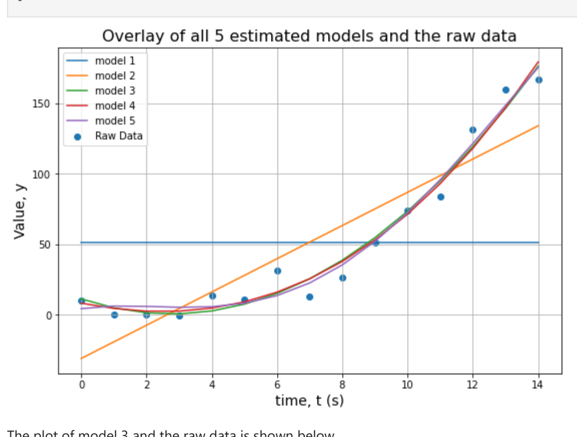
From the table above we see that model 4 is the first new model that doesn't have at least an order of magnitude imporvement. Although it is tempting to suppose that model 5 performs best, the reduced loss function is likely a result of the model learning the noise and not the trend. With this in mind, the optimal number of parameters to learn the model without introducing overfitting is 3. The model is then,

$$\hat{y}(k) = \varphi^T(k)(11.150564\ \ -7.711529\ 1.393081)^T\ +\ e(k)$$

It should be noted that the theory dictates a choise of parameters that minimize the loss function, but with noise added to this system (especially with the large standard deviation) too many parameters leads to overfitting.

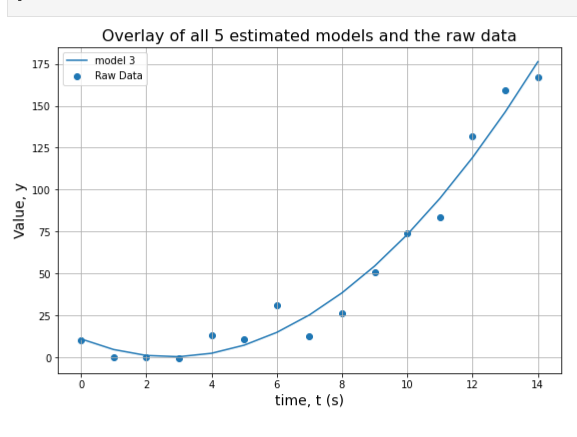
This block is dedicated to printing the different estimated outputs overlayn on eachother for comparison.

In [16]:
```python
plt.figure(figsize=(9, 6))
plt.scatter(time, vals)
plt.plot(time, y1)
plt.plot(time, y2)
plt.plot(time, y3)
plt.plot(time, y4)
plt.plot(time, y5)
plt.xlabel('time, t (s)', fontsize=14)
plt.ylabel('Value, y', fontsize=14)
plt.title('Overlay of all 5 estimated models and the raw data', fontsize=16)
plt.legend(['model 1',
            'model 2',
            'model 3',
            'model 4',
            'model 5',
            'Raw Data'])
plt.grid()
plt.show()
```



The plot of model 3 and the raw data is shown below

In [17]:
```python
plt.figure(figsize=(9, 6))
plt.scatter(time, vals)
plt.plot(time, y3)
plt.xlabel('time, t (s)', fontsize=14)
plt.ylabel('Value, y', fontsize=14)
plt.title('Overlay of all 5 estimated models and the raw data', fontsize=16)
plt.legend(['model 3',
            'Raw Data'])
plt.grid()
plt.show()
```



Q1.3

To find the unbiased estimate of the standard deviation for each model we take the square root of (iii) from Theorem 2.2 in the textbook. This is done as follows:

$$\hat{\sigma} = \sqrt{\frac{2V(\hat{\theta}, k)}{t - n}}$$

In [18]:
```python
N = np.arange(1, 6)
sigma_hat = []
for n, _2V in zip(N, 2*loss):
    sigma_hat.append(np.sqrt(_2V/(len(time) - n)))
sigma_hat = np.array(sigma_hat).reshape(-1,)
```

In [19]:
```python
data_1['std. dev.'] = pd.Series(data=sigma_hat, index=data_1.index)
data_1
```

Out[19]:

Model	b0	b1	b2	b3	b4	V	std. dev.
1	51.435013	0.000000	0.000000	0.000000	0.000000	24103.092840	58.679630
2	-31.106229	11.791606	0.000000	0.000000	0.000000	4637.216740	26.709885
3	11.150564	-7.711529	1.393081	0.000000	0.000000	634.251379	10.281467
4	8.137128	-4.576673	0.813574	0.027596	0.000000	612.440363	10.552383
5	4.234310	3.497340	-1.992023	0.346007	-0.011372	563.290183	10.614049

These results represent the unbiased estimation of the standard devaition of the signals. The expected value of this estimation should be equal to the standard deviation of the signals from the true function, which is given at $\sigma = 11$. Given this, we choose model 5 as it is the closest to the true value of the standard deviation associated with the data.

Question 2

In [20]:
```python
import matplotlib.pyplot as plt
import pandas as pd
import numpy as np

from numpy import transpose as trans
from numpy.random import normal
from numpy.linalg import inv
from numpy import cos, sin
```

Q2.1

This block is an initialization block that is used to define the inputs, true values and all the constants used for the code.

In [21]:
```python
a1_true, a2_true, b0_true, b1_true, std_dev = 1.3, 0.75, 1.1, -0.35, 0.65

recursion_length = 1500
time = np.arange(2*recursion_length + 1)

# Delayed Kronecker delta input
delta = np.zeros(time.shape)
delta[np.where(time == 100)] = 1

# Delayed step input
step = np.zeros((99))
step = np.concatenate((step, np.ones(2*recursion_length - 98,)))
```

In [22]:
```python
time[100]
```

Out[22]: 100

This function recursively executes the equations in Theorem 2.3 performing the following calculations

$$\hat{\theta}(t) = \hat{\theta}(t-1) + K(t)\big(y(t) - \phi^T(t)\hat{\theta}(t-1)\big)$$

$$K(t) = P(t-1)\phi(t)\big(I + \phi^T(t)P(t-1)\phi(t)\big)^{-1}$$

$$P(t) = \big(I - K(t)\phi^T(t)\big)P(t-1)$$

The function takes in $\hat{\theta}(t-1)$ and $P(t-1)$ and returns $\hat{\theta}(t)$ and $P(t)$.

In [23]:
```python
def rls(theta_m1, P_m1, a1=a1_true, a2 = a2_true, b0=b0_true, b1=b1_true, sig=std_dev):
    """This function takes in the previous values of theta and P and recursively finds the next values
    in accordance with Theorem 2.3 in the textbook.
    """
    global idx, N, y, u, a1_hat, a2_hat, b0_hat, b1_hat, Phi

    y[idx] = -a1*y[idx - 1] -a2*y[idx - 2] + b0*u[idx - 1] + b1*u[idx - 2] + normal(scale=sig)

    phi = np.array([[-y[idx - 1]],
                    [-y[idx - 2]],
                    [u[idx - 1]],
                    [u[idx - 2]]])

    K = P_m1 @ phi @ inv(np.eye(1) + trans(phi) @ P_m1 @ phi)
    P = (np.eye(len(phi)) - K @ trans(phi)) @ P_m1
    theta = theta_m1 + K @ (y[idx] - trans(phi) @ theta_m1)

    a1_hat.append(float(theta[0]))
    a2_hat.append(float(theta[1]))
    b0_hat.append(float(theta[2]))
    b1_hat.append(float(theta[3]))


    idx += 1
    if idx == N + 1:
        return theta, P
    else:
        return rls(theta, P)
```

In [ ]:

In [ ]:

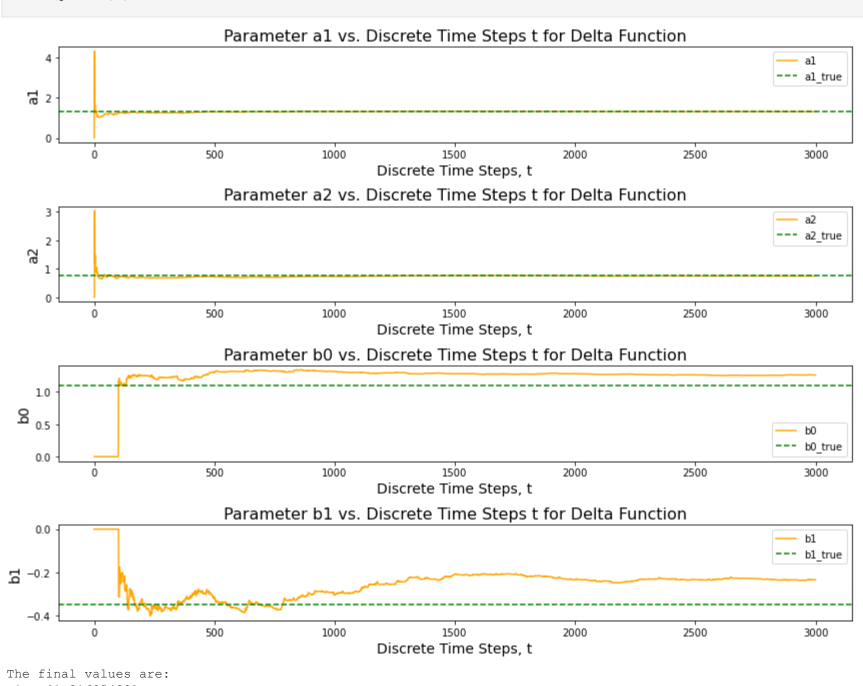
In [ ]:

Parameter Estimation with Delayed Kronecker Delta Input

This block takes the input $\delta(t - 100)$ and attempts to estimate the parameters. Note that since the recursive function only accepts values where all the terms are defined, the first two outputs and associated value are defined outside the function.

In [24]:
```python
# Initial conditions for t = 0
y = np.zeros(time.shape)
y[0] = normal(scale=std_dev)
theta_naught = np.zeros((4,1))
P_naught = 100 * np.eye(4)
Phi = np.array([])

# Initial conditions for t = 1
y[1] = -a1_true*y[0] + normal(scale=std_dev)
phi = np.array([[-y[0]],
                [0],
                [0],
                [0]]).reshape(4, -1)
K_1 = P_naught @ phi @ inv(np.eye(1) + trans(phi) @ P_naught @ phi)
P_1 = (np.eye(4) - K_1 @ trans(phi)) @ P_naught
theta_1 = theta_naught + K_1 @ (y[1] - trans(phi) @ theta_naught)
```

In [25]:
```python
idx = 1
N = recursion_length
u = delta
a1_hat, a2_hat, b0_hat, b1_hat = [0], [0], [0], [0]

params, P = rls(theta_1, P_1)
N += recursion_length
params, P_delta = rls(params, P)

plt.figure(figsize=(12, 9))
plt.subplot(4, 1, 1)
plt.plot(time, a1_hat, 'orange')
plt.axhline(y=a1_true, color ="green", linestyle ="--")
plt.title('Parameter a1 vs. Discrete Time Steps t for Delta Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('a1', fontsize=14)
plt.legend(['a1', 'a1_true'])

plt.subplot(4, 1, 2)
plt.plot(time, a2_hat, 'orange')
plt.axhline(y=a2_true, color ="green", linestyle ="--")
plt.title('Parameter a2 vs. Discrete Time Steps t for Delta Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('a2', fontsize=14)
plt.legend(['a2', 'a2_true'])

plt.subplot(4, 1, 3)
plt.plot(time, b0_hat, 'orange')
plt.axhline(y=b0_true, color ="green", linestyle ="--")
plt.title('Parameter b0 vs. Discrete Time Steps t for Delta Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('b0', fontsize=14)
plt.legend(['b0', 'b0_true'])

plt.subplot(4, 1, 4)
plt.plot(time, b1_hat, 'orange')
plt.axhline(y=b1_true, color ="green", linestyle ="--")
plt.title('Parameter b1 vs. Discrete Time Steps t for Delta Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('b1', fontsize=14)
plt.legend(['b1', 'b1_true'])

plt.tight_layout()
plt.show()

print(f'''The final values are:
a1 = {params[0]}
a2 = {params[1]}
b0 = {params[2]}
b1 = {params[3]}''')
```



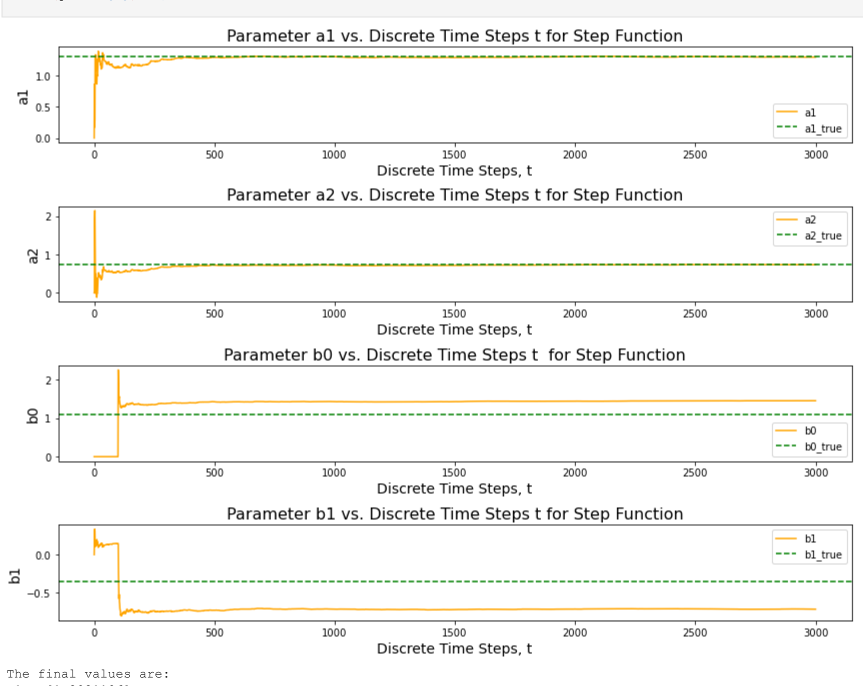

```
The final values are:
a1 = [1.31605483]
a2 = [0.75147522]
b0 = [1.25967174]
b1 = [-0.23420351]
```

Parameter Estimation with Delayed Step Input

This block takes the input $u(t - 100)$ and attempts to estimate the parameters. Note that since the recursive function only accepts values where all the terms are defined, the first two outputs and associated value are defined outside the function.

In [26]:
```python
# Initial conditions for t = 0
y = np.zeros(time.shape)
y[0] = normal(scale=std_dev)
theta_naught = np.zeros((4,1))
P_naught = 100 * np.eye(4)

# Initial conditions for t = 1
y[1] = -a1_true*y[0] + normal(scale=std_dev)
phi = np.array([[-y[0]],
                [0],
                [0],
                [0]]).reshape(4, -1)
K_1 = P_naught @ phi @ inv(np.eye(1) + trans(phi) @ P_naught @ phi)
P_1 = (np.eye(4) - K_1 @ trans(phi)) @ P_naught
theta_1 = theta_naught + K_1 @ (y[1] - trans(phi) @ theta_naught)
```

In [27]:
```python
idx = 1
N = recursion_length
u = step
a1_hat, a2_hat, b0_hat, b1_hat = [0], [0], [0], [0]

params, P = rls(theta_1, P_1)
N += recursion_length
params, P_step = rls(params, P)

plt.figure(figsize=(12, 9))
plt.subplot(4, 1, 1)
plt.plot(time, a1_hat, 'orange')
plt.axhline(y=a1_true, color ="green", linestyle ="--")
plt.title('Parameter a1 vs. Discrete Time Steps t for Step Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('a1', fontsize=14)
plt.legend(['a1', 'a1_true'])

plt.subplot(4, 1, 2)
plt.plot(time, a2_hat, 'orange')
plt.axhline(y=a2_true, color ="green", linestyle ="--")
plt.title('Parameter a2 vs. Discrete Time Steps t for Step Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('a2', fontsize=14)
plt.legend(['a2', 'a2_true'])

plt.subplot(4, 1, 3)
plt.plot(time, b0_hat, 'orange')
plt.axhline(y=b0_true, color ="green", linestyle ="--")
plt.title('Parameter b0 vs. Discrete Time Steps t  for Step Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('b0', fontsize=14)
plt.legend(['b0', 'b0_true'])

plt.subplot(4, 1, 4)
plt.plot(time, b1_hat, 'orange')
plt.axhline(y=b1_true, color ="green", linestyle ="--")
plt.title('Parameter b1 vs. Discrete Time Steps t for Step Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('b1', fontsize=14)
plt.legend(['b1', 'b1_true'])

plt.tight_layout()
plt.show()

print(f'''The final values are:
a1 = {params[0]}
a2 = {params[1]}
b0 = {params[2]}
b1 = {params[3]}''')
```




```
The final values are:
a1 = [1.2921106]
a2 = [0.73317369]
b0 = [1.45334332]
b1 = [-0.72381185]
```

Q2.2

In [ ]:

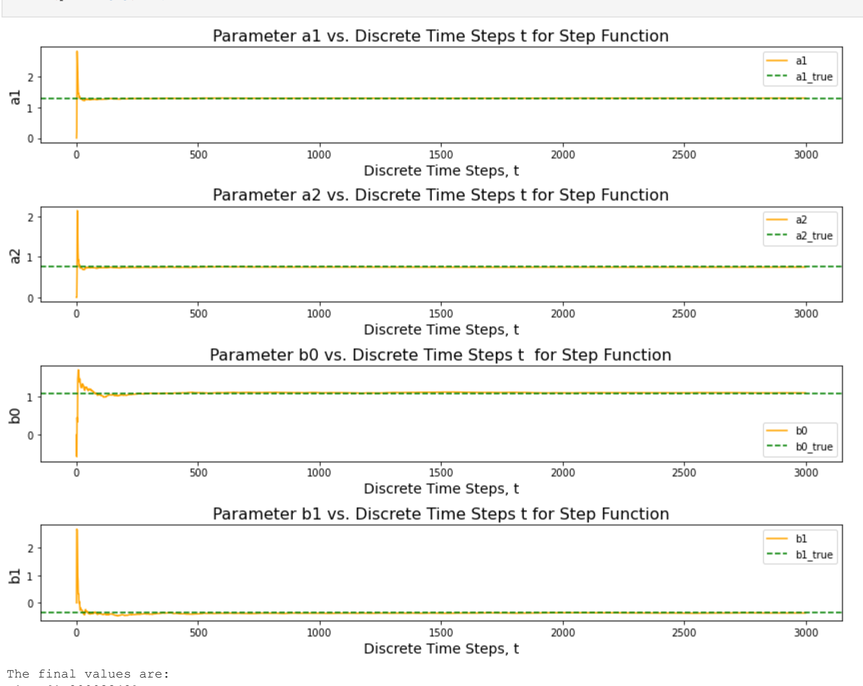
In [ ]:

In [ ]:

Q2.3

In [28]:
```python
# Multi-frequency input signal
mfs = np.array([1])
for t in time[1:]:
    mfs = np.concatenate((mfs, np.array([sin((2*np.pi)/5*t) + cos((4*np.pi)/5*t)])), axis=0)
```

In [29]:
```python
idx = 1
N = recursion_length
u = mfs
a1_hat, a2_hat, b0_hat, b1_hat = [0], [0], [0], [0]

# Initial conditions for t = 0
y = np.zeros(time.shape)
y[0] = normal(scale=std_dev)
theta_naught = np.zeros((4,1))
P_naught = 100 * np.eye(4)

# Initial conditions for t = 1
y[1] = -a1_true*y[0] + b0_true*u[0] + normal(scale=std_dev)
phi = np.array([[-y[0]],
                [0],
                [u[0]],
                [0]]).reshape(4, -1)
K_1 = P_naught @ phi @ inv(np.eye(1) + trans(phi) @ P_naught @ phi)
P_1 = (np.eye(4) - K_1 @ trans(phi)) @ P_naught
theta_1 = theta_naught + K_1 @ (y[1] - trans(phi) @ theta_naught)

params, P = rls(theta_1, P_1)
N += recursion_length
params, P_mfs = rls(params, P)

plt.figure(figsize=(12, 9))
plt.subplot(4, 1, 1)
plt.plot(time, a1_hat, 'orange')
plt.axhline(y=a1_true, color ="green", linestyle ="--")
plt.title('Parameter a1 vs. Discrete Time Steps t for Step Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('a1', fontsize=14)
plt.legend(['a1', 'a1_true'])

plt.subplot(4, 1, 2)
plt.plot(time, a2_hat, 'orange')
plt.axhline(y=a2_true, color ="green", linestyle ="--")
plt.title('Parameter a2 vs. Discrete Time Steps t for Step Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('a2', fontsize=14)
plt.legend(['a2', 'a2_true'])

plt.subplot(4, 1, 3)
plt.plot(time, b0_hat, 'orange')
plt.axhline(y=b0_true, color ="green", linestyle ="--")
plt.title('Parameter b0 vs. Discrete Time Steps t  for Step Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('b0', fontsize=14)
plt.legend(['b0', 'b0_true'])

plt.subplot(4, 1, 4)
plt.plot(time, b1_hat, 'orange')
plt.axhline(y=b1_true, color ="green", linestyle ="--")
plt.title('Parameter b1 vs. Discrete Time Steps t for Step Function', fontsize=16)
plt.xlabel('Discrete Time Steps, t', fontsize=14)
plt.ylabel('b1', fontsize=14)
plt.legend(['b1', 'b1_true'])

plt.tight_layout()
plt.show()

print(f'''The final values are:
a1 = {params[0]}
a2 = {params[1]}
b0 = {params[2]}
b1 = {params[3]}''')
```




```
The final values are:
a1 = [1.29993342]
a2 = [0.74728446]
b0 = [1.104098]
b1 = [-0.36859987]
```

Q2.4

To find the standard deviation of the parameters, we use the following,

$$\sigma_{\hat{j}} = \sigma\sqrt{P_{m,m}}$$

Where $j\,\epsilon\,\{a,\, b\}$ and m represents the associated index in the P matrix

In [30]:
```python
sigma_delta = std_dev*np.sqrt(np.array([[P_delta[2][2]],
                                        [P_delta[3][3]]]))
sigma_step = std_dev*np.sqrt(np.array([[P_step[2][2]],
                                       [P_step[3][3]]]))
sigma_mfs = std_dev*np.sqrt(np.array([[P_mfs[2][2]],
                                      [P_mfs[3][3]]]))
```

In [31]:
```python
data_2 = pd.DataFrame(np.concatenate((sigma_delta, sigma_step, sigma_mfs), axis=1),
                      columns=['Delta', 'Step', 'MFS'],
                      index=['b0', 'b1'])
```

In [32]:
```python
data_2
```

Out[32]:

	Delta	Step	MFS
b0	0.647202	0.457426	0.014289
b1	0.647371	0.457407	0.017248

Q2.5

In [33]:
```
sigma_step
```
Out[33]:
```
array([[0.45742577],
       [0.45740709]])
```
In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]: