In [1]:

```
import matplotlib.pyplot as plt
import control as co
import pandas as pd
import numpy as np
import sympy as sp
import time

from sympy.abc import a, b, q, s, z, omega, zeta
from numpy.linalg import inv
```

In [2]:

```
u, uc, vs, y, ys, Gs, Hz, Hq = sp.symbols('u(k),u_c{(k)},v(s),y(k),y(s),G(s),H(z),H(q)')
Ts = sp.symbols('T_s')

am1, am2, bm0, bm1 = sp.symbols('a_{m_1},a_{m_2},b_{m_0},b_{m_1}')
ao, a1, a2, b0, b1 = sp.symbols('a_o,a_1,a_2,b_0,b_1')
r0, s0, s1, t0, t1 = sp.symbols('r_0,s_0,s_1,t_0,t_1')

Gs_eq = sp.Eq(Gs, b/(s*(s + b)))
```

The model is given by $G(s) = \frac{b}{s\left(b + s\right)}$. To discretize,

$$G(z) = (1 - z^{-1})\mathscr{Z}\{\tfrac{b}{s^2(s+b)}\}$$

$$G(z) = (1 - z^{-1})\mathscr{Z}\{\tfrac{1}{s^2} - \tfrac{1}{bs} + \tfrac{1}{b(b+s)}\}$$

$$G(z) = (1 - z^{-1})\Big(\tfrac{T_s z}{(z-1)^2} - \tfrac{1}{b}\tfrac{z}{z-1} + \tfrac{1}{b}\tfrac{z}{z-e^{-bT_s}}\Big)$$

In [3]:

```
Gz_eq_Ts_b = sp.simplify(sp.expand((1-1/z)*(Ts*z/(z-1)**2 - z/(b*(z-1)) + z/(b*(z-sp.exp(-b*Ts))))))

num, den = sp.fraction(Gz_eq_Ts_b)
den_poly = sp.Poly(den, z)
mono_div = den_poly.coeffs()[0]
sp.Eq(Hz, sp.collect(sp.simplify(Gz_eq_Ts_b), z))
```

Out[3]:

$$H(z) = \frac{-T_s b + z\left(T_s b e^{T_s b} - e^{T_s b} + 1\right) + e^{T_s b} - 1}{b\left(z^2 e^{T_s b} + z\left(-e^{T_s b} - 1\right) + 1\right)}$$

Pulse Function

In [4]:

```
radius = 38/2000  # 38 mm diameter to radius in m
mass = 4/7000  # grams per ball
area_ball = np.pi * radius**2
volume_ball = 4/3*np.pi*radius**3
density = 1.2  # kg/m^3
veq = 2.8 # m/s

Ts_val = 0.2
b_nom = 2*9.81*(mass-density*volume_ball)/(mass*veq)
```

In [5]:

```
## Example 3.1: since it has the same structure TF I used it to validate my approach, it's consistent
# sub_vals = [(Ts, 0.5), (b, 1)]

## This is for our project
sub_vals = [(Ts, Ts_val), (b, b_nom)]

num_mono_z = sp.collect(num/mono_div, z).subs(sub_vals)
den_mono_z = sp.collect(den/mono_div, z).subs(sub_vals)

Hq_eq = sp.Eq(Hq, num_mono_z/den_mono_z)
Hq_eq
# np.roots(sp.Poly(num_mono_z).coeffs())
```

Out[5]:

$$H(q) = \frac{0.0888233086497573z + 0.0575823297146081}{1.0z^2 - 1.26797180817817z + 0.267971808178173}$$

In [6]:

```
n_coeffs_nom, d_coeffs_nom = sp.fraction(Hq_eq.rhs)
pulse_coeffs = sp.Poly(d_coeffs_nom).coeffs()[1:]
for bi in sp.Poly(n_coeffs_nom).coeffs():
    pulse_coeffs.append(bi)

zeros = np.roots(sp.Poly(n_coeffs_nom).coeffs())
poles = np.roots(sp.Poly(d_coeffs_nom).coeffs())
display(pulse_coeffs)
```

```
[-1.26797180817817, 0.267971808178173, 0.0888233086497573, 0.0575823297146081]
```

This is a sanity check with the nominal values to check the stability of the zeros. Here, the zero is unstable for a nominal value of $b =$ -0.6482794954381095 and the poles are 1.0 and 0.26797180817817307.

Control Parameter Derivation

Given that the zeros are unstable, the parameters are derived without zero cancelation.

Here we know from the compatability conditions that,

$$\mathrm{degA}_m = \mathrm{degA} = 2$$

$$\mathrm{degB}_m = \mathrm{degB} = 1$$

Since the zeros in B are unstable then $B^+ = 1$ and $B^- = B = b_0 q + b_1$

Then,

$$\mathrm{degA}_o = \mathrm{degA} - \mathrm{degB}^+ - 1 = 2 - 0 - 1 = 1$$

Using the Diophantine equation we get,

$$AR + BS = A_o A_m$$

Let $A_o = q + a_0$

Since the process is second order, then,

$$\mathrm{degR} = \mathrm{degS} = \mathrm{degT} = 1,$$

with R being monic.

Control Parameters

In [7]:

```
# Process Values
_A = q**2 + a1*q + a2
_B = b0*q + b1

# Model Values
_Am = q**2 + am1*q + am2
_beta = (_Am/_B).subs(q, 1)
_Bm = sp.simplify(sp.expand(_beta*_B))
_Ao = q + ao

# Control Values
_R = q + r0
_S = s0*q + s1
_T = t0*q + t1
```

Diophantine equation:

In [8]:

```
diophantine = sp.Eq((_A*_R + _B*_S), (_Ao*_Am))
dio_LHS_coeffs = sp.Poly((_A*_R + _B*_S), q).coeffs()
dio_RHS_coeffs = sp.Poly(_Ao*_Am, q).coeffs()
```

In [9]:

```
diophantine
```

Out[9]:

$$(q + r_0)\left(a_1 q + a_2 + q^2\right) + (b_0 q + b_1)(q s_0 + s_1) = (a_o + q)\left(a_{m_1} q + a_{m_2} + q^2\right)$$

Finding r_0

In [10]:

```
_s0 = sp.solve(dio_LHS_coeffs[1] - dio_RHS_coeffs[1], s0)[0]
_s1 = sp.solve((dio_LHS_coeffs[2] - dio_RHS_coeffs[2]).subs(s0, _s0), s1)[0]
_r0 = sp.solve((dio_LHS_coeffs[3] - dio_RHS_coeffs[3]).subs(s1, _s1), r0)[0]
_r0
```

Out[10]:

$$\frac{-a_1 b_1^2 + a_2 b_0 b_1 - a_o a_{m_1} b_0 b_1 + a_o a_{m_2} b_0^2 + a_o b_1^2 + a_{m_1} b_1^2 - a_{m_2} b_0 b_1}{-a_1 b_0 b_1 + a_2 b_0^2 + b_1^2}$$

Finding s_0

In [11]:

```
_s0 = sp.solve((dio_LHS_coeffs[1] - dio_RHS_coeffs[1]).subs(r0, _r0), s0)[0]
_s0
```

Out[11]:

$$\frac{a_1^2 b_1 - a_1 a_2 b_0 - a_1 a_o b_1 - a_1 a_{m_1} b_1 + a_2 a_o b_0 + a_2 a_{m_1} b_0 - a_2 b_1 + a_o a_{m_1} b_1 - a_o a_{m_2} b_0 + a_{m_2} b_1}{-a_1 b_0 b_1 + a_2 b_0^2 + b_1^2}$$

Finding s_1

In [12]:

```
_s1 = sp.solve((dio_LHS_coeffs[2] - dio_RHS_coeffs[2]).subs([(s0, _s0), (r0, _r0)]), s1)[0]
_s1
```

Out[12]:

$$\frac{a_1 a_2 b_1 - a_1 a_o a_{m_2} b_0 - a_2^2 b_0 + a_2 a_o a_{m_1} b_0 - a_2 a_o b_1 - a_2 a_{m_1} b_1 + a_2 a_{m_2} b_0 + a_o a_{m_2} b_1}{-a_1 b_0 b_1 + a_2 b_0^2 + b_1^2}$$

Finding T

In [13]:

```
T = sp.collect(sp.simplify(sp.expand(_Bm/_B*_Ao)), q)
num_T, den_T = sp.fraction(T)
num_T_coeffs = sp.Poly(num_T, q).coeffs()
_t0 = num_T_coeffs[0]/den_T
_t1 = num_T_coeffs[1]/den_T
display(sp.Eq(t0, _t0))
display(sp.Eq(t1, _t1))
```

$$t_0 = \frac{a_{m_1} + a_{m_2} + 1}{b_0 + b_1}$$

$$t_1 = \frac{a_o a_{m_1} + a_o a_{m_2} + a_o}{b_0 + b_1}$$

Control Action

In [14]:

```
uk1,  uck1, yk1 = sp.symbols('u(k-1),u_c{(k-1)},y(k-1)')
control_action = (sp.expand(sp.Eq((_R*u)/q, (_T*uc - _S*y)/q)))
control_action
control_subs = [(1/q*u, uk1), (1/q*uc, uck1), (1/q*y, yk1)]
control_action_sol = sp.Eq(u, sp.solve(control_action.subs(control_subs), u)[0])
control_action_sol
```

Out[14]:

$$u(k) = -r_0 u(k-1) - s_0 y(k) - s_1 y(k-1) + t_0 u_c(k) + t_1 u_c(k-1)$$

Simulation

Process Deivation

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k-1) + b_1 u(k-2) = \phi(t-1)^T \theta$$

where,

$$\phi(t-1) = \left[-y(k-1) \;\; -y(k-2) \;\; u(k-1) \;\; u(k-2)\right]^T$$

and

$$\theta = \left[a_1 \; a_2 \; b_0 \; b_1\right]^T$$

In [15]:

```
final_time = 90
t = np.arange(0, final_time + Ts_val, Ts_val)
def reference_signal(end_time=final_time, Ts_func=Ts_val, lower_set=0.2, upper_set=0.1, period=30):
    uc_func = []
    time = np.arange(0, end_time + Ts_func, Ts_func)
    for _t in time:
        rat = 2*np.pi/period
        if np.sin(rat*_t) >= 0:
            uc_func.append(upper_set)
        else:
            uc_func.append(lower_set)
    return np.array(uc_func, float)
uc_val = reference_signal()
# plt.plot(np.arange(0, 60 + Ts_val, Ts_val), uc)
# plt.show()
```

In [16]:

```
omega_n = 1
zeta = 1
std = 0.0
# std = 0.001
Bmz_tf, Amz_tf = co.tfdata(co.sample_system(co.tf([1], [1, 2*zeta*omega_n, omega_n**2]), method='zoh', Ts=Ts_va
AM1 = Amz_tf[0][0][1]
AM2 = Amz_tf[0][0][2]
A0 = 0.5
T0_num = AM1 + AM2 + 1
T1_num = A0*(T0_num)
lam = 0.98
initial_P_weights = [10000]*4
# initial_P_weights = [1000, 100, 10, 10]
theta = np.array(pulse_coeffs, float).reshape(4, -1)

# display([AM1, AM2])
# display(pulse_coeffs)
# np.roots([1, AM1, AM2])
```

In [17]:

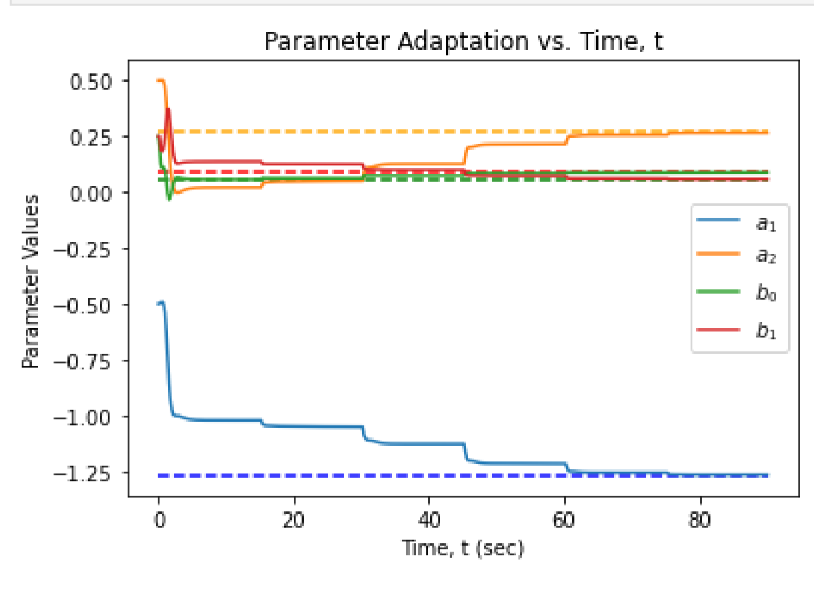
```
    # Estimates k = 0
time_ns = time.time_ns()
# theta_hat = np.array(pulse_coeffs, float).reshape(4, -1) # a1, a2, b0, b1 THIS WILL BE USED FOR THE REAL CON
# theta_hat = np.array([-1.1, 0.25, 0.1, 0.05], float).reshape(4, -1)
theta_hat = np.array([-0.5, 0.5, 0.25, 0.25], float).reshape(4, -1)
theta_arr = theta_hat
P = np.diag(initial_P_weights)
phi = np.zeros((4,1))

y_measure = (phi.T@theta + np.random.normal(0, std)).reshape(-1,)

a1, a2, b0, b1 = theta_hat[0], theta_hat[1], theta_hat[2], theta_hat[3]
den_rs = ((-a1*b0*b1) + (a2*b0**2) + b1**2)
den_t = b0 + b1
r0_val = 1/den_rs*((A0*AM2)*b0**2 + (-a1 + A0 + AM1)*b1**2 + (a2 - A0*AM1 - AM2)*b0*b1)
s0_val = 1/den_rs*((-a1*a2 + a2*(A0 + AM1) - A0*AM2)*b0 + (a1**2 - a1*(A0 + AM1) - a2 + A0*AM1 + AM2)*b1)
s1_val = 1/den_rs*((-a2**2 + A0*(a2*AM1 - a1*AM2) + a2*AM2)*b0 + (a2*(a1 - A0 - AM1) + A0*AM2)*b1)
t0_val = T0_num/den_t
t1_val = T1_num/den_t

M = np.array([r0_val, s0_val, s1_val, t0_val, t1_val], float).reshape(-1, 1)
N = np.array([0, -y_measure[0], 0, uc_val[0], 0], float).reshape(M.shape)
u_val = (N.T@M).reshape(-1, )

    # Estimates k = 1
phi = np.array([-y_measure[0], 0, u_val[0], 0], float).reshape(-1,1) # phi of 0
K = P@phi@inv(lam + phi.T@P@phi)
y_measure = np.concatenate((y_measure,
                            (phi.T@theta + np.random.normal(0, std)).reshape(-1,)))
theta_hat = theta_hat + K@(y_measure[-1] - phi.T@theta_hat)
theta_arr = np.concatenate((theta_arr,
                            theta_hat.reshape(-1, 1)), axis=1)
P = (np.eye(len(phi)) - K@phi.T) @P/lam


a1, a2, b0, b1 = theta_hat[0], theta_hat[1], theta_hat[2], theta_hat[3]
den_rs = ((-a1*b0*b1) + (a2*b0**2) + b1**2)
den_t = b0 + b1
r0_val = 1/den_rs*((A0*AM2)*b0**2 + (-a1 + A0 + AM1)*b1**2 + (a2 - A0*AM1 - AM2)*b0*b1)
s0_val = 1/den_rs*((-a1*a2 + a2*(A0 + AM1) - A0*AM2)*b0 + (a1**2 - a1*(A0 + AM1) - a2 + A0*AM1 + AM2)*b1)
s1_val = 1/den_rs*((-a2**2 + A0*(a2*AM1 - a1*AM2) + a2*AM2)*b0 + (a2*(a1 - A0 - AM1) + A0*AM2)*b1)
t0_val = T0_num/den_t
t1_val = T1_num/den_t

M = np.array([r0_val, s0_val, s1_val, t0_val, t1_val], float).reshape(-1, 1)
N = np.array([-u_val[0], -y_measure[1], -y_measure[0], uc_val[1], uc_val[0]],float).reshape(M.shape)
u_val = np.concatenate((u_val,
                        (N.T@M).reshape(-1,)))

for k in range(2, len(t)):
        phi = np.array([-y_measure[k-1], -y_measure[k-2], u_val[k-1], u_val[k-2]], float).reshape(-1,1)
        K = P@phi@inv(lam + phi.T@P@phi)
        y_measure = np.concatenate((y_measure,
                                    (phi.T@theta + np.random.normal(0, std)).reshape(-1,)))
        theta_hat = theta_hat + K@(y_measure[-1] - phi.T@theta_hat)
        theta_arr = np.concatenate((theta_arr,
                                    theta_hat.reshape(-1, 1)), axis=1)
        P = (np.eye(len(phi)) - K@phi.T)@P/lam

        a1, a2, b0, b1 = theta_hat[0], theta_hat[1], theta_hat[2], theta_hat[3]
        den_rs = ((-a1*b0*b1) + (a2*b0**2) + b1**2)
        den_t = b0 + b1
        r0_val = 1/den_rs*((A0*AM2)*b0**2 + (-a1 + A0 + AM1)*b1**2 + (a2 - A0*AM1 - AM2)*b0*b1)
        s0_val = 1/den_rs*((-a1*a2 + a2*(A0 + AM1) - A0*AM2)*b0 + (a1**2 - a1*(A0 + AM1) - a2 + A0*AM1 + AM2)*b
        s1_val = 1/den_rs*((-a2**2 + A0*(a2*AM1 - a1*AM2) + a2*AM2)*b0 + (a2*(a1 - A0 - AM1) + A0*AM2)*b1)
        t0_val = T0_num/den_t
        t1_val = T1_num/den_t

        M = np.array([r0_val, s0_val, s1_val, t0_val, t1_val], float).reshape(-1, 1)
        N = np.array([-u_val[k-1], -y_measure[k], -y_measure[k-1], uc_val[k], uc_val[k-1]]).reshape(M.shape)
        u_val = np.concatenate((u_val,
                                (N.T@M).reshape(-1,)))
(time.time_ns() - time_ns)*1e-9/len(t)*1e6
```

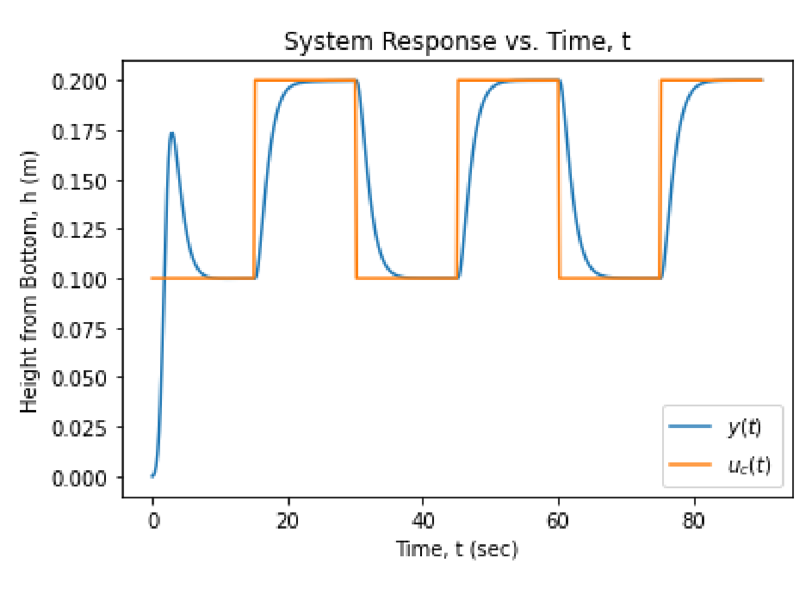
Out[17]:

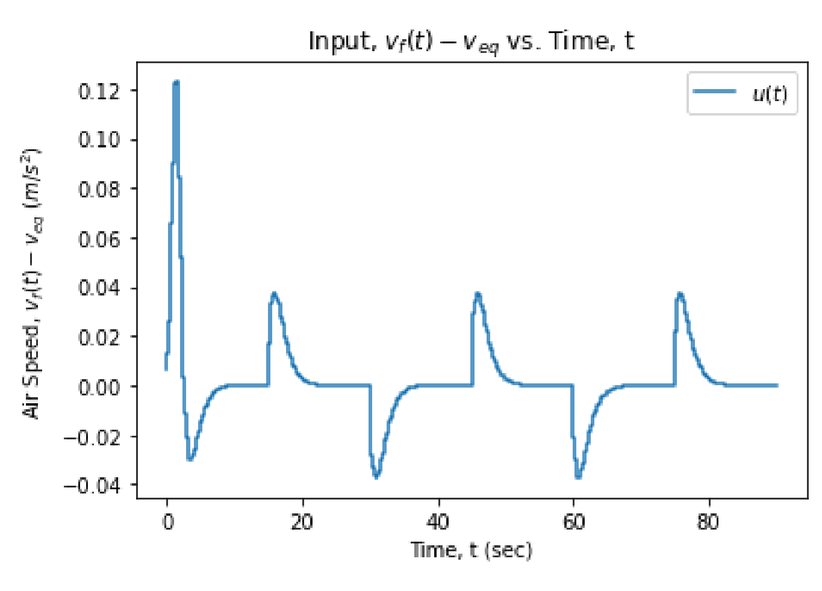
```
114.19711751662972
```

In [18]:

```
filepath = 'C:/Users/dboas065/University of Ottawa/ELG7113/system_models/derek_model/without_pictures/'
colors = ['blue', 'orange', 'red', 'green']
for row in range(len(theta_arr)):
    plt.plot(t, theta_arr[row,:])
    plt.hlines(theta[row][0], 0, t[len(t) - 1], colors=colors[row], linestyles='--')
plt.title('Parameter Adaptation vs. Time, t')
plt.xlabel('Time, t (sec)')
plt.ylabel('Parameter Values')
plt.legend(['$a_1$', '$a_2$', '$b_0$','$b_1$'])
# plt.savefig(filepath+'adaptation.pdf', format='pdf')
plt.savefig(filepath+'adaptation_noise.pdf', format='pdf')
plt.show()

st = 0
et = len(t)

plt.plot(t[st:et], y_measure[st:et])
plt.plot(t[st:et], uc_val[st:et])
plt.title('System Response vs. Time, t')
plt.xlabel('Time, t (sec)')
plt.ylabel('Height from Bottom, h (m)')
plt.legend(['$y(t)$', '$u_c(t)$'])
# plt.savefig(filepath+'output.pdf', format='pdf')
plt.savefig(filepath+'output_noise.pdf', format='pdf')
plt.show()

plt.step(t[st:et], u_val[st:et])
plt.title('Input, $v_f(t) - v_{eq}$ vs. Time, t')
plt.xlabel('Time, t (sec)')
plt.ylabel('Air Speed, $v_f(t) - v_{eq}$ ($m/s^2$)')
plt.legend(['$u(t)$'])
# plt.savefig(filepath+'input.pdf', format='pdf')
plt.savefig(filepath+'input_noise.pdf', format='pdf')
plt.show()

# for row in range(len(theta_arr)):
#     plt.plot(t, theta_arr[row,:])
# plt.show()

# et = len(t)
# plt.plot(t[0:et], y_measure[0:et])
# plt.plot(t[0:et], uc_val[0:et])
# plt.show()

# plt.step(t[0:et], u_val[0:et])
# plt.show()
```







In [ ]:

In [ ]: