A Continuous-Order Integral Operator for Maclaurin-type Reconstruction

Derek C. Braun

25 December 2025

Abstract

I introduce a continuous-order analog of the Maclaurin expansion to reconstruct an analytic function $f(x)$. This continuous-order Maclaurin-type operator replaces the discrete sum of integer-order derivatives in the classical Maclaurin expansion with an integral over fractional derivative orders, weighted by $x^r/\Gamma(r+1)$. Numerical experiments on a representative set of analytic functions show that the uncorrected operator reliably tracks the global shape of $f(x)$ with a systematic offset and an additional localized deviation near the origin. Low-order correction terms, motivated by the classical Euler–Maclaurin summation formula, reduce this discrepancy. With these corrections, the operator reconstructs the target functions accurately in the tested domains. The operator reproduces monomials exactly, reflecting the collapse of derivative information to a single order as with the classical Maclaurin expansion of monomials. This motivates a Taylor-centered extension, predicted to have smooth order dependence. Taken together, these results suggest that this continuous-order integration operator with low-order corrections may provide a coherent framework for reconstructing analytic functions and generalizing the classical Taylor–Maclaurin expansion.

1 Introduction

The classical Taylor–Maclaurin series is one of the most elegant tools in classical analysis, expressing analytic functions as infinite sums of derivatives scaled by factorials. If f is analytic near zero, then the Maclaurin series is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, x^k.$$

The classical representation sums derivatives over integer orders. That discreteness motivated me to ask: could a Maclaurin-type reconstruction also be formulated if we integrated fractional derivatives over a continuous order parameter? Formally, I asked if the discrete expansion could be replaced by

$$\int_0^\infty \frac{D^r f(0)\, x^r}{\Gamma(r+1)}\, dr, \tag{1}$$

where the factorial $k!$ is replaced by its analytic continuation $\Gamma(r+1)$.

The idea of fractional derivatives dates back to at least a 1695 letter from Leibniz to l'Hôpital speculating about the meaning of a half-order derivative [7]. More than a century later, Liouville, then Riemann, developed analytic constructions extending differentiation and integration to non-integer order, forming what is now the Riemann–Liouville framework. The Grünwald–Letnikov definition is based on discrete limits, and the Caputo definition is better suited to problems with initial and boundary conditions [2].

Fractional derivatives have been combined with discrete Taylor- or Maclaurin-type representations before. Osler [6] replaced integer-order derivatives with fractional derivatives while retaining a discrete summation. Later work developed fractional Taylor and Maclaurin series based on fixed fractional derivatives of order α, leading to expansions of the form $(x-a)^{m\alpha}/\Gamma(m\alpha+1)$ [8, 1, 9]. These constructions generalize classical power series but remain discrete in derivative order. Related approaches include series representations of fixed-order fractional derivatives in terms of integer-order derivatives [4], as well as distributed-order operators that integrate fractional operators against a prescribed weight.

Here, I propose a continuous-order integral representation based on integrating fractional derivatives over all orders, weighted by a Maclaurin-type kernel. This paper focuses on characterizing the behavior of this continuous-order operator on analytic functions and on understanding its relationship to classical Maclaurin behavior.

2 The Continuous-Order Maclaurin-type Integral Operator

Definition 2.1. Let f be analytic in a neighborhood of the origin, and let $D^r f(0)$ denote fractional derivative data of order r evaluated at the origin. For real $x > 0$, the *continuous-order Maclaurin-type operator* is defined as

$$\mathcal{T}[f](x) = \int_0^\infty \frac{D^r f(0)\, x^r}{\Gamma(r+1)}\, dr. \tag{2}$$

This definition replaces the discrete derivative orders $n \in \mathbb{N}$ in the classical Maclaurin series with a continuum of orders $r \in [0, \infty)$, and replaces the factorials $n!$ with their analytic continuation $\Gamma(r+1)$.

If $D^r f(0)$ were concentrated at integer values of r, the integral in (2) would heuristically collapse back to

$$\int_0^\infty \frac{D^r f(0)\, x^r}{\Gamma(r+1)}\, dr \;\longrightarrow\; \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, x^k,$$

recovering the formal structure of the classical Maclaurin expansion.

3 Sum–integral correction via the Euler–Maclaurin formula

Replacing a convergent infinite sum by its corresponding integral typically introduces systematic discrepancies. This observation is formalized by the Euler–Maclaurin summation formula, developed independently first by Maclaurin, then Euler [5]. This formula expresses the difference between a sum and its associated integral. It is an infinite series of correction terms with coefficients based on derivatives evaluated at the lower endpoint of the summation.

Euler–Maclaurin corrections have also appeared in fractional calculus for numerical evaluation or error estimation [3]. Here, they are naturally inspired from the sum–integral mismatch implicit in the continuous-order operator.

Definition 3.1. The kernel of the continuous-type operator $\mathcal{T}[f]$ depends on the fractional derivative of the reconstructed function evaluated at the origin:

$$k(r;x) = \frac{D^r f(0)\, x^r}{\Gamma(r+1)}. \tag{3}$$

For this kernel $k(r;x)$, I introduce a correction operator $\mathcal{E}[f]$ for correcting the sum–integral mismatch for $\mathcal{T}[f]$, defined as

$$\mathcal{E}[f](x) = \sum_{n=0}^{\infty} c_n \frac{\partial^n}{\partial r^n} \left[\frac{D^r f(0)\, x^r}{\Gamma(r+1)} \right]_{r=0}, \tag{4}$$

where the coefficients c_n are the Euler–Maclaurin coefficients determined by the Bernoulli numbers (with $c_{2n+1} = 0$ for $n \geq 1$). We denote by $\mathcal{E}_n$ the nth Euler–Maclaurin correction term in the operator expansion $\mathcal{E}[f](x)$.

4 Methodology

I performed all numerical evaluations using `mpmath` at a working precision of 100 decimal digits. I evaluated the integrals defining $\mathcal{T}[f](x)$ over the infinite interval $r \in [0, \infty)$ using `mpmath.quad`, which applies adaptive quadrature with automatic handling of infinite intervals. The routine truncates the tail when the remaining contribution falls below the working precision.

For each function tested, the correction operator $\mathcal{E}[f]$ given in (4) requires derivatives of the kernel of $\mathcal{T}[f]$ with respect to the order variable r, evaluated at $r = 0$. For three functions, I was able to calculate closed-form fractional derivatives, and for the latter three, I was not able to calculate closed-form expressions. For these, I obtained derivative data using high-precision numerical differentiation using `mpmath.diff`. Details on the fractional derivatives or analytic continuations for each function are given in Results 5.

5 Results

I tested a representative sample of analytic functions, showing the behavior of the continuous-order integral operator $\mathcal{T}$ and the effect of adding the low-order corrections $\mathcal{E}_0$, $\mathcal{E}_1$, and $\mathcal{E}_2$. The goal is to identify consistent qualitative features, including systematic offsets, deviations near the origin, and the impact of successive corrections.

I tested six analytic functions, chosen to represent distinct and complementary behaviors:

$$\begin{aligned}
&f(x) = e^x && \text{(entire function with uniform exponential growth)},\\
&f(x) = \frac{1}{1-x} && \text{(rational function with finite radius of convergence)},\\
&f(x) = \sin x && \text{(oscillatory entire function)},\\
&f(x) = e^{x^2} && \text{(quadratic exponential with rapid growth)},\\
&f(x) = e^{-x^2} && \text{(rapidly decaying entire function)},\\
&f(x) = J_0(x) && \text{(special function with oscillatory decay)}.
\end{aligned}$$

The exponential and rational cases are canonical benchmarks with simple derivative structure and well-understood Maclaurin behavior. The sine function tests whether the continuous-order operator can track sign changes and oscillations across zeros. The quadratic exponential e^{x^2} provides a stress test for rapid growth. The Gaussian e^{-x^2} tests the complementary case of rapid decay. Finally, the Bessel function $J_0(x)$ combines oscillatory behavior with decaying amplitude and nontrivial combinatorial structure in its Maclaurin coefficients.

5.1 Exponential function: $f(x) = e^x$

The exponential function provides a canonical benchmark, since all fractional derivative definitions agree at the origin and the Maclaurin series has the simplest possible structure. This case isolates the operator's behavior without complications from singularities, oscillations, or alternating coefficients.

For e^x, the Riemann–Liouville, Caputo, and Grünwald–Letnikov definitions agree that

$$D^r e^x = e^x, \qquad \text{so that} \qquad D^r e^x\big|_{x=0} = 1.$$

The kernel of $\mathcal{T}[f]$ in (3) simplifies to

$$k(r;x) = \frac{x^r}{\Gamma(r+1)}.$$

Evaluating the derivatives in (4) yields the corrections through the second order,

$$\mathcal{E}[e^x](x) = \underbrace{\frac{1}{2}}_{\mathcal{E}_0} + \underbrace{-\frac{1}{12}(\log x + \gamma)}_{\mathcal{E}_1} + \underbrace{\frac{1}{720}\left[(\log x + \gamma)^3 + \frac{\pi^2}{2}(\log x + \gamma) + 2\zeta(3)\right]}_{\mathcal{E}_2} + \cdots.$$

5.2 Rational function with a pole: $f(x) = \frac{1}{1-x}$

This rational function tests the operator in the presence of a finite radius of convergence and a nearby singularity. It allows direct comparison between the continuous-order reconstruction and the classical Maclaurin series up to the boundary of analyticity.

For $f(x) = \frac{1}{1-x}$, the Riemann–Liouville, Caputo, and Grünwald–Letnikov definitions agree that

$$D^r\left(\frac{1}{1-x}\right) = \frac{\Gamma(r+1)}{(1-x)^{r+1}}, \qquad \text{so that} \qquad D^r f(x)\big|_{x=0} = \Gamma(r+1).$$

The kernel of $\mathcal{T}[f]$ in (3) simplifies to

$$k(r;x) = \Gamma(r+1)\,\frac{x^r}{\Gamma(r+1)} = x^r.$$

Evaluating the derivatives in (4) yields the corrections through the second order,

$$\mathcal{E}\left[\tfrac{1}{1-x}\right](x) = \underbrace{\frac{1}{2}}_{\mathcal{E}_0} + \underbrace{-\frac{1}{12}(\log x + \gamma)}_{\mathcal{E}_1} + \underbrace{\frac{1}{720}\left[(\log x + \gamma)^3 + \frac{\pi^2}{2}(\log x + \gamma) + 2\zeta(3)\right]}_{\mathcal{E}_2} + \cdots.$$

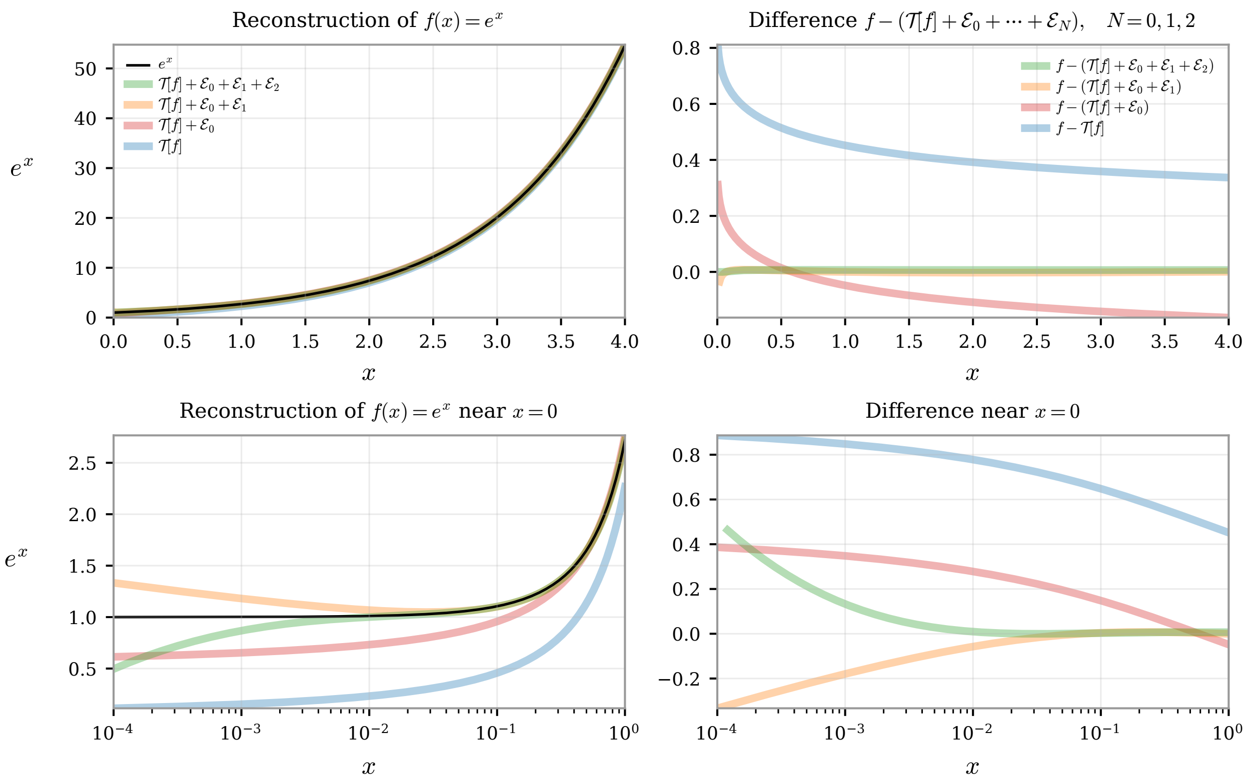

Figure 1: Reconstruction of $f(x) = e^x$ using $\mathcal{T}[f]$ with corrections $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$. Top panels show the reconstruction and residuals over a representative interval; bottom panels show the same near the origin.

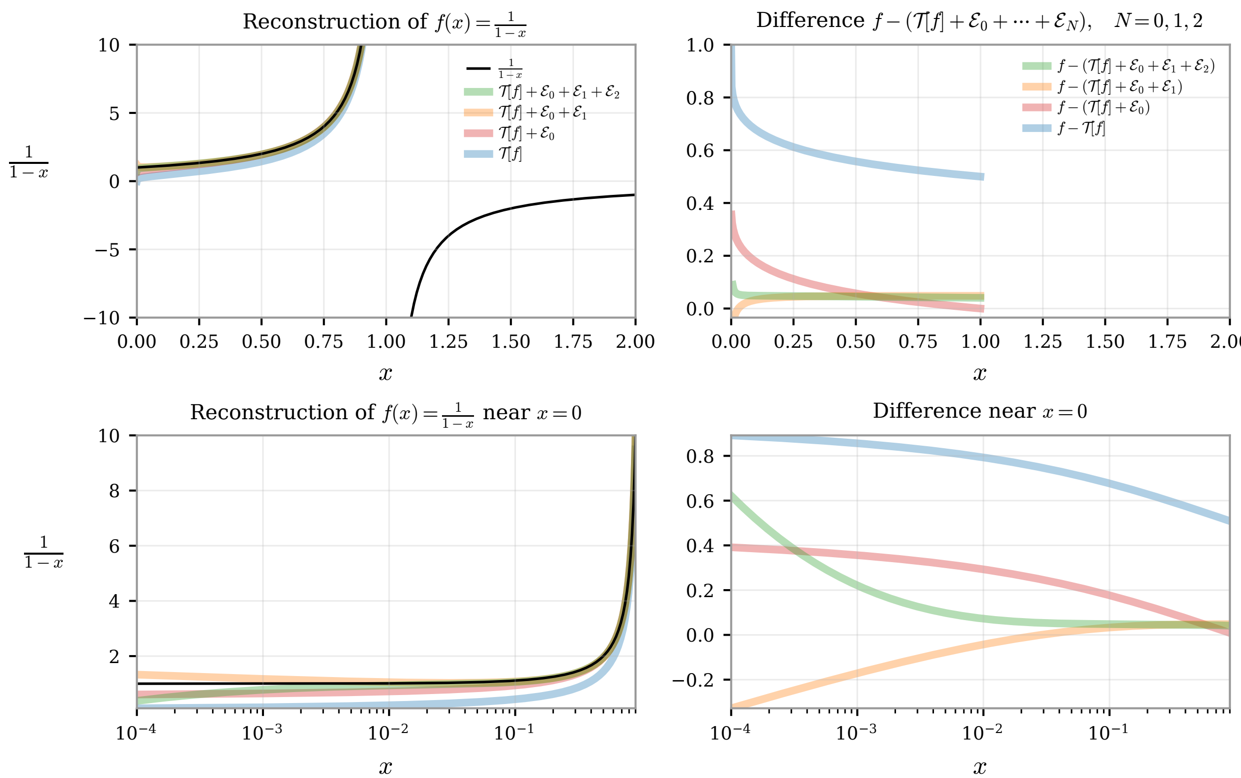

Figure 2: Reconstruction of $f(x) = \frac{1}{1-x}$ using $\mathcal{T}[f]$ with corrections $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$. Top panels show the reconstruction and residuals over a representative interval; bottom panels show the same near the origin.

5.3 Oscillatory entire function: $f(x) = \sin x$

The sine function tests whether the continuous-order operator can track oscillatory behavior and sign changes across multiple zeros. This case highlights how the reconstruction behaves when derivative data alternate in sign and magnitude.

For $\sin x$, the Riemann–Liouville, Caputo, and Grünwald–Letnikov definitions agree that

$$D^r \sin x = \sin\left(x + \frac{\pi r}{2}\right), \qquad \text{so that} \qquad D^r \sin x\big|_{x=0} = \sin\left(\frac{\pi r}{2}\right).$$

The kernel of $\mathcal{T}[f]$ in (3) simplifies to

$$k(r;x) = \sin\left(\frac{\pi r}{2}\right)\frac{x^r}{\Gamma(r+1)}.$$

Evaluating the derivatives in (4) yields the corrections through the second order,

$$\begin{aligned}\mathcal{E}[\sin](x) = {} & \underbrace{-\frac{\pi}{24}}_{\mathcal{E}_0} + \underbrace{\frac{\pi}{1920}\Big(4(\log x + \gamma)^2 - \pi^2\Big)}_{\mathcal{E}_1} \\ & \underbrace{-\frac{\pi}{967680}\Big(80(\log x + \gamma)^4 - 120\pi^2(\log x + \gamma)^2 + 640\zeta(3)(\log x + \gamma) + 9\pi^4\Big)}_{\mathcal{E}_2} \\ & + \cdots .\end{aligned}$$

5.4 Quadratic exponential function: $f(x) = e^{x^2}$

The function e^{x^2} serves as a stress test for rapid growth, with Maclaurin coefficients that increase quickly in magnitude. It reveals how the operator and its corrections respond when higher-order contributions dominate the expansion.

For $f(x) = e^{x^2}$, none of the standard definitions of the fractional derivative (Riemann–Liouville, Caputo, or Grünwald–Letnikov) yields a simple closed-form expression. However, we know the integer-order derivatives at the origin from the Maclaurin expansion:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!},$$

which implies

$$f^{(2n)}(0) = \frac{(2n)!}{n!}, \qquad f^{(2n+1)}(0) = 0.$$

I extended this discrete derivative data to non-integer orders by introducing a smooth analytic continuation in r, chosen to reproduce the integer order derivatives exactly: for $r = 2n$ it reduces to $(2n)!/n!$ for even orders and vanishes for odd integers. The cosine factor enforces parity, while the gamma ratio interpolates the factorial growth. This analytic continuation is not unique and is not derived from a specific fractional differentiation theory. Thus,

$$D^r e^{x^2}\big|_{x=0} = \frac{\Gamma(r+1)}{\Gamma(\frac{r}{2}+1)}\cos^2\left(\frac{\pi r}{2}\right).$$

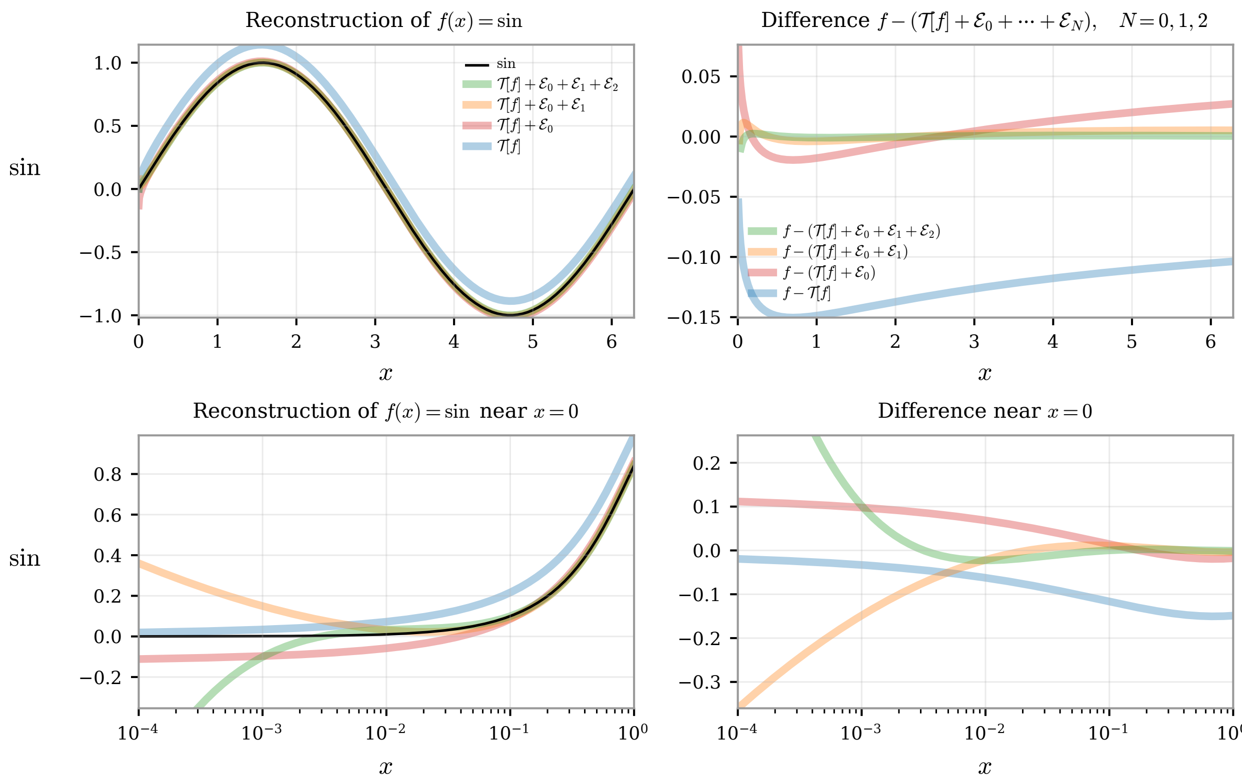

Figure 3: Reconstruction of $f(x) = \sin x$ using $\mathcal{T}[f]$ with corrections $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$. Top panels show the reconstruction and residuals over a representative interval; bottom panels show the same near the origin.

Using this continuation, the kernel of $\mathcal{T}[f]$ given in (3) simplifies to

$$k(r;x) = \frac{\cos^2\left(\frac{\pi r}{2}\right)}{\Gamma\left(\frac{r}{2}+1\right)}\, x^r.$$

The correction operator (4) requires derivatives of $k(r;x)$ with respect to r evaluated at $r = 0$. I did not find a closed-form expression for these derivatives, so I evaluated them numerically as described in the Methodology 4. This yielded a numerically equivalent correction, denoted by $\mathcal{E}^{num}[f](x)$ to distinguish it from closed-form corrections.

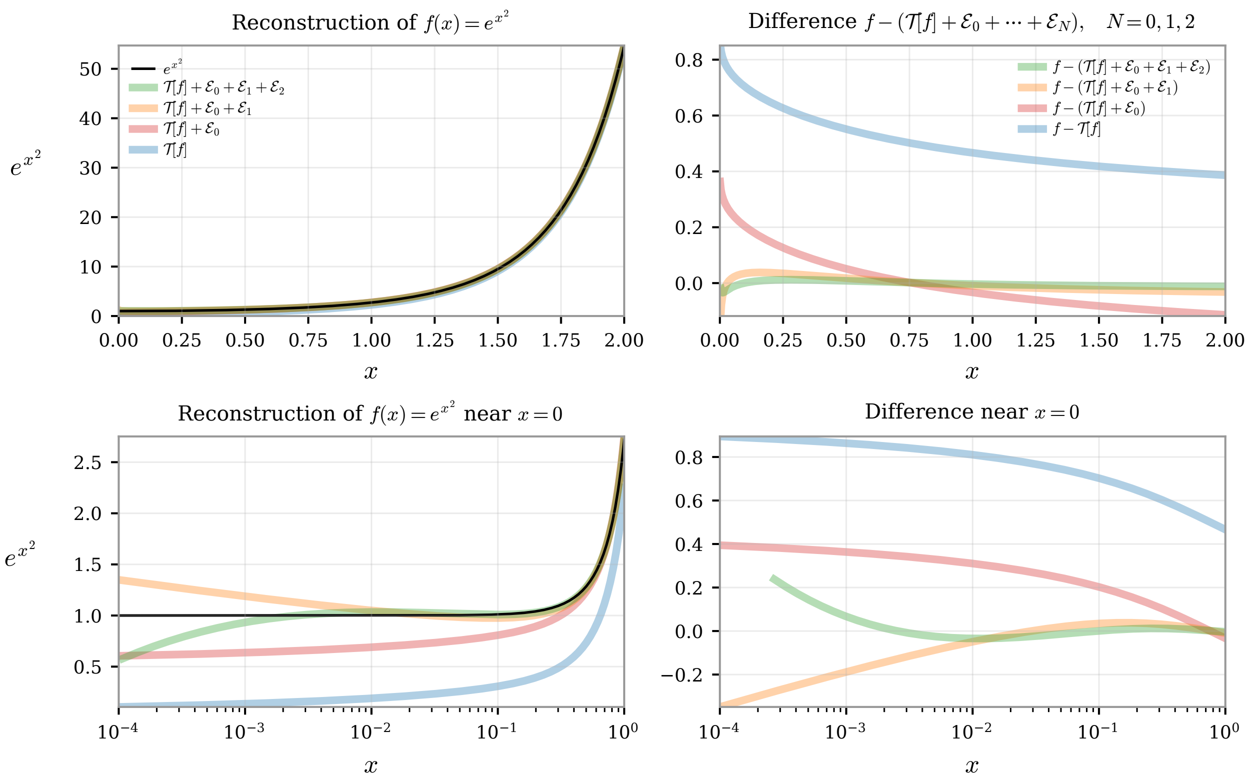

Figure 4: Reconstruction of the quadratic exponential function $f(x) = e^{x^2}$ using $\mathcal{T}[f]$ with corrections $\mathcal{E}_0^{num}$, $\mathcal{E}_1^{num}$ and $\mathcal{E}_2^{num}$. An analytic continuation was used here for derivative data. Top panels show the reconstruction and residuals over a representative interval; bottom panels show the same near the origin.

5.5 Rapidly decaying entire function: $f(x) = e^{-x^2}$

The Gaussian represents the complementary case of rapid decay, despite having a Maclaurin structure closely related to that of e^{x^2}. This function tests whether the operator behaves symmetrically under growth versus decay.

For the Gaussian $f(x) = e^{-x^2}$, the treatment is nearly identical to that of the quadratic exponential e^{x^2}. In particular, the Maclaurin expansion differs only in that it has alternating signs,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!},$$

so that the integer-order derivatives at the origin satisfy

$$f^{(2n)}(0) = (-1)^n \frac{(2n)!}{n!}, \qquad f^{(2n+1)}(0) = 0,$$

with the same parity structure and factorial growth as in the quadratic exponential case.

Accordingly, we adopt the same analytic continuation in the order parameter r, modified only by the alternating sign structure. The continuation reproduces the integer derivatives exactly, vanishing for odd orders and reducing to $(-1)^n(2n)!/n!$ for even orders $r = 2n$. As before, the cosine factor enforces parity and the gamma ratio interpolates the factorial growth; the continuation is not unique and is not tied to a specific fractional differentiation theory. We define

$$D^r e^{-x^2}\big|_{x=0} = (-1)^{r/2} \frac{\Gamma(r+1)}{\Gamma\left(\frac{r}{2}+1\right)} \cos^2\left(\frac{\pi r}{2}\right).$$

With this choice, the kernel of $\mathcal{T}[f]$ in (3) differs from that of the quadratic exponential only by the additional factor $(-1)^{r/2}$. It simplifies to:

$$k(r;x) = \frac{(-1)^{r/2} \cos^2\left(\frac{\pi r}{2}\right)}{\Gamma\left(\frac{r}{2}+1\right)}\, x^r.$$

The correction operator (4) requires derivatives of $k(r;x)$ with respect to r evaluated at $r = 0$. These derivatives are evaluated numerically, yielding $\mathcal{E}^{num}[f](x)$.

5.6 Special function with oscillatory decay: $f(x) = J_0(x)$

I concluded the numerical tests with the order-zero Bessel function of the first kind. $J_0(x)$ combines oscillations with a decaying envelope and a nontrivial combinatorial structure in its Maclaurin coefficients. This case probes the operator's performance on a special function outside the elementary exponential and trigonometric classes.

Because $J_0(0) = 1$, the left-sided Riemann–Liouville fractional derivative diverges when evaluated at the origin. I therefore work with derivative data that remain finite at the origin, consistent with the Caputo interpretation and in agreement with the classical derivatives at integer orders.

From the Maclaurin expansion:

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n}}{(n!)^2},$$

we know that the integer derivatives at $x = 0$ are

$$J_0^{(2n)}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}, \qquad J_0^{(2n+1)}(0) = 0.$$

I extend this discrete derivative data to non-integer order by introducing a smooth analytic continuation in r to reproduce the integer order derivatives exactly. As with the Gaussian case, this continuation is not unique and is not derived from a specific fractional differentiation theory. Thus,

$$D^r J_0(x)\big|_{x=0} = \frac{\Gamma(r+1)}{2^r \Gamma\left(\frac{r}{2}+1\right)^2} \cos\left(\frac{\pi r}{2}\right).$$

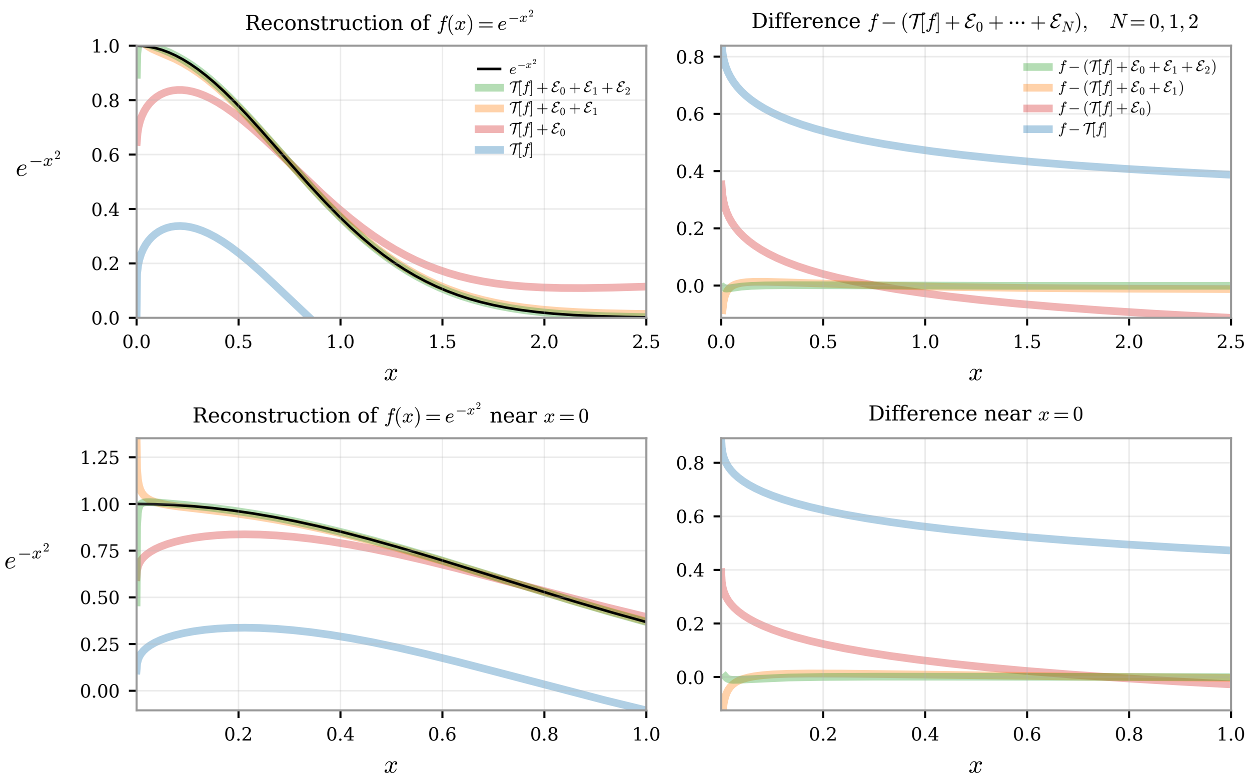

Figure 5: Reconstruction of the Gaussian function $f(x) = e^{-x^2}$ using $\mathcal{T}[f]$ with corrections $\mathcal{E}_0^{num}$, $\mathcal{E}_1^{num}$ and $\mathcal{E}_2^{num}$. An analytic continuation was used here for derivative data. Top panels show the reconstruction and residuals over a representative interval; bottom panels show the same near the origin.

Using this continuation, the kernel of the continuous-order operator in (3) simplifies to

$$k(r;x) = \frac{\cos\left(\frac{\pi r}{2}\right)}{2^r \Gamma\left(\frac{r}{2}+1\right)^2}\, x^r.$$

The correction operator (4) requires derivatives of $k(r;x)$ with respect to r evaluated at $r=0$. These derivatives are evaluated numerically yielding $\mathcal{E}^{num}[f](x)$.

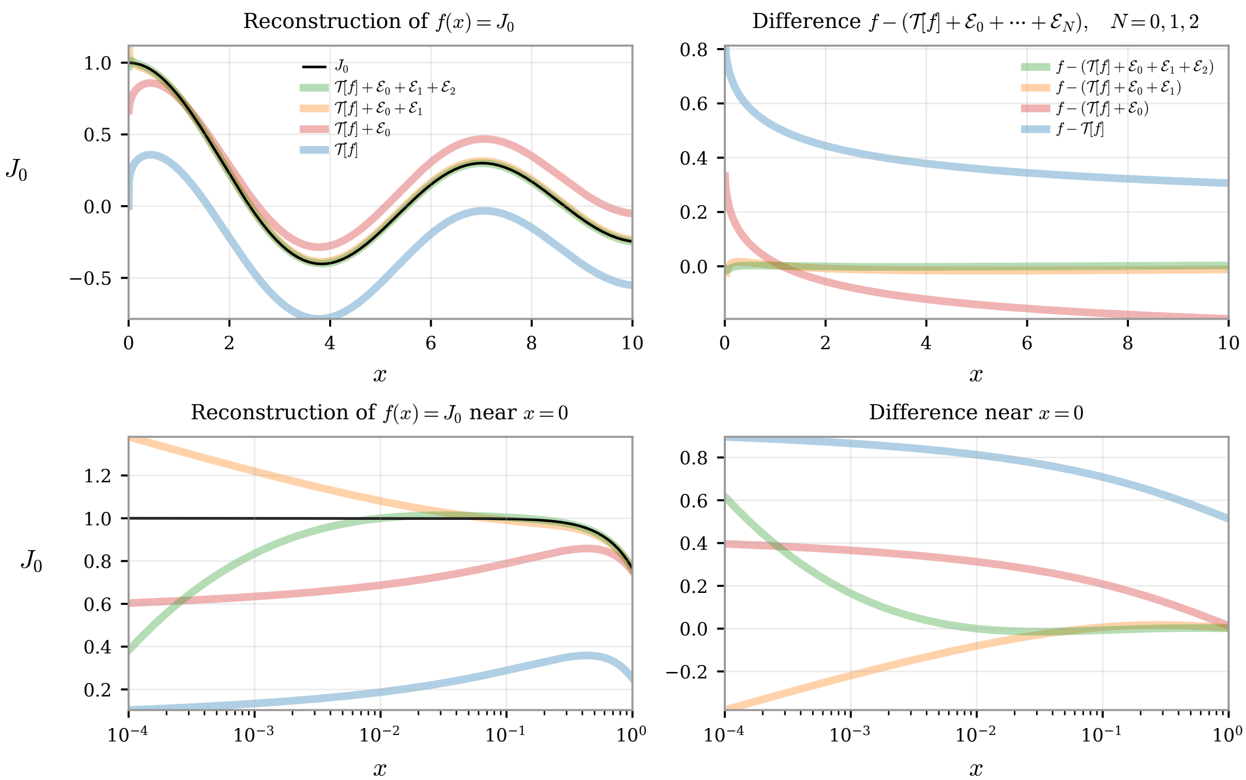

Figure 6: Reconstruction of the Bessel function $f(x) = J_0(x)$ using $\mathcal{T}[f]$ with corrections $\mathcal{E}_0^{num}$, $\mathcal{E}_1^{num}$ and $\mathcal{E}_2^{num}$. An analytic continuation was used here for derivative data. Top panels show the reconstruction and residuals over a representative interval; bottom panels show the same near the origin.

6 The special and degenerate case of monomials: $f(x) = x^k$

Monomials form a special case for the continuous-order integral operator that I analyze symbolically rather than through numerical reconstruction. This case is useful because it shows correspondence with the behavior of the classical Maclaurin series, and it also isolates the role of boundary behavior at the expansion point.

The Maclaurin expansion of x^k is maximally degenerate:

$$x^k = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \qquad f^{(n)}(0) = \begin{cases} k! & n = k, \\ 0 & n \neq k. \end{cases}$$

All information is concentrated at a single derivative order $n = k$. There is no tail of higher-order coefficients and no smooth dependence of derivative data on the order index.

For $x > 0$, the left-sided Riemann–Liouville, Grünwald–Letnikov, and Caputo definitions coincide for monomials in the bulk, differing only in their boundary behavior, yielding

$$D^r x^k = \frac{\Gamma(k+1)}{\Gamma(k-r+1)}\, x^{k-r}.$$

If we attempt to use this expression at the expansion point, the behavior changes qualitatively: for $r > k$, the factor x^{k-r} diverges as $x \to 0^+$, so the resulting boundary value $D^r x^k(0)$ becomes singular. Because the continuous-order operator $\mathcal{T}$ requires fractional derivative data evaluated at the expansion point, this singular behavior obstructs a direct Maclaurin-centered reconstruction over all orders r.

The same issue arises for the left-sided Grünwald–Letnikov definition in this setting. Grünwald–Letnikov is a first-principles difference-limit construction and, for sufficiently regular functions, it coincides with (and is frequently used to approximate) the Riemann–Liouville derivative, so it inherits the same lower-terminal singular behavior at $x = 0$ for $r > k$.

To maintain consistency with the classical Maclaurin series at the expansion point, I therefore impose the Caputo-style boundary behavior, which explicitly annihilates polynomials above their degree. In particular, at $x = 0$, we take

$$D^r x^k(0) = \begin{cases} 0 & r < k, \\ k! & r = k, \\ 0 & r > k \quad \text{(Caputo)}. \end{cases}$$

This is not used here as an assertion that Caputo is “more correct” in general, but rather as a compatibility condition with the classical Maclaurin expansion. Without this truncation, the continuous-order operator $\mathcal{T}$ forces singular boundary data for monomials.

Substituting the Caputo values into the continuous-order integral operator,

$$\mathcal{T}[x^k](x) = \int_0^\infty D^r x^k(0)\, \frac{x^r}{\Gamma(r+1)}\, dr,$$

the operator reduces, under the Caputo truncation, to a single contributing order at $r = k$, reproducing x^k exactly.

6.0.1 Correction terms

As a basic internal consistency check, I checked whether correction terms arise for monomials. Because the Maclaurin expansion of x^k collapses to a single contributing order, there is no sum–integral mismatch, and hence I do not expect a correction term.

For $f(x) = x^k$ with $k \geq 1$, the kernel of the continuous-order integral operator in (3),

$$k(r; x) = D^r x^k(0)\, \frac{x^r}{\Gamma(r+1)},$$

vanishes identically at $r = 0$. Consequently, all derivatives of the kernel with respect to r also vanish at $r = 0$:

$$\frac{\partial^n}{\partial r^n} k(r;x)\bigg|_{r=0} = 0 \qquad \text{for all } n \geq 0.$$

Substituting this into (4) yields no correction terms,

$$\mathcal{E}[x^k](x) = 0 \qquad \text{for all } k \geq 1.$$

Thus, monomials of positive degree do not generate correction terms, confirming the internal consistency of the continuous-order operator framework in this degenerate case.

7 Conclusions and Discussion

We saw the same qualitative pattern across all six analytic functions examined in this study. The continuous-order integral operator $\mathcal{T}[f](x)$ tracks the global structure of $f(x)$ with a nearly constant offset and additional deviation near the origin. The zeroth-order correction term $\mathcal{E}_0[f]$ removes the dominant offset, while the higher-order corrections $\mathcal{E}_1$ and $\mathcal{E}_2$ make successively smaller contributions, further reducing the deviation near the origin. This hierarchy matches the structure of the classical Euler–Maclaurin summation formula, in which higher-order corrections contribute diminishing improvements.

We identified monomials $f(x) = x^k$ as a special case that reinforces the symmetry between the continuous-order construction and the classical Maclaurin series. For these functions, both approaches reduce to a single contributing order. The continuous-order operator reconstructs the function exactly, and all correction terms also vanish, showing internal consistency. This case imposes a constraint on acceptable definitions of fractional differentiation at the expansion point. Definitions that annihilate polynomials above degree k, such as Caputo, preserve exact reconstruction without correction. Definitions that do not, such as Riemann–Liouville, produce singular behavior at the origin..

The continuous-order construction worked with analytic continuations as well, even though they were not derived from a fractional derivative definition. For the quadratic exponential e^{x^2}, the Gaussian e^{-x^2} and the Bessel function $J_0(x)$, we relied on analytic continuations rather than fractional derivatives. These continuations produced stable and consistent reconstructions, suggesting that the continuous-order operator may be a tool for assessing candidate definitions of fractional derivatives.

Taken together, these results indicate that integrating derivative information over continuous order may be a flexible generalization of the classical Maclaurin expansion. Like the Maclaurin series, this continuous-order operator distinguishes naturally between functions with smooth order dependence and singular cases. Because the operator depends explicitly on $D^r f(0)$, it inherits ambiguities from fractional differentiation definitions, but for the analytic functions examined here, commonly used definitions coincide at the expansion point.

We attribute the singular behavior observed for monomials under the Maclaurin-centered operator to the choice of expansion point itself. This observation motivates a Taylor-centered extension,

$$\mathcal{T}_a[f](x) = \int_0^\infty \frac{D^r f(a)(x-a)^r}{\Gamma(r+1)}\, dr, \qquad a \neq 0,$$

which we expect to show smoother order dependence for polynomial functions. We leave a systematic analysis of this extension for future work.

Open questions. Several questions remain unresolved. Under what conditions does the fully corrected operator reproduce $f(x)$ exactly? For which classes of analytic functions does the continuous-order integral converge? Does the operator define a genuinely new form of fractional calculus, or does it fit within an existing transform framework? Can classical complex-analytic methods applied to the Maclaurin series derive the operator from first principles?

References

[1] Marwan Alquran. The amazing fractional Maclaurin series for solving different types of fractional mathematical problems that arise in physics and engineering. *Partial Differential Equations in Applied Mathematics*, 7:100506, 2023.

[2] Michele Caputo. Linear models of dissipation whose Q is almost frequency independent—II. *Geophysical Journal of the Royal Astronomical Society*, 13(5):529–539, 1967.

[3] Bengt Fornberg and Cécile M. Piret. Computation of fractional derivatives of analytic functions. *Journal of Scientific Computing*, 96(3), 2023.

[4] Anastasia Gladkina, Gavriil Shchedrin, U. Al Khawaja, and Lincoln D. Carr. Expansion of fractional derivatives in terms of an integer derivative series: Physical and numerical applications. *arXiv preprint arXiv:1710.06297*, 2017.

[5] NIST Digital Library of Mathematical Functions. Euler–Maclaurin summation formula. `https://dlmf.nist.gov/24`, 2023. Release 1.1.11.

[6] Thomas J. Osler. Taylor's series generalized for fractional derivatives. *SIAM Journal on Mathematical Analysis*, 2(1):37–48, 1971.

[7] Bertram Ross. The development of fractional calculus 1695–1900. *Historia Mathematica*, 4(1):75–89, 1977.

[8] D. Usero. A fractional Taylor series for Caputo fractional derivatives. *Journal of Mathematical Analysis and Applications*, 2010.

[9] Yong Wei, Yongxiang Chen, and Zhi-Zhong Sun. A fractional Taylor series method for solving fractional differential equations. *arXiv preprint arXiv:1901.11134*, 2020. Version 3.