Operator Algebras Generated by Left Invertibles

Derek Desantis

University of Nebraska, Lincoln

GPOTS, May 2018

Background

- A sequence $\{f_n\}$ in a Hilbert space $\mathscr{H}$ is called a **frame** if there exists constants $0 < A < B$ such that for each $x \in \mathscr{H}$,

$$A\|x\|^2 \leq \sum_n |\langle x, f_n \rangle|^2 \leq B\|x\|^2$$

Background

- A sequence $\{f_n\}$ in a Hilbert space $\mathscr{H}$ is called a **frame** if there exists constants $0 < A < B$ such that for each $x \in \mathscr{H}$,

$$A\|x\|^2 \leq \sum_n |\langle x, f_n \rangle|^2 \leq B\|x\|^2$$

- We can associate to each frame $\{f_n\}$ a dual frame $\{g_n\}$ such that

$$x = \sum_n \langle x, g_n \rangle f_n$$

Background

- A sequence $\{f_n\}$ in a Hilbert space $\mathscr{H}$ is called a **frame** if there exists constants $0 < A < B$ such that for each $x \in \mathscr{H}$,

$$A\|x\|^2 \leq \sum_n |\langle x, f_n\rangle|^2 \leq B\|x\|^2$$

- We can associate to each frame $\{f_n\}$ a dual frame $\{g_n\}$ such that

$$x = \sum_n \langle x, g_n\rangle f_n$$

- If $\{f_k\}$ frame for $\mathscr{H}$, and T has closed range, then $\{Tf_k\}$ is a frame for $T\mathscr{H}$.

Definition

Let $T \in \mathscr{B}(\mathscr{H})$ have closed range. There is a unique operator $T^\dagger \in \mathscr{B}(\mathscr{H})$ called the **Moore-Penrose inverse of T** such that

1. $T^\dagger T x = x$ for all $x \in \ker(T)^\perp$
2. $T^\dagger y = 0$ for all $y \in (T\mathscr{H})^\perp$.

Definition

Let $T \in \mathscr{B}(\mathscr{H})$ have closed range. There is a unique operator $T^\dagger \in \mathscr{B}(\mathscr{H})$ called the **Moore-Penrose inverse of T** such that

1. $T^\dagger T x = x$ for all $x \in \ker(T)^\perp$
2. $T^\dagger y = 0$ for all $y \in (T\mathscr{H})^\perp$.

Example

- Let $T \in \mathscr{B}(\ell^2)$ be given by $Te_n = w_n e_n$, $n \geq 0$. If $0 < c < |w_n|$, then T is left invertible and

$$T^\dagger e_n = \begin{cases} 0 & n = 0 \\ w_n^{-1} e_{n-1} & n \geq 1 \end{cases}$$

Definition

Let $T \in \mathscr{B}(\mathscr{H})$ have closed range. There is a unique operator $T^\dagger \in \mathscr{B}(\mathscr{H})$ called the **Moore-Penrose inverse of T** such that

1. $T^\dagger T x = x$ for all $x \in \ker(T)^\perp$
2. $T^\dagger y = 0$ for all $y \in (T\mathscr{H})^\perp$.

Example

- Let $T \in \mathscr{B}(\ell^2)$ be given by $Te_n = w_n e_n$, $n \geq 0$. If $0 < c < |w_n|$, then T is left invertible and

$$T^\dagger e_n = \begin{cases} 0 & n = 0 \\ w_n^{-1} e_{n-1} & n \geq 1 \end{cases}$$

- If T is an isometry, then $T^\dagger = T^*$.

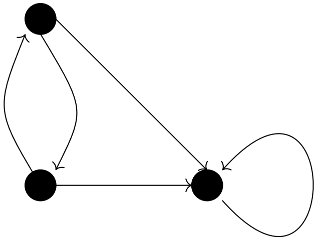
Remark

C*-algebras generated by partial isometries (graph algebras) are well studied.

Remark

C*-algebras generated by partial isometries (graph algebras) are well studied.

$$E = \{r, s, E^0, E^1\}:$$



Program

Choose a closed range operator T_e for each directed edge $e \in E^1$, subject to constraints of directed graph. What is the structure of the operator algebra

$$\overline{\mathrm{Alg}}(T_e, T_e^{\dagger})$$

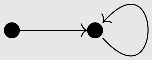
Program

Choose a closed range operator T_e for each directed edge $e \in E^1$, subject to constraints of directed graph. What is the structure of the operator algebra

$$\overline{\mathrm{Alg}}(T_e, T_e^\dagger)$$

Remark

Our focus is on representations afforded by the graph



Notation

Given a left (but not right) invertible $T \in \mathscr{B}(\mathscr{H})$, let

$$\mathfrak{A}_T := \overline{\mathrm{Alg}}(T, T^\dagger)$$

Notation

Given a left (but not right) invertible $T \in \mathscr{B}(\mathscr{H})$, let

$$\mathfrak{A}_T := \overline{\mathrm{Alg}}(T, T^\dagger)$$

Example

If $T = M_z$ on $H^2(\mathbb{T})$, then $\mathfrak{A}_T$ is the classic Toeplitz algebra

$$\mathcal{T} = \{T_f + K : f \in C(\mathbb{T}), K \in \mathscr{K}(H^2(\mathbb{T}))\}$$

Notation

Given a left (but not right) invertible $T \in \mathscr{B}(\mathscr{H})$, let

$$\mathfrak{A}_T := \overline{\mathrm{Alg}}(T, T^\dagger)$$

Example

If $T = M_z$ on $H^2(\mathbb{T})$, then $\mathfrak{A}_T$ is the classic Toeplitz algebra

$$\mathcal{T} = \{T_f + K : f \in C(\mathbb{T}), K \in \mathscr{K}(H^2(\mathbb{T}))\}$$

Remark

General left invertibles have no Wold decomposition:

$$\mathscr{H} \neq \left(\bigcap_n T^n \mathscr{H}\right) \oplus \left(\bigvee_n T^n \ker(T^*)\right)$$

Definition

A left invertible operator T is called **analytic** if

$$\bigcap_n T^n \mathscr{H} = 0$$

Definition

A left invertible operator T is called **analytic** if

$$\bigcap_n T^n \mathscr{H} = 0$$

Theorem (D-)

Let T be an analytic left invertible with ind$(T) = -n$ for some positive integer n. Let $\{x_{i,0}\}_{i=1}^n$ be an orthonormal basis for $\ker(T^*)$. *Then*

Definition

A left invertible operator T is called **analytic** if

$$\bigcap_n T^n \mathscr{H} = 0$$

Theorem (D-)

Let T be an analytic left invertible with ind$(T) = -n$ for some positive integer n. Let $\{x_{i,0}\}_{i=1}^n$ be an orthonormal basis for $\ker(T^*)$. *Then*

$$x_{i,j} := (T^{\dagger^*})^j (x_{i,0})$$

$i = 1, \dots n$, $j = 0, 1, \dots$ *is a Schauder basis for* $\mathscr{H}$

Definition

An operator $R \in \mathscr{B}(\mathscr{H})$ is called **Cowen-Douglas** if there exists open subset $\Omega \subset \sigma(R)$ such that

1. $(R - \lambda)\mathscr{H} = \mathscr{H}$ for all $\lambda \in \Omega$
2. $\dim(\ker(R - \lambda)) = n$ for all $\lambda \in \Omega$.
3. $\bigvee_{\lambda \in \Omega} \ker(R - \lambda) = \mathscr{H}$

We denote this by $R \in B_n(\Omega)$.

Definition

An operator $R \in \mathscr{B}(\mathscr{H})$ is called **Cowen-Douglas** if there exists open subset $\Omega \subset \sigma(R)$ such that

1. $(R-\lambda)\mathscr{H} = \mathscr{H}$ for all $\lambda \in \Omega$
2. $\dim(\ker(R-\lambda)) = n$ for all $\lambda \in \Omega$.
3. $\bigvee_{\lambda\in\Omega} \ker(R-\lambda) = \mathscr{H}$

We denote this by $R \in B_n(\Omega)$.

Theorem (D-)

Let $T \in \mathscr{B}(\mathscr{H})$ be left invertible operator with ind$(T) = -n$, for $n \geq 1$. Then the following are equivalent:

Definition

An operator $R \in \mathscr{B}(\mathscr{H})$ is called **Cowen-Douglas** if there exists open subset $\Omega \subset \sigma(R)$ such that

1. $(R - \lambda)\mathscr{H} = \mathscr{H}$ for all $\lambda \in \Omega$
2. $\dim(\ker(R - \lambda)) = n$ for all $\lambda \in \Omega$.
3. $\bigvee_{\lambda \in \Omega} \ker(R - \lambda) = \mathscr{H}$

We denote this by $R \in B_n(\Omega)$.

Theorem (D-)

Let $T \in \mathscr{B}(\mathscr{H})$ be left invertible operator with $ind(T) = -n$, for $n \geq 1$. Then the following are equivalent:

1. *T is an analytic*

Definition

An operator $R \in \mathscr{B}(\mathscr{H})$ is called **Cowen-Douglas** if there exists open subset $\Omega \subset \sigma(R)$ such that

1. $(R - \lambda)\mathscr{H} = \mathscr{H}$ for all $\lambda \in \Omega$
2. $\dim(\ker(R - \lambda)) = n$ for all $\lambda \in \Omega$.
3. $\bigvee_{\lambda \in \Omega} \ker(R - \lambda) = \mathscr{H}$

We denote this by $R \in B_n(\Omega)$.

Theorem (D-)

Let $T \in \mathscr{B}(\mathscr{H})$ be left invertible operator with $ind(T) = -n$, for $n \geq 1$. Then the following are equivalent:

1. *T is an analytic*
2. *There exists $\epsilon > 0$ such that $T^* \in B_n(\Omega)$ for $\Omega = \{z : |z| < \epsilon\}$*

Definition

An operator $R \in \mathscr{B}(\mathscr{H})$ is called **Cowen-Douglas** if there exists open subset $\Omega \subset \sigma(R)$ such that

1. $(R - \lambda)\mathscr{H} = \mathscr{H}$ for all $\lambda \in \Omega$
2. $\dim(\ker(R - \lambda)) = n$ for all $\lambda \in \Omega$.
3. $\bigvee_{\lambda \in \Omega} \ker(R - \lambda) = \mathscr{H}$

We denote this by $R \in B_n(\Omega)$.

Theorem (D-)

Let $T \in \mathscr{B}(\mathscr{H})$ be left invertible operator with ind$(T) = -n$, for $n \geq 1$. Then the following are equivalent:

1. *T is an analytic*
2. *There exists $\epsilon > 0$ such that $T^* \in B_n(\Omega)$ for $\Omega = \{z : |z| < \epsilon\}$*
3. *There exists $\epsilon > 0$ such that $T^\dagger \in B_n(\Omega)$ for $\Omega = \{z : |z| < \epsilon\}$*

Theorem

If $R \in B_n(\Omega)$, then R is unitarily equivalent to M_z^* on a RKHS of analytic functions $\widehat{\mathscr{H}}$ on $\Omega^* = \{\overline{z} : z \in \Omega\}$.

Theorem

If $R \in B_n(\Omega)$, then R is unitarily equivalent to M_z^* on a RKHS of analytic functions $\widehat{\mathscr{H}}$ on $\Omega^* = \{\overline{z} : z \in \Omega\}$.

Analytic Model

Let T be an analytic left invertible with $\mathrm{ind}(T) = -n$ for some positive integer n, $\{x_{i,j}\}$ the basis associated with $T^{\dagger *}$, and $\Omega = \{z : |z| < \epsilon\}$ as in previous theorem.

Theorem

If $R \in B_n(\Omega)$, then R is unitarily equivalent to M_z^* on a RKHS of analytic functions $\widehat{\mathscr{H}}$ on $\Omega^* = \{\overline{z} : z \in \Omega\}$.

Analytic Model

Let T be an analytic left invertible with $\operatorname{ind}(T) = -n$ for some positive integer n, $\{x_{i,j}\}$ the basis associated with $T^{\dagger *}$, and $\Omega = \{z : |z| < \epsilon\}$ as in previous theorem. Then for each $\lambda \in \Omega$,

$$x_\lambda = \sum_{i=1}^{n} \sum_{j \geq 0} \lambda^j x_{i,j}$$

exists in $\mathscr{H}$.

Theorem

If $R \in B_n(\Omega)$, then R is unitarily equivalent to M_z^* on a RKHS of analytic functions $\widehat{\mathscr{H}}$ on $\Omega^* = \{\overline{z} : z \in \Omega\}$.

Analytic Model

Let T be an analytic left invertible with $\mathrm{ind}(T) = -n$ for some positive integer n, $\{x_{i,j}\}$ the basis associated with $T^{\dagger *}$, and $\Omega = \{z : |z| < \epsilon\}$ as in previous theorem. Then for each $\lambda \in \Omega$,

$$x_\lambda = \sum_{i=1}^{n} \sum_{j \geq 0} \lambda^j x_{i,j}$$

exists in $\mathscr{H}$. Moreover, for each $f \in \mathscr{H}$,

$$\hat{f}(\lambda) = \langle f, x_{\overline{\lambda}} \rangle = \sum_{i=1}^{n} \sum_{j \geq 0} \lambda^j \langle f, x_{i,j} \rangle$$

Assumption

- The Fredholm index: $\mathrm{ind}(T) = -1$
- Analytic: $\bigcap T^n \mathscr{H} = 0$.

Assumption

- The Fredholm index: $\mathrm{ind}(T) = -1$
- Analytic: $\bigcap T^n \mathscr{H} = 0$.

Theorem (D-)

If T is a left invertible, then $\mathfrak{A}_T$ contains the compact operators $\mathscr{K}(\mathscr{H})$. Moreover, $\mathscr{K}(\mathscr{H})$ is a minimal ideal of $\mathfrak{A}_T$.

Assumption

- The Fredholm index: $\mathrm{ind}(T) = -1$
- Analytic: $\bigcap T^n \mathscr{H} = 0$.

Theorem (D-)

If T is a left invertible, then $\mathfrak{A}_T$ contains the compact operators $\mathscr{K}(\mathscr{H})$. Moreover, $\mathscr{K}(\mathscr{H})$ is a minimal ideal of $\mathfrak{A}_T$.

Corollary

Let L be any left inverse of T. Then

$$\mathfrak{A}_T = \overline{Alg}(T, L)$$

Theorem (D-)

Let T_i, $i = 1, 2$ *be left invertible with* $\mathfrak{A}_i := \mathfrak{A}_{T_i}$. *Suppose that* $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ *is a bounded isomorphism. Then* $\phi = Ad_V$ *for some invertible* $V \in \mathscr{B}(\mathscr{H})$. *That is, for all* $A \in \mathfrak{A}_1$,

$$\phi(A) = VAV^{-1}$$

Theorem (D-)

Let T_i, $i = 1, 2$ be left invertible with $\mathfrak{A}_i := \mathfrak{A}_{T_i}$. Suppose that $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a bounded isomorphism. Then $\phi = Ad_V$ for some invertible $V \in \mathscr{B}(\mathscr{H})$. That is, for all $A \in \mathfrak{A}_1$,

$$\phi(A) = VAV^{-1}$$

Remark

To distinguish these algebras by isomorphism classes, we need to classify the similarity orbit:

$$\mathcal{S}(T) := \{VTV^{-1} : V \in \mathscr{B}(\mathscr{H}) \text{ is invertible}\}$$

Remark

- To determine $\mathcal{S}(T)$, suffices to identify $\mathcal{S}(T^*)$.

Remark

- To determine $\mathcal{S}(T)$, suffices to identify $\mathcal{S}(T^*)$.
- Recall that $T^* \in B_1(\Omega)$ for some disc Ω centered at the origin.

Remark

- To determine $\mathcal{S}(T)$, suffices to identify $\mathcal{S}(T^*)$.
- Recall that $T^* \in B_1(\Omega)$ for some disc Ω centered at the origin.
- Determining the similarity orbit of Cowen-Douglas operators is a classic problem.

Remark

- To determine $\mathcal{S}(T)$, suffices to identify $\mathcal{S}(T^*)$.
- Recall that $T^* \in B_1(\Omega)$ for some disc Ω centered at the origin.
- Determining the similarity orbit of Cowen-Douglas operators is a classic problem.

Theorem (Jiang, Wang, Guo, Ji)

Let $A, B \in B_1(\Omega)$. Then A is similar to B if and only if

$$K_0(\{A \oplus B\}') \cong \mathbb{Z}$$

Definition

An operator $S \in \mathscr{B}(\mathscr{H})$ is **subnormal** if it has a normal extension:

$$N = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix} \in \mathscr{B}(\mathscr{K})$$

Definition

An operator $S \in \mathscr{B}(\mathscr{H})$ is **subnormal** if it has a normal extension:

$$N = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix} \in \mathscr{B}(\mathscr{K})$$

The operator N is said to be a **minimal normal extension** if $\mathscr{K}$ has no proper subspace reducing N and containing $\mathscr{H}$.

Definition

An operator $S \in \mathscr{B}(\mathscr{H})$ is **subnormal** if it has a normal extension:

$$N = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix} \in \mathscr{B}(\mathscr{K})$$

The operator N is said to be a **minimal normal extension** if $\mathscr{K}$ has no proper subspace reducing N and containing $\mathscr{H}$.

Definition

Let μ be a scalar-valued spectral measure associated to N, and $f \in L^\infty(\sigma(N), \mu)$.

Definition

An operator $S \in \mathscr{B}(\mathscr{H})$ is **subnormal** if it has a normal extension:

$$N = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix} \in \mathscr{B}(\mathscr{K})$$

The operator N is said to be a **minimal normal extension** if $\mathscr{K}$ has no proper subspace reducing N and containing $\mathscr{H}$.

Definition

Let μ be a scalar-valued spectral measure associated to N, and $f \in L^\infty(\sigma(N), \mu)$. Define $T_f \in \mathscr{B}(\mathscr{H})$ via

$$T_f := P(f(N)) |_{\mathscr{H}}$$

where P is the orthogonal projection of $\mathscr{K}$ onto $\mathscr{H}$.

Theorem (Keough, Olin and Thomson)

If S is an irreducible, subnormal, essentially normal operator, then:

$$C^*(S) = \{T_f + K : f \in C(\sigma(N)), K \in \mathscr{K}(\mathscr{H})\}$$

Moreover, if $\sigma(N) = \sigma_e(S)$, then each element has $A \in C^*(S)$ has a unique representation of the form $T_f + K$.

Theorem (D-)

Let S be an analytic left invertible, $ind(S) = -1$, essentially normal, subnormal operator with $N := mne(S)$ such that $\sigma(N) = \sigma_e(S)$.

Theorem (D-)

Let S be an analytic left invertible, $ind(S) = -1$, essentially normal, subnormal operator with $N := mne(S)$ such that $\sigma(N) = \sigma_e(S)$. Set

$$\mathscr{B} = \overline{Alg}\{z, z^{-1}\}$$

on $\sigma_e(S)$. Then

Theorem (D-)

Let S be an analytic left invertible, $ind(S) = -1$, essentially normal, subnormal operator with $N := mne(S)$ such that $\sigma(N) = \sigma_e(S)$. Set

$$\mathscr{B} = \overline{Alg}\{z, z^{-1}\}$$

on $\sigma_e(S)$. Then

$$\mathfrak{A}_S = \{T_f + K : f \in \mathscr{B}, K \in \mathscr{K}(\mathscr{H})\}$$

Moreover, the representation of each element as $T_f + K$ is unique.

Future Work:

- Are the spectral pictures "general", and do they determine the isomorphism classes?

Future Work:

- Are the spectral pictures “general”, and do they determine the isomorphism classes?
- Does there exist a representing measure for $\partial\Omega$?

Future Work:

- Are the spectral pictures “general”, and do they determine the isomorphism classes?
- Does there exist a representing measure for $\partial\Omega$?
- Determine the isomorphism classes for $\operatorname{ind}(T) < -1$.

Future Work:

- Are the spectral pictures "general", and do they determine the isomorphism classes?
- Does there exist a representing measure for $\partial\Omega$?
- Determine the isomorphism classes for $\mathrm{ind}(T) < -1$.
- Any hope for non-analytic left invertibles?

Future Work:

- Are the spectral pictures “general”, and do they determine the isomorphism classes?
- Does there exist a representing measure for $\partial\Omega$?
- Determine the isomorphism classes for $\mathrm{ind}(T) < -1$.
- Any hope for non-analytic left invertibles?
- Investigate other algebras that arise from graphs - e.g. “Cuntz algebra”.