```python
def main():
    Print_Function()
    (x, y, z) = xyz = symbols('x,y,z',real=True)
    (o3d, ex, ey, ez) = Ga.build('e_x e_y e_z', g=[1, 1, 1], coords=xyz)
    grad = o3d.grad
    (u, v) = uv = symbols('u,v',real=True)
    (g2d, eu, ev) = Ga.build('e_u e_v', coords=uv)
    grad_uv = g2d.grad
    v_xyz = o3d.mv('v','vector')
    A_xyz = o3d.mv('A','vector',f=True)
    A_uv = g2d.mv('A','vector',f=True)
    print '#3d orthogonal ($A$ is vector function)'
    print 'A =', A_xyz
    print '%A^{2} =', A_xyz * A_xyz
    print 'grad|A =', grad | A_xyz
    print 'grad*A =', grad * A_xyz
    print 'v|(grad*A) =',v_xyz|(grad*A_xyz)
    print '#2d general ($A$ is vector function)'
    print 'A =', A_uv
    print '%A^{2} =', A_uv * A_uv
    print 'grad|A =', grad_uv | A_uv
    print 'grad*A =', grad_uv * A_uv
    A = o3d.lt('A')
    print '#3d orthogonal ($A,\\;B$ are linear transformations)'
    print 'A =', A
    print '\\f{\\det}{A} =', A.det()
    print '\\overline{A} =', A.adj()
    print '\\f{\\Tr}{A} =', A.tr()
    print '\\f{A}{e_x^e_y} =', A(ex^ey)
    print '\\f{A}{e_x}^\\f{A}{e_y} =', A(ex)^A(ey)
    B = o3d.lt('B')
    print 'A + B =', A + B
    print 'AB =', A * B
    print 'A - B =', A - B
    print '#2d general ($A,\\;B$ are linear transformations)'
    A2d = g2d.lt('A')
    print 'A =', A2d
    print '\\f{\\det}{A} =', A2d.det()
    print '\\overline{A} =', A2d.adj()
    print '\\f{\\Tr}{A} =', A2d.tr()
    print '\\f{A}{e_u^e_v} =', A2d(eu^ev)
    print '\\f{A}{e_u}^\\f{A}{e_v} =', A2d(eu)^A2d(ev)
    B2d = g2d.lt('B')
    print 'B =', B2d
    print 'A + B =', A2d + B2d
    print 'AB =', A2d * B2d
    print 'A - B =', A2d - B2d
    a = g2d.mv('a','vector')
    b = g2d.mv('b','vector')
    print r'a|\f{\overline{A}}{b}-b|\f{\underline{A}}{a} =',((a|A2d.adj()(b))-(b|A2d(a))).simplify()
    m4d = Ga('e_t e_x e_y e_z', g=[1, -1, -1, -1],coords=symbols('t,x,y,z',real=True))
    T = m4d.lt('T')
    print 'g =', m4d.g
    print r'\underline{T} =',T
    print r'\overline{T} =',T.adj()
    print r'\f{\det}{\underline{T}} =',T.det()
    print r'\f{\mbox{tr}}{\underline{T}} =',T.tr()
    a = m4d.mv('a','vector')
    b = m4d.mv('b','vector')
```

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    print r'a|\f{\overline{T}}{b}-b|\f{\underline{T}}{a} =',((a|T.adj())(b))-(b|T(a))).simplify()
    coords = (r, th, phi) = symbols('r,theta,phi', real=True)
    (sp3d, er, eth, ephi) = Ga.build('e_r e_th e_ph', g=[1, r**2, r**2*sin(th)**2], coords=coords)
    grad = sp3d.grad
    sm_coords = (u, v) = symbols('u,v', real=True)
    smap = [1, u, v]  # Coordinate map for sphere of r = 1
    sph2d = sp3d.sm(smap,sm_coords,norm=True)
    (eu, ev) = sph2d.mv()
    grad_uv = sph2d.grad
    F = sph2d.mv('F','vector',f=True)
    f = sph2d.mv('f','scalar',f=True)
    print 'f =',f
    print 'grad*f =',grad_uv * f
    print 'F =',F
    print 'grad*F =',grad_uv * F
    tp = (th,phi) = symbols('theta,phi',real=True)
    smap = [sin(th)*cos(phi),sin(th)*sin(phi),cos(th)]
    sph2dr = o3d.sm(smap,tp,norm=True)
    (eth, ephi) = sph2dr.mv()
    grad_tp = sph2dr.grad
    F = sph2dr.mv('F','vector',f=True)
    f = sph2dr.mv('f','scalar',f=True)
    print 'f =',f
    print 'grad*f =',grad_tp * f
    print 'F =',F
    print 'grad*F =',grad_tp * F
    return
```

Code Output: 3d orthogonal (A is vector function)

$$A = A^x \boldsymbol{e}_x + A^y \boldsymbol{e}_y + A^z \boldsymbol{e}_z$$

$$A^2 = (A^x)^2 + (A^y)^2 + (A^z)^2$$

$$\boldsymbol{\nabla} \cdot A = \partial_x A^x + \partial_y A^y + \partial_z A^z$$

$$\boldsymbol{\nabla} A = (\partial_x A^x + \partial_y A^y + \partial_z A^z) + (-\partial_y A^x + \partial_x A^y)\boldsymbol{e}_x \wedge \boldsymbol{e}_y + (-\partial_z A^x + \partial_x A^z)\boldsymbol{e}_x \wedge \boldsymbol{e}_z + (-\partial_z A^y + \partial_y A^z)\boldsymbol{e}_y \wedge \boldsymbol{e}_z$$

$$v \cdot (\boldsymbol{\nabla} A) = (v^y \partial_y A^x - v^y \partial_x A^y + v^z \partial_z A^x - v^z \partial_x A^z)\boldsymbol{e}_x + (-v^x \partial_y A^x + v^x \partial_x A^y + v^z \partial_z A^y - v^z \partial_y A^z)\boldsymbol{e}_y + (-v^x \partial_z A^x + v^x \partial_x A^z - v^y \partial_z A^y + v^y \partial_y A^z)\boldsymbol{e}_z$$

2d general (A is vector function)

$$A = A^u \boldsymbol{e}_u + A^v \boldsymbol{e}_v$$

$$A^2 = (e_u \cdot e_u)(A^u)^2 + 2(e_u \cdot e_v) A^u A^v + (e_v \cdot e_v)(A^v)^2$$

$$\boldsymbol{\nabla} \cdot A = \partial_u A^u + \partial_v A^v$$

$$\boldsymbol{\nabla} A = (\partial_u A^u + \partial_v A^v) + \frac{-(e_u \cdot e_u)\partial_v A^u + (e_u \cdot e_v)\partial_u A^u - (e_u \cdot e_v)\partial_v A^v + (e_v \cdot e_v)\partial_u A^v}{(e_u \cdot e_u)(e_v \cdot e_v) - (e_u \cdot e_v)^2}\boldsymbol{e}_u \wedge \boldsymbol{e}_v$$

3d orthogonal (A, B are linear transformations)

$$A = \left\{ \begin{array}{ll} L(\boldsymbol{e}_x) = & A_{xx}\boldsymbol{e}_x + A_{yx}\boldsymbol{e}_y + A_{zx}\boldsymbol{e}_z \\ L(\boldsymbol{e}_y) = & A_{xy}\boldsymbol{e}_x + A_{yy}\boldsymbol{e}_y + A_{zy}\boldsymbol{e}_z \\ L(\boldsymbol{e}_z) = & A_{xz}\boldsymbol{e}_x + A_{yz}\boldsymbol{e}_y + A_{zz}\boldsymbol{e}_z \end{array} \right\}$$

$$\det(A) = A_{xz}(A_{yx}A_{zy} - A_{yy}A_{zx}) - A_{yz}(A_{xx}A_{zy} - A_{xy}A_{zx}) + A_{zz}(A_{xx}A_{yy} - A_{xy}A_{yx})$$

$$\overline{A} = \left\{ \begin{array}{ll} L(\boldsymbol{e}_x) = & A_{xx}\boldsymbol{e}_x + A_{xy}\boldsymbol{e}_y + A_{xz}\boldsymbol{e}_z \\ L(\boldsymbol{e}_y) = & A_{yx}\boldsymbol{e}_x + A_{yy}\boldsymbol{e}_y + A_{yz}\boldsymbol{e}_z \\ L(\boldsymbol{e}_z) = & A_{zx}\boldsymbol{e}_x + A_{zy}\boldsymbol{e}_y + A_{zz}\boldsymbol{e}_z \end{array} \right\}$$

$$\text{Tr}(A) = A_{xx} + A_{yy} + A_{zz}$$

$$A(e_x \wedge e_y) = (A_{xx}A_{yy} - A_{xy}A_{yx})\boldsymbol{e}_x \wedge \boldsymbol{e}_y + (A_{xx}A_{zy} - A_{xy}A_{zx})\boldsymbol{e}_x \wedge \boldsymbol{e}_z + (A_{yx}A_{zy} - A_{yy}A_{zx})\boldsymbol{e}_y \wedge \boldsymbol{e}_z$$

$$A(e_x) \wedge A(e_y) = (A_{xx}A_{yy} - A_{xy}A_{yx})\, \boldsymbol{e}_x \wedge \boldsymbol{e}_y + (A_{xx}A_{zy} - A_{xy}A_{zx})\, \boldsymbol{e}_x \wedge \boldsymbol{e}_z + (A_{yx}A_{zy} - A_{yy}A_{zx})\, \boldsymbol{e}_y \wedge \boldsymbol{e}_z$$

$$A + B = \left\{ \begin{array}{ll} L(\boldsymbol{e}_x) = & (A_{xx} + B_{xx})\, \boldsymbol{e}_x + (A_{yx} + B_{yx})\, \boldsymbol{e}_y + (A_{zx} + B_{zx})\, \boldsymbol{e}_z \\ L(\boldsymbol{e}_y) = & (A_{xy} + B_{xy})\, \boldsymbol{e}_x + (A_{yy} + B_{yy})\, \boldsymbol{e}_y + (A_{zy} + B_{zy})\, \boldsymbol{e}_z \\ L(\boldsymbol{e}_z) = & (A_{xz} + B_{xz})\, \boldsymbol{e}_x + (A_{yz} + B_{yz})\, \boldsymbol{e}_y + (A_{zz} + B_{zz})\, \boldsymbol{e}_z \end{array} \right\}$$

$$AB = \left\{ \begin{array}{ll} L(\boldsymbol{e}_x) = & (A_{xx}B_{xx} + A_{xy}B_{yx} + A_{xz}B_{zx})\, \boldsymbol{e}_x + (A_{yx}B_{xx} + A_{yy}B_{yx} + A_{yz}B_{zx})\, \boldsymbol{e}_y + (A_{zx}B_{xx} + A_{zy}B_{yx} + A_{zz}B_{zx})\, \boldsymbol{e}_z \\ L(\boldsymbol{e}_y) = & (A_{xx}B_{xy} + A_{xy}B_{yy} + A_{xz}B_{zy})\, \boldsymbol{e}_x + (A_{yx}B_{xy} + A_{yy}B_{yy} + A_{yz}B_{zy})\, \boldsymbol{e}_y + (A_{zx}B_{xy} + A_{zy}B_{yy} + A_{zz}B_{zy})\, \boldsymbol{e}_z \\ L(\boldsymbol{e}_z) = & (A_{xx}B_{xz} + A_{xy}B_{yz} + A_{xz}B_{zz})\, \boldsymbol{e}_x + (A_{yx}B_{xz} + A_{yy}B_{yz} + A_{yz}B_{zz})\, \boldsymbol{e}_y + (A_{zx}B_{xz} + A_{zy}B_{yz} + A_{zz}B_{zz})\, \boldsymbol{e}_z \end{array} \right\}$$

$$A - B = \left\{ \begin{array}{ll} L(\boldsymbol{e}_x) = & (A_{xx} - B_{xx})\, \boldsymbol{e}_x + (A_{yx} - B_{yx})\, \boldsymbol{e}_y + (A_{zx} - B_{zx})\, \boldsymbol{e}_z \\ L(\boldsymbol{e}_y) = & (A_{xy} - B_{xy})\, \boldsymbol{e}_x + (A_{yy} - B_{yy})\, \boldsymbol{e}_y + (A_{zy} - B_{zy})\, \boldsymbol{e}_z \\ L(\boldsymbol{e}_z) = & (A_{xz} - B_{xz})\, \boldsymbol{e}_x + (A_{yz} - B_{yz})\, \boldsymbol{e}_y + (A_{zz} - B_{zz})\, \boldsymbol{e}_z \end{array} \right\}$$

2d general (A, B are linear transformations)

$$A = \left\{ \begin{array}{ll} L(\boldsymbol{e}_u) = & A_{uu}\boldsymbol{e}_u + A_{vu}\boldsymbol{e}_v \\ L(\boldsymbol{e}_v) = & A_{uv}\boldsymbol{e}_u + A_{vv}\boldsymbol{e}_v \end{array} \right\}$$

$$\det(A) = \frac{A_{uu}A_{vv}\boldsymbol{e}_u \wedge \boldsymbol{e}_v - A_{uv}A_{vu}\boldsymbol{e}_u \wedge \boldsymbol{e}_v}{\sqrt{(e_u \cdot e_u)(e_v \cdot e_v) - (e_u \cdot e_v)^2}}$$

$$\overline{A} = \left\{ \begin{array}{ll} L(\boldsymbol{e}_u) = & \frac{1}{(e_u \cdot e_u)(e_v \cdot e_v) - (e_u \cdot e_v)^{\wedge 2}}\left(-(e_u \cdot e_u)(e_u \cdot e_v)A_{uv} + (e_u \cdot e_u)(e_v \cdot e_v)A_{uu} - (e_u \cdot e_v) \wedge 2A_{vv} + (e_u \cdot e_v)(e_v \cdot e_v)A_{vu}\right)\boldsymbol{e}_u + \frac{1}{(e_u \cdot e_u)(e_v \cdot e_v) - (e_u \cdot e_v)^{\wedge 2}}\left((e_u \cdot e_u) \wedge 2A_{uv} - (e_u \cdot e_u)(e_u \cdot e_v)A_{uu} + (e_u \cdot e_u)(e_v \cdot e_v)A_{vv} - (\\ L(\boldsymbol{e}_v) = & \frac{1}{(e_u \cdot e_u)(e_v \cdot e_v) - (e_u \cdot e_v)^2}\left(-(e_u \cdot e_v)^2 A_{uv} + (e_u \cdot e_v)(e_v \cdot e_v)A_{uu} - (e_u \cdot e_v)(e_v \cdot e_v)A_{vv} + (e_v \cdot e_v)^2 A_{vu}\right)\boldsymbol{e}_u + \frac{1}{(e_u \cdot e_u)(e_v \cdot e_v) - (e_u \cdot e_v)^2}\left((e_u \cdot e_u)(e_u \cdot e_v)A_{uv} + (e_u \cdot e_u)(e_v \cdot e_v)A_{vv} - (e_u \cdot e_v)^2 A_{uu} - (e_u \cdot e_v) \end{array} \right.$$

$$\text{Tr}(A) = -\frac{(e_u \cdot e_u)(e_v \cdot e_v)A_{uu}}{-(e_u \cdot e_u)(e_v \cdot e_v) + (e_u \cdot e_v)^2} - \frac{(e_u \cdot e_u)(e_v \cdot e_v)A_{vv}}{-(e_u \cdot e_u)(e_v \cdot e_v) + (e_u \cdot e_v)^2} + \frac{(e_u \cdot e_v)^2 A_{uu}}{-(e_u \cdot e_u)(e_v \cdot e_v) + (e_u \cdot e_v)^2} + \frac{(e_u \cdot e_v)^2 A_{vv}}{-(e_u \cdot e_u)(e_v \cdot e_v) + (e_u \cdot e_v)^2}$$

$$A(e_u \wedge e_v) = (A_{uu}A_{vv} - A_{uv}A_{vu})\, \boldsymbol{e}_u \wedge \boldsymbol{e}_v$$

$$A(e_u) \wedge A(e_v) = (A_{uu}A_{vv} - A_{uv}A_{vu})\, \boldsymbol{e}_u \wedge \boldsymbol{e}_v$$

$$B = \left\{ \begin{array}{ll} L(\boldsymbol{e}_u) = & B_{uu}\boldsymbol{e}_u + B_{vu}\boldsymbol{e}_v \\ L(\boldsymbol{e}_v) = & B_{uv}\boldsymbol{e}_u + B_{vv}\boldsymbol{e}_v \end{array} \right\}$$

$$A + B = \left\{ \begin{array}{ll} L(\boldsymbol{e}_u) = & (A_{uu} + B_{uu})\boldsymbol{e}_u + (A_{vu} + B_{vu})\boldsymbol{e}_v \\ L(\boldsymbol{e}_v) = & (A_{uv} + B_{uv})\boldsymbol{e}_u + (A_{vv} + B_{vv})\boldsymbol{e}_v \end{array} \right\}$$

$$AB = \left\{ \begin{array}{ll} L(\boldsymbol{e}_u) = & (A_{uu}B_{uu} + A_{uv}B_{vu})\boldsymbol{e}_u + (A_{vu}B_{uu} + A_{vv}B_{vu})\boldsymbol{e}_v \\ L(\boldsymbol{e}_v) = & (A_{uu}B_{uv} + A_{uv}B_{vv})\boldsymbol{e}_u + (A_{vu}B_{uv} + A_{vv}B_{vv})\boldsymbol{e}_v \end{array} \right\}$$

$$A - B = \left\{ \begin{array}{ll} L(\boldsymbol{e}_u) = & (A_{uu} - B_{uu})\boldsymbol{e}_u + (A_{vu} - B_{vu})\boldsymbol{e}_v \\ L(\boldsymbol{e}_v) = & (A_{uv} - B_{uv})\boldsymbol{e}_u + (A_{vv} - B_{vv})\boldsymbol{e}_v \end{array} \right\}$$

$$a \cdot \overline{A}(b) - b \cdot \underline{A}(a) = 0$$

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\underline{T} = \left\{ \begin{array}{ll} L(\boldsymbol{e}_t) = & T_{tt}\boldsymbol{e}_t + T_{xt}\boldsymbol{e}_x + T_{yt}\boldsymbol{e}_y + T_{zt}\boldsymbol{e}_z \\ L(\boldsymbol{e}_x) = & T_{tx}\boldsymbol{e}_t + T_{xx}\boldsymbol{e}_x + T_{yx}\boldsymbol{e}_y + T_{zx}\boldsymbol{e}_z \\ L(\boldsymbol{e}_y) = & T_{ty}\boldsymbol{e}_t + T_{xy}\boldsymbol{e}_x + T_{yy}\boldsymbol{e}_y + T_{zy}\boldsymbol{e}_z \\ L(\boldsymbol{e}_z) = & T_{tz}\boldsymbol{e}_t + T_{xz}\boldsymbol{e}_x + T_{yz}\boldsymbol{e}_y + T_{zz}\boldsymbol{e}_z \end{array} \right\}$$

$$\overline{T} = \left\{ \begin{array}{ll} L(\boldsymbol{e}_t) = & T_{tt}\boldsymbol{e}_t - T_{tx}\boldsymbol{e}_x - T_{ty}\boldsymbol{e}_y - T_{tz}\boldsymbol{e}_z \\ L(\boldsymbol{e}_x) = & -T_{xt}\boldsymbol{e}_t + T_{xx}\boldsymbol{e}_x + T_{xy}\boldsymbol{e}_y + T_{xz}\boldsymbol{e}_z \\ L(\boldsymbol{e}_y) = & -T_{yt}\boldsymbol{e}_t + T_{yx}\boldsymbol{e}_x + T_{yy}\boldsymbol{e}_y + T_{yz}\boldsymbol{e}_z \\ L(\boldsymbol{e}_z) = & -T_{zt}\boldsymbol{e}_t + T_{zx}\boldsymbol{e}_x + T_{zy}\boldsymbol{e}_y + T_{zz}\boldsymbol{e}_z \end{array} \right\}$$

$$\det(\underline{T}) = -T_{tz}\left(T_{xt}T_{yx}T_{zy} - T_{xt}T_{yy}T_{zx} - T_{xx}T_{yt}T_{zy} + T_{xx}T_{yy}T_{zt} + T_{xy}T_{yt}T_{zx} - T_{xy}T_{yx}T_{zt}\right) + T_{xz}\left(T_{tt}T_{yx}T_{zy} - T_{tt}T_{yy}T_{zx} - T_{tx}T_{yt}T_{zy} + T_{tx}T_{yy}T_{zt} + T_{ty}T_{yt}T_{zx} - T_{ty}T_{yx}T_{zt}\right) - T_{yz}\left(T_{tt}T_{xx}T_{zy} - T_{tt}T_{xy}T_{zx} - T_{tx}T_{xt}T_{zy} + T_{tx}T_{xy}T_{z}\right.$$

$$\text{tr}(\underline{T}) = T_{tt} + T_{xx} + T_{yy} + T_{zz}$$

$$a \cdot \overline{T}(b) - b \cdot \underline{T}(a) = 0$$

$$f = f$$

$$\boldsymbol{\nabla} f = \partial_u f \boldsymbol{e}_u + \frac{\partial_v f}{\sin{(u)}} \boldsymbol{e}_v$$

$$F = F^u \boldsymbol{e}_u + F^v \boldsymbol{e}_v$$

$$\boldsymbol{\nabla} F = \left(\frac{F^u}{\tan{(u)}} + \partial_u F^u + \frac{\partial_v F^v}{\sin{(u)}} \right) + \left(\frac{F^v}{\tan{(u)}} + \partial_u F^v - \frac{\partial_v F^u}{\sin{(u)}} \right) \boldsymbol{e}_u \wedge \boldsymbol{e}_v$$

$$f = f$$

$$\boldsymbol{\nabla} f = \partial_\theta f \boldsymbol{e}_\theta + \frac{\partial_\phi f}{\sin{(\theta)}} \boldsymbol{e}_\phi$$

$$F = F^\theta \boldsymbol{e}_\theta + F^\phi \boldsymbol{e}_\phi$$

$$\boldsymbol{\nabla} F = \left(\frac{F^\theta}{\tan{(\theta)}} + \partial_\theta F^\theta + \frac{\partial_\phi F^\phi}{\sin{(\theta)}} \right) + \left(\frac{F^\phi}{\tan{(\theta)}} + \partial_\theta F^\phi - \frac{\partial_\phi F^\theta}{\sin{(\theta)}} \right) \boldsymbol{e}_\theta \wedge \boldsymbol{e}_\phi$$