Calculation of Fourier Series Equations

The equations used to calculate the Fourier Series

The equation of a Fourier series is as follows:

$$s(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{nx\,\pi}{P}\right)+b_n\sin\left(\frac{nx\,\pi}{P}\right)\right) \qquad [1]$$

where:

a_0, a_n and b_n are coefficients that influence the shape of the curve we are looking for

n = number of cycles of the n^{th} harmonic

P = half of the period of the function

The coefficients

The coefficients of the Fourier Series, to be plugged into Equation [1] above, are:

$$a_n=\frac{1}{P}\int_{-P}^{P} s(x)\cos\left(\frac{\pi\,n\,x}{P}\right)dx \qquad [2]$$

$$b_n=\frac{1}{P}\int_{-P}^{P} s(x)\sin\left(\frac{\pi\,n\,x}{P}\right)dx \qquad [3]$$

Any continuous function can be used as *s(x)*, but for our purposes, we are only going to use a straight line function, because it is relatively straightforward to integrate.

$$s(x)=mx+c \qquad [4]$$

We will plug Equation [4] into Equations [2] and [3], as follows:

$$a_n=\frac{1}{P}\int_{-P}^{P} (mx+c)\cos\left(\frac{\pi\,n\,x}{P}\right)dx \qquad [5]$$

$$b_n=\frac{1}{P}\int_{-P}^{P} (mx+c)\sin\left(\frac{\pi\,n\,x}{P}\right)dx \qquad [6]$$

We can split the integrals in Equations [5] and [6], to make them more manageable, putting the constants to the left of the integral sign, like this:

$$a_n = \frac{m}{P} \int_{-P}^{P} x \cos\left(\frac{\pi n x}{P}\right) dx + \frac{c}{P} \int_{-P}^{P} \cos\left(\frac{\pi n x}{P}\right) dx \qquad [7]$$

$$b_n = \frac{m}{P} \int_{-P}^{P} x \sin\left(\frac{\pi n x}{P}\right) dx + \frac{c}{P} \int_{-P}^{P} \sin\left(\frac{\pi n x}{P}\right) dx \qquad [8]$$

We use integration by parts to integrate the first part of each of Equations [7] and [8], so that:

$$a \int_{-P}^{P} x \cos(nx)\, dx = a \left[\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_{-P}^{P} \qquad [9]$$

$$a \int_{-P}^{P} x \sin(nx)\, dx = a \left[\frac{\sin(nx)}{n^2} - \frac{x \cos(nx)}{n} \right]_{-P}^{P} \qquad [10]$$

The a-coefficient

Let’s start calculating the a-coefficient.

We plug in the Equation [9] into Equation [7], to get the following equation, as follows:

$$a_n = \frac{m}{P} \left[\frac{x \sin\left(\frac{\pi nx}{P}\right)}{\left(\frac{\pi n}{P}\right)} + \frac{\cos\left(\frac{\pi nx}{P}\right)}{\left(\frac{\pi n}{P}\right)^2} \right] + \frac{c}{P} \sin\left(\frac{\pi nx}{P}\right)\left(\frac{P}{\pi n}\right) \qquad [11]$$

We now simplify the above equation, by cancelling variables where we can:

$$a_n = \frac{m}{P} \left[\frac{P x \sin\left(\frac{\pi nx}{P}\right)}{\pi n} + \frac{P^2 \cos\left(\frac{\pi nx}{P}\right)}{\pi^2 n^2} \right] + \frac{c}{P}\left(\frac{P}{\pi n}\right) \sin\left(\frac{\pi nx}{P}\right)$$

Here is the final equation for the a-coefficient:

$$a_n = m\left[\frac{x\sin\left(\frac{\pi\, nx}{P}\right)}{\pi\, n} + \frac{P\cos\left(\frac{\pi\, nx}{P}\right)}{\pi^2 n^2}\right] + \frac{c}{\pi\, n}\sin\left(\frac{\pi\, nx}{P}\right) \qquad [12]$$

The b-coefficient

In the same way, we calculate the b-coefficient, by plugging Equation [10] into Equation [8]

$$b_n = \frac{m}{P}\left[\frac{\sin\left(\frac{\pi\, nx}{P}\right)}{\left(\frac{\pi\, n}{P}\right)^2} - \frac{x\cos\left(\frac{\pi\, nx}{P}\right)}{\left(\frac{\pi\, n}{P}\right)}\right] - \frac{c}{P}\cos\left(\frac{\pi\, nx}{P}\right)\left(\frac{P}{\pi\, n}\right) \qquad [13]$$

Cancelling out variables will give a final equation for the b-coefficient:

$$b_n = m\left[\frac{P\sin\left(\frac{\pi\, nx}{P}\right)}{\pi^2 n^2} - \frac{x\cos\left(\frac{\pi\, nx}{P}\right)}{\pi\, n}\right] - \frac{c}{\pi\, n}\cos\left(\frac{\pi\, nx}{P}\right) \qquad [14]$$