Numerical solution of the convection diffusion equation

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Introduction

The convection diffusion equation is solved using the finite volume method. Central differencing is applied to the diffusion terms and the Min-Mod scheme is used to determine the convective flux at the node, i.e.: control volume, boundaries.

Model equations

The convection diffusion equation is:

$$\Gamma \frac{d^2\phi}{dx^2} - \rho U \frac{d\phi}{dx} = 0 \tag{1}$$

Where Γ is the diffusion coefficient, ρ the density, U the fluid velocity, x the axial coordinate and ϕ the transport property. Dirichlet boundary conditions apply to (1).

Discretization

The discrete form of equation (1) for 'central' nodes is:

$$\begin{aligned} &-\frac{\Gamma}{\Delta x}(\phi_i - \phi_{i-1}) + \frac{\Gamma}{\Delta x}(\phi_{i+1} - \phi_i) + \rho U \phi_{i-1} \\ &+ \rho U \frac{1}{2}\psi(r_w)(\phi_i - \phi_{i-1}) - \rho U \phi_i - \rho U \frac{1}{2}\psi(r_e)(\phi_{i+1} - \phi_i) = 0 \end{aligned} \tag{2}$$

Where ϕ_i is the value of the transport property ϕ at node i, r_w is the ratio of the upstream to downstream gradients of the transport property for the west face of node i, r_e the ratio of the upstream to downstream gradients for the east face of node i and $\psi(r)$ the Min-Mod limiter function. For the left most node the discrete form of (1) is:

$$-\frac{\Gamma}{1/2\Delta x}(\phi_0 - \phi_a) + \frac{\Gamma}{\Delta x}(\phi_1 - \phi_0) + \rho U \phi_a - \rho U \phi_0 - \rho U \frac{1}{2}\psi(r_e)(\phi_1 - \phi_0) = 0 \tag{3}$$

Where ϕ_a is the value of the transport property at the domain inlet. Leonard mirror node extrapolation is used to estimate r_e. For the right most, or last, node the discrete form of (1) is:

$$\begin{aligned} &-\frac{\Gamma}{\Delta x}(\phi_{n-1} - \phi_{n-2}) \\ &+ \frac{\Gamma}{1/2\Delta x}(\phi_b - \phi_{n-1}) + \rho U \phi_{n-2} + \rho U \frac{1}{2}\psi(r_w)(\phi_{n-1} - \phi_{n-2}) - \rho U \phi_b = 0 \end{aligned} \tag{4}$$

Where ϕ_b is the value of the transport property at the domain outlet.

Verification

In order to verify that computation of the transport property is performed correctly results obtained numerically are compared with the analytical solution to (1):

$$\frac{\phi - \phi_a}{\phi_b - \phi_a} = \frac{\exp\left(\rho U x/\Gamma\right) - 1}{\exp\left(\rho U L/\Gamma\right) - 1} \tag{5}$$

The discrepancy between numerical and analytical results for a system of 50 nodes is 0.14 %