Derivation of the equation of motion for fluids

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Introduction

The equation of motion for fluids is derived based on the momentum balance for an arbitrary element of fluid.

Derivation

An element of fluid experiences viscous, convective, pressure and gravitational forces. The sum of these forces changes the momentum of the fluid element:

$$\int_V \frac{\partial \rho \mathbf{v}}{\partial t} dV = \sum_i F_i \tag{1}$$

Where ρ is the local density, $\mathbf{v}$ the velocity, t the time coordinate, V the volume of the fluid element and F_i the various forces acting on the fluid element.

The change in momentum due to convective transport across the boundaries of the fluid element is:

$$F_c = -\int_{\partial V} (\mathbf{n} \cdot \rho \mathbf{v}) \mathbf{v} dS \tag{2}$$

Where the subscript c denotes convective transport, $\mathbf{n}$ is the unit vector normal to the boundary and S the area of the boundary. The local convective flux $(\mathbf{n} \cdot \rho \mathbf{v})\mathbf{v}$ in the integral in (2) is mathematically equivalent to $\mathbf{n} \cdot \rho \mathbf{v}\mathbf{v}$:

$$(\mathbf{n} \cdot \rho \mathbf{v})\mathbf{v} = \mathbf{n} \cdot \rho \mathbf{v}\mathbf{v} \tag{3}$$

Which gives the force due to convective transport in a more familiar form:

$$F_c = -\int_{\partial V} \mathbf{n} \cdot \rho \mathbf{v}\mathbf{v} dS \tag{4}$$

Application of the divergence theorem to (4) results in:

$$F_c = -\int_V \nabla \cdot \rho \mathbf{v}\mathbf{v} dV \tag{5}$$

The contribution to the change in momentum of the fluid element due to viscous forces is:

$$F_\tau = -\int_{\partial V} \mathbf{n} \cdot \tau dS \tag{6}$$

Where the subscript τ denotes the viscous forces and τ is the viscous stress tensor. Again, application of the divergence theorem to (6) gives:

$$F_\tau = -\int_V \nabla \cdot \tau dV \tag{7}$$

The change in momentum of the fluid element due to pressure forces is:

$$F_p = -\int_{\partial V} p \mathbf{n} dS \tag{8}$$

Where the subscript p denotes the pressure forces and p the local pressure. Applying the divergence theorem to (8) gives:

$$F_p = -\int_V \nabla p dV \tag{9}$$

The total gravitational force acting on the fluid element is:

$$F_g = \int_V \rho \mathbf{g} dV \tag{10}$$

Where the subscript g denotes gravitational forces and $\mathbf{g}$ the acceleration vector.

The total momentum change of the fluid element is therefore:

$$\int_V \frac{\partial \rho \mathbf{v}}{\partial t} dV = -\int_V \nabla \cdot \rho \mathbf{v}\mathbf{v} dV - \int_V \nabla \cdot \tau dV - \int_V \nabla p dV + \int_V \rho \mathbf{g} dV \tag{11}$$

Dropping the integrals in (11) gives:

$$\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \cdot \rho \mathbf{v}\mathbf{v} - \nabla \cdot \tau - \nabla p + \rho \mathbf{g} \tag{12}$$

Which concludes the derivation.