

# Derivation of the ideal gas law

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## Introduction

The ideal gas law is derived based on the assumptions of kinetic theory.

## Derivation

The derivation is based on a cubic domain which contains the gas. The time it takes  $\Delta t_i$  for some particle to travel a distance  $2L$ , with  $L$  the length of one of the sides of the domain, is:

$$\Delta t_i = \frac{2L}{|v_{x,i}|} \quad (1)$$

Where  $v_{x,i}$  is the  $x$ -component of the velocity of particle  $i$ . The frequency  $f_i$  with which particle  $i$  collides with the west boundary is:

$$f_i = \frac{1}{\Delta t_i} \quad (2)$$

The momentum particle  $i$  imparts on the west boundary is:

$$\Delta p_i = 2m|v_{x,i}| \quad (3)$$

Where  $m$  is the mass of the particle. The momentum imparted by particle  $i$  on the west boundary per unit time is:

$$\frac{\Delta p_i}{\Delta t_i} = f_i \Delta p_i = \frac{mv_{x,i}^2}{L} \quad (4)$$

Note the absolute signs have been dropped since the square of a real number is always greater or equal to zero. The total momentum imparted on the west boundary by all particles per unit time is equal to the average force exerted on the boundary:

$$F = \sum_i^N \frac{\Delta p_i}{\Delta t_i} = \sum_i^N \frac{mv_{x,i}^2}{L} \quad (5)$$

With  $F$  the force exerted on the boundary by the particles and  $N$  the number of particles. The pressure at the boundary is therefore:

$$p = \frac{F}{L^2} = \frac{1}{L^2} \sum_i^N \frac{mv_{x,i}^2}{L} \quad (6)$$

Equation (6) can be rewritten as:

$$p = \frac{F}{L^2} = \frac{Nm}{L^3} \frac{\sum_i^N v_{x,i}^2}{N} \quad (7)$$

Where the factor containing the summation is:

$$\frac{\sum_i^N v_{x,i}^2}{N} = \langle v_x^2 \rangle \quad (8)$$

Where  $\langle v_x^2 \rangle$  is the average squared velocity. And so equation (7) becomes:

$$p = \frac{Nm}{L^3} \langle v_x^2 \rangle \quad (9)$$

Since velocities are random it holds that:

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle \quad (10)$$

The speed  $v_i$  of particle  $i$  is:

$$v_i^2 = v_{x,i}^2 + v_{y,i}^2 + v_{z,i}^2 \quad (11)$$

And therefore the average squared speed of all particles  $\langle v^2 \rangle$  is:

$$\langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle \quad (12)$$

Since the average squared velocity is the same along each coordinate equation (9) can be reformulated as:

$$p = \frac{Nm}{3L^3} \langle v^2 \rangle \quad (13)$$

With  $L^3$  the volume  $V$  of the domain equation (13) becomes:

$$pV = \frac{Nm}{3} \langle v^2 \rangle \quad (14)$$

Which concludes the derivation. Equation (14) shows that the product  $pV$  is proportional with the total kinetic energy of the particles.

Experimentally it has been shown that the product  $pV$  is equal to:

$$pV = NkT \quad (15)$$

Where  $k$  is the Boltzmann constant. It therefore follows that the temperature  $T$  is proportional with the kinetic energy of the system, which can be shown by combining (14) and (15):

$$\frac{3}{2}kT = \frac{1}{2}m\langle v^2 \rangle \quad (16)$$

## Conclusion

The ideal gas law is derived based on the assumptions of kinetic theory. The derived equation is then compared with the ideal gas law, which follows from experimental results. Comparison of the derived relation with the experimentally determined relation shows that kinetic energy of the system is proportional with temperature.