Machine Learning & Data Mining
Derek Ing Homework 6

Caltech CS/CNS/EE 155
February 2022

Policies

- **Due 9 PM, March** 2^{nd}, via Gradescope.
- You are free to collaborate on all of the problems, subject to the collaboration policy stated in the syllabus.
- In this course, we will be using Google Colab for code submissions. You will need a Google account.
- This set uses PyTorch, a Python package for neural networks. We recommend using Google Colab, which comes with PyTorch already installed.

Submission Instructions

- Submit your report as a single .pdf file to Gradescope, under "Set 6 Report".
- In the report, **include any images generated by your code** along with your answers to the questions.
- Submit your code by **sharing a link in your report** to your Google Colab notebook for each problem (see naming instructions below). Make sure to set sharing permissions to at least "Anyone with the link can view". **Links that can not be run by TAs will not be counted as turned in.** Check your links in an incognito window before submitting to be sure.
- For instructions specifically pertaining to the Gradescope submission process, see https://www.gradescope.com/get_started#student-submission.

Google Colab Instructions

For each notebook, you need to save a copy to your drive.

1. Open the github preview of the notebook, and click the icon to open the colab preview.
2. On the colab preview, go to File → Save a copy in Drive.
3. Edit your file name to "lastname_firstname_set_problem", e.g."yue_yisong_set6_prob2.ipynb"

1 Class-Conditional Densities for Binary Data [25 Points, 8 EC Points]

This problem will test your understanding of probabilistic models, especially Naive Bayes. Consider a generative classifier for C classes, with class conditional density $p(x|y)$ and a uniform class prior $p(y)$. Suppose all the D features are binary, $x_j \in \{0, 1\}$. If we assume all of the features are conditionally independent, as in Naive Bayes, we can write:

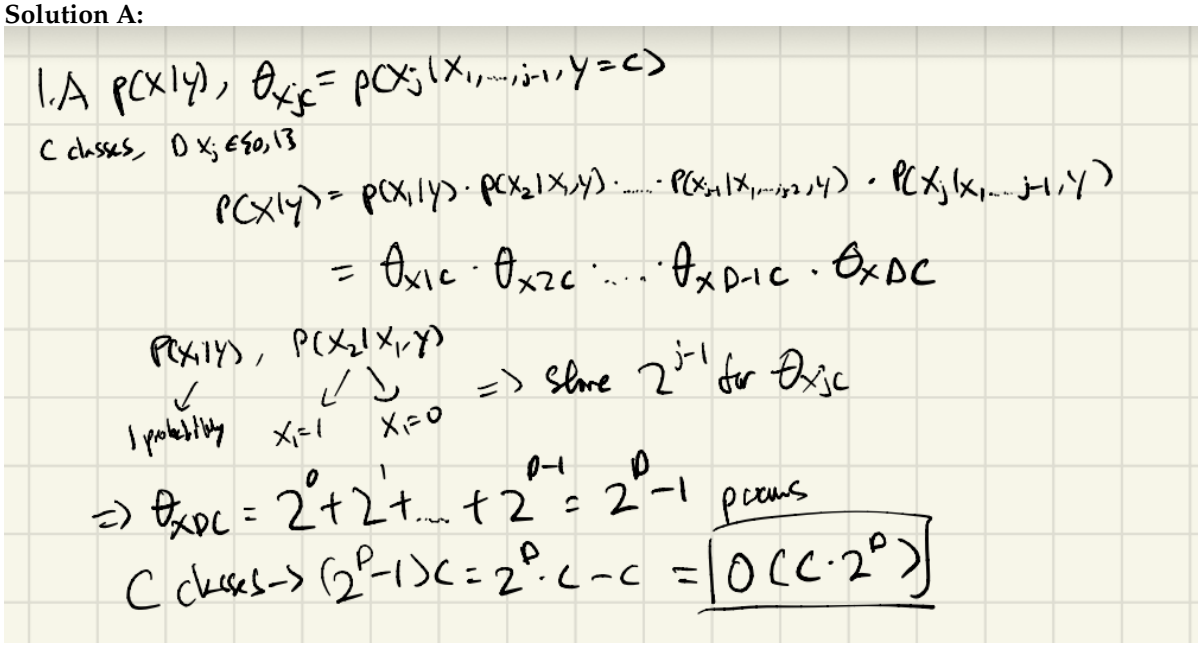
$$p(x \mid y = c) = \prod_{j=1}^{D} p(x_j \mid y = c)$$

This requires storing DC parameters.

Now consider a different model, which we will call the 'full' model, in which all the features are fully *dependent*.

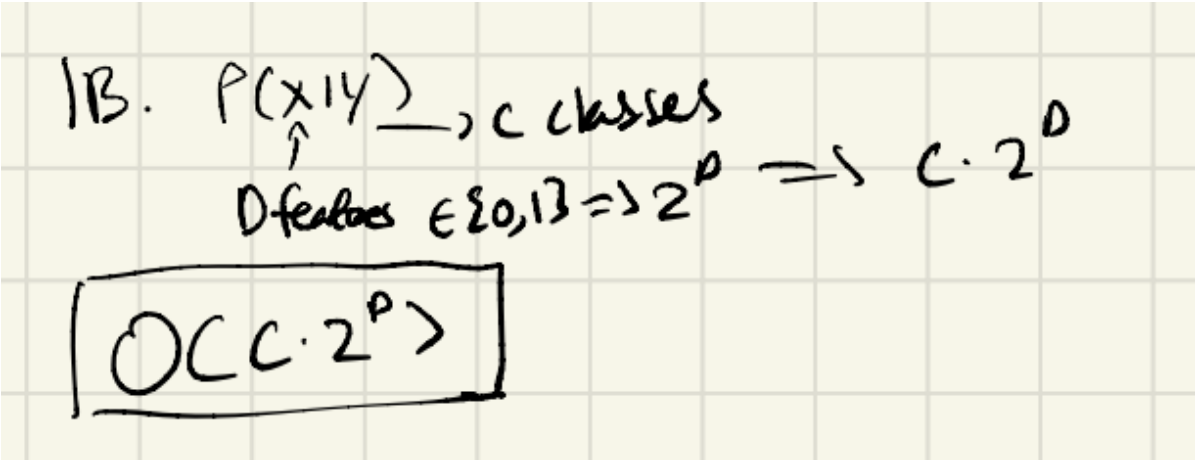
Problem A [9 points]: Use the chain rule of probability to factorize $p(x \mid y)$, and let $\theta_{xjc} = p(x_j|x_{1,\ldots,j-1}, y = c)$. Assuming we store each θ_{xjc}, how many parameters are needed to represent this factorization? Use big-O notation.

Solution A:



Problem B [8 points]: Assume we did no such factorization, and just used the joint probability $p(x \mid y = c)$. How many parameters would we need to estimate in order be able to compute $p(x|y = c)$ for arbitrary x and c? How does this compare to your answer from the previous part? Again, use big-O notation.

Solution B:



We need to estimate the same amount of parameters for the joint probability $p(x \mid y = c)$ as with factorization.

Problem C [4 points]: Assume the number of features D is fixed. Let there be N training cases. If the sample size N is very small, which model (Naive Bayes or full) is likely to give lower test set error, and why?

Solution C: *For small N, the Naive Bayes model is likely to give lower test set error. The full model tries to learn many more parameters than the Naive Bayes model. Learning relationships between features would not generalize well with a small amount of training examples. Thus, the full model is more likely to overfit, so the Naive Bayes model would give a lower test set error.*

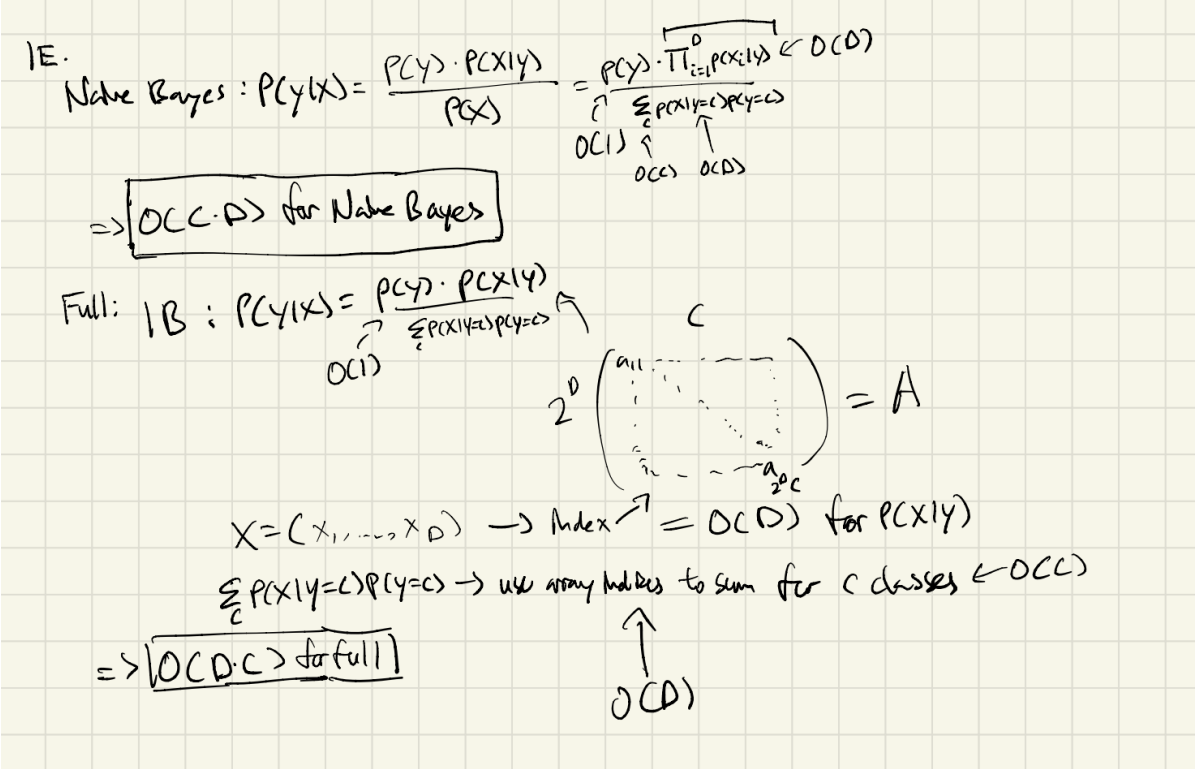
Problem D [4 points]: If the sample size N is very large, which model (Naive Bayes or full) is likely to give lower test set error, and why?

Solution D: *The full model is likely to give a lower test set error for large N. Assuming all the features are dependent, the full model would be able to learn relationships between features with very large N. The Naive Bayes model assumes all the features are independent, thus the Naive Bayes would not be able to learn the dependencies of the features. However, the full model would be able to and would better represent the true probabilities. Thus, the Naive Bayes could underfit, and the full model would give a lower test set error.*

Problem E [8 EC points]: Assume all the parameter estimates have been computed. What is the computational complexity of making a prediction, i.e. computing $p(y \mid x)$, using Naive Bayes for a single test case? What is the computation complexity of making a prediction with the full model? In justifying your answer for the full model, choose either the implementation in 1A or 1B and state your choice. For the full-model

case, assume that converting a D-bit vector to an array index is an $O(D)$ operation. Also, recall that we have assumed a uniform class prior.

Solution E:



2 Sequence Prediction [75 Points]

In this problem, we will explore some of the various algorithms associated with Hidden Markov Models (HMMs), as discussed in lecture. We have also uploaded a note on HMMs to the github that might be helpful.

Sequence Prediction

These next few problems will require extensive coding, so be sure to start early!

- You will write an implementation for the hidden Markov model in the cell for `HMM Code` in the notebook given to you, within the appropriate functions where indicated. There should be no need to write additional functions or use NumPy in your implementation, but feel free to do so if you would like.
- You can (and should!) use the helper cells for each of the subproblems in the notebook, namely `2A`, `2Bi`, `2Bii`, `2C`, `2D`, and `2F`. These can be used to run and check your implementations for each of the corresponding problems. The cells provide useful output in an easy-to-read format. There is no need to modify these cells.
- Lastly, the cell for `Utility` contains some functions used for loading data directly from the class github repository. There is no need to modify this cell.

The supplementary data folder of the class github repository contains 6 files titled `sequence_data0.txt`, `sequence_data1.txt`, ... , `sequence_data5.txt`. Each file specifies a **trained** HMM. The first row contains two tab-delimited numbers: the number of states Y and the number of types of observations X (i.e. the observations are $0, 1, ..., X-1$). The next Y rows of Y tab-delimited floating-point numbers describe the state transition matrix. Each row represents the current state, each column represents a state to transition to, and each entry represents the probability of that transition occurring. The next Y rows of X tab-delimited floating-point numbers describe the output emission matrix, encoded analogously to the state transition matrix. The file ends with 5 possible emissions from that HMM.

The supplementary data folder also contains one additional file titled `ron.txt`. This is used in problems 2C and 2D and is explained in greater detail there.

Problem A [10 points]: For each of the six trained HMMs, find the max-probability state sequence for each of the five input sequences at the end of the corresponding file. To complete this problem, you will have to implement the Viterbi algorithm (in `viterbi()` of the `HiddenMarkovModel` object). Write your implementation well, as we will be reusing it in a later problem. See the end of problem 2B for a big hint! Note that you do not need to worry about underflow in this part.

In your report, show your results on the 6 files. (Copy-pasting the results of the cell for `2A` suffices.)

Solution A:

```
File #0:
Emission Sequence             Max Probability State Sequence
########################################################################
25421                         31033
01232367534                   22222100310
5452674261527433              1031003103222222
7226213164512267255           1310331000033100310
024712060235205101025524l     22222222222222222222222103

File #1:
Emission Sequence             Max Probability State Sequence
########################################################################
77550                         22222
7224523677                    2222221000
505767442426747               222100003310031
72134131645536112267          10310310000310333100
47336677714500510602530 41    22210000032222233103222223

File #2:
Emission Sequence             Max Probability State Sequence
########################################################################
60622                         11111
4687981156                    2100202111
815833657775062               021011111111111
21310222515963505015          02020111111111111021
650319945257127400632002 5    11102021111111102021110211

  File #3:
  Emission Sequence             Max Probability State Sequence
  ########################################################################
  13661                         00021
  2102213421                    3131310213
  166066262165133               133333133133100
  53164662112162634156          20000021313131002133
  152354100512323022630625 6    13100213331331333131331 33

  File #4:
  Emission Sequence             Max Probability State Sequence
  ########################################################################
  23664                         01124
  3630535602                    0111201112
  350201162150142               011244012441112
  00214005402015146362          11201112412444011112
  21112665246651435625344 50    20120124241240111124112411 24

  File #5:
  Emission Sequence             Max Probability State Sequence
  ########################################################################
  68535                         10111
  4546566636                    1111111111
  638436858181213               110111010000011
  13240338308444514688          00010000000111111100
  01116644344413825336326 26    21111111111111100111110101
```

Problem B [17 points]: For each of the six trained HMMs, find the probabilities of emitting the five input sequences at the end of the corresponding file. To complete this problem, you will have to implement the Forward algorithm and the Backward algorithm. You may assume that the initial state is randomly selected along a uniform distribution (the starting state transition probabilities are defined in `self.A_start` in `HiddenMarkovModel`). Again, write your implementation well, as we will be reusing it in a later problem.

Note that the probability of emitting an input sequence can be found by using either the α vectors from the Forward algorithm or the β vectors from the Backward algorithm. You don't need to worry about this, as it is done for you in `probability_alphas()` and `probability_betas()`.

Implement the Forward algorithm. In your report, show your results on the 6 files.
Implement the Backward algorithm. In your report, show your results on the 6 files.

After you complete problems 2A and 2B, you can compare your results (the probabilities from the forward and backward algorithms should be the same) for the file titled `sequence_data0.txt` with the values given in the table below:

Dataset	Emission Sequence	Max-probability State Sequence	Probability of Sequence
0	25421	31033	4.537e-05
0	01232367534	22222100310	1.620e-11
0	5452674261527433	1031003103222222	4.348e-15
0	7226213164512267255	1310331000033100310	4.739e-18
0	0247120602352051010255241	22222222222222222222222103	9.365e-24

Solution B:
Forward:

```
File #0:
Emission Sequence             Probability of Emitting Sequence
######################################################################
25421                         4.537e-05
01232367534                   1.620e-11
5452674261527433              4.348e-15
7226213164512267255           4.739e-18
024712060235205101025524l     9.365e-24

File #1:
Emission Sequence             Probability of Emitting Sequence
######################################################################
77550                         1.181e-04
7224523677                    2.033e-09
505767442426747               2.477e-13
72134131645536112267          8.871e-20
473366777145005106025304l     3.740e-24

File #2:
Emission Sequence             Probability of Emitting Sequence
######################################################################
60622                         2.088e-05
4687981156                    5.181e-11
815833657775062               3.315e-15
21310222515963505015          5.126e-20
650319945257127400632002l     1.297e-25
```

```
File #3:
Emission Sequence             Probability of Emitting Sequence
######################################################################
13661                         1.732e-04
2102213421                    8.285e-09
166066262165133               1.642e-12
53164662112162634156          1.063e-16
152354100512323022630625l     4.535e-22

File #4:
Emission Sequence             Probability of Emitting Sequence
######################################################################
23664                         1.141e-04
3630535602                    4.326e-09
350201162150142               9.793e-14
00214005402015146362          4.740e-18
211126652466514356253445l     5.618e-22

File #5:
Emission Sequence             Probability of Emitting Sequence
######################################################################
68535                         1.322e-05
4546566636                    2.867e-09
638436858181213               4.323e-14
13240338308444514688          4.629e-18
011166443444138253363262l     1.440e-22
```

Backward:

```
File #0:
Emission Sequence             Probability of Emitting Sequence
######################################################################
25421                         4.537e-05
01232367534                   1.620e-11
5452674261527433              4.348e-15
7226213164512267255           4.739e-18
024712060235205101025524l     9.365e-24

File #1:
Emission Sequence             Probability of Emitting Sequence
######################################################################
77550                         1.181e-04
7224523677                    2.033e-09
505767442426747               2.477e-13
72134131645536112267          8.871e-20
473366777145005106025304l     3.740e-24

File #2:
Emission Sequence             Probability of Emitting Sequence
######################################################################
60622                         2.088e-05
4687981156                    5.181e-11
815833657775062               3.315e-15
21310222515963505015          5.126e-20
650319945257127400632002l     1.297e-25
```

```
File #3:
Emission Sequence             Probability of Emitting Sequence
######################################################################
13661                         1.732e-04
2102213421                    8.285e-09
166066262165133               1.642e-12
53164662112162634156          1.063e-16
152354100512323022630625l     4.535e-22

File #4:
Emission Sequence             Probability of Emitting Sequence
######################################################################
23664                         1.141e-04
3630535602                    4.326e-09
350201162150142               9.793e-14
00214005402015146362          4.740e-18
211126652466514356253445l     5.618e-22

File #5:
Emission Sequence             Probability of Emitting Sequence
######################################################################
68535                         1.322e-05
4546566636                    2.867e-09
638436858181213               4.323e-14
13240338308444514688          4.629e-18
011166443444138253363262l     1.440e-22
```

HMM Training

Ron is an avid music listener, and his genre preferences at any given time depend on his mood. Ron's possible moods are happy, mellow, sad, and angry. Ron experiences one mood per day (as humans are known to do) and chooses one of ten genres of music to listen to that day depending on his mood.

Ron's roommate, who is known to take to odd hobbies, is interested in how Ron's mood affects his music selection, and thus collects data on Ron's mood and music selection for six years (2190 data points). This data is contained in the supplementary file `ron.txt`. Each row contains two tab-delimited strings: Ron's mood and Ron's genre preference that day. The data is split into 12 sequences, each corresponding to half a year's worth of observations. The sequences are separated by a row containing only the character `-`.

Problem C [10 points]: Use a single M-step to train a supervised Hidden Markov Model on the data in `ron.txt`. What are the learned state transition and output emission matrices?

Tip: the $(1, 1)$ entry of your transition matrix should be `2.833e-01`, and the $(1, 1)$ entry of your observation matrix should be `1.486e-01`.

Solution C:

```
Transition Matrix:
######################################################################
2.833e-01   4.714e-01   1.310e-01   1.143e-01
2.321e-01   3.810e-01   2.940e-01   9.284e-02
1.040e-01   9.760e-02   3.696e-01   4.288e-01
1.883e-01   9.903e-02   3.052e-01   4.075e-01


Observation Matrix:
######################################################################
1.486e-01   2.288e-01   1.533e-01   1.179e-01   4.717e-02   5.189e-02   2.830e-02   1.297e-01   9.198e-02   2.358e-03
1.062e-01   9.653e-03   1.931e-02   3.089e-02   1.699e-01   4.633e-02   1.409e-01   2.394e-01   1.371e-01   1.004e-01
1.194e-01   4.299e-02   6.529e-02   9.076e-02   1.768e-01   2.022e-01   4.618e-02   5.096e-02   7.803e-02   1.274e-01
1.694e-01   3.871e-02   1.468e-01   1.823e-01   4.839e-02   6.290e-02   9.032e-02   2.581e-02   2.161e-01   1.935e-02
```

Problem D [15 points]: Now suppose that Ron has a third roommate who is also interested in how Ron's mood affects his music selection. This roommate is lazier than the other one, so he simply steals the first roommate's data. Unfortunately, he only manages to grab half the data, namely, Ron's choice of music for each of the 2190 days.

In this problem, we will train an unsupervised Hidden Markov Model on this data. Recall that unsupervised HMM training is done using the Baum-Welch algorithm and will require repeated EM steps. For this problem, we will use 4 hidden states and run the algorithm for 1000 iterations. The transition and observation matrices are initialized for you in the helper functions `supervised_learning()` and `unsupervised_learning()` such that they are random and normalized.

What are the learned state transition and output emission matrices? Please report the result using random seed state 1, as is done by default in the notebook.

Tips for debugging:

- The rows of the state transition and output emitting matrices should sum to 1.
- Your matrices should not change drastically every iteration.
- After many iterations, your matrices should converge.
- If you used random seed 1 for this computation (as is done by default in the notebook), the $(1, 1)$ entry of the state transition matrix should be `5.075e-01`, and the $(1, 1)$ entry of the output emission matrix should be `1.117e-01`.

Solution D:

```
Transition Matrix:
######################################################################
5.075e-01   4.596e-01   6.533e-09   3.292e-02
3.127e-03   2.107e-04   9.964e-01   2.733e-04
1.195e-09   6.886e-02   9.686e-16   9.311e-01
6.203e-01   3.796e-01   1.555e-05   1.579e-04


Observation Matrix:
######################################################################
1.117e-01   1.525e-01   7.740e-02   1.975e-02   1.594e-01   4.574e-13   3.556e-16   2.475e-01   1.139e-01   1.180e-01
1.205e-01   2.548e-15   1.103e-01   1.751e-01   3.656e-04   2.190e-01   1.002e-01   6.178e-02   1.323e-01   8.053e-02
1.276e-01   2.665e-02   5.788e-02   1.682e-01   1.700e-01   6.969e-02   1.254e-01   3.940e-02   1.627e-01   5.244e-02
1.918e-01   8.206e-02   1.376e-01   8.725e-02   1.152e-01   1.209e-01   1.033e-01   3.101e-02   1.308e-01   5.847e-38
```

Problem E [5 points]: How do the transition and emission matrices from 2C and 2D compare? Which do you think provides a more accurate representation of Ron's moods and how they affect his music choices? Justify your answer. Suggest one way that we may be able to improve the method (supervised or unsupervised) that you believe produces the less accurate representation.

Solution E: *There is a much larger range of probabilities in the matrices in 2D compared to 2C. The transition matrix in 2C has probabilities to the order of* 10^{-1} *to* 10^{-2}*, and the transition matrix in 2D has probabilities ranging from as low as* 10^{-16} *to* 10^{-1}*. This is the same for the emission matrix. The emission matrix in 2C has probabilities from* 10^{-3} *to* 10^{-1}*, while the emission matrix in 2D has probabilities from* 10^{-38} *to* 10^{-1}*. I think that the representation from 2C provides a more accurate representation of Ron's mood and how they affect his music choices. With the supervised learning, we can get the transition and emission matrices with a closed form optimal solution. With unsupervised learning, however, the final matrices depends on the initial matrices, which are randomly initialized, and does not always give the global optimal solution. Thus, 2C should be a more accurate representation. To improve the unsupervised method, we could initialize the transition and emission matrices based on some prior knowledge of Ron's mood and music choices.*

Sequence Generation

Hidden Markov Models fall under the umbrella of generative models and therefore can be used to not only predict sequential data, but also to generate it.

Problem F [5 points]: Run the cell for this problem. The code in the cell loads the trained HMMs from the files titled `sequence_data0.txt,...,sequence_data5.txt` and uses the six models to probabilistically generate five sequences of emissions from each model, each of length 20. In your report, show your results.

Solution F:

```
File #0:
Generated Emission
############################################################################
22622574427144751176
77335550570413755205
07022657247234605243
32755775465722564524
31744470747207064055

File #1:
Generated Emission
############################################################################
45717427644502453774
54407062276374777067
70542023571620776742
45200776600545547747
25562571765045464162

File #2:
Generated Emission
############################################################################
84079935655582063926
72979739357229591137
27048135691704127778
01075725601122276760
07481777632295535677

File #3:
Generated Emission
############################################################################
61363636441316325132
42341512311502063611
55036060524026211112
22541365521453266201
26432163111103036640

File #4:
Generated Emission
############################################################################
30433314222355166456
32024253106001041446
51310564534245335026
11661324011526646544
52563235463564616636

File #5:
Generated Emission
############################################################################
03331254341636464236
63616137603266037211
56454181361803154364
13684401484482332518
40816061344003836642
```

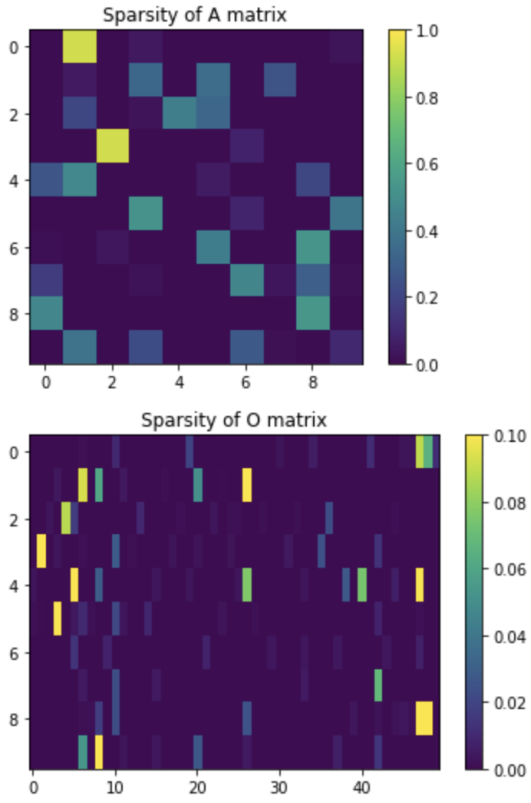
Visualization & Analysis

Once you have implemented the HMM code part of the notebook, load and run the cells for the following subproblems. Here you will apply the HMM you have implemented to the Constitution. There is no coding required for this part, only analysis.

Answer the following problems in the context of the visualizations in the notebook.

Problem G [3 points]: What can you say about the sparsity of the trained A and O matrices? How does this sparsity affect the transition and observation behaviour at each state?

Solution G:



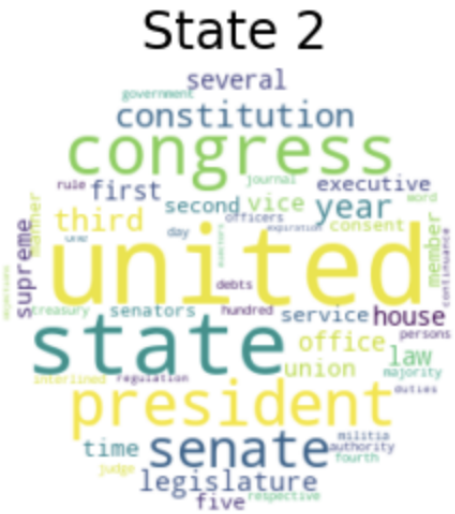
The trained A and O matrices are quite sparse, but O is more sparse than A. Probabilities of 0 mean that it is impossible to transition to another state or observe some behavior at a given state. Thus, at some state, it is only possible to transition to a few other states, and it is only possible to observe a few behaviors out of the total possible behaviors. Since O is more sparse than A, this could suggest that some states and behaviors are more closely related than the transitions between different states.

Problem H [5 points]: How do the sample emission sentences from the HMM change as the number of hidden states is increased? What happens in the special case where there is only one hidden state? In general, when the number of hidden states is unknown while training an HMM for a fixed observation set, can we increase the training data likelihood by allowing more hidden states?

Solution H: *As the number of hidden states increases, the sample emission sentences begin to make more sense and sound more like actual English sentences. When there is only one hidden state, the model is stuck in one state and does not make any transitions. Thus, the probabilities in the model come solely from probabilities of different observations. Thus, the probabilities of any word being generated in the sentence only depends on the number of times the word occurs in the dataset. Having one state also means that the order in which words occur is completely random. In general, we can increase the training data likelihood by allowing more hidden states. With more hidden states, we can generate more sparse A and O matrices. As a result, the relationships between some states and words and relationships for some transitions between states will be very strong.*

Problem I [5 points]: Pick a state that you find semantically meaningful, and analyze this state and its wordcloud. What does this state represent? How does this state differ from the other states? Back up your claim with a few key words from the wordcloud.

Solution I:

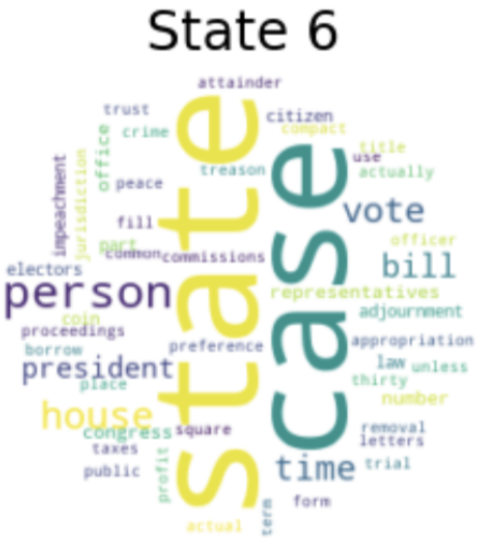


I find state 2 semantically meaningful because it contains several words describing the government and other words that are closely related to the Constitution. Some key words include united, state, president, congress, senate, constitution, and legislature, which are all words related to different parts of the government. Some other

states have some words that are related, but there are some states with key words that are not related at all. For example, state 3 has numbers two and three as two of its largest key words, but the other words in the state are not very related to numbers. State 8, however, has the words "may" and "elected" as two of its largest keywords, which I don't think are related at all.

Code link: Problem 2

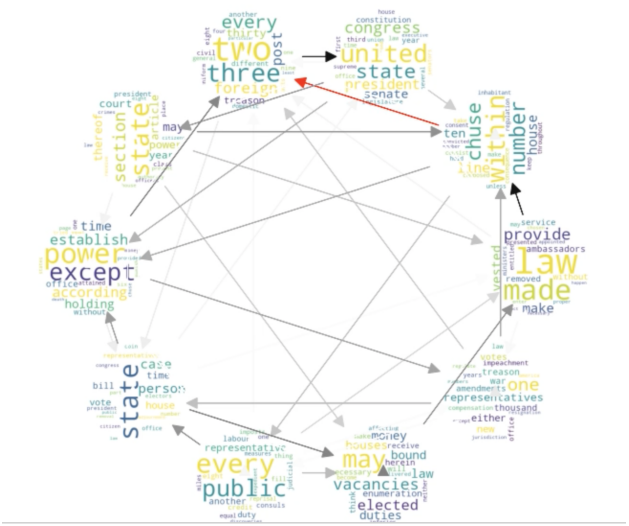

State 5
without
affirmation
according
among
chuse
attained
power
page
equal
states
one
provided
six
office
prescribed
property
establish
tried
except
trust
pass
ex
time
subject
holding
officer


State 6
attainder
trust
citizen
crime
office
treason
vote
peace
fill
bill
electors
person
representatives
adjournment
proceedings
appropriation
president
law
unless
house
number
congress
square
removal
letters
taxes
public
time
trial
form


State 7
judicial
affirm
fill
subjects
engagements
representative
miles
consuls
credit
every
duty
reprisal
water
eight
public
support
exports
labour
effect
equal
paid
collect
another
silver
neither
imposts
direct


State 8
neither
bound
officer
enumeration.
likewise
written
vacancies
senator
executed
danger
delivered
safety
may
make
law
partly
necessary
powers
affecting
receive
faithfully
elected
herein
houses
think
money
become
duties

Case nominate 1 or call two



Case nominate 1 or call two confederation of