5.2 Exploring Quotients of Polynomial Functions

YOU WILL NEED

- graph paper
- coloured pencils or pens
- graphing calculator or graphing software

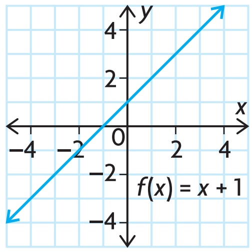
GOAL

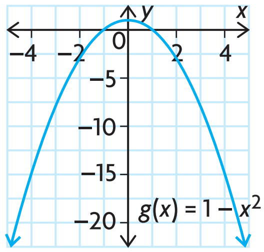
Explore graphs that are created by dividing polynomial functions.

EXPLORE the Math

Each row shows the graphs of two polynomial functions.

A.





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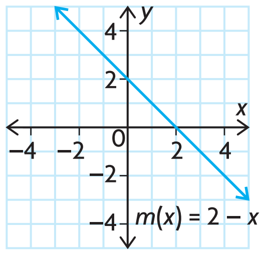

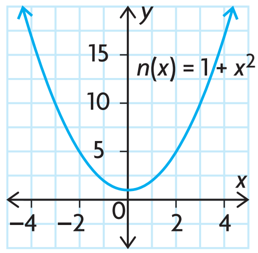

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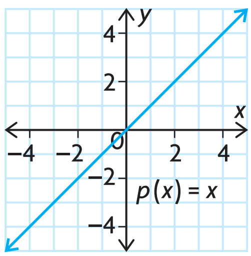
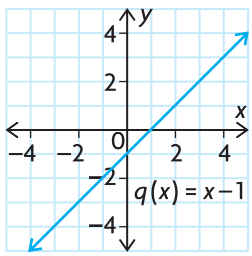


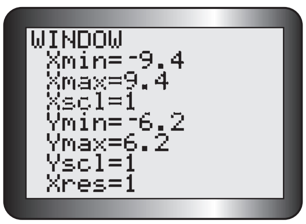


rational function

a function that can be expressed as $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions, $q(x) \neq 0$ (e.g., $f(x) = \frac{3x^2 - 1}{x + 1}$, $x \neq -1$, and $f(x) = \frac{1 - x}{x^2}$, $x \neq 0$, are rational functions, but $f(x) = \frac{1 + x}{\sqrt{2 - x}}$, $x \neq 2$, is not because its denominator is not a polynomial)

? What are the characteristics of the graphs that are created by dividing two polynomial functions?

A. Using the given functions, write the equation of the rational function $y = \frac{f(x)}{g(x)}$. Enter this equation into Y1 of the equation editor of a graphing calculator. Graph this equation using the window settings shown, and draw a sketch.



B. Describe the characteristics of the graph you created in part A by answering the following questions:

- **i)** Where are the zeros?
- **ii)** Are there any asymptotes? If so, where are they?
- **iii)** What are the domain and range of this function?
- **iv)** Is it a **continuous function**? Explain.
- **v)** Are there any values of $y = \frac{f(x)}{g(x)}$ that are undefined? What feature(s) of the graph is (are) related to these values?
- **vi)** Describe the end behaviours of this function.
- **vii)** Is the resulting graph a function? Explain.

C. Write the equation defined by $y = \frac{g(x)}{f(x)}$. Predict how the graph of this function will differ from the graph of $y = \frac{f(x)}{g(x)}$. Graph this function using your graphing calculator, and draw a sketch.

D. Describe the characteristics of the graph you created in part C by answering the questions in part B.

E. Repeat parts A through D for the functions in the other two rows.

F. Using graphing technology, and the same window settings you used in part A, explore the graphs of the following rational functions. Sketch each graph on separate axes, and note any holes or asymptotes.

- **i)** $f(x) = \dfrac{x^2 - 1}{x - 1}$
- **ii)** $f(x) = \dfrac{3}{x + 1}$
- **iii)** $f(x) = \dfrac{x + 1}{x^2 - 2x - 3}$
- **iv)** $f(x) = \dfrac{x + 1}{x + 2}$
- **v)** $f(x) = \dfrac{0.5x^2 + 1}{x - 1}$
- **vi)** $f(x) = \dfrac{x^2 + 2x}{x + 1}$
- **vii)** $f(x) = \dfrac{9x}{1 + x^2}$
- **viii)** $f(x) = \dfrac{2x^2 - 3}{x^2 + 1}$

G. Examine the graphs of the functions in parts i) and v) of part F at the point where $x = 1$. Explain why $f(x) = \frac{x^2 - 1}{x - 1}$ has a hole where $x = 1$, but $f(x) = \frac{0.5x^2 + 1}{x - 1}$ has a vertical asymptote. Identify the other functions in part F that have holes and the other functions that have vertical asymptotes.

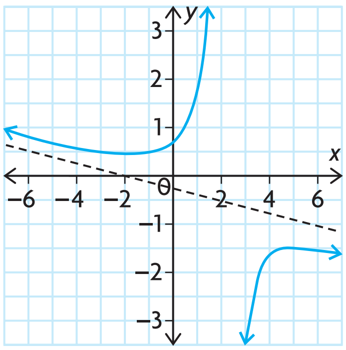
Tech Support

When entering a rational function into a graphing calculator, use brackets around the expression in the numerator and the expression in the denominator.

H. Redraw the graph of the rational function $f(x) = \frac{0.5x^2 + 1}{x - 1}$. Then enter the equation $y = 0.5x + 0.5$ into Y2 of the equation editor. What do you notice? Examine all your other sketches in this exploration to see if any of the other functions have an oblique asymptote.

oblique asymptote

an asymptote that is neither vertical nor horizontal, but slanted



I. Examine the equations with graphs that have horizontal asymptotes in part F. Compare the degree of the expression in the numerator with the degree of the expression in the denominator. Is there a connection between the degrees in the numerator and denominator and the existence of horizontal asymptotes? Explain. Repeat for functions with oblique asymptotes.

J. Investigate several functions of the form $f(x) = \frac{ax + b}{cx + d}$. Note similarities and differences. Without graphing, how can you predict where a horizontal asymptote will occur?

K. Investigate graphs of quotients of quadratic functions. How are they different from graphs of quotients of linear functions?

L. Summarize the different characteristics of the graphs of rational functions.

Reflecting

M. How do the zeros of the function in the numerator help you graph the rational function? How do the zeros of the function in the denominator help you graph the rational function?

N. Explain how you can use the expressions in the numerator and the denominator of a rational function to decide if the graph has

- **i)** a hole
- **ii)** a vertical asymptote
- **iii)** a horizontal asymptote
- **iv)** an oblique asymptote

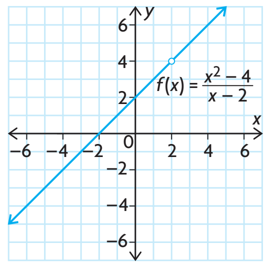
In Summary

Key Ideas

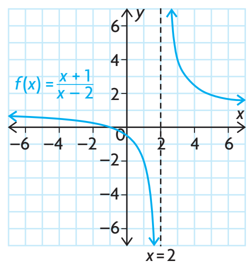
- The quotient of two polynomial functions results in a rational function which often has one or more discontinuities.
- The breaks or discontinuities in a rational function occur where the function is undefined. The function is undefined at values where the denominator is equal to zero. As a result, these values must be restricted from the domain of the function.
- The values that must be restricted from the domain of a rational function result in key characteristics that define the shape of the graph. These characteristics include a combination of vertical asymptotes (also called infinite discontinuities) and holes (also called point discontinuities).
- The end behaviours of many rational functions are determined by either horizontal asymptotes or oblique asymptotes.

Need to Know

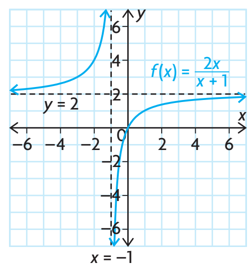
- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a hole at $x = a$ if $\frac{p(a)}{q(a)} = \frac{0}{0}$. This occurs when $p(x)$ and $q(x)$ contain a common factor of $(x - a)$.
  For example, $f(x) = \frac{x^2 - 4}{x - 2}$ has the common factor of $(x - 2)$ in the numerator and the denominator. This results in a hole in the graph of $f(x)$ at $x = 2$.



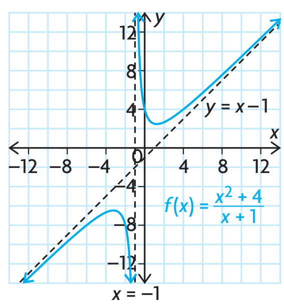
- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a vertical asymptote at $x = a$ if $\frac{p(a)}{q(a)} = \frac{p(a)}{0}$.
  For example, $f(x) = \frac{x + 1}{x - 2}$ has a vertical asymptote at $x = 2$.



- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a horizontal asymptote only when the degree of $p(x)$ is less than or equal to the degree of $q(x)$. For example, $f(x) = \frac{2x}{x + 1}$ has a horizontal asymptote at $y = 2$.



- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has an oblique (slant) asymptote only when the degree of $p(x)$ is greater than the degree of $q(x)$ by exactly 1. For example, $f(x) = \frac{x^2 + 4}{x + 1}$ has an oblique asymptote.



FURTHER Your Understanding

1. Without using graphing technology, match each equation with its corresponding graph. Explain your reasoning.

a) $y = \dfrac{-1}{x - 3}$

b) $y = \dfrac{x^2 - 9}{x - 3}$

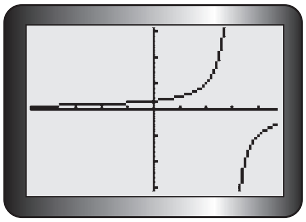
c) $y = \dfrac{1}{(x + 3)^2}$

d) $y = \dfrac{x}{(x - 1)(x + 3)}$

e) $y = \dfrac{1}{x^2 + 5}$

f) $y = \dfrac{x^2}{x - 3}$

A



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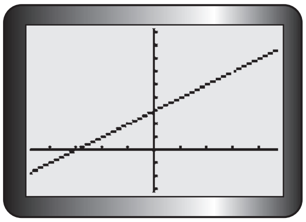

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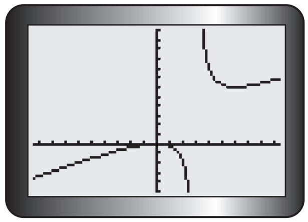

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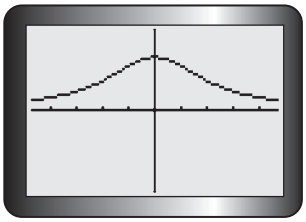

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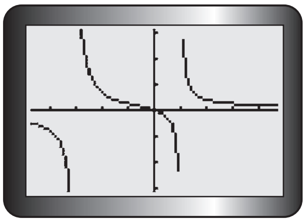

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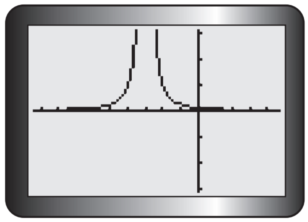

2. For each function, determine the equations of any vertical asymptotes, the locations of any holes, and the existence of any horizontal or oblique asymptotes.

a) $y = \dfrac{x}{x + 4}$

b) $y = \dfrac{1}{2x + 3}$

c) $y = \dfrac{2x + 5}{x - 6}$

d) $y = \dfrac{x^2 - 9}{x + 3}$

e) $y = \dfrac{1}{(x + 3)(x - 5)}$

f) $y = \dfrac{-x}{x + 1}$

g) $y = \dfrac{3x - 6}{x - 2}$

h) $y = \dfrac{-4x + 1}{2x - 5}$

i) $y = \dfrac{8x}{4x + 1}$

j) $y = \dfrac{x + 4}{x^2 - 16}$

k) $y = \dfrac{x}{5x - 3}$

l) $y = \dfrac{-3x + 1}{2x - 8}$

3. Write an equation for a rational function with the properties as given.

a) a hole at $x = 1$

b) a vertical asymptote anywhere and a horizontal asymptote along the x-axis

c) a hole at $x = -2$ and a vertical asymptote at $x = 1$

d) a vertical asymptote at $x = -1$ and a horizontal asymptote at $y = 2$

e) an oblique asymptote, but no vertical asymptote