5.4 Solving Rational Equations

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Connect the solution to a rational equation with the graph of a rational function.

LEARN ABOUT the Math

When they work together, Stuart and Lucy can deliver flyers to all the homes in their neighbourhood in 42 min. When Lucy works alone, she can finish the deliveries in 13 min less time than Stuart can when he works alone.

? When Stuart works alone, how long does he take to deliver the flyers?

EXAMPLE 1 Selecting a strategy to solve a rational equation

Determine the time that Stuart takes to deliver the flyers when he works alone.

Solution A: Creating an equation and solving it using algebra

Let s minutes be the time that Stuart takes to deliver the flyers when working alone.

Lucy takes $(s - 13)$ minutes when working alone.

Choose a variable to represent Stuart's time and use it to write an expression for Lucy's time.

Lucy delivers the flyers in 13 min less time than Stuart.

The fraction of deliveries made in one minute

- by Stuart working alone is $\frac{1}{s}$
- by Lucy working alone is $\frac{1}{s - 13}$
- by Stuart and Lucy working together is $\frac{1}{42}$

$$\frac{1}{s} + \frac{1}{s - 13} = \frac{1}{42}$$

Compare the rates at which they work.
For example, if Stuart took 80 min to deliver all the flyers, he would deliver $\frac{1}{80}$ of the flyers per minute.
$s > 13$ because Stuart takes longer than Lucy to deliver the flyers, and it is not possible for the denominators to be zero.

Multiply by the LCD.

$$42s(s-13)\left(\frac{1}{s}+\frac{1}{s-13}\right)=42s(s-13)\left(\frac{1}{42}\right)$$

$$\frac{42s(s-13)}{s}+\frac{42s(s-13)}{s-13}=\frac{42s(s-13)}{42}$$

$$\frac{42\cancel{s}^{1}(s-13)}{\cancel{s}_{1}}+\frac{42s\cancel{(s-13)}^{1}}{\cancel{s-13}_{1}}=\frac{\cancel{42}^{1}s(s-13)}{\cancel{42}_{1}}$$

$$42(s-13)+42s=s(s-13)$$

There are no common factors in the denominators, so the LCD (lowest common denominator) is the product of the three denominators $42s(s-13)$. Multiply each term by the LCD, and then simplify the resulting rational expressions to remove all the denominators.

$$\begin{aligned}42s-546+42s&=s^2-13s\\0&=s^2-97s+546\\0&=(s-91)(s-6)\\s&=6\text{ or }91\end{aligned}$$

Solve the quadratic equation by factoring or by using the quadratic formula.

$s > 13$ so 6 is not an admissible solution.

Remember to look for inadmissible solutions by carefully considering both the context and the information given in the problem.

$$\begin{aligned}\text{LS}&=\frac{1}{s}+\frac{1}{s-13}\\&=\frac{1}{91}+\frac{1}{91-13}\\&=\frac{1}{91}+\frac{1}{78}\\&=\frac{78+91}{7098}\\&=\frac{169}{7098}\\&=\frac{1}{42}\end{aligned}$$

Check the solution, $s = 91$, by substituting it into the original equation.

$$\text{RS}=\frac{1}{42}$$

Since LS = RS, $s = 91$ is the solution.

It will take Stuart 91 min to deliver the flyers when working alone.

Solution B: Using the graph of a rational function to solve a rational equation

The equation that models the problem is $\frac{1}{s} + \frac{1}{s - 13} = \frac{1}{42}$, where s represents the time, in minutes, that Stuart takes to deliver the flyers when working alone.

$$\frac{1}{s} + \frac{1}{s - 13} - \frac{1}{42} = 0$$

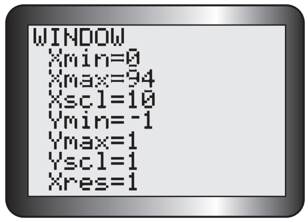
Subtract $\frac{1}{42}$ from each side.

To solve the equation, find the zeros of the function $f(s) = \frac{1}{s} + \frac{1}{s - 13} - \frac{1}{42}$.

Graph $f(s) = \frac{1}{s} + \frac{1}{s - 13} - \frac{1}{42}$.

From the equation, you can expect the graph to have vertical asymptotes at $s = 0$ and $s = 13$.

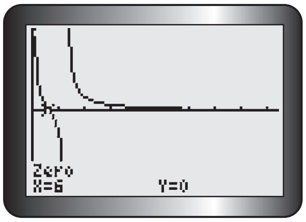
Since you are only interested in finding the zeros, you can limit the y-values to those close to zero.



Use the zero operation to determine the zeros.

Tech Support

For help determining the zeros on a graphing calculator, see Technical Appendix, T-8.

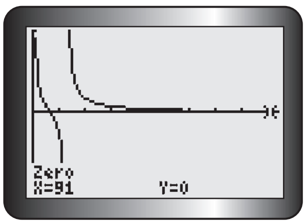


The first zero for $f(s)$ is $s = 6$. Reject this solution since $s > 13$.

You know that Lucy takes 13 min less time than Stuart takes, so Stuart must take longer than 13 min.

The solution $s = 6$ is inadmissible.

Determine the other zero.



The solution is $s = 91$.

Stuart takes 91 min to deliver the flyers when working alone.

Reflecting

A. In Solution A, explain how a rational equation was created using the times given in the problem.

B. In Solution B, explain how finding the zeros of a rational function provided the solution to the problem.

C. Where did the inadmissible root obtained in Solution A show up in the graphical solution in Solution B? How was this root dealt with?

APPLY the Math

EXAMPLE 2 | Using an algebraic strategy to solve simple rational equations

Solve each rational equation.

a) $\dfrac{x-2}{x-3} = 0$ **b)** $\dfrac{x+3}{x-4} = \dfrac{x-1}{x+2}$

Solution

a) $\dfrac{x-2}{x-3} = 0,\ x \neq 3$

Determine any restrictions on the value of x.

$$\cancel{(x-3)}^{1}\dfrac{(x-2)}{\cancel{(x-3)}_{1}} = 0(x-3)$$

$$x - 2 = 0$$

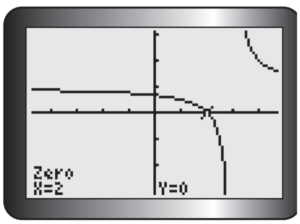
$$x = 2$$

Multiply both sides of the equation by the LCD, $(x-3)$.

Add 2 to each side.

To verify, graph $f(x) = \dfrac{x-2}{x-3}$ and use the zero operation to determine the zero.

From the equation, the graph will have a vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 1$.



The solution is $x = 2$.

b) $\dfrac{x+3}{x-4} = \dfrac{x-1}{x+2}, x \neq -2, 4$

Note the restrictions.

$$(x-4)(x+2)\left(\frac{x+3}{x-4}\right) = (x-4)(x+2)\left(\frac{x-1}{x+2}\right)$$

Multiply each side of the equation by the LCD, $(x-4)(x+2)$.

$$\overset{1}{\cancel{(x-4)}}(x+2)\left(\frac{x+3}{\underset{1}{\cancel{x-4}}}\right) = (x-4)\overset{1}{\cancel{(x+2)}}\left(\frac{x-1}{\underset{1}{\cancel{x+2}}}\right)$$

Simplify.

$$(x+2)(x+3) = (x-4)(x-1)$$
$$x^2 + 5x + 6 = x^2 - 5x + 4$$

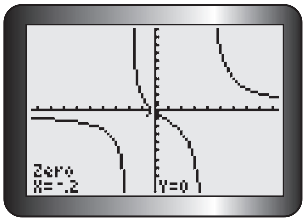
Expand. Subtract x^2, and add $5x$ to both sides.

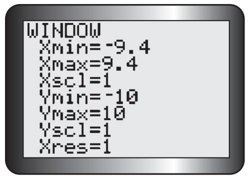
$$10x + 6 = 4$$
$$10x = -2$$
$$x = -0.2$$

Solve the resulting linear equation.

To verify, graph $f(x) = \dfrac{x+3}{x-4} - \dfrac{x-1}{x+2}$ and use the zero operation to determine the zero.

Adjust the window settings so you can view enough of the graph to see all the possible zeros.





The solution is $x = -0.2$.

EXAMPLE 3 Connecting the solution to a problem with the zeros of a rational function

Salt water is flowing into a large tank that contains pure water. The concentration of salt, c, in the tank at t minutes is given by $c(t) = \dfrac{10t}{25+t}$, where c is measured in grams per litre. When does the salt concentration in the tank reach 3.75 g/L?

Solution

If the salt concentration is 3.75, $c(t) = 3.75$.

$$\frac{10t}{25+t} = 3.75$$

$$(25+t)\left(\frac{10t}{25+t}\right) = 3.75(25+t)$$

$$\overset{1}{\cancel{(25+t)}}\frac{10t}{\underset{1}{\cancel{(25+t)}}} = 3.75(25+t)$$

$$10t = 93.75 + 3.75t$$

Set the function expression equal to 3.75. $25 + t \neq 0$, and because t measures the time since the salt water started flowing, $t \geq 0$.

Multiply both sides of the equation by the LCD, $(25+t)$, and solve the resulting linear equation.

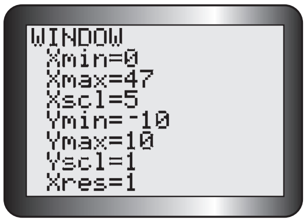
$$10t - 3.75t = 93.75$$
$$6.25t = 93.75$$
$$\frac{6.25t}{6.25} = \frac{93.75}{6.25}$$
$$t = 15$$

Use inverse operations to solve for t.

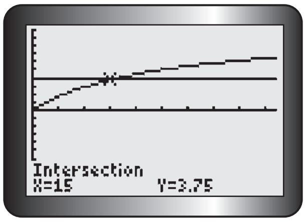
It takes 15 min for the salt concentration to reach 3.75 g/L.

To verify, graph $f(t) = \frac{10t}{25 + t}$ and $g(t) = 3.75$, and determine where the functions intersect.

Use an appropriate window setting, based on the domain, $t \geq 0$.



Use the intersect operation.



The salt concentration reaches 3.75 g/L after 15 min.

Tech ***Support***

For help determining the point of intersection between two functions, see Technical Appendix, T-12.

EXAMPLE 4 Using a rational function to model and solve a problem

Rima bought a case of concert T-shirts for \$450. She kept two T-shirts for herself and sold the rest for \$560, making a profit of \$10 on each T-shirt. How many T-shirts were in the case?

Solution

Let the number of T-shirts in the case be x.

Buying price per T-shirt $= \frac{450}{x}$

Selling price per T-shirt $= \frac{560}{x - 2}$

Rima paid \$450 for x T-shirts, so each T-shirt cost her $\$\frac{450}{x}$.

She kept two for herself, which left $x - 2$ T-shirts for her to sell.

Rima sold $x - 2$ T-shirts for \$560, so she charged $\$\frac{560}{x - 2}$ for each one.

$$\frac{560}{x-2} - \frac{450}{x} = 10$$

She made a profit of \$10 on each T-shirt, so the difference between the selling price and the buying price was \$10.

$$x(x-2)\left(\frac{560}{x-2} - \frac{450}{x}\right) = 10x(x-2)$$

Multiply both sides of the equation by the LCD, $x(x-2)$.

$$\frac{560x\cancel{(x-2)}^{1}}{\cancel{x-2}_{1}} - \frac{450\cancel{x}^{1}(x-2)}{\cancel{x}_{1}} = 10x(x-2)$$

$$\begin{aligned} 560x - 450(x-2) &= 10x(x-2) \\ 560x - 450x + 900 &= 10x^2 - 20x \\ 0 &= 10x^2 - 130x - 900 \end{aligned}$$

Expand and collect all terms to one side of the equation.

$$\begin{aligned} 0 &= 10(x^2 - 13x - 90) \\ 0 &= 10(x-18)(x+5) \\ x &= 18 \text{ or } -5 \end{aligned}$$

Solve the resulting quadratic equation by factoring.

-5 is inadmissible since $x \geq 0$.
There were 18 T-shirts in the case.

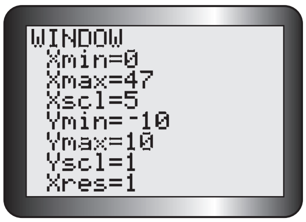
You cannot have a negative number of T-shirts in the case.

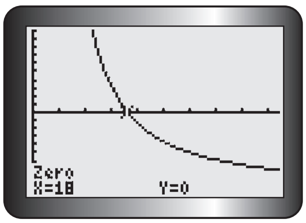
To verify, graph $f(x) = \frac{560}{x-2} - \frac{450}{x} - 10$ and determine the zeros using the zero operation.

If $\frac{560}{x-2} - \frac{450}{x} = 10$, then $\frac{560}{x-2} - \frac{450}{x} - 10 = 0$.

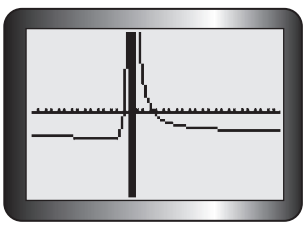
Zeros for $f(x)$ are possible solutions to the problem.

Use an appropriate window setting, based on the domain, $x \geq 0$.





The zero occurs when $x = 18$.
Zoom out to check that there are no other zeros in the domain.



The other zero is for a negative value of x, which is inadmissible in the context of this problem.

There is no other zero in the domain.
There were 18 T-shirts in the case.

In Summary

Key Ideas

- You can solve a rational equation algebraically by multiplying each term in the equation by the lowest common denominator and solving the resulting polynomial equation.
- The root of the equation $\frac{ax + b}{cx + d} = 0$ is the zero (x-intercept) of the function $f(x) = \frac{ax + b}{cx + d}$.
- You can use graphing technology to solve a rational equation or verify the solution. Determine the zeros of the corresponding rational function, or determine the intersection of two functions.

Need to Know

- The zeros of a rational function are the zeros of the function in the numerator.
- Reciprocal functions do not have zeros. All functions of the form $f(x) = \frac{1}{g(x)}$ have the x-axis as a horizontal asymptote. They do not intersect the x-axis.
- When solving contextual problems, it is important to check for inadmissible solutions that are outside the domain determined by the context.
- When using a graphing calculator to determine a zero or intersection point, you can avoid inadmissible roots by matching the window settings to the domain of the function in the context of the problem.

CHECK Your Understanding

1. Are $x = 3$ and $x = -2$ solutions to the equation $\frac{2}{x} = \frac{x - 1}{3}$? Explain how you know.

2. Solve each equation algebraically. Then verify your solution using graphing technology.

a) $\frac{x + 3}{x - 1} = 0$ **c)** $\frac{x + 3}{x - 1} = 2x + 1$

b) $\frac{x + 3}{x - 1} = 2$ **d)** $\frac{3}{3x + 2} = \frac{6}{5x}$

3. For each rational equation, write a function whose zeros are the solutions.

a) $\frac{x - 3}{x + 3} = 2$ **c)** $\frac{x - 1}{x} = \frac{x + 1}{x + 3}$

b) $\frac{3x - 1}{x} = \frac{5}{2}$ **d)** $\frac{x - 2}{x + 3} = \frac{x - 4}{x + 5}$

4. Solve each equation in question 3 algebraically, and verify your solution using a graphing calculator.

PRACTISING

5. Solve each equation algebraically.

a) $\frac{2}{x} + \frac{5}{3} = \frac{7}{x}$

b) $\frac{10}{x + 3} + \frac{10}{3} = 6$

c) $\frac{2x}{x - 3} = 1 - \frac{6}{x - 3}$

d) $\frac{2}{x + 1} + \frac{1}{x + 1} = 3$

e) $\frac{2}{2x + 1} = \frac{5}{4 - x}$

f) $\frac{5}{x - 2} = \frac{4}{x + 3}$

6. Solve each equation algebraically.

a) $\frac{2x}{2x + 1} = \frac{5}{4 - x}$

b) $\frac{3}{x} + \frac{4}{x + 1} = 2$

c) $\frac{2x}{5} = \frac{x^2 - 5x}{5x}$

d) $x + \frac{x}{x - 2} = 0$

e) $\frac{1}{x + 2} + \frac{24}{x + 3} = 13$

f) $\frac{-2}{x - 1} = \frac{x - 8}{x + 1}$

7. Solve each equation using graphing technology. Round your answers to two decimal places, if necessary.

a) $\frac{2}{x + 2} = \frac{3}{x + 6}$

b) $\frac{2x - 5}{x + 10} = \frac{1}{x - 6}$

c) $\frac{1}{x - 3} = \frac{x + 2}{7x + 14}$

d) $\frac{1}{x} - \frac{1}{45} = \frac{1}{2x - 3}$

e) $\frac{2x + 3}{3x - 1} = \frac{x + 2}{4}$

f) $\frac{1}{x} = \frac{2}{x} + 1 + \frac{1}{1 - x}$

8. K **a)** Use algebra to solve $\frac{x + 1}{x - 2} = \frac{x + 3}{x - 4}$. Explain your steps.

b) Verify your answer in part a) using substitution.

c) Verify your answer in part a) using a graphing calculator.

9. The Greek mathematician Pythagoras is credited with the discovery of the Golden Rectangle. This is considered to be the rectangle with the dimensions that are the most visually appealing. In a Golden Rectangle, the length and width are related by the proportion $\frac{l}{w} = \frac{w}{l - w}$. A billboard with a length of 15 m is going to be built. What must its width be to form a Golden Rectangle?

10. The Turtledove Chocolate factory has two chocolate machines. Machine A takes s minutes to fill a case with chocolates, and machine B takes $s + 10$ minutes to fill a case. Working together, the two machines take 15 min to fill a case. Approximately how long does each machine take to fill a case?

11. Tayla purchased a large box of comic books for \$300. She gave 15 of the comic books to her brother and then sold the rest on an Internet website for \$330, making a profit of \$1.50 on each one. How many comic books were in the box? What was the original price of each comic book?

12. **A** Polluted water flows into a pond. The concentration of pollutant, c, in the pond at time t minutes is modelled by the equation $c(t) = 9 - 90\,000\left(\frac{1}{10\,000 + 3t}\right)$, where c is measured in kilograms per cubic metre.

a) When will the concentration of pollutant in the pond reach $6\ \text{kg/m}^3$?

b) What will happen to the concentration of pollutant over time?

13. **T** Three employees work at a shipping warehouse. Tom can fill an order in s minutes. Paco can fill an order in $s - 2$ minutes. Carl can fill an order in $s + 1$ minutes. When Tom and Paco work together, they take about 1 minute and 20 seconds to fill an order. When Paco and Carl work together, they take about 1 minute and 30 seconds to fill an order.

a) How long does each person take to fill an order?

b) How long would all three of them, working together, take to fill an order?

14. **C** Compare and contrast the different methods you can use to solve a rational equation. Make a list of the advantages and disadvantages of each method.

Extending

15. Solve $\frac{x^2 - 6x + 5}{x^2 - 2x - 3} = \frac{2 - 3x}{x^2 + 3x + 3}$ correct to two decimal places.

16. Objects A and B move along a straight line. Their positions, s, with respect to an origin, at t seconds, are modelled by the following functions:

Object A: $s(t) = \dfrac{7t}{t^2 + 1}$

Object B: $s(t) = t + \dfrac{5}{t + 2}$

a) When are the objects at the same position?

b) When is object A closer to the origin than object B?