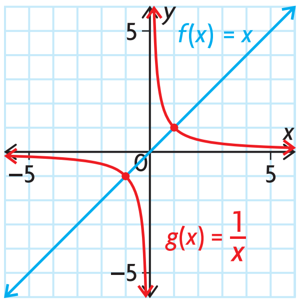
5.1 Graphs of Reciprocal Functions

YOU WILL NEED

- graph paper
- coloured pencils or pens
- graphing calculator or graphing software



GOAL

Sketch the graphs of reciprocals of linear and quadratic functions.

INVESTIGATE the Math

Owen has noted some connections between the graphs of $f(x) = x$ and its reciprocal function $g(x) = \frac{1}{x}$.

- Both graphs are in the same quadrants for the same x-values.
- When $f(x) = 0$, there is a vertical asymptote for $g(x)$.
- $f(x)$ is always increasing, and $g(x)$ is always decreasing.

? How are the graphs of a function and its reciprocal function related?

A. Explain why the graphs of $f(x) = x$ and $g(x) = \frac{1}{x}$ are in the same quadrants over the same intervals. Does this relationship hold for $m(x) = -x$ and $n(x) = -\frac{1}{x}$? Does this relationship hold for any function and its reciprocal function? Explain.

B. What graphical characteristic in the reciprocal function do the zeros of the original function correspond to? Explain.

C. Explain why the reciprocal function $g(x) = \frac{1}{x}$ is decreasing when $f(x) = x$ is increasing. Does this relationship hold for $n(x) = -\frac{1}{x}$ and $m(x) = -x$? Explain how the increasing and decreasing intervals of a function and its reciprocal are related.

D. What are the y-coordinates of the points where $f(x)$ and $g(x)$ intersect? Will the points of intersection for any function and its reciprocal always have the same y-coordinates? Explain.

E. Explain why the graph of $g(x)$ has a horizontal asymptote. What is the equation of this asymptote? Will all reciprocal functions have the same horizontal asymptote? Explain.

F. On graph paper, draw the graph of $p(x) = x^2 - 4$. In a table like the one below, note the characteristics of the graph of $p(x)$ and use this information to help you determine the characteristics of the reciprocal function $q(x) = \frac{1}{x^2 - 4}$.

Characteristics	$p(x) = x^2 - 4$	$q(x) = \frac{1}{x^2 - 4}$
zeros and/or vertical asymptotes		
interval(s) on which the graph is above the x-axis (all values of the function are positive)		
interval(s) on which the graph is below the x-axis (all values of the function are negative)		
interval(s) on which the function is increasing		
interval(s) on which the function is decreasing		
point(s) where the y-value is 1		
point(s) where the y-value is -1		

G. On the same graph, draw the vertical asymptotes for the reciprocal function. Then use the rest of the information determined in part F to draw the graph for $q(x) = \frac{1}{x^2 - 4}$.

H. Verify your graphs by entering $p(x)$ and $q(x)$ in a graphing calculator using the "friendly" window setting shown.

I. Repeat parts F to H for the following pairs of functions.

a) $p(x) = x + 2$ and $q(x) = \frac{1}{x + 2}$

b) $p(x) = 2x - 3$ and $q(x) = \frac{1}{2x - 3}$

c) $p(x) = (x - 2)(x + 3)$ and $q(x) = \frac{1}{(x - 2)(x + 3)}$

d) $p(x) = (x - 1)^2$ and $q(x) = \frac{1}{(x - 1)^2}$

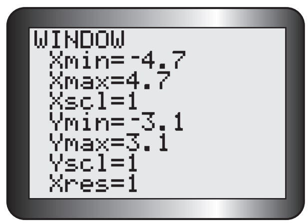
J. Write a summary of the relationships between the characteristics of the graphs of

a) a linear function and its reciprocal function

b) a quadratic function and its reciprocal function

Tech *Support*

On a graphing calculator, the length of the display screen contains 94 pixels, and the width contains 62 pixels. When the domain, *X*max-*X*min, is cleanly divisible by 94, and the range, *Y*max-*Y*min, is cleanly divisible by 62, the window is friendly. This means that you can trace without using "ugly" decimals. A friendly window is useful when working with rational functions.



Use brackets when entering reciprocal functions in the $Y=$ editor of a graphing calculator. For example, to graph the function $f(x) = \frac{1}{x^2 - 4}$, enter $Y1 = \frac{1}{(x^2 - 4)}$.

Reflecting

K. How did knowing the positive/negative intervals and the increasing/decreasing intervals for $p(x) = x^2 - 4$ help you draw the graph for $p(x) = \frac{1}{x^2 - 4}$?

L. Why are some numbers in the domain of a function excluded from the domain of its reciprocal function? What graphical characteristic of the reciprocal function occurs at these values?

M. What common characteristics are shared by all reciprocals of linear and quadratic functions?

APPLY the Math

EXAMPLE 1 Connecting the characteristics of a linear function to its corresponding reciprocal function

Given the function $f(x) = 2 - x$,

a) determine the domain and range, intercepts, positive/negative intervals, and increasing/decreasing intervals

b) use your answers for part a) to sketch the graph of the reciprocal function

Solution

a) $f(x) = 2 - x$ is a linear function.

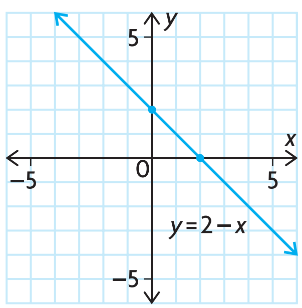
$D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R}\}$

The domain and range of most linear functions are the set of real numbers.

From the equation, the y-intercept is 2.
$f(x) = 0$ when $0 = 2 - x$
$x = 2$

The x-intercept is 2.

A linear function $f(x) = mx + b$ has y-intercept b.
The x-intercept occurs where $f(x) = 0$.



Sketch the graph of $f(x)$ to determine the positive and negative intervals. The line $y = 2 - x$ is above the x-axis for all x-values less than 2 and below the x-axis for all x-values greater than 2.

$f(x)$ is positive when $x \in (-\infty, 2)$ and negative when $x \in (2, \infty)$.
$f(x)$ is decreasing when $x \in (-\infty, \infty)$.

This is a linear function with a negative slope, so it is decreasing over its entire domain.

b) The reciprocal function is $g(x) = \frac{1}{2 - x}$.

All the y-values of points on the reciprocal function are reciprocals of the y-values on the original function.

$D = \{x \in \mathbf{R} | x \neq 2\}$

$R = \{y \in \mathbf{R} | y \neq 0\}$

The y-intercept is 0.5.

The vertical asymptote is $x = 2$ and the horizontal asymptote is $y = 0$.

There is a vertical asymptote at the zero of the original function. The reciprocals of a linear function always have the x-axis as a horizontal asymptote.

The reciprocal function is positive when $x \in (-\infty, 2)$ and negative when $x \in (2, \infty)$.

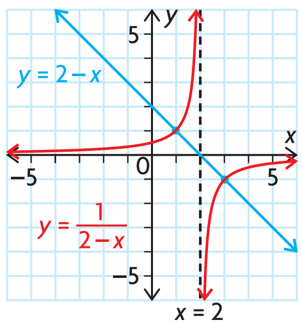
The positive/negative intervals are always the same for both functions.

It is increasing when $x \in (-\infty, 2)$ and when $x \in (2, \infty)$.

Because the original function is always decreasing, the reciprocal function is always increasing.

The graph of $g(x) = \frac{1}{2 - x}$ intersects the graph of $g(x) = 2 - x$ at $(1, 1)$ and $(3, -1)$.

The reciprocal of 1 is 1, and the reciprocal of -1 is -1. Thus, the two graphs intersect at any points with these y-values.



Use all this information to sketch the graph of the reciprocal function.

EXAMPLE 2 Connecting the characteristics of a quadratic function to its corresponding reciprocal function

Given the function $f(x) = 9 - x^2$

a) determine the domain and range, intercepts, positive/negative intervals, and increasing/decreasing intervals

b) use your answers for part a) to sketch the graph of the reciprocal function

Solution

a) $f(x) = 9 - x^2$ is a quadratic function.

$D = \{x \in \mathbf{R}\}$

The domain of a quadratic function is the set of real numbers.

$f(0) = 9$, so the y-intercept is 9.

$R = \{y \in \mathbf{R} | y \leq 9\}$

The graph of $f(x)$ is a parabola that opens down. The vertex is at (0, 9), so $y \leq 9$.

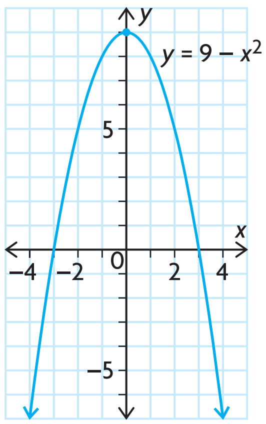
$f(x) = 0 \Rightarrow 9 - x^2 = 0$

$(3 - x)(3 + x) = 0$

$x = \pm 3$

Factor and determine the x-intercepts.

The x-intercepts are -3 and 3.



The parabola $y = 9 - x^2$ is above the x-axis for x-values between -3 and 3. The graph is below the x-axis for x-values less than -3 and for x-values greater than 3.

$f(x)$ is positive when $x \in (-3, 3)$ and negative when $x \in (-\infty, -3)$ and when $x \in (3, \infty)$.

$f(x)$ is increasing when $x \in (-\infty, 0)$ and decreasing when $x \in (0, \infty)$.

The y-values increase as x increases from $-\infty$ to 0. The y-values decrease as x increases from 0 to ∞.

b) The reciprocal function is $g(x) = \frac{1}{9 - x^2}$.

The vertical asymptotes are $x = -3$ and $x = 3$.

$D = \{x \in \mathbf{R} \mid x \neq \pm 3\}$

Vertical asymptotes occur at each zero of the original function, so these numbers must be excluded from the domain.

The horizontal asymptote is $y = 0$.

The y-intercept is $\frac{1}{9}$.

The reciprocals of all quadratic functions have the x-axis as a horizontal asymptote. The y-intercept of the original function is 9, so the y-intercept of the reciprocal function is $\frac{1}{9}$.

There is a local minimum value at $\left(0, \frac{1}{9}\right)$.

$R = \left\{y \in \mathbf{R} \mid y < 0 \text{ or } y \geq \frac{1}{9}\right\}$

When the original function has a local maximum point, the reciprocal function has a corresponding local minimum point.

The reciprocal function is positive when $x \in (-3, 3)$ and negative when $x \in (-\infty, -3)$ and when $x \in (3, \infty)$. It is decreasing when $x \in (-\infty, -3)$ and when $x \in (-3, 0)$, and increasing when $x \in (0, 3)$ and when $x \in (3, \infty)$.

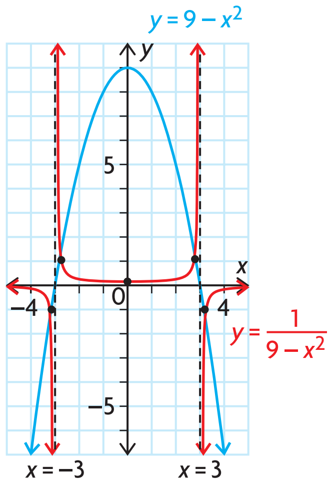
The positive/negative intervals are always the same for both functions. Where the original function is decreasing, excluding the zeros, the reciprocal function is increasing (and vice versa).

$$f(x) = 1 \text{ when } 9 - x^2 = 1 \qquad \text{and} \qquad f(x) = -1 \text{ when } 9 - x^2 = -1$$

$$\begin{aligned} -x^2 &= 1 - 9 \\ -x^2 &= -8 \\ x^2 &= 8 \\ x &= \pm 2\sqrt{2} \end{aligned} \qquad\qquad \begin{aligned} -x^2 &= -1 - 9 \\ -x^2 &= -10 \\ x^2 &= 10 \\ x &= \pm \sqrt{10} \end{aligned}$$

A function and its reciprocal intersect at points where $y = \pm 1$. Solve the corresponding equations to determine the x-coordinates of the points of intersection.

The graph of $g(x) = \frac{1}{9 - x^2}$ intersects the graph of $f(x) = 9 - x^2$ at $(-2\sqrt{2}, 1)$, $(2\sqrt{2}, 1)$ and at $(-\sqrt{10}, -1)$, $(\sqrt{10}, -1)$.



Use all this information to sketch the graph of the reciprocal function.

In Summary

Key Idea

- You can use key characteristics of the graph of a linear or quadratic function to graph the related reciprocal function.

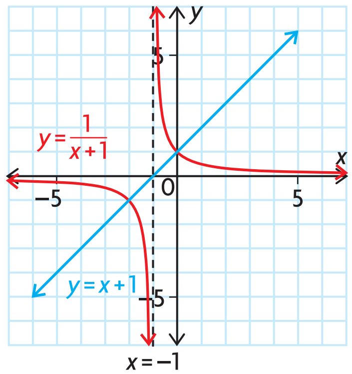
Need to Know

- All the y-coordinates of a reciprocal function are the reciprocals of the y-coordinates of the original function.
- The graph of a reciprocal function has a vertical asymptote at each zero of the original function.
- A reciprocal function will always have $y = 0$ as a horizontal asymptote if the original function is linear or quadratic.
- A reciprocal function has the same positive/negative intervals as the original function.

(continued)

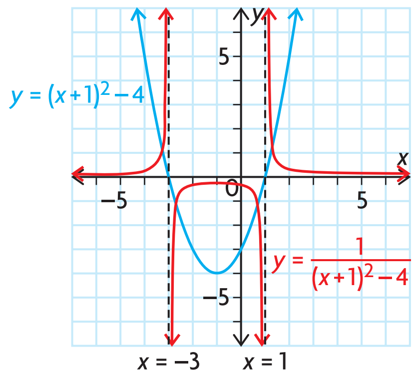
- Intervals of increase on the original function are intervals of decrease on the reciprocal function. Intervals of decrease on the original function are intervals of increase on the reciprocal function.
- If the range of the original function includes 1 and/or -1, the reciprocal function will intersect the original function at a point (or points) where the y-coordinate is 1 or -1.
- If the original function has a local minimum point, the reciprocal function will have a local maximum point at the same x-value (and vice versa).

A linear function and its reciprocal



Both functions are negative when $x \in (-\infty, -1)$ and positive when $x \in (-1, \infty)$. The original function is increasing when $x \in (-\infty, \infty)$. The reciprocal function is decreasing when $x \in (-\infty, -1)$ or $(-1, \infty)$.

A quadratic function and its reciprocal

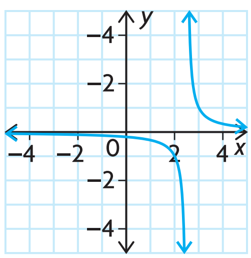


Both functions are negative when $x \in (-3, 1)$ and positive when $x \in (-\infty, -3)$ or $(1, \infty)$. The original function is decreasing when $x \in (-\infty, -1)$ and increasing when $x \in (-1, \infty)$. The reciprocal function is increasing when $x \in (-\infty, -3)$ or $(-3, -1)$ and decreasing when $x \in (-1, 1)$ or $(1, \infty)$.

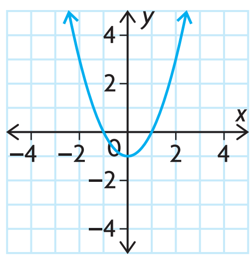
CHECK Your Understanding

1. Match each function with its equation on the next page. Then identify which function pairs are reciprocals.

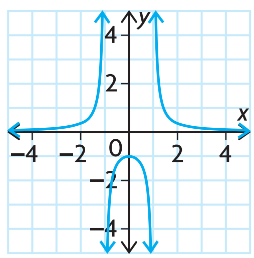
a)



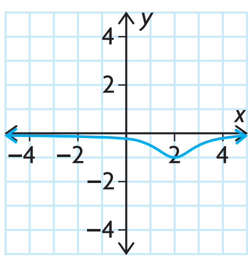
c)



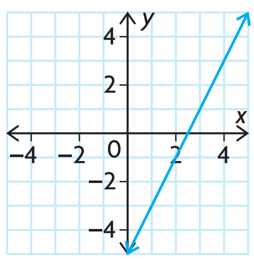
e)



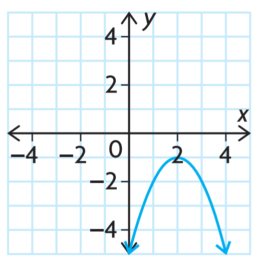
b)



d)



f)



A $y = \dfrac{1}{-(x-2)^2 - 1}$

B $y = \dfrac{1}{x^2 - 1}$

C $y = \dfrac{1}{2x - 5}$

D $y = x^2 - 1$

E $y = -(x-2)^2 - 1$

F $y = 2x - 5$

2. For each pair of functions, determine where the zeros of the original function occur and state the equations of the vertical asymptotes of the reciprocal function, if possible.

a) $f(x) = x - 6,\ g(x) = \dfrac{1}{x-6}$

b) $f(x) = 3x + 4,\ g(x) = \dfrac{1}{3x+4}$

c) $f(x) = x^2 - 2x - 15,\ g(x) = \dfrac{1}{x^2 - 2x - 15}$

d) $f(x) = 4x^2 - 25,\ g(x) = \dfrac{1}{4x^2 - 25}$

e) $f(x) = x^2 + 4,\ g(x) = \dfrac{1}{x^2 + 4}$

f) $f(x) = 2x^2 + 5x + 3,\ g(x) = \dfrac{1}{2x^2 + 5x + 3}$

3. Sketch the graph of each function. Use your graph to help you sketch the graph of the reciprocal function.

a) $f(x) = 5 - x$ **b)** $f(x) = x^2 - 6x$

PRACTISING

4. a) Copy and complete the following table.

x	−4	−3	−2	−1	0	1	2	3	4	5	6	7
$f(x)$	16	14	12	10	8	6	4	2	0	−2	−4	−6
$\dfrac{1}{f(x)}$												

b) Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

c) Find equations for $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

5. State the equation of the reciprocal of each function, and determine the equations of the vertical asymptotes of the reciprocal. Verify your results using graphing technology.

a) $f(x) = 2x$

b) $f(x) = x + 5$

c) $f(x) = x - 4$

d) $f(x) = 2x + 5$

e) $f(x) = -3x + 6$

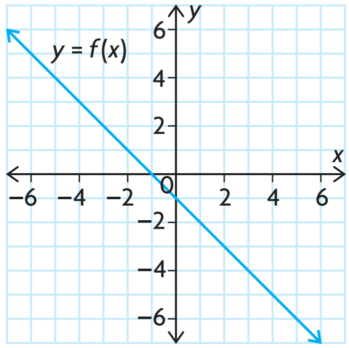
f) $f(x) = (x-3)^2$

g) $f(x) = x^2 - 3x - 10$

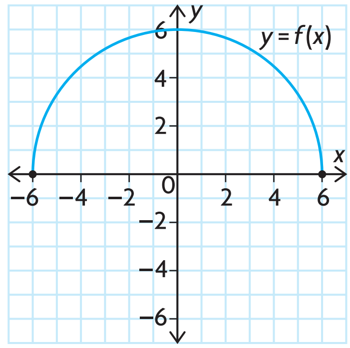
h) $f(x) = 3x^2 - 4x - 4$

6. Sketch the graph of the reciprocal of each function.

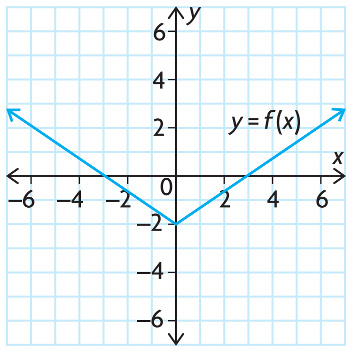
a)



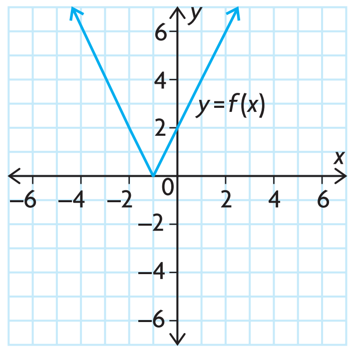
c)



b)



d)



7. Sketch each pair of graphs on the same axes. State the domain and range of each reciprocal function.

a) $y = 2x - 5, y = \dfrac{1}{2x - 5}$ **b)** $y = 3x + 4, y = \dfrac{1}{3x + 4}$

8. Draw the graph of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same axes.

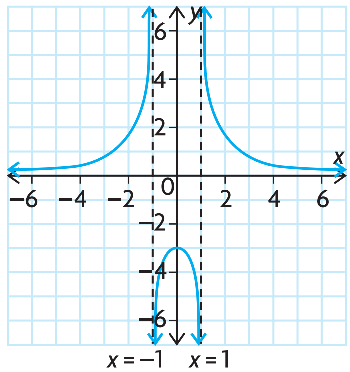
a) $f(x) = x^2 - 4$
b) $f(x) = (x - 2)^2 - 3$
c) $f(x) = x^2 - 3x + 2$
d) $f(x) = (x + 3)^2$
e) $f(x) = x^2 + 2$
f) $f(x) = -(x + 4)^2 + 1$

9. **K** For each function, determine the domain and range, intercepts, positive/negative intervals, and increasing/decreasing intervals. State the equation of the reciprocal function. Then sketch the graphs of the original and reciprocal functions on the same axes.

a) $f(x) = 2x + 8$
b) $f(x) = -4x - 3$
c) $f(x) = x^2 - x - 12$
d) $f(x) = -2x^2 + 10x - 12$

10. Why do the graphs of reciprocals of linear functions always have vertical asymptotes, but the graphs of reciprocals of quadratic functions sometimes do not? Provide sketches of three different reciprocal functions to illustrate your answer.

11. An equation of the form $y = \frac{k}{x^2 + bx + c}$ has a graph that closely matches the graph shown. Find the equation. Check your answer using graphing technology.



12. **A** A chemical company is testing the effectiveness of a new cleaning solution for killing bacteria. The test involves introducing the solution into a sample that contains approximately 10 000 bacteria. The number of bacteria remaining, $b(t)$, over time, t, in seconds is given by the equation $b(t) = 10\,000\frac{1}{t}$.

a) How many bacteria will be left after 20 s?
b) After how many seconds will only 5000 bacteria be left?
c) After how many seconds will only one bacterium be left?
d) This model is not always accurate. Determine what sort of inaccuracies this model might have. Assume that the solution was introduced at $t = 0$.
e) Based on these inaccuracies, what should the domain and range of the equation be?

13. **T** Use your graphing calculator to explore and then describe the key characteristics of the family of reciprocal functions of the form $g(x) = \frac{1}{x + n}$. Make sure that you include graphs to support your descriptions.

a) State the domain and range of $g(x)$.
b) For the family of functions $f(x) = x + n$, the y-intercept changes as the value of n changes. Describe how the y-intercept changes and how this affects $g(x)$.
c) If graphed, at what point would the two graphs $f(x)$ and $g(x)$ intersect?

14. **C** Due to a basketball tournament, your friend has missed this class. Write a concise explanation of the steps needed to graph a reciprocal function using the graph of the original function (without using graphing technology). Use an example, and explain the reason for each step.

Extending

15. Sketch the graphs of the following reciprocal functions.

a) $y = \frac{1}{\sqrt{x}}$ **c)** $y = \frac{1}{2^x}$

b) $y = \frac{1}{x^3}$ **d)** $y = \frac{1}{\sin x}$

16. Determine the equation of the function in the graph shown.