5.5 Solving Rational Inequalities

YOU WILL NEED

- graphing calculator

GOAL

Solve rational inequalities using algebraic and graphical approaches.

rational inequality

a statement that one rational expression is less than or greater than another rational expression $\left(\text{e.g., } \frac{2x}{x+3} > \frac{x-1}{5x}\right)$

LEARN ABOUT the Math

The function $P(t) = \frac{20t}{t+1}$ models the population, in thousands, of Nickelford, t years after 1997. The population, in thousands, of nearby New Ironfield is modelled by $Q(t) = \frac{240}{t+8}$.

? How can you determine the time period when the population of New Ironfield exceeded the population of Nickelford?

EXAMPLE 1 Selecting a strategy to solve a problem

Determine the interval(s) of t where the values of $Q(t)$ are greater than the values of $P(t)$.

Solution A: Using an algebraic strategy to solve an inequality

$$\frac{240}{t+8} > \frac{20t}{t+1}$$

The population of New Ironfield exceeds the population of Nickelford when $Q(t) > P(t)$.
$t \geq 0$ in the context of this problem. There are no other restrictions on the expressions in the rational inequality since the values that make both expressions undefined are negative numbers.

$$(t+8)(t+1)\left(\frac{240}{t+8}\right) > (t+8)(t+1)\left(\frac{20t}{t+1}\right)$$

$$\overset{1}{\cancel{(t+8)}}(t+1)\left(\frac{240}{\underset{1}{\cancel{t+8}}}\right) > (t+8)\overset{1}{\cancel{(t+1)}}\left(\frac{20t}{\underset{1}{\cancel{t+1}}}\right)$$

Multiply both sides of the inequality by the LCD. The value of the LCD is always positive, since $t \geq 0$, so the inequality sign is unchanged.

$$\begin{aligned}
240(t+1) &> 20t(t+8) \\
240t + 240 &> 20t^2 + 160t \\
0 &> 20t^2 + 160t - 240t - 240 \\
0 &> 20t^2 - 80t - 240 \\
0 &> 20(t^2 - 4t - 12) \\
0 &> 20(t-6)(t+2)
\end{aligned}$$

Expand and simplify both sides. Then subtract $240t$ and 240 from both sides.

Factor the resulting quadratic expression.

Examine the sign of the factored polynomial expression on the right side of the inequality.

	$t < -2$	$-2 < t < 6$	$t > 6$
$20(t-6)$	$-$	$-$	$+$
$t+2$	$-$	$+$	$+$
$20(t-6)(t+2)$	$(-)(-) = +$	$(-)(+) = -$	$(+)(+) = +$

The inequality $0 > 20(t-6)(t+2)$ is true when the expression on the right side is negative. The sign of the factored quadratic expression changes when $t = -2$ and when $t = 6$, because the expression is zero at these values. Use a table to determine when the sign of the expression is negative on each side of these values.

The inequality $0 > 20(t-6)(t+2)$ is true when $-2 < t < 6$.

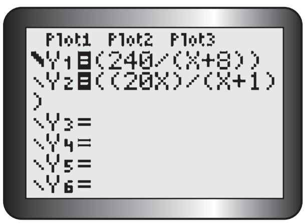
The population of New Ironfield exceeded the population of Nickelford for six years after 1997, until 2003.

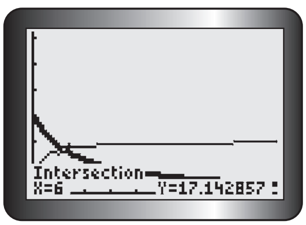
Since the domain is $t \geq 0$, however, numbers that are negative cannot be included. Therefore, the solution is $0 \leq t < 6$.

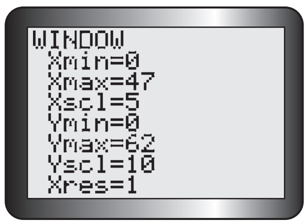
Solution B: Solving a rational inequality by graphing two rational functions

To solve $Q(t) > P(t)$, graph $Q(t) = \frac{240}{t+8}$ and $P(t) = \frac{20t}{t+1}$ using graphing technology, and determine the value of t at the intersection point(s).

It helps to bold the graph of $Q(t)$ so you can remember which graph is which. Use window settings that reflect the domain of the functions.







There is only one intersection within the domain of the functions.

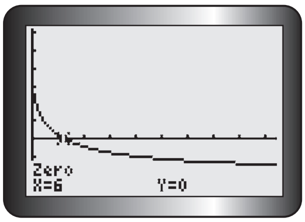
From the graphs, $Q(t) > P(t)$ for $0 \leq t < 6$. The population of New Ironfield exceeded the population of Nickelford until 2003.

If $Q(t) > P(t)$, the graph of $Q(t)$ lies above the graph of $P(t)$. Looking at the graphs, this is true for the parts of the graph of $Q(t)$ up to the intersection point at $t = 6$. The graphs will not intersect again because each graph is approaching a different horizontal asymptote. From the defining equations, the graph of $Q(t)$ is approaching the line $Q = 0$ while the graph of $P(t)$ is approaching the line $P = 20$.

Solution C: Solving a rational inequality by determining the zeros of a combined function

When $Q(t) > P(t)$, $Q(t) - P(t) > 0$.

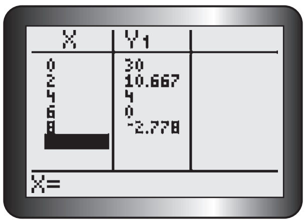
Graph $f(t) = Q(t) - P(t) = \dfrac{240}{t + 8} - \dfrac{20t}{t + 1}$ and use the zero operation to locate the zero.



Combine the two population functions into a single function, $f(t) = Q(t) - P(t)$. When $Q(t) > P(t)$, $f(t)$ will have positive values.

When a function has positive values, its graph lies above the x-axis.

The graph is above the x-axis for $0 \leq t < 6$.



By examining the values of $f(t)$ in a table, you can verify that the function continues to decrease but remains positive when $0 \leq t < 6$.

$f(t)$ has positive values for $0 \leq t < 6$.
For the six years after 1997, the population of New Ironfield exceeded the population of Nickelford.

Reflecting

A. How is the solution to an inequality different from the solution to an equation?

B. In Solution A, how was the rational inequality manipulated to obtain a simpler quadratic inequality?

C. In Solution B, how were the graphs of the related rational functions used to find the solution to an inequality?

D. In Solution C, how did creating a new function help to solve the inequality?

APPLY the Math

EXAMPLE 2 Selecting a strategy to solve an inequality that involves a linear function and a reciprocal function

Solve $x - 2 < \frac{8}{x}$.

Solution A: Using an algebraic strategy and a sign chart

$$x - 2 < \frac{8}{x},\ x \neq 0$$

Determine any restrictions on x.

Subtract $\frac{8}{x}$ from both sides.

$$x - 2 - \frac{8}{x} < 0$$

$$\frac{x^2}{x} - \frac{2x}{x} - \frac{8}{x} < 0$$

x is the LCD and it can be positive or negative. Multiplying both sides by x would require that two cases be considered, since the inequality sign must be reversed when multiplying by a negative. The alternative is to create an expression with a common denominator, x.

$$\frac{x^2 - 2x - 8}{x} < 0$$

Combine the terms to create a single rational expression.

$$\frac{(x - 4)(x + 2)}{x} < 0$$

Factor the numerator.

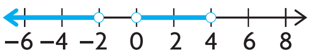
Examine the sign of the rational expression.

	$x < -2$	$-2 < x < 0$	$0 < x < 4$	$x > 4$
$x - 4$	$-$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$\dfrac{(x-4)(x+2)}{x}$	$\dfrac{(-)(-)}{-} = -$	$\dfrac{(-)(+)}{-} = +$	$\dfrac{(-)(+)}{+} = -$	$\dfrac{(+)(+)}{+} = +$

The sign of a rational expression changes each time the sign of one of its factors changes. Choose a test value in each interval to determine the sign of each part of the expression. Then determine the intervals where the overall expression is negative.

The overall expression is negative when $x < -2$ or when $0 < x < 4$.

The inequality is true when $x \in (-\infty, -2)$ or $x \in (0, 4)$.



Write the solution in interval or set notation, and draw the solution set on a number line.

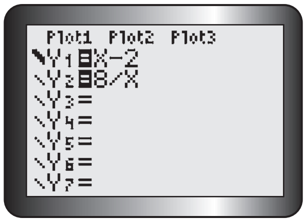
Solution B: Using graphing technology

$x - 2 < \dfrac{8}{x}, x \neq 0$

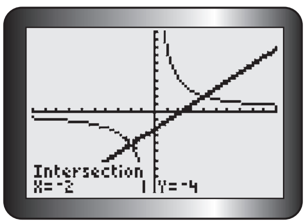
Let $f(x) = x - 2$ and $g(x) = \dfrac{8}{x}$.

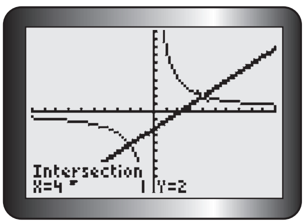
The solution set for the inequality will be all x-values for which $f(x) < g(x)$.

Write each side of the inequality as its own function. Enter both functions in the equation editor, using a bold line for $f(x)$.



Graph $f(x)$ and $g(x)$ on the same axes, and use the intersect operation to determine the intersection points.





$f(x) < g(x)$ where the bold graph of $f(x)$ lies beneath the graph of $g(x)$. Notice that the bold linear function is above the reciprocal function on the left side and close to the vertical asymptote, $x = 0$. It is below the reciprocal function on the right side and close to this asymptote.

$f(x) < g(x)$ when $x < -2$ or when $0 < x < 4$.

The solution set is $\{x \in \mathbf{R} \mid x < -2 \text{ or } 0 < x < 4\}$.

You can also use interval notation or a number line to describe the solution set, as in Solution A.

EXAMPLE 3 Determining the solution set for an inequality that involves two rational functions

Determine the solution set for the inequality $\frac{x+3}{x+1} \geq \frac{x-2}{x-3}$.

Solution A: Using algebra and a sign chart

Rewrite $\frac{x+3}{x+1} \geq \frac{x-2}{x-3}$, $x \neq -1, 3$,

as $\frac{x+3}{x+1} - \frac{x-2}{x-3} \geq 0$.

$$\frac{(x-3)(x+3)}{(x-3)(x+1)} - \frac{(x-2)(x+1)}{(x-3)(x+1)} \geq 0$$

$$\frac{x^2 - 9 - (x^2 - x - 2)}{(x-3)(x+1)} \geq 0$$

$$\frac{x^2 - 9 - x^2 + x + 2}{(x-3)(x+1)} \geq 0$$

Note the restrictions on x.

Subtract $\frac{x-2}{x-3}$ from both sides to create an inequality with zero on the right side.

Subtract the rational expressions on the left side using a common denominator.

Expand and simplify the numerator.

$$\frac{x-7}{(x-3)(x+1)} \geq 0$$

A rational expression is zero when its numerator is zero.

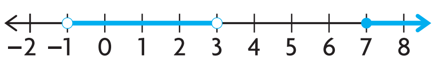
The rational expression is equal to zero when $x = 7$, so 7 is included in the solution set.

Examine the sign of the simplified rational expression on the intervals shown to determine where the rational expression is greater than zero.

	$x < -1$	$-1 < x < 3$	$3 < x < 7$	$x > 7$
$x - 7$	$-$	$-$	$-$	$+$
$x - 3$	$-$	$-$	$+$	$+$
$x + 1$	$-$	$+$	$+$	$+$
$\frac{(x-7)}{(x-3)(x+1)}$	$\frac{-}{(-)(-)} = -$	$\frac{-}{(-)(+)} = +$	$\frac{-}{(+)(+)} = -$	$\frac{+}{(+)(+)} = +$

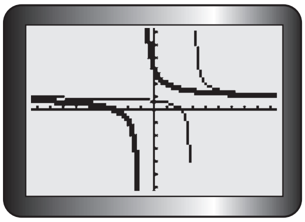
The expression is undefined at $x = -1$ and $x = 3$. It is equal to 0 at $x = 7$. These numbers create four intervals to consider. Choose a test value in each interval to determine the sign of each part of the expression. Then determine the intervals where the overall expression is positive.

The solution set is $\{x \in \mathbf{R} \mid -1 < x < 3 \text{ or } x \geq 7\}$.



Solution B: Using graphing technology

$$\frac{x+3}{x+1} \geq \frac{x-2}{x-3}, x \neq -1, 3$$

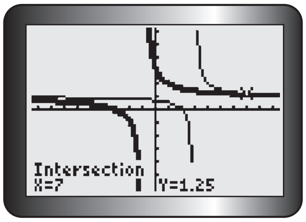


Use each side of the inequality to define a function. Graph $f(x) = \frac{x+3}{x+1}$ with a bold line and $g(x) = \frac{x-2}{x-3}$ with a regular line.

The graph of $f(x)$ has a vertical asymptote at $x = -1$. The graph for $g(x)$ has a vertical asymptote at $x = 3$. Both graphs have $y = 1$ as a horizontal asymptote.

Determine the equations of the asymptotes from the equations of the functions.

Use the intersect operation to locate any intersection points.



It looks as though the graphs might intersect on the left side of the screen, as well as on the right side. No matter how far you trace along the left branches, however, you never reach a point where the y-value is the same on both curves.

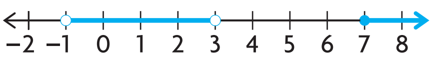
The functions are equal when $x = 7$.

$f(x) > g(x)$ between the asymptotes at $x = -1$ and $x = 3$, and for $x > 7$.

$f(x) = g(x)$ when $x = 7$.

The bold graph of $f(x)$ is above the graph of $g(x)$ between the two vertical asymptotes and then after the intersection point.

The solution set for $\frac{x+3}{x+1} \geq \frac{x-2}{x-3}$ is



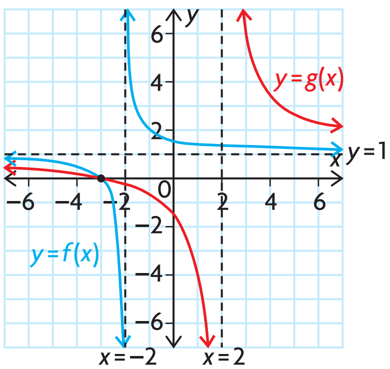
In Summary

Key Ideas

- Solving an inequality means finding all the possible values of the variable that satisfy the inequality.
- To solve a rational inequality algebraically, rearrange the inequality so that one side is zero. Combine the expressions on the no-zero side using a common denominator. Make a table to examine the sign of each factor and the sign of the entire expression on the intervals created by the zeros of the numerator and the denominator.
- Only when you are certain that each denominator is positive can you multiply both sides by the lowest common denominator to make the inequality easier to solve.
- You can always solve a rational inequality using graphing technology.

Need to Know

- When multiplying or dividing both sides of an inequality by a negative it is necessary to reverse the inequality sign to maintain equivalence.
- You can solve an inequality using graphing technology by graphing the functions on each side of the inequality sign and then identifying all the intervals created by the vertical asymptotes and points of intersection. For x-values that satisfy $f(x) > g(x)$, identify the specific intervals where the graph of $f(x)$ is above the graph of $g(x)$. For x-values that satisfy $f(x) < g(x)$, identify the specific intervals where the graph of $f(x)$ is below the graph of $g(x)$.
  Consider the following graph:



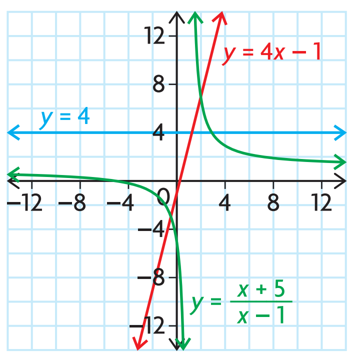
In this graph, there are four intervals to consider: $(-\infty, -3)$, $(-3, -2)$, $(-2, 2)$ and $(2, \infty)$. In these intervals, $f(x) > g(x)$ when $x \in (-\infty, -3)$ or $(-2, 2)$, and $f(x) < g(x)$ when $x \in (-3, -2)$ or $(2, \infty)$.

- You can also solve an inequality using graphing technology by creating an equivalent inequality with zero on one side and then identifying the intervals created by the zeros on the graph of the new function. Finding where the graph lies above the x-axis (where $f(x) > 0$) or below the x-axis (where $f(x) < 0$) defines the solutions to the inequality.

CHECK *Your Understanding*

1. Use the graph shown to determine the solution set for each of the following inequalities.

a) $\dfrac{x+5}{x-1} < 4$ **b)** $4x - 1 > \dfrac{x+5}{x-1}$



2. a) Show that the inequality $\dfrac{6x}{x+3} \le 4$ is equivalent to the inequality $\dfrac{2(x-6)}{(x+3)} \le 0$.

b) Sketch the solution on a number line.

c) Write the solution using interval notation.

3. a) Show that the inequality $x + 2 > \frac{15}{x}$ is equivalent to the inequality $\frac{(x+5)(x-3)}{x} > 0$.

b) Use a table to determine the positive/negative intervals for $f(x) = \frac{(x+5)(x-3)}{x}$.

c) State the solution to the inequality using both set notation and interval notation.

PRACTISING

4. Use algebra to find the solution set for each inequality. Verify your answer using graphing technology.

a) $\frac{1}{x+5} > 2$

b) $\frac{1}{2x+10} < \frac{1}{x+3}$

c) $\frac{3}{x-2} < \frac{4}{x}$

d) $\frac{7}{x-3} \geq \frac{2}{x+4}$

e) $\frac{-6}{x+1} > \frac{1}{x}$

f) $\frac{-5}{x-4} < \frac{3}{x+1}$

5. Use algebra to obtain a factorable expression from each inequality, if necessary. Then use a table to determine interval(s) in which the inequality is true.

a) $\frac{t^2 - t - 12}{t-1} < 0$

b) $\frac{t^2 + t - 6}{t-4} \geq 0$

c) $\frac{6t^2 - 5t + 1}{2t+1} > 0$

d) $t - 1 < \frac{30}{5t}$

e) $\frac{-2t - 10}{t} > t + 5$

f) $\frac{-t}{4t-1} \geq \frac{2}{t-9}$

6. Use graphing technology to solve each inequality.

a) $\frac{x+3}{x-4} \geq \frac{x-1}{x+6}$

b) $x + 5 < \frac{x}{2x+6}$

c) $\frac{x}{x+4} \leq \frac{1}{x+1}$

d) $\frac{x}{x+9} \geq \frac{1}{x+1}$

e) $\frac{x-8}{x} > 3 - x$

f) $\frac{x^2 - 16}{(x-1)^2} \geq 0$

7. K **a)** Find all the values of x that make the following inequality true: $\frac{3x-8}{2x-1} > \frac{x-4}{x+1}$

b) Graph the solution set on a number line. Write the solution set using interval notation and set notation.

8. **a)** Use an algebraic strategy to solve the inequality $\frac{-6t}{t-2} < \frac{-30}{t-2}$.
b) Graph both inequalities to verify your solution.
c) Can these rational expressions be used to model a real-world situation? Explain.

9. **A** The equation $f(t) = \frac{5t}{t^2 + 3t + 2}$ models the bacteria count, in thousands, for a sample of tap water that is left to sit over time, t, in days. The equation $g(t) = \frac{15t}{t^2 + 9}$ models the bacteria count, in thousands, for a sample of pond water that is also left to sit over several days. In both models, $t > 0$. Will the bacteria count for the tap water sample ever exceed the bacteria count for the pond water? Justify your answer.

10. Consider the inequality $0.5x - 2 < \frac{5}{2x}$.
a) Rewrite the inequality so that there is a single, simplified expression on one side and a zero on the other side.
b) List all the factors of the rational expression in a table, and determine on which intervals the inequality is true.

11. **T** An economist for a sporting goods company estimates the revenue and cost functions for the production of a new snowboard. These functions are $R(x) = -x^2 + 10x$ and $C(x) = 4x + 5$, respectively, where x is the number of snowboards produced, in thousands. The average profit is defined by the function $AP(x) = \frac{P(x)}{x}$, where $P(x)$ is the profit function. Determine the production levels that make $AP(x) > 0$.

12. **C** **a)** Explain why the inequalities $\frac{x+1}{x-1} < \frac{x+3}{x+2}$ and $\frac{x+5}{(x-1)(x+2)} < 0$ are equivalent.
b) Describe how you would use a graphing calculator to solve these inequalities.
c) Explain how you would use a table to solve these inequalities.

Extending

13. Solve $\left|\frac{x}{x-4}\right| \geq 1$.

14. Solve $\frac{1}{\sin x} < 4$, $0° \leq x \leq 360°$.

15. Solve $\frac{\cos(x)}{x} > 0.5$, $0° < x < 90°$.