

Rates of Change in Rational Functions

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Determine average rates of change, and estimate instantaneous rates of change for rational functions.

LEARN ABOUT the Math

The instantaneous rate of change at a point on a revenue function is called the *marginal revenue*. It is a measure of the estimated additional revenue from selling one more item.

For example, the demand equation for a toothbrush is $p(x) = \frac{5}{2+x}$, where x is the number of toothbrushes sold, in thousands, and p is the price, in dollars.

? What is the marginal revenue when 1500 toothbrushes are sold? When is the marginal revenue the greatest? When is it the least?

EXAMPLE 1

Selecting a strategy to determine instantaneous rates of change

Determine the marginal revenue when 1500 toothbrushes are sold and when it is the greatest and the least.

Solution A: Calculating the average rate of change by squeezing centred intervals around $x = 1.5$

$$\begin{aligned} \text{Revenue } R(x) &= xp(x) \\ &= \frac{5x}{2+x} \end{aligned} \quad \leftarrow \text{Revenue = Number of items sold} \times \text{Price}$$

The average rate of change close to $x = 1.5$ is shown in the following table.

Centred Intervals	Average Rate of Change $\frac{R(x_2) - R(x_1)}{x_2 - x_1}$
$1.4 \leq x \leq 1.6$	0.817
$1.45 \leq x \leq 1.55$	0.816
$1.49 \leq x \leq 1.51$	0.816
$1.499 \leq x \leq 1.501$	0.816

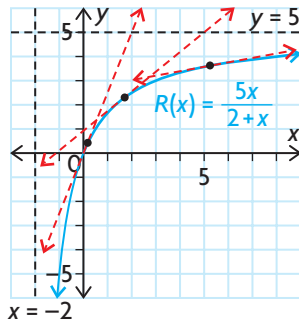
x is measured in thousands, so when 1500 toothbrushes are sold, $x = 1.5$.

The average rate of change from $(x_1, R(x_1))$ to $(x_2, R(x_2))$ is the slope of the secant that joins each pair of endpoints.

The average rate of change approaches 0.816. The marginal revenue when 1500 toothbrushes are sold is \$0.82 per toothbrush.

Sketch the graph of $R(x) = \frac{5x}{2+x}$.

The graph starts at $(0, 0)$ and has a horizontal asymptote at $y = 5$.



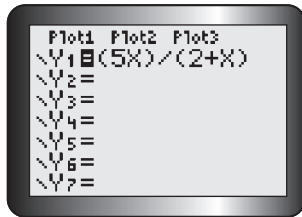
The marginal revenue is the greatest when $x = 0$ and then decreases from there, approaching zero, as the graph gets closer to the horizontal asymptote.

When x_1 and x_2 are very close to each other, the slope of the secant is approximately the same as the slope of the tangent. The slopes of the secants near the point where $x = 1.5$ approach 0.816.

The vertical asymptote at $x = -2$ is not in the domain of $R(x)$, since $x \geq 0$, so it can be ignored.

In the context of the problem, x and $R(x)$ have only positive values. Examine the slope of the tangent lines at various points along the domain of the revenue graph. The slope is the greatest at the beginning of the graph and then decreases as x increases.

Solution B: Using the difference quotient and graphing technology to analyze the revenue function



Enter the revenue function into a graphing calculator.

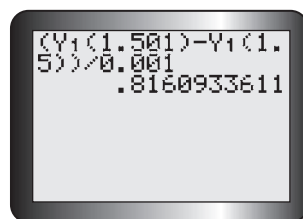
$$\text{Average rate of change} = \frac{R(a+h) - R(a)}{h}$$

Let $h = 0.001$

$$\begin{aligned} &= \frac{R(1.5 + 0.001) - R(1.5)}{0.001} \\ &= \frac{R(1.501) - R(1.5)}{0.001} \end{aligned}$$

Use the difference quotient and a very small value for h , where $a = 1.5$, to estimate the instantaneous rate of change in revenue when 1500 toothbrushes are sold.

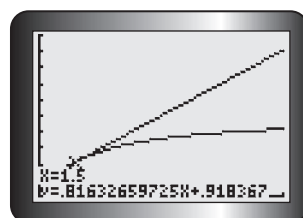




Enter the rate of change expression into the graphing calculator to determine its value, using the equation entered into Y1.

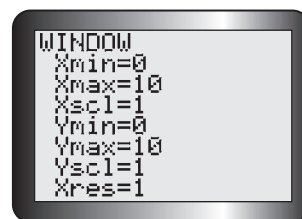
The average rate of change is about 0.816. The marginal revenue when 1500 toothbrushes are sold is \$0.82 per toothbrush.

To verify, graph the revenue function $R(x) = xp(x) = \frac{5x}{2+x}$ and draw a tangent line at $x = 1.5$.

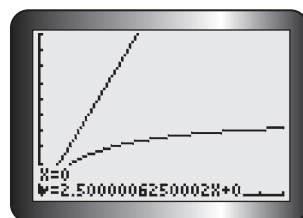


When 1500 toothbrushes are sold, the marginal revenue is \$0.82 per toothbrush.

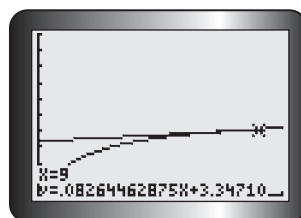
Since x and $R(x)$ only have positive values, graph the function in the first quadrant.



Use the DRAW feature of the graphing calculator to draw a tangent line where $x = 1.5$.



The marginal revenue is the greatest when $x = 0$.



The marginal revenue decreases to very small values as x increases.

The tangent lines to this curve are steepest at the beginning of the curve. Their slopes decrease as x increases.

Reflecting

- In Solution A, how were average rates of change used to estimate the instantaneous rate of change at a point?
- In Solutions A and B, how were graphs used to estimate the instantaneous rate of change at a point?
- In each solution, how was it determined where the marginal revenue was the greatest? Why was it not possible to determine the least marginal revenue?
- What are the advantages and disadvantages of each method to determine the instantaneous rate of change?

APPLY the Math

EXAMPLE 2

Connecting the instantaneous rate of change to the slope of a tangent

a) Estimate the slope of the tangent to the graph of $f(x) = \frac{x}{x+3}$ at the point where $x = -5$.

b) Why can there not be a tangent line where $x = -3$?

Solution

a) $f(x) = \frac{x}{x+3}$

$$\text{average rate of change} = \frac{f(a+h) - f(a)}{h}$$

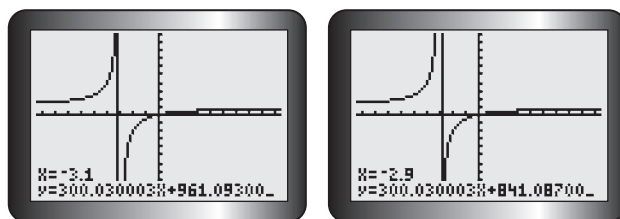
Let $h = 0.001$

$$\begin{aligned} &= \frac{f(-5 + 0.001) - f(-5)}{0.001} \\ &= \frac{f(-4.999) - f(-5)}{0.001} \\ &= \frac{\left(\frac{-4.999}{-4.999 + 3}\right) - \left(\frac{-5}{-5 + 3}\right)}{0.001} \\ &= \frac{2.500750375 - 2.5}{0.001} \\ &\doteq 0.7504 \end{aligned}$$

Use the difference quotient and a very small value for h close to $a = -5$ to estimate the slope of the tangent where $x = -5$.

The slope of the tangent at $x = -5$ is 0.75.

b) The value -3 is not in the domain of $f(x)$, so no tangent line is possible there. The graph of $f(x)$ has a vertical asymptote at $x = -3$.



As x approaches -3 from the left and from the right, the tangent lines are very steep. The tangent lines approach a vertical line, but are never actually vertical. There is no point on the graph with an x -coordinate of -3 , so there is no tangent line there.

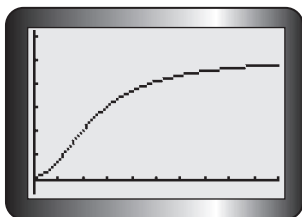
EXAMPLE 3

Selecting a graphing strategy to solve a problem that involves average and instantaneous rates of change

The snowshoe hare population in a newly created conservation area can be predicted over time by the model $p(t) = 50 + \frac{2500t^2}{25 + t^2}$ where p represents the population size and t is the time in years since the opening of the conservation area. Determine when the hare population will increase most rapidly, and estimate the instantaneous rate of change in population at this time.

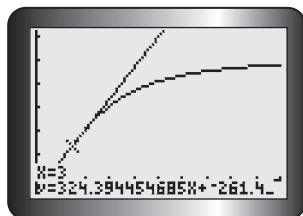
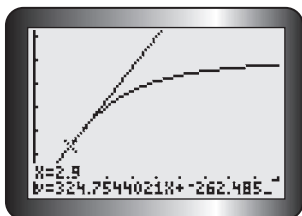
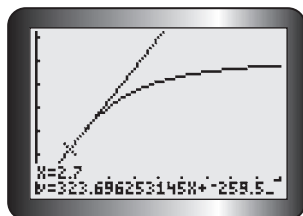
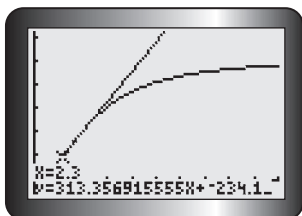
Solution

Graph $p(t) = 50 + \frac{2500t^2}{25 + t^2}$ for $0 \leq t \leq 20$.



The slopes of the tangent lines increase slowly at the beginning of the graph. The slopes start to increase more rapidly around $t = 2$. They begin to decrease after $t = 3$.

Draw tangent lines between 2 and 3, and look for the tangent line that has the greatest slope.



$p(t) \geq 0$, but you can set the minimum y-value to a negative number to allow some space for displayed values.

```
WINDOW
Xmin=0
Xmax=20
Xscl=2
Ymin=-300
Ymax=3100
Yscl=500
Xres=1
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The slopes of the tangent lines increase until 2.9, and then decrease. The tangent line at $t = 2.9$ has the greatest slope.

The average rate of change is greatest when t is close to 2.9. The hare population will increase most rapidly about 2 years and 11 months after the conservation area is opened. The instantaneous rate of change in population at this time is approximately 325 hares per year.

0.9×12 months is approximately 11 months.

In Summary

Key Ideas

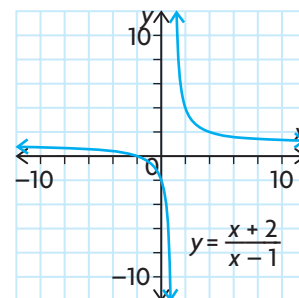
- The methods that were previously used to calculate the average rate of change and estimate the instantaneous rate of change can be used for rational functions.
- You cannot determine the average and instantaneous rates of change of a rational function at a point where the graph is discontinuous (that is, where there is a hole or a vertical asymptote).

Need to Know

- The average rate of change of a rational function, $y = f(x)$, on the interval from $x_1 \leq x \leq x_2$ is $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$. Graphically, this is equivalent to the slope of the secant line that passes through the points (x_1, y_1) and (x_2, y_2) on the graph of $y = f(x)$.
- The instantaneous rate of change of a rational function, $y = f(x)$, at $x = a$ can be approximated using the difference quotient $\frac{f(a+h) - f(a)}{h}$ and a very small value of h . Graphically, this is equivalent to estimating the slope of the tangent line that passes through the point $(a, f(a))$ on the graph of $y = f(x)$.
- The instantaneous rate of change at a vertical asymptote is undefined. The instantaneous rates of change at points that are approaching a vertical asymptote become very large positive or very large negative values. The instantaneous rate of change near a horizontal asymptote approaches zero.

CHECK Your Understanding

- The graph of a rational function is shown.
 - Determine the average rate of change of the function over the interval $2 \leq x \leq 7$.
 - Copy the graph, and draw a tangent line at the point where $x = 2$. Determine the slope of the tangent line to estimate the instantaneous rate of change at this point.
- Estimate the instantaneous rate of change of the function in question 1 at $x = 2$ by determining the slope of a secant line from the point where $x = 2$ to the point where $x = 2.01$. Compare your answer with your answer for question 1, part b).
- Use graphing technology to estimate the instantaneous rate of change of the function in question 1 at $x = 2$.



PRACTISING

- Estimate the instantaneous rate of change of $f(x) = \frac{x}{x-4}$ at the point $(2, -1)$.

5. Select a strategy to estimate the instantaneous rate of change of each function at the given point.

a) $y = \frac{1}{25 - x}$, where $x = 13$

b) $y = \frac{17x + 3}{x^2 + 6}$, where $x = -5$

c) $y = \frac{x + 3}{x - 2}$, where $x = 4$

d) $y = \frac{-3x^2 + 5x + 6}{x + 6}$, where $x = -3$

6. Determine the slope of the line that is tangent to the graph of each function at the given point. Then determine the value of x at which there is no tangent line.

a) $f(x) = \frac{-5x}{2x + 3}$, where $x = 2$

b) $f(x) = \frac{x - 6}{x + 5}$, where $x = -7$

c) $f(x) = \frac{2x^2 - 6x}{3x + 5}$, where $x = -2$

d) $f(x) = \frac{5}{x - 6}$, where $x = 4$

7. When polluted water begins to flow into an unpolluted pond, the concentration of pollutant, c , in the pond at t minutes is modelled by $c(t) = \frac{27t}{10\,000 + 3t}$, where c is measured in kilograms per cubic metre. Determine the rate at which the concentration is changing after

a) 1 h b) one week

8. The demand function for snack cakes at a large bakery is given by the function $p(x) = \frac{15}{2x^2 + 11x + 5}$. The x -units are given in thousands of cakes, and the price per snack cake, $p(x)$, is in dollars.

- a) Find the revenue function for the cakes.
- b) Estimate the marginal revenue for $x = 0.75$. What is the marginal revenue for $x = 2.00$?

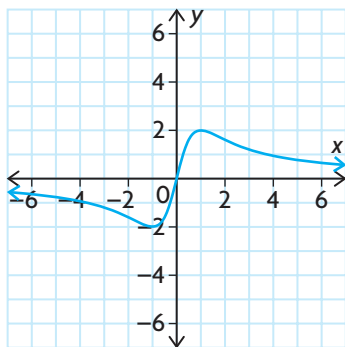
9. At a small clothing company, the estimated average cost function for producing a new line of T-shirts is $C(x) = \frac{x^2 - 4x + 20}{x}$, where x is the number of T-shirts produced, in thousands. $C(x)$ is measured in dollars.

- a) Calculate the average cost of a T-shirt at a production level of 3000 pairs.
- b) Estimate the rate at which the average cost is changing at a production level of 3000 T-shirts.

10. Suppose that the number of houses in a new subdivision after t months of development is modelled by $N(t) = \frac{100t^3}{100 + t^3}$, where N is the number of houses and $0 \leq t \leq 12$.
- Calculate the average rate of change in the number of houses built over the first 6 months.
 - Calculate the instantaneous rate of change in the number of houses built at the end of the first year.
 - Graph the function using a graphing calculator. Discuss what happens to the rate at which houses were built in this subdivision during the first year of development.
11. Given the function $f(x) = \frac{x-2}{x-5}$, determine an interval and a point where the average rate of change and the instantaneous rate of change are equal.
12. a) The position of an object that is moving along a straight line at t seconds is given by $s(t) = \frac{3t}{t+4}$, where s is measured in metres. Explain how you would determine the average rate of change of $s(t)$ over the first 6 s.
- What does the average rate of change mean in this context?
 - Compare two ways that you could determine the instantaneous rate of change when $t = 6$. Which method is easier? Explain. Which method is more accurate? Explain.
 - What does the instantaneous rate of change mean in this context?

Extending

13. The graph of the rational function $f(x) = \frac{4x}{x^2 + 1}$ has been given the name Newton's Serpentine. Determine the equations for the tangents at the points where $x = -\sqrt{3}$, 0, and $\sqrt{3}$.



14. Determine the instantaneous rate of change of Newton's Serpentine at points around the point (0, 0). Then determine the instantaneous rate of change of this instantaneous rate of change.