SPH4U-C

Lesson 2

Projectile Motion

Introduction

In the last lesson, you read about how airplanes take off and land into the wind, shortening the runway distance necessary. However, what if an airplane only had access to a very short runway, such as the length of a ship? How could it possibly get the takeoff speed required? The answer is an aircraft catapult—a device first used in 1915 to launch an airplane from a moving ship.

Aircraft catapults are often used on aircraft carriers to project, or throw, an aircraft into the air with a motion similar to that achieved with a crossbow and arrow, a slingshot, or even an arm pitching a ball. It is a device that gives the aircraft the initial velocity it requires to take flight.

Catapults, however, are more widely known for their use in warfare. They were used to launch objects that could destroy walls and castles, and were considered to be the engineering feat of the Middle Ages. In many ways, they changed the meaning and effect of physical conflict by increasing the amount of destruction possible. In the fourteenth century, catapults were replaced by cannons, which used explosions to launch heavy cannonballs to do the necessary damage. From then on, warfare projectiles became more and more technologically advanced, until eventually, rockets and missiles could be launched to travel large distances. With these advancements, the possibilities for destruction and devastation reached unimaginable levels.

The result of using a catapult, cannon, or bow and arrow, or even of kicking or throwing a ball in a sport, is the same: an object is launched into the air with some velocity and follows a curved path toward the ground, due to the influence of gravity. In this lesson, you will learn the physics and mathematics behind this motion, and will be able to project an object and perform an analysis of its motion.

Planning Your Study

You may find this time grid helpful in planning when and how you will work through this lesson.

Suggested Timing for This Lesson (hours)	
Acceleration and Acceleration Due to Gravity	1
Projectile Motion	½
Projectiles with Horizontal Projection	1
Projectiles with Projection at an Angle	1
Projectile Motion Investigation	½
Key Questions	½

What You Will Learn

After completing this lesson, you will be able to

- use component analysis and knowledge of acceleration due to gravity to solve problems involving projectile motion
- collect observations and data in a projectile motion investigation and use this information to perform relevant calculations
- identify safety concerns and sources of error in an inquiry situation

Acceleration

In the last lesson, you worked with velocities—vector quantities that describe the motion of an object with respect to time. In this lesson, you're going to be working with measurements of change in velocity with respect to time, known as acceleration.

Definition of Acceleration

Acceleration ($\vec{a}$) is a measure of change in velocity ($\Delta\vec{v}$) with respect to time (Δt). In equation form:

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t}$$

where $\vec{v}_2$ represents the object's final velocity and $\vec{v}_1$ represents the object's initial velocity. As long as you know the time interval and the initial and final velocities of an object, this equation can be used in any situation to determine the object's average acceleration. In this lesson, you are concerned with situations in which acceleration is uniform, or constant, due to gravity.

Acceleration Due to Gravity

If you were to drop a ball in mid-air, it would fall to the ground. Due to the force of gravity near the earth's surface, the ball's velocity increases as it falls, with a uniform acceleration of 9.8 m/s^2 [down]. This means that in every second, the ball's velocity increases by 9.8 m/s. If you were to throw that same ball upward, it would rise, but it would still experience this uniform acceleration downward. As a result, the ball's velocity would decrease as it went higher and eventually reach a maximum height and instantaneous vertical velocity of zero. At this point, it would change direction to fall back to its starting point, with its velocity increasing as it falls.

Note: The rate and direction of acceleration due to gravity is constant throughout all parts of a free fall, regardless of whether the object is on the upward path, at maximum height, or on the way back down. It is equal to 9.8 m/s^2 [down], assuming that air resistance is negligible (so small that it can be ignored or doesn't matter) and you are relatively close to the surface of the earth. You will learn more about this in the next lesson.

Useful Equations for Uniform Acceleration Problems

Listed below are a few equations and variables that are useful when working with uniform acceleration. The examples that follow this section will remind you how to use them. Keep these equations handy throughout the rest of this lesson. Note that equation 3 does not have vector symbols on the variables. This is because there is no process involving pure multiplication of vectors.

1. $\Delta\vec{d} = \vec{v}_1\Delta t + \frac{1}{2}\vec{a}(\Delta t)^2$

2. $\Delta\vec{d} = \frac{1}{2}(\vec{v}_1 + \vec{v}_2)\Delta t$

3. $v_2^2 = v_1^2 + 2a\,\Delta d$

4. $\Delta\vec{d} = \vec{v}_2\Delta t - \frac{1}{2}\vec{a}(\Delta t)^2$

5. $\vec{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t}$

The variables:

$\Delta\vec{d}$ = displacement

$\vec{v}_1$ = initial velocity

$\vec{v}_2$ = final velocity

$\vec{a}$ = acceleration

Δt = time interval

As you work through the examples that follow, keep in mind that there is often more than one way to solve a problem involving acceleration due to gravity. One method is shown here, but feel free to explore other possible methods. Regardless of the path you take, the answers will be the same.

Example 1

As you stand on the edge of a cliff 10.0 m over a body of water, you prepare to jump in. Assuming that you allow yourself to fall from the edge of the cliff, how long will it take you to reach the surface of the water, and with what velocity will you reach the water's surface?

Solution

Given:

$\Delta\vec{d} = 10.0 \text{ m [down]}$

$\vec{a} = 9.8 \text{ m/s}^2 \text{ [down]}$

$\vec{v}_1 = 0 \text{ m/s}$

Note that whenever something *falls* or *is dropped* from some height, the initial velocity of that object in the vertical direction is 0 m/s.

Required: Δt and $\vec{v}_2$

Analysis and solution: Considering the variables you know and those which you are looking to solve for Δt, the first equation in the numbered list on the previous page will be useful here.

$$\Delta\vec{d} = \vec{v}_1 \Delta t + \frac{1}{2}\vec{a}(\Delta t)^2$$

Let down be positive. Thus, both Δd and a will be positive.

$$10.0 \text{ m} = 0 \text{ m/s}\Delta t + \frac{1}{2}(9.8 \text{ m/s}^2)(\Delta t)^2$$

$$10.0 \text{ m} = \frac{1}{2}(9.8 \text{ m/s}^2)(\Delta t)^2$$

Notice that because the initial velocity is 0 m/s, this equation simplifies. If this were not the case, the solution would require solving a quadratic. (This will be discussed later on.)

Rearranging to solve for the unknown, you get:

$$\Delta t = \sqrt{\frac{10.0 \text{ m}}{\frac{1}{2}(9.8 \text{ m/s}^2)}}$$

$$\Delta t = \pm 1.43 \text{ s}$$

Square roots: Whenever the square root of a number is calculated, the answer can be either positive or negative. You must figure out which answer makes sense. Because time can never be negative, the answer must be positive.

As you solve for $\vec{v}_2$, remember that [down] is the positive direction.

$$v_2^2 = v_1^2 + 2a\Delta d$$

$$v_2^2 = (0 \text{ m/s})^2 + 2(9.8 \text{ m/s}^2)(10.0 \text{ m})$$

$$v_2 = \sqrt{2(9.8 \text{ m/s}^2)(10.0 \text{ m})}$$

$$v_2 = \pm 14.0 \text{ m/s}$$

Choose the positive root because you are travelling [down]; thus, the final velocity is positive.

Paraphrase: Therefore, it will take you 1.43 s to reach the surface of the water and your velocity will be 14.0 m/s [down], just before you enter the water.

From now on, we will use the scalar components to solve all problems.

Example 2

In trying to see how high you can throw a ball straight up into the air, you manage to get it to the height of your roof (7.0 m). Determine the following:

a) What is the velocity of the ball at maximum height?

b) What was the ball's initial velocity as it left your hand, assuming that it left your hand at a height of 2.0 m above the ground?

c) How long does it take the ball to reach maximum height?

d) How long does it take the ball to arrive back at your hand (in the same position as when you started)?

e) What is the final velocity of the ball as it arrives back at your hand?

Solution

Given:

Height of roof = 7.0 m

Initial height of ball = 2.0 m

Let up be positive.

Therefore,

$\Delta\vec{d} = 5$ m [up]

$\Delta d = 5$ m

$\vec{a} = 9.8$ m/s^2 [down]

a) Required: $\vec{v}$ of the ball at maximum height (that is, 7.0 m)

Analysis and solution:

As the ball rises, it slows down and eventually comes to a stop. After this instant, it begins to fall downward. Therefore, at the very top, the velocity is 0 m/s.

Paraphrase:

The velocity of the ball at maximum height is 0 m/s.

b) Required: $\vec{v}_1$

Analysis and solution:

The third equation from the list is useful here.

$v_2^2 = v_1^2 + 2a\,\Delta d$

Rearranging to solve for the unknown, you get:

$$v_1^2 = v_2^2 - 2a\,\Delta d$$

$$v_1^2 = (0\text{ m/s})^2 - 2(-9.8\text{ m/s}^2)(5.0\text{ m})$$

$$v_1^2 = (0\text{ m/s})^2 + 2(9.8\text{ m/s}^2)(5.0\text{ m})$$

$$v_1 = \sqrt{2(9.8\text{ m/s}^2)(5.0\text{ m})}$$

$$v_1 = \pm 9.9\text{ m/s}$$

Because the ball was thrown upwards, and you've let [up] be the positive direction, you know that the correct solution here is positive.

Paraphrase: Therefore, the initial velocity of the ball is 9.9 m/s [up].

c) Required: Δt to reach maximum height

Analysis and solution: Here, the fourth equation from the list is useful.

$$\Delta d = v_2\,\Delta t - \frac{1}{2}a(\Delta t)^2$$

$$5.0\text{ m} = (0\text{ m/s})\Delta t - \frac{1}{2}(-9.8\text{ m/s}^2)(\Delta t)^2$$

$$5.0\text{ m} = \frac{1}{2}(9.8\text{ m/s}^2)(\Delta t)^2$$

Rearranging:

$$(\Delta t)^2 = \frac{5.0\text{ m}}{\frac{1}{2}(9.8\text{ m/s}^2)}$$

$$\Delta t = \sqrt{\frac{5.0\text{ m}}{\frac{1}{2}(9.8\text{ m/s}^2)}}$$

$$\Delta t = \pm 1.0\text{ s}$$

Choose the positive root.

Paraphrase: Therefore, it would take the ball 1.0 s to reach the height of the roof.

d) Required: Δt required for the ball to travel up and back

Analysis and solution: Let up be positive. You know the time it takes for the ball to travel upwards. Therefore, you simply need to solve for the time required for the ball to make the downward trip. Starting at the top, where the velocity for an instant is 0 m/s:

$$\Delta d = v_1 \Delta t + \frac{1}{2}a(\Delta t)^2$$

$$-5.0 \text{ m} = 0 \text{ m/s}\,\Delta t + \frac{1}{2}(-9.8 \text{ m/s}^2)(\Delta t)^2$$

$$5.0 \text{ m} = \frac{1}{2}(9.8 \text{ m/s}^2)(\Delta t)^2$$

$$\Delta t = \sqrt{\frac{5.0 \text{ m}}{\frac{1}{2}(9.8 \text{ m/s}^2)}}$$

$$\Delta t = \pm 1.0 \text{ s}$$

Choose the positive root, as time cannot be negative.

Notice that the time it takes for the ball to come back down is equal to the time required for it to travel up. This will always be the case, as long as the start and finish positions are the same.

The time for the total trip will be 1.0 s + 1.0 s = 2.0 s.

Paraphrase: Therefore, the time required for the ball to travel up and back down to the same point is 2.0 s.

e) Required: $\vec{v}_2$, as the ball arrives back at your hand

Analysis and solution:

You can solve this by using the equations provided in the above examples (and listed earlier on). Go ahead and try this yourself.

However, logically, if the displacement, time, and acceleration are constant, the change in velocity must be constant as well. Therefore, as long as there is no air resistance, the magnitude of the velocity of the ball as it arrives back at your hand will equal the magnitude of velocity of the ball when you initially tossed it. The direction, though, would be opposite.

Paraphrase: The ball will have a velocity of 9.9 m/s [down] as it arrives back at your hand.

Support Questions

Be sure to try the Support Questions on your own before looking at the suggested answers provided. Click on each "Suggested answer" button to check your work.

10. Imagine that you are tossing a rock up and down. You throw the rock up and then catch it at the same height from which you threw it. Air resistance is negligible (so small that it can be ignored).

a) How does the time it takes for the upward trip compare to the time it takes for the downward trip?

b) How do the initial and final velocities compare?

c) What will the rock's instantaneous velocity be at the maximum height?

d) What is the acceleration of the rock as it is rising, as it is falling, and at maximum height?

e) The rock is thrown from the edge of a 14.0 m high cliff with a velocity of magnitude 7.0 m/s. Find the velocity with which the rock hits the ground and the total time that it was in the air, if it was initially launched (i) upward, and (ii) downward. Compare your answers.

Suggested answer

Projectile Motion

A ball dropped in mid-air accelerates to the ground, due to the force of gravity. What about a ball that is thrown horizontally or at some angle, so that it has horizontal, as well as vertical motion? It must experience acceleration due to gravity, but what happens in the horizontal direction?

A ball that is thrown, once it has left your hand, is an example of a projectile. A projectile is an object travelling through the air under the influence of gravity and nothing else (neglecting, or ignoring friction). The object does not have a propulsion system, so a ball you have thrown (once it has left your hand) would be a good example of a projectile. However, an airplane flying through the air would not be an example of projectile motion, since it has mechanical structures helping it to fly.

Projectile motion, then, is the motion of an object with constant horizontal velocity and constant vertical acceleration due to gravity. The horizontal uniform motion is independent of the accelerated vertical motion but, happening simultaneously, they result in a smooth, curved path of motion, or trajectory. In this course, the effect of air resistance (friction) will be ignored in projectile motion problem solving, even though it often has an effect in real life.

Projectile Motion Simulation

Open this simulation called Projectile Motion. It will enable you to become more familiar with the concepts related to projectile motion. The program allows you to change the following variables:

- launch (firing) angle
- initial speed of projectile (magnitude of the velocity)
- height of cannon
- air resistance
- mass of object
- diameter of object
- object itself

It also enables you to set up a target to hit, and has a measuring device you can use to set up specific distances and heights to fire from. The measuring tool is important, as it allows you to measure the height at various positions in a trajectory. There is also a sound option.

The program will calculate the range (horizontal distance), height values, and time of flight for each setup. At this stage, you will only need to consult the values to help you answer the Support Questions that follow.

Since we are now dealing with 2-dimensional motion, we will not consider the case of firing the projectile straight up (in other words, at a 90° angle).

Instructions

1. Familiarize yourself with the program. Try various options and become comfortable with all the features of the program.
2. Keep the angle constant. Remove air resistance, and vary the initial speed of the projectile.
3. Keeping the speed of the projectile fixed, vary the firing angle.
4. Keeping the angle and initial speed fixed, vary the mass.
5. Keeping the angle, initial speed, and mass constant, vary the diameter of the object.
6. Keeping all variables constant, move the cannon up to different heights and observe the effect.
7. Set the cannon at some arbitrary height above the ground and set the angle to 0° (horizontal position). Vary the initial speed, and observe the effect on the time of flight and range.
8. Optional: Repeat steps 2–7 with air resistance. Try changing the amount of air resistance and observe the effect on the trajectory. Ignore the altitude setting.

Support Questions

11. **a)** How does changing the magnitude of the initial velocity change the

i) range?

ii) maximum height?

iii) time of flight?

Suggested answer

b) How does changing the initial angle change the

i) range?

ii) maximum height?

iii) time of flight?

Suggested answer

c) How does changing the mass change the

i) range?

ii) maximum height?

iii) time of flight?

Suggested answer

d) How does changing the diameter of the object change the

i) range?

ii) maximum height?

iii) time of flight?

Suggested answer

e) Which launch angle gives the maximum

i) range?

ii) height?

iii) time of flight?

Suggested answer

f) When firing horizontally, how does the speed of the projectile affect

i) range?

ii) time of flight?

Suggested answer

g) How does the height of the cannon above level ground affect

i) range?

ii) maximum height?

iii) time of flight?

Suggested answer

h) Summarize the effect of air resistance on

- **i)** range
- **ii)** maximum height
- **iii)** time of flight
- **iv)** flight pattern

Suggested answer

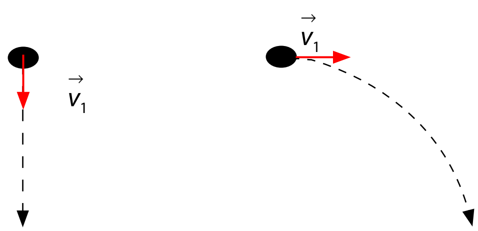
i) What factors affect air resistance?

Suggested answer

Note: When solving problems involving projectile motion, it is important to remember the following points:

- There is constant acceleration due to gravity in the vertical direction (*always* 9.8 m/s^2 [down]).
- There is no acceleration in the horizontal direction. In other words, the initial horizontal (x) velocity remains the same throughout.
- The horizontal (x) and vertical (y) components are completely independent of each other. Giving an object some horizontal motion, in addition to its vertical falling motion, does not affect its rate of fall.

For example, if you were to drop a ball from a certain height and, at the same time, project another ball horizontally from that same height, which ball would hit the ground first?



Obviously, the ball that is projected horizontally travels a longer path. However, it turns out that both balls will hit the ground at the same time. This is because it is the vertical motion that determines the length of time that it takes for an object to fall. The horizontal motion does not affect this whatsoever.

Now watch this simulation called Effect of Horizontal Velocity on a Falling Object.

Projectiles with Horizontal Projection

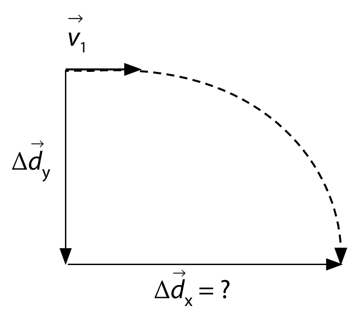
As you work through these examples, notice that the equations being used are the same ones used for acceleration due to gravity. The only difference is that you are using x and y subscripts to talk about components of vectors.

Example 1

A downhill skier skiing at a speed of 7.0 m/s skis over the side of a 5.0 m high cliff and lands safely at the bottom. How far from the base of the cliff will the skier land?

Solution

Given: In considering your "Given" information and how to work with that information, it is useful to create a sketch of the situation. In fact, your "Given" section can consist of a well-labelled sketch, as opposed to a list of information.



Required: Δd_x

Analysis and solution: When solving projectile motion problems, it can be helpful to divide your work into two columns. In one column, you list the information related to the vertical motion and perform calculations involving this information, and in the other column, you list and work with information related to the horizontal motion. It helps to keep your work and thoughts organized.

Let [forward] be the positive x-direction and [down] be the positive y-direction.

Vertically	Horizontally
$v_{1_y} = 0.0$ m/s	$v_{1_x} = 7.0$ m/s
$a_y = 9.8$ m/s^2	$a_x = 0$ m/s^2
$\Delta d_y = 5.0$ m	$\Delta d_x = ?$

Here is the thought process used for beginning to solve this problem:

- You want to solve for the horizontal displacement.
- You know the initial horizontal velocity and you know that it does not change throughout the motion.
- You know that, in the absence of acceleration, $v = \frac{\Delta d}{\Delta t}$.
- Rearranging to solve for the unknown variable, you get $\Delta d = v\Delta t$.
- Therefore, in order to determine Δd_x, you need to know the time interval. You can find this by using the information you know about the vertical motion.

Vertically	Horizontally
1. Using the vertical component to find time: $\Delta d_y = v_{1_y}\Delta t + \frac{1}{2}a_y\Delta t^2$ $5.0\text{ m} = (0.0\text{ m/s})(\Delta t) + \frac{1}{2}(9.8\text{ m/s}^2)(\Delta t)^2$ $\Delta t = \sqrt{\frac{2(5.0\text{ m})}{9.8\text{ m/s}^2}}$ $\Delta t = +1.01\text{ s}$ Because this value for time will be used in further calculations, it is good practice to keep an extra digit beyond what is considered significant, to reduce error due to rounding. When you state the final answer, though, you must state the answer with the correct number of significant digits.	2. Once you know time, you can solve for Δd_x. $\Delta d_x = v_x\Delta t$ $\Delta d_x = (7.0\text{ m/s})(1.01\text{ s})$ $\Delta d_x = 7.07\text{ m}$ $\Delta d_x = 7.1\text{ m}$

Paraphrase: Therefore, the skier will land 7.1 m from the base of the cliff.

Example 2

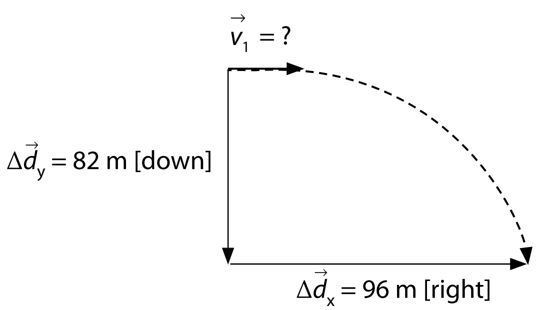
An airplane is flying horizontally at a height of 82 m above the ground when it releases a package of supplies to an earthquake-ravaged city. The package travels a horizontal distance of 96 m before landing on the ground. Assuming that there is negligible wind or air resistance, what was the package's (and therefore the airplane's)

a) initial velocity with respect to the ground?

b) final velocity before hitting the ground?

Solution

Given:

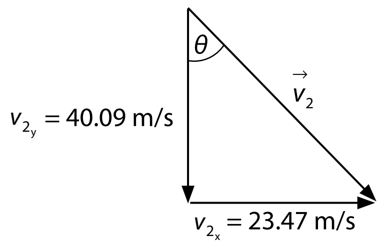


Required: $\vec{v}_1$ and $\vec{v}_2$

Analysis and solution: Let [forward] be the positive *x*-direction and [down] be the positive *y*-direction.

Vertically	Horizontally
$v_{1_y} = 0.0$ m/s $a_y = 9.8$ m/s^2 $\Delta d_y = 82$ m 1. Using the vertical component to find time: $\Delta d_y = v_{1_y}\Delta t + \frac{1}{2}a_y\Delta t^2$ $82\text{ m} = (0.0\text{ m/s})(\Delta t) + \frac{1}{2}(9.8\text{ m/s}^2)(\Delta t)^2$ $\Delta t = \sqrt{\frac{2(82\text{ m})}{9.8\text{ m/s}^2}}$ $\Delta t = +4.09$ s We choose the positive root, as time cannot be negative. **Note:** If you know the appropriate sign for the correct answer for certain (for example, in this case, time must be positive), then it is fine to omit the ± symbol and jump ahead to include the correct sign. Calculating the *y*-component of the final velocity: $v_{2_y}^2 = v_{1_y}^2 + 2a_y\Delta d_y$ $v_{2_y}^2 = (0\text{ m/s})^2 + 2(9.8\text{ m/s}^2)(82\text{ m})$ $v_{2_y} = \sqrt{2(9.8\text{ m/s}^2)(82\text{ m})}$ $v_{2_y} = 40.09$ m/s	$v_{1_x} = ?$ $a_x = 0$ m/s^2 $\Delta d_x = 96$ m 2. Once you know time, you can solve for v_x. $v_x = \frac{\Delta d_x}{\Delta t}$ $v_x = \frac{96\text{ m}}{4.09\text{ s}}$ $v_x = 23.47\text{ m/s} = 23\text{ m/s}$ (significant digits) Since the initial velocity was strictly horizontal, you now know the initial velocity. The final velocity, however, has a horizontal component and a vertical component. Since there is no acceleration in the horizontal direction, the final horizontal velocity is the same as the initial velocity. Therefore, $v_{x_2} = 23$ m/s. However, the *y*-component of the velocity must be calculated.

Putting the scalar x- and y-components of the final velocities together in a sketch, you get:



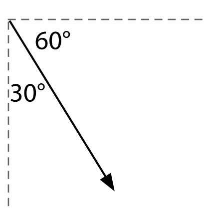
Using the Pythagorean theorem to find the magnitude of the final velocity, you get:

$$v_2^2 = v_{2_x}^2 + v_{2_y}^2$$
$$v_2^2 = (23.47\ \text{m/s})^2 + (40.09\ \text{m/s})^2$$
$$v_2 = \sqrt{(23.47\ \text{m/s})^2 + (40.09\ \text{m/s})^2}$$
$$v_2 = 46.45\ \text{m/s} = 46\ \text{m/s (significant digits)}$$

To find the direction of the velocity, you use trigonometry:

$$\tan\theta = \frac{23.47\ \text{m}}{40.09\ \text{m}}$$
$$\theta = \tan^{-1}\left(\frac{23.47\ \text{m}}{40.09\ \text{m}}\right)$$
$$\theta = 30°$$

This direction can be communicated as [30° forward from vertical]. However, directions are most commonly communicated as being "above or below the horizontal." Given that the complementary angle here is 60°, the direction is [60° below horizontal].



This "[60° below horizontal]" direction would also have been apparent had you drawn the x-component first, and then "added" the y-component to it. Now that you know that it is perfectly correct either way, you will do this consistently in future examples.

Paraphrase: Therefore, the initial velocity of the package was 23 m/s [forward] and the final velocity of the package was 46 m/s [60° below horizontal].

Support Questions

12. A rock was thrown horizontally from a 25.0 m high cliff and landed 15.0 m from the base of the cliff. Determine the initial speed with which the rock was thrown, as well as its final velocity.

Suggested answer

Projectiles with Projection at an Angle

So far, you have been mathematically analyzing situations in which the initial velocity had a horizontal component only. However, objects are often projected at an angle that has both horizontal and vertical components. These types of motions involve more component analysis and may, when the starting and landing heights are different, require you to solve a quadratic equation. You may wish to take some time to review quadratic equations before moving forward. Once you are comfortable with this new step, the rest is just more of what you have already been doing.

Example 1

A football player kicks a football so that the angle of incidence is 50° and the initial magnitude of velocity of the ball is 15 m/s. Find the

a) ball's maximum height.

b) time of flight.

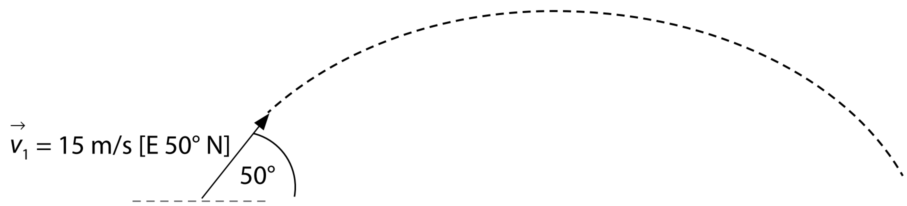
c) time when the ball reaches the maximum height.

d) horizontal distance.

e) velocity at impact.

Solution

Given:

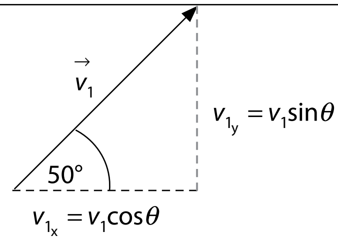


Required:

- Maximum height (Δd_y when v_y = 0 m/s)
- Time of flight (Δt)
- Time when ball reaches maximum height ($\Delta t_{\text{maximum height}}$)
- Horizontal distance (Δd_x)
- Velocity at impact ($\vec{v}_2$)

Analysis and solution:

Let [up] be the positive y-direction and [forward] be the positive x-direction. We will round off the answers to significant digits at the end of the solution.



<u>Vertically</u>

To find the vertical component of the initial velocity, you can use the sin function (because the *y*-component is opposite the known angle). Because it is directed upwards, it will be negative.

$v_{1_y} = +15 \text{ m/s}(\sin 50) = +11.49 \text{ m/s}$

$a_y = -9.8 \text{ m/s}^2$

$\Delta d_{y\text{ total}} = 0 \text{ m/s}$

1. Using the vertical component to find time:

$$\Delta d_y = v_{1_y}\Delta t + \frac{1}{2}a_y\Delta t^2$$

$$0 \text{ m} = (+11.49 \text{ m/s})(\Delta t) + \frac{1}{2}(-9.8 \text{ m/s}^2)(\Delta t)^2$$

Rearranging to solve for Δt:

$$(11.49 \text{ m/s})\Delta t = \frac{1}{2}(9.8 \text{ m/s}^2)(\Delta t)^2$$

Notice that one of the Δt variables can cancel, saving you from having to solve a quadratic this time. $t = 0$ at the beginning of the problem, when the ball was originally on the ground.

Cancelling and rearranging to solve for Δt:

$$(11.49 \text{ m/s})\Delta \cancel{t} = \frac{1}{2}(9.8 \text{ m/s}^2)(\Delta t)^{\cancel{2}}$$

$$(11.49 \text{ m/s}) = \frac{1}{2}(9.8 \text{ m/s}^2)(\Delta t)$$

$$\Delta t = \frac{2(11.49 \text{ m/s})}{9.8 \text{ m/s}^2}$$

$$\Delta t = 2.34 \text{ s}$$

<u>Horizontally</u>

To find the horizontal component of the initial velocity, you can use the cos function (because the *x*-component is adjacent to the known angle).

$v_{1_x} = 15 \text{ m/s}(\cos 50) = 9.64 \text{ m/s}$

$a_x = 0 \text{ m/s}^2$

2. Once you know time, you can solve for Δd_x.

$$v_x = \frac{\Delta d_x}{\Delta t}$$

Rearranging:

$\Delta d_x = v_x \Delta t$

$\Delta d_x = (9.64 \text{ m/s})(2.34 \text{ s})$

$\Delta d_x = 22.6 \text{ m}$

3. To find maximum height $\left(\Delta d_{\text{y maximum height}}\right)$:

$$v_{2_y}^2 = v_{1_y}^2 + 2a_y \Delta d_y$$

$$(0\ \text{m/s})^2 = (-11.49\ \text{m/s})^2 + 2(9.8\ \text{m/s}^2)\left(\Delta d_{\text{y maximum height}}\right)$$

Rearranging to solve for $\Delta d_{\text{y maximum height}}$:

$$\Delta d_{\text{y maximum height}} = -\frac{(+11.49\ \text{m/s})^2}{2(9.8\ \text{m/s}^2)}$$

$$\Delta d_{\text{y maximum height}} = +6.74\ \text{m}$$

4. To calculate the time taken for the ball to reach maximum height, remember that at maximum height, the vertical component of velocity is instantaneously 0 m/s. Therefore:

$$\Delta d_{\text{y maximum height}} = v_{2_y}\Delta t - \frac{1}{2}a_y \Delta t^2$$

$$+6.74\ \text{m} = (0\ \text{m/s})\Delta t - \frac{1}{2}(-9.8\ \text{m/s}^2)\Delta t^2$$

$$-6.74\ \text{m} = -\frac{1}{2}(9.8\ \text{m/s}^2)\Delta t^2$$

Rearranging to solve for Δt:

$$\Delta t = \sqrt{\frac{+6.74\ \text{m}}{+\frac{1}{2}(9.8\ \text{m/s}^2)}}$$

$$\Delta t = 1.17\ \text{s}$$

Notice that the time taken to get to the maximum height is equal to half of the total time interval. This will always be true if the object projects from, and lands at the same vertical height.

5. Calculating the y-component of the final velocity:

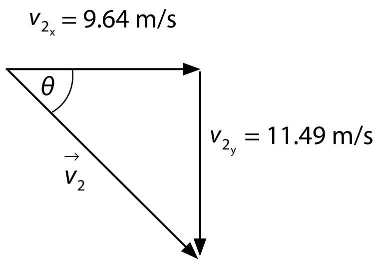
$$v_{2_y}^2 = v_{1_y}^2 + 2a_y \Delta d_y$$

$$v_{2_y}^2 = (+11.49\ \text{m/s})^2 + 2(-9.8\ \text{m/s}^2)(0\ \text{m})$$

$$v_{2_y}^2 = (-11.49\ \text{m/s})^2$$

$$v_{2_y} = 11.49\ \text{m/s}$$

6. Putting the scalar x- and y-components of the final velocity together, you get:



Applying the Pythagorean theorem:

$$v_2^2 = (9.64 \text{ m/s})^2 + (11.49 \text{ m/s})^2$$
$$v_2 = \sqrt{(9.64 \text{ m/s})^2 + (11.49 \text{ m/s})^2}$$
$$v_2 = 15 \text{ m/s}$$

Notice that the final velocity is the same in magnitude as the initial velocity. This will always be true when the object lands at the same height from which it was projected. The angle will be the same as well, but the direction will be opposite, in the vertical direction.

Paraphrase: Therefore, the

a) ball's maximum height is 6.7 m.

b) time of flight is 2.3 s.

c) time when the ball reaches maximum height is 1.2 s.

d) horizontal distance is 23 m.

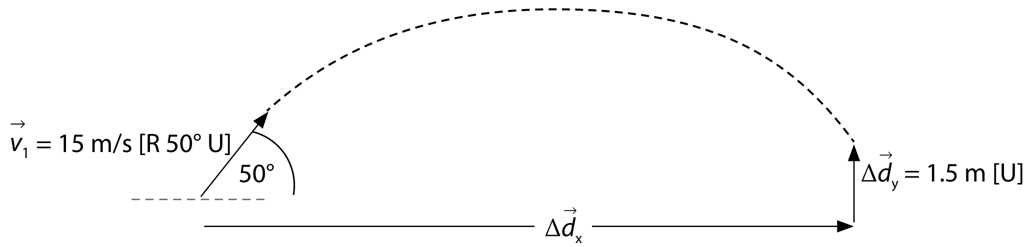
e) velocity at impact is 15 m/s [50° below horizontal].

Example 2

Now imagine that the situation is exactly the same, except that this time, the ball is caught by a player when it is at a height of 1.5 m above the ground. Determine the horizontal range of the ball, the length of time that it was in the air, and the final velocity.

Solution

Given:



Required:

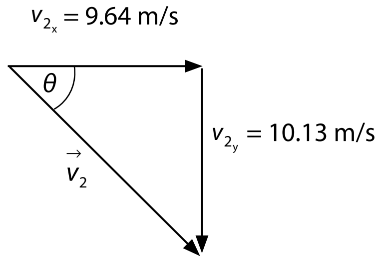
- Horizontal range (Δd_x)
- Time of flight (Δt)
- Final velocity ($\vec{v}_2$)

Analysis and solution:

Let [up] be the positive y-direction and [forward] be the positive x-direction.

Vertically	Horizontally
From the previous example, you know that: $v_{1_y} = +11.49\text{ m/s}$ You also know that: $a_y = -9.8\text{ m/s}^2$ $\Delta d_{y\,\text{total}} = +1.5\text{ m}$ 1. Using the vertical component to find time: $\Delta d_y = v_{1_y}\Delta t + \frac{1}{2}a_y\Delta t^2$ $+1.5\text{ m} = (+11.49\text{ m/s})(\Delta t) + \frac{1}{2}(-9.8\text{ m/s}^2)(\Delta t)^2$ This time, because there is no "zero" term, you end up with a quadratic equation to solve and proceed as follows: $0 = (4.9\text{ m/s}^2)(\Delta t)^2 + (-11.49\text{ m/s})(\Delta t) + 1.5\text{ m}$ For the quadratic: $a = 4.9$ m, $b = -11$ m/s, and $c = 1.5$ m. For simplicity, you will omit the units for this operation. $\Delta t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\Delta t = \frac{-(-11.49) \pm \sqrt{(-11.49)^2 - 4(4.9)(1.5)}}{2(4.9)}$ $\Delta t = \frac{11.49 + \sqrt{(-11.49)^2 - 4(4.9)(1.5)}}{2(4.9)}$ or $\frac{11.49 - \sqrt{(-11.49)^2 - 4(4.9)(1.5)}}{2(4.9)}$ $\Delta t = 2.21\text{ s or } 0.14\text{ s}$ The correct time is 2.21 s. The smaller time is when the ball reaches 1.5 m on the way up. The larger time is when the ball hits the 1.5 m mark on the way down, which is what you were looking for. 3. Calculating the y-component of the final velocity: $v_{2_y}^2 = v_{1_y}^2 + 2a_y\Delta d_y$ $v_{2_y}^2 = (+11.49\text{ m/s})^2 + 2(-9.8\text{ m/s}^2)(+1.5\text{ m})$ $v_{2_y} = \sqrt{(+11.49\text{ m/s})^2 + 2(-9.8\text{ m/s}^2)(+1.5\text{ m})}$ $v_{2_y} = 10.13\text{ m/s}$ Choose the positive root.	To find the horizontal component of the initial velocity, you can use the cosine function (because the x-component is adjacent to the known angle). $v_{1_x} = 9.64\text{ m/s}$ $a_x = 0\text{ m/s}^2$ 2. Once you know time, you can solve for Δd_x. $v_x = \frac{\Delta d_x}{\Delta t}$ Rearranging: $\Delta d_x = v_x\Delta t$ $\Delta d_x = (9.64\text{ m/s})(2.21\text{ s})$ $\Delta d_x = 21.3\text{ m}$

4. Putting the scalar x- and y-components of the final velocity together, you get:



Applying the Pythagorean theorem:

$$v_2^2 = (9.64\ \text{m/s})^2 + (10.13\ \text{m/s})^2$$
$$v_2 = \sqrt{(9.64\ \text{m/s})^2 + (10.13\ \text{m/s})^2}$$
$$v_2 = 14.0\ \text{m/s}$$

$$\tan\theta = \frac{10.1\ \text{m/s}}{9.64\ \text{m/s}}$$
$$\theta = \tan^{-1}\left(\frac{10.13\ \text{m/s}}{9.64\ \text{m/s}}\right)$$
$$\theta = 46.4°$$

Paraphrase:

Therefore, the

- horizontal range is 21 m.
- time of flight is 2.2 s.
- final velocity is 14 m/s [46° below horizontal], or [R 46° D].

Example 3

If the football player had kicked the ball as hard as he could, was there anything else he could have done to give the ball a longer horizontal range?

Solution

Yes; he could have kicked the ball at, or at least closer to, a 45° angle!

Now use the Projectile Motion simulation again to set up your own problems (you set the initial conditions) and solve for range, maximum height, and time of flight. Then check your answers using the simulation. Don't forget that you need to measure the maximum height using the measuring tool in the program, and that it always fires the projectile at a minimum height of 1.2 m above the ground. This exercise is good practice for the Final Test.

Support Questions

13. A projectile is launched so that its point of launch is higher than its landing point.

a) When is the vertical velocity at a maximum?

b) When is the horizontal velocity at a maximum?

c) When is the vertical velocity at a minimum?

d) What is the acceleration of the object at the very top of its path?

e) Which will take longer: the upward motion or the downward motion?

Suggested answer

14. Shereen throws her pencil out of her classroom window, 9.0 m above the ground, with a velocity of 7.0 m/s [43° above the horizontal].

a) What is the time of flight?

b) What is the horizontal distance travelled?

c) What is the velocity on impact?

d) At what angle could Shereen have thrown the pencil to get a greater horizontal displacement?

Suggested answer

15. Imagine that you are using a bow and arrow to shoot a falling target. Where should you aim? Should you aim slightly above the target, slightly below the target, or directly at it? Explain.

If you are still having trouble with this, or to help confirm that your answer is correct, take a look at this target practice simulation.

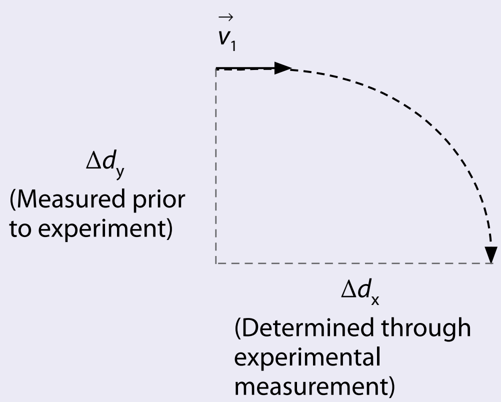
Suggested answer

Projectile Motion Investigation

Note: You will be asked to submit your "Analysis" and "Conclusion" for this investigation in the Key Questions for this lesson.

Objective: The objective of this investigation is to project an object horizontally from some point above ground level and use the measurements to determine the object's initial and final velocity.

Your object may be a ball projected from a window, an eraser projected from a desk, and so on.



You may decide to make this a personal challenge to see how far you can possibly kick or throw a ball, using a horizontal aim. Regardless, you will want to make sure that no one will get hurt and nothing will get damaged, when you kick the ball. For example, ask yourself:

- Is the object travelling along a clear path?
- Is there any chance that someone might walk into the path of the object?
- Can the object itself or the surface on which it is falling get damaged?

Hypothesis: A hypothesis is a statement in which you state what is expected. In this case, try to estimate the velocity your object will have, initially. For example, if you think that you can kick a ball a horizontal distance of 20 m and that it will land in 2 s, your estimated initial velocity might be 10 m/s. Then, you need to estimate whether you think that the final velocity will be higher or lower than the initial velocity. Look back at some of the examples you've completed. What do you think? Making estimations like this makes experimentation more meaningful.

Materials: The materials that you use are up to you. You must have an object that can be safely projected and you must have a measuring tool.

Procedure:

1. Measure the initial height from which the object will be projected (Δd_y). When taking measurements, be sure to read the measuring tool at eye level to avoid error due to a parallax—an apparent shifting of lines due to observations made at an angle.
2. Project the object horizontally from the raised surface using a technique that can be measured. (This might be as simple as a controlled push.) Observe and mark where the object lands.
3. Measure and record the object's horizontal range (Δd_x), again making sure that the measurement is made by standing directly above the mark and measuring tool, as opposed to at an angle.
4. Repeat this process five times, using the same procedure. Use the average of these measurements in further calculations.
5. In this investigation, there may be a great deal of variation in your results, depending on the method used for launching. However, it is important to note that in scientific experimentation, repeating a procedure multiple times and averaging the results is a good method of reducing error due to erroneous results. Keep this in mind for future investigations. Now, complete this investigation by answering the Key Questions that follow.

Key Questions

Now work on your Key Questions in the online submission tool. You may continue to work at this task over several sessions, but be sure to save your work each time. When you have answered all the unit's Key Questions, submit your work to the ILC.

(33 marks)

Note: Key Question 4 is based on the projectile motion investigation from the previous section.

4. Using the data that you have just collected, complete the Analysis questions and Conclusion statement for the projectile motion investigation.

Analysis:

a) Construct a labelled drawing of your investigation, indicating the object projected, the height from which it was projected, and its horizontal range. **(4 marks)**

b) Using the known vertical height (Δd_y) and the average horizontal range (Δd_x), perform the necessary calculations to determine the initial velocity of your object. **(3 marks)**

c) State two possible sources of error (reviewed here, or others that you can think of) and how you worked to reduce the possibility or effects of these errors. **(2 marks)**

d) Describe any safety precautions taken when carrying out this investigation. **(1 mark)**

Conclusion:

Write a conclusion by filling in the blanks in the following statement. Note that the statement is concise and is directly related to the objective of the investigation.

When projecting a ______________________ from a height of ______________________, the average horizontal range was measured to be ______________________ and the magnitude of the initial velocity was calculated to be ______________________.

(2 marks for completed conclusion)

Note: Key Questions 5, 6, and 7 are not based on the projectile motion investigation from the previous section.

5. A projectile is launched so that its point of launch is lower than its landing point. **(5 marks: 1 mark each)**

a) When is the vertical velocity at a maximum?

b) When is the horizontal velocity at a maximum?

c) When is the vertical velocity at a minimum?

d) What is the acceleration of the object at the very top of its path? Explain.

e) Which will take longer: the upward motion or the downward motion?

6. A child sitting in a tree throws his apple core from where he is perched (4.0 m high) with a velocity of 5.0 m/s [35° above the horizontal], and it hits the ground right next to his friend.

a) How long does it take for the apple core to hit the ground? **(3 marks)**

b) How far from the base of the tree will the apple core land? **(3 marks)**

c) What is the velocity of the apple core on impact? **(4 marks)**

7. Describe three ways that understanding projectile motion and relative velocity could help you improve your success in a basketball game. **(6 marks)**

Save your answers to the Key Questions in the online submission tool. You'll be able to submit them when you've finished all of the Key Questions for this unit. Now go on to Lesson 3!