SPH4U-C

Unit 1

Dynamics

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Introduction

Over the last century, transportation has progressed from the rickety "horseless carriage"—an early term for an automobile—to the sleek, quiet, and very fast cars and planes that we enjoy travelling in today. Not only has our means of transportation changed for the better, but the way in which we navigate has improved dramatically as well. The Global Positioning System (GPS), for instance, can safely guide you to your destination in cities you have never visited before, helping you to avoid traffic jams and preventing you from driving the wrong way down one-way streets.

These advances are the result of our knowledge of physics. Our ability to travel safely and quickly around the world depends on knowledge first discovered by scientists like Sir Isaac Newton and Albert Einstein. In this unit, you will learn about the physics that explain forces and motion, and govern transportation.

Overall Expectations

After completing this unit, you will be able to

- solve problems involving projectile motion
- establish frames of reference and explain their importance
- solve problems related to relative motion using vector components and mathematics
- solve problems involving forces and motion
- describe the relationships between variables affecting uniform circular motion
- distinguish between inertial and non-inertial frames of reference and real and fictitious (apparent) forces

SPH4U-C

Lesson 1

Relative Motion

Introduction

Imagine that you are taking part in a canoe race across a river. You need to reach the other side first, regardless of where you land on the opposite side. How can you do this? Should you aim to cross the river somewhat upstream, to counteract the effect of the current? Should you aim to cross somewhat downstream, to travel with the current? Or, should you travel straight across, perpendicular to the current? By the end of this lesson, you will know the answer. After studying the lesson's explanation of relative motion and frames of reference, you will be able to figure out how to win this race!

This lesson will also help you to understand how airplanes navigate—a technology that has made the world as we know it very different from the world that our ancestors knew. Now, with technology like the Global Positioning System (GPS), air travel has greatly advanced. Although we are often amazed at the way that technology like the Internet has changed our lives, it might be argued that the airplane brought about even more dramatic change, in its time. While all kinds of information can "travel" worldwide over the Internet, air travel created a world in which real things can travel around the globe at a surprising pace. Whether a package needs to be delivered overnight or a relative has to travel halfway around the world to join you, they almost always arrive quickly and safely.

There are potential drawbacks to air travel though; it uses a lot of energy and spreads pollution into the upper atmosphere. Airports and airplanes are also places where infectious diseases can easily spread among travellers. However, one of air travel's many advantages is that it enables aid workers to fly to disaster-stricken countries all over the globe, just hours after earthquakes and other events occur.

Safe air travel simply relies on an understanding of some basic physics, including the concept of relative velocity.

Planning Your Study

You may find this time grid helpful in planning when and how you will work through this lesson.

Suggested Timing for This Lesson (hours)	
Frames of Reference	½
Scalar and Vector Quantities	½
Vector Addition and Relative Velocity	1
Vector Subtraction and Relative Velocity	1
Activity: More Practice with Vector Addition and Subtraction	½
Navigation and Our Society and Environment	½
Key Questions	½

What You Will Learn

After completing this lesson, you will be able to

- establish frames of reference and explain their importance
- resolve a vector into its components
- add and subtract vectors using their components
- solve problems related to relative motion using vector components and mathematics
- explain applications of relative velocity in the world and their potential impacts on the environment and society

Frames of Reference

Did you know that airplanes use different runways on different days so that they can take off into the wind? Doing so increases the airplane's airspeed (that is, its speed with respect to the air) so that it requires a lower groundspeed in order to lift off the ground. It therefore requires less time and distance for acceleration. Airplanes also land into the wind. The effect of the wind makes the velocity of the plane, with respect to the ground, slower, enabling the plane to slow to a stop over a shorter distance. Headwinds, tailwinds, and crosswinds are all important to a pilot, just as winds and currents are important to sailors. Proper navigation requires knowing that the motion of the vehicle you are operating, with respect to the ground, can be altered by other forces.

As you are reading this, are you moving or are you stationary? Perhaps you are in a moving car, stationary with respect to the person next to you, but moving with respect to the ground. Perhaps you are sitting very still at home, totally stationary, unless, of course, you consider the planets, stars, and sun, as you are moving with respect to them. Or, are they moving with respect to you? Perhaps they are, depending on the frame of reference used.

A frame of reference is the point of view from which observations and measurements of motion are made. Establishing a frame of reference for a motion is important. It clarifies how the motion appears, relative to a particular set of coordinates. For example, imagine that you are standing at a curb, watching a bus drive by. A passenger on the bus happens to be tossing a ball up and down as she rides along. Through the bus's windows, you can see the ball going up and down, just as the passenger on the bus sees it. However, because the bus is moving to your right as it passes by, you see the ball moving not only up and down, but to your right, as well. It has horizontal motion, as well as vertical motion, relative to you and the ground. Relative to the passenger and the bus, though, the ball only has vertical motion. It is staying above that same spot on the floor of the bus. Because the frame of reference of the bus, in which the ball is being tossed, is in motion with respect to you, the ball's motion appears different to you than it does to the passenger.

Example

Watch the following demonstration called Frame of Reference. Then, consider the following question:

Is the path of the ball, as observed by the sailor, the same as the path observed by the observer on the shore? Why do they differ?

Solution

From the point of view (or frame of reference) of the sailor:

The sailor is stationary, with respect to the boat. He is initially holding the ball stationary, relative to the boat. When he drops the ball, it falls straight downward, with respect to him and the boat. He observes a vertical downward path, relative to the boat.

From the point of view (or frame of reference) of the observer on the shore:

Initially, the sailor, ball, and boat are all moving to the right, relative to the shore. The ball, having the same horizontal velocity as the boat, continues to have that horizontal velocity when it is dropped. Therefore, relative to the observer on the shore, the ball is moving to the right, as it moves downward.

It is obvious that frames of reference are important in clarifying your observation and measurements of motion, especially when one frame of reference is in motion, relative to another.

Support Questions

Be sure to try the Support Questions on your own before looking at the suggested answers provided. Click on each "Suggested answer" button to check your work.

1. Define "frame of reference" and explain why establishing a frame of reference for observations is important. Use an example to aid your explanation. Suggested answer
2. Ask someone to stand in front of you while repeatedly tossing something straight up and down. Describe the path of the object. Now ask the person to continue tossing the object up and down as they walk across the room in front of you. Observe the path of the object now. How does the path of the object, relative to the person tossing the object, compare to the path of the object relative to you, the stationary observer? Why does it appear different to you? Suggested answer

Scalar and Vector Quantities

In fully describing motion, both scalar and vector quantities are used because both magnitude (that is, size) and direction are important. Scalar quantities, such as time, mass, speed, and distance have numerical values and a unit of measurement. For example:

Time = t = 4 s

Speed = v = 80 m/s

Distance = Δd = 10 m

Mass = m = 10 kg

Vector quantities, such as displacement, velocity, and acceleration, take direction into account and therefore have a number, unit, and direction. For example:

Displacement = $\Delta\vec{d}$ = 40 m [east]

Velocity = $\vec{v}$ = 80 m/s [south]

Acceleration = $\vec{a}$ = 9.8 m/s^2 [down]

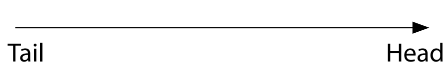
Notice that variables representing vectors have arrows above them. Whenever you specify a vector quantity using a value and a direction, you add the vector symbol (arrow) above the variable.

Vector quantities have very practical uses. For example, if you went on a hike and ended up getting lost, you would be worried about your position, relative to your starting point. You may use a compass to help you get your bearings. It is your position—the path length and direction from your starting point—that matters. Similarly, if you were in an airplane travelling west from Ottawa to Vancouver and then took the return trip, you might discover that one trip took less time than the other, due to wind directions. The velocities of the airplane and the winds, as well as their directions with respect to one another, matter.

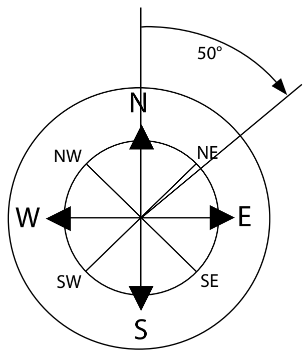
There are, of course, situations where direction doesn't really matter and scalar quantities are useful, instead. For example, if you were driving too fast on your way to work and passed a parked police cruiser, no one would care much about your direction; it would be your speed—the distance travelled, with respect to time—that would get you into trouble!

Describing Vectors

A vector is represented as an arrow, with a "head" and a "tail," as shown below. The arrowhead shows the vector's direction.



You will often work with vectors that do not point directly north, south, east, or west. Instead, the vectors will include angles. There are several acceptable ways to communicate a vector's direction. In the diagram shown below, the vector points 50° east of due north, or [N 50° E]; throughout the rest of this course, we will use this convention. (It could also be expressed as [E 40° N], because of complementary angles.)

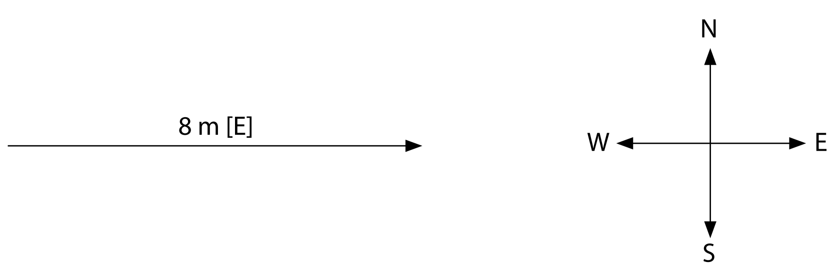


Note: Diagrams accompanying the examples and questions in this unit are not drawn to scale. This is common practice when solving problems involving vectors, since math, instead of actual measurements, is primarily used for problem solving.

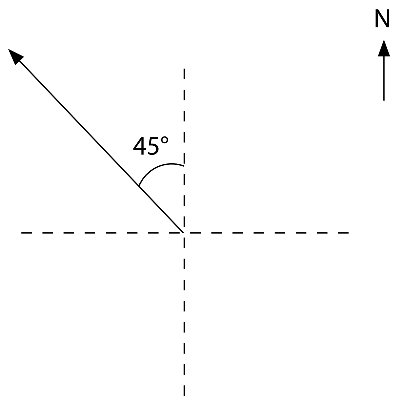
There are situations in which it is useful to draw scale diagrams of vectors. When you do this, always include a statement such as, "Let 1 cm = 2 m," to communicate the size of the vector being drawn. For example, to draw a vector that is 10 m long, you may draw a vector that is 5 cm long on your paper and include a statement that "1 cm = 2 m."

In this lesson, however, you will not need to draw vectors to scale as you will find answers using math, as opposed to measurement. Make sure that you always include direction. (A compass may be included in these drawings, either showing four directions, as illustrated below, or just an arrow pointing upward to the letter "N" for north.) Here are some examples.

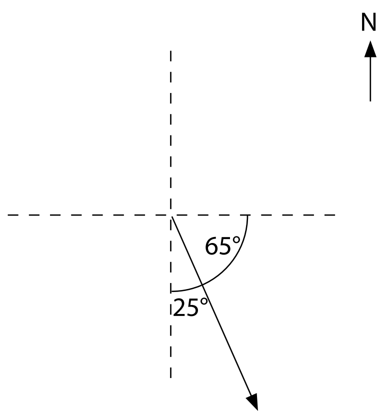
Example 1: A displacement of 8 m [E]



Example 2: A velocity of 80 km/h [NW] or [N 45° W]



Example 3: An acceleration of 5 m/s^2 [S 25° E] or [E 65° S]

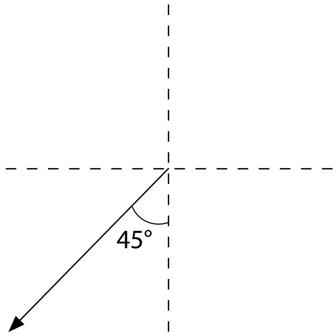


There are many types of vector quantities (for example, position, displacement, velocity, acceleration, force, and so on), but this lesson will focus on velocity vectors, since they are most relevant to the topic of this lesson. The skills you learn, however, are ones that you can use with all types of vectors.

Support Questions

3. Describe the following vector directions.

a)



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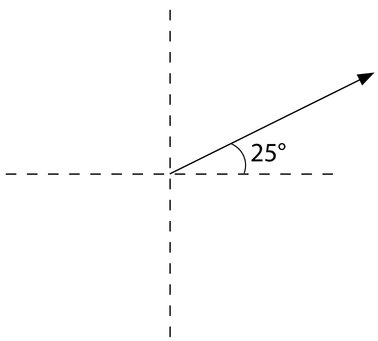
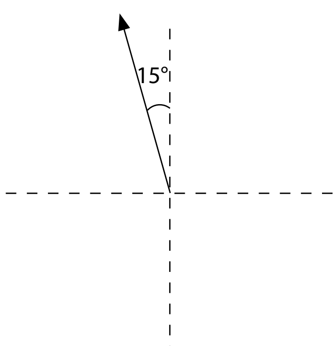


c)



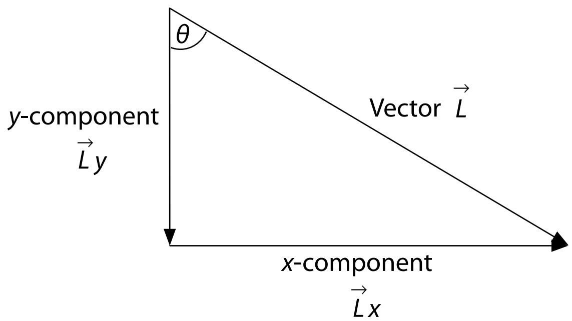
Suggested answer

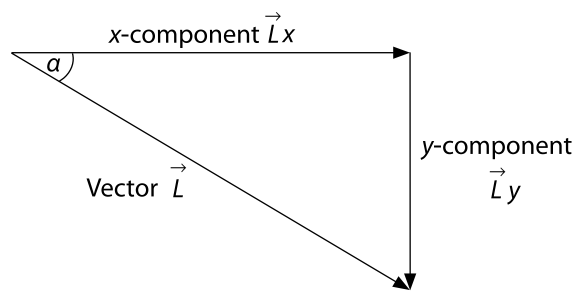
Resolving a Vector into its Components

In order to analyze and solve problems related to vectors, you will often need to add and subtract them. While this can be done using graphical techniques and scale diagrams or more complex trigonometry and geometry, in this unit you will use vector components.

A vector that does not lie directly along a horizontal or vertical axis can be thought of as being made up of two parts, called components. It has a vertical part and a horizontal part. These two components and the vector together create the shape of a right triangle, in which

- the horizontal part is the x-component of the vector.
- the vertical part is the y-component.
- the vector itself is the hypotenuse.





Notice from the diagrams that components add together to create a vector. Each component has its own scalar value (magnitude) and direction. The scalar value can be positive or negative, depending on the way directions are assigned on the Cartesian plane. For example, it's standard practice to designate up and right as positive. However, if the problem only involves actions moving down and to the right, it makes more sense to designate down and right as positive.

Notice also that you can draw the vertical component first, followed by the horizontal one, or draw the horizontal component first, followed by the vertical one. The final vector (hypotenuse) will be the same; only the angles will change. Remember that $\theta + \alpha = 90°$.

Because a vector and its two components make up a right triangle, trigonometric functions (sine, cosine, and tangent) can be used to mathematically resolve a vector into its components. Here's a quick review:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{Lx}{L}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{Ly}{L}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{Lx}{Ly}$$

where θ is the known angle of the vector, the "opposite side" is the side opposite the known angle, and the "adjacent side" is the side helping to make up the angle.

Similarly, if we use angle α from the second triangle, the functions become:

$$\sin\alpha = \frac{Ly}{L}$$

$$\cos\alpha = \frac{Lx}{L}$$

$$\tan\alpha = \frac{Lx}{Ly}$$

Example 1

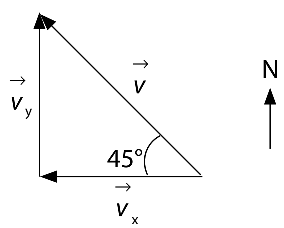
Resolve a velocity vector of 25 m/s [N 45° W] into its components.

Solution

Given: $\vec{v} = 25$ m/s [N 45° W]

Required: v_x and v_y

Analysis and solution: The first step is to draw the vector showing the *x*- and *y*-components. Trigonometric functions can then be used to solve for the components.



You must now apply the appropriate trigonometric function to determine the *x*-component of the vector. Because the *x*-component is “adjacent” to the known angle, you use the cos function.

Note regarding symbols: Straight lines bracketing a velocity vector symbol, $|\vec{v}|$, indicate that only the magnitude (that is, the scalar component) of the vector is being considered in these calculations. A simpler way to show this is to use the letter representing the vector without the arrow above it; thus, the scalar component of $\vec{v}$ is simply v.

When dealing with one-dimensional scalar components, we designate a positive (+) direction and a negative (–) direction for the values on a number line. At the end of the problem, we use those designations to determine the actual direction of the resultant (final) vector.

Since $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$,

$$\cos\theta = \frac{v_x}{v}.$$

Rearranging and substituting:

$$v_x = v\cos\theta$$

$$v_x = (25\text{ m/s})\cos 45°$$

$$v_x = 18\text{ m/s}$$

The next step is to apply the appropriate trigonometric function to determine the *y*-component of the vector. Because the *y*-component is located "opposite" the known angle, you use the sin function.

Since $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$,

$\sin\theta = \dfrac{v_y}{v}$.

Rearranging and substituting:

$$v_y = v\sin\theta$$

$$v_y = (25\text{ m/s})\sin 45°$$

$$v_y = 18\text{ m/s}$$

Notice that the components are equal, when the angle is 45°.

Paraphrase: Therefore, the *x*-component of this vector, v_x, is 18 m/s and the *y*-component, v_y, is 18 m/s. Symbolically, this is written as $\vec{v}_x$ = 18 m/s [W] and $\vec{v}_y$ = 18 m/s [N].

Example 2

Resolve a velocity vector of 4 m/s [W] into its components.

Solution

Given: $\vec{v}$ = 4 m/s [W]

Required: v_x and v_y

Analysis and solution: Because the vector falls along the east–west horizontal line, there is no vertical component. The vector only has a horizontal component.

Paraphrase: Therefore, v_x = 4 m/s and v_y = 0 m/s.

Support Questions

4. Resolve the following vectors into their components.

a) 17 m/s [N]

b) 40 m/s [S 45° E]

c) 95 m/s [N 20° W]

Suggested answer

Vector Addition and Relative Velocity

Think back to the situation described at the beginning of the lesson, in which a ball was being tossed up and down relative to a passenger on a bus, with the bus moving relative to the ground. As well as introducing frames of reference, this example can also be used to demonstrate the concept of relative velocity—the velocity of an object relative to its frame of reference. Problem solving with relative velocity involves an analysis of the motion of an object within a frame of reference that is also in motion. Consider, for example, the ball and bus. Because the bus was moving, the total (that is, the net) velocity of the ball relative to the ground would be:

$\vec{v}_{\text{net}} = \vec{v}_{\text{ball}} + \vec{v}_{\text{bus}}$.

In vector algebra, the plus sign (+) actually describes how the vectors are connected. It means "connect the head of the first vector to the tail of the second vector." In the example shown above, the head of $\vec{v}_{\text{ball}}$ is connected to the tail of $\vec{v}_{\text{bus}}$.

Relative velocity problems can be tricky. The systematic use of subscripts with your variables is a technique that can help you to reason through these problems. Using this method, each variable in a relative motion problem is given two subscripts. The first subscript refers to the object being discussed and the second refers to its frame of reference.

To understand this use of subscripts, consider a situation where there is a boat travelling on a river in which there is a current, and you want to know the velocity of the boat with respect to the shore. Using "B" to represent the boat, "W" to represent the water, and "S" to represent the shore:

- $\vec{v}_{\text{BW}}$ represents the velocity of the boat with respect to (or relative to) the water.
- $\vec{v}_{\text{WS}}$ represents the velocity of the water with respect to the shore.
- $\vec{v}_{\text{BS}}$ represents the velocity of the boat with respect to the shore. This is sometimes referred to as "velocity relative to ground." It is the *net* velocity, which is the sum of the other two velocities.

The equation that would help you to find the total velocity of the boat relative to the shore is written below. Notice the "subscript rules."

The inner subscripts match each other.

$$\vec{v}_{\text{BS}} = \vec{v}_{\text{BW}} + \vec{v}_{\text{WS}}$$

The outer subscripts match the subscripts of the resultant.

The next four examples involve collinear vectors (vectors that share the same line) and non-collinear vectors (vectors that do not fall on the same line). They will all use the following scenario:

Carl the canoeist can consistently row to maintain a speed of 1.5 m/s in still water. Right now, though, he is travelling in a river that has a current of 1.0 m/s [S].

Example 1: Adding Collinear Vectors

If the canoe is heading downstream in the river, what will Carl's velocity be, relative to a stationary observer on the shore?

Solution

Given: Let W represent water, C represent the canoe, and S represent the shore.

The current is a measure of the velocity of the water relative to the shore. Therefore, $\vec{v}_{\text{WS}} = 1.0 \text{ m/s [S]}$.

The canoe's velocity relative to water, still or otherwise, is 1.5 m/s. Since the canoe is moving downstream, it is moving in the direction of the current. Therefore, $\vec{v}_{\text{CW}} = 1.5 \text{ m/s [S]}$.

Required: $\vec{v}_{\text{CS}}$

Analysis and solution:

Since these vectors have the same direction, they can simply be added together.

$$\vec{v}_{\text{CS}} = \vec{v}_{\text{CW}} + \vec{v}_{\text{WS}}$$

We now designate the direction south as positive, and use the scalar components of the vectors to solve the problem.

$$\begin{aligned} v_{\text{CS}} &= 1.5 \text{ m/s} + 1.0 \text{ m/s} \\ &= 2.5 \text{ m/s} \end{aligned}$$

Paraphrase: Therefore, Carl's velocity relative to a stationary observer on the shore will be 2.5 m/s [S].

Example 2: Adding Collinear Vectors with Opposite Directions

If the canoe is heading upstream in the river, what will Carl's velocity be, according to an observer on the shore?

Solution

Given: Since the canoe is moving upstream, this time, it is moving in an opposite direction to the current. Therefore,

$$\vec{v}_{\text{CW}} = 1.5 \text{ m/s [N]}$$

$$\vec{v}_{\text{WS}} = 1.0 \text{ m/s [S]}$$

Required: $\vec{v}_{CS}$

Analysis and solution:

We will once again designate the direction north as positive, and use the scalar components of the vectors to solve the problem. Notice that the 1.0 m/s is subtracted. This is because its direction is south, which is negative using our direction designation.

$$v_{CS} = 1.5 \text{ m/s} - 1.0 \text{ m/s}$$
$$= 0.5 \text{ m/s}$$

Since the answer was positive, the final direction is north.

Paraphrase: Therefore, when Carl is headed upstream, relative to an observer on the shore, his velocity will be 0.5 m/s [N].

Example 3: Adding Non-collinear Vectors that Form a Right Triangle

If the canoe heads west, directly across the river perpendicular to the current, what will Carl's velocity be, according to an observer on the shore?

Solution

Given:

$\vec{v}_{CW} = 1.5$ m/s [W]

$\vec{v}_{WS} = 1.0$ m/s [S]

Required: $\vec{v}_{CS}$

Analysis and solution:

$$\vec{v}_{CS} = \vec{v}_{CW} + \vec{v}_{WS}$$
$$\vec{v}_{CS} = 1.5 \text{ m/s [W]} + 1.0 \text{ m/s [S]}$$

Notice here that there are two directions that differ, but they are not simply opposites; they are non-collinear, meaning that they don't fall along the same line. Whenever this happens, the vectors must be added graphically.

To graphically add two vectors, follow this process:

Sketch the first vector.

Starting at the head of the first vector, sketch the second vector.

Draw the resultant vector by starting at the tail of the first vector (where you started drawing) and creating a straight line to the head of the last vector (where you finished drawing). When adding vectors, the head (arrow) of the resultant always points to the head that finishes the sketch of added vectors. This method works no matter how many vectors are being added. In Grade 11, you would have drawn this diagram to scale and measured the resultant vector in order to find the resultant. This year, we will use trigonometric methods to solve the problem.

Because the two vectors being added and the resultant form a right triangle, the Pythagorean theorem ($a^2 + b^2 = c^2$ where "c" is the hypotenuse) can be used to determine the magnitude of the resultant vector.

$$v_{CS}^2 = v_{CW}^2 + v_{WS}^2$$

$$v_{CS} = \sqrt{v_{CW}^2 + v_{WS}^2}$$

$$v_{CS} = \sqrt{(1.5\text{ m/s})^2 + (1.0\text{ m/s})^2}$$

$$v_{CS} = 1.8\text{ m/s}$$

You now know the magnitude of the resultant velocity, but a vector requires direction. You can see that it is pointing in a direction between west and south, but you still need to know the angle. The required angle is always the angle that is partly formed by the tail of the resultant vector. That angle, in this situation, is indicated with the symbol "θ" (theta), shown below:

Because you know the magnitudes of all three sides, any of the three trigonometric functions can be used to find this angle. It is safest, though, to use the tan function, because you are then using numbers that were given in the question. Therefore:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{1.0\text{ m/s}}{1.5\text{ m/s}}$$

$$\theta = \tan^{-1}\left(\frac{1.0\text{ m/s}}{1.5\text{ m/s}}\right)$$

$$\theta = 34°$$

Paraphrase: Therefore, Carl's velocity, relative to an observer on the shore, will be 1.8 m/s [W 34° S].

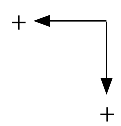
We can also solve this problem using scalar components, as we did earlier. This method will be much more useful in situations where the triangle formed does not contain a 90° angle.

The equation $\vec{v}_{cs} = \vec{v}_{cw} + \vec{v}_{ws}$ is in two dimensions. We will break the vectors into their x- and y-components, then solve the equation twice: once in the horizontal direction and once in the vertical direction.

The answers to each equation generate the components of the final answer. These get combined using the Pythagorean theorem and the angle obtained by using the inverse tangent function.

Solution

Let left and down be positive.



x-component:

$$v_{CS_x} = v_{CW_x} + v_{WS_x}$$
$$v_{CS_x} = 1.5 \text{ m/s} + 0 \text{ m/s}$$
$$= 1.5 \text{ m/s}$$

y-component:

$$v_{CS_y} = v_{CW_y} + v_{WS_y}$$
$$v_{CS_y} = 0 \text{ m/s} + 1.0 \text{ m/s}$$
$$= 1.0 \text{ m/s}$$

Now find the magnitude of v_{cs}:

$$v_{CS}^2 = v_{CS_x}^2 + v_{CS_y}^2$$
$$v_{CS} = \sqrt{v_{CS_x}^2 + v_{CS_y}^2}$$
$$v_{CS} = \sqrt{(1.5 \text{ m/s})^2 + (1.0 \text{ m/s})^2}$$
$$v_{CS} = 1.8 \text{ m/s}$$

To find the angle, use $\theta = \tan^{-1} \dfrac{v_{CS_y}}{v_{CS_x}}$

$$= \frac{1.0 \text{ m/s}}{1.5 \text{ m/s}}$$
$$= 34°$$

Because the value of the x-component is positive, its direction is west.

Because the value of the y-component is positive, its direction is south.

Thus, the direction is written as [W 34° S].

Example 4: Adding Non-collinear Vectors that Do Not Form a Right Triangle

Now, Carl heads his canoe at an angle of 35° upstream. What will his velocity be, this time, with respect to the observer on the shore?

Solution

Given:

$\vec{v}_{CW} = 1.5$ m/s [W 35° N]

$\vec{v}_{WS} = 1.0$ m/s [S]

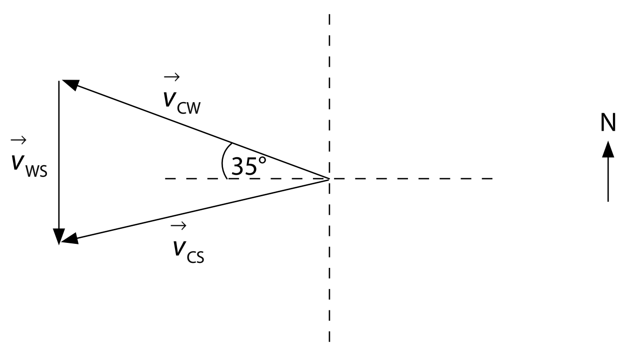
Required: $\vec{v}_{CS}$

Analysis and solution:

$\vec{v}_{CS} = \vec{v}_{CW} + \vec{v}_{WS}$

$\vec{v}_{CS} = 1.5$ m/s [W 35° N] + 1.0 m/s [S]

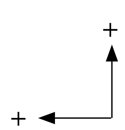
Here, again, you have non-collinear vectors, so you can graphically add them according to the process shown in the previous example.



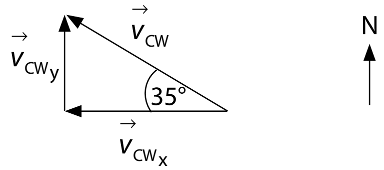
In this situation, you end up with a triangle that is not a right triangle. You could solve for the unknown side and angle using the sine and cosine laws (*not* functions!). However, in this part of the lesson, you will continue to use components. We will solve the problem using the sine and cosine laws after using the component method.

Once you figure out that the vectors need to be added together, the next step is to find the x- and y-components of each vector.

In this situation, [N] will be the positive y-direction and [W] will be the positive x-direction. If the components point in the opposite directions, they are considered to be negative.



Components of $\vec{v}_{\text{CW}}$:

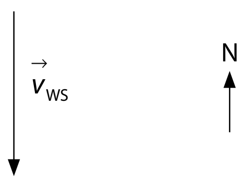


$$v_{\text{CW}_x} = v_{\text{CW}} \cos 35°$$
$$v_{\text{CW}_x} = (1.5 \text{ m/s})\cos 35°$$
$$v_{\text{CW}_x} = 1.23 \text{ m/s}$$

$$v_{\text{CW}_y} = v_{\text{CW}} \sin 35°$$
$$v_{\text{CW}_y} = (1.5 \text{ m/s})\sin 35°$$
$$v_{\text{CW}_y} = 0.860 \text{ m/s}$$

We are keeping an extra significant digit during calculations and will round off later.

Components of $\vec{v}_{\text{WS}}$:



$$v_{\text{WS}_x} = 0 \text{ m/s}$$
$$v_{\text{WS}_y} = -1.0 \text{ m/s}$$

Notice that $v_{\text{WS}_y} = -1.0$ m/s because it represents a southerly direction, as opposed to a northerly one, which was chosen to be "positive."

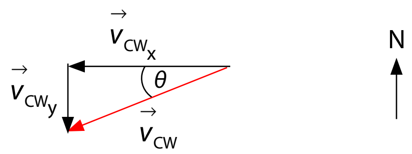
Now, add the individual x-components to get the resultant x-component:

$$v_{\text{CS}_x} = v_{\text{CW}_x} + v_{\text{WS}_x}$$
$$v_{\text{CS}_x} = 1.23 \text{ m/s} + 0 \text{ m/s}$$
$$v_{\text{CS}_x} = 1.23 \text{ m/s}$$

Then, add the y-components to get the resultant y-component:

$$v_{\text{CS}_y} = v_{\text{CW}_y} + v_{\text{WS}_y}$$
$$v_{\text{CS}_y} = 0.860 \text{ m/s} + (-1.0 \text{ m/s})$$
$$v_{\text{CS}_y} = -0.140 \text{ m/s}$$

Next, graphically add the resultant x- and y-components to get the resultant vector:



Because you now have a right triangle (which you always get when you use the component method for adding vectors), you can use the Pythagorean theorem to find the magnitude of the resultant vector.

$$v_{CS}^2 = (1.23 \text{ m/s})^2 + (0.140 \text{ m/s})^2$$
$$v_{CS} = \sqrt{(1.23 \text{ m/s})^2 + (0.140 \text{ m/s})^2}$$
$$v_{CS} = 1.24 \text{ m/s}$$

When rounded to the correct number of significant digits, the answer becomes 1.2 m/s.

Finally, you can use trigonometry to determine the direction of the resultant velocity. (The desired angle is always the one partly composed of the resultant's tail.)

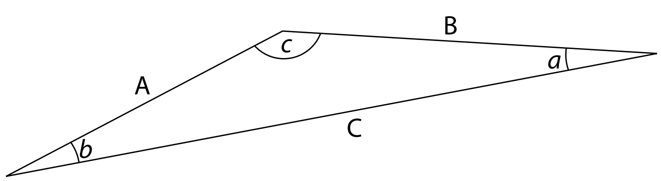
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan\theta = \frac{0.14 \text{ m/s}}{1.2 \text{ m/s}}$$
$$\theta = \tan^{-1}\left(\frac{0.14 \text{ m/s}}{1.2 \text{ m/s}}\right)$$
$$\theta = 7°$$

Since the x-direction was positive, it is west. Since the y-direction was negative, it is south. Therefore, the final direction is [W 7° S].

Paraphrase: Therefore, Carl's velocity, relative to the stationary observer, will be 1.2 m/s [W 7° S].

As mentioned at the beginning of this problem, it can also be solved using the sine law and cosine law. A quick review of these laws follows.

Given a non-90° triangle,

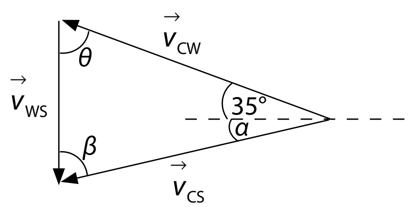


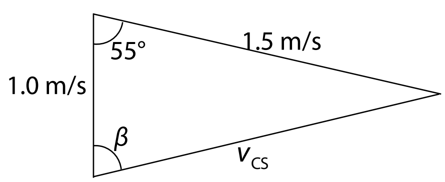
Cosine Law $A^2 = B^2 + C^2 - 2BC\cos a$

$B^2 = A^2 + C^2 - 2AC\cos b$

$C^2 = A^2 + B^2 - 2AB\cos c$

Sine Law $\dfrac{A}{\sin a} = \dfrac{B}{\sin b} = \dfrac{C}{\sin c}$





From the diagram $\theta = 90° - 35° = 55°$

Now use the cosine law to obtain v_{CS}

$(v_{CS})^2 = (1.5 \text{ m/s})^2 + (1.0 \text{ m/s})^2 - 2(1.5 \text{ m/s})(1.0 \text{ m/s})\cos 55°$

$= 1.44 \text{ (m/s)}^2$

$v_{CS} = 1.2 \text{ m/s}$

The angle we need is specified by α. But first we need angle β.

Using the sine law,

$$\frac{v_{CW}}{\sin\beta} = \frac{1.2 \text{ m/s}}{\sin 55°}$$

$\beta = 83°$

Because the sum of the angles of a triangle is always 180°, we can find α as follows:

$\beta + 55° = (35° + \alpha) = 180°$

$\alpha = 6.6°$

Rounded to the correct number of significant digits, the answer is 7°, as was the angle found using the component method.

Support Questions

5. You can only add two vectors if their units and directions match. What must you do if their directions are

a) collinear (sharing the same line), but opposite?

b) non-collinear?

Suggested answer

6. A package of food is being delivered by airplane to a group of people stranded on an island. The package is travelling in an airplane with a velocity of 900.0 km/h [W]. A wind is blowing with a velocity of 75.0 km/h [SW]. What will the velocity of the package be, with respect to the ground, once it is released from the airplane (given that it maintains the velocity of the airplane *and* is affected by the wind)?

Suggested answer

Summary: Vector Addition

Using the component method:

1. Consult the defining equation. The + sign indicates which vectors are to be added.
2. Find the horizontal and vertical components for all the given vectors. Remember to choose a positive and negative system for each direction.
3. Solve the defining equation twice: once in each direction.
4. Combine the two components of the answer using the Pythagorean theorem. Find the angle using the tangent function.

Using the sine and cosine law method:

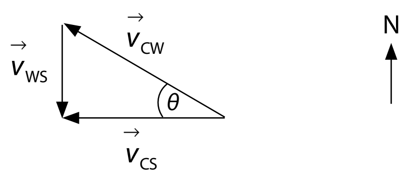
1. Draw a vector diagram illustrating the situation. Consult the defining equation in order to connect the vectors correctly. Your diagram should include all given angles and magnitudes.
2. If necessary, use geometry to determine the value of an interior angle.
3. Solve using the appropriate laws.
4. Express the final angle in terms of standard compass directions.

Vector Subtraction and Relative Velocity

Up to this point, you have been adding vectors. There are, however, times when you need to subtract vectors. To determine whether addition or subtraction is necessary, problem solving is required.

Example

Going back to the canoeing scenario from the previous section, suppose that Carl, in his canoe, must cross the river to a point on the west bank, directly opposite the starting point. At what angle must he head upstream? It was found that he was moving at 1.12 m/s in his canoe. We must also find the velocity of the canoe relative to the water.



Solution

Given: The canoe must follow a path due west, perpendicular to the river bank.

$\vec{v}_{WS} = 1.0$ m/s [S]

$\vec{v}_{CS} = 1.12$ m/s [W]

Required: Heading and speed required to maintain a path due west

Analysis and solution:

Method 1: You know that $\vec{v}_{CS} = \vec{v}_{CW} + \vec{v}_{WS}$. However, the unknown in this situation is the angle that corresponds with the velocity of the canoe, with respect to the water ($\vec{v}_{CW}$). Therefore, you must rearrange the equation to isolate $\vec{v}_{CW}$.

You get:

$\vec{v}_{CW} = \vec{v}_{CS} - \vec{v}_{WS}$

In the standard algebraic vector method, you only *add* vectors. To change this "subtraction" to an "addition," you change the direction of the subtracted vector to its opposite direction, thus creating a new vector. The following sequence illustrates this method:

$\vec{v}_{CW} = \vec{v}_{CS} - \vec{v}_{WS}$

$\vec{v}_{CW} = \vec{v}_{CS} - 1.0$ m/s [S]

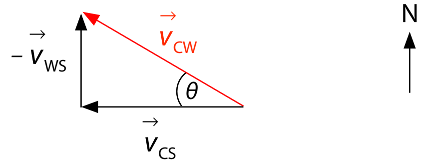
This becomes $\vec{v}_{CW} = \vec{v}_{CS} + (-\vec{v}_{WS})$.

It can also be written as:

$\vec{v}_{CW} = \vec{v}_{CS} + 1.0$ m/s [N]

Notice that the negative vector is just like any other vector and is connected head to tail as before. It has effectively become a new vector, pointing north but called $-\vec{v}_{WS}$.

You know that the direction of the canoe, with respect to the shore, must be due west. Therefore, you can construct your vector addition diagram. The angle of the canoe's heading is indicated.



You know that the magnitude of $\vec{v}_{CS}$ is 1.12 m/s and you also know that the magnitude of $\vec{v}_{WS}$ (the opposite side) is 1.0 m/s. Therefore, you can use trigonometry to calculate the angle.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{1.0 \text{ m/s}}{1.12 \text{ m/s}}$$

$$\theta = \tan^{-1}\left(\frac{1.0 \text{ m/s}}{1.12 \text{ m/s}}\right)$$

$$\theta = 42°$$

Paraphrase: To maintain a path due west across the river, Carl would have to head his canoe 42° upstream.

To find the speed relative to the water, use the Pythagorean theorem to find v_{cw}.

$$v_{CS}^2 + v_{WS}^2 = v_{CW}^2$$

$$(1.12 \text{ m/s})^2 + (1.10 \text{ m/s})^2 = v_{CW}^2$$

Thus $v_{cw} = 1.5$ m/s.

Method 2: You can also use the scalar components method. In this case, you don't need to worry about changing the vector direction and redefining it.

Use the vector equation $\vec{v}_{cw} = \vec{v}_{cs} - \vec{v}_{ws}$ in the *x*-direction and the *y*-direction with components. Then use the Pythagorean theorem and the tan inverse function.

Thus, in the *x*-direction:

$$v_{cw_x} = -1.12 \text{ m/s} - 0 = -1.12 \text{ m/s}$$

In the *y*-direction:

$$v_{cw_y} = 0 - (-1.0 \text{ m/s}) = +1.0 \text{ m/s}$$

Using the Pythagorean theorem:

$$v_{CW} = \sqrt{v_{CW_x}^2 + v_{CW_y}^2}$$
$$= 1.5 \text{ m/s as was found previously.}$$

To find the angle, you would use

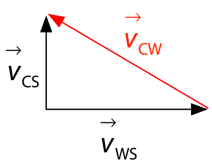
$$\tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\left(\frac{1.0}{1.12}\right) = 42°$$

and seeing that the *y*-value is positive (north) and the *x*-value is negative (west), the direction is [W 42° N].

Other Methods

If you wish to use the sine law and cosine law method, construct the triangle as described earlier. This involves redefining the negative vector. However, a subtraction of vectors can also be done by connecting the two vectors tail to tail. Then, the final vector completing the triangle touches the first vector mentioned in the equation.

Thus $\vec{v}_{cw} = \vec{v}_{cs} - \vec{v}_{ws}$ is constructed by attaching v_{CS} and v_{WS} tail to tail. Then complete the triangle (v_{CW}) and have the head touch $\vec{v}_{CS}$.



You can also connect vectors head to head for a subtraction. But the final vector's head will touch the second vector in the equation. Try it on the problem from the most recent example, above.

Support Questions

7. An airplane flies with an airspeed of 50.0 m/s [E 40° N]. If the velocity of the airplane, according to an observer on the ground, is 30.0 m/s [SE], what is the wind velocity?

Suggested answer

8. Look again at the scenario introduced at the beginning of the lesson, taking into consideration what you have learned so far about relative velocities. The scenario involved a canoe race across a river. Imagine, now, that you are taking part in the race and desperately want to reach the other side first, regardless of where you land on the opposite side. How can you do it? Should you aim to cross the river somewhat upstream, to counteract the effect of the current? Should you aim to cross somewhat downstream, to travel with the current? Or, should you travel straight across, perpendicular to the current?

Suggested answer

Activity: More Practice with Vector Addition and Subtraction

Now open the following activity called Vector Addition and Subtraction.

The purpose of this interactive online activity is to provide experience and practice with vectors and their components, and with the addition and subtraction of these vectors. You will be able to

- create vectors and see their components
- add and subtract vectors instantaneously
- graphically construct situations that you can solve mathematically (on paper) to practise resolving vectors into components, and adding and subtracting them

Note: When using this program, you may find it easier to use vectors with lengths that are whole numbers. Try to adjust the lengths and angles so that you minimize the extra decimals used by the program.

1. Practice with components:

a) Using your mouse, drag to adjust the velocity vector. Note the magnitude and direction of this vector ($\vec{v}_{PA}$).

b) Calculate the components of the vector.

c) Click on the button that says "Show Components." The components of the vector will be displayed. Verify that your calculations are correct.

2. Practice with vector addition:

a) Check the "Vector addition" box and adjust the second vector ($\vec{v}_{AG}$).

b) Calculate the components of the second vector.

c) Calculate the resultant of the two vectors added together, that is, the velocity of the plane, relative to the ground ($\vec{v}_{PG}$).

d) Click on the button that says "Show Resultant" to see the resultant vector. Verify that your calculations are correct.

3. Practice with vector subtraction:

a) Click the Reset button in the top right-hand corner of the applet.

b) Adjust the first vector and check the "Vector subtraction" box. Calculate the components of each vector and subtract the second vector from the first.

c) Click on the button that says "Show subtracted vector" and check your answer.

Navigation and Our Society and Environment

As you are now aware, the concepts you have studied in this lesson have many practical applications, especially in navigation. Understanding the best way to take off in an airplane and navigate the wind currents is just one of the ways in which our world is able to become a faster, and seemingly smaller, place. We have come a long way since 1911, when the first coast-to-coast flight in the United States lasted 49 days (with many stops, of course). Today, we can travel around the world within hours, thanks to thousands of flights reaching all corners of the globe. As mentioned in the introduction, though, flight has its disadvantages, as well as its advantages. For example, a commercial aircraft such as the Boeing 747 can burn about four litres of fuel per second. Such consumption has consequences for our environment, in terms of pollution and use of energy resources. Fortunately, scientists are continuing to discover techniques and develop technology that can allow us to be as energy-efficient as possible.

Support Questions

9. One way in which air travel has become more efficient is through the use of jet streams. Do research to find out what jet streams are and how they influence flight times and fuel consumption.

Suggested answer

Key Questions

Now work on your Key Questions in the online submission tool. You may continue to work at this task over several sessions, but be sure to save your work each time. When you have answered all the unit's Key Questions, submit your work to the ILC.

(18 marks)

1. A ladybug with a velocity of 10.0 mm/s [W] crawls on a chair that is being pulled [W 50° N] at 40.0 mm/s. What is the velocity of the ladybug relative to the ground? **(9 marks)**

2. An airplane is flying to a city due west from its current location. If there is a slight wind blowing to the southwest, in what direction must the plane head (that is, in what direction must it point)? Explain your answer using a diagram. **(3 marks)**

3. Do research to find out how relative velocity is related to the direction in which rockets are launched to send them into space. Explain the benefits (to society and/or the environment) of using a specific direction for launching rockets. **(6 marks: 3 marks for explanation of relative velocity and 3 marks for explanation of benefits)**

Save your answers to the Key Questions in the online submission tool. You'll be able to submit them when you've finished all of the Key Questions for this unit. Now go on to Lesson 2!