lecture_10

February 21, 2017

```
In [1]: %plot --format svg
```

```
In [2]: setdefaults
```

1 Gauss Elimination

1.0.1 Solving sets of equations with matrix operations

The number of dimensions of a matrix indicate the degrees of freedom of the system you are solving.

If you have a set of known output, $y_1,\ y_2,\ ...y_N$ and a set of equations that relate unknown inputs, $x_1,\ x_2,\ ...x_N$, then these can be written in a vector matrix format as:

$y = Ax$

Consider a problem with 2 DOF:

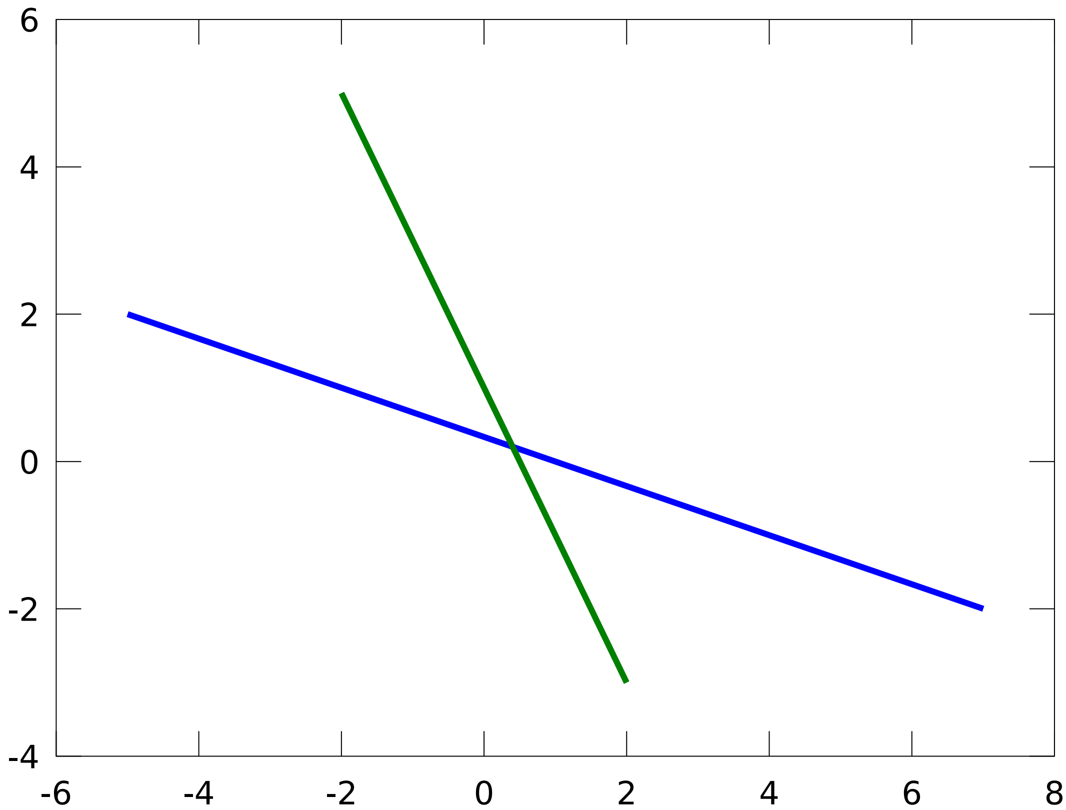
$x_1 + 3x_2 = 1$

$2x_1 + x_2 = 1$

$\left[\begin{array}{cc} 1 & 3 \\ 2 & 1 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$

The solution for x_1 and x_2 is the intersection of two lines:

```
In [3]: x21=[-2:2];
        x11=1-3*x21;
        x21=[-2:2];
        x22=1-2*x21;
        plot(x11,x21,x21,x22)
```




```
In [4]: A=[1,3;2,1]; y=[1;1];
        A\y % matlab's Ax=y solution for x
```

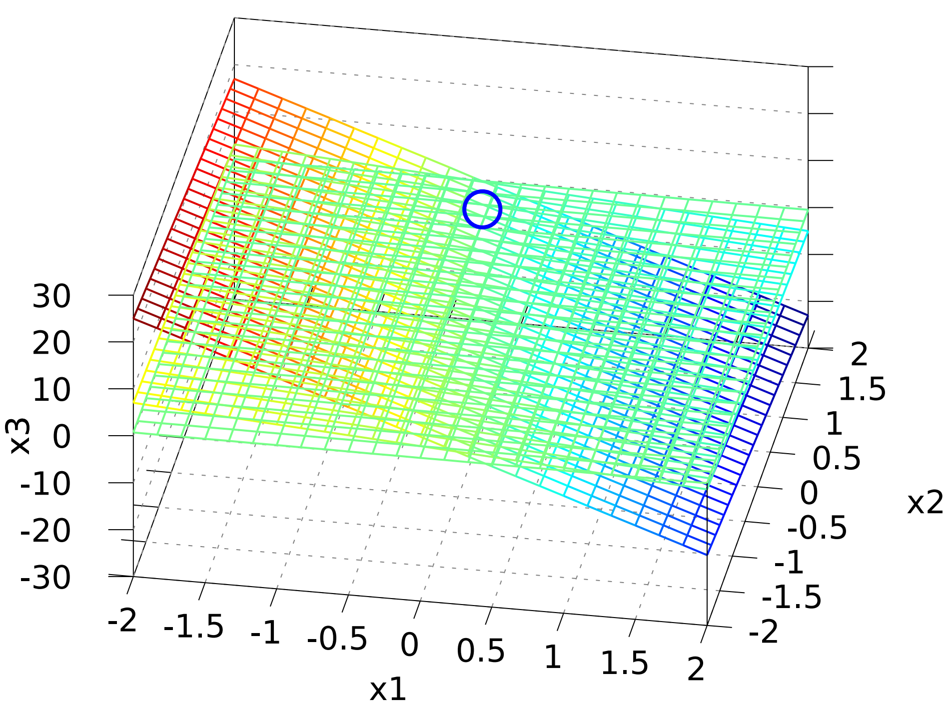
```
ans =

   0.40000
   0.20000
```

For a 3×3 matrix, the solution is the intersection of the 3 planes.
$10x_1 + 2x_2 + x_3 = 1$
$2x_1 + x_2 + x_3 = 1$
$x_1 + 2x_2 + 10x_3 = 1$

$$\begin{bmatrix} 10 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

```
In [9]: N=25;
        x11=linspace(-2,2,N);
        x12=linspace(-2,2,N);
```

```
[X11,X12]=meshgrid(x11,x12);
X13=1-10*X11-2*X12;

x21=linspace(-2,2,N);
x22=linspace(-2,2,N);
[X21,X22]=meshgrid(x21,x22);
X23=1-2*X11-X22;

x31=linspace(-2,2,N);
x32=linspace(-2,2,N);
[X31,X32]=meshgrid(x31,x32);
X33=1/10*(1-X31-2*X32);

mesh(X11,X12,X13);
hold on;
mesh(X21,X22,X23)
mesh(X31,X32,X33)
x=[10,2, 1;2,1, 1; 1, 2, 10]\[1;1;1];
plot3(x(1),x(2),x(3),'o')
xlabel('x1')
ylabel('x2')
zlabel('x3')
view(10,45)
```



After 3 DOF problems, the solutions are described as *hyperplane* intersections. Which are even harder to visualize

1.1 Gauss elimination

1.1.1 Solving sets of equations systematically

$$\begin{array}{ccc|c} & A & & y \\ \hline \end{array}\quad \left[\begin{array}{ccc|c} 10 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 10 & 1 \end{array}\right]$$

Ay(2,:)-Ay(1,:)/5 = ([2 1 1 1]-1/5[10 2 1 1])

$$\begin{array}{ccc|c} & A & & y \\ \hline \end{array}\quad \left[\begin{array}{ccc|c} 10 & 2 & 1 & 1 \\ 0 & 3/5 & 4/5 & 4/5 \\ 1 & 2 & 10 & 1 \end{array}\right]$$

Ay(3,:)-Ay(1,:)/10 = ([1 2 10 1]-1/10[10 2 1 1])

$$\begin{array}{ccc|c} & A & & y \\ \hline \end{array}\quad \left[\begin{array}{ccc|c} 10 & 2 & 1 & 1 \\ 0 & 3/5 & 4/5 & 4/5 \\ 0 & 1.8 & 9.9 & 0.9 \end{array}\right]$$

Ay(3,:)-1.8*5/3*Ay(2,:) = ([0 1.8 9.9 0.9]-3*[0 3/5 4/5 4/5])

$$\begin{array}{ccc|c} & A & & y \\ \hline \end{array}\quad \left[\begin{array}{ccc|c} 10 & 2 & 1 & 1 \\ 0 & 3/5 & 4/5 & 4/5 \\ 0 & 0 & 7.5 & -1.5 \end{array}\right]$$

now, $7.5x_3 = -1.5$ so $x_3 = -\frac{1}{5}$

then, $3/5x_2 + 4/5(-1/5) = 1$ so $x_2 = \frac{8}{5}$

finally, $10x_1 + 2(8/5) + 1(-\frac{1}{5}) = 1$

Consider the problem again from the intro to Linear Algebra, 4 masses are connected in series to 4 springs with K=10 N/m. What are the final positions of the masses?

Springs-masses

The masses haves the following amounts, 1, 2, 3, and 4 kg for masses 1-4. Using a FBD for each mass:

$m_1g + k(x_2 - x_1) - kx_1 = 0$

$m_2g + k(x_3 - x_2) - k(x_2 - x_1) = 0$

$m_3g + k(x_4 - x_3) - k(x_3 - x_2) = 0$

$m_4g - k(x_4 - x_3) = 0$

in matrix form:

$$\left[\begin{array}{cccc} 2k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & k \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] = \left[\begin{array}{c} m_1g \\ m_2g \\ m_3g \\ m_4g \end{array}\right]$$

```
In [10]: k=10; % N/m
         m1=1; % kg
         m2=2;
         m3=3;
         m4=4;
         g=9.81; % m/s^2
         K=[2*k -k 0 0; -k 2*k -k 0; 0 -k 2*k -k; 0 0 -k k]
         y=[m1*g;m2*g;m3*g;m4*g]
```

```
K =

   20  -10    0    0
  -10   20  -10    0
    0  -10   20  -10
    0    0  -10   10

y =

    9.8100
   19.6200
   29.4300
   39.2400
```

```
In [11]: K1=[K y];
         K1(2,:)=K1(1,:)/2+K1(2,:)
```

```
K1 =

   20.00000  -10.00000    0.00000    0.00000    9.81000
    0.00000   15.00000  -10.00000    0.00000   24.52500
    0.00000  -10.00000   20.00000  -10.00000   29.43000
    0.00000    0.00000  -10.00000   10.00000   39.24000
```

```
In [12]: K2=K1;
         K2(3,:)=K1(2,:)*2/3+K1(3,:)
```

```
K2 =

   20.00000  -10.00000    0.00000    0.00000    9.81000
    0.00000   15.00000  -10.00000    0.00000   24.52500
    0.00000    0.00000   13.33333  -10.00000   45.78000
    0.00000    0.00000  -10.00000   10.00000   39.24000
```

```
In [13]: K2(4,:)=-K2(3,:)*K2(4,3)/K2(3,3)+K2(4,:)
```

```
K2 =

   20.00000  -10.00000    0.00000    0.00000    9.81000
    0.00000   15.00000  -10.00000    0.00000   24.52500
    0.00000    0.00000   13.33333  -10.00000   45.78000
    0.00000    0.00000    0.00000    2.50000   73.57500
```

```
In [14]: yp=K2(:,5);
         x4=yp(4)/K2(4,4)
         x3=(yp(3)+10*x4)/K2(3,3)
         x2=(yp(2)+10*x3)/K2(2,2)
         x1=(yp(1)+10*x2)/K2(1,1)
```

```
x4 =  29.430
x3 =  25.506
x2 =  18.639
x1 =  9.8100
```

```
In [15]: K\y
```

```
ans =

    9.8100
   18.6390
   25.5060
   29.4300
```

1.2 Automate Gauss Elimination

We can automate Gauss elimination with a function whose input is A and y:
`x=GaussNaive(A,y)`

```
In [16]: x=GaussNaive(K,y)
```

```
x =

    9.8100
   18.6390
   25.5060
   29.4300
```

1.3 Problem (Diagonal element is zero)

If a diagonal element is 0 or very small either:

1. no solution found
2. errors are introduced

Therefore, we would want to pivot before applying Gauss elimination
Consider:

(a) $\begin{bmatrix} 0 & 2 & 3 \\ 4 & 6 & 7 \\ 2 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 5 \end{bmatrix}$

(b) $\begin{bmatrix} 0.0003 & 3.0000 \\ 1.0000 & 1.0000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.0001 \\ 1.0000 \end{bmatrix}$

In [17]:
```
format short
Aa=[0,2,3;4,6,7;2,-3,6]; ya=[8;-3;5];
GaussNaive(Aa,ya)
Aa\ya
```

```
warning: division by zero
warning: called from
    GaussNaive at line 16 column 12
warning: division by zero
warning: division by zero
ans =

   NaN
   NaN
   NaN

ans =

  -5.423913
   0.021739
   2.652174
```

In [32]:
```
[x,Aug,npivots]=GaussPivot(Aa,ya)
```

```
x =

  -5.423913
   0.021739
   2.652174

Aug =
```

```
    4.00000    6.00000    7.00000   -3.00000
    0.00000   -6.00000    2.50000    6.50000
    0.00000    0.00000    3.83333   10.16667

npivots =  2
```

```
In [33]: format long
         Ab=[0.3E-13,3.0000;1.0000,1.0000];yb=[2+0.1e-13;1.0000];
         GaussNaive(Ab,yb)
         Ab\yb
```

```
ans =

   0.325665420556713
   0.666666666666667

ans =

   0.333333333333333
   0.666666666666667
```

```
In [34]: [x,Aug,npivots]=GaussPivot(Ab,yb)
         Ab\yb
         format short
```

```
x =

   0.333333333333333
   0.666666666666667

Aug =

   1.000000000000000   1.000000000000000   1.000000000000000
   0.000000000000000   2.999999999999970   1.999999999999980

npivots =  1
ans =

   0.333333333333333
   0.666666666666667
```

```
In [36]: % determinant is (-1)^(number_of_pivots)*diagonal_elements
         det(Ab)
         Aug(1,1)*Aug(2,2)
```

```
ans = -3.0000
ans =  3.0000
```

1.3.1 Spring-Mass System again

Now, 4 masses are connected in series to 4 springs with K_1=10 N/m, K_2=5 N/m, K_3=2 N/m and K_4=1 N/m. What are the final positions of the masses?

Springs-masses

The masses have the following amounts, 1, 2, 3, and 4 kg for masses 1-4. Using a FBD for each mass:

$m_1 g + k_2(x_2 - x_1) - k_1 x_1 = 0$
$m_2 g + k_3(x_3 - x_2) - k_2(x_2 - x_1) = 0$
$m_3 g + k_4(x_4 - x_3) - k_3(x_3 - x_2) = 0$
$m_4 g - k_4(x_4 - x_3) = 0$
in matrix form:

$$\left[\begin{array}{cccc} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] = \left[\begin{array}{c} m_1 g \\ m_2 g \\ m_3 g \\ m_4 g \end{array}\right]$$

In [24]:
```
k1=10; k2=5;k3=2;k4=1; % N/m
m1=1; % kg
m2=2;
m3=3;
m4=4;
g=9.81; % m/s^2
K=[k1+k2 -k2 0 0; -k2, k2+k3, -k3 0; 0 -k3, k3+k4, -k4; 0 0 -k4 k4]
y=[m1*g;m2*g;m3*g;m4*g]
```

```
K =

   15   -5    0    0
   -5    7   -2    0
    0   -2    3   -1
    0    0   -1    1

y =

    9.8100
   19.6200
   29.4300
   39.2400
```

1.4 Tridiagonal matrix

This matrix, K, could be rewritten as 3 vectors e, f and g

$$e = \begin{bmatrix} 0 \\ -5 \\ -2 \\ -1 \end{bmatrix}$$

$$f = \begin{bmatrix} 15 \\ 7 \\ 3 \\ 1 \end{bmatrix}$$

$$g = \begin{bmatrix} -5 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

Where all other components are 0 and the length of the vectors are n and the first component of e and the last component of g are zero

`e(1)=0`

`g(end)=0`

No need to pivot and number of calculations reduced enormously.

method	Number of Floating point operations for n×n-matrix
Gauss	n-cubed
Tridiagonal	n

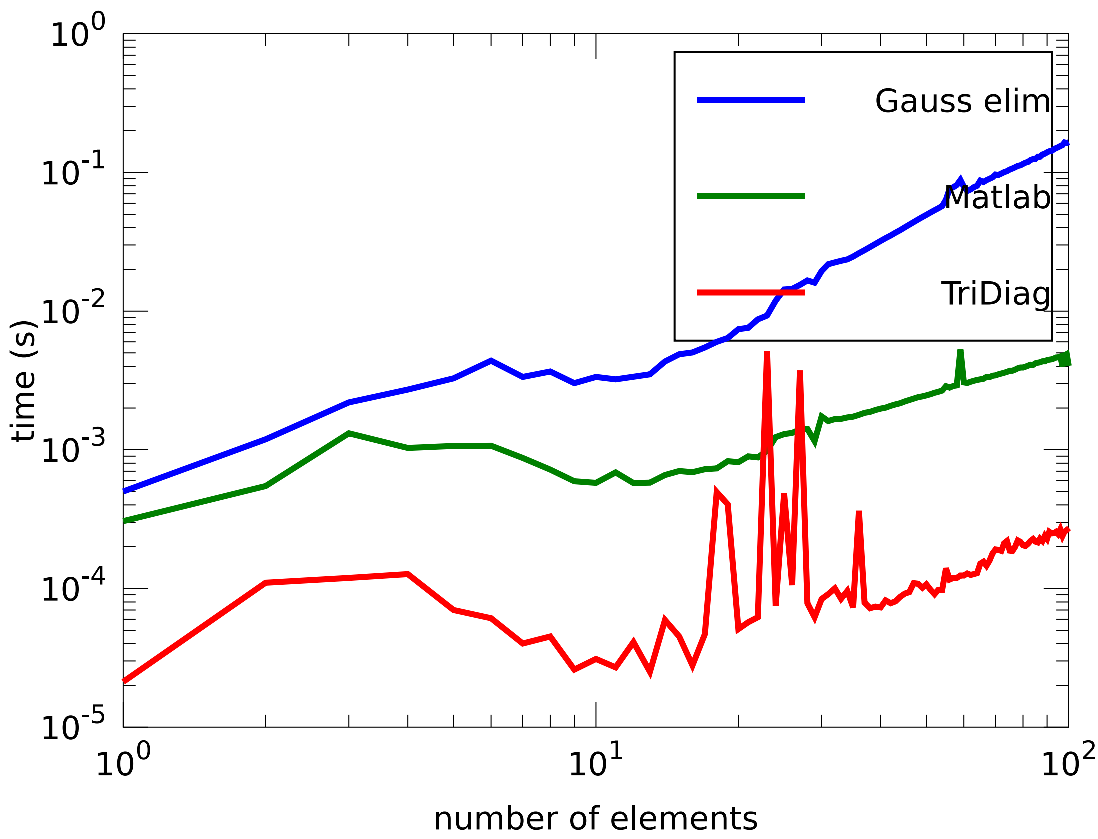
```
In [25]: e=[0;-5;-2;-1];
         g=[-5;-2;-1;0];
         f=[15;7;3;1];
         Tridiag(e,f,g,y)
         K\y
```

```
ans =

     9.8100    27.4680    61.8030   101.0430

ans =

     9.8100
    27.4680
    61.8030
   101.0430
```

```
In [26]: % tic ... t=toc
         % is Matlab timer used for debugging programs
         t_GE = zeros(1,100);
         t_GE_tridiag = zeros(1,100);
```

```
t_TD = zeros(1,100);
%for n = 1:200
for n=1:100
    A = rand(n,n);
    e = rand(n,1); e(1)=0;
    f = rand(n,1);
    g = rand(n,1); g(end)=0;
    Atd=diag(f, 0) - diag(e(2:n), -1) - diag(g(1:n-1), 1);
    b = rand(n,1);
    tic;
    x = GaussPivot(A,b);
    t_GE(n) = toc;
    tic;
    x = A\b;
    t_GE_tridiag(n) = toc;
    tic;
    x = Tridiag(e,f,g,b);
    t_TD(n) = toc;
end
```

In [28]:
```
n=1:100;
loglog(n,t_GE,n,t_TD,n,t_GE_tridiag)
legend('Gauss elim','Matlab \','TriDiag')
xlabel('number of elements')
ylabel('time (s)')
```




```
In [29]: [x,Aug,npivots]=GaussPivot(K,y)

x =

     9.8100
    27.4680
    61.8030
   101.0430

Aug =

   15.00000   -5.00000    0.00000    0.00000    9.81000
    0.00000    5.33333   -2.00000    0.00000   22.89000
    0.00000    0.00000    2.25000   -1.00000   38.01375
    0.00000    0.00000    0.00000    0.55556   56.13500

npivots = 0


In [30]: A=Aug(1:4,1:4)
```

```
A =

   15.00000   -5.00000    0.00000    0.00000
    0.00000    5.33333   -2.00000    0.00000
    0.00000    0.00000    2.25000   -1.00000
    0.00000    0.00000    0.00000    0.55556
```

In [31]:
```
det(A)
detA=A(1,1)*A(2,2)*A(3,3)*A(4,4)
```

```
ans =  100.00
detA =  100.00
```

In [ ]: