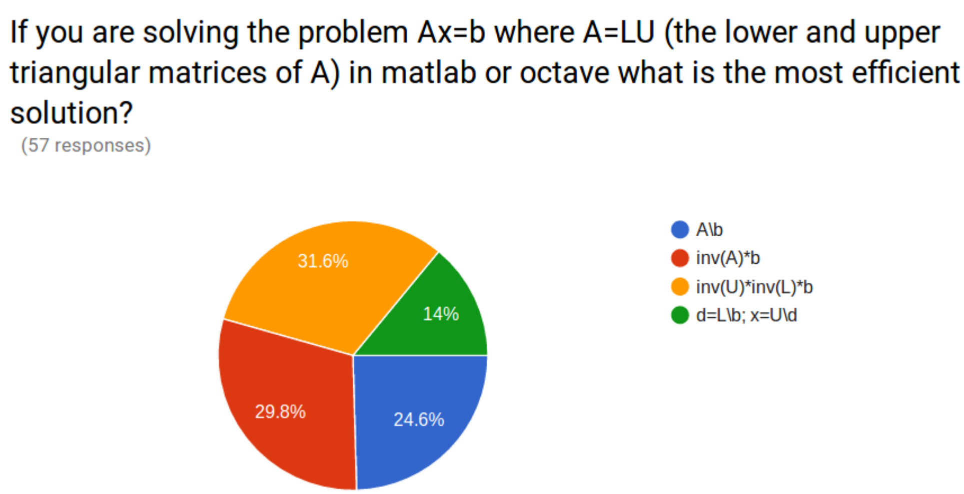
lecture_13

March 2, 2017

```
In [1]: %plot --format svg
```

```
In [2]: setdefaults
```

0.1 My question from last class



q1

$$A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 5 & 1 \\ 3 & 4 & 5 \end{bmatrix}$$

0.2 Your questions from last class

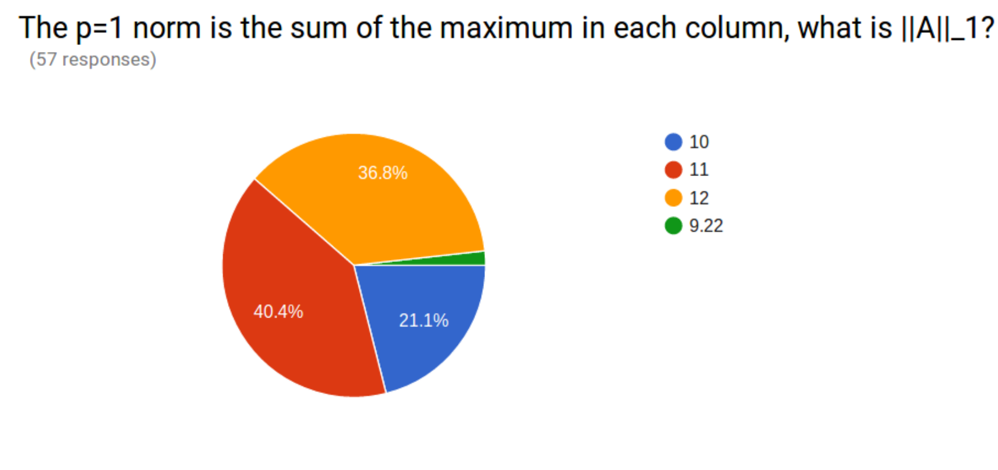
1. Do we have to submit a link for HW #4 somewhere or is uploading to Github sufficient?

   -no, your submission from HW 3 is sufficient

2. How do I get the formulas/formatting in markdown files to show up on github?

   -no luck for markdown equations in github, this is an ongoing request

3. Confused about the p=1 norm part and ||A||_1



q2

4. When's the exam?

   -next week (3/9)

5. What do you recommend doing to get better at figuring out the homeworks?

   -time and experimenting (try going through the lecture examples, verify my work)

6. Could we have an hw or extra credit with a video lecture to learn some simple python?

   -Sounds great! how simple?

   -Installing Python and Jupyter Notebook (via Anaconda) - https://www.continuum.io/downloads

   -Running Matlab kernel in Jupyter - https://anneurai.net/2015/11/12/matlab-based-ipython-notebooks/

   -Running Octave kernel in Jupyter - https://anaconda.org/pypi/octave_kernel

1 Markdown examples

" ' ‘

```
x=linspace(0,1);
y=x.^2;
plot(x,y)
for i = 1:10
    fprintf('markdown is pretty')
end
```

1.1 Condition of a matrix

1.1.1 *just checked in to see what condition my condition was in*

1.1.2 Matrix norms

The Euclidean norm of a vector is measure of the magnitude (in 3D this would be: $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$) in general the equation is:

$||x||_e = \sqrt{\sum_{i=1}^{n} x_i^2}$

For a matrix, A, the same norm is called the Frobenius norm:

$||A||_f = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} A_{i,j}^2}$

In general we can calculate any p-norm where

$||A||_p = \sqrt{\sum_{i=1}^{n} \sum_{i=1}^{m} A_{i,j}^p}$

so the p=1, 1-norm is

$||A||_1 = \sqrt{\sum_{i=1}^{n} \sum_{i=1}^{m} A_{i,j}^1} = \sum_{i=1}^{n} \sum_{i=1}^{m} |A_{i,j}|$

$||A||_\infty = \sqrt{\sum_{i=1}^{n} \sum_{i=1}^{m} A_{i,j}^\infty} = \max_{1 \le i \le n} \sum_{j=1}^{m} |A_{i,j}|$

1.1.3 Condition of Matrix

The matrix condition is the product of

$Cond(A) = ||A|| \cdot ||A^{-1}||$

So each norm will have a different condition number, but the limit is $Cond(A) \ge 1$

An estimate of the rounding error is based on the condition of A:

$\frac{||\Delta x||}{x} \le Cond(A) \frac{||\Delta A||}{||A||}$

So if the coefficients of A have accuracy to 10^{-t}

and the condition of A, $Cond(A) = 10^c$

then the solution for x can have rounding errors up to 10^{c-t}

In [7]:
```
A=[1,1/2,1/3;1/2,1/3,1/4;1/3,1/4,1/5]
[L,U]=LU_naive(A)
```

```
A =

   1.00000   0.50000   0.33333
   0.50000   0.33333   0.25000
   0.33333   0.25000   0.20000


L =

   1.00000   0.00000   0.00000
   0.50000   1.00000   0.00000
   0.33333   1.00000   1.00000


U =

   1.00000   0.50000   0.33333
```

```
   0.00000    0.08333    0.08333
   0.00000   -0.00000    0.00556
```

Then, $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$
$Ld_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, Ux_1 = d_1$ ...

```
In [8]: invA=zeros(3,3);
        d1=L\[1;0;0];
        d2=L\[0;1;0];
        d3=L\[0;0;1];
        invA(:,1)=U\d1; % shortcut invA(:,1)=A\[1;0;0]
        invA(:,2)=U\d2;
        invA(:,3)=U\d3
        invA*A
```

```
invA =

     9.0000   -36.0000    30.0000
   -36.0000   192.0000  -180.0000
    30.0000  -180.0000   180.0000

ans =

   1.0000e+00   3.5527e-15   2.9976e-15
  -1.3249e-14   1.0000e+00  -9.1038e-15
   8.5117e-15   7.1054e-15   1.0000e+00
```

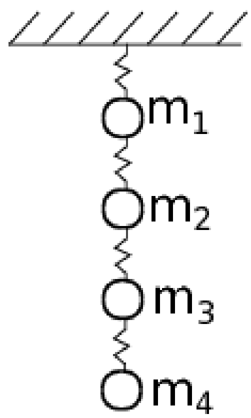
Find the condition of A, $cond(A)$

```
In [9]: % Frobenius norm
        normf_A = sqrt(sum(sum(A.^2)))
        normf_invA = sqrt(sum(sum(invA.^2)))

        cond_f_A = normf_A*normf_invA

        norm(A,'fro')

        % p=1, column sum norm
        norm1_A = max(sum(A,2))
        norm1_invA = max(sum(invA,2))
        norm(A,1)

        cond_1_A=norm1_A*norm1_invA
```

```
% p=inf, row sum norm
norminf_A = max(sum(A,1))
norminf_invA = max(sum(invA,1))
norm(A,inf)

cond_inf_A=norminf_A*norminf_invA
```

```
normf_A =  1.4136
normf_invA =  372.21
cond_f_A =  526.16
ans =  1.4136
norm1_A =  1.8333
norm1_invA =  30.000
ans =  1.8333
cond_1_A =  55.000
norminf_A =  1.8333
norminf_invA =  30.000
ans =  1.8333
cond_inf_A =  55.000
```

Consider the problem again from the intro to Linear Algebra, 4 masses are connected in series to 4 springs with spring constants K_i. What does a high condition number mean for this problem?



Springs-masses

The masses haves the following amounts, 1, 2, 3, and 4 kg for masses 1-4. Using a FBD for each mass:

$m_1 g + k_2(x_2 - x_1) - k_1 x_1 = 0$
$m_2 g + k_3(x_3 - x_2) - k_2(x_2 - x_1) = 0$
$m_3 g + k_4(x_4 - x_3) - k_3(x_3 - x_2) = 0$
$m_4 g - k_4(x_4 - x_3) = 0$
in matrix form:

$$\left[\begin{array}{cccc} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] = \left[\begin{array}{c} m_1 g \\ m_2 g \\ m_3 g \\ m_4 g \end{array}\right]$$

In [10]:
```
k1=10; % N/m
k2=100000;
k3=10;
k4=1;
m1=1; % kg
m2=2;
m3=3;
m4=4;
g=9.81; % m/s^2
K=[k1+k2 -k2 0 0; -k2 k2+k3 -k3 0; 0 -k3 k3+k4 -k4; 0 0 -k4 k4]
y=[m1*g;m2*g;m3*g;m4*g]
```

```
K =

   100010  -100000        0        0
  -100000   100010      -10        0
        0      -10       11       -1
        0        0       -1        1

y =

    9.8100
   19.6200
   29.4300
   39.2400
```

In [11]:
```
cond(K,inf)
cond(K,1)
cond(K,'fro')
cond(K,2)
```

```
ans =    3.2004e+05
ans =    3.2004e+05
ans =    2.5925e+05
ans =    2.5293e+05
```

In [26]:
```
e=eig(K)
max(e)/min(e)
```

```
e =

   7.9078e-01
   3.5881e+00
   1.7621e+01
   2.0001e+05
```

```
ans =    2.5293e+05
```

1.2 P=2 norm is ratio of biggest eigenvalue to smallest eigenvalue!

no need to calculate the inv(K)

2 Iterative Methods

2.1 Gauss-Seidel method

If we have an intial guess for each value of a vector x that we are trying to solve, then it is easy enough to solve for one component given the others.

Take a 3×3 matrix

$Ax = b$

$$\left[\begin{array}{ccc} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = \left[\begin{array}{c} 7.85 \\ -19.3 \\ 71.4 \end{array}\right]$$

$x_1 = \frac{7.85+0.1x_2+0.2x_3}{3}$

$x_2 = \frac{-19.3-0.1x_1+0.3x_3}{7}$

$x_3 = \frac{71.4+0.1x_1+0.2x_2}{10}$

```
In [12]: A=[3 -0.1 -0.2;0.1 7 -0.3;0.3 -0.2 10]
         b=[7.85;-19.3;71.4]

         x=A\b
```

```
A =

    3.00000   -0.10000   -0.20000
    0.10000    7.00000   -0.30000
    0.30000   -0.20000   10.00000

b =

    7.8500
  -19.3000
   71.4000

x =

   3.0000
  -2.5000
   7.0000
```

2.1.1 Gauss-Seidel Iterative approach

As a first guess, we can use $x_1 = x_2 = x_3 = 0$

$x_1 = \frac{7.85+0.1(0)+0.3(0)}{3} = 2.6167$
$x_2 = \frac{-19.3-0.1(2.6167)+0.3(0)}{7} = -2.7945$
$x_3 = \frac{71.4+0.1(2.6167)+0.2(-2.7945)}{10} = 7.0056$
Then, we update the guess:
$x_1 = \frac{7.85+0.1(-2.7945)+0.3(7.0056)}{3} = 2.9906$
$x_2 = \frac{-19.3-0.1(2.9906)+0.3(7.0056)}{7} = -2.4996$
$x_3 = \frac{71.4+0.1(2.9906)+0.2(-2.4966)}{10} = 7.00029$
The results are conveerging to the solution we found with \ of $x_1 = 3,\ x_2 = -2.5,\ x_3 = 7$
We could also use an iterative method that solves for all of the x-components in one step:

2.1.2 Jacobi method

$x_1^i = \frac{7.85+0.1x_2^{i-1}+0.3x_3^{i-1}}{3}$
$x_2^i = \frac{-19.3-0.1x_1^{i-1}+0.3x_3^{i-1}}{7}$
$x_3^i = \frac{71.4+0.1x_1^{i-1}+0.2x_2^{i-1}}{10}$
Here the solution is a matrix multiplication and vector addition

$$\left[\begin{array}{c} x_1^i \\ x_2^i \\ x_3^i \end{array}\right] = \left[\begin{array}{c} 7.85/3 \\ -19.3/7 \\ 71.4/10 \end{array}\right] - \left[\begin{array}{ccc} 0 & 0.1/3 & 0.2/3 \\ 0.1/7 & 0 & -0.3/7 \\ 0.3/10 & -0.2/10 & 0 \end{array}\right] \left[\begin{array}{c} x_1^{i-1} \\ x_2^{i-1} \\ x_3^{i-1} \end{array}\right]$$

x_{j}	Jacobi method	vs	Gauss-Seidel
$x_1^i =$	$\frac{7.85+0.1x_2^{i-1}+0.3x_3^{i-1}}{3}$		$\frac{7.85+0.1x_2^{i-1}+0.3x_3^{i-1}}{3}$
$x_2^i =$	$\frac{-19.3-0.1x_1^{i-1}+0.3x_3^{i-1}}{7}$		$\frac{-19.3-0.1x_1^{i}+0.3x_3^{i-1}}{7}$
$x_3^i =$	$\frac{71.4+0.1x_1^{i-1}+0.2x_2^{i-1}}{10}$		$\frac{71.4+0.1x_1^{i}+0.2x_2^{i}}{10}$

```
In [14]: ba=b./diag(A) % or ba=b./[A(1,1);A(2,2);A(3,3)]
         sA=A-diag(diag(A)) % A with zeros on diagonal
         sA(1,:)=sA(1,:)/A(1,1);
         sA(2,:)=sA(2,:)/A(2,2);
         sA(3,:)=sA(3,:)/A(3,3)
         x0=[0;0;0];
         x1=ba-sA*x0
         x2=ba-sA*x1
         x3=ba-sA*x2
         fprintf('solution is converging to [3,-2.5,7]]\n')
```

```
ba =

   2.6167
  -2.7571
   7.1400
```

```
sA =

   0.00000  -0.10000  -0.20000
   0.10000   0.00000  -0.30000
   0.30000  -0.20000   0.00000

sA =

   0.000000  -0.033333  -0.066667
   0.014286   0.000000  -0.042857
   0.030000  -0.020000   0.000000

x1 =

   2.6167
  -2.7571
   7.1400

x2 =

   3.0008
  -2.4885
   7.0064

x3 =

   3.0008
  -2.4997
   7.0002

solution is converging to [3,-2.5,7]]
```

```
In [16]: diag(A)
         diag(diag(A))
```

```
ans =

    3
    7
   10

ans =

Diagonal Matrix

    3    0    0
    0    7    0
```

```
   0     0    10
```

This method works if problem is diagonally dominant,
$|a_{ii}| > \sum_{j=1,j\neq i}^{n} |a_{ij}|$
If this condition is true, then Jacobi or Gauss-Seidel should converge

```
In [15]: A=[0.1,1,3;1,0.2,3;5,2,0.3]
         b=[12;2;4]
         A\b
```

```
A =

   0.10000   1.00000   3.00000
   1.00000   0.20000   3.00000
   5.00000   2.00000   0.30000

b =

   12
    2
    4

ans =

  -2.9393
   9.1933
   1.0336
```

```
In [16]: ba=b./diag(A) % or ba=b./[A(1,1);A(2,2);A(3,3)]
         sA=A-diag(diag(A)) % A with zeros on diagonal
         sA(1,:)=sA(1,:)/A(1,1);
         sA(2,:)=sA(2,:)/A(2,2);
         sA(3,:)=sA(3,:)/A(3,3)
         x0=[0;0;0];
         x1=ba-sA*x0
         x2=ba-sA*x1
         x3=ba-sA*x2
         fprintf('solution is not converging to [-2.93,9.19,1.03]\n')
```

```
ba =

   120.000
    10.000
    13.333
```

```
sA =

   0   1   3
   1   0   3
   5   2   0

sA =

    0.00000   10.00000   30.00000
    5.00000    0.00000   15.00000
   16.66667    6.66667    0.00000

x1 =

   120.000
    10.000
    13.333

x2 =

   -380.00
   -790.00
  -2053.33

x3 =

   6.9620e+04
   3.2710e+04
   1.1613e+04

solution is not converging to [-2.93,9.19,1.03]
```

2.2 Gauss-Seidel with Relaxation

In order to force the solution to converge faster, we can introduce a relaxation term λ.

where the new x values are weighted between the old and new:

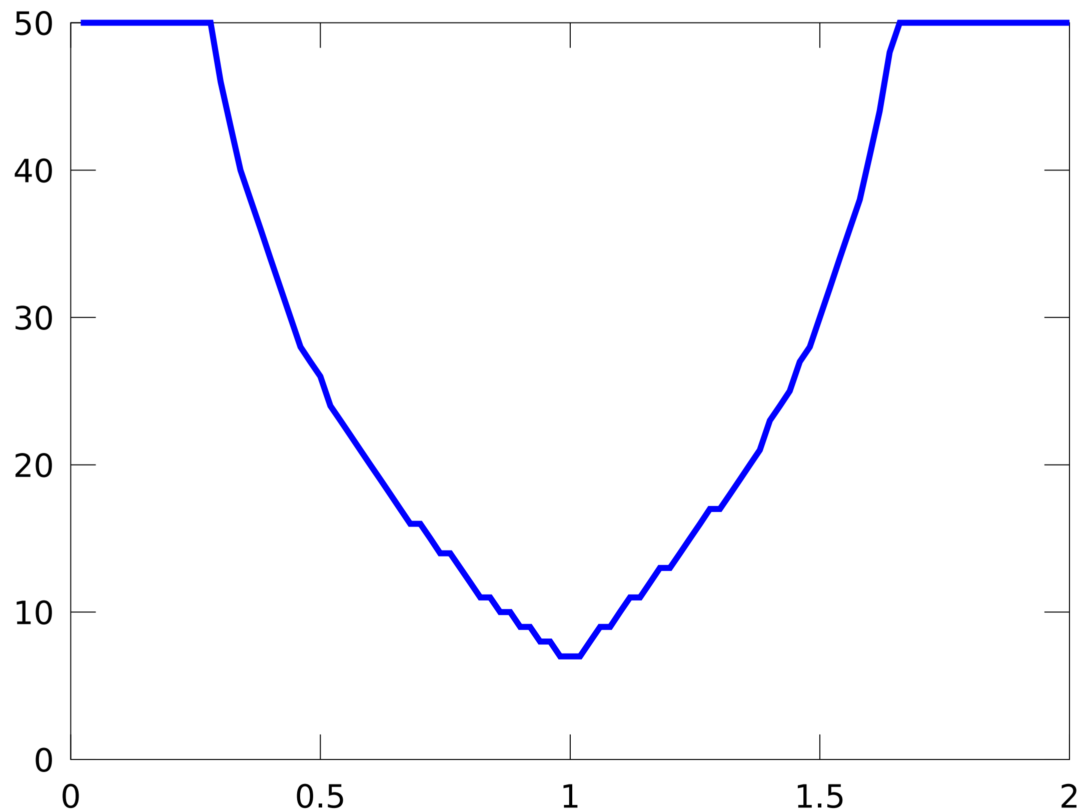
$x^i = \lambda x^i + (1-\lambda)x^{i-1}$

after solving for x, lambda weights the current approximation with the previous approximation for the updated x

```
In [17]: % rearrange A and b
         A=[3 -0.1 -0.2;0.1 7 -0.3;0.3 -0.2 10]
         b=[7.85;-19.3;71.4]

         iters=zeros(100,1);
         for i=1:100
             lambda=2/100*i;
```

```
        [x,ea,iters(i)]=Jacobi_rel(A,b,lambda);
    end
    plot([1:100]*2/100,iters)
```

```
A =

    3.00000   -0.10000   -0.20000
    0.10000    7.00000   -0.30000
    0.30000   -0.20000   10.00000

b =

    7.8500
  -19.3000
   71.4000
```




```
In [107]: l=fminbnd(@(l) lambda_fcn(A,b,l),0.5,1.5)
```

```
l =  0.99158
```

```
In [108]: A\b
```

```
ans =

   3.0000
  -2.5000
   7.0000
```

```
In [109]: [x,ea,iter]=Jacobi_rel(A,b,l,0.000001)
          [x,ea,iter]=Jacobi_rel(A,b,1,0.000001)
```

```
x =

   3.0000
  -2.5000
   7.0000

ea =

   1.8289e-07
   2.1984e-08
   2.3864e-08

iter =  8
x =

   3.0000
  -2.5000
   7.0000

ea =

   1.9130e-08
   7.6449e-08
   3.3378e-08

iter =  8
```

2.3 Nonlinear Systems

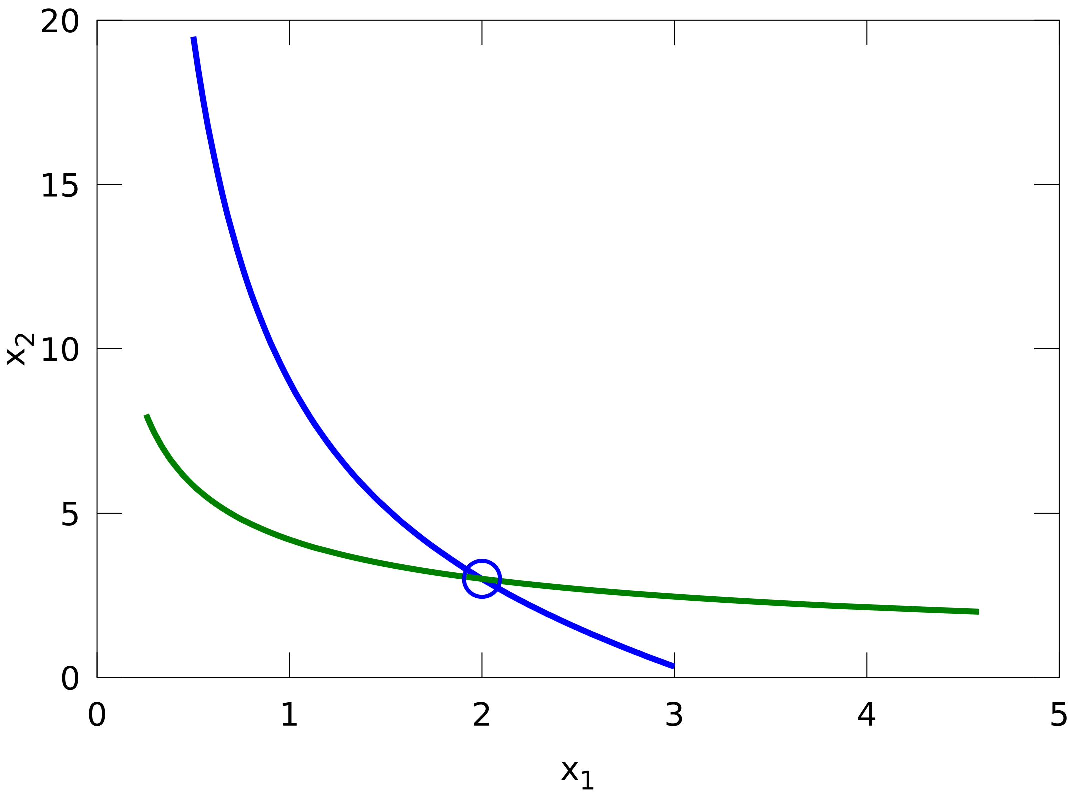
Consider two simultaneous nonlinear equations with two unknowns:

$x_1^2 + x_1 x_2 = 10$

$x_2 + 3x_1 x_2^2 = 57$

Graphically, we are looking for the solution:

```
In [19]: x11=linspace(0.5,3);
         x12=(10-x11.^2)./x11;

         x22=linspace(2,8);
         x21=(57-x22).*x22.^-2/3;

         plot(x11,x12,x21,x22)
         % Solution at x_1=2, x_2=3
         hold on;
         plot(2,3,'o')
         xlabel('x_1')
         ylabel('x_2')
```



2.4 Newton-Raphson part II

Remember the first order approximation for the next point in a function is:

$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i)f'(x_i)$

then, $f(x_{i+1}) = 0$ so we are left with:

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

We can use the same formula, but now we have multiple dimensions so we need to determine the Jacobian

$$[J] = \begin{bmatrix} \frac{\partial f_{1,i}}{\partial x_1} & \frac{\partial f_{1,i}}{\partial x_2} & \cdots & \frac{\partial f_{1,i}}{\partial x_n} \\ \frac{\partial f_{2,i}}{\partial x_1} & \frac{\partial f_{2,i}}{\partial x_2} & \cdots & \frac{\partial f_{2,i}}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_{n,i}}{\partial x_1} & \frac{\partial f_{n,i}}{\partial x_2} & \cdots & \frac{\partial f_{n,i}}{\partial x_n} \end{bmatrix}$$

$$\begin{bmatrix} f_{1,i+1} \\ f_{2,i+1} \\ \vdots \\ f_{n,i+1} \end{bmatrix} = \begin{bmatrix} f_{1,i} \\ f_{2,i} \\ \vdots \\ f_{n,i} \end{bmatrix} + \begin{bmatrix} \frac{\partial f_{1,i}}{\partial x_1} & \frac{\partial f_{1,i}}{\partial x_2} & \cdots & \frac{\partial f_{1,i}}{\partial x_n} \\ \frac{\partial f_{2,i}}{\partial x_1} & \frac{\partial f_{2,i}}{\partial x_2} & \cdots & \frac{\partial f_{2,i}}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_{n,i}}{\partial x_1} & \frac{\partial f_{n,i}}{\partial x_2} & \cdots & \frac{\partial f_{n,i}}{\partial x_n} \end{bmatrix} \left(\begin{bmatrix} x_{i+1} \\ x_{i+1} \\ \vdots \\ x_{i+1} \end{bmatrix} - \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ \vdots \\ x_{n,i} \end{bmatrix} \right)$$

2.4.1 Solution is again in the form Ax=b

$[J]([x_{i+1}] - [x_i]) = -[f]$

so

$[x_{i+1}] = [x_i] - [J]^{-1}[f]$

2.5 Example of Jacobian calculation

2.5.1 Nonlinear springs supporting two masses in series

Two springs are connected to two masses, with m_1=1 kg and m_2=2 kg. The springs are identical, but they have nonlinear spring constants, of k_1=100 N/m and k_2=-10 N/m

We want to solve for the final position of the masses (x_1 and x_2)

$m_1 g + k_1(x_2 - x_1) + k_2(x_2 - x_1)^2 + k_1 x_1 + k_2 x_1^2 = 0$

$m_2 g - k_1(x_2 - x_1) - k_2(x_2 - x_1)^2 = 0$

$J(1,1) = \frac{\partial f_1}{\partial x_1} = -k_1 - 2k_2(x_2 - x_1) + k_1 + 2k_2 x_1$

$J(1,2) = \frac{\partial f_1}{\partial x_2} = k_1 + 2k_2(x_2 - x_1)$

$J(2,1) = \frac{\partial f_2}{\partial x_1} = k_1 + 2k_2(x_2 - x_1)$

$J(2,2) = \frac{\partial f_2}{\partial x_2} = -k_1 - 2k_2(x_2 - x_1)$

In [ ]:
```
m1=1; % kg
m2=2; % kg
k1=100; % N/m
k2=-10; % N/m^2
```

In [20]:
```
function [f,J]=mass_spring(x)
    % Function to calculate function values f1 and f2 as well as Jacobian
    % for 2 masses and 2 identical nonlinear springs
    m1=1; % kg
    m2=2; % kg
    k1=100; % N/m
    k2=-10; % N/m^2
    g=9.81; % m/s^2
    x1=x(1);
    x2=x(2);
    J=[-k1-2*k2*(x2-x1)-k1-2*k2*x1,k1+2*k2*(x2-x1);
        k1+2*k2*(x2-x1),-k1-2*k2*(x2-x1)];
```

```
        f=[m1*g+k1*(x2-x1)+k2*(x2-x1).^2-k1*x1-k2*x1^2;
           m2*g-k1*(x2-x1)-k2*(x2-x1).^2];
    end
```

```
In [21]: [f,J]=mass_spring([1,0])
```

```
f =

  -190.19
   129.62

J =

  -200   120
   120  -120
```

```
In [22]: x0=[3;2];
         [f0,J0]=mass_spring(x0);
         x1=x0-J0\f0
         ea=(x1-x0)./x1
         [f1,J1]=mass_spring(x1);
         x2=x1-J1\f1
         ea=(x2-x1)./x2
         [f2,J2]=mass_spring(x2);
         x3=x2-J2\f2
         ea=(x3-x2)./x3
         x=x3
         for i=1:3
             xold=x;
             [f,J]=mass_spring(x);
             x=x-J\f;
             ea=(x-xold)./x
         end
```

```
x1 =

  -1.5142
  -1.4341

ea =

   2.9812
   2.3946

x2 =

   0.049894
```

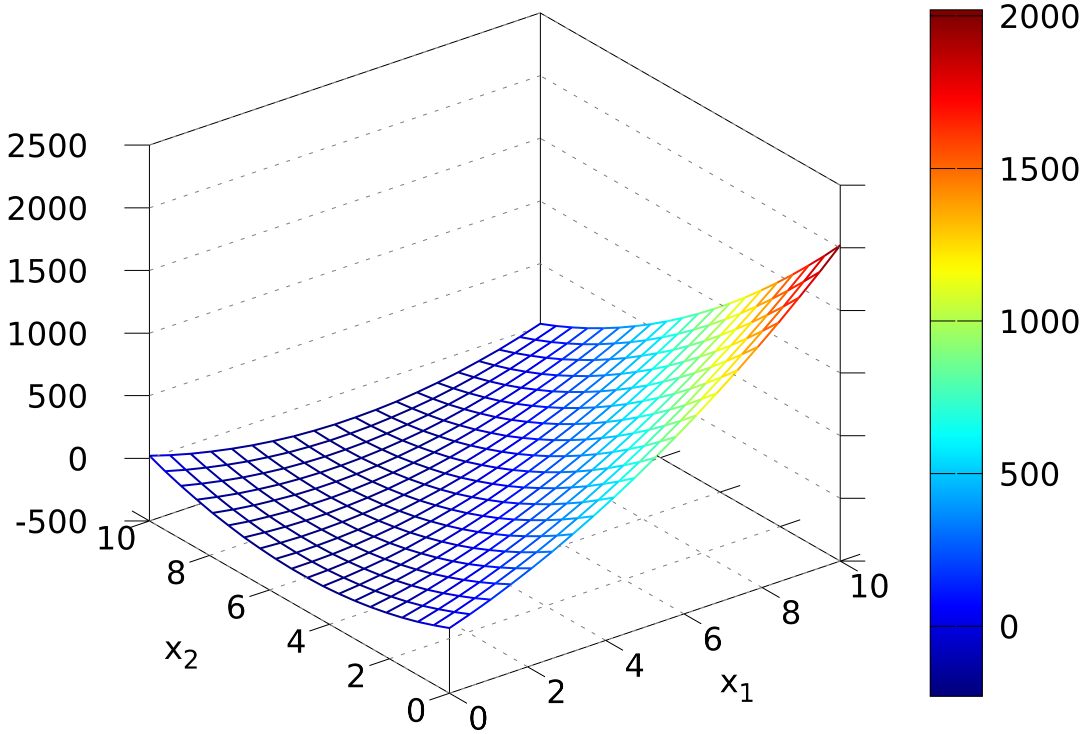
```
   0.248638

ea =

   31.3492
    6.7678

x3 =

   0.29701
   0.49722

ea =

   0.83201
   0.49995

x =

   0.29701
   0.49722

ea =

   0.021392
   0.012890

ea =

   1.4786e-05
   8.9091e-06

ea =

   7.0642e-12
   4.2565e-12
```

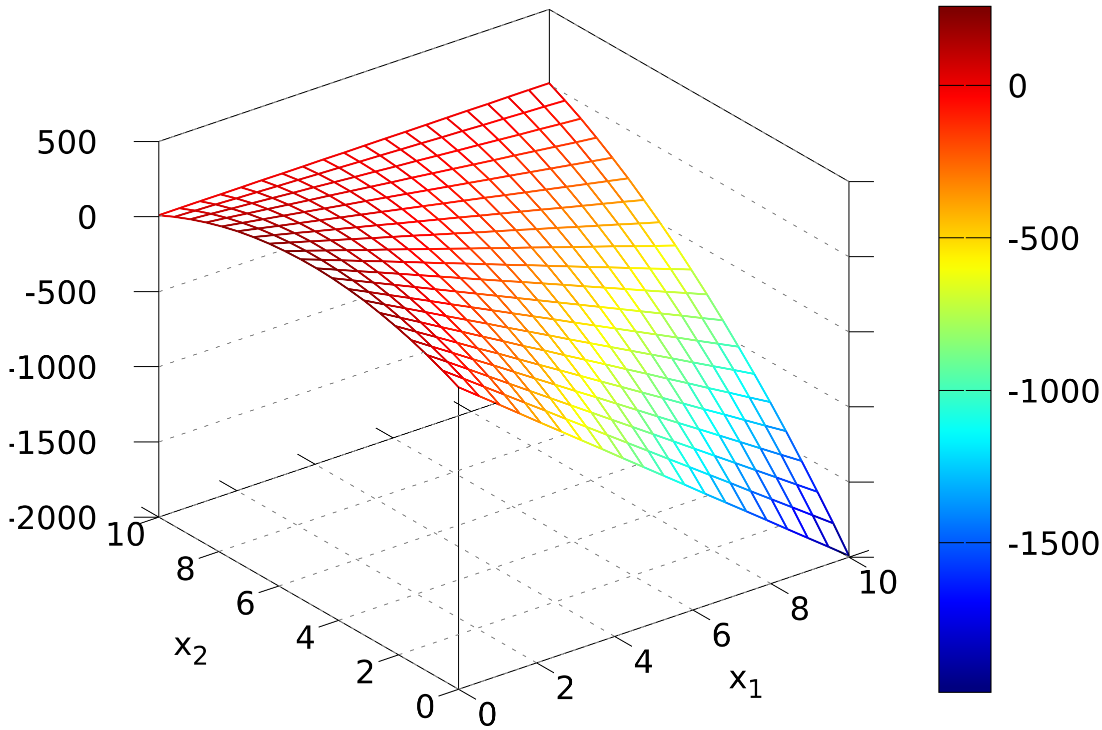
In [23]:
```
x
X0=fsolve(@(x) mass_spring(x),[3;5])
```

```
x =

   0.30351
   0.50372

X0 =
```

```
0.30351
0.50372
```

```
In [26]: [X,Y]=meshgrid(linspace(0,10,20),linspace(0,10,20));
         [N,M]=size(X);
         F=zeros(size(X));
         for i=1:N
             for j=1:M
                 [f,~]=mass_spring([X(i,j),Y(i,j)]);
                 F1(i,j)=f(1);
                 F2(i,j)=f(2);
             end
         end
         mesh(X,Y,F1)
         xlabel('x_1')
         ylabel('x_2')
         colorbar()
         figure()
         mesh(X,Y,F2)
         xlabel('x_1')
         ylabel('x_2')
         colorbar()
```


2500
2000
1500
1000
500
0
-500
10
8
6
4
2
0
x_2
0
2
4
6
8
10
x_1
2000
1500
1000
500
0



In [ ]: