Sources of Error

usually ↑ precision = ↑ accuracy

for numerical approx of analytical model

1. Truncation ← due to approx. of function (Taylor series)
2. Roundoff ← due to fixed # of bits

Roundoff

base 2	base 10
11011	$2^4+2^3+0\cdot2^2+2^1+2^0=27$
1.011	$2^0+0\cdot2^{-1}+2^{-2}+2^{-3}=$ $1+\frac{1}{4}+\frac{1}{8}=1.375$
1.00001	$1+2^{-5}=1+\frac{1}{32}=33/32$
1.000001	$1+2^{-6}=1.015625$

if we have 8 bits, how many integers can we store?

→ max int (w/ 0 + positives = negatives)

max 2^7-1

min -2^7

# of integers stored is 2^8

base 2

$1.0001 + 1.0001$
$= 10.0010$

base 10

$= 2^1 + 2^{-3} = 2 + \frac{1}{8} = 2.125$

$2E{-}5 + 1E1$

$0.00002 + 1.00000$

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Truncation Error

$f(x) = \cos(x)$

$f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$

1st order approx. of $\cos(x)$

$f(x) \approx 1 - 0 \cdot x$

$f(x) = 1 + \underline{O(x^2)}$

2nd order approx. of $\cos(x)$

$f(x) = 1 - \frac{x^2}{2} + \underline{O(x^3)}$

$O(\dots)$ is order of error expected from truncation

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<u>base 10</u>

$2321 \rightarrow 2\times10^3 + 3\times10^2 + 2\times10^1 + 1\times10^0$

2E10^3 3E10^2 2E10^1 1E10^0

Matlab & Octave

<u>script</u> &	<u>function</u>
· list of commands	- given input, gives output
- all var's stored in memory	- all var's deleted after execution
	- help file is everything % % after function defined

Root finding $f(x_0)=0$

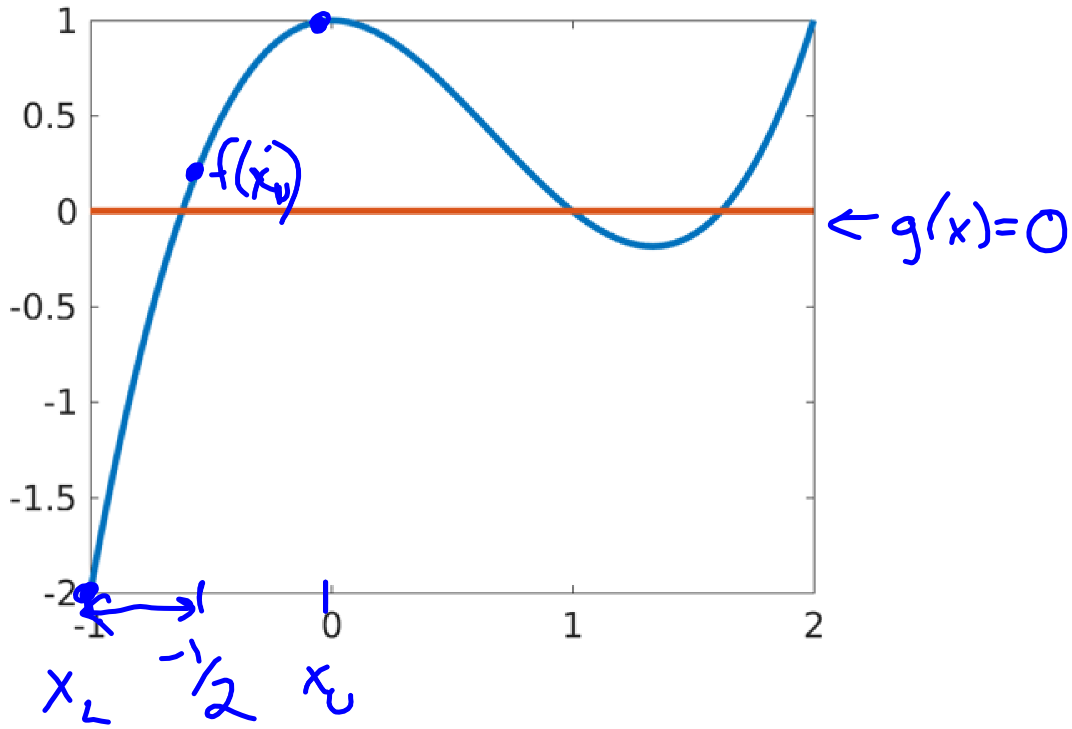
Bracketing

- incremental search
- bisection
- false positive

Open

- Newton-Raphson
- modified secant

$f(x) = x^3 - 2x^2 + 1$



$f(-1) = -1 - 2 + 1 = -2$

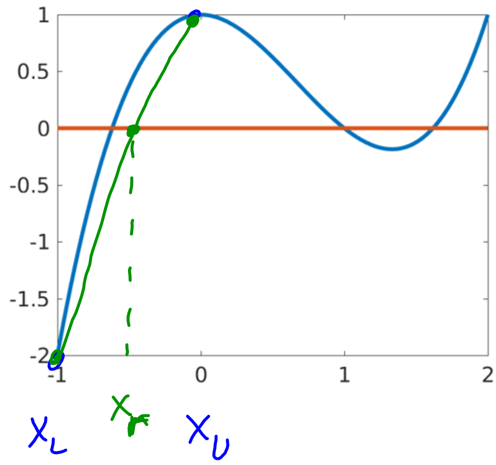
$f(0) = 0 - 0 + 1 = 1$

$f(-\frac{1}{2}) = -\frac{1}{8} - \frac{2}{4} + 1 = 0.375$

new bounds

$-1 \to -\frac{1}{2}$

$f(x) = x^3 - 2x^2 + 1$



false pos.

$$\frac{f(x_U)-f(x_L)}{x_U - x_L} = \frac{0 - f(x_L)}{x_r - x_L}$$

$$\frac{0 - -1}{1 - -2} = \frac{0 - -2}{x_r - -1}$$

$$\frac{2}{x_r+1} = \frac{1}{3}$$

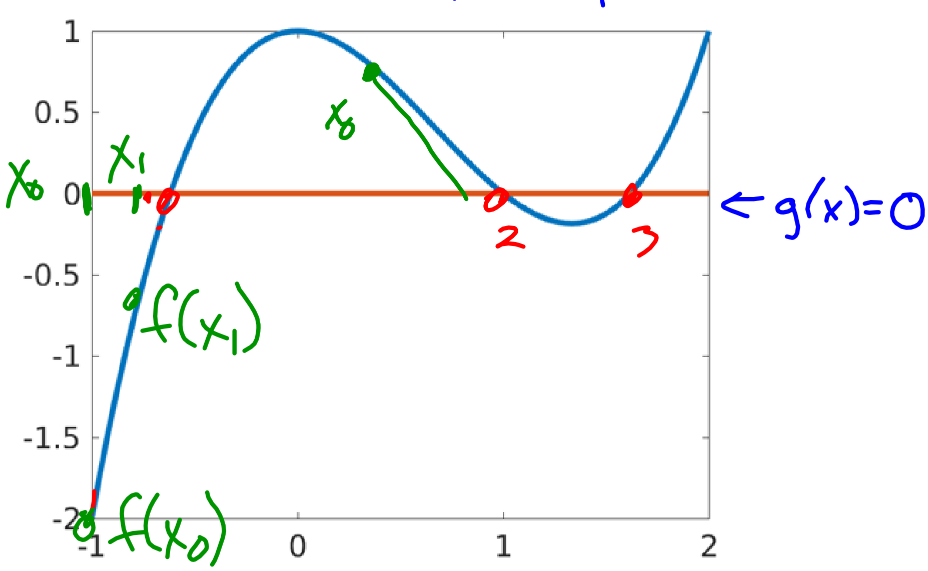
$$x_r = -1/3$$

$$f(x_r) = -\frac{1}{27} - \frac{3}{9} + 1 = 0.741$$

new bounds

$x_L \to x_r$ $\boxed{-1 \to -1/3}$

$$f(x) = x^3 - 2x^2 + 1$$



Newton-Raphson

$$x_0 = -1$$

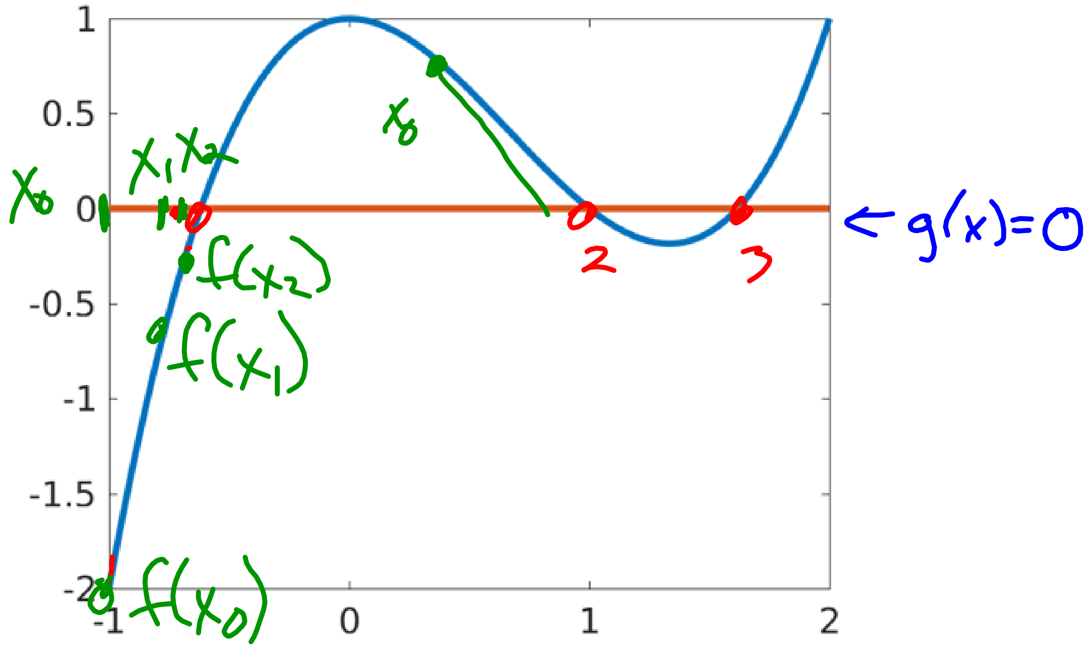
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{N-R eqn}$$

$$= -1 - \frac{-1-2+1}{3(-1)^2 - 4(-1)} = -1 + \frac{2}{7}$$

$$= -0.7143$$

$$e_a = \left| \frac{x_1 - x_0}{x_1} \right| = \left| \frac{-0.7143 + 1}{-0.7143} \right| = 0.4$$

$f(x) = x^3 - 2x^2 + 1$



Newton-Raphson

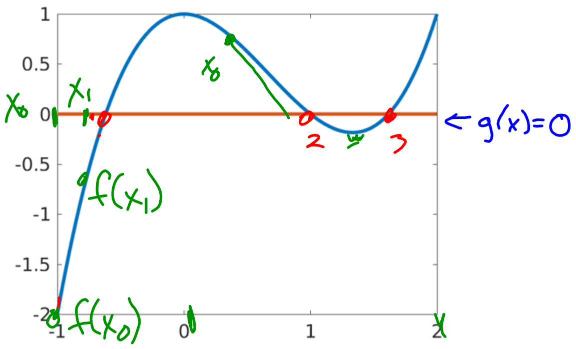
$x_1 = -0.7143$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -0.7143 - \frac{-0.3849}{4.388}$$

$x_2 = -0.6266$

$$\epsilon_a = \left| \frac{-0.6266 + 0.7143}{-0.6266} \right| = 0.14$$

$f(x) = x^3 - 2x^2 + 1$



Mod. Secant

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \approx x_i - \frac{f(x_i)\delta_x}{f(x_i+\delta x) - f(x_i)}$$

$$f'(x_i) \approx \frac{f(x_i+\delta x) - f(x_i)}{\delta x}$$

if $\delta x = 0.1$

$f'(x_0) = 7$

$$\frac{f(x_0 + \delta x) - f(x_0)}{\delta x} = \frac{-2.751 - -2}{0.1}$$

$f'(x_0) \approx 7.51$ X

if $\delta x = 0.001$

$$\frac{f(x_0 + \delta x) - f(x_0)}{\delta x} = \frac{-2.007 - -2}{0.001}$$

$f'(x_0) \approx 7$ ✓

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Optimization

if $\frac{df}{dx}$ known, then min/max of $f(x)$

ⓐ $f'(x)=0$

$f(x)=x^3-2x^2+1$

$f'(x)=3x^2-4x$

$f''(x)=6x-4$

$$x_{i+1}=x_i-\frac{f'(x_i)}{f''(x_i)}$$

w/o derivative a little harder

- Golden Ratio method

$x_L=0,\ x_U=2$

$\phi=1.61803$ soln to $\frac{l_1+l_2}{l_1}=\frac{l_1}{l_2}$

or $1+\frac{l_2}{l_1}=\frac{l_1}{l_2}$

$\phi=\frac{l_1}{l_2}$,

$\phi^2-\phi-1=0$

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now

$x_L = 0$

$x_U = 2$

$d = (1-\phi)(x_U - x_L) = 0.61803(2-0)$

$x_1 = x_L + d = 0 + 1.236$

$x_2 = x_U - d = 2 - 1.236 = 0.7639$

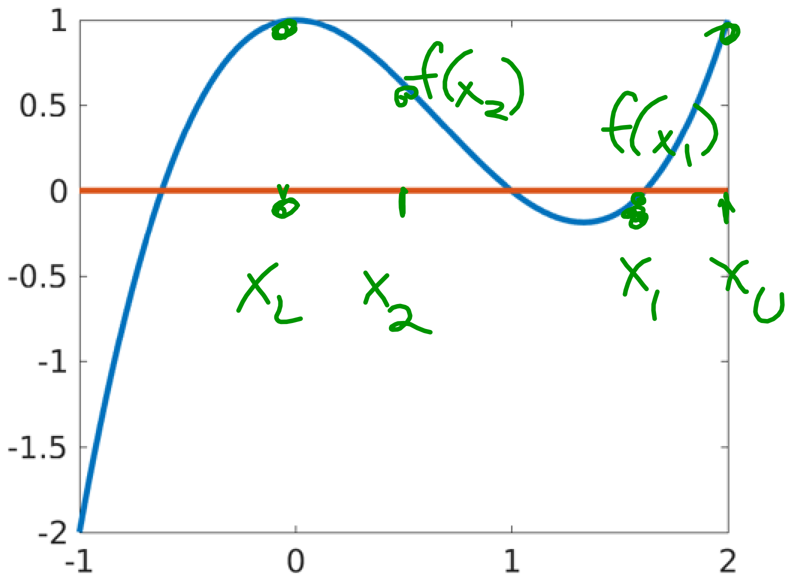
$f(x_2) = 0.2787$

$f(x_1) = -0.1672$

$f(x_1) < f(x_2)$ so

$x_U = x_1$

$f(x_U) = f(x_1)$



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