lecture_12

February 28, 2017

```
In [27]: %plot --format svg

In [28]: setdefaults

In [29]: A=rand(4,4)

         [L,U,P]=lu(A)

         det(L)
```

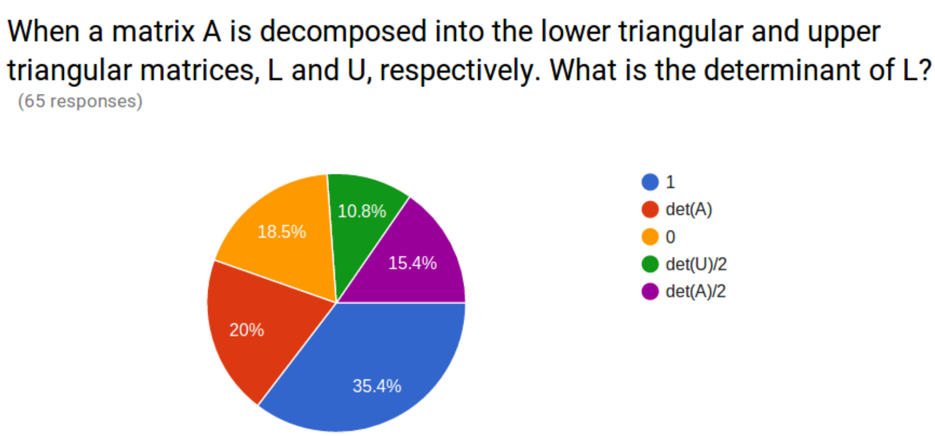
```
A =

   0.447394   0.357071   0.720915   0.499926
   0.648313   0.323276   0.521677   0.288345
   0.084982   0.581513   0.466420   0.142342
   0.576580   0.658089   0.916987   0.923165

L =

   1.00000   0.00000   0.00000   0.00000
   0.13108   1.00000   0.00000   0.00000
   0.69009   0.24851   1.00000   0.00000
   0.88935   0.68736   0.68488   1.00000

U =

   0.64831   0.32328   0.52168   0.28834
   0.00000   0.53914   0.39804   0.10455
   0.00000   0.00000   0.26199   0.27496
   0.00000   0.00000   0.00000   0.40655

P =

Permutation Matrix

   0   1   0   0
   0   0   1   0
   1   0   0   0
```

```
   0   0   0   1

ans =  1
```

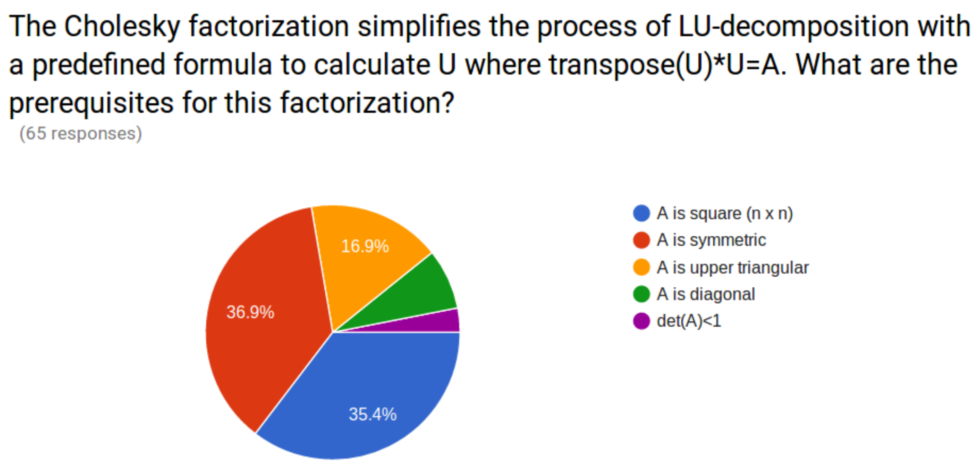
```
In [44]: A=rand(4,100)';
         A=A'*A;
         size(A)
         min(min(A))
         max(max(A))
         cond(A)
         C=chol(A)
```

```
ans =

   4   4

ans =  23.586
ans =  35.826
ans =  14.869
C =

   5.98549   4.28555   4.35707   4.31359
   0.00000   3.63950   1.35005   1.45342
   0.00000   0.00000   3.62851   1.50580
   0.00000   0.00000   0.00000   3.21911
```

0.1 My question from last class



q1



q2

0.2 Your questions from last class

1. Will the exam be more theoretical or problem based?
2. Writing code is difficult
3. What format can we expect for the midterm?
4. Could we go over some example questions for the exam?
5. Will the use of GitHub be tested on the Midterm exam? Or is it more focused on linear algebra techniques/what was covered in the lectures?
6. This is not my strong suit, getting a bit overwhelmed with matrix multiplication.
7. I forgot how much I learned in linear algebra.
8. What's the most exciting project you've ever worked on with Matlab/Octave?

1 Matrix Inverse and Condition

Considering the same solution set:
$y = Ax$
If we know that $A^{-1}A = I$, then
$A^{-1}y = A^{-1}Ax = x$
so
$x = A^{-1}y$
Where, A^{-1} is the inverse of matrix A.
$2x_1 + x_2 = 1$
$x_1 + 3x_2 = 1$
$Ax = y$
$\left[\begin{array}{cc} 2 & 1 \\ 1 & 3 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$

$$A^{-1} = \frac{1}{2*3-1*1}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}$$

```
In [45]: A=[2,1;1,3]
         invA=1/5*[3,-1;-1,2]

         A*invA
         invA*A
```

```
A =

   2   1
   1   3

invA =

   0.60000  -0.20000
  -0.20000   0.40000

ans =

   1.00000   0.00000
   0.00000   1.00000

ans =

   1.00000   0.00000
   0.00000   1.00000
```

How did we know the inverse of A?
for 2×2 matrices, it is always:

$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$A^{-1} = \frac{1}{det(A)}\begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$

$AA^{-1} = \frac{1}{A_{11}A_{22}-A_{21}A_{12}}\begin{bmatrix} A_{11}A_{22} - A_{21}A_{12} & -A_{11}A_{12} + A_{12}A_{11} \\ A_{21}A_{22} - A_{22}A_{21} & -A_{21}A_{12} + A_{22}A_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

What about bigger matrices?
We can use the LU-decomposition

$A = LU$

$A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$

if we divide A^{-1} into n-column vectors, a_n, then

$Aa_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \; Aa_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \; Aa_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

Which we can solve for each a_n with LU-decomposition, knowing the lower and upper triangular decompositions, then

$$A^{-1} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

$$Ld_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ;\ Ua_1 = d_1$$

$$Ld_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} ;\ Ua_2 = d_2$$

$$Ld_n = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ n \end{bmatrix} ;\ Ua_n = d_n$$

Consider the following matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Note on solving for A^{-1} column 1 $Aa_1 = I(:,1)$

$LUa_1 = I(:,1)$
$(LUa_1 - I(:,1)) = 0$
$L(Ua_1 - d_1) = 0$
$I(:,1) = Ld_1$

```
In [56]: A=[2,-1,0;-1,2,-1;0,-1,1]
         U=A;
         L=eye(3,3);
         U(2,:)=U(2,:)-U(2,1)/U(1,1)*U(1,:)
         L(2,1)=A(2,1)/A(1,1)
```

```
A =

   2  -1   0
  -1   2  -1
   0  -1   1

U =

   2.00000  -1.00000   0.00000
   0.00000   1.50000  -1.00000
   0.00000  -1.00000   1.00000
```

```
L =

   1.00000   0.00000   0.00000
  -0.50000   1.00000   0.00000
   0.00000   0.00000   1.00000
```

```
In [57]: L(3,2)=U(3,2)/U(2,2)
         U(3,:)=U(3,:)-U(3,2)/U(2,2)*U(2,:)
```

```
L =

   1.00000   0.00000   0.00000
  -0.50000   1.00000   0.00000
   0.00000  -0.66667   1.00000

U =

   2.00000  -1.00000   0.00000
   0.00000   1.50000  -1.00000
   0.00000   0.00000   0.33333
```

Now solve for d_1 then a_1, d_2 then a_2, and d_3 then a_3

$$Ld_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} d1(1) \\ d1(2) \\ d1(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; Ua_1 = d_1$$

```
In [58]: d1=zeros(3,1);
         d1(1)=1;
         d1(2)=0-L(2,1)*d1(1);
         d1(3)=0-L(3,1)*d1(1)-L(3,2)*d1(2)
```

```
d1 =

   1.00000
   0.50000
   0.33333
```

```
In [59]: a1=zeros(3,1);
         a1(3)=d1(3)/U(3,3);
         a1(2)=1/U(2,2)*(d1(2)-U(2,3)*a1(3));
         a1(1)=1/U(1,1)*(d1(1)-U(1,2)*a1(2)-U(1,3)*a1(3))
```

```
a1 =

   1.00000
   1.00000
   1.00000
```

```
In [60]: d2=zeros(3,1);
         d2(1)=0;
         d2(2)=1-L(2,1)*d2(1);
         d2(3)=0-L(3,1)*d2(1)-L(3,2)*d2(2)
```

```
d2 =

   0.00000
   1.00000
   0.66667
```

```
In [61]: a2=zeros(3,1);
         a2(3)=d2(3)/U(3,3);
         a2(2)=1/U(2,2)*(d2(2)-U(2,3)*a2(3));
         a2(1)=1/U(1,1)*(d2(1)-U(1,2)*a2(2)-U(1,3)*a2(3))
```

```
a2 =

   1.0000
   2.0000
   2.0000
```

```
In [62]: d3=zeros(3,1);
         d3(1)=0;
         d3(2)=0-L(2,1)*d3(1);
         d3(3)=1-L(3,1)*d3(1)-L(3,2)*d3(2)
```

```
d3 =

   0
   0
   1
```

```
In [63]: a3=zeros(3,1);
         a3(3)=d3(3)/U(3,3);
         a3(2)=1/U(2,2)*(d3(2)-U(2,3)*a3(3));
         a3(1)=1/U(1,1)*(d3(1)-U(1,2)*a3(2)-U(1,3)*a3(3))
```

```
a3 =

   1.00000
   2.00000
   3.00000
```

Final solution for A^{-1} is $[a_1\ a_2\ a_3]$

In [69]:
```
invA=[a1,a2,a3]
I_app=A*invA
I_app(2,3)
eps

2^-8
```

```
invA =

   1.00000   1.00000   1.00000
   1.00000   2.00000   2.00000
   1.00000   2.00000   3.00000

I_app =

   1.00000   0.00000   0.00000
   0.00000   1.00000  -0.00000
  -0.00000  -0.00000   1.00000

ans =   -4.4409e-16
ans =    2.2204e-16
ans =  0.0039062
```

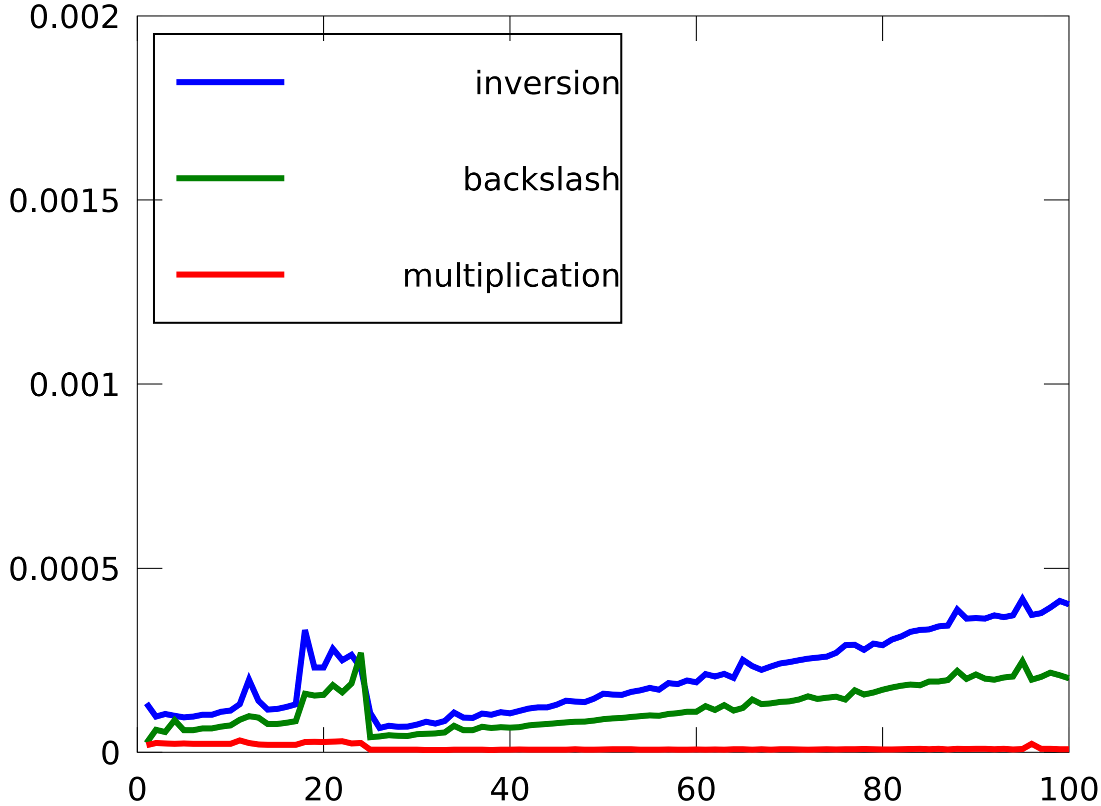
Now the solution of x to $Ax = y$ is $x = A^{-1}y$

In [70]:
```
y=[1;2;3]
x=invA*y
xbs=A\y
x-xbs
eps
```

```
y =

   1
   2
   3

x =
```

```
    6.0000
   11.0000
   14.0000

xbs =

    6.0000
   11.0000
   14.0000

ans =

  -3.5527e-15
  -8.8818e-15
  -1.0658e-14

ans =    2.2204e-16
```

```
In [71]: N=100;
         n=[1:N];
         t_inv=zeros(N,1);
         t_bs=zeros(N,1);
         t_mult=zeros(N,1);
         for i=1:N
             A=rand(i,i);
             tic
             invA=inv(A);
             t_inv(i)=toc;
             b=rand(i,1);
             tic;
             x=A\b;
             t_bs(i)=toc;
             tic;
             x=invA*b;
             t_mult(i)=toc;
         end
         plot(n,t_inv,n,t_bs,n,t_mult)
         axis([0 100 0 0.002])
         legend('inversion','backslash','multiplication','Location','NorthWest')
```



1.1 Condition of a matrix

1.1.1 *just checked in to see what condition my condition was in*

1.1.2 Matrix norms

The Euclidean norm of a vector is measure of the magnitude (in 3D this would be: $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$) in general the equation is:

$||x||_e = \sqrt{\sum_{i=1}^{n} x_i^2}$

For a matrix, A, the same norm is called the Frobenius norm:

$||A||_f = \sqrt{\sum_{i=1}^{n} \sum_{i=1}^{m} A_{i,j}^2}$

In general we can calculate any p-norm where

$||A||_p = \sqrt{\sum_{i=1}^{n} \sum_{i=1}^{m} A_{i,j}^p}$

so the p=1, 1-norm is

$||A||_1 = \sqrt{\sum_{i=1}^{n} \sum_{i=1}^{m} A_{i,j}^1} = \sum_{i=1}^{n} \sum_{i=1}^{m} |A_{i,j}|$

$||A||_\infty = \sqrt{\sum_{i=1}^{n} \sum_{i=1}^{m} A_{i,j}^\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^{m} |A_{i,j}|$

1.1.3 Condition of Matrix

The matrix condition is the product of

$Cond(A) = ||A|| \cdot ||A^{-1}||$

So each norm will have a different condition number, but the limit is $Cond(A) \geq 1$

An estimate of the rounding error is based on the condition of A:

$\frac{||\Delta x||}{x} \leq Cond(A) \frac{||\Delta A||}{||A||}$

So if the coefficients of A have accuracy to $10^{-t}

and the condition of A, $Cond(A) = 10^c$

then the solution for x can have rounding errors up to 10^{c-t}

In [72]:
```
A=[1,1/2,1/3;1/2,1/3,1/4;1/3,1/4,1/5]
[L,U]=LU_naive(A)
```

```
A =

   1.00000   0.50000   0.33333
   0.50000   0.33333   0.25000
   0.33333   0.25000   0.20000

L =

   1.00000   0.00000   0.00000
   0.50000   1.00000   0.00000
   0.33333   1.00000   1.00000

U =

   1.00000   0.50000   0.33333
   0.00000   0.08333   0.08333
   0.00000  -0.00000   0.00556
```

Then, $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$

$Ld_1 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right], Ux_1 = d_1$ ...

In [75]:
```
invA=zeros(3,3);
d1=L\[1;0;0];
d2=L\[0;1;0];
d3=L\[0;0;1];
invA(:,1)=U\d1;
invA(:,2)=U\d2;
invA(:,3)=U\d3
invA*A
```

```
invA =

     9.0000   -36.0000    30.0000
   -36.0000   192.0000  -180.0000
    30.0000  -180.0000   180.0000

ans =

   1.0000e+00   3.5527e-15   2.9976e-15
  -1.3249e-14   1.0000e+00  -9.1038e-15
   8.5117e-15   7.1054e-15   1.0000e+00
```

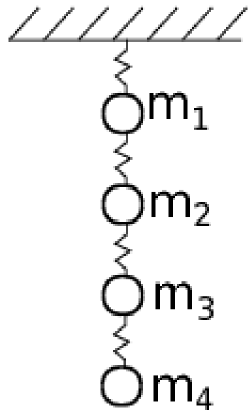
Find the condition of A, $cond(A)$

In [74]:
```
% Frobenius norm
normf_A = sqrt(sum(sum(A.^2)))
normf_invA = sqrt(sum(sum(invA.^2)))

cond_f_A = normf_A*normf_invA

norm(A,'fro')

% p=1, column sum norm
norm1_A = max(sum(A,2))
norm1_invA = max(sum(invA,2))
norm(A,1)

cond_1_A=norm1_A*norm1_invA

% p=inf, row sum norm
norminf_A = max(sum(A,1))
norminf_invA = max(sum(invA,1))
norm(A,inf)

cond_inf_A=norminf_A*norminf_invA
```

```
normf_A =  1.4136
normf_invA =  372.21
cond_f_A =  526.16
ans =  1.4136
norm1_A =  1.8333
norm1_invA =  30.000
ans =  1.8333
cond_1_A =  55.000
norminf_A =  1.8333
norminf_invA =  30.000
```

```
ans =  1.8333
cond_inf_A =  55.000
```

Consider the problem again from the intro to Linear Algebra, 4 masses are connected in series to 4 springs with spring constants K_i. What does a high condition number mean for this problem?



Springs-masses

The masses haves the following amounts, 1, 2, 3, and 4 kg for masses 1-4. Using a FBD for each mass:

$m_1 g + k_2(x_2 - x_1) - k_1 x_1 = 0$
$m_2 g + k_3(x_3 - x_2) - k_2(x_2 - x_1) = 0$
$m_3 g + k_4(x_4 - x_3) - k_3(x_3 - x_2) = 0$
$m_4 g - k_4(x_4 - x_3) = 0$
in matrix form:

$$\left[\begin{array}{cccc} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right] = \left[\begin{array}{c} m_1 g \\ m_2 g \\ m_3 g \\ m_4 g \end{array} \right]$$

```
In [21]: k1=10; % N/m
         k2=100000;
         k3=10;
         k4=1;
         m1=1; % kg
         m2=2;
         m3=3;
         m4=4;
         g=9.81; % m/s^2
         K=[k1+k2 -k2 0 0; -k2 k2+k3 -k3 0; 0 -k3 k3+k4 -k4; 0 0 -k4 k4]
         y=[m1*g;m2*g;m3*g;m4*g]
```

```
K =

   100010  -100000        0        0
  -100000   100010      -10        0
```

```
        0       -10        11        -1
        0         0        -1         1

y =

    9.8100
   19.6200
   29.4300
   39.2400
```

In [25]:
```
cond(K,inf)
cond(K,1)
cond(K,'fro')
cond(K,2)
```

```
ans =    3.2004e+05
ans =    3.2004e+05
ans =    2.5925e+05
ans =    2.5293e+05
```

In [26]:
```
e=eig(K)
max(e)/min(e)
```

```
e =

   7.9078e-01
   3.5881e+00
   1.7621e+01
   2.0001e+05

ans =    2.5293e+05
```

In [ ]: