CS787: Advanced Algorithms	**Homework 4**
Out: 3/19/22	**Due: 3/31/22**

1 An LP for Minimum Mean Cycle

In the Goldberg-Tarjan minimum mean cost cycle canceling algorithm we sought a cycle C that minimizes:

$$\sum_{(u,v)\in C} \frac{c(u,v)}{|C|}$$

Consider a linear program with a variable $x_{u,v}$ for each of the $m = |E|$ edges in the graph $G = (V, E)$:

$$\min_{x_{u,v}} \sum_{(u,v)\in E} c(u,v)x_{u,v}$$

such that:

$$\begin{aligned} \sum_u x_{u,v} &= \sum_u x_{v,u} && \text{for all nodes } v \in V \\ \sum_{(u,v)\in E} x_{u,v} &= 1 \\ x_{u,v} &\geq 0 && \text{for all edges } (u,v) \in E \end{aligned}$$

(a) Show that the minimum value of the LP is equal to the mean cost of the minimum mean cost cycle. Hint: Show that the values of $x_{u,v}$ form a circulation in G. A circulation is like a flow, but flow is conserved at every node – i.e. there are no special source/sink nodes s and t. As in flow decomposition, we can decompose circulations, but the decomposition will only contain cycles.

(b) Write down a dual linear program. Show that the optimum of the this dual program is equivalent to $\mathcal{E}$ from our original analysis of Goldberg-Tarjan.

(c) Conclude that $\mathcal{E}$ is equal to the mean cost of the minimum mean cost cycle in H. We proved this statement in class without going through linear programming.

2 Stationary distributions

When playing Monopoly, if you are at the i-th square and roll the dice you will end up at a random square. This might be the square that the dice tell you to go but in some cases you draw a card that might also move you to a different location. Thus for every square i, you end up at square j with probability P_{ij} after your turn. This is an example of a Markov chain which is described by the matrix P with non-negative entries and rows that sum to 1. Such matrices are called stochastic.

It is known that for every stochastic matrix P there is a stationary distribution π with $\pi \geq 0$ and $\sum_i \pi_i = 1$ such that $\pi^T P = \pi^T$. In the Monopoly example, there is a distribution over squares such that if you start at a random such square the distribution of the location after one turn is exactly the same. This is the distribution that you expect to see if you looked at the game board at a random time a long time after the game has started. For Monopoly, it can be shown that the most probable square is JAIL with probability 6.24%.

Use duality to prove that for any stochastic matrix P, a stationary distribution π such that $\pi^T P = \pi^T$ exists.

3 Simultaneous Zero-Sum Games

Players 1 and 2 are playing a zero-sum game where player 1 must pay player 2, A_{ij} if they play strategies i and j respectively. At the same time players 1 and 3 are playing another zero-sum game where player 1 must pay player 3, B_{ik} if they play strategies i and k respectively. Instead of using different strategies in the two games, player 1 must use the same strategy in both games. All players have n distinct strategies but may randomize among them.

Using linear programming duality, show that there exists a Nash equilibrium in this game, i.e. randomized strategies $x, y, z \in \Delta_n \triangleq \{\pi : \sum_{i=1}^{n} \pi_i = 1, \pi \geq 0\}$ for the three players so that:

$$x^T Ay + x^T Bz \leq x'^T Ay + x'^T Bz \text{ for any } x' \in \Delta_n$$
$$x^T Ay \geq x^T Ay' \text{ for any } y' \in \Delta_n$$
$$x^T Bz \geq x^T Bz' \text{ for any } z' \in \Delta_n$$

4 Renting Houses

You want to start a time-sharing business. You plan to buy many houses around the world and rent them out for a week to your clients. You have identified m potential properties and n clients. Every client i has a set $S_{i,t} \subseteq [m]$ of properties he wishes to stay on week $t \in [T]$. What is the minimum number of properties you need to purchase so that for each of the T weeks all clients stay at some location. No two clients can stay at the same location during the same week.

The following integer program with indicator variables $x_j \in \{0, 1\}$, gives a solution.

$$\min_x \sum_{j=1}^{m} x_j$$

under the constraint that for all weeks t and subsets of clients $S \subseteq [n]$

$$\sum_{j \in \cup_{i \in S} S_{i,t}} x_j \geq |S|$$

You want to solve the above program as an LP by relaxing the constraints on the variables to be $x_j \in [0, 1]$. Let OPT_{LP} be the resulting fractional solution. You now want to round the solution x of the LP to an integral one so that all clients are matched to a property every week without buying a lot more properties than OPT_{LP}.

You incrementally build your collection of properties by randomly buying every time a single property j with probability proportional to x_j, i.e. x_j/OPT_{LP}. You repeat this process until all clients are satisfied in all weeks.

(a) Show that the above LP can be solved efficiently despite having $O(2^n)$ constraints.

(b) Show that if k clients remain unmatched in week t, the probability that the matching can increase by 1 after a random property is bought is at least k/OPT_{LP}.
HINT: For the matching to increase by 1, we need either

- a property that one of the k unmatched clients prefer to be selected or,
- a property that one of the matched clients prefer to be selected as long as there is a path in the residual graph to that client from the unmatched clients.

(c) Conclude with a polynomial-time algorithm which is $O(\log n \log T)$ approximate by showing that the expected number of properties that you will buy until all clients are satisfied in all weeks is $O(\log n \log T) OPT_{LP}$.