Excercise2 part a,d,e,f

2. Suppose we live at a place where days are either sunny, cloudy, or rainy. The weather transition function is a Markov chain with the following transition table:

		tomorrow will be...		
		sunny	cloudy	rainy
today it's...	sunny	.8	.2	0
	cloudy	.4	.4	.2
	rainy	.2	.6	.2

(a) Suppose Day 1 is a sunny day. What is the probability of the following sequence of days: Day2 = *cloudy*, Day3 = *cloudy*, Day4 = *rainy*?

(b) Write a simulator that can randomly generate sequences of "weathers" from this state transition function.

(c) Use your simulator to determine the stationary distribution of this Markov chain. The stationary distribution measures the probability that a random day will be sunny, cloudy, or rainy.

(d) Can you devise a closed-form solution to calculating the stationary distribution based on the state transition matrix above?

(e) What is the entropy of the stationary distribution?

(f) Using Bayes rule, compute the probability table of yesterday's weather given today's weather. (It is okay to provide the probabilities numerically, and it is also okay to rely on results from previous questions in this exercise.)

(g) Suppose we added seasons to our model. The state transition function above would only apply to the Summer, whereas different ones would apply to Winter, Spring, and Fall. Would this violate the Markov property of this process? Explain your answer.

2. (a) $P = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$

$1 \times 0.2 \times 0.4 \times 0.2 = 0.016$

$\uparrow$ Day1 Sunny, $\uparrow$ Day2 cloudy, $\uparrow$ Day3 Cloudy, $\uparrow$ Day4 Rainy

(d) Stationary Distribution is $\pi = [\pi_0\ \pi_1\ \pi_2]$

$\pi = \pi P$, $\pi = [\pi_0, \pi_1, \pi_2]$

$\pi I = \pi P$

$\pi P - \pi I = 0$

$\pi(P-I) = 0$

$$[\pi_0\ \pi_1\ \pi_2] \begin{bmatrix} -0.2 & 0.2 & 0 \\ 0.4 & -0.6 & 0.2 \\ 0.2 & 0.6 & -0.8 \end{bmatrix} = 0$$

$$\begin{cases} -2\pi_0 + 4\pi_1 + 2\pi_2 = 0 \\ 2\pi_0 - 6\pi_1 + 6\pi_2 = 0 \\ 2\pi_1 - 8\pi_2 = 0 \end{cases}$$

$$\rightarrow \begin{cases} \pi_1 = 4\pi_2 \\ -\pi_0 + 2\pi_1 + \pi_2 = 0 \\ \pi_0 - 3\pi_1 + 3\pi_2 = 0 \end{cases}$$

$$\begin{cases} \pi_0 + \pi_1 + \pi_2 = 1. \\ \pi_0 - 9\pi_2 = 0 \end{cases}$$

$\Rightarrow 9\pi_2 + 4\pi_2 + \pi_2 = 1$

$\Rightarrow \pi_2 = \frac{1}{14} \approx 0.071429$

$\pi_1 = \frac{4}{14} \approx 0.285714$

$\pi_0 = \frac{9}{14} \approx 0.642857$ #

(e) Entropy of the system is $-\sum_i \pi_i \log_2(\pi_i)$

$= -\pi_0 \log_2 \pi_0 - \pi_1 \log_2 \pi_1 - \pi_2 \log_2 \pi_2$

$\fallingdotseq 0.409776 + 0.516387 + 0.271955$

≈ 1.198118 #

(f) Bayes' Theorem :

$$P(y|x,H) = \frac{P(x|y,H)\,P(y|H)}{P(x|H)} = \frac{P(x|y,H)\,P(y|H)}{\sum_{y'} P(x|y',H)\,P(y'|H)}$$

$$\Rightarrow p(x_{t-1}|x_t) = \frac{P(x_t|x_{t-1})\,P(x_{t-1})}{P(x_t)} = \frac{P(x_t|x_{t-1})\,P(x_{t-1})}{\sum_{x_{t-1}} P(x_t|x'_{t-1})\,P(x'_{t-1})}$$

① let Sunny = s, Cloudy = c, Rainy = r,

$$P(x_{t-1}=s|x_t=s) = \frac{P(x_t=s|x_{t-1}=s)\,P(x_{t-1}=s)}{P(x_t=s)}$$

$\Leftrightarrow$ If Sunny, Cloudy, Rainy is uniform distribution

$$P(x_{t-1}=s|x_t=s) = \frac{4/5 \times 1/3}{1/3 \times 4/5 + \frac{1}{3}\times 2/5 + \frac{1}{3}\times 1/5} = \frac{4}{7}$$

$$\begin{cases} P(x_{t-1}=s) = P(x_{t-1}=c) = P(x_{t-1}=r) = \frac{1}{3} \\ P(x_t=s) = \frac{1}{3}\times\frac{4}{5} + \frac{1}{3}\times\frac{2}{5} + \frac{1}{3}\times\frac{1}{5} = \frac{7}{15} \\ P(x_t=c) = \frac{1}{3}\times\frac{1}{5} + \frac{1}{3}\times\frac{2}{5} + \frac{1}{3}\times\frac{3}{5} = \frac{6}{15} \\ P(x_t=r) = \frac{1}{3}\times 0 + \frac{1}{3}\times\frac{1}{5} + \frac{1}{3}\times\frac{1}{5} = \frac{2}{15} \end{cases}$$

$$\Rightarrow P(x_{t-1}=c|x_t=s) = \frac{2/5 \times 1/3}{7/15} = \frac{2}{7}$$

$$P(x_{t-1}=r|x_t=s) = \frac{1/5 \times 1/3}{7/15} = \frac{1}{7}$$

$$P(x_{t-1}=s|x_t=c) = \frac{1/5 \times 1/3}{6/15} = \frac{1}{6}$$

$$P(x_{t-1}=c|x_t=c) = \frac{2/5 \times 1/3}{6/15} = \frac{2}{6}$$

$$P(x_{t-1}=r|x_t=c) = \frac{3/5 \times 1/3}{6/15} = \frac{3}{6}$$

$$P(x_{t-1}=s|x_t=r) = \frac{0 \times 1/3}{2/15} = 0$$

$$P(x_{t-1}=c|x_t=r) = \frac{1/5 \times 1/3}{2/15} = \frac{1}{2}$$

$$P(x_{t-1}=r|x_t=r) = \frac{1/5 \times 1/3}{2/15} = \frac{1}{2}$$

Using uniform distribution as prior:

		yesterday		
		sunny	cloudy	rainy
today	sunny	4/7	2/7	1/7
	cloudy	1/6	1/3	1/2
	rainy	0	1/2	1/2

#

Using stationary distribution as prior:

$$\begin{cases} P(X_{t-1}=s) = \pi_0 = 9/14 \\ P(X_{t-1}=c) = \pi_1 = 4/14 \\ P(X_{t-1}=r) = \pi_2 = 1/14 \end{cases} \qquad \begin{cases} P(X_t=s) = \frac{9}{14}\times\frac{4}{5}+\frac{4}{14}\times\frac{2}{5}+\frac{1}{14}\times\frac{1}{5} = \frac{45}{70} \\ P(X_t=c) = \frac{9}{14}\times\frac{1}{5}+\frac{4}{14}\times\frac{2}{5}+\frac{1}{14}\times\frac{3}{5} = \frac{20}{70} \\ P(X_t=r) = \frac{9}{14}\times 0+\frac{4}{14}\times\frac{1}{5}+\frac{1}{14}\times\frac{1}{5} = \frac{5}{70} \end{cases}$$

$$P(X_{t-1}=s \mid X_t=s) = \frac{P(X_t=s \mid X_{t-1}=s)\,P(X_{t-1}=s)}{P(X_t=s)} = \frac{\frac{4}{5}\times\frac{9}{14}}{\frac{45}{70}} = \frac{4}{5}$$

$$P(X_{t-1}=c \mid X_t=s) = \frac{\frac{2}{5}\times\frac{4}{14}}{\frac{45}{70}} = \frac{8}{45}$$

$$P(X_{t-1}=r \mid X_t=s) = \frac{\frac{1}{5}\times\frac{1}{14}}{\frac{45}{70}} = \frac{1}{45}$$

$$P(X_{t-1}=s \mid X_t=c) = \frac{\frac{1}{5}\times\frac{9}{14}}{\frac{20}{70}} = \frac{9}{20}$$

$$P(X_{t-1}=c \mid X_t=c) = \frac{\frac{2}{5}\times\frac{4}{14}}{\frac{20}{70}} = \frac{2}{5}$$

$$P(X_{t-1}=r \mid X_t=c) = \frac{\frac{3}{5}\times\frac{1}{14}}{\frac{20}{70}} = \frac{3}{20}$$

$$P(X_{t-1}=s \mid X_t=r) = 0$$

$$P(X_{t-1}=c \mid X_t=r) = \frac{\frac{1}{5}\times\frac{4}{14}}{\frac{5}{70}} = \frac{4}{5}$$

$$P(X_{t-1}=r \mid X_t=r) = \frac{\frac{1}{5}\times\frac{1}{14}}{\frac{5}{70}} = \frac{1}{5}$$

		yesterday		
		sunny	cloudy	rainy
today	sunny	4/5	8/45	1/45
	cloudy	9/20	2/5	3/20
	rainy	0	4/5	1/5

#

Date: /
NO:

Excercise2 part g

馬可夫性質的定義是：未來狀態的條件機率分佈只依賴於當前狀態。我認為加入季節的特徵只是令馬可夫鏈多了一個特徵(條件)而已，也就是說當考慮明天的天氣時，也要考慮到當前的季節是什麼。以前面的表格舉例：

		tomorrow will be...		
		sunny	cloudy	rainy
today it's...	sunny	.8	.2	0
	cloudy	.4	.4	.2
	rainy	.2	.6	.2

$P(X_t=\text{sunny} \mid X_{t-1}=\text{sunny}) = 0.8$ 若考慮季節，就會寫成：

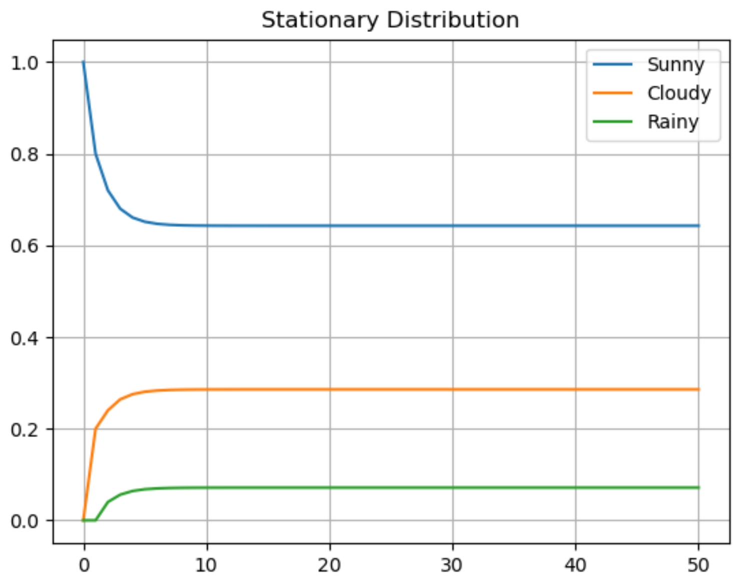
$P(X_t=\text{sunny} \mid X_{t-1}=\text{sunny}, \text{Summer}) = 0.8$

因此加入季節特徵不違背馬可夫性質。

Excercise2 part b,c

Result:

```
Start state: Sunny
Possible states: ['Sunny', 'Cloudy', 'Rainy', 'Rainy', 'Rainy', 'Rainy']
End state after 5 days: Rainy
Probability of the possible sequence of states: 0.00032000000000000013
Stationary Distribution (Pi vector): [0.64285714 0.28571429 0.07142857]
```



CODE:

```
import numpy as np
import pandas as pd
import random as rm
import matplotlib.pyplot as plt

# The statespace
states = ["Sunny","Cloudy","Rainy"]

# Possible sequences of events
transitionName = [["SS","SC","SR"],["CS","CC","CR"],["RS","RC","RR"]]

# Probabilities matrix (transition matrix)
transitionMatrix = [[0.8, 0.2, 0.0],[0.4, 0.4, 0.2],[0.2, 0.6, 0.2]]

# # Check transition matrix's probibilities
# if sum(transitionMatrix[0])+sum(transitionMatrix[1])+sum(transitionMatrix[1]) != 3:
#     print("Somewhere, something went wrong. Transition matrix, perhaps?")
# else: print("All is gonna be okay, you should move on!! ;)")

# A function that implements the Markov model to forecast the state/mood.
def weather_forecast(days, startWeather):
    # Choose the starting state
    weatherToday = startWeather
    print("Start state: " + weatherToday)
    # Shall store the sequence of states taken. So, this only has the starting state for now.
    weatherList = [weatherToday]
    # To calculate the probability of the weatherList
    prob = 1

    for i in range(days):
        if weatherToday == "Sunny":
```

```python
            change = np.random.choice(transitionName[0],replace=True,p=
transitionMatrix[0])
            if change == "SS":
                prob = prob * 0.8
                weatherList.append("Sunny")
                pass    # 繼續執行迴圈剩下的程式碼
            elif change == "SC":
                prob = prob * 0.2
                weatherToday = "Cloudy"
                weatherList.append("Cloudy")
            else:
                prob = prob * 0.0
                weatherToday = "Rainy"
                weatherList.append("Rainy")
        elif weatherToday == "Cloudy":
            change = np.random.choice(transitionName[1],replace=True,p=
transitionMatrix[1])
            if change == "CS":
                prob = prob * 0.4
                weatherList.append("Sunny")
                pass    # 繼續執行迴圈剩下的程式碼
            elif change == "CC":
                prob = prob * 0.4
                weatherToday = "Cloudy"
                weatherList.append("Cloudy")
            else:
                prob = prob * 0.2
                weatherToday = "Rainy"
                weatherList.append("Rainy")
        else:
            change = np.random.choice(transitionName[0],replace=True,p=
transitionMatrix[0])
            if change == "RS":
                prob = prob * 0.2
                weatherList.append("Sunny")
                pass    # 繼續執行迴圈剩下的程式碼
            elif change == "RC":
                prob = prob * 0.6
```

```
                weatherToday = "Cloudy"
                weatherList.append("Cloudy")
            else:
                prob = prob * 0.2
                weatherToday = "Rainy"
                weatherList.append("Rainy")

    print("Possible states: " + str(weatherList))
    print("End state after "+ str(days) + " days: " + weatherToday)
    print("Probability of the possible sequence of states: " + str(prob
))

def Markov_chain_stationary_distribution(P):
    state = np.array([[1.0, 0.0, 0.0]])
    stateHist = state
    dfStateHist = pd.DataFrame(state)
    distr_hist = [[0 ,0, 0]]

    for x in range(50):
        state = np.dot(state, P)
        # print(state)
        stateHist = np.append(stateHist, state, axis=0)
        dfDistrHist = pd.DataFrame(stateHist)

    print("Stationary Distribution (Pi vector):", state[0])
    dfDistrHist.plot()
    plt.grid()
    plt.title('Stationary Distribution')
    plt.legend(['Sunny', 'Cloudy', 'Rainy'])
    plt.show()

### Main Function ###

# inputWeather = input("Please input the weather today, type in \"Sunny
\", \"Cloudy\", or \"Rainy\": ")
# days = int(input("Please input how many days you want to forecast: ")
)
weather_forecast(days = 5, startWeather = "Sunny")
```

```
### 判斷穩態分佈存在的條件
# 1. 不可約性 Irreducible: 任何狀態都可能轉移到任意狀態
# 2. 非週期性 Aperiodic: 才會有穩態存在
# 3. 時間均勻性 Homogeneous
# 4. 有限狀態 Finite States
Markov_chain_stationary_distribution(P = transitionMatrix)
```

Excercise3

No:
Date: /

s, c, c, r, s

3. (a) By bayer's rule and Markov assumption

$P(x_5 | x_1, z_{2:5})$ = Day 5 is indeed sunny and detected sunny

$$= \frac{P(z_5 | x_5, x_1, z_{2:4}) P(x_5 | x_1, z_{2:4})}{P(z_5 | x_1, z_{2:4})}$$

* x means true weather
z means sensor data

$$= \eta \cdot P(z_5 | x_5, x_1, z_{2:4}) P(x_5 | x_1, z_{2:4})$$

$$= \eta \cdot P(z_5 | x_5) P(x_5 | x_1, z_{2:4}) \quad \cdots ①$$

$$P(x_5 | x_1, z_{2:4}) = \sum_{x_4} P(x_4, x_5 | x_1, z_{2:4}) \quad \text{(Sum Rule)}$$

$$= \sum_{x_4} P(x_5 | x_4, x_1, z_{2:4}) P(x_4 | x_1, z_{2:4}) \quad \text{Chain Rule}$$

$$= \sum_{x_4} P(x_5 | x_4) P(x_4 | x_1, z_{2:4})$$

$\because$ z_4 = rainy $\Rightarrow$ x_4 = rainy

$$= P(x_5 | x_4 = \text{rainy}) \cdot 1 \quad \cdots ②$$

So, with ①, ②, $P(x_5 | x_1, z_{2:5})$

$$= \eta \cdot P(z_5 | x_5) \cdot P(x_5 | x_4 = r) \cdot 1$$

$$= \frac{P(z_5 = s | x_5 = s) \cdot P(x_5 = s | x_4 = r)}{\sum_{x_5'} P(z_5 | x_5', z_{2:4}) P(x_5' | z_{2:4})}$$

$$= \frac{P(z_5 = s | x_5 = s) \cdot P(x_5 = s | x_4 = r)}{\sum_{x_5'} P(z_5 | x_5') P(x_5' | x_4)}$$

$$= \frac{P(z_5 = s | x_5 = s) \cdot P(x_5 = s | x_4 = r)}{\sum_{x_5'} P(z_5 = s | x_5') P(x_5' | x_4 = r)}$$ $\Rightarrow$ x_5' 含 sunny, cloudy, rainy

$$= \frac{0.6 \times 0.2}{0.6 \times 0.2 + 0.3 \times 0.6 + 0} = 0.4$$

Date: 5, 5, 5, r

3.(b) Future data not used: Day 2

$$P(x_2|x_1,z_2) = P(z_2|x_2,x_1)P(x_2|x_1)/P(z_2|x_1)$$

$$= \frac{P(z_2|x_2,x_1)P(x_2|x_1)}{\sum_{x_2'} P(z_2|x_2',x_1)P(x_2'|x_1)} = \eta \cdot P(z_2|x_2,x_1)P(x_2|x_1)$$

$$= \eta \cdot \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix} = \eta \begin{pmatrix} 0.48 \\ 0.06 \\ 0 \end{pmatrix} = \begin{pmatrix} 8/9 \\ 1/9 \\ 0 \end{pmatrix}$$

$\Rightarrow$ Sunny is most possible on Day 2 ☆

Future considered: Day 2

$$P(x_2|x_1,z_{2:4}) = \eta P(x_2|x_1)P(z_{2:4}|x_2,x_1)$$

$$= \eta P(x_2|x_1)P(z_{2:4}|x_2) \quad \text{↴ chain Rule}$$

$$= \eta P(x_2|x_1)P(z_2|z_{3:4},x_2)P(z_{3:4}|x_2)$$

$$= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(x_3,z_{3:4}|x_2)$$

$$= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(z_{3:4}|x_3,x_2)P(x_3|x_2)$$

$$= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(x_3|x_2)P(z_4|z_3,x_3)P(z_3|x_3)$$

$$= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(x_3|x_2)P(z_3|x_3)\sum_{x_4} P(z_4,x_4|x_3)$$

$$= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(x_3|x_2)P(z_3|x_3)\sum_{x_4} P(z_4|x_4,x_3)P(x_4|x_3)$$

$$= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(x_3|x_2)P(z_3|x_3)\sum_{x_4} P(x_4|x_3)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{P'(x_4|z_4)}$$

$$= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(x_3|x_2)P(z_3|x_3)\begin{pmatrix} 0 \\ 0.2 \\ 0.2 \end{pmatrix}_{P'(x_3|z_4)}$$

$$= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(x_3|x_2)\begin{pmatrix} 0 \\ 0.06 \\ 0 \end{pmatrix}_{P'(x_3|z_3 z_4)}$$

$$= \eta P(x_2|x_1)P(z_2|x_2)\begin{pmatrix} 0.012 \\ 0.024 \\ 0.036 \end{pmatrix}_{P'(x_2|z_3,z_4)}$$

$$= \eta \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix}_{P'(x_2|x_1)} \cdot \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix}_{P'(x_2|z_2)} \cdot \begin{pmatrix} 0.012 \\ 0.024 \\ 0.036 \end{pmatrix}_{P'(x_2|z_3,z_4)}$$

$$= \eta \begin{pmatrix} 0.00576 \\ 0.00144 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix}$$ $\Rightarrow$ Sunny is most possible on Day 2 ☆

s, s, s, r

Day 3 : $P(x_3|x_1, z_{2:3}) = \eta P(z_3|x_3, x_1, z_2) P(x_3|x_1, z_2)$

$= \eta P(z_3|x_3) \sum_{x_2} P(x_3, x_2|x_1, z_2) = \eta P(z_3|x_3) \sum_{x_2} P(x_3|x_2) P(x_2|x_1, z_2)$

$= \eta P(z_3|x_3) \sum_{x_2} P(x_3|x_2) \begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix} P'(x_2|x_1, z_2)$

$= \eta P(z_3|x_3) \begin{pmatrix} 6.8 \\ 2.0 \\ 0.2 \end{pmatrix} P'(x_3|x_1, z_2) = \eta \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 6.8 \\ 2.0 \\ 0.2 \end{pmatrix}$

$= \begin{pmatrix} 408/468 \\ 60/468 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.8718 \\ 0.1282 \\ 0 \end{pmatrix} \Rightarrow$ 87.2% sunny, 12.8% cloudy, 0% rainy

Day 3 with future data :

$P(x_3|x_1, z_{2:4}) = \eta P(z_4|x_3, x_1, z_{2:3}) P(x_3|x_1, z_{2:3})$

$= \eta P(x_4|x_3) P(x_3|x_2, z_3)$

$= \eta \begin{pmatrix} 0 \\ 0.2 \\ 0.2 \end{pmatrix} \cdot \begin{pmatrix} 0.8395 \\ 0.1605 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$ 100% cloudy

Day 4 : $P(x_4|x_1, z_{2:4}) = P(x_4|x_3, z_4)$

$= \eta P(z_4|x_4, x_3) P(x_4|x_3)$

$= \eta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0.2 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow$ 100% rainy

3. (c) $s \to s \to s \to r$

The probability of the sequence of weather is given by

$P(x_{2:4}|x_1, z_{2:4}) = \eta P(z_{2:4}|x_1, x_{2:4}) P(x_{2:4}|x_1)$

where

$P(x_{2:4}|x_1) = P(x_4|x_3) P(x_3|x_2) P(x_2|x_1)$

and

$P(z_{2:4}|x_1, x_{2:4}) = P(z_4|x_4) P(z_3|x_3) P(z_2|x_2)$

Hence, the most likely sequence of weather is sunny, cloudy, rainy which has 0.00576/(0.00576+0.00144) = 80% of occurring. There is 20% probability of cloudy, cloudy, rainy, and 0% for others.