PROPERTIES of CONTEXT-FREE LANGUAGES

Chap. 8

Summary

- The general properties for CFL to compare
  with the properties of Regular Languages (RL).
- The property of closure and decision algorithms
  – more complicated for CFL.
- More complicate for the two Pumping Lemmas,
  – one for CFL, the other for Linear Languages ($\supset$ RL).
- While the basic idea is the same as for Regular languages, specific arguments normally involve much more detail.

Learning Objectives

- Apply the *Pumping Lemma* to show that a language is ***not*** *Context-Free*.
- State the *Closure properties* applicable to CFL.
- Prove that CFLs are *closed* under the operations of *union, concatenation,* and *star-closure*.
- Prove that CFLs are ***not*** *closed* under either *intersection* or *complementation*.
- Describe a *Membership Algorithm* for CFL.
- Describe an *algorithm* to determine if a CFL is *empty*.
- Describe an *algorithm* to determine if a CFL is *infinite*.

A Pumping Lemma for Context-Free Language

- Theorem 8.1: Given an *infinite* Context-Free Laguage L, there exists some positive integer m such that any sufficiently long string $w \in L$ with $|w| \geq m$ can be decomposed as
  - $w = uvxyz$, $u, v, x, y, z \in T^*$ with
  - $|vy| \geq 1$ and $|vxy| \leq m$
    where m is an arbitrary integer, $0 < m \leq |w|$
  - an arbitrary, but equal number of repetitions of v and y yields another string in L: $w_i = uv^ixy^iz \in L,\ i \geq 0.$
- The "pumped" string consists of two separate parts (v and y) and can occur anywhere in the string.
- The Pumping Lemma can be used to show, by contradiction, that a certain language is *not context-free*.

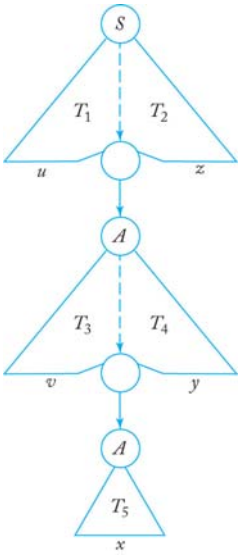
An Illustration of the Pumping Lemma for CFL

As shown in Fig. 8.1,

the pumping lemma for CFLs can be illustrated by sketching

a general derivation tree that shows a decomposition of the string into the required components.

Proof: ?



Proof: Pumping Lemma for CFL

Proof: Consider L – {λ} and assume a CFG G without unit-productions or λ-productions.

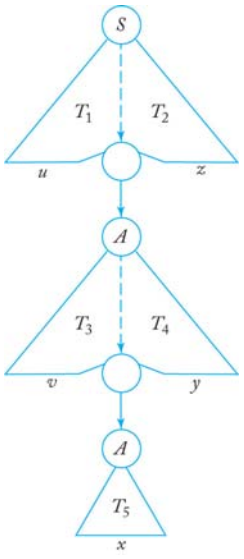
Since the length of string on the right side of any production is bounded, say by *k*, the length of the derivation of any $w \in L$ must be at least $|w|/k$. Thus, since L is infinite, there exist arbitrarily long derivations and corresponding derivation trees of arbitrary height.

Consider such a high derivation tree and some sufficiently long path from the root to a leaf. Since the number of variables in G is finite, there must be some variable that repeats on this path.

Corresponding to the derivation tree in Fig.,

$\exists\ S \Rightarrow^* uAz \Rightarrow^* uvAyz \Rightarrow^* uvxyz$ for $u,v,x,y,z \in T^*$

From this, we see $A \Rightarrow^* vAy$ and $A \Rightarrow^* x$, so all the strings uv^ixy^iz, $i = 0,1,2, ..$, can be generated by the grammar and are therefore in *L*.



Proof: Pumping Lemma for CFL

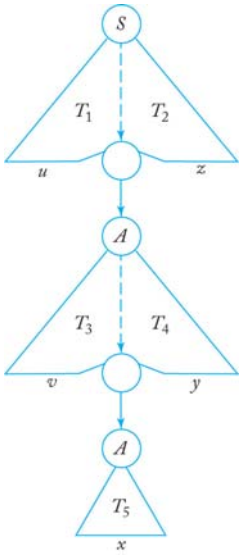
Cont.: $S \Rightarrow^* uAz \Rightarrow^* uvAyz \Rightarrow^* uvxyz,\ u,v,x,y,z \in T^*$

In $A \Rightarrow^* vAy$ and $A \Rightarrow^* x$, assume that no variable repeats. In Fig., no variable repeats in T_5. Similarly, assume no variable repeats in T_3 and T_4. Thus, the lengths of *v, x* and *y* depend only on the productions of the grammar and can be bounded independently of w, so $|vxy| \leq m$ holds.

Since there are no unit/λ-productions, *v* and *y* can't both be empty, so $|vy| \geq 1$.

This completes the argument that w = *uvxyz* and

$w_i = uv^ixy^iz \in L,\ i \geq 0$ holds.



Example: Pumping Lemma

- Example 8.1: $L = \{a^nb^nc^n \mid n \geq 0\}$ is not context free.

Pf) Let's choose *n=m* for an arbitrary *m,* so $w=a^mb^mc^m \in L$.

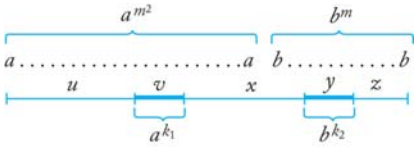
For *w=uvxyz,* if we choose *vxy* contains only *a's* since $|vxy| \leq m$, then $u=a^j,\ v=a^k,\ x=a^p,\ y=a^q,\ z=a^{m-j-k-p-q}b^mc^m$.

Then, by Pumping Lemma, $w_i = uv^ixy^iz$.

With *i*=0, $w_0 = uv^0xy^0z = uxz = a^{j+p}\ a^{m-j-k-p-q}b^mc^m = a^{m-k-q}b^mc^m \notin L$.

So, $L = a^nb^nc^n$ is *not* a *CFL.*

- Example 8.3: $L = \{a^{n!} \mid n \geq 0\}$ is not context free.
- Example 8.4: $L = \{a^nb^j \mid n = i^2\}$ is not context free.



Example: Pumping Lemma

- Example 8.2: L = {ww | $w \in \{a,b\}^*$ } is not context free.

Pf) Let's choose $ww = a^m b^m a^m b^m \in L$.

For $w = uvxyz$, let's choose each substring

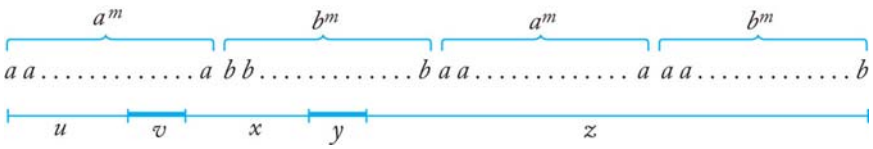
$u = a^j$, $v = a^k$, $x = a^{m-j-k} b^p$, $y = b^q$, $z = b^{m-p-q} a^m b^m$

where $m \geq j, k, p, q > 0$ and $|vxy| = m-j+p+q = m - (j-p-q) \leq m$.

By Pumping Lemma, $w_i = uv^i xy^i z$.

Thus, with i=0, $w_0 = uv^0 xy^0 z = uxz = a^{m-k} b^{m-q} a^m b^m \notin L$.

So, L = ww is *not* a *CFL.*



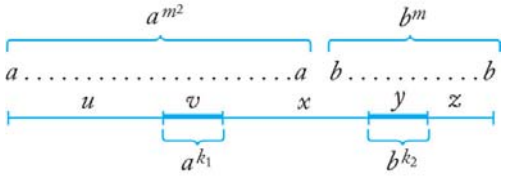
Example: Pumping Lemma

- Example 8.3: L = {$a^{n!}$ | $n \geq 0$ } is not context free.

Pf) Similar to the Examples 8.1 & 8.2. See textbook.

- Example 8.4: L = {$a^n b^j$ | $n = j^2$ } is not context free.

Pf) See the textbook.



Pumping Lemma for Linear Language

- Definition 8.1: A CFL L is said to be linear
  if there exists a *linear* CFG G s.t. $L = L(G)$.
  (*at most one nonterminal* in the right side of production)

Cf) A grammar G = (V, T, S, P) is *linear CFG*
if all productions in P are linear with the form
$A \rightarrow xBy$, where $A,B \in V$ and $x, y \in T^*$.

- Example 8.5(A):
  $L = \{a^n b^n \mid n \geq 0\}$ is a linear language, generated by a linear CFG,
  G = ({S}, {*a, b*}, S, P) with P given by { $S \rightarrow aSb \mid \lambda$ }.
- Example 8.5(B):
  $L = \{w \mid n_a(w) = n_b(w)\}$ is not linear because the grammar G,
  s.t. L = L(G), is not linear where G = ({S}, {*a, b*}, S, P)
  with P given by { $S \rightarrow SS \mid aSb \mid bSa \mid \lambda$ }.

Pumping Lemma for Linear Language

- Theorem 8.2: Let L be an infinite linear language.

Then, $\exists\, m > 0$, s.t. for any $w \in L$, with $|w| \geq m$ can be decomposed as $w = uvxyz$ with

$|vy| \geq 1$ and $|uvyz| \leq m$,

s.t. $w_i = uv^i xy^i z \in L \;\; \forall i = 0, 1, 2, \ldots$.

- The updated condition, $|uvyz| \leq m$, implies v and y must be within m symbols of the left & right ends of w, respectively.
- x can be of arbitrary length.

Pumping Lemma for Linear Language

- Theorem 8.2: Let L be an infinite linear language.

Then, $\exists\ m > 0$, s.t. for any $w \in L$, with $|w| \geq m$ can be decomposed as $w = uvxyz$ with $|uvyz| \leq m$, $|vy| \geq 1$, s.t. $w_i = uv^ixy^iz \in L\ \forall i$.

Proof: Since the language is linear there exists a linear grammar G for it. For the argument it is convenient to assume that G has no unit-productions and no λ-productions. An examination of the proofs of Theorem 6.3 and 6.4 makes it clear that removing unit-productions and λ-productions does not destroy the linearity of the grammar. We can therefore assume that G has the required property.

Consider now the derivation of a string $w \in L(G)$

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uv\,Ayz \overset{*}{\Rightarrow} uvxyz = w.$$

Assume, for the moment, that for every $w \in L(G)$, there is a variable A, such that

1. in the partial derivation $S \overset{*}{\Rightarrow} uAz$ no variable is repeated,
2. in the partial derivation $S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uv\,Ayz$ no variable except A is repeated,
3. the repetition of A must occur in the first m steps, where m can depend on the grammar, but not on w.

If this is true, then the lengths of u, v, y, z must be bounded independent of w. This in turn implies that (8.5), (8.6), and (8.7) must hold.

Pumping Lemma for Linear Language

- Example 8.6: $L = \{w \mid n_a(w) = n_b(w)\}$ is not linear.

Assume that L is linear and apply Pumping Lemma in Thm. 8.2 to $w = a^m b^{2m} a^m$.

$|uvyz| \leq m$ shows that the substrings u, v, y, z must all consist of all a's: $u=a^j$, $v=a^k$, $y=a^p$, $z=a^q$, $x=a^{m-j-k}b^{2m}a^{m-p-q}$.

Then, by Pumping lemma for Linear language, $w_i = uv^ixy^iz$.

With i=0, $w_0 = uv^0xy^0z = uxz = a^j\, a^{m-j-k}b^{2m}a^{m-p-q}\, a^q = a^{m-k}b^{2m}a^{m-p} \notin L$.

So, $L = a^nb^nc^n$ is not a CFL.

Closure Properties for CFL

- Theorem 8.3: If L_1 and L_2 are context-free languages, so are the languages that result from the following operations:
  - $L_1 \cup L_2$: union
  - $L_1 \cdot L_2$: concatenation.
  - L_1^* : star-closure
- In other words, the family of CFLs is closed under union, intersection, and star-closure.
- To prove these properties, we assume the existence of two context-free grammars G_1 and G_2 that generate the respective languages

Proof of Closure under Union

- Let L_1 and L_2 are CFLs generated by the Context-Free Grammars $G_1 = (V_1, T_1, S_1, P_1)$ and $G_2 = (V_2, T_2, S_2, P_2)$, respectively.
  Assume V_1 and V_2 are disjoint: $V_1 \cap V_2 = \varnothing$.
- Create a new variable $S_3 \notin V_1 \cup V_2$.
- Construct a new grammar $G_3 = (V_3, T_3, S_3, P_3)$ so that
  - $V_3 = V_1 \cup V_2 \cup \{ S_3 \}$
  - $T_3 = T_1 \cup T_2$
  - $P_3 = P_1 \cup P_2$
- Add to P_3 a production that allows the new start symbol to derive either of the start symbols for L_1 and L_2
  $S_3 \rightarrow S_1 \mid S_2$. So, $P_3 = P_1 \cup P_2 \cup \{S_3 \rightarrow S_1 \mid S_2 \}$
- Clearly, G_3 is Context-Free grammar so that the generated language, $L(G_3)$ is a CFL.

Proof of Closure under Union (cont.)

- Clearly, G_3 is CFG so that the generated language, $L(G_3)$, is a CFL.
- Let's show that $L(G_3) = L_1 \cup L_2$.

$\leftarrow$) if $w \in L_1$ or $w \in L_2$ (i.e. $S_1 \Rightarrow_{G1}^* w$ or $S_2 \Rightarrow_{G2}^* w$)
then w is derived by G_3, i.e. $S_3 \Rightarrow_{G3}^* w$.

Suppose $w \in L_1$. Then, $S_3 \Rightarrow S_1 \Rightarrow^* w$ is a possible derivation in G_3.
Similarly, for $w \in L_2$, $S_3 \Rightarrow S_2 \Rightarrow^* w$ is a possible derivation in G_3.
Thus, $w \in L(G_3)$.

$\rightarrow$) If $w \in L(G_3)$, then *either* $S_3 \Rightarrow S_1$ *or* $S_3 \Rightarrow S_2$ must be the 1st step of derivation.

- If $S_3 \Rightarrow S_1$ *is used, since sentential forms* derived from S_1 have variables in V_1, the derivation $S_1 \Rightarrow^* w$ can involve productions in P_1 only. Thus, w must be in L_1.
- Similarly, If $S_3 \Rightarrow S_2$ *is used,* w must be in L_2.
- Thus, $w \in L_1 \cup L_2$. *Therefore,* $L(G_3) = L_1 \cup L_2$.

Proof of Closure under Concatenation

- Let L_1 and L_2 are CFLs generated by the Context-Free Grammars $G_1 = (V_1, T_1, S_1, P_1)$ and $G_2 = (V_2, T_2, S_2, P_2)$, respectively.
  Assume V_1 and V_2 are disjoint: $V_1 \cap V_2 = \varnothing$.
- Create a new variable $S_4 \notin V_1 \cup V_2$.
- Construct a new grammar $G_4 = (V_4, T_4, S_4, P_4)$ so that
  - $V_4 = V_1 \cup V_2 \cup \{ S_4 \}$
  - $T_4 = T_1 \cup T_2$
  - $P_4 = P_1 \cup P_2$
- Add to P_4 a production that allows the new start symbol to derive the concatenation of the start symbols for L_1 and L_2
  $S_4 \rightarrow S_1 S_2$. So, $P_4 = P_1 \cup P_2 \cup \{S_4 \rightarrow S_1 S_2\}$
- Clearly, G_4 is context-free and generates the concatenation of L_1 and L_2, i.e. $L(G_4) = L(G_1)L(G_2)$.

Proof of Closure under Star-Closure

- Let L_1 be CFL generated by $G_1 = (V_1, T_1, S_1, P_1)$.
- Create a new variable $S_5 \notin V_1$.
- Construct a new grammar $G_5 = (V_5, T_5, S_5, P_5)$ so that
  - $V_5 = V_1 \cup \{ S_5 \}$
  - $T_5 = T_1$
  - $P_5 = P_1$
- Add to P_5 a production that allows the new start symbol S_5 to derive the repetition of the start symbol for L_1 any number of times

  $S_5 \rightarrow S_1S_5 \mid \lambda$. So, $P_5 = P_1 \cup P_2 \cup \{S_5 \rightarrow S_1S_5 \mid \lambda\}$
- Clearly, G_5 is context-free and generates the star-closure of L_1, i.e. $L(G_5) = L(G_1)^*$.

No Closure under Intersection

- Unlike regular languages, the intersection of two CFLs L_1 & L_2 does not necessarily produce a context-free language.
- As a counter example, consider the CFLs

  $L_1 = \{ a^n b^n c^m \mid n \geq 0, m \geq 0 \}$ where a grammar for L_1 is

  $S \rightarrow S_1S_2,\ S_1 \rightarrow aS_1b \mid \lambda,\ S_2 \rightarrow cS_2 \mid \lambda.$

  $L_2 = \{ a^n b^m c^m \mid n \geq 0, m \geq 0 \}$ whose CFL is, similarly,

  $S \rightarrow S_1S_2,\ S_1 \rightarrow aS_1 \mid \lambda,\ S_2 \rightarrow bS_2c \mid \lambda.$
- We note that L_1 is the concatenation of two CFLs, $\{a^n b^n \mid n \geq 0\}$ and $\{c^m \mid m \geq 0\}$, L_1 is context free by Th^m. 8.3. Similarly, so is L_2.
- However, the intersection of L_1 and L_2

  $L_3 = L_1 \cap L_2 = \{ a^n b^n c^n \mid n \geq 0 \}$

  has been proven *not context-free* by Pumping Lemma in Ex. 8.1.
- Therefore, CFL is not closed under intersection.

No Closure under Complementation

- The complement of a context-free language L_1 does not necessarily produce a context-free language.
- Proof by Contradiction

  Given CFLs L_1 and L_2, assume that their complements are also context-free: i.e. $\overline{L_1}$, $\overline{L_2}$ are CFLs.

  By Theorem 8.3, the union of $\overline{L_1}$ and $\overline{L_2}$ must also produce a CFL L_3 since CFLs are closed under union: $\overline{L_1} \cup \overline{L_2} = L_3$.

  Using our assumption, the complement of L_3 is also context-free.

  By De Morgan's law, however, we conclude that the complement of L_3 is the intersection of L_1 and L_2, which has been shown not to be context-free, thus contradicting our assumption.

  $$\overline{L_3} = \overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2.$$

  Therefore, CFLs are not closed under complementation.

Closure under Regular Intersection: Intersection of CFL and Regular Language

- Theorem 8.5: Let L_1 be a CFL and L_2 be a Regular language. Then, $L_1 \cap L_2$ is context free.

Proof) Let $M_1 = (Q, \Sigma, \Gamma, \delta_1, q_0, z, F_1)$ be an NPDA for L_1, $L(M_1)=L_1$, and $M_2 = (P, \Sigma, \delta_2, p_0, F_2)$ be a DFA for L_2, $L(M_2)=L_2$.

We construct a PDA $M' = (Q', \Sigma, \Gamma, \delta', q_0', z, F')$ that simulates the parallel action of M_1 and M_2: Whenever a symbol is read from the input string, M' simultaneously executes the moves of M_1 & M_2.

To this end, let $Q' = Q \times P$, $q_0'=(q_0, p_0)$, $F' = F_1 \times F_2$, and define δ' s.t. $((q_k, p_l), x) \in \delta'((q_i, p_j), a, b)$

iff $(q_k, x) \in \delta_1(q_i, a, b)$ and $\delta_2(p_j, a) = p_l$.

We also require that if $a = \lambda$, then $p_j = p_l$. *i.e.*

Intersection of CFL and Regular Language

- Theorem 8.5: Let L_1 be a CFL and L_2 be a Regular language. Then, $L_1 \cap L_2$ is context free.

Proof (cont.): The states of M' are labeled with pairs (q_k, p_l), representing the respective states in which M1 and M2 can reach after reading an input string.

By straightforward induction on the number of transitions,

we can show that

$((q_0, p_0), w, z) \vdash_M^* ((q_f, p_f), \lambda, x)$, where $q_f \in F_1$ and $p_f \in F_2$

iff $(q_0, w, z) \vdash_{M_1}^* (q_f, \lambda, x)$ and $\delta^*(p_0, w) = p_f$.

Thus, a string is accepted by M' iff it's accepted by M_1 and M_2,

i.e. $w \in L(M')$ iff $w \in L(M_1) \cap L(M_2) = L_1 \cap L_2$.

Thus, the family of CFL is closed under regular intersection. Q.E.D.

Intersection of CFL and Regular Language

- Example 8.7: $L = \{a^n b^n \mid n \geq 0, n \neq 100\}$ is context free.

Proof) Let $L_1 = \{a^{100} b^{100}\}$.

Then, L_1 is regular since it's finite.

Then, L may be defined as $L = \{a^n b^n \mid n \geq 0\} \cap \bar{L}_1$

By the closure of regular languages (L_1) under complementation and the closure of CFL under regular intersection,

L is context free. Q.E.D.

Intersection of CFL and Regular Language

- Theorem 8.5: Let L_1 be a CFL and L_2 be a Regular language. Then, $L_1 \cap L_2$ is context free.
- Example 8.7: $L = \{a^n b^n : n \geq 0, n \neq 100\}$ is context free.
- Example 8.8: $L = \{w \in \{a, b, c\}^* \mid n_a(w) = n_b(w) = n_c(w)\}$ is *not* context free.

Proof) Suppose L is context free. Then,

$L \cap L(a^*b^*c^*) = \{a^n b^n c^n \mid n \geq 0\}$

would be context free since CFL is closed under regular intersection, i.e. CFL $\cap$ RegL = CFL.

But, it's proven that $\{a^n b^n c^n \mid n \geq 0\}$ is not CFL in Ex. 8.1.

Contradiction!

Thus, L is not context free. Q.E.D.

More Properties of Closures: (N)CFL, DCFL, Regular Lang, and Linear Lang.

$\forall L_1, L_2 \in$ (N)CFL, $\forall L_3 \in$ RegL, $\forall L_4 \in$ DCFL, $\forall L_5 \in$ LinearL

- The family of (N)CFL is closed under:
  - Homomorphism, Reversal.
- The family of (N)CFL is closed with the Regular lang. under:
  - Regular Difference: $L_1 - L_2 = L_1 \cap \overline{L_2} \notin$ CFL,
    but $L_1 - L_3 = L_1 \cap \overline{L_3} \in$ CFL.
    Also, If L_4 is deterministic CFL, then $L_4 - L_3 \in$ DCFL.
- The family of DCFL is closed under:
  - Close under: homomorphism, regular difference.
  - Not closed under: Union, intersection, *reversal*

More Properties of Closures: (N)CFL, DCFL, Regular Lang, and Linear Lang. (cont.)

$\forall L_1, L_2 \in$ (N)CFL, $\forall L_3 \in$ RegL, $\forall L_4 \in$ DCFL, $\forall L_5 \in$ LinearL

- The family of Linear Language is closed under:
  - Union, regular concatenation ($L_5 \cdot L_3 \in L_5$) but
  - Not closed under: concatenation, intersection
- The family of unambiguous CFL is not closed under:
  - Union, intersection

Elementary Questions about CFL

- Given a CFL *L* and an arbitrary string *w*,

  is there an algorithm to determine whether or not *w* is in *L*? :

  $w \in L(G)$? *Membership*
- Given a CFL *L*, is there an algorithm to determine if *L* is empty? : $L = \varnothing$? *Emptiness*
- Given a CFL *L*, is there an algorithm to determine if *L* is infinite? : *(In)finiteness*
- Given two CFGs G_1 and G_2, is there an algorithm to determine if $L(G_1) = L(G_2)$? : *Equality of grammar*

A Membership Algorithm for CFL

- The combination of Theorems 5.2 and 6.5 confirms the existence of a membership algorithm for CFL.
- By Theorem 5.2, exhaustive parsing is guaranteed to give the correct result for any context-free grammar that contains neither λ-productions nor unit-productions.
- By Theorem 6.5, such a grammar can always be produced if the language does not include λ.
- Alternatively, a NDPA to accept the language can be constructed as established by Theorem 7.1

Theorem 5.2:
Exhaustive parsing is guaranteed to yield all $w \in L(G)$ eventually, but may fail to stop for strings $w \notin L(G)$, unless we restrict the productions in the grammar to avoid the forms

$A \rightarrow \lambda$ (λ production) and $A \rightarrow B$ (unit production).

i.e. If the CFG G with *no* rule of the form

$A \rightarrow \lambda$ (λ production) or $A \rightarrow B$ (unit production),

the exhaustive parsing decides/stops for $w \notin L(G)$.

Theorem 6.5: For any CFL that *does not include λ*, there exists a CFG *without useless, λ-, or unit-productions.*

Theorem 7.1: $\forall$CFL L, $\exists$ NDPA M s.t. L(M) = L.
For any CFL L, there exists a NDPA that accepts L.

A Membership Algorithm for CFG: CYK algorithm (Sec. 6.3)

- J. Cocke, D.H. Younger, and T. Kasami
- Membership and parsing algorithms for CFG exist that requires approximately $|w|^3$ steps to parse a string w.
- Requirement: CFG in Chomsky Normal Form (CNF).
- Idea: One problem is broken into a sequence of smaller ones.

Proof)

Assume we have a CFG G=(V, T, S, P) in CNF and string $w= a_1a_2...a_n$.

We define substrings $w_{ij}= a_i...a_j$, and subsets of V, $V_{ij} = \{A \in V \mid A \Rightarrow^* w_{ij}\}$.

Clearly, $w \in L(G)$ iff $S \in V_{1n}$.

To compute V_{ij}, observe that $A \in V_{ii}$ iff G contains production $A \rightarrow a_i$.

Thus, V_{ii} can be computed for all $1 \le i \le n$ by inspection of w and the productions of the grammar. To continue, notice that for $j > i$,

$A \Rightarrow^* w_{ij}$ iff $A \rightarrow BC$, with $A \Rightarrow^* w_{ik}$ and $C \Rightarrow^* w_{k+1j}$ for some k, $i \le k < j$.

A Membership Algorithm for CFG: CYK algorithm (Sec. 6.3)

Proof, cont.) Notice that for $j > i$,

$A \Rightarrow^* w_{ij}$ iff $A \rightarrow BC$, with $B \Rightarrow^* w_{ik}$ and $C \Rightarrow^* w_{k+1j}$ for some k, $i \le k < j$.

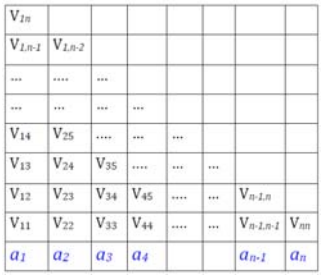
i.e.

$$V_{ij} = \bigcup_{k \in \{i, i+1, \ldots, j-1\}} \{A : A \rightarrow BC, \text{ with } B \in V_{ik}, C \in V_{k+1,j}\}.$$

An inspection of the indices above shows that it can be used to compute all the V_{ij} if we proceed in the sequence:

1. Compute V_{11}, V_{22}, ..., V_{nn},
2. Compute V_{12}, V_{23}, ..., V_{n-1n},
3. Compute V_{13}, V_{24}, ..., V_{n-2n},

and so on.



A Membership Algorithm for CFG: CYK algorithm (Sec. 6.3)

Proof, cont.) To fill the table, we work row by row, upwards.

Notice that each row corresponds to one length of substrings the bottom row is for strings of length 1, the second-from-bottom row for strings of length 2, and so on, until the top row corresponds to the one substring of length n, which is w itself. It takes O(n) time to compute any one entry of the table, by a method below.

Since there are n (n+1)/2 entries, the whole table construction process takes O(n^3) times. The algorithm for computing the V_{ij}'s is:

Basis: Compute the 1st row. Since the string beginning and ending at position i is just the terminal a_i, and the grammar is in CNF, the only way to derive the string a_i is to use a production of the form A $\rightarrow a_i$.

Thus, V_{ii} is the set of variables A s.t. A $\rightarrow a_i$ is a production of G.

Inductive Hypothesis: Suppose we want to compute V_{ij}, which is in row j-i+1, and we have computed all the V's in the rows below.

A Membership Algorithm for CFG: CYK algorithm (Sec. 6.3)

Proof, cont.) Inductive Hypothesis: Suppose we want to compute V_{ij}, which is in row j-i+1, and we have computed all the V's in the rows below.

i.e. we know about all strings shorter than $w_{ij} = a_i..a_j$, and in particular,

We know about all proper prefixes and proper suffixes of that string.

As $j - i > 0$ may be assumed, we know that any derivation A $\Rightarrow^*$ $w_{ij} = a_i..a_j$ must start out with some step A $\Rightarrow$ BC. Then, B derives some prefix of w_{ij}, say, B $\Rightarrow^*$ $w_{ik} = a_i..a_k$, for some $k < j$. Also, C must then derive the remainder of of w_{ij}, that is, C $\Rightarrow^*$ $w_{kj} = a_k..a_j$.

We conclude that in order for A to be in V_{ij} i.e. A$\in V_{ij}$,

we must find variables B and C, and integer k s.t.

1. $i \leq k < j$,
2. B is in V_{ik},
3. C is in $V_{k+1,j}$,
4. A $\rightarrow$ BC is a production of G.

A Membership Algorithm for CFG: CYK algorithm (Sec. 6.3)

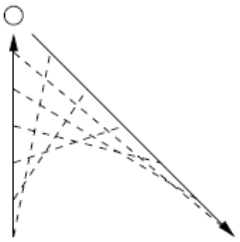
Proof, cont.) We conclude that in order for A to be in V_{ij} i.e. $A \in V_{ij}$, we must find variables B and C, and integer *k* s.t.

1. $i \leq k < j$, 2. B is in V_{ik}, 3. C is in $V_{k+1,j}$, 4. $A \rightarrow BC$ is a production of G.

Finding such variables A requires to compare at most n pairs of previously computed sets: (V_{ii}, $V_{i+1,j}$), ($V_{i,i+1}$, $V_{i+2,j}$), and so on, until ($V_{i,j-1}$, $V_{j,j}$).

The pattern, in which we go up the column below V_{ij} at the same time we go down the diagonal, is suggested by Figure below.

Computation of V_{ij} at requires matching the column below with the diagonal to the right.



A Membership Algorithm for CFG: CYK algorithm (Sec. 6.3)

Theorem:

The algorithm described correctly computes V_{ij} for all *i* and *j*, thus *w* is in L(G) if and only if S is in V_{1n}.

Moreover, the running time of the algorithm is $O(n^3)$.

PROOF: The reason the algorithm finds the correct sets of variables was explained as we introduced the basis and inductive parts of the algorithm. For the running time, note that there are $O(n^2)$ entries to compute, and each involves comparing and computing with n pairs of entries. It is important to remember that, although there can be many variables in each set X_{ij}, the grammar G is fixed and the number of its variables does not depend on n, the length of the string w whose membership is being tested. Thus, the time to compare two entries X_{ik} and $X_{k+1,j}$, and find variables to go into X_{ij} is $O(1)$. As there are at most n such pairs for each X_{ij}, the total work is $O(n^3)$. □

A Membership Algorithm for CFG: CYK algorithm (Sec. 6.3)

Example 6.11: Decide whether $w = aabbb \in L(G)$ where the production rules of G are: $S \to AB$, $A \to BB \mid a$, $B \to AB \mid b$.

First, note that $w_{11} = a$, so V_{11} is the set of all variables immediately derive a, i.e. $V_{11} = \{A\}$. Since $w_{22} = a$, so $V_{22} = \{A\}$ and, similarly,

$V_{11} = \{A\}$, $V_{22} = \{A\}$, $V_{33} = \{B\}$, $V_{44} = \{B\}$, $V_{55} = \{B\}$.

Now, by algorithm, get $V_{12} = \{A \mid A \to BC, B \in V_{11}, C \in V_{22}\}$.

Since $V_{11} = \{A\}$, $V_{22} = \{A\}$, the set consists of all variables that occur on the left side of a production whose right side is AA.

Since there are none, $V_{12} = \varnothing$. Next, $V_{23} = \{A \mid A \to BC, B \in V_{22}, C \in V_{33}\}$.

So, the required right side is AB, and we have $V_{23} = \{S, B\}$.

Thus, $V_{12} = \varnothing$, $V_{23} = \{S, B\}$, $V_{34} = \{A\}$, $V_{45} = \{A\}$,
$V_{13} = \{S, B\}$, $V_{24} = \{A\}$, $V_{35} = \{S, B\}$,
$V_{14} = \{A\}$, $V_{25} = \{S, B\}$,
$V_{15} = \{S, B\}$, so that $w \in L(G)$.

Deciding Whether a CFL is Empty

- Theorem 8.6: Given a CFL G=(V, T, S, P), there exists an algorithm to decide if a context-free language $L(G) = \varnothing$.

Proof) For simplicity, assume that $\lambda \notin L(G)$

- Apply the algorithm for removing useless symbols and productions.
- If the start symbol, S, is found to be useless, then $L(G)$ is empty; otherwise, $L(G)$ contains at least one string.

Deciding Whether a CFL is Infinite

- Theorem 8.7: Given a CFL $G=(V, T, S, P)$, there exists an algorithm to decide if a CFL $L(G)$ is infinite.

Proof) Apply the algorithms for removing λ-productions, unit-productions, and useless productions.

- Suppose G has a repeating variable $A \in V$ for which there is a derivation $A \Rightarrow^{*1} xAy$. Since G has no λ-productions and no unit-productions, x and y can't be simultaneously empty.
- Since A is neither nullable nor a useless symbol, we have
  $S \Rightarrow^{*2} uAv \Rightarrow^{*} w$ and $A \Rightarrow^{*3} z$, where $u, v, z \in T^*$.
  But, then $S \Rightarrow^{*2} uAv \Rightarrow^{*1} ux^nAy^nv \Rightarrow^{*3} ux^nzy^nv$, $\forall n$ *is possible.*
  So, *L(G)* is infinite.
- If no variable in G repeats, the length of any derivation is bounded by $|V|$. So, L(G) is finite.
- Note: The grammar has a repeating variable iff the dependency graph has a cycle as well as any variable that is at the base of a cycle is repeating variable. So, it can be an algorithm to decide if a grammar has a repeating variable to be used for the algorithm of infiniteness/finiteness.

Deciding Whether Two CFLs are Equal

- Given two CFGs G_1 and G_2, is there an algorithm to decide if $L(G_1) = L(G_2)$?
- If the languages are finite, the answer can be found by performing a string-by-string comparison.
- However, for general CFLs, *no algorithm exists to decide an equality of CFGs, thus an equality of CFLs.*