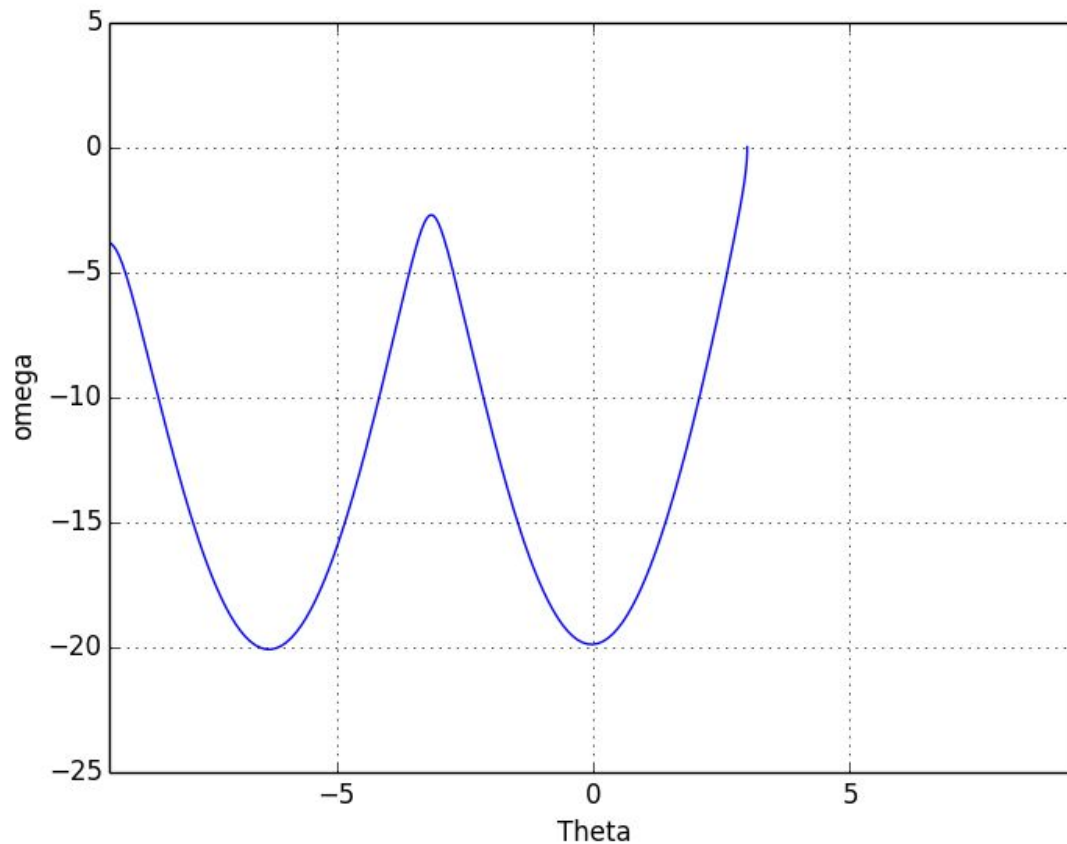


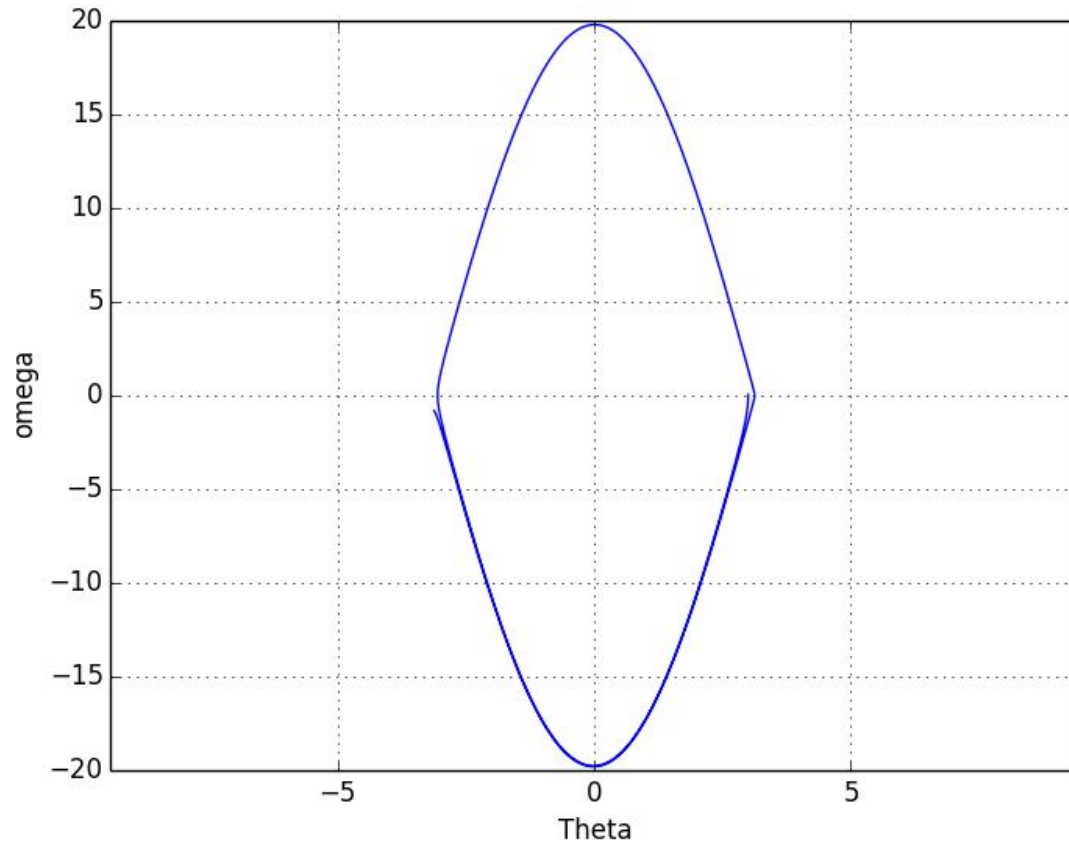
1. ---

2.

a.



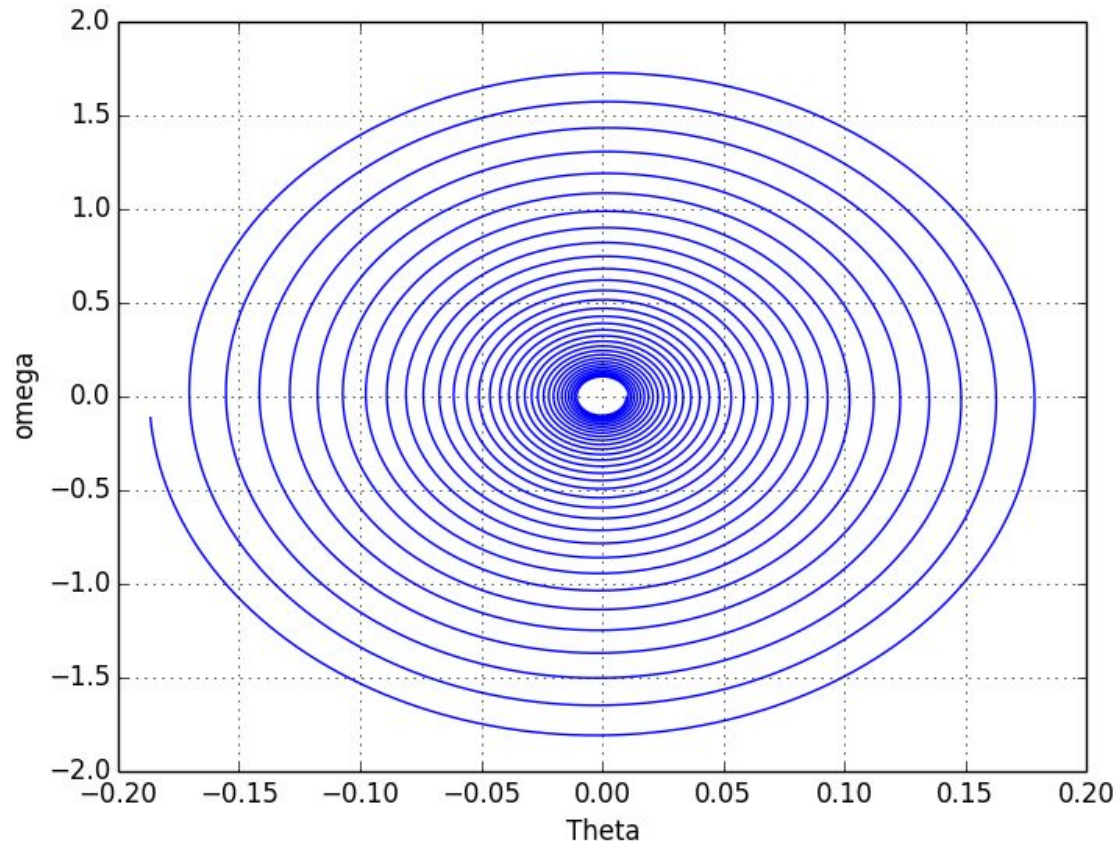
The above shows the starting trajectory from  $[\theta, \omega] = [3, 0.1]$  for two cycles. This should be upsetting to the reader. Because this system currently has no friction, I *should* see an ellipsoid centered around an elliptical fixed point -- not a "stable" fixed point, since we are modelling a conservative system. The issue here is the time step of 0.005. I found that if I go down to 0.00321, I got something that looks like:



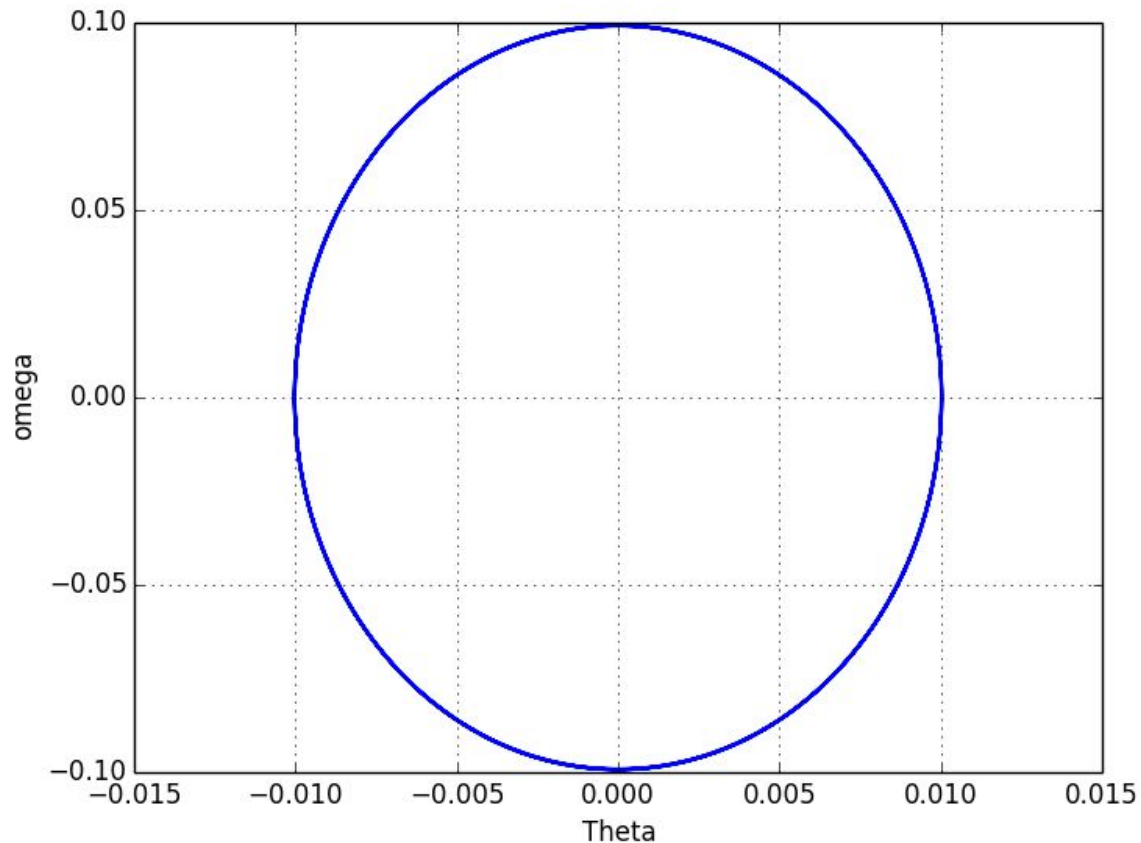
There it is. Of note is the slight warp from a perfect ellipsoid. This (and knowledge of the physical system), tells me that this starting point is indeed near a fixed point -- specifically the one at  $[\pi, 0]$ . Because this is a conservative system, this fixed point is neither stable nor unstable, but instead **hyperbolic**.

**b.**

Below is the plot produced by the initial condition  $[.01, 0]$  with the time step found in part (a): 0.00321. I've exaggerated the duration of my simulation.



This is again concerning. The system is growing outwards, contradicting the model of no forcing, no dissipation. Again, I made my time step smaller, finding that the smaller I went, the less deviation I had from a stable ellipse:

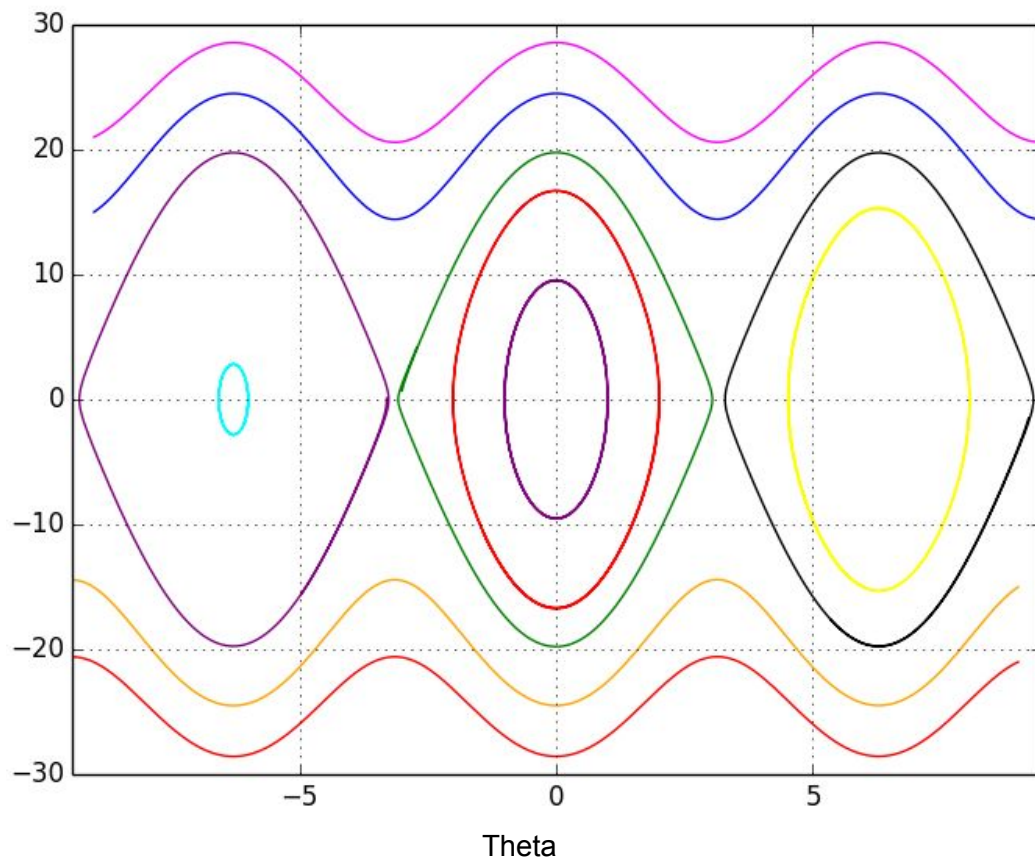


Same duration, 100 times smaller time step.

This trajectory does indeed look more like a perfect ellipse than in part (a). I'd venture to guess that this is because the trajectory in part (a) was being acted on by the hyperbolic fixed point -- physically, the pendulum was being stalled at the top of its orbit just slightly, before continuing its swing. Oppositely, when swinging through the center point, the pendulum swings smoothly, creating the perfect curve we see around the  $\Theta = 0$  axis.

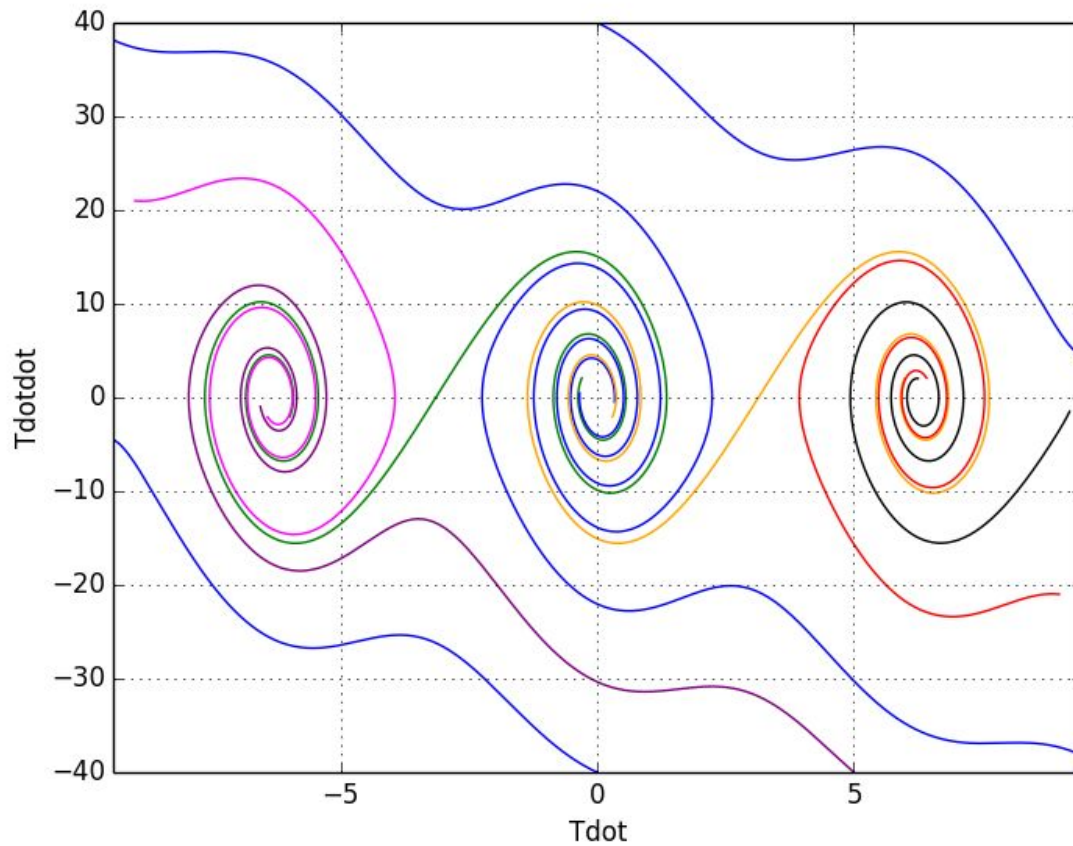
**Please Note:** from here on I am using a time step of  $10^{-4}$  as I found it to be quite consistent and not too small as to be computationally infeasible.

3.



Phase portrait of the conservative, non-forced pendulum. This clearly exemplifies both the elliptical and hyperbolic fixed points of this system.

4.

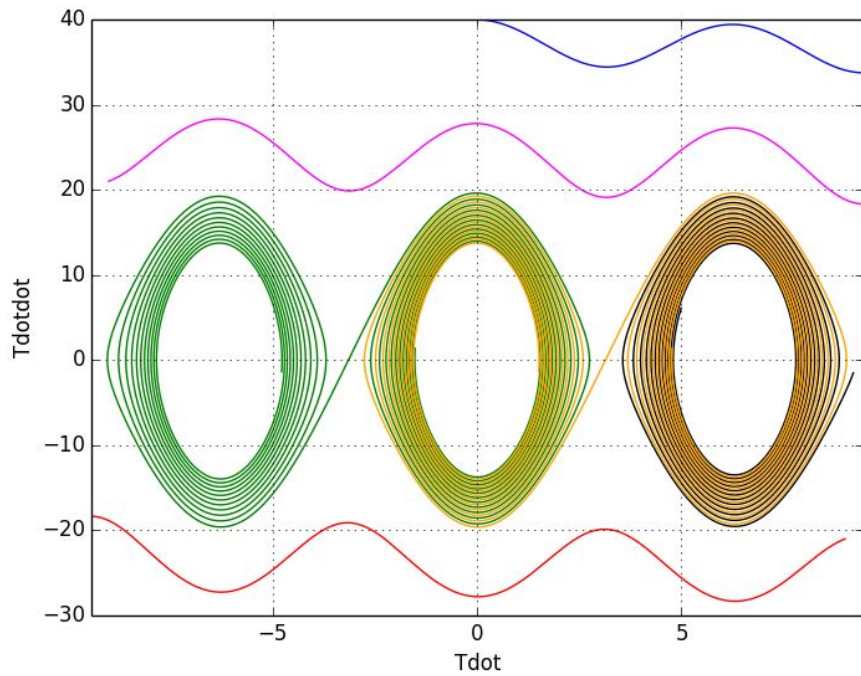
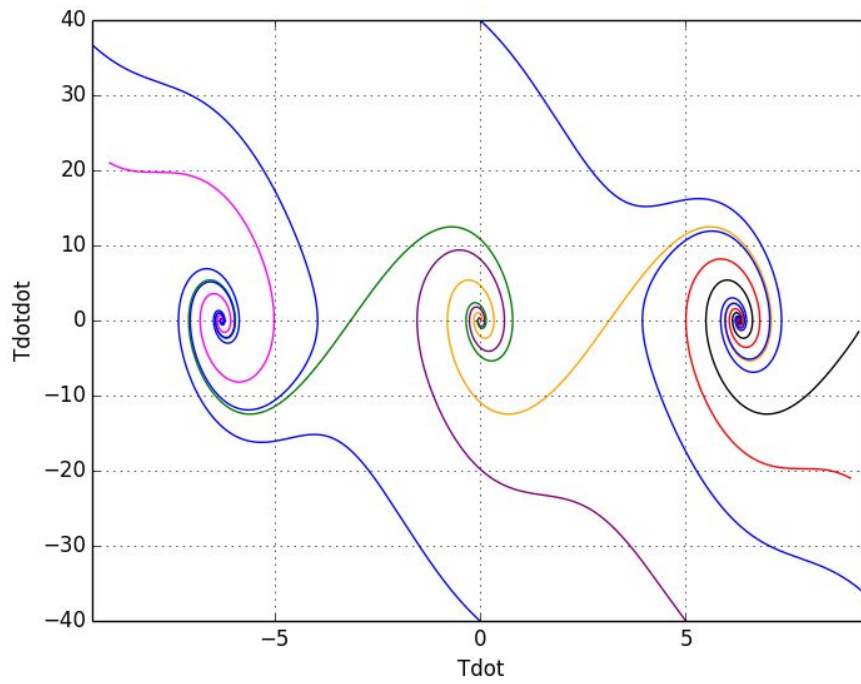


This is the same plot but with  $\beta = 0.25$ . Two things immediately merit comment: First, the trajectories now converge (the spiralling) onto a stable fixed point, instead of orbiting an elliptical one. And second, the trajectories seen before the spirals seem to follow an oblique angle uniformly -- that is, until they get close enough to a fixed point that they fall into its well. This tells me that the system is now dissipative, and that each trajectory will eventually end at rest:  $[\theta, \omega] = [2k\pi, 0]$ , where  $k$  is an integer.

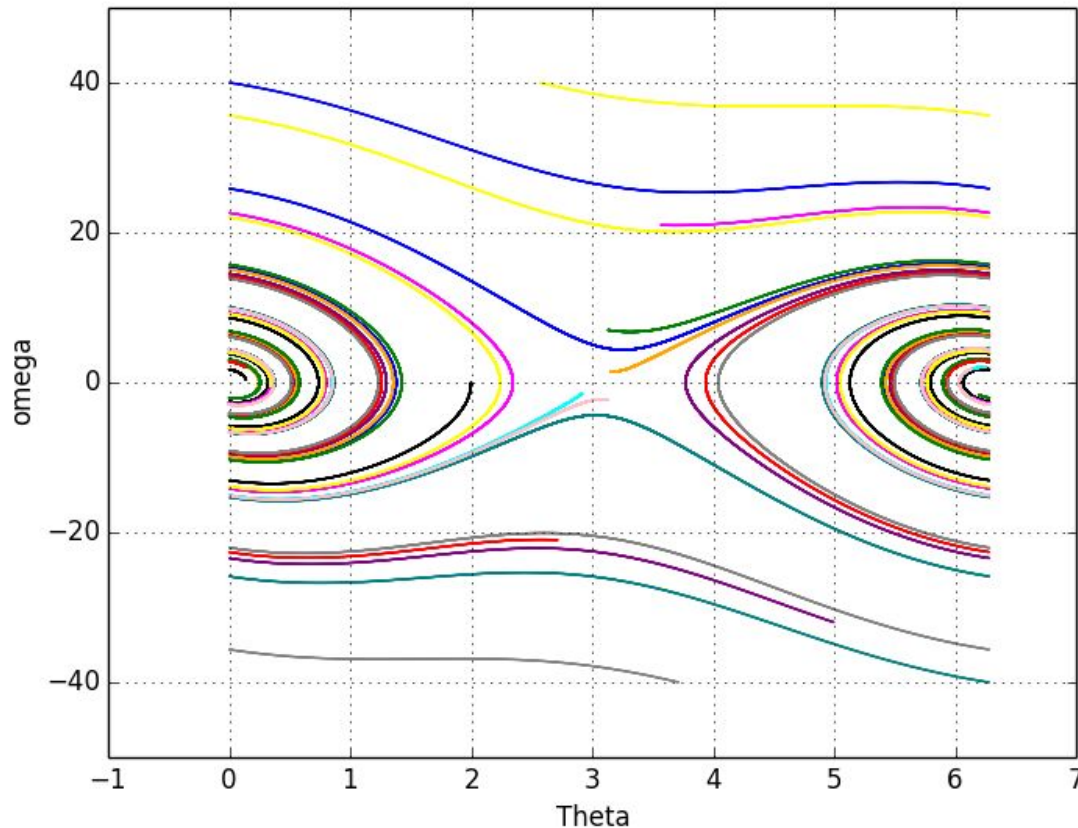
At higher  $\beta$ , I predict two things: that the trajectories will converge to stable fixed points faster, and that the oblique angles will get steeper. I also predict the opposite for smaller  $\beta$ . This of



course does not need to remain a hypothesis, so I have included two plots below:



5.



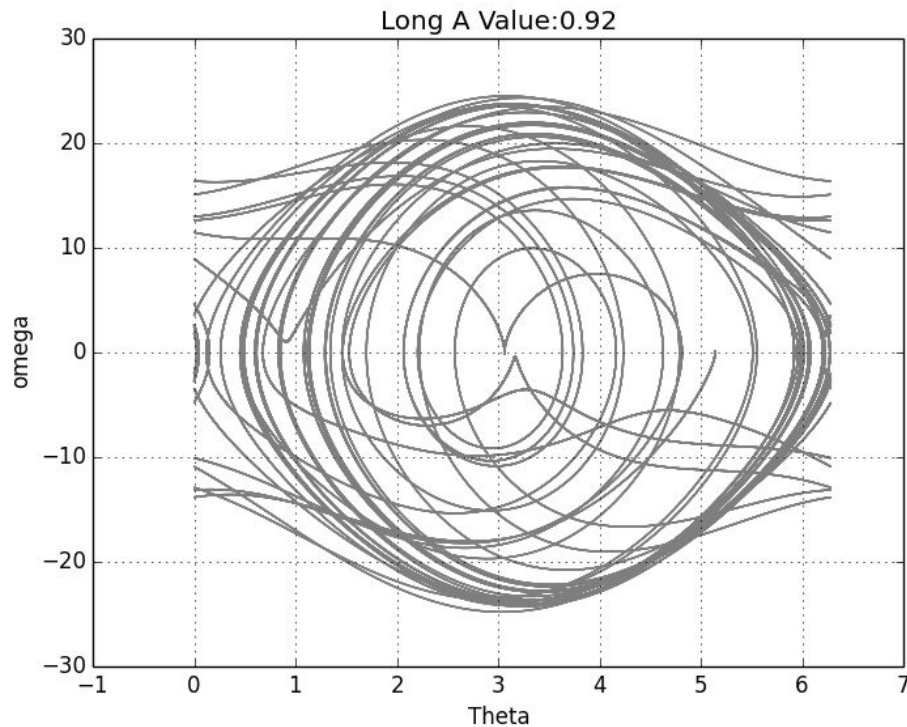
The modulus operator simply wraps around so that the plot I see is within 0 and  $2\pi$ . This is somewhat undesirable, though, since the eyes are drawn to the stable fixed points. Thus, the plots you will see from here on will be centered around  $\pi$ , but really the middle is the 0, and the left and right extremes are  $\pm \pi$ .

6.

Setting the alpha term to  $\alpha = .75 * \sqrt{\frac{g}{l}} \approx 7.425$ , and then sweeping across the amplitude parameter A, I saw the usual things in the parameter space of a dynamical system. First, stable attractors, then bifurcations into 2-cycles, and n-cycles. The system would descend into chaos and then a couple decimals of A later, a 1-cycle would coalesce. I also saw what appeared to be unstable periodic orbits.

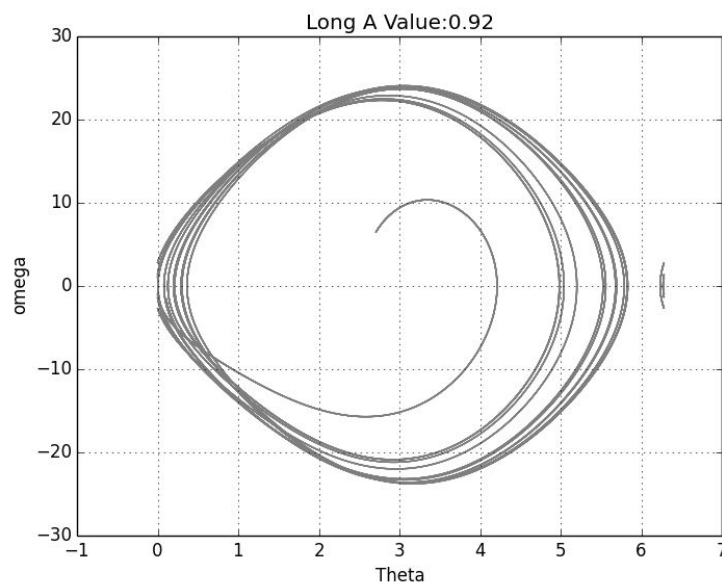


There are so many fun graphs that I could share, but I will just show you what I believe to be the most chaotic trajectory I found. This happened at the  $\alpha$  listed above, and at  $A = 0.92$ . This first graph shows a 60-second trajectory with all the points:

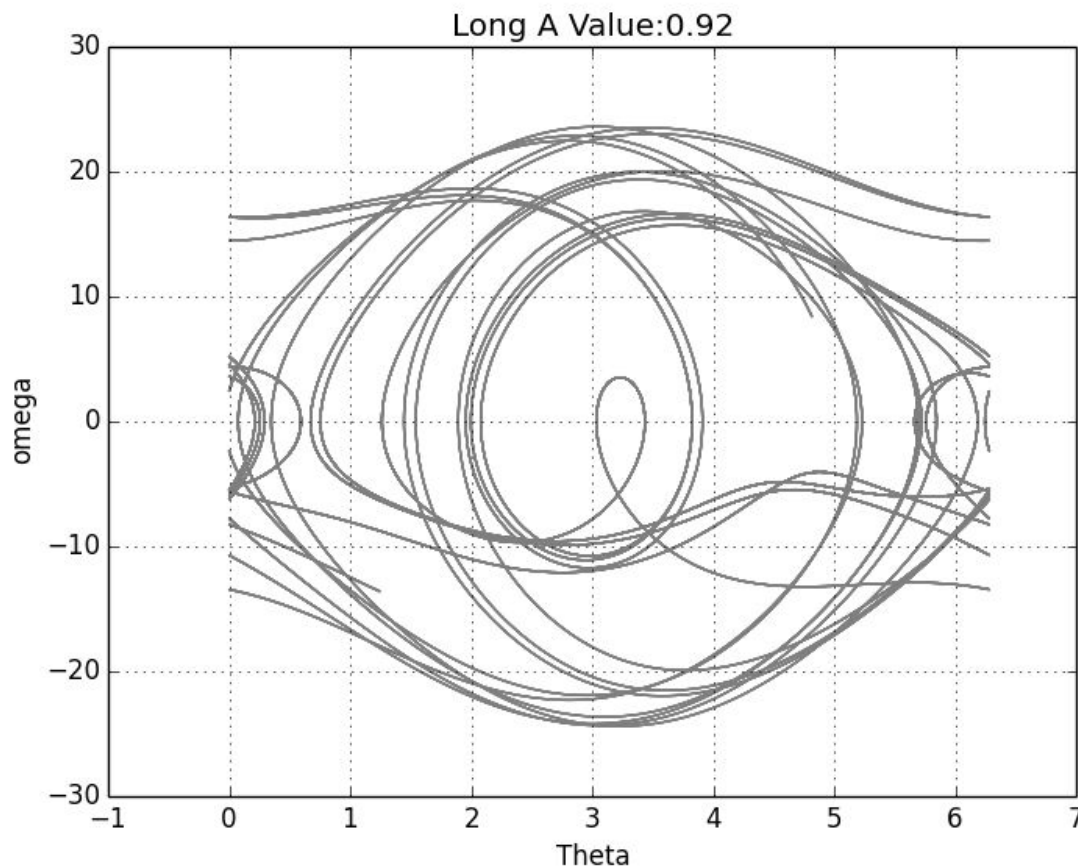


[Remember: Though this looks centered around  $\pi$ , it is only because I shifted my data to be friendly to look at]

Next I attempted to get rid of the transient by running an 80-second duration and only taking the last 5 seconds of it:



It *almost* gets into a stable orbit, but the loops are not repeating. I went to a 90-second trajectory and started at 75 seconds again:



Back to chaos! This is an excellent example of an unstable periodic orbit. As we might say in class, the pendulum “rode the volcano rim” for a good 5 seconds but could not stay there.

## 7.

We've seen how the time step affects the response already from question (2). With too big a time step for both the large and small ellipse, it is shown that it is thrown on a different trajectory -- specifically, a higher energy trajectory, because the large one's appears to just wrap around and around, never changing signs of its angular velocity, while the small ellipse spirals outward, unstably diverging from its elliptical fixed point. This is thus similar to the error you might see in a Forward Euler implementation. An overestimate of next state causes the solution to become farther and farther off as time goes on.

Formalizing a little bit, I have attached some plots (still shifting by  $\pi$ ) of the  $[3.1 \ 0]$  initial condition at time steps of 0.1, 0.01, 0.001, 0.0001, and finally 0.00001. There is a wide range of solutions:

