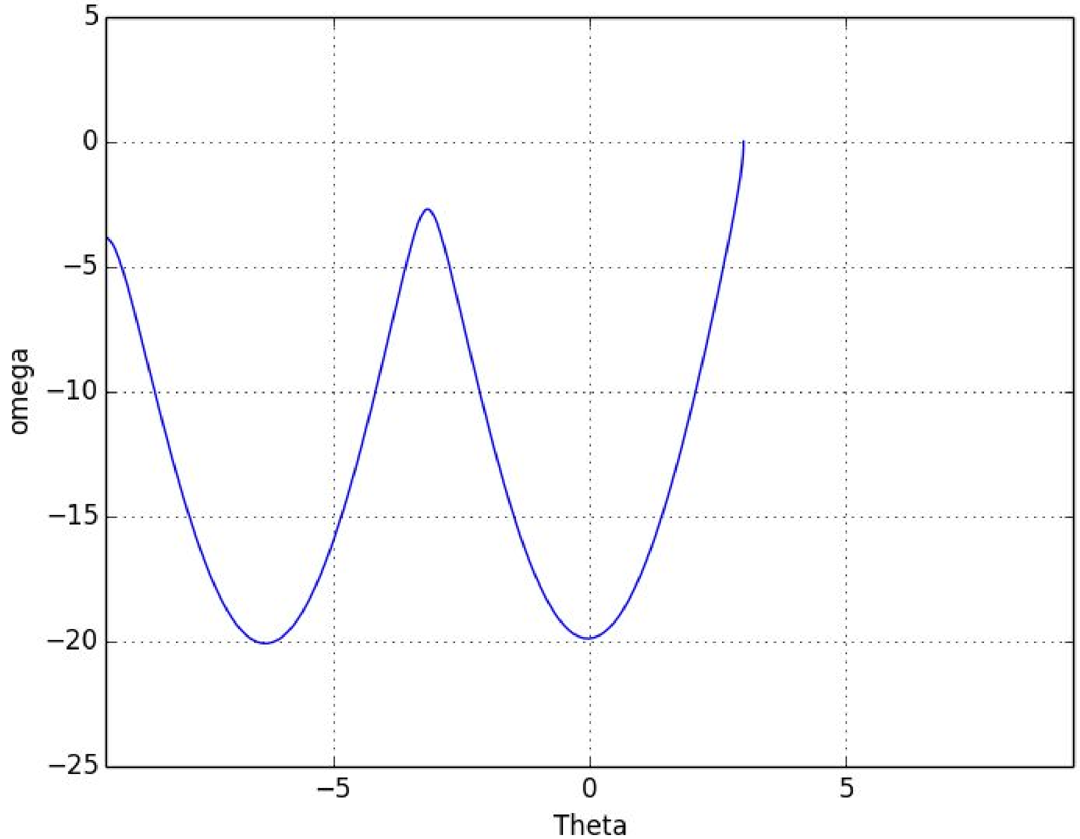
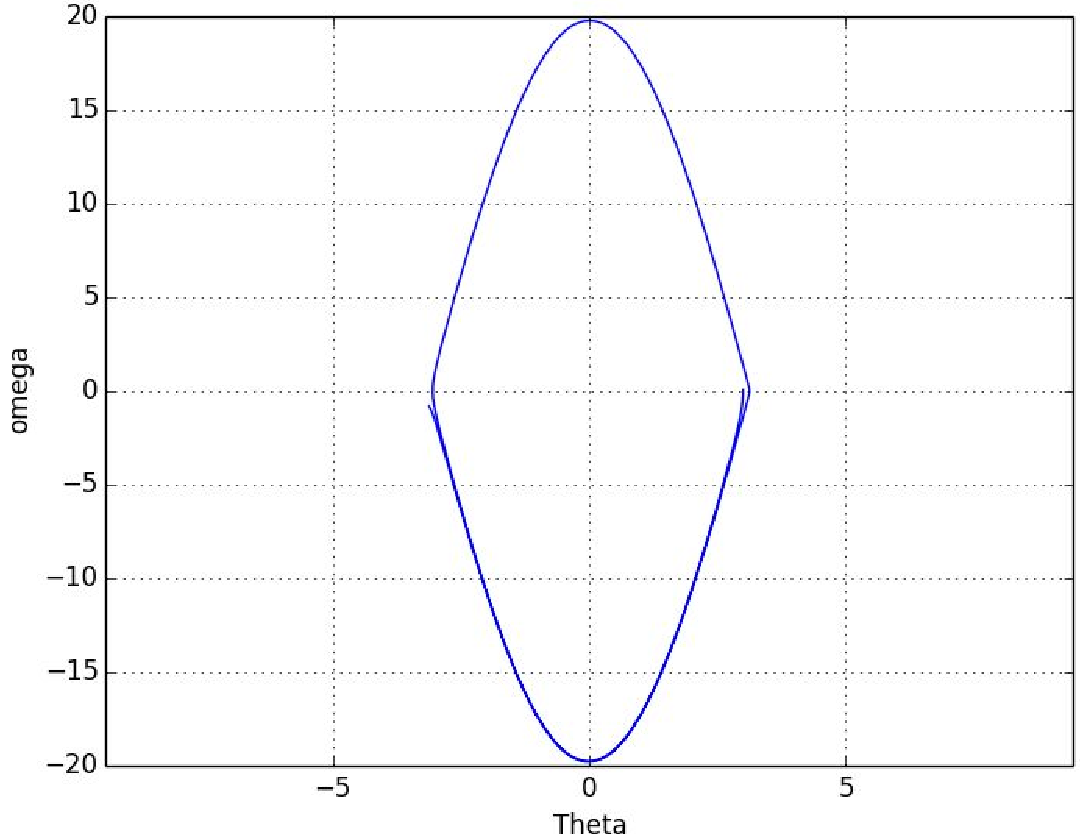
1. ---
2.
   a.



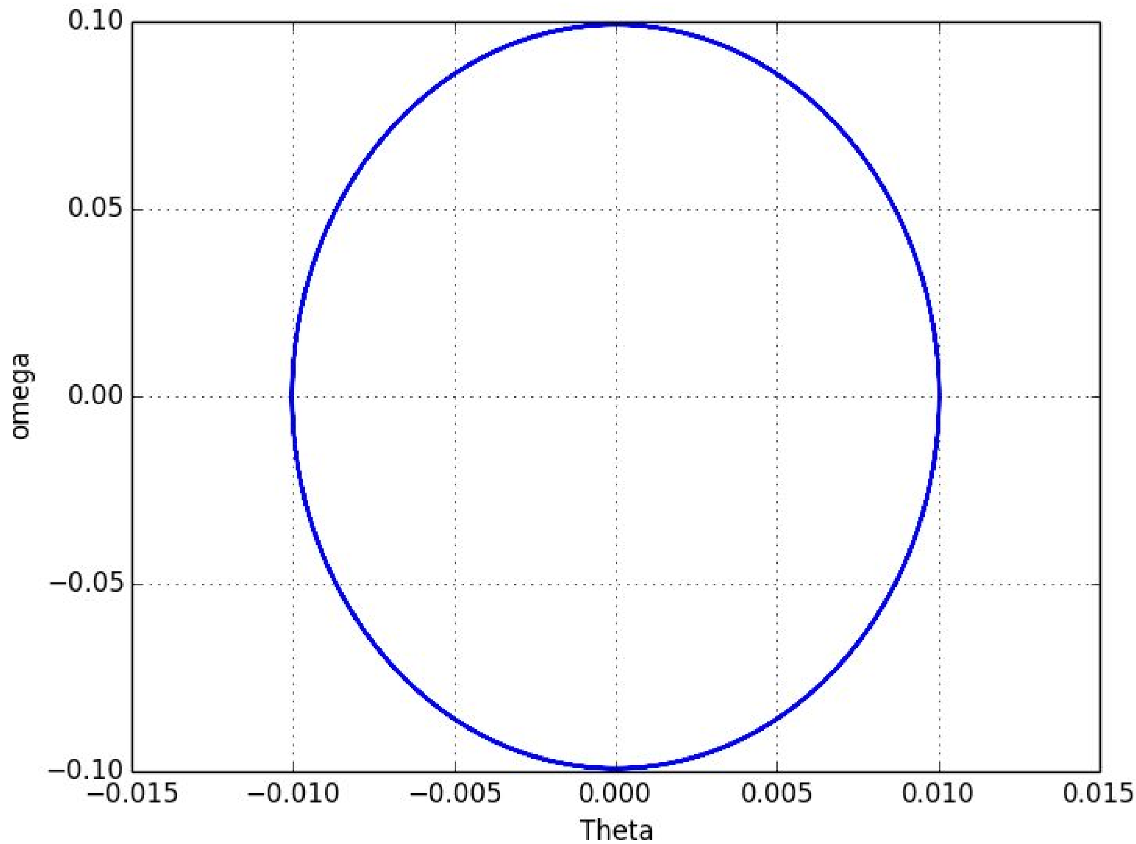
The above shows the starting trajectory from $[\theta,\ \omega] = [3,\ 0.1]$ for two cycles. This should be upsetting to the reader. Because this system currently has no friction, I *should* see a ellipsoid centered around an elliptical fixed point -- not a “stable” fixed point, since we are modelling a conservative system. The issue here is the time step of 0.005. I found that if I go down to 0.00321, I got something that looks like:



There it is. Of note is the slight warp from a perfect ellipsoid. This (and knowledge of the physical system), tells me that this starting point is indeed near a fixed point -- specifically the one at $[\pi,\ 0]$. Because this is a conservative system, this fixed point is neither stable nor unstable, but instead **hyperbolic**. A hyperbolic fixed point is, of course, the conservative analog to an unstable fixed point in a dissipative system. Semantics…

b.

Below is the plot produced by the initial condition [.01, 0].



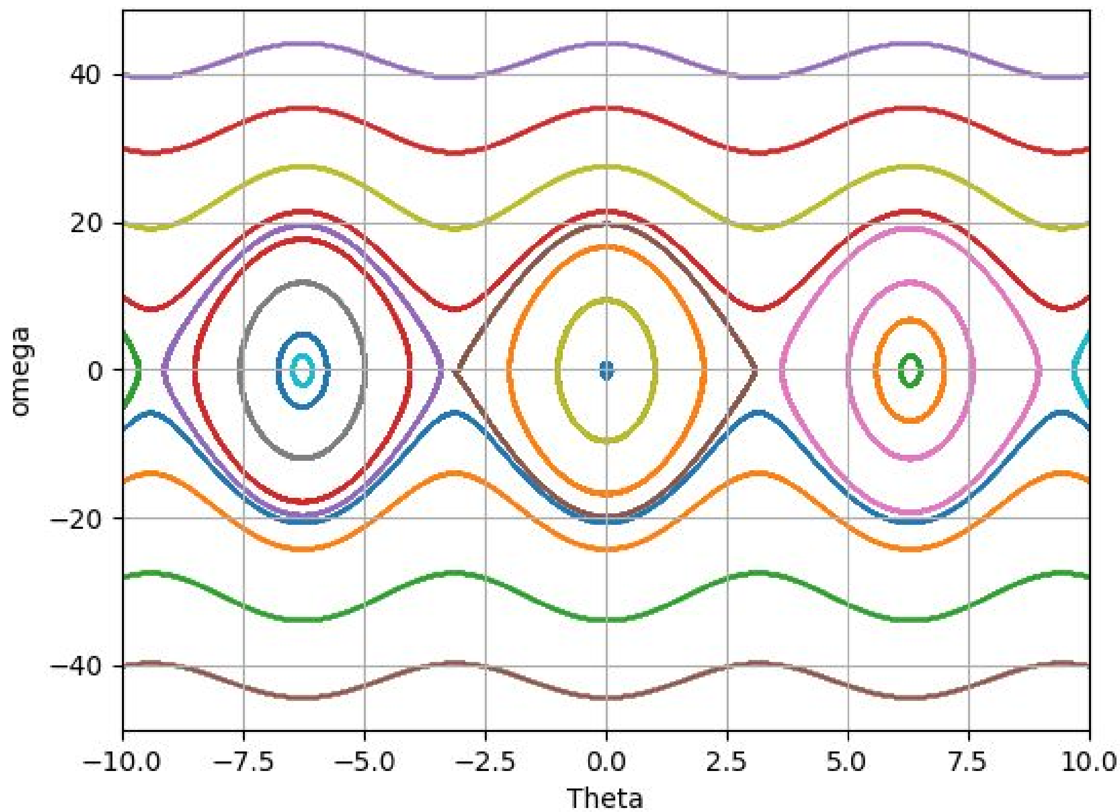
This ellipse looks much more exact and is what I am expecting.

This trajectory does indeed look more like a perfect ellipse than in part (a). I'd venture to guess that this is because the trajectory in part (a) was being acted on by the hyperbolic fixed point -- physically, the pendulum was being stalled at the top of its orbit just slightly, before continuing is swing. Oppositely, when swinging through the center point, the pendulum swings smoothly, creating the perfect curve we see around the Theta = 0 axis.

Please Note: from here on I am using a time step of 10^{-4} as I found it to be quite consistent and not too small as to be computationally infeasible.

CSCI 4446 Spring 2019 PS4.1
Derek Wright

3.

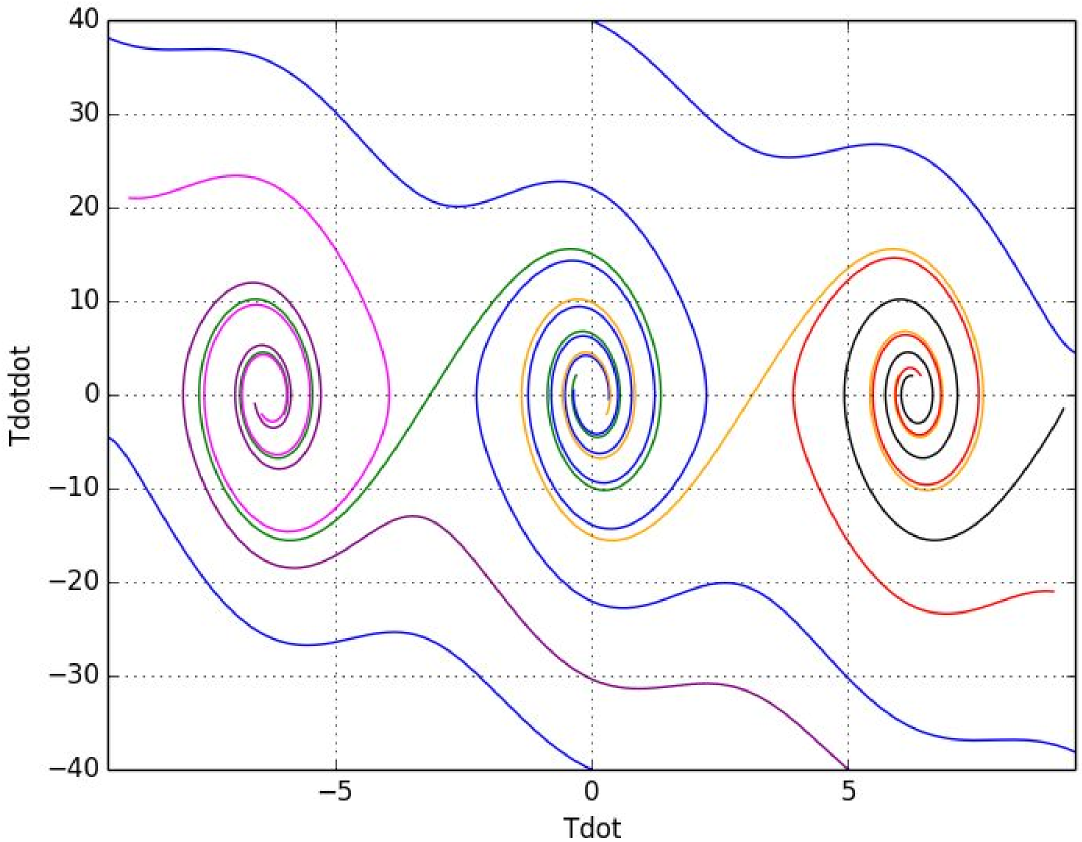


Theta

Phase portrait of the conservative, non-forced pendulum. This clearly exemplifies both the elliptical and hyperbolic fixed points of this system. The elliptical fixed points are at $2\pi k$ for any integer k, and the hyperbolic fixed points are at $(2k+1)\pi$.

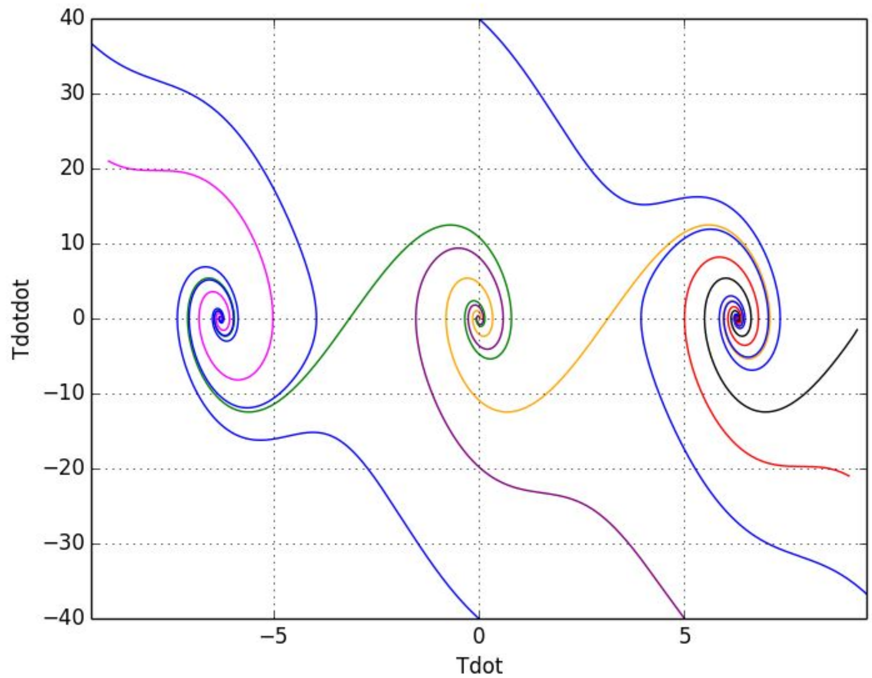
My original submission contained a sparser phase portrait, and the fixed points were not explicitly valued like I have here.

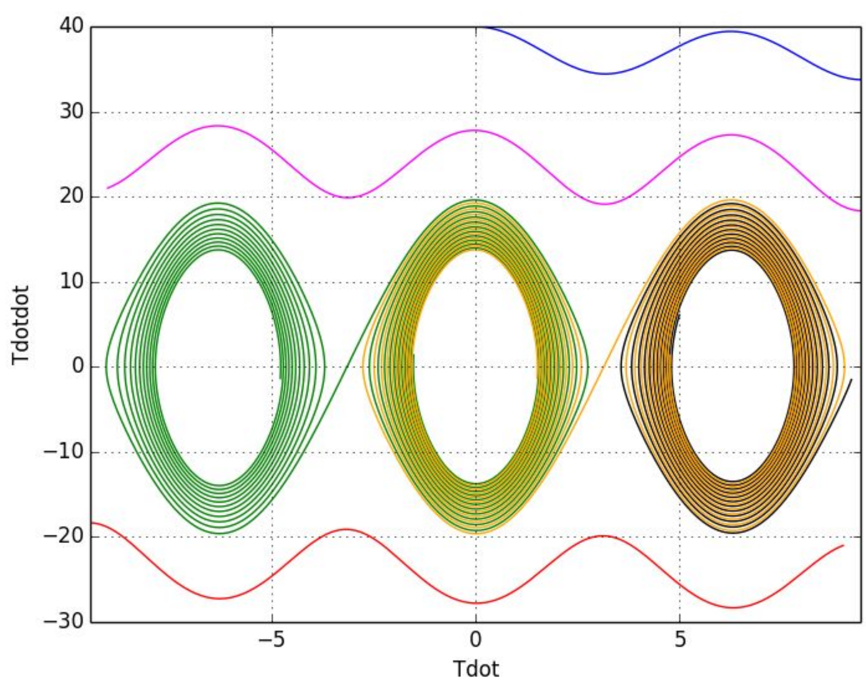
4.



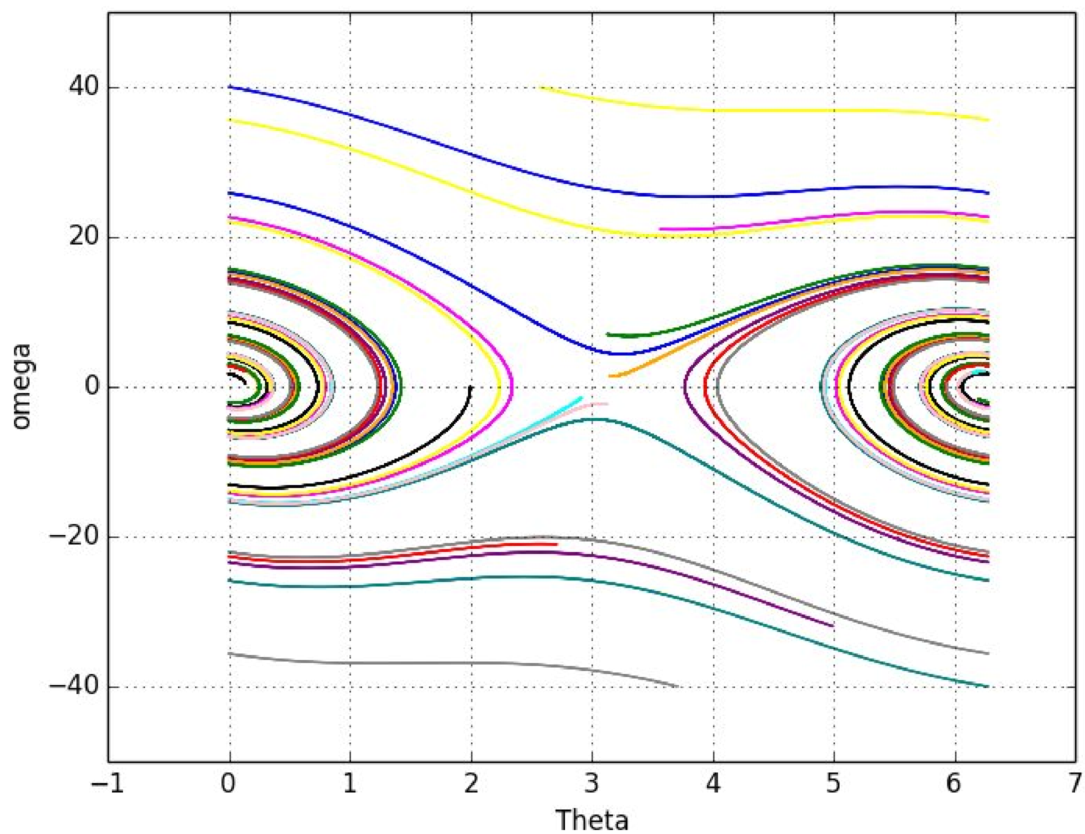
This is the same plot but with β = 0.25. Two things immediately merit comment: First, the trajectories now converge (the spiralling) onto a stable fixed point, instead of orbiting an elliptical one. And second, the trajectories seen before the spirals seem to follow an oblique angle uniformly -- that is, until they get close enough to a fixed point that they fall into its well.
This tells me that the system is now dissipative, and that each trajectory will eventually end at rest: $[\theta,\ \omega]\ =\ [2k\pi,\ 0]$, where k is an integer.

At higher β, I predict two things: that the trajectories will converge to stable fixed points faster, and that the oblique angles will get steeper. I also predict the opposite for smaller β. This of course does not need to remain a hypothesis, so I have included two plots below:





5.

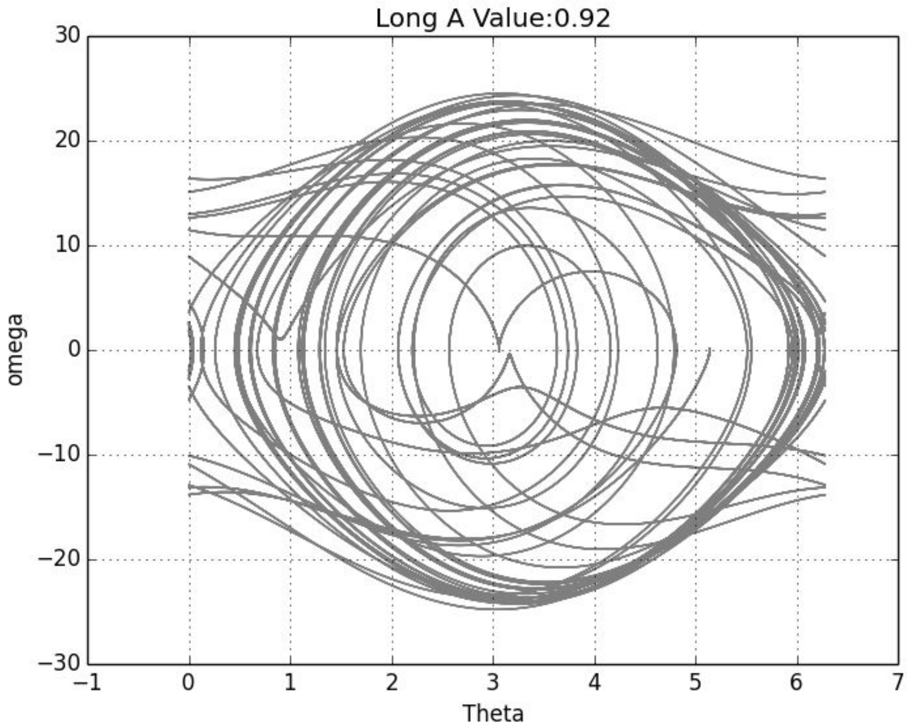


The modulus operator simply wraps around so that the plot I see is within 0 and 2π. This is somewhat undesirable, though, since the eyes are drawn to the stable fixed points. Thus, the plots you will see from here on will be centered around π, but really the middle is the 0, and the left and right extremes are $\pm\ \pi$.

6.

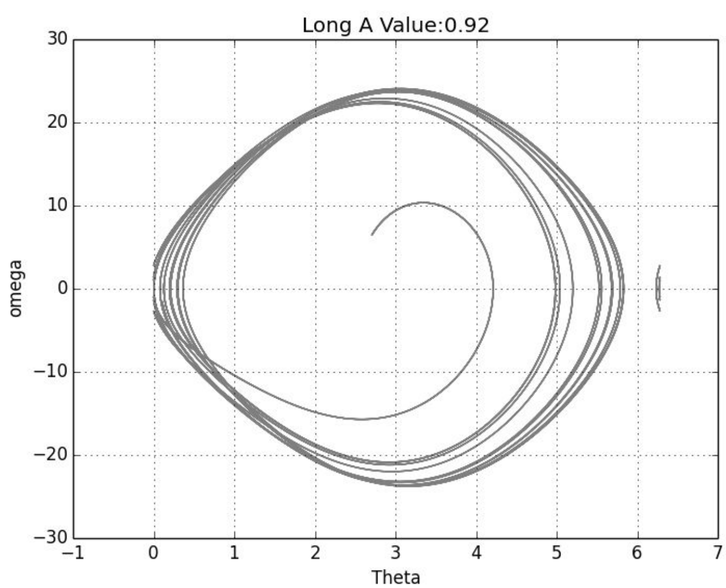
Setting the alpha term to $\alpha\ =\ .75\ *\ \sqrt{\frac{g}{l}} \approx 7.425$, and then sweeping across the amplitude parameter A, I saw the usual things in the parameter space of a dynamical system. First, stable attractors, then bifurcations into 2-cycles, and n-cycles. The system would descend into chaos and then a couple decimals of A later, a 1-cycle would coalesce. I also saw what appeared to be unstable periodic orbits.

There are so many fun graphs that I could share, but I will just show you what I believe to be the most chaotic trajectory I found. This happened at the α listed above, and at A = 0.92. This first graph shows a 60-second trajectory with all the points:

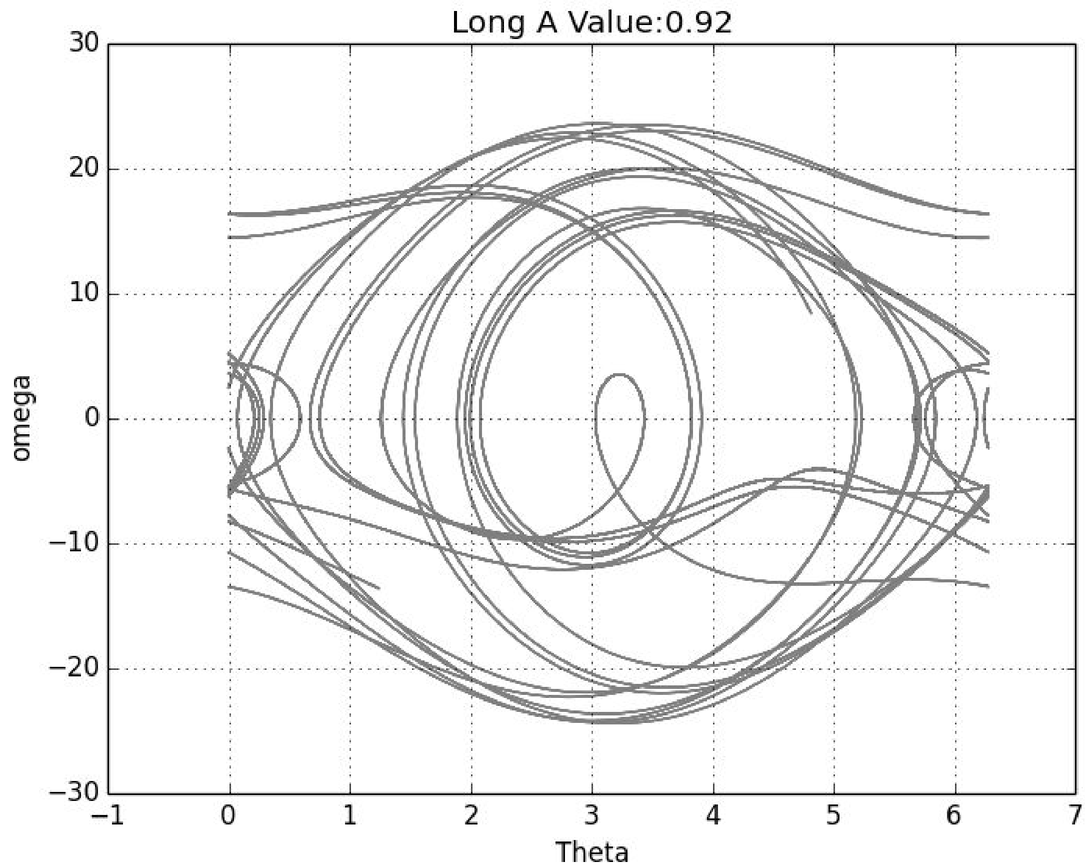


[Remember: Though this looks centered around π, it is only because I shifted my data to be friendly to look at]

Next I attempted to get rid of the transient by running an 80-second duration and only taking the last 5 seconds of it:



It *almost* gets into a stable orbit, but the loops are not repeating. I went to a 90-second trajectory and started at 75 seconds again:



Back to spread-out chaos! This is an excellent example of an unstable periodic orbit. As we might say in class, the pendulum “rode the volcano rim” for a good 5 seconds but could not stay there.

7.

We’ve seen how the time step affects the response already from question (2). With too big a time step, it is shown that the integration is thrown onto a different trajectory. This is thus similar to the error you might see in a Forward Euler implementation. The error in both cases (FE and RK4) happen in the same way. An overestimate of next state causes the solution to become farther and farther off as time goes on. Specific to RK4, we are time-stepping past the trajectory and accidentally get bumped to a different trajectory -- this includes the h/2. So even a half-timestep can’t accurately give us the dynamics for big enough timesteps. This can be solved by using an adaptive RK4 algorithm, that recognizes this problem, and halves the time step accordingly.

Formalizing a little bit, I have attached some plots (still shifting by π) of the [3.1 0] initial condition at time steps of 0.1, 0.01, 0.001, 0.0001, and finally 0.00001. There is a wide range of solutions: