1.a.

```
A = [-1 2001;-1 0];
eig(A)
```

```
ans =

  -0.5000 +44.7297i
  -0.5000 -44.7297i
```

This system is asymptotically stable because the real part of the eigen values are negative. Even though there are large imaginary components and there is damped oscillation, the negative real parts guarantee that the system will converge over time

1.b.

```
A = [-1 0; 0 1];
eig(A)
```

```
ans =

    -1
     1
```

This system is unstable because all roots of a system need to be negative in order for the system to be asymptotically stable

1.c.

```
A = [1i 1; 0 1i];
eig(A)
```

```
ans =

   0.0000 + 1.0000i
   0.0000 + 1.0000i
```

The system is marginally stable because it has neither positive or negative roots, and only has imaginary components. The system will oscillate forever.

However, this A matrix is defective, and R(λ) = 0 which also defines this system as unbounded or unstable

1.d.

```
A = [-2 0;0 0.5];
eig(A)
```

```
ans =

   -2.0000
    0.5000
```

Criteria for stability in discrete systems:
$|\lambda| <= 1$
And if $|\lambda| = 1$, λ must be non-defective

Since absolute value of -2 is 2, which is greater than 1, system is unstable

2.

With MATLAB:

```
A = [0.5 1 -0.5 0; -1 0.5 0 -0.5; 0.5 0 -0.5 1;0 0.5 -1 -0.5];
eig(A)
```

```
ans =

   0.0000 + 1.0000i
   0.0000 - 1.0000i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

With MATLAB, the eigen vectors of matrix A are purely imaginary. That means the system is marginally stable.

A is not defective.

By hand:

$$\lambda_A = \det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} \frac{1}{2}-\lambda & & & \\ & \frac{1}{2}-\lambda & & \\ & & -\frac{1}{2}-\lambda & \\ & & & -\frac{1}{2}-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A-\lambda I) = {} & (0.5-\lambda)\Big((0.5-\lambda)\big[((-0.5-\lambda)(-0.5-\lambda)-(-1)) + (-\tfrac{1}{2})(0-(0.5-\lambda)(-0.5))\big]\Big) \\ & - 1\big(-1\big[(-0.5-\lambda)(-0.5-\lambda)-(-1)\big) - 0 + (-0.5)((0.5)(-1))\big]\big) \\ & - 0.5\big(-1\big[0-(0.5)(1) - (0.5)((0.5)(-0.5-\lambda)-0) - \tfrac{1}{2}(0.5)(0.5)\big]\big) \end{aligned}$$

Solve for lambda...
Then, if λ<0, the system is stable.
If λ>0, the system is unstable.
If λ = imaginary, the system is marginally stable.
If there are a mix of positive and negative λ's, all λ's must be negative to be stable.

Answer should match the matlab answer above, marginally stable

$\cos(\theta) = 1$ when $\theta = 0$

$\sin(\theta) \approx \theta$ when $\theta = 0$

3.a. $\ddot{q} + (\dot{q}-1)|\dot{q}-1| + 2\sin q = 0$

$q^e = \pi/6 \quad \ddot{q} = 0 \quad \dot{q} = 0$

$\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} \rightarrow f = \ddot{q} = -(\dot{q}-1)|\dot{q}-1| - 2\sin q = \delta\ddot{q} = -(-1)|-1| - 2\delta q$

$x_1 = \delta q, \; x_2 = \delta\dot{q}, \; \dot{x}_1 = x_2$

$\ddot{q} = -(x_2-1)|x_2-1| - 2\sin x_1$

$\frac{\partial f}{\partial x_1} = -2(\pi/6) = -\frac{\pi}{3}$

$\frac{\partial f}{\partial x_2} \Rightarrow$ for $x_2 > 1$, $|x_2-1| = (x_2-1) \rightarrow f = -(x_2-1)^2 - 2\sin x_1$

$\frac{\partial f}{\partial x_2} = -2(x_2-1)$

for $x_2 < 1$, $|x_2-1| = -(x_2-1) \rightarrow f = (x_2-1)^2 - 2\sin x_1$

$\frac{\partial f}{\partial x_2} = 2(x_2-1)$

$x_1 = q^e = \pi/6 \quad x_2 = \dot{q} = 0$

So, for $x_2 = \dot{q} > 0 \rightarrow A = \begin{bmatrix} 0 & 1 \\ -\pi/3 & -2(0-1) \end{bmatrix}$

$A_1 = \begin{bmatrix} 0 & 1 \\ -\pi/3 & 2 \end{bmatrix} \rightarrow \mathrm{eig}(A_1) = \begin{matrix} 1 + 0.2172i \\ 1 - 0.2172i \end{matrix}$

for $x_2 = \dot{q} < 0 \rightarrow A_2 = \begin{bmatrix} 0 & 1 \\ -1.732 & -2 \end{bmatrix} \rightarrow \mathrm{eig}(A_2) = \begin{matrix} -1 + 0.8556i \\ -1 - 0.8556i \end{matrix}$

For $\dot{q}$ greater than 0, the system is unstable as there is a positive root.

For $\dot{q}$ less than 0, the system is asympotically stable as there are only negative real roots.

3.b. $\dot{q}_1 = e^{q_1} q_2 - q_1^3 \qquad x_1 = q_1$

$\dot{q}_2 = -q_1 \cos q_2 \qquad x_2 = q_2$

$f_1 \rightarrow \dot{x}_1 = e^{x_1} x_2 - x_1^3$

$f_2 \rightarrow \dot{x}_2 = -x_1 \cos x_2$

3.b. $\dot{q}_1 = e^{q_1} q_2 - q_1^3$ $\qquad x_1 = q_1$

$\dot{q}_2 = -q_1 \cos q_2$ $\qquad x_2 = q_2$

$f_1 \rightarrow \dot{x}_1 = e^{x_1} x_2 - x_1^3$

$f_2 \rightarrow \dot{x}_2 = -x_1 \cos x_2$

$$q_1^e = q_2^e = 0$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} e^{x_1} x_2 - 3x_1^2 & e^{x_1} \\ -\cos x_2 & x_1 \sin x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\mathrm{eig}(A) = i, -i$$

System is <u>marginally stable</u> because the roots are imaginary with no real parts.
A is not defective.

3.c. $\ddot{q}_1 = q_2$ $\qquad x_1 = q_1 \qquad \dot{x}_1 = x_2 \leftarrow f_1$

$\ddot{q}_2 = \sin q_1$ $\qquad x_2 = \dot{q}_1$

$x_3 = q_2 \qquad \dot{x}_3 = x_4 \leftarrow f_3$

$x_4 = \dot{q}_2$

$\dot{x}_2 = x_3 \leftarrow f_2$

$\dot{x}_4 = \sin x_1 \leftarrow f_4$

$$q_1 = q_2 = 0$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \cdots & \\ \frac{\partial f_3}{\partial x_1} & & & \\ \frac{\partial f_4}{\partial x_1} & & & \cdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \cos x_1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\mathrm{eig}(A) =$

```
ans =

  -1.0000 + 0.0000i
   0.0000 + 1.0000i
   0.0000 - 1.0000i
   1.0000 + 0.0000i
```

System is <u>unstable</u> because there is a root with a positive real part.

$x(k) \approx 0$

$\cos(x(k)) \approx 1$

$\sin(x(k)) \approx x(k)$

$x_1(k) = 0 \qquad x_2(k) = 0$

Higher order term

4. $x_1(k+1) = \cancel{x_1(k)^3} + \sin(x_2(k))$

$x_1(k+1) = x_2(k)$

$$x_2(k+1) = -\frac{1}{2}\underbrace{\cancel{\cos(x_2(k))}}_{1}\, x_1(k) + \underbrace{\cancel{x_2(k)^3}}_{\text{H.O.T.}}$$

$x_2(k+1) = -\frac{1}{2} x_1(k)$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$\mathrm{eig}(A) =$

```
ans =

   0.0000 + 0.7071i
   0.0000 - 0.7071i
```

System is marginally stable because it only has purely imaginary roots

5.

Computing the A matrices of the linearized two pendulum two cart system with L1 and L2 properties, and then computing the eigen values:

```
A =

         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
   -0.2500   -0.5000   -0.5000   -2.5000         0         0
   -0.2500   -1.5000   -0.5000   -2.5000         0         0
   -0.2500   -0.5000   -1.5000   -2.5000         0         0

L1
  -1.9694 + 0.0000i
  -0.1095 + 0.0000i
  -0.2105 + 1.0557i
  -0.2105 - 1.0557i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

We see that for L1, there are only negative real part roots but also imaginary components. Therefore, the system is stable

```
A =

         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
   -0.2500   -0.5000   -0.5000   -2.5000         0         0
    0.2500    1.5000    0.5000    2.5000         0         0
    0.2500    0.5000    1.5000    2.5000         0         0

L2
   -2.8174
    1.1380
   -0.1097
   -0.7109
   -1.0000
    1.0000
```

We see that for L2, there are positive real part roots, therefore the system is unstable.