Homework 2

1. To compute the observations of the object at 0:30 AM local time on Sep 6, 2024, we must first propagate the object's orbital elements forward in time by 16.5 hours to the local time since they were given at noon, Sep 5 2024 UTC. See attached Matlab files.

After propagating satellite position with:

```
[Tout, Z] = ode45(@two_body_ode,t,[r_ijk v_ijk],options);
```
(1)

Where:

```
[r_ijk,v_ijk] = keplerian2ijk(a,e,i,RAAN,w,v);
```
(2)

Final position of satellite that is observed is at the end of the solution:

$$Position_{Final,Satellite} = \mathrm{Z}(\mathrm{end}, 1{:}3) \tag{3}$$

Compute satellite geocentric declination:

$$\delta' = atan2\left(Position_{Final,Satellite}Z, norm\left(Position_{Final,Satellite}X, Position_{Final,Satellite}Y\right)\right) \tag{4}$$

Compute satellite geocentric right ascension:

$$\alpha' = atan2\left(Position_{Final,Satellite}Y, Position_{Final,Satellite}X\right) \tag{5}$$

Armstrong Hall's latitude and longitude coordinates are selected as:

$$\phi = latitude = 40.431°, \lambda = longitude = -86.915 \tag{6}$$

ECI coordinates of the station are computed via Matlab function:

$$Position_{ECI,Station} = lla2eci(\,LLA, [2024 \;\; 9 \;\; 6 \;\; 4 \;\; 30 \;\; 00]) \tag{7}$$

To compute geocentric latitude of the station:

$$Latitude_{geocentric} = atan2(Position_{ECI,Station}Y, Position_{ECI,Station}X) \tag{8}$$

Compute R vector from geocenter to the station:

$$R_{ECI} = \sqrt{Position_{ECI,Station}X^2 + Position_{ECI,Station}Y^2 + Position_{ECI,Station}Z^2} \tag{9}$$

Compute range magnitude to satellite [Meters]:

$$r = \sqrt{Satellite_{ECI}X^2 + Satellite_{ECI}Y^2 + Satellite_{ECI}Z^2} \tag{10}$$

Compute vector from station to satellite:

$$\rho = vecnorm(Satellite_{ECI} - Position_{ECI,Station}, 2, 2) \tag{11}$$

Compute Julian Date (JD) given Gregorian Date of September 6, 2024, 4:30 AM ET:

```
if month <= 2
    year = year - 1;
    month = month + 12;
end

A = floor(year / 100);
B = 2 - A + floor(A / 4);
```

$$JD = floor\big(365.25 * (year + 4716)\big) + floor\big(30.6001 * (month + 1)\big) + day + B - 1524.5 \tag{12}$$

$$JD = JD + \left(hour + \frac{minute}{60}\right) / \, 24 \tag{13}$$

Compute Julian Centuries (T1):

$$T1 = \frac{(JD - 2451545)}{36525} \tag{14}$$

Compute JD0:

$$JD0 = floor(JD) \tag{15}$$

Compute T0:

$$T0 = \frac{JD0 - 2451545}{36525} \tag{16}$$

Compute Mean Sidereal Time of Greenwich 0h Earth time (UT):

$$UT = 24110.54841 + 8640184.812866 * T0 + 0.093104 * T1^2 - 0.0000062 * T1^3 + (1.0027279093 * (mod(JD, 1) - 5) * 24) * 3600; \quad (17)$$

Convert UT to Hours:

$$UT = UT * \frac{1\ hr}{3600\ sec} \quad (18)$$

Compute sidereal time:

$$Sidereal\ Time = mod(UT + Station\ Geocentric\ Longitude * \frac{1\ hr}{15\ deg}, 24\ hours) \quad (19)$$

Compute sidereal angle:

$$Sidereal\ Angle = Sidereal\ Time * \frac{15\ deg}{1\ hr} \quad (20)$$

We are assuming we are starting in the **True of Date frame**, so we proceed to rotate to **J2000 frame**:

$$Position_{ECI,Station,J2000} = N^T(t) * P^T(t) * Position_{ECI,Station} \quad (21)$$

Compute nutation rotation terms:

$$T = \frac{JD_{TT} - 2451545.0}{36525} \quad (22)$$

$$l = 134°\, 57'46.733'' + 477198°\, 52'02.633" * T + 31.31'' * T^2 + 0.064'' * T^3 \quad (23)$$

$$l' = 357°31'39.804'' + 35999°03'01.244'' * T - 0.577'' * T^2 - 0.012'' * T^3 \quad (24)$$

$$F = 93°16'18.877'' + 483202°01'03.137'' * T - 13.257'' * T^2 + 0.011'' * T^3 \quad (25)$$

$$D = 297°51'01.307'' + 445267°06'41.328'' * T - 6.891'' * T^2 + 0.019'' * T^3 \quad (26)$$

$$\Omega = 125°02'40.280'' - 1934°08'10.539'' * T + 7.455'' * T^2 + 0.008'' * T^3 \quad (27)$$

Ecliptic is calculated via:

$$\epsilon = 23.43929111° - 46.8150'' * T - 0.00059'' * T^2 + 0.001813'' * T^3 \quad (28)$$

$\Delta\psi$ (longitude of the mean vernal equinox in relation to the true vernal equinox) is computed via:

$$\Delta\psi = \sum_{i=1}^{106} (\Delta\psi)_i \sin(\phi_i) \quad (29)$$

$\Delta\epsilon$ (difference between the true and the mean obliquity of the ecliptic) is computed via:

$$\Delta\epsilon = \sum_{i=1}^{106} (\Delta\epsilon)_i \cos(\phi_i) \quad (30)$$

Where ϕ_i is:

$$\phi_i = p_{l,i} * I + p_{l',i} * I' + p_{F,i} * F + p_{D,i} * D + p_{\Omega,\mathrm{i}} * \Omega \quad (31)$$

Nutation is computed as:

$$N = R_x(-\epsilon - \Delta\epsilon)\, R_z(-\Delta\psi) R_x(\epsilon) \quad (32)$$

Expanded out:

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\epsilon - \Delta\epsilon) & \sin(-\epsilon - \Delta\epsilon) \\ 0 & -\sin(-\epsilon - \Delta\epsilon) & \cos(-\epsilon - \Delta\epsilon) \end{bmatrix} \begin{bmatrix} \cos(-\Delta\psi) & \sin(-\Delta\psi) & 0 \\ -\sin(-\Delta\psi) & \cos(-\Delta\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\epsilon) & \sin(\epsilon) \\ 0 & -\sin(\epsilon) & \cos(\epsilon) \end{bmatrix} \quad (33)$$

Compute precession:

$$\zeta = 2306.2181 * T1^2 + 0.30188 * T1^2 + 0.017998 * T1^3 \quad (34)$$

$$\theta = 2004.3109 * T1 - 0.42665 * T1^2 - 0.041833 * T1^3 \quad (35)$$

$$z = \zeta + 0.79280 * T1^2 + 0.000205 * T1^3 \quad (36)$$

$$P(t) = R_3(z) * R_2(\theta) * R_3(\zeta) \quad (37)$$

$$P(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(z) & \sin(z) \\ 0 & -\sin(z) & \cos(z) \end{bmatrix} \begin{bmatrix} \cos(-\theta) & 0 & -\sin(-\theta) \\ 0 & 1 & 0 \\ \sin(-\theta) & 0 & \sin(-\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\zeta) & \sin(\zeta) \\ 0 & -\sin(\zeta) & \cos(\zeta) \end{bmatrix} \quad (38)$$

$$\boldsymbol{Position_{ECI,Station,J2000} = P^T(t) * N^T(t) * Position_{ECI,Station}}$$

Compute topocentric declination of satellite. Starting with these relations:

$$\begin{aligned} r\cos(\delta')\cos(\alpha') &= \rho\cos(\delta)\cos(\alpha) + R\cos(\phi')\cos(\theta) \\ r\cos(\delta')\sin(\alpha') &= \rho\cos(\delta)\sin(\alpha) + R\cos(\phi')\sin(\theta) \\ r\sin(\delta') &= \rho\sin(\delta) + R\sin(\phi') \end{aligned} \quad (39)$$

Solve for topocentric declination δ. Rearrange Eqn () to isolate δ:

$$\delta = sin^{-1}\left(\frac{r\sin(\delta') - R\sin(\phi')}{\rho}\right) \quad (40)$$

Solve for topocentric right ascension α. Rearrange Eqn(39) . These are the x and y components of topocentric right ascension:

$$\cos(\alpha) = \frac{r\cos(\delta')\cos(\alpha') - R\cos(\phi')\cos(\theta)}{\rho\cos(\delta)} \quad (41)$$

$$\sin(\alpha) = \frac{r\cos(\delta')\sin(\alpha') - R\cos(\phi')\sin(\theta)}{\rho\cos(\delta)} \quad (42)$$

Solve for topocentric right ascension by computing arctan:

$$\alpha = tan^{-1}\left(\frac{\sin(\alpha)}{\cos(\alpha)}\right) \tag{43}$$

Compute hour angle, τ [deg]:

$$\tau = \theta - \alpha \tag{44}$$

The relations between topocentric azimuth (a) and elevation (h) and hour angle (τ), geodetic latitude (ϕ), and topocentric declination are given:

$$\begin{gathered}\cos(h)cos(a) = sin(\phi)\cos(\delta)cos(\tau) - \cos(\phi)sin(\phi) \\ \cos(h)sin(a) = \cos(\delta)sin(\tau) \\ \sin(h) = sin(\phi)\sin(\delta) + \cos(\phi)cos(\delta)\cos(\tau)\end{gathered} \tag{45}$$

Solve for elevation (h) by rearranging Eqn (45):

$$h = sin^{-1}\big(sin(\phi)sin(\delta) + cos(\phi)cos(\delta)cos(\tau)\big) \tag{46}$$

Solve for azimuth (a) by first rearranging Eqn (45):

$$cos(a) = \frac{sin(\phi)\cos(\delta)cos(\tau) - \cos(\phi)sin(\phi)}{cos(h)} \tag{47}$$

$$sin(a) = \frac{\cos(\delta)sin(\tau)}{cos(h)} \tag{48}$$

Solve for topocentric azimuth by computing arctan:

$$a = tan^{-1}\left(\frac{\sin(a)}{\cos(a)}\right) \tag{49}$$

Obtaining:

$$h_{J2000} = Elevation = -48.2937258012151°$$

$$a_{J2000} = Azimuth = 25.9432402027009°$$

1.b. Solving for light travel time, τ using fixed-point iteration:

$$\tau^{(k+1)} = \frac{1}{c}|\boldsymbol{r}(t-\tau) - \boldsymbol{R}(t)| \tag{49}$$

We obtain:

$$\tau = 0.0334\ seconds$$

We search through the vector of propagated orbit positions to $\boldsymbol{r}(t-\tau)$ to find:

$$\boldsymbol{r}(t-\tau) = [915199.599535056 \quad -4125874.42824264 \quad -5307927.11150869]\ [m] \tag{50}$$

Which is slightly different than the final satellite position of the propagated orbit:

$$Position_{Final,Satellite} = [915067.875841307 \quad -4125750.86163945 \quad -5307745.37099989]\ [m] \tag{51}$$

Using $\boldsymbol{r}(t-\tau)$ to re-solve for Azimuth and Elevation to account for this light travel time, we obtain:

$$h = Elevation = -48.2942617888165°$$

$$a = Azimuth = 25.9466136419182°$$

1.c. When making observations from Earth, you should be in the International Terrestrial Reference Frame (ITRF) frame because you will be experiencing nutation, precession and polar motion as you are physically standing on the Earth. Therefore, if you want to be in the J2000 frame, you must rotate your observations first by Polar motion to get GTOD, then rotate by Sidereal Angle to get TOD, then Nutation to get mean of date (MOD) and then Precession to get to J2000.

2. We are given station coordinates in the ITRF frame and want to rotate to the J2000 frame to validate the data.

From IERS[1], one can gather x_p and y_p for the date of June 19-20, 2017, given to us in the validation spreadsheet:

$$x_p = 125.744\ [mas], \quad y_p = 455.691\ [mas]$$

The polar motion matrix is:

$$\Pi = \begin{bmatrix} 1 & 0 & x_p \\ 0 & 1 & -y_p \\ -x_p & y_p & 1 \end{bmatrix} \tag{50}$$

Compute nutation terms:

$$T = \frac{JD_{TT} - 2451545.0}{36525} \tag{50}$$

$$I = 134°\,57'46.733'' + 477198°\,52'02.633" * T + 31.31'' * T^2 + 0.064'' * T^3 \tag{51}$$

$$I' = 357°31'39.804'' + 35999°03'01.244'' * T - 0.577'' * T^2 - 0.012'' * T^3 \tag{52}$$

$$F = 93°16'18.877'' + 483202°01'03.137'' * T - 13.257'' * T^2 + 0.011'' * T^3 \tag{53}$$

$$D = 297°51'01.307'' + 445267°06'41.328'' * T - 6.891'' * T^2 + 0.019'' * T^3 \tag{54}$$

$$\Omega = 125°02'40.280'' - 1934°08'10.539'' * T + 7.455'' * T^2 + 0.008'' * T^3 \tag{55}$$

Ecliptic is calculated via:

$$\epsilon = 23.43929111° - 46.8150'' * T - 0.00059'' * T^2 + 0.001813'' * T^3 \tag{56}$$

$\Delta\psi$ (longitude of the mean vernal equinox in relation to the true vernal equinox) is computed via:

$$\Delta\psi = \sum_{i=1}^{106} (\Delta\psi)_i \sin(\phi_i) \tag{57}$$

$\Delta\epsilon$ (difference between the true and the mean obliquity of the ecliptic) is computed via:

$$\Delta\epsilon = \sum_{i=1}^{106} (\Delta\epsilon)_i \cos(\phi_i) \tag{58}$$

Where ϕ_i is:

$$\phi_i = p_{l,i} * l + p_{l',i} * l' + p_{F,i} * F + p_{D,i} * D + p_{\Omega,\mathrm{i}} * \Omega \tag{59}$$

Nutation is computed as:

$$N = R_x(-\epsilon - \Delta\epsilon)\, R_z(-\Delta\psi) R_x(\epsilon) \tag{60}$$

Expanded out:

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\epsilon-\Delta\epsilon) & \sin(-\epsilon-\Delta\epsilon) \\ 0 & -\sin(-\epsilon-\Delta\epsilon) & \cos(-\epsilon-\Delta\epsilon) \end{bmatrix} \begin{bmatrix} \cos(-\Delta\psi) & \sin(-\Delta\psi) & 0 \\ -\sin(-\Delta\psi) & \cos(-\Delta\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\epsilon) & \sin(\epsilon) \\ 0 & -\sin(\epsilon) & \cos(\epsilon) \end{bmatrix} \tag{61}$$

Compute precession:

$$\zeta = 2306.2181 * T1^2 + 0.30188 * T1^2 + 0.017998 * T1^3 \tag{61}$$

$$\theta = 2004.3109 * T1 - 0.42665 * T1^2 - 0.041833 * T1^3 \tag{62}$$

$$z = \zeta + 0.79280 * T1^2 + 0.000205 * T1^3 \tag{63}$$

$$P(t) = R_3(z) * R_2(\theta) * R_3(\zeta) \tag{64}$$

$$P(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(z) & \sin(z) \\ 0 & -\sin(z) & \cos(z) \end{bmatrix} \begin{bmatrix} \cos(-\theta) & 0 & -\sin(-\theta) \\ 0 & 1 & 0 \\ \sin(-\theta) & 0 & \sin(-\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\zeta) & \sin(\zeta) \\ 0 & -\sin(\zeta) & \cos(\zeta) \end{bmatrix} \tag{65}$$

Form the Θ rotation matrix:

$$\Theta = \begin{bmatrix} \cos(\Theta) & \sin(\Theta) & 0 \\ -\sin(\Theta) & \cos(\Theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{66}$$

Where Θ is the computed GMST sidereal time.

$$\boldsymbol{Position_{ECI,Station,J2000}} = \boldsymbol{P^T(t)} * \boldsymbol{N^T(t)} * \boldsymbol{\Theta^T(t)} * \boldsymbol{\Pi^T(t)} * \boldsymbol{Position_{ITRF,Station}} \tag{67}$$

Computed Station J2000 Coordinates		
Station X [km]	Station X [km]	Station X [km]

-2039.05021	-5133.983789	-3177.61164
-347.7729739	-5514.779569	-3174.742713
1378.26128	-5352.824436	-3171.812957
2969.180904	-4664.057578	-3169.110712
4268.411814	-3516.265602	-3166.901931
5148.087195	-2022.411123	-3165.404006
5521.63175	-329.5152935	-3164.76437
5352.282221	1395.811571	-3165.04599
4656.705536	2983.767439	-3166.221163
3503.358512	4278.069907	-3168.174244
2005.750481	5151.337198	-3170.713028
311.2720139	5517.624681	-3173.587662
-1413.310811	5340.883312	-3176.515234
-2998.269183	4638.507503	-3179.207619
-4287.615701	3479.623216	-3181.399835
-5154.456335	1978.284744	-3182.876119
-5513.478944	282.2497918	-3183.491168
-5329.258883	-1441.897571	-3183.184283
-4619.999456	-3023.789442	-3181.985775
-3455.521812	-4308.086982	-3180.013615
-1950.430722	-5168.393076	-3177.461887

Table 1: Computed J2000 Station Coordinates

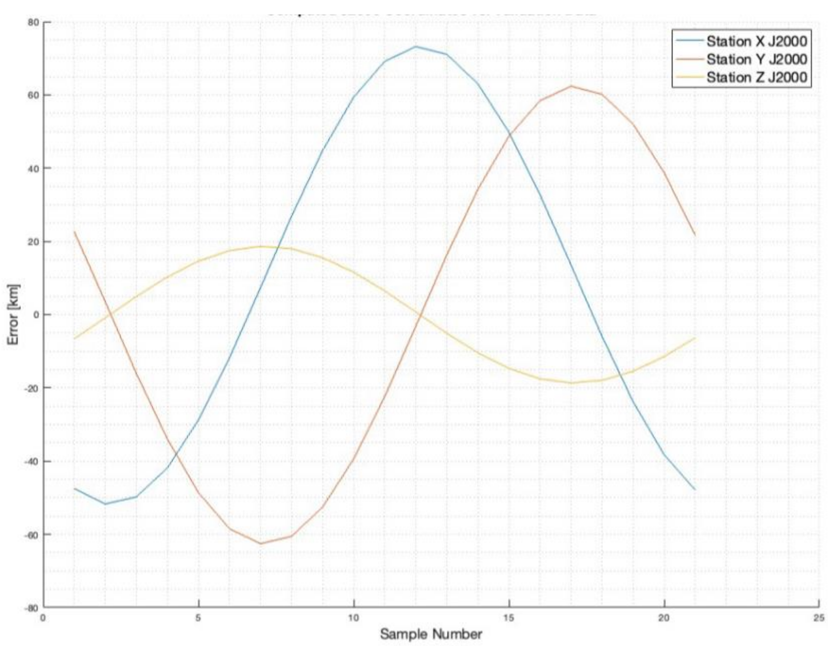

Figure 1: Computed J2000 Values vs. Validation Spreadsheet

3.a. To compute the refraction of a geostationary object at 90°, 75° and 25° elevation, we use the following equation to compute refraction:

$$\Delta z_{\geq 15^\circ} = 16.271'' * \tan(z')\left[1 + 0.0000394\tan^2(z')\left(\frac{p - 0.156e}{T}\right)\right]\left(\frac{p - 0.156e}{T}\right) - 0.0749''(\tan^3(z') + \tan(z'))\left(\frac{p}{1000}\right)$$

Where e is computed via:

$$e = 0.01(RH)\exp\left(AT^2 + BT + C + \frac{D}{T}\right)\ [mbar] \quad (68)$$

Where:

$$T = 293.15\ °K\ (Local\ average\ temperature)$$

$$p = 1013.25\ (Sealevel\ atmospheric\ pressure\ [millibar])$$

$$RH = Relative\ humidty = 0.72\ (Average\ RH\ in\ Indiana)$$

$$A = 1.2378847e - 5$$

$$B = -1.9121316e - 5$$

$$C = 33.93711047$$

$$D = -6.3431645e3$$

The computed $\Delta z_{\geq 15^\circ}$ for the elevation angles given in the problem are:

$$\Delta z_{\geq 15^\circ} = [\ 0^\circ \quad 0.0012452^\circ \quad 0.0097631^\circ]$$

It makes sense that from a 90° elevation, there would be negligible refraction, and as you lower elevation, the refraction gets more and more significant as there is more atmosphere for the light to travel through.

Computing absolute position differences caused by refraction…

Compute True Elevation (h) from $\Delta z_{\geq 15^\circ}$ and Observed Elevation (h'):

$$h = h' - \Delta z_{\geq 15^\circ}$$

Compute Observed distances:

$$Distance_{Horz,Obs} = Altitude_{GEO} * \cos(h') \quad (69)$$

$$Distance_{Vert,Obs} = Altitude_{GEO} * \sin(h') \quad (70)$$

$$Altitude_{GEO} = 36{,}000\ km$$

Compute True distances:

$$Distance_{Horz,True} = Altitude_{GEO} * \cos(h') \tag{71}$$

$$Distance_{Vert,True} = Altitude_{GEO} * \sin(h) \tag{72}$$

Computing the difference between True and Observed distances, we obtain:

$$Error_{X,Refraction} = [0 \quad 0.7557266 \quad 2.592] \quad [km]$$

$$Error_{Y,Refraction} = [0 \quad 41.309 \quad 348.14697] \quad [km]$$

3.b. To compute the positional difference caused by light speed travel time, we can first compute the time it takes for light to travel from GEO altitudes to an observer on the surface of Earth:

$$Time_{Signal} = Altitude_{GEO}/c \ = 0.12\ sec$$

$$c = speed\ of\ light = 3E8\ m/s$$

The average orbital velocity of objects at GEO altitude is 3000 m/s.

Computing potential position error due to light travel time:

$$Error_{Light\ Travel} = Velocity_{satellite} * Time_{Signal} = 0.36\ km$$

3.c. Computing aberration error:

$$Error_{Angular} = \ Time_{Signal} * \omega_{Earth} \tag{73}$$

$$\omega_{Earth} = 15°/hour$$

The potential angular error is caused by Earth spinning while the signal is traveling down to the observer. Converting angular error to position error:

$$Error_{Aberration} = Altitude_{GEO} * \ Error_{Angular} = 0.31416\ km$$

Summing up errors accumulated from 25° of elevation (refraction), light travel time and aberration, we obtain:

$$Total\ Error = \sqrt{Error_{X,Refraction}^2 + Error_{Y,Refraction}^2} + \ Error_{Light\ Travel} + Error_{Aberration} = 3.26617\ km$$

References

1. *https://datacenter.iers.org/data/207/bulletinb-353.txt*