0.1 Mathematical Statements

A *mathematical statement* is an English statement that has a truth value.

Types of Statements Compound statements, Implications, Double implications, Converse of an Implications, Negations, and Quantifiers.

Compound Statement (P and Q)

A *compound statement* is a statement constructed from two statements joined be the words "and" or "or".

Example 0.1 (Compound Statement)**.** Let x be a real number.

P : Then number x is greater than 3.

Q : Then number x is even.

P and Q : The number x is greater than 3 **and** x is even.

P and Q : The number x is greater than 3 **or** x is even.

Question: What are the truth values of P and Q and P and Q?

1. If $x = 6$? P and Q, P or Q are true.
2. If $x = 5$? P and Q is false. P or Q is true.
3. If $x < 3$? P and Q is false. , P or Q depends on the value of x.

Implication ($P \to Q$)

The mathematical statement "P implies Q" is an implication where P is the hypothesis and Q is the conclusion. Other forms of an implication are "If P then Q.", " $P \to Q$", and "$P \Rightarrow Q$".

Example 0.2. If x is greater than 0, then x^2 is greater than 0. Here "x is greater than 0" is the hypothesis and "x^2 is greater than 0" is the conclusion.

Converse of an implication ($Q \to P$)

The converse of " P implies Q" is " Q implies P".

Example 0.3. Converse of previous example

If x^2 is greater than 0, then x is greater than 0.

Double implication ($Q \leftrightarrow P$)

The statement "“P implies Q” and “Q implies P”' is a double implication. Other forms of a double implication are " $P \leftrightarrow Q$", "$P \Leftrightarrow Q$", "P iff Q", where 'iff' means 'if and only if'.

Example 0.4. (Double implication) The value of x is greater than 0 if and only if x^2 is greater than 0.

Negation

The negation of P is not P. The notation for the negation of P is denoted by $\neg P$

Example 0.5. (Negation) Let P : The value of x is greater than 0. Then $\neg P$: The value of x is less than or equal to 0.

Contrapositive

The contrapositive of an implication $P \to Q$ is an equivalent statement of the following form $\neg Q \to \neg P$.

Example 0.6. The following statement is the contrapositive of the statement from example Example 0.2. If x^2 is less than or equal to 0 then x is less than or equal to 0.

Problem 0.7. *Is the statement "Let x be a real number." a mathematical statement?*

Solution 0.8. This is not a mathematical statement because it does not have a truth value. Statements similar to the statement "Let x be a real number." are *commands* and are typically used to define variables.

Quantifiers

Expressions that quantify statements. Common quantifiers are "for all" and "there exists" denoted by $\forall$ and $\exists$ respectively.

Example 0.9. (Quantifiers)

- For all integers x, $x^2 \geq 0$. Here the expression "For all" quantifies for which integers the statement $x^2 \geq 0$ is true.
- There exists an integer x such that $x^2 - 1 = 0$.
- For all x there exists y such that x is less than y.

1.1 Truth Tables

Truth tables help us determine the validity of a statement. Truth tables give us a way to construct equivalent statements.

Let P and Q be mathematical statements. A table listing the possible truth values of each statement is called a truth table. For simplicity, instead of writing the word "and"("or") to join to statements we will used the symbols $\wedge(\vee)$ respectively.

Example 0.10. The following is a truth table which can be used to determine the value of the statement $P \wedge Q$.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 0.11. Below are the truth values of some frequently used statements.

P	Q	$\neg P$	$P \vee Q$	$P \rightarrow Q$
T	T	F	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	T

Consider the more complexed statement $\neg Q \rightarrow \neg P$ which is the contrapositive of $P \rightarrow Q$. A truth table can be used to show that the two statements are actually equivalent. To do this we must show that the two statements have the exact same true values which are independent of the values of P and Q.

Example 0.12. The statements $\neg Q \rightarrow \neg P$ and $P \rightarrow Q$ are equivalent statements because they have the same values in their columns.

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

In some cases, proving an equivalent statements may be easier than proving the actual statement.

Review

1. The product of nonzero real numbers is nonzero. For example, if $xy \neq 0$ then $x \neq 0$ and $y \neq 0$.
2. If x is a nonzero real number then $x^2 > 0$.
3. If x is an even integer then $x = 2k$ for some integer k.
4. If x is an odd integer then $x = 2k + 1$ for some integer k.
5. If x and y are even integers, then xy is an even integer.