Contradiction

Proof by way of contradiction is a proof that uses the negation of the original statement to prove the validity of the original statement. For example, let suppose P is a statement that is to be proven. One way to use a proof by contradiction is to disprove $\neg P$ constructing an implication $\neg P \to Q$ and showing that the value of Q is always false. This implies that $\neg P$ must be false to guarantee that $\neg P \to Q$ is a true statement.

Theorem 1.21. *The real number $\sqrt{2}$ is irrational.*

To prove the following theorem we will show that the negation, "The real number $\sqrt{2}$ is rational" is false.

Idea of proof: Let P : The real number $\sqrt{2}$ is irrational., $\neg P$: The real number $\sqrt{2}$ is rational., and Q : There exists integers m and n such that $\sqrt{2} = \frac{m}{n}$ where $n \neq 0$ and m and n have no common factors. If $\neg P$ is false then P is true. Assume $\neg P$ is true. Then Q is true. However in the proof Q is shown to be false. Therefore $\neg P$ must be false which implies that P is true.

Proof. Assume $\sqrt{2}$ is rational. Then there exists integers m and n such that $\sqrt{2} = \frac{m}{n}$ where $n \neq 0$ and m and n have no common factors. Since m and n have no common factors we know that $\frac{m}{n}$ is in lowest terms so both m and n can not be even. We have $\sqrt{2} = \frac{m}{n}$ implies $2 = \frac{m^2}{n^2}$ which implies $2n^2 = m^2$. Since m^2 is even it must be the case that m is even. Hence $m = 2k$ for some integer k. Moreover, $2n^2 = m^2 = (2k)^2$ which implies $n^2 = 2k^2$. This contradicts that both m and n are not even. Therefore, $\sqrt{2}$ is irrational. □

Problem 1.22. *Prove that no odd integer can be expressed as the sum of three even integers.*

Counter Example

Counter examples are examples which show that a statement is false. For instance, $x = 3$ is a counter example to the statement "For all integer x, x^2 is even." Evaluation of the example suffices when showing that the statement is false. For instance, "For all integer x, x^2 is even." is false, since $3^2 = 9$ is odd.

Problem 1.23. *Disprove the following statement. For all positive integers x, if $\frac{x(x+1)}{2}$ is odd then $\frac{(x+1)(x+2)}{2}$ is odd.*