

## Contradiction

Proof by way of contradiction is a proof that uses the negation of the original statement to prove the validity of the original statement. For example, let suppose  $P$  is a statement that is to be proven. One way to use a proof by contradiction is to disprove  $\neg P$  constructing an implication  $\neg P \rightarrow Q$  and showing that the value of  $Q$  is always false. This implies that  $\neg P$  must be false to guarantee that  $\neg P \rightarrow Q$  is a true statement.

**Theorem 1.0.21.** *The real number  $\sqrt{2}$  is irrational.*

To prove the following theorem we will show that the negation, “The real number  $\sqrt{2}$  is rational” is false.

**Idea of proof:** Let  $P$  : The real number  $\sqrt{2}$  is irrational.,  $\neg P$  : The real number  $\sqrt{2}$  is rational., and  $Q$  : There exists integers  $m$  and  $n$  such that  $\sqrt{2} = \frac{m}{n}$  where  $n \neq 0$  and  $m$  and  $n$  have no common factors. If  $\neg P$  is false then  $P$  is true. Assume  $\neg P$  is true. Then  $Q$  is true. However in the proof  $Q$  is shown to be false. Therefore  $\neg P$  must be false which implies that  $P$  is true.

*Proof.* Assume  $\sqrt{2}$  is rational. Then there exists integers  $m$  and  $n$  such that  $\sqrt{2} = \frac{m}{n}$  where  $n \neq 0$  and  $m$  and  $n$  have no common factors. Since  $m$  and  $n$  have no common factors we know that  $\frac{m}{n}$  is in lowest terms so both  $m$  and  $n$  can not be even. We have  $\sqrt{2} = \frac{m}{n}$  implies  $2 = \frac{m^2}{n^2}$  which implies  $2n^2 = m^2$ . Since  $m^2$  is even it must be the case that  $m$  is even. Hence  $m = 2k$  for some integer  $k$ . Moreover,  $2n^2 = m^2 = (2k)^2$  which implies  $n^2 = 2k^2$ . This contradicts that both  $m$  and  $n$  are not even. Therefore,  $\sqrt{2}$  is irrational.  $\square$

**Problem 1.0.22.** *Prove that no odd integer can be expressed as the sum of three even integers.*

## Counter Example

Counter examples are examples which show that a statement is false. For instance,  $x = 3$  is a counter example to the statement “For all integer  $x$ ,  $x^2$  is even.” Evaluation of the example suffices when showing that the statement is false. For instance, “For all integer  $x$ ,  $x^2$  is even.” is false, since  $3^2 = 9$  is odd.

**Problem 1.0.23.** *Disprove the following statement. For all positive integers  $x$ , if  $\frac{x(x+1)}{2}$  is odd then  $\frac{(x+1)(x+2)}{2}$  is odd.*