Chapter 2

Sets and Relations

Sets

A *set* is a collection of elements. Sets are typically represented by a left curly brace before the first element of the list and a right curly brace after the last element of the list. A definition for the elements of a set can also be used to describe a set. The *empty set* is a set with no elements. The empty set can be denoted by $\emptyset$ or $\{\}$.

Example 2.0.1. The following are sets.

$$\{1, 2, 3\} \quad \{\{1, w\}, \pi, x^2 + x, \text{'proofs'}\} \quad \{x | x^2 - 1 = 0\}$$

The symbols $\in$ can be used to indicate that the element x is in the set A. Write $x \in A$. If the element x is not in the set A write, $x \notin A$. The symbols $\setminus$ can be used to construct a set which is the difference between two set. For instance, the set containing elements in the set A which are not in the set B can be represented by $A \setminus B$.

Example 2.0.2. Let $A = \{1, 2, 3\}$ and $B = \{\{\}, 2, \{1, 2, 3\}\}$. Then $1 \in A$ and $A \in B$. Moreover $B \setminus A = \{1, 3, \emptyset, A\}$.

Problem 2.0.3. *Write out the elements of the following sets.*

1. $\{x | x^2 + 2x - 3 = 0\}$
2. $\{\{\}, 1, \{1, 2, 3\}\}$

Common Sets

- The natural numbers (denoted by $\mathbb{N}$) $= \{1, 2, 3, \dots\}$

- The integers (denoted by $\mathbb{Z}$) $= \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- The rational numbers (denoted by $\mathbb{Q}$) $= \{\frac{m}{n} | m, n \in \mathbb{Z} \text{ and } n \neq 0\}$
- The irrational numbers (denoted by $\mathbb{I}$) $= \mathbb{R} \setminus \mathbb{Q}$

Subsets

A set A is a *subset* of the set B, denoted by $A \subseteq B$, if and only if every element in A is an element in B. In this case, the set B is called *superset* of the set A. If A is not a subset of B write $A \nsubseteq B$.

Example 2.0.4. The set $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ and $\{\} \subseteq \{1, 2\}$.

Some of the common sets are subsets of each other.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

Problem 2.0.5. *Which of the common sets are supersets of* $\mathbb{I}$*? Which of the common sets are subsets of* $\mathbb{I}$*?*

The *power set* of the set A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A.

Example 2.0.6. Let $A = \{1, 2, 3\}$. Then

$$\mathcal{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Since the empty set and the set A are subsets of A they are listed in $\mathcal{P}(A)$.

Theorem 2.0.7. *For any set A, $A \subseteq A$ and $\{\} \subseteq A$.*

Proof. Direct Proof

For every $x \in A$ it is the case that $x \in A$. Therefore, by definition of subsets, $A \subseteq A$.

Proof by Contradiction

Suppose $\{\} \nsubseteq A$. By definition, there exists $x \in \{\}$ such that $x \notin A$. However, this contradicts that the empty set has no elements. This shows that $\{\} \nsubseteq A$ is false which implies that $\{\} \subseteq A$ is true. $\square$

The set A and B are equal, denote by $A = B$, whenever every element of A is in B and every element of B is in A.

Theorem 2.0.8. *Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

Since Theorem 2.0.8 is a double implication it is necessary to prove two implications. While this can be done in a linear fashion(The proof only uses double implications.), it is not always easy to construct a proof which only uses double implications.

Proof. ($\Rightarrow$) We want to show, if $A = B$ then $A \subseteq B$ and $B \subseteq A$.

Direct Proof

Suppose $A = B$. Let $x \in A$. Then $x \in B$ since $A = B$. By definition $A \subseteq B$. Now let $x \in B$. Then $x \in A$ since $A = B$. Therefore $B \subseteq A$.

($\Leftarrow$) We want to show, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.

Direct Proof

Suppose $A \subseteq B$ and $B \subseteq A$. Then every element in A is in B and every element in B is in A. By definition, $A = B$. $\square$