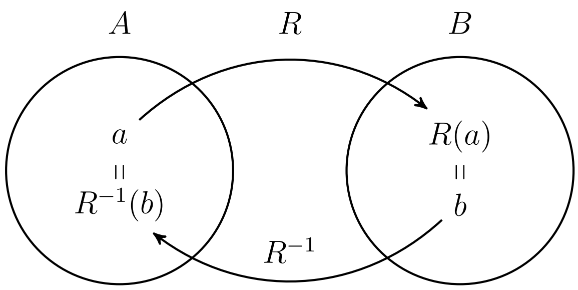
Inverse and Composition

Inverse Relations

Let R be a relation from the set A to the B. The *inverse relation*, denoted by R^{-1}, from B to A is the set

$$\{(b,a) \mid (a,b) \in R\} \subseteq B \times A.$$

Inverse Relation



Example 3.10. Let $R = \{(a,1),(a,3),(c,2),(c,3),(d,1)\}$ be a function from A to B. Then the inverse relation of R is $R^{-1} = \{(1,a),(3,a),(2,c),(3,c),(1,d)\}$.

In Example 3.10, the inverse relation is not a function from B to A because the element 3 gets mapped to two distinct elements, namely a and c. This leads to the question of when is the inverse relation a function from B to A.

Proposition 3.11. *Let $f : A \to B$. Then the inverse relation f^{-1} is a function from B to A if and only if f is a bijection.*

Proof. We want to show f^{-1} is a function from B to A if and only if f is a bijection(one-to-one and onto).

($\Rightarrow$) Assume f^{-1} is a function from B to A.

<u>One-to-one, Proof by Contradiction</u>

Suppose $x_1, x_2 \in A$, $x_1 \neq x_2$, and $f(x_1) = f(x_1)$. Let $f(x_1) = b_1$ and $f(x_2) = b_2$. Then $(x_1,b_1),(x_2,b_2) \in f$ and $b_1 = b_2$. So $(b_1,x_1),(b_2,x_2) \in f^{-1}$ which implies $(b_1,x_1),(b_2,x_2) \in f^{-1}$ since $b_1 = b_2$. This contradicts that f^{-1} is a function. Therefore f is one-to-one.

<u>Onto, Direct Proof</u>

Since f^{-1} is a function from B to A, $\forall b \in B$ there exists $a \in A$ such that $(b, a) \in f^{-1}$. Since f^{-1} is the inverse relation of f it is the case that $(a, b) \in f$. Hence $\forall b \in B$ there exists $a \in A$ such that $(a, b) \in f$ which implies that f is onto.

This shows that f is bijective.

($\Leftarrow$) Assume f is bijective.

<u>The inverse relation maps B to A.</u>

Since f is onto, for every $b \in B$ there exists $a \in A$ such that $(a, b) \in f$. Hence, for every $b \in B$ there exists $a \in A$ such that $(b, a) \in f^{-1}$. Therefore f^{-1} maps B to A.

<u>The inverse relation of f is a function, Proof by Contradiction</u>

Suppose $(b, y_1), (b, y_2) \in f^{-1}$ where $y_1 \neq y_2$. Then $(y_1, b), (y_2, b) \in f$. However this contradicts that f is one-to-one.

$\square$

The inverse relation f^{-1} is called the *inverse of f* if f is a bijection. Hence an inverse of a function is an inverse relation but an inverse relation need not be an inverse. Also, f^{-1} should not be confused with $\frac{1}{f}$.

In each step of the proof we start with the definitions of the variables in the hypothesis. Then we create implications assuming the original hypothesis is true to show that the conclusion must be true.