Recursively Defined Sequences

This section introduces recurrence relations and techniques to solve recurrence relations.

A *sequence* is a function(or list of elements) whose domain is some subset of integers and range is a set of elements(real numbers). Typically, the sequence is represented by a *list* of the elements from the range of the sequence. The *nth term* of the sequence is the element which n is mapped to.

Example 5.7. Let $f : \mathbb{N} \to \mathbb{Z}$. The following list of numbers $1, 4, 9, 16, 25, \ldots$ is a list representation of the sequence f where the nth term of the sequence is n^2 for $n \geq 1$.

Problem 5.8. *Write the list representation of the sequence $f : \mathbb{N} \to \mathbb{Z}$ such that $f : a \mapsto 2a$.*

A *recurrence relation* is an equation where one term of a sequence is expressed in terms of the other terms of the sequence. The *solution* of a recurrence relation is the sequence described by the recurrence relation.

Example 5.9. The sequence $1, 2, 4, 8, 16, \ldots$ has the recurrence relation $a_k = 2a_{k-1}$ where $a_0 = 1$.

In Example 5.9, $a_0 = 1$ is an *initial condition*. It is given because a_0 does not satisfy the recurrence relation. The function $a_n = 2^n$ corresponding to the sequence is the *solution* to the recurrence relation.

Example 5.10. Let $a_0 = 1$ and $a_1 = 4$ be the initial conditions for the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$ where $n \geq 2$.

1. Find the first 6 terms of the sequence.

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 4 \\
a_2 &= 4a_1 - 4a_0 = 4(4) - 4(1) = 12 \\
a_3 &= 4a_2 - 4a_1 = 4(12) - 4(4) = 32 \\
a_4 &= 4a_3 - 4a_2 = 4(32) - 4(12) = 80 \\
a_5 &= 4a_4 - 4a_3 = 4(80) - 4(32) = 192
\end{aligned}$$

The sequence is starts with the terms $1, 4, 12, 32, 80, 192, \ldots$.

2. Conjecture a function/formula for a_n where $n \geq 0$.

Note that $1 = 2^0 \cdot$, $4 = 2^1 \cdot 2$, $12 = 2^2 \cdot 3$, $32 = 2^3 \cdot 4$, $80 = 2^4 \cdot 5$, and $192 = 2^5 \cdot 6$. We can conjecture that the solution to the recurrence relation is given by

$$a_n = 2^n(n+1)$$

where $n \geq 0$.

Problem 5.11. *Let $a_0 = 1$ and $a_n = 2a_{n-1}$ for $n \geq 1$. Write out the first 6 terms. Conjecture a function for the recurrence relation.*

Theorem 5.12. *(Strong Principle of Mathematical Induction)*

Let $P(n)$ be a statement concerning the integer n. Suppose

(1) $P(n_0)$ is a true statement for some integer n_0, and

(2) if $P(k)$ is true for $k \in \{n_0, n_0+1, n_0+2, \ldots, k\}$, then $P(k+1)$ is true.

Then the statement $P(n)$ is true for all integers n such that $n_0 \leq n$.

Theorem 5.13. *Let $a_1 = 1$, $a_2 = 0$, and $a_n = 4a_{n-1} - 4a_{n-2}$ for $n > 2$. Prove that $a_n = 2^n(1 - \frac{n}{2})$ for all $n \geq 1$.*

Proof. First note that the statement $a_n = 2^n(1 - \frac{n}{2})$ is true for $n = 1$ since $a_1 = 1$ by definition and $a_1 = 2^1(1 - \frac{1}{2}) = 1$ by the function.

Proof using Strong Mathematical Induction on n

Base step:

For $n = 2$, the left hand side (LHS) is $a_n = 0$ and the right hand side (RHS) is $2^2(1 - \frac{2}{2}) = 0$. Since the LHS is the same as the RHS the statement is true for the base step.

Induction step:

Assume $a_n = 2^n(1 - \frac{n}{2})$ for $1, \ldots, k$. Then $a_{k+1} = 4a_k - 4a_{k-1}$ and by hypothesis

$$a_{k+1} = 4\left(2^k(1 - \frac{k}{2})\right) - 4\left(2^{k-1}(1 - \frac{k-1}{2})\right).$$

Hence

$$\begin{aligned} a_{k+1} &= 2^2 2^k \left[1 - \frac{k}{2} - 2^{-1} - \frac{k-1}{2^2}\right] \\ &= 2^{k+1}\left[2 - k - 1 - \frac{k-1}{2}\right] \\ &= 2^{k+1}\left[1 - k - \frac{k-1}{2}\right] \\ &= 2^{k+1}\left[1 - \frac{2k-k+1}{2}\right] \\ &= 2^{k+1}\left[1 - \frac{k+1}{2}\right]. \end{aligned}$$

By Strong Mathematical Induction the statement is true. □

Problem 5.14. *Prove the conjecture from Example 5.9 using Strong Mathematical Induction.*

Special Sequences

Arithmetic Sequence

Let $a \in \mathbb{R}$. Then $a, a+d, a+2d, a+3d, \dots$ is a general arithmetic sequence.

Theorem 5.15. *A formula for a general arithmetic sequence is given by*

$$a_n = a + (n-1)d$$

where $n \geq 1$.

Proof. (Proof by Principle of Mathematical Induction)

Base step:

For $n = 1$, LHS $= a_1 = a$ and RHS $= a + (1-1)d = a$.

Induction step:

Assume $a_k = a + (k-1)d$. Then LHS $= a_{k+1} = a + (k+1-1)d = a + kd =$ RHS.

The statement holds by the Principle of Mathematical Induction. □

Proposition 5.16. *Consider the arithmetic sequence defined by the recurrence relation*

$$a_n = a + (n-1)d.$$

Then $\sum\limits_{m=1}^{k} a_m = \frac{n}{2}\big[2a + (n-1)d\big]$ *for* $k \geq 1$.

Proof. (Proof by Principle of Mathematical Induction)

Base step:

For $k = 1$, LHS $= \sum\limits_{m=1}^{1} a_m = a_1 = a(1-1)d = a$. Also, RHS $= \frac{1}{2}\big[2a + (1-1)d\big] = a$.

Induction step:

Assume $\sum\limits_{m=1}^{k} a_m = \frac{k}{2}\big[2a + (k-1)d\big]$. Then

$$\begin{aligned}
\text{LHS} = \sum_{m=1}^{k+1} a_m &= \sum_{m=1}^{k} a_m + a_{k+1} \\
&= ak + \frac{k}{2}(k-1)d + a + (k+1-1)d \\
&= (k+1)a + k\left(\frac{(k-1)d}{2}\right) + kd \\
&= \frac{1}{2}(2ak + 2a + k^2 d - kd + 2kd).
\end{aligned}$$

Also, RHS $= \left(\frac{k+1}{2}\right)\big[2a + kd\big] = \frac{1}{2}(2ak + 2a + k^2 d - kd + 2kd)$. Therefore LHS = RHS and by the Principle of Mathematical Induction the statement is true. □

Another Sequence

Let $a, r \in \mathbb{R}$. A geometric sequence has the form

$$a, ar, ar`ar^3, \ldots.$$

The recurrence relation for the geometric sequence is given by $a_{k+1} = ra_k$ where $a_1 = a$ and $k \geq 1$.

Problem 5.17. *Prove that* $a_n = ar^{n-1}$ *is a formula for the geometric sequence.*

Theorem 5.18. *The sum of the first* n *terms of a geometric sequence is given by*

$$S = \frac{a(1-r^n)}{1-r}.$$

The proof of Theorem 5.18 is not by induction. It uses the identity $(1-r)(r+r^2+r^3+\cdots+r^{n-1}) = 1-r^n$.

The Fibonacci Sequence

The Fibanacci Sequence is

$$f_k = f_{k-1} + f_{k-2}$$

where $f_0 = 1$, $f_1 = 1$ and $k \geq 2$.

Chapter 6

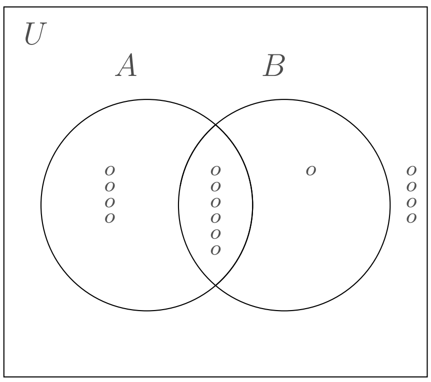
Counting

The Principle of Inclusion-Exclusion

Problem 6.1. *In a group of* 15 *pizza experts, ten like bacon, seven like mushrooms, and six like both. How many people liked at least one topping?*

Let A be the set of the experts who like bacon and B be the set of people who like mushrooms. There is more than one way to count the answer to Problem 6.1. Which of the following values should be calculated first

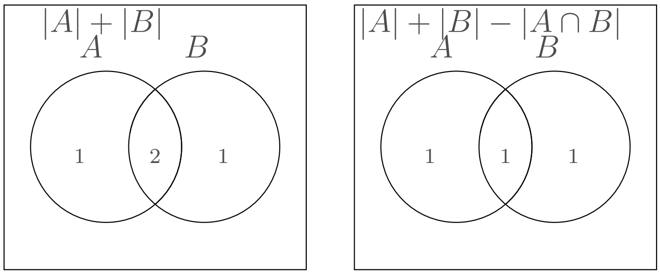
$$|A|, |B|, |A \cup B|, |A \cap B|, |B^c|, |A^c|, |A \setminus B|, |B \setminus A|, |(A \cup B)^c|?$$



One way of counting is to over count.

$$|A \cup B| = \underbrace{|A| + |B|}_{\text{include}} - \underbrace{|A \cap B|}_{\text{exclude}} = 10 + 7 - 6 = 11$$

This way of counting uses $|A| + |B|$ to count the elements in $|A \setminus B|$ once, $|B \setminus A|$ once, and $|A \cap B|$ twice. Hence to count the desired value $|A \cap B|$ must be excluded.



Let $n \in \mathbb{N}$. Define $[n]$ to be the set $\{1, 2, \dots, n\}$.

Example 6.2. How many integers in [100] are not divisible by $2, 3$, or 5? Let A_2 be the set of numbers divisible by 2, A_3 be the set of numbers divisible by 3, and A_5 be the set of numbers divisible by 5. Then $100 - |A_2 \cup A_3 \cup A_5|$ is the number of integers in [100] which are not divisible by $2, 3$, or 5.

What is $|A_2 \cup A_3 \cup A_5|$?

Consider the multi set which includes elements from A_2, A_3, and A_5. Then

$$A = \{2, 4, 6, 8, 10, \dots, 3, 6, 9, 12, \dots, 5, 10, 15, 20, \dots\}.$$

Then $|A| = |A_2| + |A_3| + |A_5| \geq |A_2 \cup A_3 \cup A_5|$ since $A_2 \cap A_3$ may not be the empty set.

Notice that the set A is the same as $A_2 \cup A_3 \cup A_5$.

Question Which elements can we delete from A so that $|A| = |A_2 \cup A_3 \cup A_5|$?

"How many times do the duplicate elements occur in A?"

Here the numbers $1, 2$, and 3 indicate the number of times that every element in the corresponding region appears in A.

For example $6 \in A_2, A_3$ so 6 appears twice in A.

"How can we construct a set from A which does not have duplicates?"

$|A| - |A_2 \cap A_3|$

$|A| - (|A_2 \cap A_3| + |A_2 \cap A_5| + |A_3 \cap A_5|)$

Deleting the elements from the intersection removes the element 30 trice.

$|A| - (|A_2 \cap A_3| + |A_2 \cap A_5| + |A_3 \cap A_5|) + |A_2 \cap A_3 \cap A_5| = |A_2 \cup A_3 \cup A_5|$

"Now we need to calculate the size of each set which typically requires proof."

$|A_2| = \lfloor \frac{100}{2} \rfloor$ is the largest number of times we can sum 2 which is less than 100 which is 50.

$$|A_3| = \lfloor \frac{100}{3} \rfloor = 33$$

$$|A_5| = \lfloor \frac{100}{5} \rfloor = 20$$

$$|A_2 \cap A_3| = \lfloor \frac{100}{6} \rfloor = 16$$

$$|A_3 \cap A_5| = \lfloor \frac{100}{15} \rfloor = 10$$

$$|A_2 \cap A_3 \cap A_5| = \lfloor \frac{100}{30} \rfloor = 3$$

$$\begin{aligned}
|A_2 \cup A_3 \cup A_5| &= |A_2| + |A_3| + |A_5| \\
&- |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| \\
&+ |A_2 \cap A_3 \cap A_5| \\
&= 50 + 33 + 20 - (16 + 6 + 10) + 3 \\
&= 74
\end{aligned}$$

Therefore the number of elements of $\{1, 2, \ldots, 100\}$ which are not divisible by $2, 3$, or 5 is $100 - 74 = 26$.

How many elements of $\{1, 2, \ldots, 100\}$ are not divisible by 2 or 3?

$$|A_2 \cup A_3| = |A_2| + |A_3| - |A_2 \cap A_3| = 50 + 33 - 16 = 67$$

There are $100 - 67 = 23$ such elements.

Problem 6.3. *The number of positive integers which is divisible by a and less than n is given by $\lfloor \frac{n}{a} \rfloor$.*