Review

1. The product of nonzero real numbers is nonzero. For example, if $xy \neq 0$ then $x \neq 0$ and $y \neq 0$.
2. If x is a nonzero real number then $x^2 > 0$.
3. If x is an even integer then $x = 2k$ for some integer k.
4. If x is an odd integer then $x = 2k + 1$ for some integer k.
5. If x and y are integers then xy is an integer.
6. If x and y are even integers, then xy is an even integer.

Proofs in Mathematics

Proofs should consist of English statements. That is, mathematical expression should be written as complete English sentences. Proofs should not have statements that begin with mathematical symbols.

Example 1.0.14. The statement "x is even." should not be used in a proof. However, using the previous statement as a hypothesis is fine.

If x is even, then x^2 is even.

Types of proofs:

1. Direct Proof - sequence of implications
2. Proof by Cases - a list of proofs that cover all possible values
3. Proof by Contrapositive - proving a equivalent implication
4. Proof by Contradiction - showing the negation is false
5. Counterexamples - using an example to disprove a statement

Direct Proof ($P_1 \to P_2 \to \cdots \to P_n$)

Direct proof can be thought of as sequence of implications used to show that the hypothesis of the first implication in the sequence implies the last hypothesis in the sequence. Typically, the hypothesises are omitted from the proof for a more elegant proof.

Theorem 1.0.15. *If x is even then x^2 is even.*

To prove the previous theorem we must show that the value of the implication is true. This means whenever "x is even" (the hypothesis) is a true statement then " x^2 is even" (the conclusion) is a true statement. Whenever the hypothesis is a false the implication is true no matter the value of conclusion. So we need only to consider the case when the hypothesis is a true statement.

Proof. Assume x is even. Then $x = 2k$ for some integer k. Hence $x^2 = x \cdot x = 2k \cdot 2k = 2(2k^2)$. Since $x^2 = 2(2k^2)$ and $2k^2$ is an integer it follows that x^2 is even. □

Notice that after the first sentence which is a command (not a mathematical statement) each of the following statements imply the next. Which leads to the less attractive proof.

Proof. If x is even then $x = 2k$ for some integer k. If $x = 2k$ for some integer k then $x^2 = x \cdot x = 2k \cdot 2k = 2(2k^2)$. If $x^2 = 2(2k^2)$ then x^2 is even. Therefore, if x is even then x^2 is even. □

Problem 1.0.16. *Prove that if x is odd then x^2 is odd.*

Cases

Proof by cases is useful when it is easier to use different proof techniques for different values of the hypothesis.

Theorem 1.0.17. *For all integers x, $x^2 + x$ is even.*

The following theorem is a quantified statement which is an indication that proof by cases might be useful.

Proof. For consistency write the quantified statement as an implication. If x is an integer then $x^2 + x$ is even.

Case 1: (x is even)

Assume x is even. Then $x = 2k$ for some integer k. Hence $x^2 + x = (2k)^2 + (2k) = 2(2k^2 + k)$. Therefore, $x^2 + x$ is even.

Case 2: (x is odd)

Assume x is odd. Then $x = 2k + 1$ for some integer k. Hence $x^2 + x = (2k + 1)^2 + (2k + 1) = 2(2k^2 + 3k + 1)$. Therefore, $x^2 + x$ is even. □

Problem 1.0.18. *Let x be an integer. Prove that $x^2 - 3x + 9$ is odd.*

Contrapositive

Recall that the contrapositive of the implication $P \to Q$ is $\neg Q \to \neg P$.

Theorem 1.0.19. *Let x be an integer. If $5x - 7$ is even, then x is odd.*

A direct proof might be the first proof in mind. However $x = \frac{2k-7}{5}$ for some integer k is not a useful form of x when considering it's parity.

Proof. Consider the contrapositive, if x is even then $5x - 7$ is odd. Assume x is even. Then $x = 2k$ for some integer k. Hence $5x - 7 = 5(2k) - 7 = 2(5k) - 2(4) + 1 = 2(5k - 4) + 1$. Therefore, $5x - 7$ is odd. $\square$

The first statement of the proof is a tip to the reader that proof by contrapositive is used.

Problem 1.0.20. *Prove that if x^2 is even then x is even.*