Recursively Defined Sequences

This section introduces recurrence relations and techniques to solve recurrence relations.

A *sequence* is a function(or list of elements) whose domain is some subset of integers and range is a set of elements(real numbers). Typically, the sequence is represented by a *list* of the elements from the range of the sequence. The *nth term* of the sequence is the element which n is mapped to.

Example 5.7. Let $f : \mathbb{N} \to \mathbb{Z}$. The following list of numbers $1, 4, 9, 16, 25, \ldots$ is a list representation of the sequence f where the nth term of the sequence is n^2 for $n \geq 1$.

Problem 5.8. *Write the list representation of the sequence* $f : \mathbb{N} \to \mathbb{Z}$ *such that* $f : a \mapsto 2a$.

A *recurrence relation* is an equation where one term of a sequence is expressed in terms of the other terms of the sequence. The *solution* of a recurrence relation is the sequence described by the recurrence relation.

Example 5.9. The sequence $1, 2, 4, 8, 16, \ldots$ has the recurrence relation $a_k = 2a_{k-1}$ where $a_0 = 1$.

In Example 5.9, $a_0 = 1$ is an *initial condition*. It is given because a_0 does not satisfy the recurrence relation. The function $a_n = 2^n$ corresponding to the sequence is the *solution* to the recurrence relation.

Example 5.10. Let $a_0 = 1$ and $a_1 = 4$ be the initial conditions for the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$ where $n \geq 2$.

1. Find the first 6 terms of the sequence.

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 4 \\
a_2 &= 4a_1 - 4a_0 = 4(4) - 4(1) = 12 \\
a_3 &= 4a_2 - 4a_1 = 4(12) - 4(4) = 32 \\
a_4 &= 4a_3 - 4a_2 = 4(32) - 4(12) = 80 \\
a_5 &= 4a_4 - 4a_3 = 4(80) - 4(32) = 192
\end{aligned}
$$

The sequence is starts with the terms $1, 4, 12, 32, 80, 192, \ldots$.

2. Conjecture a function/formula for a_n where $n \geq 0$.

Note that $1 = 2^0 \cdot$, $4 = 2^1 \cdot 2$, $12 = 2^2 \cdot 3$, $32 = 2^3 \cdot 4$, $80 = 2^4 \cdot 5$, and $192 = 2^5 \cdot 6$. We can conjecture that the solution to the recurrence relation is given by

$$a_n = 2^n(n+1)$$

where $n \geq 0$.

Problem 5.11. *Let $a_0 = 1$ and $a_n = 2a_{n-1}$ for $n \geq 1$. Write out the first 6 terms. Conjecture a function for the recurrence relation.*

Theorem 5.12. *(Strong Principle of Mathematical Induction)*

Let $P(n)$ be a statement concerning the integer n. Suppose

(1) $P(n_0)$ is a true statement for some integer n_0, and

(2) if $P(k)$ is true for $k \in \{n_0, n_0+1, n_0+2, \ldots, k\}$, then $P(k+1)$ is true.

Then the statement $P(n)$ is true for all integers n such that $n_0 \leq n$.

Theorem 5.13. *Let $a_1 = 1$, $a_2 = 0$, and $a_n = 4a_{n-1} - 4a_{n-2}$ for $n > 2$. Prove that $a_n = 2^n(1 - \frac{n}{2})$ for all $n \geq 1$.*

Proof. First note that the statement $a_n = 2^n(1 - \frac{n}{2})$ is true for $n = 1$ since $a_1 = 1$ by definition and $a_1 = 2^1(1 - \frac{1}{2}) = 1$ by the function.

Proof using Strong Mathematical Induction on n

Base step:

For $n = 2$, the left hand side (LHS) is $a_n = 0$ and the right hand side (RHS) is $2^2(1 - \frac{2}{2}) = 0$. Since the LHS is the same as the RHS the statement is true for the base step.

Induction step:

Assume $a_n = 2^n(1 - \frac{n}{2})$ for $1, \ldots, k$. Then $a_{k+1} = 4a_k - 4a_{k-1}$ and by hypothesis

$$a_{k+1} = 4\Big(2^k(1 - \frac{k}{2})\Big) - 4\Big(2^{k-1}(1 - \frac{k-1}{2})\Big).$$

Hence

$$\begin{aligned}
a_{k+1} &= 2^2 2^k \left[1 - \frac{k}{2} - 2^{-1} - \frac{k-1}{2^2}\right] \\
&= 2^{k+1}\left[2 - k - 1 - \frac{k-1}{2}\right] \\
&= 2^{k+1}\left[1 - k - \frac{k-1}{2}\right] \\
&= 2^{k+1}\left[1 - \frac{2k-k+1}{2}\right] \\
&= 2^{k+1}\left[1 - \frac{k+1}{2}\right].
\end{aligned}$$

By Strong Mathematical Induction the statement is true. □

Problem 5.14. *Prove the conjecture from Example 5.9 using Strong Mathematical Induction.*