Contents

Chapter 1

Proofs

Mathematical Statements

A *mathematical statement* is an English statement that has a truth value.

Types of Statements Compound statements, Implications, Double implications, Converse of an Implications, Negations, and Quantifiers.

Compound Statement (P and Q)

A *compound statement* is a statement constructed from two statements joined be the words "and" or "or".

Example 1.0.1 (Compound Statement)**.** Let x be a real number.

> P : Then number x is greater than 3.
>
> Q : Then number x is even.
>
> P and Q : The number x is greater than 3 **and** x is even.
>
> P and Q : The number x is greater than 3 **or** x is even.

Question: What are the truth values of P and Q and P and Q?

1. If $x = 6$? P and Q, P or Q are true.
2. If $x = 5$? P and Q is false. P or Q is true.
3. If $x < 3$? P and Q is false. , P or Q depends on the value of x.

Implication $(P \to Q)$

The mathematical statement "P implies Q" is an implication where P is the hypothesis and Q is the conclusion. Other forms of an implication are "If P then Q.", " $P \to Q$", and "$P \Rightarrow Q$".

Example 1.0.2. If x is greater than 0, then x^2 is greater than 0. Here "x is greater than 0" is the hypothesis and "x^2 is greater than 0" is the conclusion.

Converse of an implication $(Q \to P)$

The converse of " P implies Q" is " Q implies P".

Example 1.0.3. Converse of previous example

If x^2 is greater than 0, then x is greater than 0.

Double implication $(Q \leftrightarrow P)$

The statement "'P implies Q" and "Q implies P"' is a double implication. Other forms of a double implication are " $P \leftrightarrow Q$", "$P \Leftrightarrow Q$", "P iff Q", where 'iff' means 'if and only if'.

Example 1.0.4. (Double implication) The value of x is greater than 0 if and only if x^2 is greater than 0.

Negation $(\neg P)$

The negation of P is not P. The notation for the negation of P is denoted by $\neg P$

Example 1.0.5. (Negation) Let P : The value of x is greater than 0. Then $\neg P$: The value of x is less than or equal to 0.

Contrapositive $(\neg Q \to \neg P)$

The contrapositive of an implication $P \to Q$ is an equivalent statement of the following form $\neg Q \to \neg P$.

Example 1.0.6. The following statement is the contrapositive of the statement from example Example 1.0.2. If x^2 is less than or equal to 0 then x is less than or equal to 0.

Problem 1.0.7. *Is the statement "Let x be a real number." a mathematical statement?*

Solution 1.0.8. This is not a mathematical statement because it does not have a truth value. Statements similar to the statement "Let x be a real number." are *commands* and are typically used to define variables.

Quantifiers

Expressions that quantify statements. Common quantifiers are "for all" and "there exists" denoted by $\forall$ and $\exists$ respectively.

Example 1.0.9. (Quantifiers)

- For all integers x, $x^2 \geq 0$. Here the expression "For all" quantifies for which integers the statement $x^2 \geq 0$ is true.
- There exists an integer x such that $x^2 - 1 = 0$.
- For all x there exists y such that x is less than y.

Truth Tables

Truth tables help us determine the validity of a statement. Truth tables give us a way to construct equivalent statements.

Let P and Q be mathematical statements. A table listing the possible truth values of each statement is called a truth table. For simplicity, instead of writing the word "and"("or") to join to statements we will used the symbols $\wedge(\vee)$ respectively.

Example 1.0.10. The following is a truth table which can be used to determine the value of the statement $P \wedge Q$.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.0.11. Below are the truth values of some frequently used statements.

P	Q	$\neg P$	$P \vee Q$	$P \to Q$
T	T	F	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	T

Consider the more complexed statement $\neg Q \rightarrow \neg P$ which is the contrapositive of $P \rightarrow Q$. A truth table can be used to show that the two statements are actually equivalent. To do this we must show that the two statements have the exact same true values which are independent of the values of P and Q.

Example 1.0.12. The statements $\neg Q \rightarrow \neg P$ and $P \rightarrow Q$ are equivalent statements because they have the same values in their columns.

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$\neg Q \rightarrow \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

In some cases, proving an equivalent statements may be easier than proving the actual statement.

Problem 1.0.13. *Use a truth table to show that* $(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$.

Review

1. The product of nonzero real numbers is nonzero. For example, if $xy \neq 0$ then $x \neq 0$ and $y \neq 0$.
2. If x is a nonzero real number then $x^2 > 0$.
3. If x is an even integer then $x = 2k$ for some integer k.
4. If x is an odd integer then $x = 2k + 1$ for some integer k.
5. If x and y are integers then xy is an integer.
6. If x and y are even integers, then xy is an even integer.

Proofs in Mathematics

Proofs should consist of English statements. That is, mathematical expression should be written as complete English sentences. Proofs should not have statements that begin with mathematical symbols.

Example 1.0.14. The statement “x is even.” should not be used in a proof. However, using the previous statement as a hypothesis is fine.

If x is even, then x^2 is even.

Types of proofs:

1. Direct Proof - sequence of implications
2. Proof by Cases - a list of proofs that cover all possible values
3. Proof by Contrapositive - proving a equivalent implication
4. Proof by Contradiction - showing the negation is false
5. Counterexamples - using an example to disprove a statement

Direct Proof ($P_1 \to P_2 \to \cdots \to P_n$)

Direct proof can be thought of as sequence of implications used to show that the hypothesis of the first implication in the sequence implies the last hypothesis in the sequence. Typically, the hypothesises are omitted from the proof for a more elegant proof.

Theorem 1.0.15. *If x is even then x^2 is even.*

To prove the previous theorem we must show that the value of the implication is true. This means whenever "x is even" (the hypothesis) is a true statement then " x^2 is even" (the conclusion) is a true statement. Whenever the hypothesis is a false the implication is true no matter the value of conclusion. So we need only to consider the case when the hypothesis is a true statement.

Proof. Assume x is even. Then $x = 2k$ for some integer k. Hence $x^2 = x{\cdot}x = 2k{\cdot}2k = 2(2k^2)$. Since $x^2 = 2(2k^2)$ and $2k^2$ is an integer it follows that x^2 is even. $\square$

Notice that after the first sentence which is a command (not a mathematical statement) each of the following statements imply the next. Which leads to the less attractive proof.

Proof. If x is even then $x = 2k$ for some integer k. If $x = 2k$ for some integer k then $x^2 = x \cdot x = 2k \cdot 2k = 2(2k^2)$. If $x^2 = 2(2k^2)$ then x^2 is even. Therefore, if x is even then x^2 is even. $\square$

Problem 1.0.16. *Prove that if x is odd then x^2 is odd.*

Cases

Proof by cases is useful when it is easier to use different proof techniques for different values of the hypothesis.

Theorem 1.0.17. *For all integers x, $x^2 + x$ is even.*

The following theorem is a quantified statement which is an indication that proof by cases might be useful.

Proof. For consistency write the quantified statement as an implication. If x is an integer then $x^2 + x$ is even.

Case 1: (x is even)

Assume x is even. Then $x = 2k$ for some integer k. Hence $x^2 + x = (2k)^2 + (2k) = 2(2k^2 + k)$. Therefore, $x^2 + x$ is even.

Case 2: (x is odd)

Assume x is odd. Then $x = 2k+1$ for some integer k. Hence $x^2 + x = (2k+1)^2 + (2k+1) = 2(2k^2 + 3k + 1)$. Therefore, $x^2 + x$ is even. □

Problem 1.0.18. *Let x be an integer. Prove that $x^2 - 3x + 9$ is odd.*

Contrapositive

Recall that the contrapositive of the implication $P \to Q$ is $\neg Q \to \neg P$.

Theorem 1.0.19. *Let x be an integer. If $5x - 7$ is even, then x is odd.*

A direct proof might be the first proof in mind. However $x = \frac{2k-7}{5}$ for some integer k is not a useful form of x when considering it's parity.

Proof. Consider the contrapositive, if x is even then $5x - 7$ is odd. Assume x is even. Then $x = 2k$ for some integer k. Hence $5x - 7 = 5(2k) - 7 = 2(5k) - 2(4) + 1 = 2(5k - 4) + 1$. Therefore, $5x - 7$ is odd. □

The first statement of the proof is a tip to the reader that proof by contrapositive is used.

Problem 1.0.20. *Prove that if x^2 is even then x is even.*

Contradiction

Proof by way of contradiction is a proof that uses the negation of the original statement to prove the validity of the original statement. For example, let suppose P is a statement that is to be proven. One way to use a proof by contradiction is to disprove $\neg P$ constructing an implication $\neg P \to Q$ and showing that the value of Q is always false. This implies that $\neg P$ must be false to guarantee that $\neg P \to Q$ is a true statement.

Theorem 1.0.21. *The real number $\sqrt{2}$ is irrational.*

To prove the following theorem we will show that the negation, "The real number $\sqrt{2}$ is rational" is false.

Idea of proof: Let P : The real number $\sqrt{2}$ is irrational., $\neg P$: The real number $\sqrt{2}$ is rational., and Q : There exists integers m and n such that $\sqrt{2} = \frac{m}{n}$ where $n \neq 0$ and m and n have no common factors. If $\neg P$ is false then P is true. Assume $\neg P$ is true. Then Q is true. However in the proof Q is shown to be false. Therefore $\neg P$ must be false which implies that P is true.

Proof. Assume $\sqrt{2}$ is rational. Then there exists integers m and n such that $\sqrt{2} = \frac{m}{n}$ where $n \neq 0$ and m and n have no common factors. Since m and n have no common factors we know that $\frac{m}{n}$ is in lowest terms so both m and n can not be even. We have $\sqrt{2} = \frac{m}{n}$ implies $2 = \frac{m^2}{n^2}$ which implies $2n^2 = m^2$. Since m^2 is even it must be the case that m is even. Hence $m = 2k$ for some integer k. Moreover, $2n^2 = m^2 = (2k)^2$ which implies $n^2 = 2k^2$. This contradicts that both m and n are not even. Therefore, $\sqrt{2}$ is irrational. □

Problem 1.0.22. *Prove that no odd integer can be expressed as the sum of three even integers.*

Counter Example

Counter examples are examples which show that a statement is false. For instance, $x = 3$ is a counter example to the statement "For all integer x, x^2 is even." Evaluation of the example suffices when showing that the statement is false. For instance, "For all integer x, x^2 is even." is false, since $3^2 = 9$ is odd.

Problem 1.0.23. *Disprove the following statement. For all positive integers x, if $\frac{x(x+1)}{2}$ is odd then $\frac{(x+1)(x+2)}{2}$ is odd.*

Chapter 2

Sets

Sets

A *set* is a collection of elements. Sets are typically represented by a left curly brace before the first element of the list and a right curly brace after the last element of the list. A definition for the elements of a set can also be used to describe a set. The *empty set* is a set with no elements. The empty set can be denoted by $\emptyset$ or $\{\}$.

Example 2.0.1. The following are sets.

$$\{1,2,3\} \quad \{\{1,w\},\pi,x^2+x,\text{'proofs'}\} \quad \{x|x^2-1=0\}$$

The symbols $\in$ can be used to indicate that the element x is in the set A. Write $x \in A$. If the element x is not in the set A write, $x \notin A$. The symbols $\setminus$ can be used to construct a set which is the difference between two set. For instance, the set containing elements in the set A which are not in the set B can be represented by $A \setminus B$.

Example 2.0.2. Let $A = \{1,2,3\}$ and $B = \{\{\},2,\{1,2,3\}\}$. Then $1 \in A$ and $A \in B$. Moreover $B \setminus A = \{1,3,\emptyset,A\}$.

Problem 2.0.3. *Write out the elements of the following sets.*

1. $\{x|x^2+2x-3=0\}$
2. $\{\{\},1,\{1,2,3\}\}$

Common Sets

- The natural numbers (denoted by $\mathbb{N}$) $= \{1,2,3,\dots\}$

- The integers (denoted by $\mathbb{Z}$) $= \{\dots, -2, -1, 0, 1, 2, \dots\}$
- The rational numbers (denoted by $\mathbb{Q}$) $= \{\frac{m}{n} | m, n \in \mathbb{Z}$ and $n \neq 0\}$
- The real numbers (denoted by $\mathbb{R}$) is the set of all numbers
- The irrational numbers (denoted by $\mathbb{I}$) $= \mathbb{R} \setminus \mathbb{Q}$

Subsets

A set A is a *subset* of the set B, denoted by $A \subseteq B$, if and only if every element in A is an element in B. In this case, the set B is called *superset* of the set A. If A is not a subset of B write $A \nsubseteq B$.

Example 2.0.4. The set $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ and $\{\} \subseteq \{1, 2\}$.

Some of the common sets are subsets of each other.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

Problem 2.0.5. *Which of the common sets are supersets of* $\mathbb{I}$*? Which of the common sets are subsets of* $\mathbb{I}$*?*

The *power set* of the set A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A.

Example 2.0.6. Let $A = \{1, 2, 3\}$. Then

$$\mathcal{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Since the empty set and the set A are subsets of A they are listed in $\mathcal{P}(A)$.

Problem 2.0.7. *Write the power set for the set* $\{\{1, 2\}, 3, \{\}\}$.

Problem 2.0.8. *How many elements are in the power set of a set containing exactly three elements?*

To show that a set A is a subset of the set B is suffices to show that every element in A is in B. One way to do this is to pick an arbitrary element in A, say x, and show that $x \in B$. Since x is an arbitrary element in A it is the case that all elements of A are elements of B.

Theorem 2.0.9. *For any set* A*,* $A \subseteq A$ *and* $\{\} \subseteq A$*.*

Proof. Direct Proof

For every $x \in A$ it is the case that $x \in A$. Therefore, by definition of subsets, $A \subseteq A$.

Proof by Contradiction

Suppose $\{\} \nsubseteq A$. By definition, there exists $x \in \{\}$ such that $x \notin A$. However, this contradicts that the empty set has no elements. This shows that $\{\} \nsubseteq A$ is false which implies that $\{\} \subseteq A$ is true. $\square$

The set A and B are equal, denote by $A = B$, whenever every element of A is in B and every element of B is in A. A common technique to prove that two sets are equal is to show that they are subsets of each other.

Theorem 2.0.10. *Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

Since Theorem 2.0.10 is a double implication it is necessary to prove two implications. While this can be done in a linear fashion(The proof only uses double implications.), it is not always easy to construct a proof which only uses double implications.

Proof. This proof breaks the double implication into two implications and prove them individually.

($\Rightarrow$) We want to show, if $A = B$ then $A \subseteq B$ and $B \subseteq A$.

Direct Proof

Suppose $A = B$. Let $x \in A$. Then $x \in B$ since $A = B$. By definition $A \subseteq B$. Now let $x \in B$. Then $x \in A$ since $A = B$. Therefore $B \subseteq A$.

($\Leftarrow$) We want to show, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.

Direct Proof

Suppose $A \subseteq B$ and $B \subseteq A$. Then every element in A is in B and every element in B is in A. By definition, $A = B$. $\square$

Problem 2.0.11. *Let A, B, and C be sets such that $A \in B$. Prove that if $B \subseteq C$ then $A \in C$.*

Operations on Sets

The *union* of two sets A and B, denoted by $A \cup B$, is the set containing elements from A or B. The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements which are in A and B.

Example 2.0.12.

$$A = \{1, 2, 4, 6\} \quad B = \{1, 3, 5, 6\}$$

$$A \cap B = \{1, 6\} \quad A \cup B = \{1, 2, 3, 4, 5, 6\}$$

Problem 2.0.13. *Let* $A = \{a, b, c\}$ *and* $B = \{A, b, 3\}$. *Find* $A \cup B$ *and* $A \cap B$.

The union of multiple sets can be generalized in the following way.

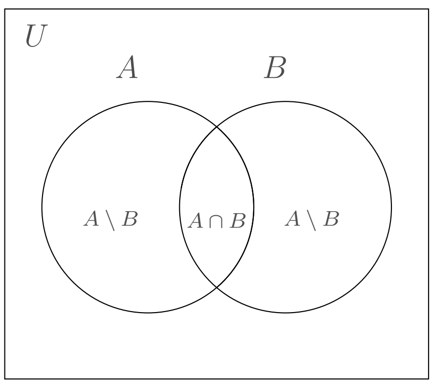
Notation

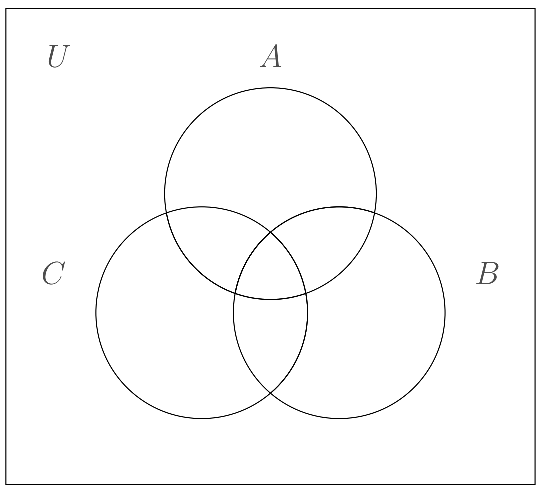
Union of n Sets	Intersection of n Sets
$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$	$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$

The *complement* of a set A with respect to the superset U, denoted by A^c, is the set containing all elements of U which are not in A.

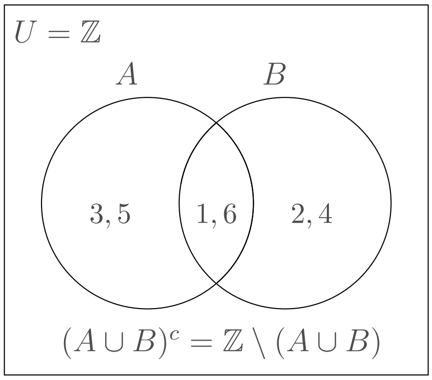
Venn Diagram

A *Venn diagram* is a diagram which shows the relationship between an element x in a set A with another set B.





Example 2.0.14. Consider the following set define in Example 2.0.12. The following Venn diagram show lists all elements from all set.



Problem 2.0.15. *Make a Venn diagram for the sets* $A = \{1, 2, 3\}$, $B = \{1, 4, 5\}$, *and* $C = \{2, 5, 7\}$.

The *Cartesian product* of the set A and B, denoted by $A \times B$, is the set

$$\{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Note that the Cartesian product of two sets is a set of order pairs. Hence $(a, b) \in A \times B$ does not imply that $(b, a) \in A \times B$. Also,

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}} = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A\,, i \in \{1, 2, \ldots, n\}\}.$$

Example 2.0.16. Consider the sets $A = \{1, 2\}$ and $B = \{x, y, z\}$. Then the Cartesian product $A \times B$ is

$$\{(1, x), (1, y), (1, z), (2, x), (2, y)(2, z)\}$$

and

$$A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}.$$

Problem 2.0.17. *Let* A *and* B *be the sets defined in Example 2.0.16. Find* $B \times A$ *and* B^2.

Theorem 2.0.18. *For any sets* A *and* B,

$$(A \cup B)^c = A^c \cap B^c$$

Proof. We want to show that $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$.

1. $(A \cup B)^c \subseteq A^c \cap B^c$

Let $x \in (A \cup B)^c$. Then $x \notin A$ and $x \notin B$. Hence $x \in A^c$ and $x \in B^c$. Thus $x \in A^c \cap B^c$. Therefore $(A \cup B)^c \subseteq A^c \cap B^c$.

Notice that this proof is nice and boring but it gets the job done.

2. $A^c \cap B^c \subseteq (A \cup B)^c$

Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$ which implies $x \notin A$ and $x \notin \cap B$. Hence $x \notin A \cap B$ and it follows that $x \in (A \cap B)^c$. Therefore $A^c \cap B^c \subseteq (A \cup B)^c$.

This proof has a little more flavor than the former but it is still kept simple.

$\square$

Problem 2.0.19. *Prove that for any set A and B, $(A \cap B)^c = A^c \cup B^c$.*

Chapter 3

Functions

Basic Terminology

Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$. A *function* (or *map*) from a set A to a set B is a binary relation, called f, from A to B with the property that

$$\forall a \in A \text{ there is exactly one } b \in B \text{ such that } (a, b) \in A \times B.$$

Write $f : A \to B$ to indication that the function f is being mapped from the set A to the set B. This notation should not be used if f is not a function. Write $f : a \mapsto b$ whenever $f(a) = b$.

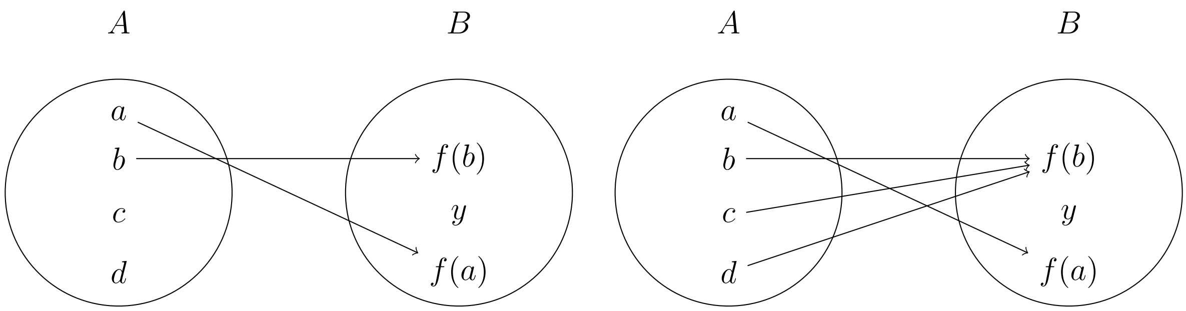
The following diagram show a relation R which is not a function from A to B since all elements in A are not being mapped to B. Whereas, the relation f is a function from A to B even though multiple elements of A are being mapped to $f(b)$.

Relation and not a function

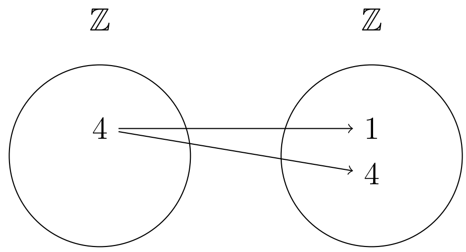
$R = \{(a, f(a)), (b, f(b)\}$

Relation and a function

$f = \{(a, f(a)), (b, f(b), (b, f(b), (b, f(b)\}$



Example 3.0.1. The set $\{(a, b) \,|\, a, b \in \mathbb{N}, a/b \in \mathbb{Z}\}$ is a binary relation from $\mathbb{N}$ to $\mathbb{N}$.



Note that the binary relation in Example 3.0.1 is not a function since both $(4,1),(4,4)$ are in the relation.

Example 3.0.2. The set $\{(x_1,x_2)\,|\,x_1,x_2 \in \mathbb{Z}, x_1^2 = x_2\} = f$ is a function. The function f could also be represented by $f(x) = x^2$ for $x \in \mathbb{Z}$.

Problem 3.0.3. *Let $A = \{1,2,3\}$ and $B = \{a,b,c,d\}$. Give an example of a relation from A to B containing exactly three elements such that the relation is not a function from A to B.*

Let f be a function from A to B.

1. The *domain* of f is A which is denoted by $\operatorname{dom} f$.
2. The *range* of f, denoted by $\operatorname{rng} f$, is the set
$$\{b \in B\,|\, f(a) = b \text{ for some } a \in A\}.$$
Note that $\operatorname{rng} f$ may not contain all elements of B.
3. The function f is one-to-one if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.
4. The function f is *onto* if $\operatorname{rng} f = B$. In other words, $\forall b \in B, \exists a \in A$ such that $f(a) = b$.
5. The function f is bijective if it is one-to-one and onto.

Example 3.0.4. The domain of the function $\{(x_1,x_2)\,|\,x_1,x_2 \in \mathbb{Z}, x_1^2 = x_2\}$ is $\mathbb{Z}$. The range of f is $\{x^2\,|\,x \in \mathbb{Z}\}$. Since $\operatorname{rng} f \neq \mathbb{Z}$ the function f is not onto. It is left a problem to show that f is one-to-one.

Problem 3.0.5. *Let $A = \{a,b,c,d\}$ and $B = \{x,y,z\}$. Then $f\{(a,y),(b,z),(c,y),(d,z)\}$ is a function from A to B. Determine* $\operatorname{dom} f$ *and* $\operatorname{rng} f$.

Problem 3.0.6. *Let $A = \{w,x,y,z\}$ and $B = \{r,s,t\}$. Give an example of a function $f : A \to B$ that is neither one-to-one nor onto.*

Theorem 3.0.7. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(x) = 3x^3 - x$.*

1. *The function f is one-to-one.*

2. *The function f is not bijective.*

Proof. 1. Proof by Contradiction

We want to show that f is not one-to-one is false.

Let $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$. Suppose $f(x_1) = f(x_2)$. Then

$$\begin{aligned} 3x_1^3 - x_1 = 3x_2^3 - x_2 &\Rightarrow 3x_1^3 - 3x_2^3 = x_1 - x_2 \\ &\Rightarrow 3(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = x_1 - x_2 \\ &\Rightarrow x_1^2 + x_1x_2 + x_2^2 = \frac{1}{3}. \end{aligned}$$

However, this contradicts that $x_1^2 + x_1x_2 + x_2^2 \in \mathbb{Z}$ which must be the case since $x_1, x_2 \in \mathbb{Z}$. Therefore, f is one-to-one.

2. Direct Proof

We want to show that f is not onto which implies that f is not bijective.

Since $x \in \mathbb{Z}$, $3x^3$ and x are of the same parity. Hence $3x^3 - x$ is always even. Thus there does not exists $x \in \mathbb{Z}$ such that $f(x) = 1$. This shows that f is not onto. Therefore f is not bijective.

□

Problem 3.0.8. *Show that the function $\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{Z}, x_1^2 = x_2\}$ is one-to-one.*

The *absolute value of* x, denoted by $|x|$, is x if $x \geq 0$ and $-x$ otherwise. The *floor of* x, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x. The *ceiling of* x, denoted by $\lceil x \rceil$, is the least integer greater than or equal to x

Example 3.0.9. It is the case that $|-3| = 3 = |3|$, $\lfloor 2.3 \rfloor = 2$, and $\lceil \pi \rceil = 4$.