

The *power set* of the set  $A$ , denoted by  $\mathcal{P}(A)$ , is the set of all subsets of  $A$ .

**Example 2.0.6.** Let  $A = \{1, 2, 3\}$ . Then

$$\mathcal{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Since the empty set and the set  $A$  are subsets of  $A$  they are listed in  $\mathcal{P}(A)$ .

**Problem 2.0.7.** Write the power set for the set  $\{\{1, 2\}, 3, \{\}\}$ .

**Problem 2.0.8.** How many elements are in the power set of a set containing exactly three elements?

To show that a set  $A$  is a subset of the set  $B$  suffices to show that every element in  $A$  is in  $B$ . One way to do this is to pick an arbitrary element in  $A$ , say  $x$ , and show that  $x \in B$ . Since  $x$  is an arbitrary element in  $A$  it is the case that all elements of  $A$  are elements of  $B$ .

**Theorem 2.0.9.** For any set  $A$ ,  $A \subseteq A$  and  $\{\} \subseteq A$ .

*Proof.* Direct Proof

For every  $x \in A$  it is the case that  $x \in A$ . Therefore, by definition of subsets,  $A \subseteq A$ .

Proof by Contradiction

Suppose  $\{\} \not\subseteq A$ . By definition, there exists  $x \in \{\}$  such that  $x \notin A$ . However, this contradicts that the empty set has no elements. This shows that  $\{\} \not\subseteq A$  is false which implies that  $\{\} \subseteq A$  is true.  $\square$

The set  $A$  and  $B$  are equal, denote by  $A = B$ , whenever every element of  $A$  is in  $B$  and every element of  $B$  is in  $A$ . A common technique to prove that two sets are equal is to show that they are subsets of each other.

**Theorem 2.0.10.** Let  $A$  and  $B$  be sets. Then  $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$ .

Since Theorem 2.0.10 is a double implication it is necessary to prove two implications. While this can be done in a linear fashion (The proof only uses double implications.), it is not always easy to construct a proof which only uses double implications.

*Proof.* This proof breaks the double implication into two implications and prove them individually.

( $\Rightarrow$ ) We want to show, if  $A = B$  then  $A \subseteq B$  and  $B \subseteq A$ .

Direct Proof

Suppose  $A = B$ . Let  $x \in A$ . Then  $x \in B$  since  $A = B$ . By definition  $A \subseteq B$ . Now let  $x \in B$ . Then  $x \in A$  since  $A = B$ . Therefore  $B \subseteq A$ .

( $\Leftarrow$ ) We want to show, if  $A \subseteq B$  and  $B \subseteq A$  then  $A = B$ .

Direct Proof

Suppose  $A \subseteq B$  and  $B \subseteq A$ . Then every element in  $A$  is in  $B$  and every element in  $B$  is in  $A$ . By definition,  $A = B$ .  $\square$

**Problem 2.0.11.** *Let  $A, B$ , and  $C$  be sets such that  $A \in B$ . Prove that if  $B \subseteq C$  then  $A \in C$ .*