The *power set* of the set A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A.

Example 2.0.6. Let $A = \{1, 2, 3\}$. Then

$$\mathcal{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Since the empty set and the set A are subsets of A they are listed in $\mathcal{P}(A)$.

Theorem 2.0.7. *For any set A, $A \subseteq A$ and $\{\} \subseteq A$.*

Proof. <u>Direct Proof</u>

For every $x \in A$ it is the case that $x \in A$. Therefore, by definition of subsets, $A \subseteq A$.

<u>Proof by Contradiction</u>

Suppose $\{\} \nsubseteq A$. By definition, there exists $x \in \{\}$ such that $x \notin A$. However, this contradicts that the empty set has no elements. This shows that $\{\} \nsubseteq A$ is false which implies that $\{\} \subseteq A$ is true. □

The set A and B are equal, denote by $A = B$, whenever every element of A is in B and every element of B is in A.

Theorem 2.0.8. *Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

Since Theorem 2.0.8 is a double implication it is necessary to prove two implications. While this can be done in a linear fashion(The proof only uses double implications.), it is not always easy to construct a proof which only uses double implications.

Proof. ($\Rightarrow$) We want to show, if $A = B$ then $A \subseteq B$ and $B \subseteq A$.

<u>Direct Proof</u>

Suppose $A = B$. Let $x \in A$. Then $x \in B$ since $A = B$. By definition $A \subseteq B$. Now let $x \in B$. Then $x \in A$ since $A = B$. Therefore $B \subseteq A$.

($\Leftarrow$) We want to show, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.

<u>Direct Proof</u>

Suppose $A \subseteq B$ and $B \subseteq A$. Then every element in A is in B and every element in B is in A. By definition, $A = B$. □