Mathematical Induction

Proposition 4.34. *Prove that for any integer $n \geq 1$ the sum of the odd integers from 1 to $2n-1$ is n^2.*

Proposition 4.34 is an example of a statement that can be proved using Mathematical Induction. Mathematical Induction is used to prove a statement is true for an infinite list of consecutive integers which contains a least element.

Theorem 4.35. *(Principle of Mathematical Induction)*

Given a statement $P(n)$ concerning the integer n, suppose

(1) $P(n_0)$ is a true statement for some integer n_0, and

(2) if k is an arbitrary integer greater than n_0 and $P(k)$ is true, then $P(k+1)$ is true.

Then the statement $P(n)$ is true for all integers n such that $n_0 \leq n$.

Proof. Proof of Proposition 4.34

Proof by induction on n

Base step:

For $n = 1$, $2n-1 = 2(1)-1 = 1$. The sum of odd integers from 1 to 1 is 1. Also $n^2 = 1^2 = 1$. Therefore the statement is true for $n = 1$.

Induction step:

Assume that the statement is true for some integer $k > 1$. Then $1 + 3 + \cdots + 2k - 1 = k^2$. Now,

$$\begin{aligned} 1 + 3 + \cdots + (2(k+1) - 1) &= [1 + 3 + \cdots + (2k+1)] + (2(k+1) - 1) \\ &= k^2 + 2(k+1) - 1 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2. \end{aligned}$$

Therefore by the Principle of Mathematical Induction the statement is true. $\square$

Proof. Proof of Theorem 4.35

Proof by contradiction

Suppose (1) and (2) are true and $P(n)$ is false for some positive integer. Let $Q = \{t \mid P(t) \text{ is false }\}$. We can assume $Q \neq \emptyset$ and $1 \notin Q$. Since $Q \subseteq \mathbb{N}$, Q contains a least element k. Hence $1, 2, \ldots, k \notin Q$ which implies $P(1), \ldots, P(k-1)$ are true. Thus by assumption $P(k)$ is true for arbitrary k. This contradicts that $k \in Q$. □

Theorem 4.36. *Principle of Mathematical Induction implies the Well Ordering Principle.*

Proof. Let P be an nonempty subset of natural numbers. Suppose P has no least element. Let Q be the set of elements of $\mathbb{N}$ which are not in P. Then $1, 2, \ldots, k, k+1 \in Q$ for an arbitrary $k \in \mathbb{N}$.

Base Step: $1 \in Q$.

Induction Step: Assume $1, 2, \ldots, k \in Q$. If $k+1 \in P$ then $k+1$ is a least element. Hence $k+1 \in Q$. By the Principle of Mathematical Induction $\mathbb{N} \subseteq Q$. This contradicts that $P \neq \emptyset$. □

The summation of the statement $P(i)$ from k to n, denoted by $\sum_{i=k}^{n} P(i)$, is

$$P(k) + P(k+1) + P(k+2) + \cdots + P(k+n).$$