

The set  $A$  and  $B$  are equal, denote by  $A = B$ , whenever every element of  $A$  is in  $B$  and every element of  $B$  is in  $A$ . A common technique to prove that two sets are equal is to show that they are subsets of each other.

**Theorem 2.10.** *Let  $A$  and  $B$  be sets. Then  $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$ .*

Since Theorem 2.10 is a double implication it is necessary to prove two implications. While this can be done in a linear fashion(The proof only uses double implications.), it is not always easy to construct a proof which only uses double implications.

*Proof.* This proof breaks the double implication into two implications and prove them individually.

( $\Rightarrow$ ) We want to show, if  $A = B$  then  $A \subseteq B$  and  $B \subseteq A$ .

Direct Proof

Suppose  $A = B$ . Let  $x \in A$ . Then  $x \in B$  since  $A = B$ . By definition  $A \subseteq B$ . Now let  $x \in B$ . Then  $x \in A$  since  $A = B$ . Therefore  $B \subseteq A$ .

( $\Leftarrow$ ) We want to show, if  $A \subseteq B$  and  $B \subseteq A$  then  $A = B$ .

Direct Proof

Suppose  $A \subseteq B$  and  $B \subseteq A$ . Then every element in  $A$  is in  $B$  and every element in  $B$  is in  $A$ . By definition,  $A = B$ .  $\square$

**Problem 2.11.** *Let  $A, B$ , and  $C$  be sets such that  $A \in B$ . Prove that if  $B \subseteq C$  then  $A \in C$ .*