Problem 3.0.3. *Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$. Give an example of a relation from A to B containing exactly three elements such that the relation is not a function from A to B.*

Let f be a function from A to B.

1. The *domain* of f is A which is denoted by $\operatorname{dom} f$.
2. The *range* of f, denoted by $\operatorname{rng} f$, is the set
$$\{b \in B \mid f(a) = b \text{ for some } a \in A\}.$$
Note that $\operatorname{rng} f$ may not contain all elements of B.
3. The function f is *one-to-one* if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. Equivalently, f is one-to-one if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
4. The function f is *onto* if $\operatorname{rng} f = B$. In other words, $\forall b \in B, \exists a \in A$ such that $f(a) = b$.
5. The function f is *bijective* if it is one-to-one and onto. The function is also called a *bijection.*

Example 3.0.4. The domain of the function $\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{Z}, x_1^2 = x_2\}$ is $\mathbb{Z}$. The range of f is $\{x^2 \mid x \in \mathbb{Z}\}$. Since $\operatorname{rng} f \neq \mathbb{Z}$ the function f is not onto. It is left a problem to show that f is one-to-one.

Problem 3.0.5. *Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$. Then $f\{(a, y), (b, z), (c, y), (d, z)\}$ is a function from A to B. Determine* $\operatorname{dom} f$ *and* $\operatorname{rng} f$*.*

Problem 3.0.6. *Let $A = \{w, x, y, z\}$ and $B = \{r, s, t\}$. Give an example of a function $f : A \to B$ that is neither one-to-one nor onto.*

Theorem 3.0.7. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(x) = 3x^3 - x$.*

1. *The function f is one-to-one.*
2. *The function f is not bijective.*

Proof. 1. Proof by Contradiction

We want to show that f is not one-to-one is false.

Let $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$. Suppose $f(x_1) = f(x_2)$. Then

$$\begin{aligned}3x_1^3 - x_1 = 3x_2^3 - x_2 &\Rightarrow 3x_1^3 - 3x_2^3 = x_1 - x_2 \\ &\Rightarrow 3(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = x_1 - x_2 \\ &\Rightarrow x_1^2 + x_1x_2 + x_2^2 = \frac{1}{3}.\end{aligned}$$

However, this contradicts that $x_1^2 + x_1x_2 + x_2^2 \in \mathbb{Z}$ which must be the case since $x_1, x_2 \in \mathbb{Z}$. Therefore, f is one-to-one.

2. Direct Proof

We want to show that f is not onto which implies that f is not bijective.

Since $x \in \mathbb{Z}$, $3x^3$ and x are of the same parity. Hence $3x^3 - x$ is always even. Thus there does not exists $x \in \mathbb{Z}$ such that $f(x) = 1$. This shows that f is not onto. Therefore f is not bijective.

$\square$

Problem 3.0.8. *Show that the function* $\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{Z}, x_1^2 = x_2\}$ *is one-to-one.*

The *absolute value of* x, denoted by $|x|$, is x if $x \geq 0$ and $-x$ otherwise. The *floor of* x, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x. The *ceiling of* x, denoted by $\lceil x \rceil$, is the least integer greater than or equal to x

Example 3.0.9. It is the case that $|-3| = 3 = |3|$, $\lfloor 2.3 \rfloor = 2$, and $\lceil \pi \rceil = 4$.