Analysis I

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These notes were prepared between October 2025 and (tentative) **(Last update: January 4, 2026)**.
If you find any mistakes or typos, please report them to **caccacpenguin@gmail.com**. I would really appreciate it.
I often use informal language to make the ideas easier to grasp. My goal is to make the material feel approachable, while still respecting the rigor that makes mathematics what it is.
I hope you will find these notes helpful :D!

References

These lecture notes closely follow the textbooks by Amann and Escher, however these two lecture notes are also useful although, they are written in German.

- Analysis I — WiSe 2016/17, by Franz Merkl, Facultät für Mathematik, Informatik und Statistik, LMU München.
- Analysis einer Veränderlichen — WiSe 2013/14, by Lars Diening, Facultät für Mathematik, Informatik und Statistik, LMU München.

These books will serve as great references. There's a mix of book English and German texts, although you can find English translations for some of them.

- Herbert Amann, Joachim Escher, Analysis I, Dritte Auflage
- Otto Forster, Florian Lindemann, Analysis 1, 13. Auflage
- Walter Rudin, Principles of Mathematical Analysis, 3rd. Edition
- K. Königsberger, Analysis 1
- W. Walter, Analysis 1
- Serge Lang, Analysis I

Contents

1 Introduction

You have already learned how to do calculus and somewhat understand why it works. So, you might wonder, "*Why study calculus again? Why do we need to justify it formally?*"

It's a fair question. We already have an intuitive idea of how *infinitesimal calculus* works: the derivative is defined as the slope of a secant line on a curve when the points on the curve come "infinitesimally close" to each other, so close that they almost overlap but are still two distinct points.

But have you ever wondered what "infinitesimally close" or "infinitesimally small" quantities even mean? You might say, "It means that the quantity is *very small*, almost close to zero but not zero *itself*, such as 10^{-40}".

But examples like that describe *concrete* small quantities, it is a *concrete* value of "how small". No matter how small a quantity you pick, "infinitesimally small" quantities are supposed to be always smaller than them.

So, infinitesimals seems like this vague, intuitive idea which we can't pin down precisely. You may wonder, "How do you *precisely* pin down something?"

See, in mathematics, we try to prove statements from first principles formally, and visual demonstrations or informal intuitive explanations are not accepted as proofs.

This topic was discussed in more detail from a philosophical perspective in the notes on *Foundations of Mathematics* and the crux of it is this: our informal intuitions are based on assumptions that are not explicitly pinned down, in formal mathematics we try to pin down every assumption explicitly.

So, the reason for studying analysis is not just to understand why calculus works and when our rules apply, it is to understand what *formally proving* something even means and why we do it.

"*Okay, so you're saying that our informal intuitions hide some assumptions but what exactly are those assumptions?*"

The assumptions for calculus actually start from very basic concepts in maths, from numbers and arithmetic itself. Because we deal with real numbers in analysis we need to discuss what real numbers mean and construct it axiomatically from primitive axioms. The need for this is that there is some ambiguity in our understanding of numbers, for example, do you know if 0.999... *exactly* equals 1 or is this just an approximation? And does infinite sum of a geometric series exactly equal to its limit? We say that it "approaches this value", but why souldn't it just keep on increasing forever, even if by extremely tiny amounts?

Our aim is to resolve such doubts and gaps in our understanding caused by the intuitive and imprecise explanation given in school. For that purpose, we are usually given the $\delta - \epsilon$ arguments for a limit and convergence, and we use similar arguments for derivatives and integrals as well.

But, if we should accept that with all these $\delta - \epsilon$ arguments convergence happens, then why are we introduced to the notion of topology later on, if convergence was already justified formally? You're usually told that, "These ideas were way too abstract for that time, so we just stuck to proving $\delta - \epsilon$." But that's not true at all, if you understand the need for topology, the idea of neighbourhoods, closure, compactness will not seem like arbitrary abstract ideas but the inevitable need to justify Newton and Leibniz.

The reason why analysis is so hard even for the best students is not just because it's the first time dealing with abstraction and proofs, but also that students are not given proper motivation. I will not try to rush through the concepts or delay topological foundations just for the sake of pedagogy but without honesty. I'll show you that analysis is not

"Calculus with Proofs" but "Justification of Calculus", a subtle but important distinction.

The **prerequisites** for this course are working knowledge of logic, sets and proof techniques, you can refer to my notes on *Mathematical Foundations* for those topics.

Some knowledge of groups, rings, fields and vector spaces would also be helpful but that can be covered alongside this course. Apart from these, only high school mathematics is expected from the reader.

In the first volume of the notes on Analysis we'll focus on the analysis of real numbers and functions of single real variables. Although we will discuss some basic concept of functions of complex variables and how functions in general form vector spaces as well, a detailed coverage will not be done here. You can refer to the notes on *Function Theory* and *Functional Analysis* for a detailed coverage. For applications to theoretical physics, calculus of vector-valued functions and a section on Fourier Analysis is also given.

2 Fields

2.1 The Field Axioms

2.2 The Real Field

2.2.1 Order Completeness

2.2.2 Dedekind Cuts

2.2.3 The Natural Order on $\mathbb{R}$

2.2.4 The Extended Number Line

2.2.5 A Characterization of Supremum and Infimum

2.2.6 The Archimedean Property

2.2.7 The Density of the Rational Numbers in R

2.2.8 nth Roots

2.3 The Density of the Irrational Numbers in R

2.3.1 Intervals

2.4 The Complex Field

2.5 Balls in $\mathbb{K}$

2.6 p-Adic Numbers*

3 General Topology

3.1 Topological Spaces

3.2 Metric Space

3.3 Compactness

3.4 Connectivity

3.5 The Hausdorff Condition

4 Sequence

4.1 Convergence of Sequence in $\mathbb{R}$ and $\mathbb{C}$

4.2 Cauchy Sequences

4.3 Infinite Limits

4.4 Completeness

5 Series

5.1 Finite and Infinite Series

5.2 Calculations with Series

5.3 Majorant, Root and Ratio Tests

5.4 Absolute Convergence

5.5 Rearrangement of Series

5.6 Power Series

6 Continuous Functions

6.1 Limits and Continuity of Functions in $\mathbb{R}$

6.2 Compactedness and Connectedness

6.3 Monotonic Functions

6.4 Limits at Infinity

6.5 Exponential and Related Functions in $\mathbb{R}$

6.6 Complex Exponential, Logarithms and Powers

7 Differentiation in One Variable

7.1 Differentiability

7.2 Rules for Differentiation

7.3 Mean Value Theorem

7.4 The Inequalities of Young, Hölder and Minkowski

7.5 L'Hospital's Rule

7.6 Taylor's Theorem

7.7 Iterative Procedures

7.8 Differentiation of Vector-valued Functions

8 Integration in One Variable

8.1 Piecewise Continuous Function

8.2 Banach Space of Piecewise Continuous Functions

8.3 Continuous Extensions

8.4 The Cauchy-Riemann Integral

8.5 The Riemann Sum

8.6 Properties of Integral

8.7 Integration Techniques

8.8 Sums

8.9 The two Fundamental Theorem of Calculus

8.10 Integration of Vector-valued Functions

8.11 Rectifiable Curves

9 Sequences and Series of Functions

9.1 Uniform Convergence

9.2 The Weierstrass Majorant Criterion

9.3 Continuity and Differentiability

9.4 Analytic Functions

9.5 The Riemann ζ-Function

9.6 Banach Algebras

9.7 The Stone-Weierstrass Theorem

9.8 Polynomial and Trigonometric Approximations

10 Fourier Analysis

10.1 The L_2-Scalar Product

10.2 Orthonormal Systems

10.3 Periodic Functions

10.4 Fourier Coefficients

10.5 Fourier Series

10.6 The Bessel Inequality

10.7 Complete Orthonormality

10.8 Dirac δ-Function

11 Improper Interals

11.1 Absolute Convergence Integral

11.2 Gamma Function

11.3 Euler β-Integral