Mathematics for Physics

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References

The course heavily follows the textbook *Alexander Altland and Jan von Delft, Mathematics for Physicist, CUP, 2019.* And the lectures taken by Prof. Dr. Jan von Delft in the WiSe 2022-23 at the Department of Physics, LMU München.

Contents

1 Introduction

These lecture notes provide a "hyperaccelerated" introduction to the mathematics needed for the beginner physics student. While there is overlap with the mathematics notes, the emphasis here is different: to develop the concepts and mathematical tools used in physics at a much faster pace rather than prove every result from first principles.

Unlike standard undergraduate textbooks, here, abstraction plays a central role. This is due to the significant gap between the mathematical background of most undergraduates and the mathematics used in modern research and graduate-level physics. To avoid getting cooked later on it's better to burn early.

For example, if your understanding of vectors is just an arrow in 3D space then later on quantum mechanics will be rawdogging you hard.

Now, a general concern of the audience would be that heavy emphasis on the structures would compromise computational fluency but this is not the case. There are plenty of examples of concrete calculations and a solid understanding of the underlying structural ideas greatly improves practical and methodological skills.

The content presented here is sophisticated but the the treated is light enough for freshers to pick up easily. Still it can feel very challenging, but do not feel discouraged if you don't get the concepts immediately. Through hard work and dedication you'll be able to get by it.

The only prerequisite is a firm grasp of high-school mathematics (no cap fr). In particular, it would be helpful to brush up your knowledge of trigonometry, complex numbers and basic calculus.

We begin with foundational concepts of sets, groups and fields, then the theory of vector spaces is treated in full generality. Then we cover one-dimentional and multi-dimensional calculus and vector calculus. We also cover matrix theory and differential equations and complex analysis. We conclude with advanced topics including Multilinear Algebra, Differentiable Manifolds, and an introduction to Riemannian and Symplectic Geometry.

Someone might claim that you need "mathematical maturity", the transition from high school algebra to group theory is significant. However, I believe it is very ironic to say that comfort with abstract concepts is somehow supposed to come from doing cookie-cutter problems. Comfort with abstract generalised concepts doesn't come from the concrete, intuitive examples, it comes from hitting it head on. Even the best mathematics students struggle a lot after years of computational learning when they encounter analysis and algebra the first time.

Intuition however given is not incentivised, at some point in your mathematical studies you run out of physical intuition and you need to think in abstraction exclusively, trying to force geometric analogies can be counterproductive in such cases.

Good luck dawg!

2 Mathematical Foundations

2.1 Sets and maps

We start with a review of intuitive set theory, but we won't study them as merely a tools, but as mathematical structures. The most basic mathematical structure is considered to be sets.

Sets

Definition 2.1 (Set)**.** A **set** is a collection of objects, called the **elements** of the set. The formula $a \in A$ indicates that "a in an element of set A".

Remark 2.2. We denote sets using the standard curly brackets {a, b, c, ...}. Oftentimes we use conditional rules to define sets, the standard notation being $A = \{x | \, E(x)\}$, where x is a "placeholder", the **free variable** and $E(x)$ is a property satisfied by x.

Remark 2.3. There is an issue with defining sets in this way when we consider infinite sets, but within restricted comprehension it is okay to study sets in this way for practical purposes.[1]

Definition 2.4 (Cartesian Product)**. Cartesian product** of two sets is defined as

$$A \times B = \{(x, y) | \, x \in A \wedge y \in B\}. \tag{1}$$

Remark 2.5. Here, the wedge $\wedge$ is the **conjuction** logical connective which essentially means that both $x \in A$ AND $y \in B$ need to be true for (x, y) to be an element of the cartesian product.

Example 2.6. For two sets $A = \{x, +, \Delta\}$ and $B = \{\circ, \square\}$, their cartesian product is:

$$A \times B = \{(x, \circ), (x, \square), (+, \circ), (+, \square), (\Delta, \circ), (\Delta, \square), \}$$

Maps

Definition 2.7 (Maps)**.** A **map** F, is a "rule" which assigns to each element a (argument) of a set A called the **domain** an element b (image) of a set B called the **codomain** of the map. Mathematically, it is denoted as:

$$F : A \to B, \quad a \mapsto F(a). \tag{2}$$

Example 2.8.

$$F : \mathbb{N}_0 \to \{-1, 1\}, \quad n \mapsto F(n) := (-1)^n$$

Definition 2.9 (Image)**. Image** of a set A under the map F is defined as,

$$F(A) = \{F(a) | \, a \in A\} \subseteq B. \tag{3}$$

Definition 2.10. A map F is called,

1. **surjective**: if $F(A) = B$.

[1] See the notes on *Mathematical Foundations* for details on Russell's Paradox.

2. **injective**: if $\forall x, y \in A : (F(x) = F(y)) \to (x = y)$.

3. **bijective**: if it is both injective and surjective, $\forall b \in B : \exists! a(b = F(a))$.

Definition 2.11 (Composition)**.** Let $F : A \to B$ and $G : B \to C$ be two functions. Then we define a new function $G \circ F$, the **composition** of F followed by G, by

$$G \circ F : A \to C, \quad a \mapsto G(F(a)).$$

Example 2.12. Let A, B, C be three sets equal to $\mathbb{Z}$. $F(a) = a + 1$, $G(b) = 2b$ then $(G \circ F)(a) = 2(a + 1)$.

Definition 2.13 (Inverse Maps)**.** For a bijective map $F : A \to B$, an **inverse map** $F^{-1} : B \to A$, such that $F(a) \mapsto F^{-1}(F(a)) = a$ for every $a \in A$ exists.

Definition 2.14 (Binary Operation)**.** A **binary operation** is a map H defined on a cartesian product $A \times B$ as

$$H : A \times B \to C, \quad (a, b) \mapsto c = H(a, b).$$

Example 2.15. The shape of sand dune can be described by a map, $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto h(x, y)$.

2.2 Groups

Groups

While sets are mundane containers of objects, they don't do much. The first structure where we can do something with the objects is a *group*.

Definition 2.16 (Group)**.** A **group** $G \coloneqq (A, \bullet)$ is a pair of a set A along with a binary operation $\bullet$ called the **group law**:

$$\bullet : A \times A \to A, \quad (a, b) \mapsto a \bullet b. \tag{4}$$

which satisfy the **group axioms**:

1. **Closure**: $\forall a, b \in A, a \bullet b \in A$.

2. **Associativity**: $\forall a, b, c \in A, (a \bullet b) \bullet c = a \bullet (b \bullet c)$.

3. **Neutral element**: $\exists e \in A : \forall a \in A, a \bullet e = e \bullet a = a$.

4. **Inverse element**: $\forall a, \exists b \in A : a \bullet b = b \bullet a = e$.

Definition 2.17 (Abelian Group)**.** For a group $G = (A, \bullet)$, if the group law is

1. **Commutative**: $\forall a, b \in A, a \bullet b = b \bullet a$, then G is an **abelian group**.

2. **Non-commutative**: $\exists a \in A \exists b \in A, a \bullet b \neq b \bullet a$, then G is a **non-abelian group**.

Example 2.18 (Rotation of three dimensional space)**.** Rotation of a book in three dimensions can be represented as succession of rotations around the coordinate axis, the set of all these rotations forms a group where the group operation is the successive application of rotations. This operation is non-commutative since rotation about z-axis followed by x-axis is not the same as in the reverse order.

Example 2.19 (Rotation of a lattice about a fixed axis)**.** Let $r(\phi)$ be rotation of a lattice about angle ϕ from its principal axis. By symmetry, $r(\phi + 360) = r(\phi)$. The set of all possible right-angle rotations, $R_{90} := \{r(\phi) | \ \phi \in \{0, 90, 180, 270\}\}$.

We define a binary operation of two rotations about a fixed axis as the sum of the angles:

$$\bullet : R_{90} \times R_{90} \to R_{90}, \quad (r(\phi), r(\phi')) \mapsto r(\phi) \bullet r(\phi') := r(\phi + \phi').$$

So, $r(0) \bullet r(90) = r(90)$, $r(90) \bullet r(180) = r(270)$, $r(270) \bullet r(90) = r(180)$. To show the rest of operations we can use a table as such:

$\bullet$	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

From the table, it is clear that $r(\phi) \bullet r(\phi') = r(\phi') \bullet r(\phi)$ for all $\phi \in R_{90}$, therefore $(R_{90}, \bullet)$ is an abelian group. The neutral element is $r(0)$ and the inverse of $r(\phi)$ is $r(360 - \phi)$.

Definition 2.20 (Modulo)**.** For $p, q \in \mathbb{Z}$, **modulo**, $p \bmod q$ is the remainder of p divided by q.

Example 2.21. $5 \bmod 3 = 2$, $6 \bmod 2 = 0$, $4 \bmod 7 = 4$, $-3 \bmod 4 = (-4 + 1) \bmod 4 = 1$.

Example 2.22 (Addition of $\mathbb{Z}$ modulo q)**.** Define $\mathbb{Z}_q := \{0, 1, 2, 3, \dots, q - 1\}$.

We define a binary operation of two integers on mod of q:

$$\bullet : \mathbb{Z}_q \times \mathbb{Z}_q \to \mathbb{Z}_q, \quad (p, p') \mapsto p \bullet p' := (p + p') \bmod q.$$

For $q = 4$,

$$0 \bullet 1 = (0 + 1) \bmod 4 = 1,$$
$$1 \bullet 2 = (1 + 2) \bmod 4 = 3,$$
$$2 \bullet 3 = (2 + 3) \bmod 4 = 1,$$

to show the rest of operations we can use a table as such:

$\bullet$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

From the table, it is clear that $p \bullet p' = p' \bullet p$ for all $p \in \mathbb{Z}_q$, therefore $(\mathbb{Z}_q, \bullet)$ is an abelian group. The neutral element is 0 and the inverse of p is $(4 - p)$.

Group Homomorphism

In Example 2.19 and Example 2.22 we saw two different groups which are structurally similar. If you replace the elements of one group with the elements of the other group in the tables in a particular way, the tables remain unchanged. This means that the two groups have the same structure, formally we say that they are "isomorphic".

Definition 2.23 (Homomorphism)**.** Let $(G, \bullet)$ and $(H, \star)$ be groups with a *priori* independent group operations. A map $\psi : G \to H$, $a \mapsto \psi(a)$ is a **group homomorphism** if for all $a, b \in G$, we have:

$$\psi(a \bullet b) = \psi(a) \star \psi(b). \tag{5}$$

Example 2.24. Let $(G, \bullet) = (H, \star) = (\mathbb{Z}, +)$. A map $\psi : G \to H$, $n \mapsto \psi(n) = 2n$ is a group homomorphism since for all $m, n \in \mathbb{Z}$:

$$\psi(n + m) = 2(n + m) \leftrightarrow \psi(n) + \psi(m) = 2n + 2m.$$

However, the map $\phi : G \to H$, $n \mapsto \phi(n) = 2n^2$ is not a group homomorphism since for all $m, n \in \mathbb{Z}$:

$$\phi(n + m) = 2(n + m)^2 \text{ is not equal to } \phi(n) + \phi(m) = 2n^2 + 2m^2.$$

Definition 2.25 (Isomorphism)**.** A map $\psi : G \to H$, is called a **group isomorphism** if it is a bijective homomorphism. We indicate it by $G \cong H$. Two isomorphic groups are essentially the same group.

Example 2.26. In Example 2.19 and Example 2.22, we have $(R_{90}, \bullet) \cong (\mathbb{Z}_4, \bullet)$. Since, we have a one-to-one correspondence such as $r(0) \leftrightarrow 0$, $r(90) \leftrightarrow 1$, $r(180) \leftrightarrow 2$, $r(270) \leftrightarrow 3$.

Permutation Group

Permutations of n distinct objects forms an important finite group, the **permutation group** or **symmetric group** S_n, containing all the $n!$ possible permutations of n objects.

Example 2.27 (Four numbered balls)**.** Consider the permutations P of four numbered balls $\{1, 2, 3, 4\}$. One possible permutation is to change $1 \mapsto 4$, $2 \mapsto 2$, $3 \mapsto 1$, $4 \mapsto 3$. We denote this permutation as $[4213] : (1, 2, 3, 4) \mapsto (4, 2, 1, 3)$.

The set of all possible permutations P forms the group S_4 with the group operation is a composition of two permutations which are applied successively. For example, let $P = [4213]$ and $P' = [2134]$, then the composition $P'' = P' \circ P$ is given by:

$$(1, 2, 3, 4) \xrightarrow{P} (4, 2, 1, 3) \xrightarrow{P'} (4, 1, 2, 3), \quad [4123] = [2134] \circ [4213],$$

$$(1, 2, 3, 4) \xrightarrow{P'} (2, 1, 3, 4) \xrightarrow{P} (2, 4, 1, 3), \quad [2413] = [4213] \circ [2134].$$

It must be obvious that permutation groups S_n (except S_2) are non-abelian.

2.3 Fields

Adding more structure to groups, we arrive at the concept of fields. These are the structures where two operations, addition and multiplication are allowed.

Field axioms

Definition 2.28 (Field)**.** A (number) **field** is a triple $\mathbb{F} \coloneqq (A, +, \cdot)$, consisting of a set A and two composition rules, addition and multiplication. Fields have the following properties:

- **Addition**$(A, +)$: forms an abelian group, with neutral element 0. Additive inverse of $a = -a$, addition of inverse is called **subtraction**, $a - b \coloneqq a + (-b)$.

- **Multiplication**$(A \backslash \{0\}, \cdot)$: forms an abelian group, with neutral element 1. Multiplicative inverse of $a = a^{-1}$, multiplication of inverse is called **division**, $a \div b \coloneqq a \cdot (b^{-1})$.

- For the neutral element of addition 0,

$$0 \cdot a \coloneqq 0, \ \forall a \in A. \tag{6}$$

- Multiplicative inverse of 0 does not exist. If it did, we would have: $0^{-1} \cdot 0 \cdot a = 0^{-1} \cdot 0 \Rightarrow a = 1, \ \forall a \in A$.

- **Distributivity axiom**:

$$a \cdot (b + c) = a \cdot b + a \cdot c. \tag{7}$$

Just like groups, it is possible to construct fields of finite sets, the **Galois fields**. However, most important fields for physics are infinite — rational numbers, real numbers and complex numbers.

The Real field

The integers, $\mathbb{Z}$, do not form a field (they rather form a **ring**, a structure which allows addition, subtraction and multiplication but not division[2]) because there are no multiplicative inverses for non-zero integers in $\mathbb{Z}$.

Definition 2.29 (Rational numbers)**.** The **rational numbers** are the simplest for of infinite field.

$$\mathbb{Q} \coloneqq \left\{ \frac{q}{p} \mid q, p \in \mathbb{Z}, p \neq 0 \right\}. \tag{8}$$

Another type of numbers are the **irrational numbers**, the numbers that can't be put in the $\frac{q}{p}$ form.

Example 2.30 (Irrational numbers)**.** $\mathbb{Q} \not\ni \sqrt{2} \simeq 1.4142... \simeq \frac{14142}{10000}$ is an irrational number.

Definition 2.31 (Real field)**.** The **real numbers** $\mathbb{R}$, contain all the rational and irrational numbers. The real numbers form a continuous line. The **real field** is then defined as the triple $(\mathbb{R}, +, \cdot)$.

[2]See Section 2.1 of *Linear Algebra I*

The Complex field

The idea of complex numbers is usually introduced as a remedy for closure of real numbers under the square root operation, such as the solutions of equations like $x^2 + 1 = 0$.

Definition 2.32 (Imaginary unit)**.** The object, $i \notin \mathbb{R}$ called the **imaginary unit** is defined as a number whose square-1 gives -1, $i^2 = -1$.

$$i \coloneqq \sqrt{-1}. \tag{9}$$

Definition 2.33 (Imaginary numbers)**.** The square roots of negative numbers called **imaginary numbers** can be defined as,

$$r > 0 : \ \sqrt{-r} \coloneqq \sqrt{-1}\sqrt{r} = i\sqrt{r} \tag{10}$$

Definition 2.34 (Complex field)**.** The **complex numbers** is the set,

$$\mathbb{C} \coloneqq \{z = x + iy |\ x, y \in \mathbb{R}\} \tag{11}$$

$x \coloneqq \Re(z)$ is the **real part**, and $y \coloneqq \Im(z)$ is the **imaginary part** of z.

Assuming the ordinary rules of arithmetic apply to complex numbers, we define the addition and multiplication of complex numbers.

Theorem 2.35 (Addition of complex numbers)**.**

$$z + z' = (x + iy) + (x' + iy') \coloneqq (x + x') + i(y + y'). \tag{12}$$

Theorem 2.36 (Multiplication of complex numbers)**.**

$$zz' = (x + iy)(x' + iy') \coloneqq (xx' - yy') + i(xy' + yx'). \tag{13}$$

Remark 2.37. Neutral element under addition is 0 and the additive inverse for all z is $-z = -x - iy$. Neutral element under multiplication is 1 and the multiplicative inverse for all $z \neq 0$ is defined using the complex conjugate.

Definition 2.38 (Complex conjugate)**.** The **complex conjugate** of $z = x + iy$,

$$z^* \coloneqq \bar{z} \coloneqq x - iy. \tag{14}$$

This yields

$$z\bar{z} = x^2 + y^2. \tag{15}$$

Therfore, $z\bar{z} \in \mathbb{R}$ and $z\bar{z} \geq 0$ (non-zero if $z \neq 0$).

We know that, $\frac{z\bar{z}}{x^2+y^2} = z \cdot \frac{\bar{z}}{x^2+y^2} = 1$. So, using this we can define the inverse of a complex number, z^{-1}.

Definition 2.39 (Inverse of a complex number)**.** For any $z = x + iy$, it's **inverse** is given vy the rational function,

$$\boxed{z^{-1} \coloneqq \tfrac{\bar{z}}{x^2+y^2} = \tfrac{x-iy}{x^2+y^2},\ \Re(z^{-1}) = \tfrac{x}{x^2+y^2},\ \Im(z^{-1}) = \tfrac{-y}{x^2+y^2}.} \tag{16}$$

Example 2.40 (Inverse of complex numbers)**.** For $z = 3 - 2i$,

$$z^{-1} = \frac{3-2i}{3^2+4^2} = \frac{3}{25} - \frac{2i}{25}.$$

A complex number can be identified with a "ordered pair" of two real numbers on the **complex plane**:

$$I : \mathbb{C} \to \mathbb{R}^2 = \{(x, y) |\ x, y \in \mathbb{R}\}, \tag{17}$$

$$z \mapsto (x, y) = \text{a point in two dimensional complex plane.} \tag{18}$$

Here, $\Re(z) = x$ is the **real axis**, and $\Im(z) = iy$ is the **imaginary axis**. The **polar representation** of complex numbers is useful for such purposes. The length of the line connecting $(0, 0)$ and (x, y) on the complex plane is given by the **modulus**:

$$|z| \coloneqq \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}. \tag{19}$$

the angle of the line from the real axis is given by its **argument**:

$$\arg(z) \coloneqq \phi = \arctan\left(\frac{\Im(z)}{\Re(z)}\right). \tag{20}$$

By using these, we define the polar form as:

$$z \coloneqq |z|(\cos\phi + i\sin\phi). \tag{21}$$

3 Vector Spaces

In school, we are taught that physical quantites can be divided into scalars (numbers of a field) such as kinetic energy, mass, temperature, volume of a body and vectors such as momentum, velocity, force. Vectors are introduces as arrows with a magnitute and a direction. However, some quantites which seem to be a vector (like torque) aren't actually vectors. Vectors are much more than just arrows in three-dimensional space.

This chapter introduces the general theory of vector spaces which is motivated by the generalised perspective of vectors algebraically since often times in physics we encounter vectors that don't have obvious geometric interpretations.

3.1 General vector spaces

We define vectors algebraically as objects that can be added to each other and multiplied by elements of a number field $\mathbb{F}$.

Definition 3.1 (Vector space)**.** An $\mathbb{F}$-**vector space** is a triple $(V, +, \cdot)$ of a set V, and two composition rules, **vector addition**:

$$+ : V \times V \to V,\ (\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{u} + \mathbf{v}) := \mathbf{u} + \mathbf{v}. \tag{22}$$

and **scalar multiplication**:

$$\cdot : \mathbb{F} \times V \to V,\ (a, \mathbf{v}) \mapsto a \cdot \mathbf{v} := a\mathbf{v}. \tag{23}$$

such that the following **vector space axioms** hold:
I. $(V, +)$ is an abelian group. It holds, $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

1. Closure: $(\mathbf{u} + \mathbf{v}) \in V$.
2. Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. Neutral element (**null vector**): $\mathbf{0}$.
4. Inverse of $\mathbf{v}$: $-\mathbf{v}$.
5. Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

II. Scalar multiplication satisfy the following rules. $\forall a, b \in \mathbb{F}, \mathbf{u}, \mathbf{v} \in V$,

1. Distributivity over scalar addition: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
2. Distributivity over vector addition: $a \cdot (\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
3. Associativity over scalar multiplication: $(ab) \cdot \mathbf{v} = a(b\mathbf{v})$.
4. Neutral element: for $1 \in \mathbb{F}$, $1 \cdot \mathbf{v} = \mathbf{v}$.

Remark 3.2. The general definition of a vector space does not make references to "components" or "dimensions" of vectors. These are secondary characteristics that must be derived from the general definition.

Remark 3.3. For $a, b, c \in \mathbb{F}$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $a\mathrm{v} + b\mathbf{u} \in V$, and $a\mathrm{v} + b\mathbf{u} + c\mathbf{w} \in V$. Expressions like these are called **linear combinations** of vectors.

Remark 3.4. Sets fulfilling the above criteria are generally called "spaces", spaces of functions, spaces of matrices, etc. The smallest possible space is the **null space**, $\{\mathbf{0}\}$.

3.2 Standard vector space $\mathbb{R}^n$

As a concrete examples of vector spaces, we now discuss the higher dimensional generalisation of the familiar real vector space, the **standard vector space** $\mathbb{R}^n$.

$$\mathbb{R}^n := \left\{ \mathbf{x} := \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \middle| \; x^1, x^2, \dots, x^n \in \mathbb{R} \right\} \tag{24}$$

Remark 3.5 (Component vector)**.** The elements of $\mathbb{R}^n$ are n**-component vectors**, their i-th component[3] $(\mathbf{x})^i := x^i$, where $i = 1, 2, \dots, n$. We introduced the **column notation** of component vectors in Equation 24, oftentimes we use the in-line or the **row notation**, $\mathbf{x} := (x^1, x^2, \dots, x^n)^{\mathrm{T}}$, where "T" is the "transpose" of the matrix, and n is the "dimension" of $\mathbb{R}^n$.

Remark 3.6 (Sum of component vectors)**.** Addition of two vectors, $\mathbf{z} = \mathbf{x} + \mathbf{y}$ can be done by the sum of their components, $z^i = x^i + y^i$.

$$\boxed{+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \; (\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x} + \mathbf{y}) := \begin{pmatrix} x^1 + y^1 \\ x^2 + y^2 \\ \vdots \\ x^n + y^n \end{pmatrix}} \tag{25}$$

Example 3.7 (Adding component vectors)**.** In $\mathbb{R}^3$,

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2+(-1) \\ 5+3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 8 \end{pmatrix}$$

Remark 3.8 (Scalar multiplication of component vector)**.** Multiplication of a vector $\mathbf{x}$ with a scalar λ, $\mathbf{x}' = \lambda \cdot \mathbf{x}$ can be defined component-wise as $(\lambda \mathbf{x})^i = \lambda x^i$.

$$\boxed{\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \; (\lambda, \mathbf{x}) \mapsto (\lambda \mathbf{x}) := \begin{pmatrix} \lambda x^1 \\ \lambda x^2 \\ \vdots \\ \lambda x^n \end{pmatrix}} \tag{26}$$

Example 3.9 (Scaling component vectors)**.** In $\mathbb{R}^3$,

$$3 \cdot \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \cdot (-1) \\ 3 \cdot 2 \\ 3 \cdot 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 15 \end{pmatrix}$$

Note that vector multiplication has not been defined yet as we need *inner product.* Division of vectors is not possible since there is not multiplicative inverse.

The standard vector space $\mathbb{R}^n$ is just one of the many vector spaces encountered in physics. One of the most important spaces in quantum mechanics is the infinite-dimensional function space — *Hilbert space.*

[3]We use superscripts instead of subscripts for components because of *covariant notation*, see Section 3.4.

3.3 Examples of vector spaces

We now introduce more examples of vector spaces that are important for theoretical physics.

Example 3.10 (Standard vector spaces)**.** For the three number fields,

- Rational numbers: $\mathbb{F} = \mathbb{Q}$: $\mathbf{q} \coloneqq (q^1, \dots, q^n)^{\mathrm{T}} \in \mathbb{Q}^n$.
- Real numbers: $\mathbb{F} = \mathbb{R}$: $\mathbf{r} \coloneqq (r^1, \dots, r^n)^{\mathrm{T}} \in \mathbb{R}^n$.
- Complex numbers: $\mathbb{F} = \mathbb{C}$: $\mathbf{z} \coloneqq (z^1, \dots, z^n)^{\mathrm{T}} \in \mathbb{C}^n$.

Remark 3.11. Note that in the **complex standard vector space** $\mathbb{C}^n$, the components $z^j = x^j + iy^j$ are more "complicated" than in rational or real vector spaces.

Affine and Euclidean spaces

We will now consider the geometrically familiar flat Euclidean space.

Definition 3.12 (Affine space)**.** An **affine space** $\mathcal{A}$ over a field $\mathbb{F}$ is a triple $(P, V, +)$, consisting of a set P, whose elements are called **points**, a vector space V over $\mathbb{F}$ and a mapping:

$$+ : P \times V \to P, \quad (p, \mathbf{v}) \mapsto p + \mathbf{v} = q, \tag{27}$$

such that the following axioms are satisfied:

1. $\forall p \in P,\ p + 0_V = p$.
2. $\forall p \in P,\ \forall \mathbf{v}, \mathbf{w} \in V,\ (p + \mathbf{v}) + \mathbf{w} = p + (\mathbf{v} + \mathbf{w})$.
3. $\forall p \in P\ \forall \mathbf{v} \in V\ \exists! q \in P :\ \mathbf{v} = \vec{pq}$.
4. $\forall p, q, r \in P,\ \vec{pq} + \vec{qr} = \vec{pr}$.
5. $\mathrm{Dim}(A) = \mathrm{Dim}(V)$.

Remark 3.13. Here, 0_V is the null vector in V, addition of this vector to any point does nothing as expected. An affine space is not exactly a vector space because we don't have a null vector, hence no additive inverses. However, by choosing an arbitrary point (origin) $O \in P$, we can assign unique **position vectors** $\vec{Or}$ to each point $r \in P$. We therefore regard V as a space of position vectors of P relative to O, then the vector space is sometimes written as $V = (A, O)$.

Remark 3.14. The mapping $+$ is a different kind of addition as it doesn't apply to two vectors, rather it is translation of a point with a vector. It could be thought of as starting from a point p and joining it with another point q with a unique vector $\vec{pq} = \mathbf{v}$. The vector $\mathbf{v}$ is sometimes called the *difference vector* for the pair (p, q), since we never add two points but their difference gives us the vector.

Remark 3.15 (Dimensions)**.** The last axiom makes use of "Dim", which will only be defined in next section, can be intuitively thought of as the dimensions of the spaces.

Remark 3.16 (Euclidean space)**.** The familiar d-dimensional **Euclidean space** is an example of an affine space $A \coloneqq \mathbb{E}^d$ when $V = \mathbb{R}^d$. This is the flat-space in which we live in (without relativistic effects), by choosing a suitable reference frame (origin) O we get the real standard vector space $\mathbb{R}^d = (\mathbb{E}^d, O)$.

Function spaces

Definition 3.17 (Function space)**.** Let $f : I \to \mathbb{R},\ t \mapsto f(t)$ and $g : I \to \mathbb{R},\ t \mapsto g(t)$ be two real functions over a finite interval, $I \subset \mathbb{R}$. The set containing all such functions that are square-intergrable[4] is called $L^2(I)$. The space $(L^2(I), +, \cdot)$ is a vector space with the operations:

$$\text{Addition: } f + g : I \to \mathbb{R},\ t \mapsto (f + g)(t) \coloneqq f(t) + g(t).$$

$$\text{Scalar multiplication: } \lambda \cdot f : I \to \mathbb{R},\ t \mapsto (\lambda \cdot f)(t) \coloneqq \lambda f(t).$$

Remark 3.18. The above definition is quite intuitive and a formal presentation of function spaces can only be given much later. Intuitively, $L^2(I)$ can be thought of as a set containing functions that are vectors.

Remark 3.19 (Discretised functions)**.** To make the vectorial interpretation of functions more concrete, think of a discretised function $f : I \to \mathbb{R}$, where $I = [0, \tau]$. We discretise the inteval I into N small intervals of width τ/N, each centered on a point $t_i \in I$, $i = 1, \ldots, N$, the vector components are, $f^i = f(t_i)$. Now, the N-dimensional component vector can be written as $\mathbf{f} \coloneqq (f^1, \ldots, f^N)^{\mathrm{T}}$. Two functions can be then added component-wise as $(\mathbf{f}+\mathbf{g})^i = (\mathbf{f})^i + (\mathbf{g})^i$. Scalar multiplication is $(\lambda \cdot \mathbf{f})^i = \lambda \cdot (\mathbf{f})^i$. The set of N-step discretised functions can be represented as $\mathcal{F} \coloneqq \{\mathbf{f} \,|\, (\mathbf{f})^i \in \mathbb{R}, i = 1, \ldots, N\}$. It should be obvious that $(\mathcal{F}, +, \cdot) \cong \mathbb{R}^N$.

Remark 3.20. Heuristically, $L^2(I)$ may be interpretated as $N \to \infty$ limit of the discretisation space $\mathbb{R}^N$, and the functions are "infinitly-high"-dimensional vectors. Later on, we equip a function space with an *inner product* and it becomes the *Hilbert space*, the space where wave functions and quantum mechanics lives.

3.4 Basis and dimensions

In the previous sections, we used the notion of *dimensions* often, informally it is the number of components required to specify a vector, here we'll formally define it.

Definition 3.21 (Linear span)**.** Given a $\mathbb{F}$-vector space V and a set S containing m of its vectors[5],

$$S \coloneqq \{\mathbf{v}_1, \ldots, \mathbf{v}_m\},\ \mathbf{v}_i \in V,\ \mathbf{v}_i \neq 0. \tag{28}$$

the **linear span** or **linear hull** of S is the set of all linear combinations of its elements,

$$\mathrm{Span}(S) \coloneqq \{\mathbf{v}_1 a^1 + \mathbf{v}_2 a^2 + \cdots + \mathbf{v}_m a^m \mid a^1, \ldots, a^m \in \mathbb{F}\} \tag{29}$$

Remark 3.22 (Subspace)**.** The span(S) is itself a vector space (verify it!). In fact, span$(S) \subseteq V$, it is embedded in V and hence, it is a **subspace**, a vector space that is a subset of another vector space. Often times, span$(S) \neq V$, in that case we call it a *true* subspace and write span$(S) \subsetneq V$. Subspace of 1 or 2 dimensions are called **lines** and **planes**, respectively.

Example 3.23 (Examples of subspaces)**.** Planes in 3-d space $(m = 2, n = 3)$, lines in 3-d space $(m = 1, n = 3)$ or 2-d space $(m = 1, n = 2)$ are some visual examples of subspaces. Higher-dimensional subspaces cannot be visualised. For e.g., the space of polynomials of degree 2 is a 3-d subspace of the infinite-dimensional $L^2(I)$ space.

[4]See, Section 17.4

[5]here, we write the indices in the contravariant notation.

We want to find how large a set has to be for its linear span to be equal to the vector space.

Definition 3.24 (Linear independence)**.** A set of vectors $S \coloneqq \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is called **linearly independent** if it is not possible to find a non-trivial linear combination for the null vector.

$$(\mathbf{v}_1 a^1 + \mathbf{v}_2 a^2 + \cdots + \mathbf{v}_m a^m = 0) \Rightarrow (a^1 = a^2 = \cdots = a^m = 0) \tag{30}$$

otherwise it is said to be **linearly dependent**, hence one of the vectors can be written as the linear combinations of the others:

$$\mathbf{v}_m = -\frac{1}{a_m}(\mathbf{v}_1 a^1 + \mathbf{v}_2 a^2 + \cdots + \mathbf{v}_{m-1} a^{m-1}) \tag{31}$$

Example 3.25 (Linear independence in $\mathbb{R}^2$)**.** Consider,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent because $\mathbf{v}_1 = -(\mathbf{v}_2 + 2\mathbf{v}_3)$. But the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{v}_1, \mathbf{v}_3\}$ are linearly independent.

Remark 3.26. If S is linearly dependent, it has some redundant vectors which may be removed until we arrive at a linearly independent set as seen in the above example.

Definition 3.27 (Completeness)**.** A set S is **complete** if

$$\operatorname{span}(S) = V. \tag{32}$$

in such case, every vector in V can be written as a linear combination of vectors in S.

Definition 3.28 (Basis)**.** If a set S is both complete and linearly independent, it forms a **basis** for V. The number of elements in S is said to be the **dimension** of V.[6]

Theorem 3.29. *Every vector $\mathbf{v} \in V$ can be written as a unique **expansion** of linear combinations of basis vectors:*

$$\mathbf{v} = \mathbf{v}_1 a^1 + \mathbf{v}_2 a^2 + \cdots + \mathbf{v}_n a^n. \tag{33}$$

where, a^i are the expansion coefficients *or* components *w.r.t. the basis.*

Proof. Suppose a different linear combination for $\mathbf{v}$ existed as, $\mathbf{v} = \mathbf{v}_1 b^1 + \mathbf{v}_2 b^2 + \cdots + \mathbf{v}_n b^n$. Subtracting this from the first linear combination we get,

$$0 = \mathbf{v} - \mathbf{v} = \mathbf{v}_1(a^1 - b^1) + \cdots + \mathbf{v}_n(a^n - b^n).$$

This contradicts the fact that S is linearly independent. Therfore, $a^j = b^j$. ☐

Theorem 3.30. *For every vector space, there exists a basis .*

Remark 3.31. All bases of a given vector space contain the same number of vectors (its dimension). All bases can be expressed in terms of each other throught *basis transformation.*

[6] While it is straightforward to assign dimension to a finite-dimensional vector space, for infinite-dimensional case it is much more complicated. Thankfully, in physics we can often make them finite-dimensional without much loss of physical information, such as in the above example of $L^2(I)$ space.

Einstein summation convention

Summations involving contravariant (superscripts) and covariant (subscripts) indices can be simpler by **Einstein summation convention** (ES).

$$A_1B^1 + A_2B^2 + \cdots + A_nB^n \coloneqq \sum_{i=1}^{n} A_iB^i \coloneqq \sum_i A_iB^i \overset{\text{ES}}{\coloneqq} A_iB^i. \tag{34}$$

Remark 3.32. In the last equation, we have applied ES to pairwise repeated indices on co- and contravariant positions. This is known as **contraction** of indices. The limit of the summation is implicitly assumed to be specified by the context. The repeated indices are "dummy" indices so their name doesn't matter and can be changed at will.

Remark 3.33. An unsummed (free) index always appear on the same position on both sides of an equation. Often times we may have 2 indices on same objects such as in: $T_{\mu 1}F^1 + \cdots + T_{\mu n}F^n \coloneqq T_{\mu\nu}F^\nu$. Here, we have not summed over μ.

Using ES, we can now write Equation 33 as:

$$\mathbf{v} \overset{\text{ES}}{\coloneqq} \mathbf{v}_i a^i \coloneqq \mathbf{v}_j a^j. \tag{35}$$

This isn't just a shorthand, it helps up check for inconsistency in calculations. Sometimes inconsistency in indices may hint at deeper meaning which we will see in later sections.

Canonical basis

Definition 3.34 (Canonical basis)**.** The **canonical basis** for *standard vector spaces*, $\mathbb{R}^n$ or $\mathbb{C}^n$:

$$\{\mathbf{e}_j \mid j = 1, \ldots, n\}, \quad \mathbf{e}_j = (0, \ldots, 1, \ldots, 0)^{\mathrm{T}}. \tag{36}$$

where 1 stands at the j-th position.

Definition 3.35 (Kronecker-delta)**.** Compact notation for the i-th component using the **Kronecker-delta**,

$$(\mathbf{e}_j)^i \coloneqq \delta^i{}_j \coloneqq \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{37}$$

$$\delta^i{}_j \coloneqq \begin{pmatrix} \delta^1{}_1 & \delta^1{}_2 & \delta^1{}_3 \\ \delta^2{}_1 & \delta^2{}_2 & \delta^2{}_3 \\ \delta^3{}_1 & \delta^3{}_2 & \delta^3{}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{38}$$

Remark 3.36. Expansion of a general vector in canonical basis can now be written as

$$\mathbf{x} = \mathbf{e}_1 x^1 + \mathbf{e}_2 x^2 + \cdots + \mathbf{e}_n x^n \overset{\text{ES}}{=} \mathbf{e}_j x^j. \tag{39}$$

3.5 Vector spaces isomorphism

For this section, we adopt a caret notation, $\hat{\mathbf{v}}, \hat{\mathbf{u}}, \hat{\mathbf{w}} \in V$, for general vectors, vectors in the standard vector spaces are denoted without the hat, $\mathbf{u} \in \mathbb{F}^n$. The field $\mathbb{F}$ can be replaced with $\mathbb{R}$ or $\mathbb{C}$ and everything applies.

For a basis, $\{\hat{\mathbf{v}}_j\} \coloneqq \{\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_n\}$ of V, expansion of a general vector can be written as

$$\hat{\mathbf{u}} = \hat{\mathbf{v}}_1 u^1 + \cdots + \hat{\mathbf{v}}_n u^n \stackrel{\text{ES}}{=} \hat{\mathbf{v}}_j u^j. \tag{40}$$

By Theorem his basis defines a bijective mapping for every vector $\hat{\mathbf{u}} \in V$ to component vectors $\mathbf{u} \in \mathbb{F}^n$.

$$\phi_{\hat{\mathbf{v}}} : V \to \mathbb{R}^n, \quad \hat{\mathbf{u}} = \hat{\mathbf{v}}_j u^j \mapsto \phi_{\hat{\mathbf{v}}}(\hat{\mathbf{u}}) \coloneqq \mathbf{u} = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix}, \tag{41}$$

where the subscript in $\phi_{\hat{\mathbf{v}}}$ indicates that the map is specific for the basis $\{\hat{\mathbf{v}}_j\}$. Basis vectors in V themselves are assigned to the canonical basis $\phi_{\hat{\mathbf{v}}}(\hat{\mathbf{v}}) \stackrel{36}{=} \mathbf{e}_j$.

The mapping $\phi_{\hat{\mathbf{v}}}$ respects the rules of vector addition and scalar multiplication

$$\phi_{\hat{\mathbf{v}}}(\hat{\mathbf{u}} + \hat{\mathbf{w}}) = \phi_{\hat{\mathbf{v}}}(\hat{\mathbf{u}}) + \phi_{\hat{\mathbf{v}}}(\hat{\mathbf{w}}), \tag{42}$$

$$\phi_{\hat{\mathbf{v}}}(a \cdot \hat{\mathbf{u}}) = a \cdot \phi_{\hat{\mathbf{v}}}(\hat{\mathbf{u}}). \tag{43}$$

The key point to note here is that operations $+$ and $\cdot$ on both sides are in different vector spaces, V and $\mathbb{R}^n$. This means that $\phi_{\hat{\mathbf{v}}}$ is an *isomorphism* between $(V, +)$ and $(\mathbb{R}^n, +)$. Hence, $V \cong \mathbb{R}^n$. However, this isomorphism is basis dependent.

Because of this isomorphism we are able to do calculations in either V or $\mathbb{R}^n$.

4 Analytical Geometry

We would now like to give our vector space more geometric properties. This chapter introduces the familiar *scalar product* in full generality and then the Euclidean vector space. Caret notation for the abstract vector is adopted here as well.

4.1 Inner product space

We will define the inner product in higher generality for applications in physics.

Let V be an $\mathbb{F}$-vector space.

Bilinear forms

Definition 4.1 (Bilinear form)**.** A **bilinear form** on V is a function,

$$\beta : V \times V \to \mathbb{F}, \quad (\hat{\mathbf{u}}, \hat{\mathbf{v}}) \mapsto \beta(\hat{\mathbf{u}}, \hat{\mathbf{v}}), \tag{44}$$

that is linear in each arguments, $\forall \hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}} \in V$ and $a \in \mathbb{F}$:

1. $\beta(\hat{\mathbf{u}} + \hat{\mathbf{v}}, \hat{\mathbf{w}}) = \beta(\hat{\mathbf{u}}, \hat{\mathbf{w}}) + \beta(\hat{\mathbf{v}}, \hat{\mathbf{w}})$
2. $\beta(a\hat{\mathbf{u}}, \hat{\mathbf{v}}) = a\beta(\hat{\mathbf{u}}, \hat{\mathbf{v}})$

A bilinear form is:

1. **symmetric**: if $\beta(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = \beta(\hat{\mathbf{v}}, \hat{\mathbf{u}})$,
2. **skew-symmetric**: if $\beta(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = -\beta(\hat{\mathbf{v}}, \hat{\mathbf{u}})$,
3. **alternating**: if $\beta(\hat{\mathbf{v}}, \hat{\mathbf{v}}) = 0$.

Definition 4.2 (Sesquilinear form)**.** Let V be a complex vector space, then the function β in Equation 44 is called a **sesquilinear form**. In such case, β is **conjugate linear**:

$$\beta(\lambda\hat{\mathbf{u}} + \mu\hat{\mathbf{v}}, \hat{\mathbf{w}}) = \overline{\lambda}\beta(\hat{\mathbf{u}}, \hat{\mathbf{w}}) + \overline{\mu}\beta(\hat{\mathbf{v}}, \hat{\mathbf{w}}). \tag{45}$$

A sesquilinear form is **Hermitian** if $\beta(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = \beta\overline{(\hat{\mathbf{v}}, \hat{\mathbf{u}})}$.

Definition 4.3 (Non-degenerate)**.** A *bilinear form* β on V is **non-degenerate** if:

1. $\forall \hat{\mathbf{v}} \neq 0 \in V, \exists \hat{\mathbf{u}} \in V : \beta(\hat{\mathbf{u}}, \hat{\mathbf{v}}) \neq 0$,
2. $\forall \hat{\mathbf{u}} \neq 0 \in V, \exists \hat{\mathbf{v}} \in V : \beta(\hat{\mathbf{u}}, \hat{\mathbf{v}}) \neq 0$.

Definition 4.4 (Positive definiteness)**.** A *bilinear form* β on V is **positive definite** if:

$$\hat{\mathbf{v}} \neq 0 \Rightarrow \beta(\hat{\mathbf{v}}, \hat{\mathbf{v}}) > 0, \ \forall \hat{\mathbf{v}} \in V \tag{46}$$

Inner products

Definition 4.5 (Inner product)**.** An **inner product** or **scalar product** of V is a *positive symmetric bilinear (Hermitian sesquilinear) form*,

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}, \quad (\hat{\mathbf{u}}, \hat{\mathbf{v}}) \mapsto \langle \hat{\mathbf{u}}, \hat{\mathbf{v}} \rangle, \tag{47}$$

with the following properties: $\forall \hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}} \in V$ and $\forall a \in \mathbb{F}$,

1. *Symmetry (Hermiticity)*: $\langle \hat{\mathbf{u}}, \hat{\mathbf{v}} \rangle = \overline{\langle \hat{\mathbf{v}}, \hat{\mathbf{u}} \rangle}$,
2. *linearity (conjugate linearity)*: $\langle \hat{\mathbf{u}} + \hat{\mathbf{v}}, \hat{\mathbf{w}} \rangle = \langle \hat{\mathbf{u}}, \hat{\mathbf{w}} \rangle + \langle \hat{\mathbf{v}}, \hat{\mathbf{w}} \rangle$ and $\langle a\hat{\mathbf{u}}, \hat{\mathbf{v}} \rangle = \overline{a} \langle \hat{\mathbf{u}}, \hat{\mathbf{v}} \rangle$,
3. *positive definiteness*: $\langle \hat{\mathbf{v}}, \hat{\mathbf{v}} \rangle \geq 0$ and $\langle \hat{\mathbf{v}}, \hat{\mathbf{v}} \rangle = 0 \Leftrightarrow \hat{\mathbf{v}} = 0$.

A vector space equipped with an inner product, $(V, \langle \cdot, \cdot \rangle)$ is called an **inner product space**.

Remark 4.6. Note that the inner product is *non-degenerate*. The inner product is a generalisation of the scalar product in $\mathbb{R}^n$. If $\mathbb{F} = \mathbb{R}$, we call the vector space a **Euclidean vector space**, and property 1. becomes: $\langle \hat{\mathbf{u}}, \hat{\mathbf{v}} \rangle = \langle \hat{\mathbf{v}}, \hat{\mathbf{u}} \rangle$ (symmetry of bilinear form) and the inner product is said to be **symmetric**. If $\mathbb{F} = \mathbb{C}$, it is called a **unitary space** and the inner product is said to be **Hermitian**.

Remark 4.7. Sometimes the positive definiteness of inner product is abandoned. In such cases, we obtain inner products that are **positive semi-definite** ($\exists \hat{\mathbf{v}} \neq 0 : \langle \hat{\mathbf{v}}, \hat{\mathbf{v}} \rangle = 0$), or **positive indefinite** ($\exists \hat{\mathbf{v}} \in V : \langle \hat{\mathbf{v}}, \hat{\mathbf{v}} \rangle < 0$). Positive indefinite inner products play an important role in physics, notably in the theory of relativity.

Metric tensor

For a given basis, inner product is calcululated by the contraction of covariant basis (co-basis) and contravariant basis (con-basis) vectors, however, we have only seen co-basis vectors, to calculate con-basis we need to "raise" the indices with a *metric tensor*.

Definition 4.8 (Metric tensor)**.** Let $(V, \langle \cdot, \cdot \rangle)$ be an *inner product space*. Given a basis $\{\hat{\mathbf{v}}_\nu\}$, the **metric tensor** $g \coloneqq \{g_{\mu\nu}\}$,

$$g_{\mu\nu} \coloneqq \langle \hat{\mathbf{v}}_\mu, \hat{\mathbf{v}}_\nu \rangle \overset{47}{=} g_{\nu\mu}, \tag{48}$$

is an $n \times n$ array containing all inner products of the co-basis vectors.

Definition 4.9 (Inverse Metric tensor)**.** For a given metric tensor, $\{g_{\mu\nu}\}$, the **inverse metric tensor** $g^{\mu\nu}$ is defined through the relation:

$$g^{\mu\upsilon} g_{\upsilon\nu} = \delta^\mu{}_\nu \Leftrightarrow g_{\nu\upsilon} g^{\upsilon\mu} = \delta_\nu{}^\mu. \tag{49}$$

For a given co-basis $\{\hat{\mathbf{v}}_\nu\}$, we can construct the con-basis $\{\hat{\mathbf{v}}^\mu\}$ via the **index raising** relation defined by the inverse metric tensor:

$$\hat{\mathbf{v}}^\mu = g^{\mu\nu} \hat{\mathbf{v}}_\nu. \tag{50}$$

Using this, the inner product of two generic vectors can be written as:

$$\langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle = x^\mu g_{\mu\nu} y^\nu. \tag{51}$$

Example 4.10 (Calculating inverse metric tensor in $\mathbb{R}^n$)**.** Let the vector space $V := \mathbb{R}^2$, the basis $\{\hat{\mathbf{v}}_\mu\} := \{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2\}$, where, $\hat{\mathbf{v}}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\hat{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Then the metric tensor is:

$$g_{\mu\nu} := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_1 \rangle & \langle \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2 \rangle \\ \langle \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_1 \rangle & \langle \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_2 \rangle \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}.$$

Now, to get the inverse metric tensor $g^{\mu\nu}$, we use Equation 49

$$\mu = 1, \nu = 1 : \quad g_{11}g^{11} + g_{12}g^{21} = \delta_1{}^1 \Rightarrow 4g^{11} + 2g^{21} = 1, \tag{52}$$

$$\mu = 2, \nu = 1 : \quad g_{11}g^{12} + g_{12}g^{22} = \delta_1{}^2 \Rightarrow 4g^{12} + 2g^{22} = 0, \tag{53}$$

$$\mu = 1, \nu = 2 : \quad g_{21}g^{11} + g_{12}g^{21} = \delta_2{}^1 \Rightarrow 2g^{11} + 2g^{21} = 0, \tag{54}$$

$$\mu = 2, \nu = 2 : \quad g_{21}g^{12} + g_{22}g^{22} = \delta_2{}^2 \Rightarrow 2g^{12} + 2g^{22} = 1. \tag{55}$$

from (52) and (54), $g^{11} = \frac{1}{2}$, $g^{21} = -\frac{1}{2} = g^{12}$ (symmetry), from (53), $g^{22} = 1$.

Hence,

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

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