

Linear Algebra I

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These notes were prepared between October 2025 and (tentative) (**Last update: January 4, 2026**).

If you find any mistakes or typos, please report them to caccacpenguin@gmail.com. I would really appreciate it.

I often use informal language to make the ideas easier to grasp. My goal is to make the material feel approachable, while still respecting the rigour that makes mathematics what it is.

I hope you find these notes helpful :D!

References

These books will serve as our main references:

- Klaus Jänich. Linear Algebra. Springer-Verlag. 1994. New York.
- Siegfried Bosch. Lineare Algebra. 5. Auflage. Springer-Verlag. 2014. Heidelberg.
- Kenneth Hoffman. Ray Kunze. Linear Algebra. 2nd. Edition

Some other recommendations:

- James B. Carell. Groups, Matrices, and Vector Spaces
- B.L. van der Waerden. Modern Algebra (Vol I)
- Stephen H. Friedberg, Arnold J. Insel, Lawrence E. Spence. Linear Algebra. 4th. Ed.
- Sheldon Axler. Linear Algebra Done Right, 4th. Edition
- Serge Lang. Linear Algebra. 3rd. Edition
- Saunders MacLane. G. Birkhoff. Algebra. 1967
- Michael Artin. Algebra
- Paul R. Halmos. Finite-Dimensional Vector Spaces

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1 Introduction

Linear Algebra is one of the most well-understood theory of mathematics, it also has very wide spread applications to several fields beyond the natural sciences, computer science, engineering, finance and social sciences. Modern age machine learning algorithm and neural network are written in matrices. It probably has more applications than calculus. It is also used for modern treatment of geometry and analysis

Just like *Analysis*, the formal foundation for calculus, linear algebra is really just the formal, general and abstract foundation for concepts like vectors, matrices and determinants. It will not be much of a new subject for most of us. We sure have had experiences of doing calculations with the objects which constitutes the matrix theory. Though many concepts presented here would still be new to some of you, as we will cover more computational techniques than done in high school.

There is a more general course of algebra sometimes called *Abstract Algebra*, a lot of the concepts of linear algebra rely on the concepts from the general theory of algebraic structures. For this reason, our treatment mirrors the continental treatment of Europe and especially Germany. We will start from the elementary concepts of groups, rings, fields and modules and define vector spaces as modules over fields.

These notes present a rigorous and abstract treatment of the concepts of linear algebra in the elementary and more general infinite-dimensional cases. But we will not miss any computation techniques which can be used for several applications linear algebra has. My goal is always to give a comprehensive, historical, philosophical, and foundational treatment and show you the most elegant, modern formalism for mathematics.

The first volume covers basic algebraic structures such as groups, rings, fields, and vector spaces; systems of linear equations; linear mappings and their relationship to matrices; bases, dimension, and linear independence; determinants and eigenvalues. We will also consider many applications, though primarily to the fields of theoretical physics and computer science. Applications to differential equations or other areas of maths are not considered here.

The **prerequisites** for this course are working knowledge of **logic, sets and proof techniques**, you can refer to the notes on *Mathematical Foundations* for those topics. Apart from that, only high school mathematics is expected from the readers.

2 Algebraic Structures

The modern treatment of linear algebra begins with the discussion of the fundamental algebraic structures of groups, rings, fields and modules. These structures are defined axiomatically in terms of one or more binary operations satisfying certain properties.

2.1 Groups

2.2 Homomorphisms

2.3 Rings

2.4 Fields

2.5 Polynomials

2.6 Modules

3 Vector Spaces

3.1 Vector Spaces

3.2 Basis and Dimensions

3.3 Direct Sums

4 Matrices and Systems of Linear Equations

- 4.1 Matrix Multiplication**
- 4.2 Systems of Linear Equations**
- 4.3 Matrices and Elementary Row Operations**
- 4.4 Row-Reduced Echelon Matrices**
- 4.5 Invertible Matrices**

5 Linear Maps

5.1 Linear Maps

5.2 Quotient Space

5.3 Dual Space