Linear Algebra I

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These notes were prepared between October 2025 and (tentative) **(Last update: January 4, 2026)**.

f you find any mistakes or typos, please report them to **caccacpenguin@gmail.com**. I would really appreciate it.

I often use informal language to make the ideas easier to grasp, but it's important to keep in mind the formalism and not get too attached to the informal ideas. My goal is to make the material feel approachable, while still respecting the rigour that makes mathematics what it is.

I hope you find these notes helpful :D!

Textbook Recommendations

These books will serve as our main references:

- Klaus Jänich. Linear Algebra. Springer-Verlag. 1994. New York.
- Siegfried Bosch. Lineare Algebra. 5. Auflage. Springer-Verlag. 2014. Heidelberg.
- Kenneth Hoffman. Ray Kunze. Linear Algebra. 2nd. Edition

Some other recommendations:

- James B. Carell. Groups, Matrices, and Vector Spaces
- B.L. van der Waerden. Modern Algebra (Vol I)
- Stephen H. Friedberg, Arnold J. Insel, Lawrence E. Spence. Linear Algebra. 4th. Ed.
- Sheldon Axler. Linear Algebra Done Right, 4th. Edition
- Serge Lang. Linear Algebra. 3rd. Edition
- Saunders MacLane. G. Birkhoff. Algebra. 1967
- Michael Artin. Algebra
- Paul R. Halmos. Finite-Dimentional Vector Spaces

Contents

1 Introduction

Linear Algebra is one of the most well-understood theory of mathematics, it also has very wide spread applications to several fields beyong just the natural scinces, computer science, engineering, finance and social sciences. Modern age machine learning algorithm and neural network are written in matrices. It probably has more applications than even calculus.

Just like Analysis, the formal foundation for calculus, Linear Algebra is a not much of a new subject for most of us. We sure have had experiences of doing calculations with vectors, metrices and determinants which constitutes the matrix theory. Linear Algebra is really just the formal, general and abstract foundation for those concepts.

Though the concepts presented here would still be new to some of you, as will cover more computational techniques than done in high school.

It is also used for modern treatment of geometry and analysis which is what I do extensively in my notes, which is why the notes of Linear Algebra are extremely important for you to cover before moving on to anything.

There is a more general course of algebra sometimes called as *Abstract Algebra* and a lot of the concepts of linear algebra rely on the concepts from the general theory of algebraic structures. For that reason, my treatment mirrors the continental treatment of Europe and especially Germany. We will start from the elementary concepts of groups, rings, fields and modules and define vector spaces as modules over fields.

These notes present a rigorous and abstract treatment of the concepts of linear algebra in the elementary and more general infinite-dimensional cases. But we will not miss any computation techniques which can be used for several applications linear algebra has. My goal is always to give a comprehensive, historical, philosophical, and foundational treatment and show you the most elegant, modern formalism for mathematics.

The first volume covers the basic concepts typically presented in a one-semester course of vector spaces, linear maps, matrix theory and eigenvalue theory. We will also consider many applications, though primarily to the fields of theoretical physics and computer science. Applications to differential equations or other maths areas however are not considered here.

The **prerequisites** for this course are working knowledge of **logic, sets** and **proof techniques**, you can refer to my notes on *Mathematical Preliminaries* for those topics. Those notes cover much more than we need for this course but you can skip some sections and cover the essentials. Apart from that, only high school mathematics is expected from the readers.

2 Algebraic Structures

This chapter assumes knowledge of Sections

The modern treatment of linear algebra begins with the discussion of the fundamental algebraic structures of gropus, rings, fields and modules. These structures are defined axiomatically in terms of one or more binary operations satisfying certain properties.

2.1 Groups

2.2 Homomorphisms

2.3 Rings

2.4 Fields

2.5 Polynomials

2.6 Modules

3 Vector Spaces

3.1 Vector Spaces

3.2 Basis and Dimensions

3.3 Direct Sums

4 Matrices and Systems of Linear Equations

4.1 Matrix Multiplication

4.2 Systems of Linear Equations

4.3 Matrices and Elementary Row Operations

4.4 Row-Reduced Echelon Matrices

4.5 Invertible Matrices

5 Linear Maps

5.1 Linear Maps

5.2 Quotient Space

5.3 Dual Space